Singapore Polytechnic

School of Maths and Science

Engineering Mathematics II

MS2216 / MS4216 / MS6216

Name: ______________________________

Class: ______________________________

SINGAPORE POLYTECHNIC
SCHOOL OF MATHEMATICS & SCIENCE

Course Information

Module	:	Engineering Mathematics II
Course	:	DAPC/DCHE/DPCS/DASE/DCPE/DEB/DEEE/DARE/DME/DMRO/2FT
Session	:	2023/2024 Semester 2
L : T	:	1 (online) : 3

Synopsis

The module is designed to provide students with further knowledge in mathematics and analytical skills to solve engineering problems encountered in their studies. Among the topics covered are Calculus, Ordinary Differential Equations, Laplace Transforms and Fourier Series.

Means of Assessment

CA1	:	10 %	(weekly quizzes)
CA2	:	15 %	(in-class participation, tutorial solving, notes annotation, assignment, etc.)
TST	:	25 %	(Mid-Semester Test)
EXM	:	50 %	(Examination)

Important Dates

Mid-Semester Test	8^{th} week
Examination	19^{th} – 20^{th} week

References

1. Paul A. Calter (2007), Technical Mathematics (5th edition), Wiley.
2. John C. Peterson (2004), Technical Mathematics with Calculus, Thomson.

Lecturers' Information

Please refer to Brightspace for detailed information.

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Teaching Schedule for Semester 2

Week #	Topics	Practice	Online Quiz
1	Chapter 1: Partial Derivatives	Tutorial 1 LearningANTS	Take-home (Best of 3 attempts)
2	Chapter 2: Further Integration Techniques	Tutorial 2.1	In-class
3		Tutorial 2.2 LearningANTS	In-class
4	Chapter 3: First Order Differential Equations and Its Applications	Tutorial 3.1	In-class
5		Tutorial 3.2	-
6	Chapter 4: Integration by Parts & Simpson's Rule	Tutorial 4 LearningANTS	In-class
7	Revision for MST		
8	**Mid-Semester Test (25%)** → Chapters 1 to 3		
9 – 11	**Term Break** (3 weeks)		
12	Chapter 5: Fourier Series	Tutorial 5.1	In-class
13		Tutorial 5.2	In-class
14	Chapter 6: Laplace Transform	Tutorial 6	In-class
15	Chapter 7: Inverse Laplace Transform	Tutorial 7	In-class
16	Chapter 8: Solving Second Order Differential Equations	Tutorial 8	In-class
17	Chapter 9: Applications of Second Order Differential Equations	Tutorial 9	-
18	Revision for Exam		
19-20	**Exam (50%)** → Chapters 1 to 9		

Chapter 1: Partial Derivatives

Objectives:

1. *Review differentiation techniques and rate of change of a function with one independent variable.*
2. *Describe functions of several variables.*
3. *Define partial derivatives.*
4. *Use partial derivatives to find rate of change of a function with more than one independent variable.*

1.1 Revision on Differentiation

Differentiation allows us to find the gradient or rate of change of a function. Let us first review standard formulae and rules for basic differentiation.

Standard Derivatives

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$
$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$
$$\frac{d}{dx}\left(e^x\right) = e^x$$
$$\frac{d}{dx}\left(\sin x\right) = \cos x$$
$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$
$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$
$$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$$
$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$
$$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$
$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

Rules of Differentiation

Let $u \equiv u(x)$, $v \equiv v(x)$ and $y \equiv y(u)$

- Constant Multiple Rule $\quad \frac{d}{dx}(ku) = k\frac{du}{dx}$
- Sum Rule $\quad \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
- Product Rule $\quad \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$
- Quotient Rule $\quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- Chain Rule $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

***Example 1*:**

(a) $\frac{d}{dx}\left(5x^2 + 3x^{-\frac{1}{2}}\right)$

(b) $\frac{d}{dx}\left[\left(x^3+7\right)^{10}\right]$

(c) $\frac{d}{dx}\left(2\ln x\right)$

(d) $\frac{d}{dx}\left(\frac{1}{e^x}\right)$

(e) $\dfrac{d}{dx}\left(\dfrac{4}{x}-6+5e^{3-4x}\right)$

***Example 2*:**

(a) $\dfrac{d}{dx}\left[7\sin(5x)\right]$

(b) $\dfrac{d}{dx}\left[12\sec(3x)\right]$

(c) $\dfrac{d}{dx}\left[5\ln(\cos x)\right]$

***Example 3*:**

(a) $\dfrac{d}{dx}(x\sin x)$

(b) $\dfrac{d}{dx}\left(\dfrac{e^x}{1+x}\right)$

1.2 Rate of Change of a Function with One Independent Variable

Let's apply the differentiation rules to find the rate of change of a function with one independent varible, using the following example.

