Fibonacci Numbers

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Editor's Note:

This article was inspired by a YouTube video by Stand-up Maths called "Complex Fibonacci Numbers?". The link to the original video is here .

Introduction

The Fibonacci sequence is a sequence in which each number is the sum of the two preceding ones. Starting from 0 and 1, the sequence begins:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

Moreover, this sequence can be defined using a series of equations:

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \quad , n \geq 2$$

Here, the function $F(n)$ is equal to the nth term Fibonacci number for all integer values of n greater than or equal to 2.

These equations can be implemented as a recursive algorithm that will generate a Fibonacci sequence upto the nth term. The following Python code generates the sequence upto 5th term, starting with 0 and proceeding with 1st term, 1.

In [ ]:
```
def fibonacci_sequence(n):
    if n == 0:
        return [0]
    elif n == 1:
        return [0, 1]
    else:
        sequence = [0, 1]
        for i in range(2, n+1):
            sequence.append(sequence[i-1] + sequence[i-2])

        return sequence

print(fibonacci_sequence(7))  # Change the "5" on this line to change the value of n.
```

```
[0, 1, 1, 2, 3, 5, 8, 13]
```

Different Fibonacci Numbers

Negative Fibonacci Numbers

Fibonacci sequences can also include negative numbers for nth terms less than 0:

$$.... -144, 89, -55, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, ...$$

Notice how all the odd nth terms less than 0 remain positive while the even terms less than 0 become negative. We can use this to create a new recurrence relation for negative nth terms:

$$F(-n) = (-1)^{n+1}\, F(n)$$

The Binet Formula

We can generalise the function $F(n)$ to include negative integer nth terms using the Binet formula:

$$F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where:

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2}$$

Note: φ is also known as the golden ratio , while ψ is the conjugate (or negative reciprocal) of φ.

Furthermore, we can use this formula to improve our Python code from earlier. The variables `n1` and `n2` represents the lower and uppper bounds, respectively, of the range of nth terms the function will generate. To demonstrate, I have set `n1` to equal -7 and `n2` to equal 3.

In [ ]:
```
phi = (1 + 5**0.5)/2
psi = (1 - 5**0.5)/2

def fibonacci_sequence(n1, n2):
    sequence = []
    for n in range(n1, n2 + 1):
        num = round(((phi**n) - (psi**n)) / (5**0.5))
        sequence.append(num)

    return sequence

print(fibonacci_sequence(-7, 3))  # Change the "-7" and "3" on this line to change the range of values of n.
```