***Example 4*:** Find the rate of change of the current after 4 milliseconds if the current is given by $i = 20\cos 500t$ Amperes.

1.3 Partial Derivatives

So far, in your previous study of calculus, you have been introduced to the notion of a derivative which measures the rate of change of $f(x)$ with respect to one independent variable x.
Henceforth, we will extend it to finding the derivatives of function of **two or more independent variables.** The following are a few more examples of functions with two or more independent variables:

- Volume of a cylinder, $V = \pi r^2 h$
- Ohms law, $V = IR$
- Cobb-Douglas production function, $Q = AL^{\alpha} K^{\beta}$

Let $f(x, y)$ be a function of the two variables x and y.
To find the rate of change of $f(x, y)$ w.r.t. both x and y, the technique called **partial differentiation** will be involved.
Since $f(x, y)$ is dependent on two variables, we have to, first of all, determine how $f(x, y)$ changes with x, keeping y constant; and how $f(x, y)$ changes with y, keeping x constant. Summing up the effects, the rate of change of $f(x, y)$ w.r.t. both x and y can be evaluated.

Notations:

The *partial derivative of* $f(x, y)$ *with respect to* x is written as $\dfrac{\partial f}{\partial x}$, or simply, f_x.

$\dfrac{\partial f}{\partial x}$ is the derivative of $f(x, y)$, where y is treated as the constant and $f(x, y)$ is treated as a function of x alone.

The *partial derivative of* $f(x, y)$ *with respect to* y is written as $\dfrac{\partial f}{\partial y}$, or simply, f_y.

$\dfrac{\partial f}{\partial y}$ is the derivative of $f(x, y)$, where x is treated as the constant and $f(x, y)$ is treated as a function of y alone.

$\boxed{\textbf{Note that } \partial \neq \mathrm{d}.}$

For a function $f(x, y, z)$ with three independent variables, there are three partial derivatives:

- $\dfrac{\partial f}{\partial x}$ is found by treating y and z as constants and differentiating with respect to x.
- $\dfrac{\partial f}{\partial y}$ is found by treating x and z as constants and differentiating with respect to y.
- $\dfrac{\partial f}{\partial z}$ is found by treating x and y as constants and differentiating with respect to z.

Example 5: Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ of the following functions:

(a) $f(x,y) = (5x^3)y^2$

(b) $f(x,y) = (4x+3y-5)^8$

(c) $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$

<u>More notations:</u>

The value of $\dfrac{\partial f}{\partial x}$ at a point $(x, y) = (a, b)$ is denoted by $\left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}}$ or $f_x(a,b)$.

The value of $\dfrac{\partial f}{\partial y}$ at point $(x, y) = (a, b)$ is denoted by $\left.\dfrac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}}$ or $f_y(a,b)$.

Example 6: Given that $f(a,b) = \dfrac{a-b}{a+b}$, evaluate $\dfrac{\partial f}{\partial a}$ and $\dfrac{\partial f}{\partial b}$ when $a = 1$ and $b = 1$.

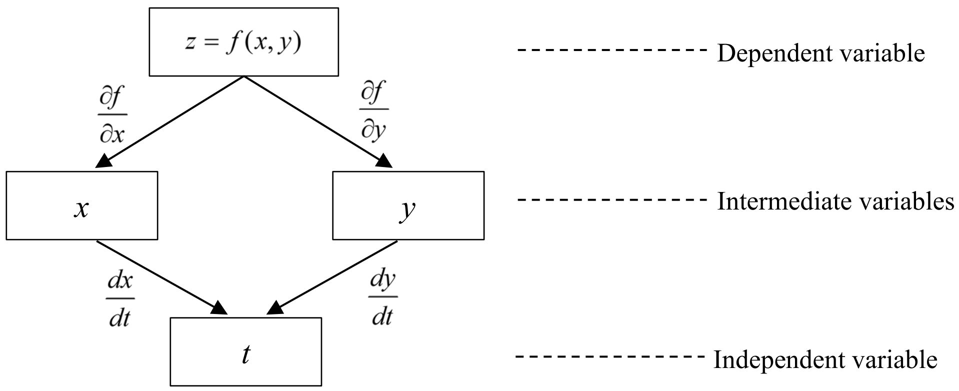
1.4 Rate of Change of a Function with More Than One Independent Variable

Sometimes, to find the derivative $\frac{dz}{dt}$, which is the rate of change of $z = f(x, y)$ with respect to t, where $x = g(t)$ and $y = h(t)$, this could generate functions whose formulae are too complicated for convenient substitution or for which formulae are not readily available.

To find derivative under circumstances like this, we will use **Chain Rule**. The Chain Rule for a function $z = f(x, y)$, expressed in terms of two variables, is given by:

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}$$

An easy way to remember this rule is to use a diagram showing the relationships between variables:



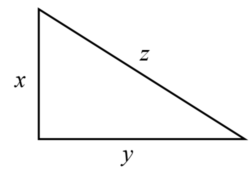
The Chain Rule for a function $w = f(x, y, z)$, expressed in terms of three variables, is given by:

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}}$$

(Sketch a similar diagram to depict this chain rule.)

***Example 7*:** If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos 2t$, find $\frac{dz}{dt}$ when $t = 0$.

***Example 8*:** In the right-angle triangle shown below, x is increasing at a rate of 3 cm/s , while y is decreasing at a rate of 4 cm/s. Calculate the rate at which z is changing when $x = 3$ cm and $y = 5$ cm .



***Example 9*:** The pressure P (in kilo-Pascals), volume V (in litres) and temperature T (in kelvins) of a mole of ideal gas are related by the equation $PV = 8.31T$.

Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of $0.1\ \text{K/s}$, given that the volume is 100 L and increasing at a rate of 0.2 L/s.

Tutorial 1

Section A: Revision on Differentiation

1. Differentiate the following with respect to x and simplify your answer wherever possible:

(a) $5x^4 + \dfrac{1}{2x^2}$ (b) $\sqrt{x} + 8$ (c) $7\sin 5x$

(d) $2\cos 3x$ (e) $\ln(7x^3 + 5)$ (f) $2\ln(e^x + 1)$

(g) e^{3x} (h) $2e^{\cos 2x}$ (i) $(x^3 + 7)^{10}$

(j) $\sqrt{5x - 7}$

2. Differentiate the following with respect to x and simplify your answer wherever possible:

(a) $x^4 \sin 2x$ (b) $e^x \ln x$ (c) $x^3 \ln x$

(d) $e^x \sin 2x$ (e) $\dfrac{2x+1}{x-3}$ (f) $\dfrac{2x}{x+3}$

(g) $\dfrac{e^x}{x}$ (h) $\dfrac{3x}{(1+\sin x)}$

Section B: Partial Derivatives

1. The volume V of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius and h is the height.

(a) Find the formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.

(b) Find the formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.

(c) Suppose that h has a constant value of 4 cm. but r varies. Find the rate of change of V with respect to r at the point where $r = 6$ cm.

(d) Suppose that r has a constant value of 8 cm but h varies. Find the instantaneous rate of change of V with respect to h at the point where $h = 10$cm.

2. Find partial derivatives of the following functions:

(a) $f(x, y) = 5x^2 - 3y^2 + 10$ (b) $f(x, y) = 4x^3 + y^3 - 5x^2 y$

(c) $z(x, y) = \sin x + \cos y$ (d) $z(x, y) = x^2\sqrt{1 - y^2}$

(e) $z = \ln(xy)$ (f) $u = \ln\sqrt{x^2 - y^2}$

(g) $z(x, y) = x^2 \sin y$ (h) $z = \left(1 + x^2 y\right)e^{3y}$

(i) $f(x, y) = x^2 \sin(xy) - 3y^3$ (j) $f(r, s) = r \cdot \ln\left(r^2 + s^2\right)$

3. Evaluate the indicated partial derivatives:

   (a) $f(x,y)=\sqrt{x^2+y^2}$, $f_x(3,4)$ (b) $f(x,y)=\dfrac{x}{y+1}$, $f_y(3,2)$

4. Determine the slope of the tangent line in the $x-$ and $y-$directions to the surface $z(x, y)$ at the stated point P. [Hint: slope of the tangent line in the $x-$direction is $\dfrac{\partial z}{\partial x}$]

   (a) $z(x, y)=x^3+2xy+2y^2$, $P(1, 2)$ (b) $z(x, y)=x\sin(xy)+3$, $P(1, \frac{\pi}{2})$

5. Find partial derivatives of the following functions:

   (a) $f(x, y, z)=2x^3y+z^2$ (b) $f(x, y, z)=xyz+xy+z$

6. The voltage V in a circuit that satisfy the law $V = IR$ is slowly dropping as battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Find how the current is changing at the instant when R = 600 Ohm, I = 0.04 Amperes, $\dfrac{dR}{dt}=0.5$ Ohm/s, and $\dfrac{dV}{dt}=-0.01$ Volt/s.