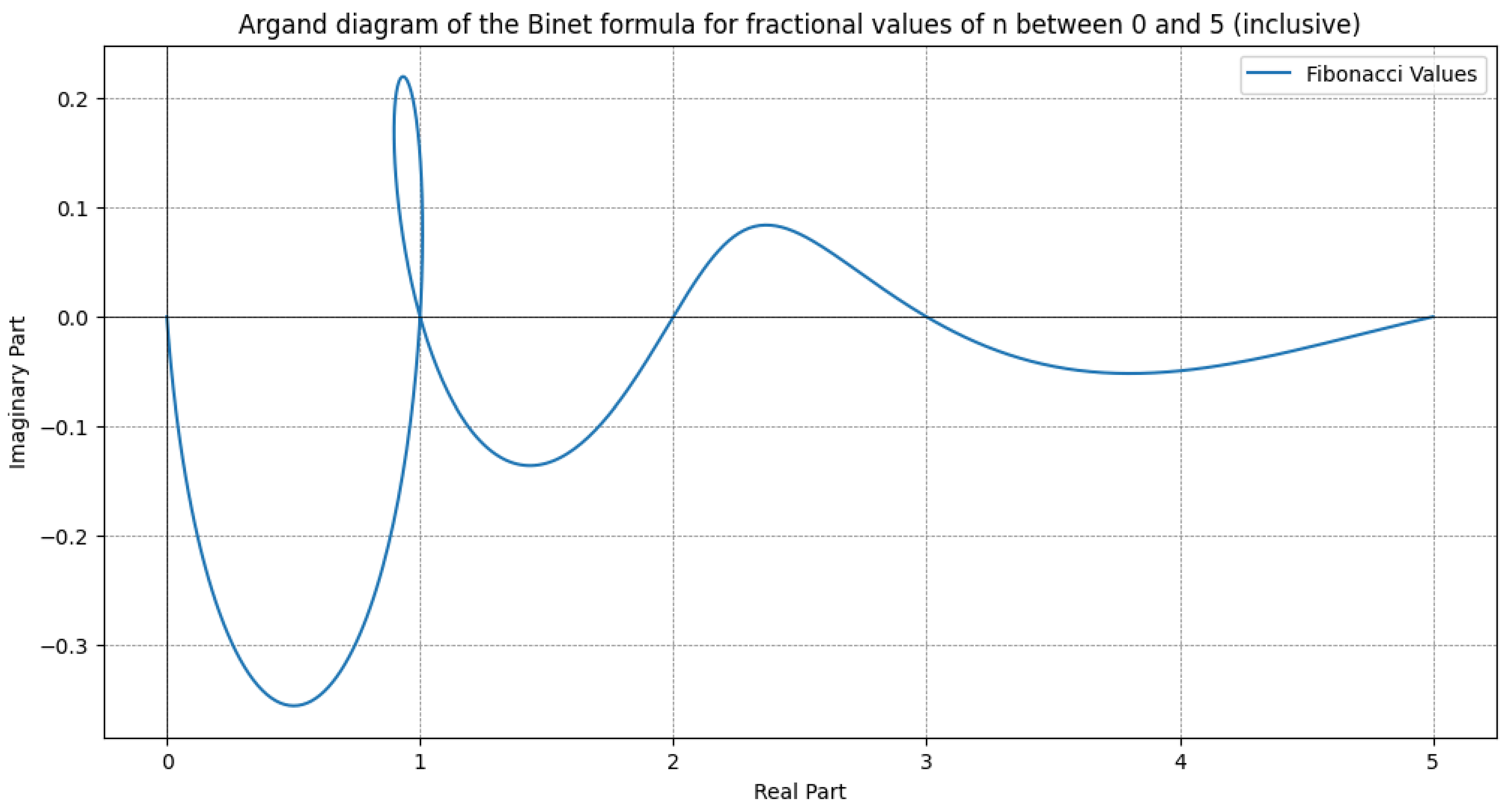
```
[13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2]
```

Complex Fibonacci Numbers

But what about using fractional values of n? Notice how ψ is a negative value as 1 is less than the square root of 5. Therefore, if we make n, lets say, 1/2, we would be square rooting a negative value which will give a complex number.

Using this knowledge, we can plot the function $F(n)$ on plot on an Argand diagram:

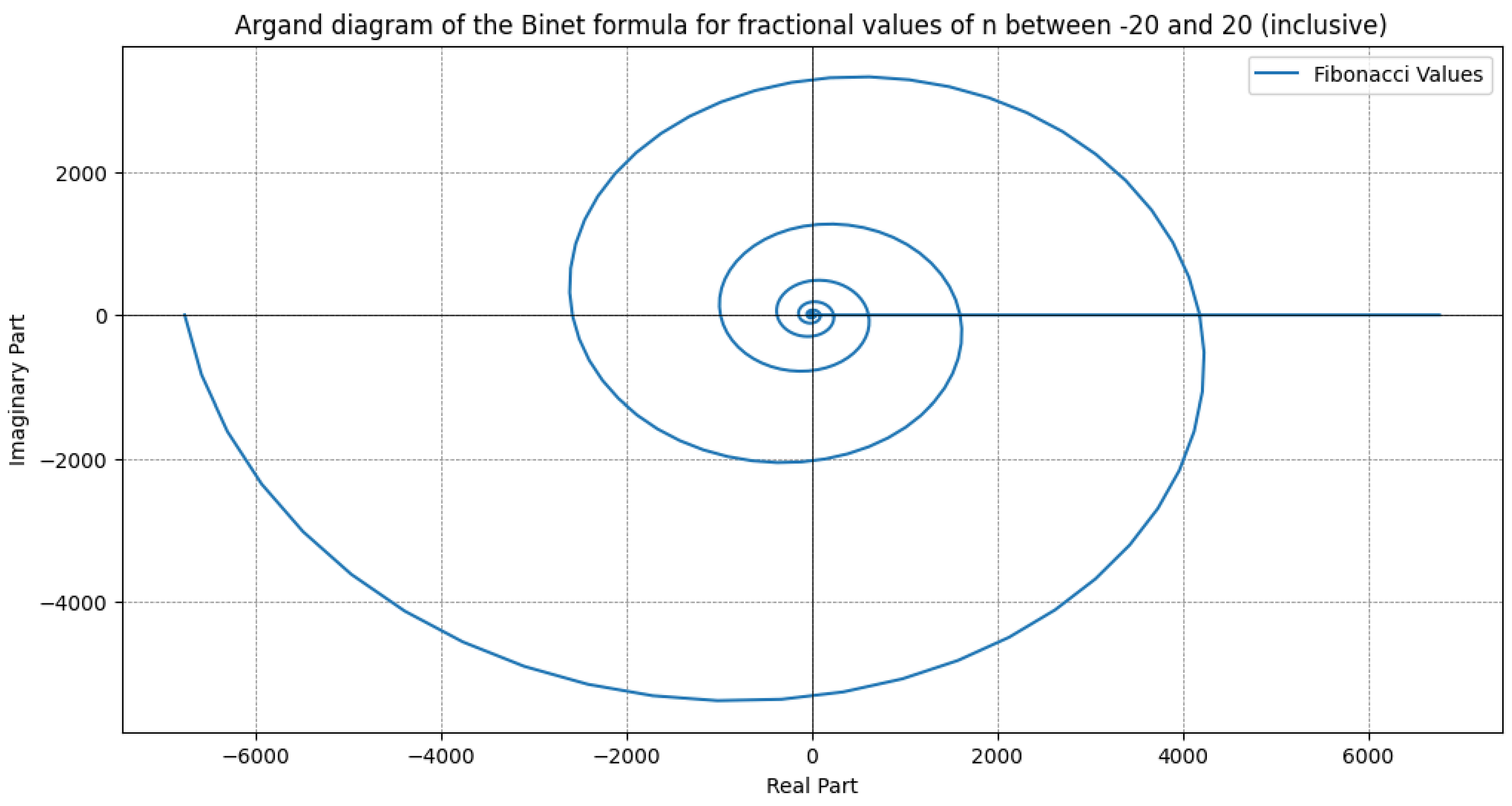
In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

# Define constants
phi = (1 + np.sqrt(5)) / 2
psi = (1 - np.sqrt(5)) / 2

def binet_formula(n):
    return (phi**(n + 0j) - psi**(n + 0j)) / np.sqrt(5)

# Generate values for n, including fractional values
n_values = np.linspace(0, 5, 400)

# Calculate Fibonacci values using the Binet formula
fibonacci_values = binet_formula(n_values)

# Plotting
plt.figure(figsize=(12, 6))

# Plotting first set of Fibonacci values
plt.plot(fibonacci_values.real, fibonacci_values.imag, label="Fibonacci Values")

plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.grid(color='gray', linestyle='--', linewidth=0.5)

plt.xlabel('Real Part')
plt.ylabel('Imaginary Part')
plt.title('Argand diagram of the Binet formula for fractional values of n between 0 and 5 (inclusive)')

plt.legend()
plt.show()
```



Notice how the values in which the graph intersects the real axis are the terms of the original Fibonacci sequence. Also notice how to graph crosses the 1 on the real axis twice, representing the two repeated 1's in the original sequence.

Generating the graph for greater and more negative values of n also gives rise to another interesting shape.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

# Define constants
phi = (1 + np.sqrt(5)) / 2
psi = (1 - np.sqrt(5)) / 2

def binet_formula(n):
    return (phi**(n + 0j) - psi**(n + 0j)) / np.sqrt(5)

# Generate values for n, including fractional values
n_values = np.linspace(-20, 20, 10**3)

# Calculate Fibonacci values using the Binet formula
fibonacci_values = binet_formula(n_values)

# Plotting
plt.figure(figsize=(12, 6))

# Plotting first set of Fibonacci values
plt.plot(fibonacci_values.real, fibonacci_values.imag, label="Fibonacci Values")

plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.grid(color='gray', linestyle='--', linewidth=0.5)

plt.xlabel('Real Part')
plt.ylabel('Imaginary Part')
plt.title('Argand diagram of the Binet formula for fractional values of n between -20 and 20 (inclusive)')

plt.legend()
plt.show()
```



Argand diagram of the Binet formula for fractional values of n between -20 and 20 (inclusive)
Fibonacci Values
Imaginary Part
Real Part
2000
0
−2000
−4000
−6000
−4000
−2000
0
2000
4000
6000