7. The total surface area S of a closed cylinder is given by $S=2\pi r^2+2\pi rh$, where r is its radius and h is its height. The radius r of a closed cylindrical can decreased at a rate of 0.02 cm/s , when r = 5 cm and h = 15 cm. Find the rate of change of h such that the surface area of the can remains unchanged.

8. The total surface area S of a cone with base radius r and perpendicular height h is given by:

$$S=\pi r^2+\pi r\sqrt{r^2+h^2}$$

   If r and h are each increasing at the rate of 0.25 cm/s, find the rate at which S is increasing at the instant when $r = 3$ cm and $h = 4$ cm.

9. The power P dissipated in a resistor is given by $P=\dfrac{E^2}{R}$, where E is voltage and R is resistance. If E = 200 Volts and is decreasing at a rate of 0.025 Volts/s , and R = 8 Ω and is increasing at a rate of 0.05 Ω/s, find the rate at which the power is changing.

10. The volume V of a right circular cylinder is given by $V=\dfrac{1}{4}\pi D^2h$, where D is its diameter and h is its height. The diameter and height of a right circular cylinder are found by measurement to be 8 cm and 12.5 cm respectively. If the diameter decreases at a rate of 0.1 cm/s and the height increases at a rate of 0.2 cm/s, use partial differentiation to find the rate at which the volume changes.

Miscellaneous Exercises

1. Two straight roads intersect at right angle. Car A, moving on one of the roads, approaches the intersection at 55 km/h, whereas car B, moving on the other road, approaches the intersection at 70km/h. At what rate is the distance between the cars decreasing when A is 3 km from the intersection and B is 4 km from the intersection?

2. The total volume of a cone is given by $V = \frac{\pi}{3}h^3\tan^2(\theta)$, where V (in m^3) is the volume of a cone, θ (in radians) is the semi-vertical angle, and h (in m) is height. If $\frac{dh}{dt} = 0.001\,\text{m/s}$ and $\frac{d\theta}{dt} = 0.002$ rad/s, find the rate of change of V when $\theta = \frac{\pi}{4}$ and $h = 0.20$ m.

3. The time rate Q of flow of fluid through a cylindrical tube (such as a windpipe) with radius r and length l is given by $Q = \frac{\pi p r^4}{8l\eta}$, where η is the viscosity of the fluid and p is the difference in the pressure at the two ends of the tube. Suppose the length of the tube remains constant, while the radius increases at the rate of $\frac{1}{10}$ m/s, and the pressure decreases at the rate of $\frac{1}{5}$ $\text{N/m}^3\text{/s}$, find the rate of change of Q with respect to time t.

4. A container with molten metal has a mass of m (kg). A force F (N) acting on an angle θ (rad) measured from the vertical is required to tilt the container. Given that $F = \frac{2.5m}{1+0.8\tan\theta}$,
   (a) find $\frac{\partial F}{\partial m}$ and $\frac{\partial F}{\partial \theta}$.
   (b) If θ is increasing at $\frac{\pi}{90}$ rad/s and m is decreasing at 1.2 kg/s, find the rate of change of F when $\theta = \frac{\pi}{4}$ rad and $m = 100$ kg.

5. For a real gas, van der Waal's equation states that:
$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$
where P is the pressure of the gas (in Nm^2), V is the volume of the gas (in m^3), T is the temperature of the gas (in Kelvin), n is the number of moles of gas, R is the universal gas constant, and a and b are constants. Determine $\frac{\partial P}{\partial V}$ and $\frac{\partial T}{\partial P}$.

6. The magnitude of the resultant force R of two forces P and Q acting on an object and inclined at an angle θ is given by:

$$R = \sqrt{P^2 + 2PQ\cos\theta + Q^2}$$

(a) Show that $\dfrac{\partial R}{\partial P} = \dfrac{P + Q\cos\theta}{R}$.

(b) Find $\dfrac{\partial R}{\partial \theta}$.

(c) Suppose the force Q remains constant at 15 N, find the rate at which the resultant force R is changing if the force P is increasing at a rate of 0.2 N/s and θ is decreasing at a rate of 0.2 rad/s at the instant when P = 25 N and $\theta = \dfrac{\pi}{3}$.

Multiple Choice Questions

1. Given that $f(x, y) = 0$, then $\dfrac{dy}{dx}$ equals to ___________.

(a) $\dfrac{f_x}{f_y}$ (b) $\dfrac{f_y}{f_x}$

(c) $-\dfrac{f_x}{f_y}$ (d) $-\dfrac{f_y}{f_x}$

2. If partial differentiation is performed on a function, then this function must have

(a) only one independent variable.

(b) more than one dependent variable.

(c) two or more independent variables.

(d) equal number of dependent and independent variables.

Answers to Selected Lecture Examples

Example 1: (a) $10x-\frac{3}{2}x^{-3/2}$ (b) $30x^2(x^3+7)^9$ (c) $\frac{2}{x}$ (d) $-e^{-x}$ (e) $-4x^{-2}-20e^{3-4x}$

Example 2: (a) $35\cos 5x$ (b) $36\sec 3x\tan 3x$ (c) $-5\tan x$

Example 3: (a) $x\cos x+\sin x$ (b) $\frac{xe^x}{(1+x)^2}$

Example 4: -9092.97 Amperes/s

Example 5: (a) $15x^2y^2$, $10x^3y$ (b) $32(4x+3y-5)^7$, $24(4x+3y-5)^7$

(c) $\frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$, $-\frac{x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$

Example 6: $\frac{1}{2}$, $-\frac{1}{2}$

Example 7: $\left.\frac{dz}{dt}\right|_{t=0}=6$

Example 8: -1.89 cm/s

Example 9: - 0.042 kPa/s
(decreasing at 0.042 kPa/s)

Answers to Tutorial 1

Section A

1. (a) $20x^3-\frac{1}{x^3}$ (b) $\frac{1}{2\sqrt{x}}$ (c) $35\cos 5x$ (d) $-6\sin 3x$

(e) $\frac{21x^2}{7x^3+5}$ (f) $\frac{2e^x}{e^x+1}$ (g) $3e^{3x}$ (h) $-4e^{\cos 2x}\sin 2x$

(i) $30x^2(x^3+7)^9$ (j) $\frac{5}{2\sqrt{5x-7}}$

2. (a) $2x^3(x\cos 2x+2\sin 2x)$ (b) $e^x\left(\frac{1}{x}+\ln x\right)$ (c) $x^2(1+3\ln x)$

(d) $e^x(\sin 2x+2\cos 2x)$ (e) $-\frac{7}{(x-3)^2}$ (f) $\frac{6}{(x+3)^2}$

(g) $\frac{e^x(x-1)}{x^2}$ (h) $\frac{3(1+\sin x-x\cos x)}{(1+\sin x)^2}$

Section B

1. (a) $2\pi rh$ (b) πr^2 (c) 48π (d) 64π

2. (a) $f_x(x,y)=10x$; $f_y(x,y)=-6y$

(b) $f_x(x,y)=12x^2-10xy$; $f_y(x,y)=3y^2-5x^2$

(c) $z_x(x,y)=\cos x$; $z_y(x,y)=-\sin y$

(d) $z_x(x,y)=2x\sqrt{1-y^2}$; $z_y(x,y)=\dfrac{-x^2y}{\sqrt{1-y^2}}$

(e) $\dfrac{\partial z}{\partial x}=\dfrac{1}{x}$; $\dfrac{\partial z}{\partial y}=\dfrac{1}{y}$

(f) $\dfrac{\partial u}{\partial x}=\dfrac{x}{x^2-y^2}$; $\dfrac{\partial u}{\partial y}=\dfrac{-y}{x^2-y^2}$ *or* $\dfrac{y}{y^2-x^2}$

(g) $z_x(x,y)=2x\sin y$; $z_y(x,y)=x^2\cos y$

(h) $\dfrac{\partial z}{\partial x}=2xye^{3y}$; $\dfrac{\partial z}{\partial y}=e^{3y}\left(3+x^2+3x^2y\right)$

(i) $f_x(x,y)=2x\sin(xy)+x^2y\cos(xy)$; $f_y(x,y)=x^3\cos(xy)-9y^2$

(j) $f_r(r,s)=\ln\left(r^2+s^2\right)+\dfrac{2r^2}{r^2+s^2}$; $f_s(r,s)=\dfrac{2rs}{r^2+s^2}$

3. (a) $\dfrac{3}{5}$ (b) $-\dfrac{1}{3}$

4. (a) $z_x(1,2)=7$; $z_y(1,2)=10$ (b) $z_x(1,\frac{\pi}{2})=1$; $z_y(1,\frac{\pi}{2})=0$

5. (a) $f_x(x,y,z)=6x^2y$; $f_y(x,y,z)=2x^3$; $f_z(x,y,z)=2z$

(b) $f_x(x,y,z)=yz+y$; $f_y(x,y,z)=xz+x$; $f_z(x,y,z)=xy+1$

6. –0.00005 Amperes/s 7. 0.1 cm/s 8. 11.94 cm²/s

9. –32.5 W/s 10. –5.65 cm^3/s

Miscellaneous

1. 89 km/h 2. 0.00019 m^3/s 3. $\dfrac{\pi r^3}{40l\eta}(2p-r)$ m^3/s

4. (a) $\dfrac{2.5}{1+0.8\tan\theta}$; $-\dfrac{2m\sec^2\theta}{(1+0.8\tan\theta)^2}$ (b) –5.98 N/s

5. $-\dfrac{nRT}{(V-nb)^2}+\dfrac{2n^2a}{V^3}$ N/m^5 ; $\dfrac{1}{nR}(V-nb)$ Kelvin m^2/N

6. (b) $-\dfrac{PQ\sin\theta}{\sqrt{P^2+2PQ\cos\theta+Q^2}}$ or $-\dfrac{PQ\sin\theta}{R}$ (c) 2.05 N/s

MCQ

1. (c) 2. (c)

Chapter 2: Further Integration Techniques

Objectives:

1. *Integrate functions of a linear function.*
2. *Integrate trigonometric functions using trigonometric identities.*
3. *Derive root-mean-square value of a given function.*
4. *Integrate rational functions by resolving into partial fractions.*
5. *Integrate by using an appropriate substitution.*

2.1 Revision on Integration

2.1.1 Integration

Integration is the process of finding anti-derivatives:

If $\dfrac{d}{dx}\big(F(x)\big) = f(x)$,

then $\int f(x)\,dx = F(x) + C$, where C is an arbitrary constant.

2.1.2 Standard Integrals

Standard Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$$

2.1.3 Properties of Indefinite integral

- $\int k \cdot f(x)\,dx = k\int f(x)\,dx$, where k is a constant.
- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

***Example 1*:**

(a) $\int \left(x - \frac{1}{x^2} - 3 \right) dx$

(b) $\int \left(\frac{5}{x} + \sqrt{x} - 5e^{2x} \right) dx$

(c) $\int \left(\sin x - \cos x + 3\sec^2 x \right) dx$

(d) $\int \tan x \left(1 + \sec x \right) dx$

2.1.4 Definite Integrals

If $\int f(x)dx = F(x) + C$, then $\int_a^b f(x)\ dx = F(b) - F(a)$

Note that in definite integral, whenever trigonometric functions, such as $\sin x$ or $\cos x$ is involved, the limits for x are measured in **radians**. Therefore, when evaluating a definite integral involving trigonometric function, your calculator has to be in radian mode.

***Example 2*:**

Evaluate the following:

(a) $\int_0^2 (x+2)\,dx$

(b) $\int_0^{0.5} \left(3e^{-2t} - \frac{1}{2}\cos \pi t \right) dt$

Note

This is a wrong move!

$$\int \cos(\pi t)\ dt = \frac{\sin(\cancel{\pi} t)}{\cancel{\pi}} + C$$

$$\neq \sin(t) + C$$

2.2 Integration of Functions of a Linear Function

2.2.1 Linear Function

A function $f(x) = ax + b$, where a and b are constants and $a \neq 0$, is known as a linear function of x. For examples, $2x + 1$, $-5x + 1$, $\frac{x}{\sqrt{2}} - 3$ are all linear functions of x.

2.2.2 Functions of a Linear Function

Functions of a linear function are functions that are expressed in terms of $(ax + b)$. For examples:

- $(5x+2)^3$ is a cubic function of the linear function $5x + 2$
- $\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$ is a cosine function of the linear function $\frac{x}{2} - \frac{\pi}{4}$, also written as $\frac{1}{2}x - \frac{\pi}{4}$

2.2.3 Integration of Functions of a Linear Function

By reversing the process of differentiation of functions of a linear function, the following results can be obtained:

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad , \quad n \neq -1$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

Let's demonstrate with the first result.
Consider the differentiation of the function $(ax+b)^{n+1}$:

$$\begin{aligned} \frac{d}{dx}(ax+b)^{n+1} &= (n+1)\cdot(ax+b)^n \cdot \frac{d}{dx}(ax+b) \\ &= (n+1)\cdot(ax+b)^n \cdot a \\ &= a(n+1)\cdot(ax+b)^n \end{aligned}$$

The **reverse process** (i.e. integration) gives:

$$\int a(n+1)\cdot(ax+b)^n \, dx = (ax+b)^{n+1} + C_1$$

$$a(n+1)\int (ax+b)^n \, dx = (ax+b)^{n+1} + C_1$$

$$\therefore \int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad , \quad n \neq -1$$

Example 3:

(a) $\int (3x+1)^9 \, dx$

(b) $\int \frac{1}{(3x+4)^2} \, dx$

(c) $\int \frac{2}{\sqrt{1-3u}} \, du$

Example 4:

(a) $\int_0^1 \frac{1}{3x+4} dx$

(b) $\int \frac{2}{1-4x} dx$

(c) $\int \frac{x+2}{x+1} \, dx$

Example 5:

(a) $\int e^{7x+2} \, dx$

(b) $\int \frac{dx}{e^{3+x}}$

(c) $\int \sqrt{e^{x+3}} \, dx$

2.2.4 Summary

Generally,

If	$\int f(x)\, dx = F(x) + C \;\leftarrow \{\text{Standard Integral}\}$
Then	$\int f(ax+b)\, dx = \frac{1}{a} \cdot F(ax+b) + C$

That is, the steps to integrate function of a linear function are outlined as follow:

1. Apply the standard integral, replacing "x" with "$ax+b$"
2. Include the extra factor "$\frac{1}{a}$" in the result

Example 6: Find $\int 3^{2x+7}\, dx$.

Standard Integral (given in formulae card)

$\int k^x \;\; dx = \frac{k^x}{\ln k} + C$, where k is a constant

Example 7:

Find the following integrals:

(a) $\int \cos(2x+\pi)\, dx$

Standard Integral
(given in formulae card)

$\int \cos(x)\, dx = \sin(x) + C$

$\int \sin(x)\, dx = -\cos(x) + C$

$\int \sec^2(x)\, dx = \tan(x) + C$

(b) $\int_0^1 \sin(3x+1)\, dx$

(c) $\int 2\sec^2(\pi t - 1)\, dt$

2.3 Integration using Trigonometric Identities

2.3.1 Integrals of Product of Sine and Cosine Functions

Apply the following **Product to Sum Identities** so that linearity property can be used in integration:

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$

Remember also that:

$$\sin(-A) = -\sin A \quad \text{and} \quad \cos(-A) = \cos A$$

Example 8: Find $\int \sin 2x \cos 3x \, dx$.

2.3.2 Integrals of Even Powers of Sine and Cosine Functions

If the integral is of the form $\int \sin^m x \cos^n x \, dx$ such that both m and n are even numbers, we use the **Formulae for Reducing Power** so that linearity property can be used in integration:

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Note that one of the exponents, m or n , may be zero.

Example 9: Find $\int 5\sin^2 3x \, dx$.

Example 10: Find $\int \sin^2 x \cos^2 x \, dx$.

2.4 Application: Root-Mean-Square (RMS) Value

The **root-mean-square (rms) value** of a function $y = f(x)$ over an interval $x = a$ to $x = b$ is defined as:

$$\boxed{y_{rms} = \sqrt{\frac{1}{b-a}\int_a^b \left[f(x)\right]^2 dx}}$$

Note that y_{rms} is non-negative.

Example 11: Find the rms value of the function $y = 2x + 1$ over the interval $x = 1$ to $x = 4$.

Example 12: Find the rms value of the voltage $v = 3\sin 2t$.

2.5 Integration of Rational Functions

2.5.1 Proper and Improper Fractions

A function of the form $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ is a polynomial in x of degree n. Given the polynomial $f(x)$ of degree n, and another polynomial $g(x)$ of degree m, then the **rational expression** $\dfrac{f(x)}{g(x)}$ is a **proper** fraction if $n < m$, and an **improper** fraction if $n \geq m$.

2.5.2 Partial Fractions

A **proper fraction** $\dfrac{f(x)}{g(x)}$ can be expressed as a sum of simpler fractions if the denominator $g(x)$ can be factorised. These simpler fractions are called **partial fractions**. Each partial fraction corresponds to a factor of $g(x)$.

The rules of partial fractions are as follows:

Rule 1 The fraction $\dfrac{f(x)}{g(x)}$ must be a proper fraction. (If it is not, then first divide out by long division.)

Rule 2 Factorise the denominator $g(x)$ into its prime factors. This is important since the factors obtained determine the form of the partial fractions.

Rule 3 Corresponding to each **linear factor** $\boxed{ax+b}$ in the denominator, there is a partial fraction of the form $\boxed{\dfrac{A}{ax+b}}$.

For example, $\dfrac{3x-2}{(x+1)(2x-1)}=\dfrac{A}{x+1}+\dfrac{B}{2x-1}$. The constants A and B can then be determined by "Cover-up" method or substitution method.

Example 13:

Find the following integrals using partial fractions:

(a) $\displaystyle\int \frac{3}{(x+1)(x-2)}\,dx$

(b) $\displaystyle\int \frac{5x+3}{x^2-3x}\,dx$

2.6 Integration by Substitution

Look at the following integrals:

- $\int 3x^2(x^3+1)^8\,dx$
- $\int \dfrac{3x^2-1}{x^3-x}\,dx$
- $\int x\,e^{x^2}\,dx$

These integrals might look complicated, but they can be integrated using the technique of ***"Integration by Substitution"***.

Integration by substitution enables us to reduce a given integral to one with which we are familiar. The technique is very powerful and covers a great range of problems. Unfortunately, it is not possible to give a general rule for choosing the required substitution, but this will come with experience gained through practice.

2.6.1 Differential of a Function

The **differential** of $y = f(x)$ is defined as:

$$\boxed{dy = \frac{dy}{dx}\cdot dx} \qquad \text{or} \qquad \boxed{dy = f'(x)\cdot dx}$$

Example 14: Find the differential of the following functions:

(a) $y = 4x^2 + 3x - 7$ (b) $u = 3\sin 4t$

Solution:

(a) $\dfrac{dy}{dx} =$

Hence, the differential of y is

(b) $\dfrac{du}{dt} =$

Hence, the differential of u is

2.6.2 Integration by Substitution of the form $\int [f(x)]^n \cdot f'(x)\,dx$

We notice that one function of the product is the differential coefficient of the other function. We can solve the problem by using a substitution so that the integral becomes a standard integral.

Let $u = f(x)$, then $\dfrac{du}{dx} = f'(x)$.

Expressing in differential form, we get : $du = f'(x)\,dx$

Substitute x by u completely: $\int \left[f(x)\right]^n \cdot f'(x)\,dx = \int u^n\,du$ (this is a standard integral)

$$= \frac{u^{n+1}}{n+1} + C, \text{ where } n \neq -1$$

Finally, we substitute back $u = f(x)$ to get the following result:

$$\boxed{\int \left[f(x)\right]^n \cdot f'(x)\,dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + C \quad , \text{ where } n \neq -1}$$

Example 15: Find $\int \left(x^2+3\right)^5 2x\,dx$.

Solution:

Choose substitution → Let $u =$

Find differential → $du =$

Substitute x by u completely → $\int \left(x^2+3\right)^5 2x\,dx =$

(Integrate, then substitute x back)

Example 16: Find the following integrals by choosing appropriate substitutions:

(a) $\int 3x\sqrt{1-2x^2}\,dx$ (b) $\int (e^x+1)^3\,e^x\,dx$

2.6.3 Integration by Substitution of the form $\int \frac{f'(x)}{f(x)}\, dx$

Let us look at an integral in which the numerator is the differential of the denominator.

Let $u = f(x)$, then $du = f'(x)\,dx$.

Substitute x by u completely, $\int \frac{f'(x)}{f(x)}\, dx = \int \frac{1}{u}\, du = \ln|u| + C$ (standard integral)

That is:

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C}$$

***Example 17*:** Find the following integrals by choosing appropriate substitutions:

(a) $\int \frac{2x^2}{x^3 - 4}\, dx$ (b) $\int \frac{e^{2x}}{e^{2x} + 1}\, dx$

2.6.4 Integration by Substitution of the form $\int e^{f(x)} \cdot f'(x)\,dx$

Let $u = f(x)$, then $du = f'(x)\,dx$.

Substitute x by u completely, $\int e^{f(x)} f'(x)\,dx = \int e^u\, du = e^u + C$ (standard integral)

That is:

$$\boxed{\int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)} + C}$$

***Example 18*:** Find $\int 3x^2 e^{x^3}\, dx$.

2.6.5 Summary of Steps

Integration by substitution can be summarised in the following steps:

	Recommended Procedure	Example: $\int 6x^2\left(2x^3-3\right)^7 dx$	Example: $\int \frac{6x^2}{2x^3-3}\,dx$	Example: $\int 6x^2 e^{2x^3-3} dx$
Step 1	Choose u as some expression that appears in the integrand. (This may require some trial and error to find the correct expression for u.)	Let $u=2x^3-3$	Let $u=2x^3-3$	Let $u=2x^3-3$
Step 2	Find $\frac{du}{dx}$ and obtain the differential of u.	$\frac{du}{dx}=6x^2$ $\Rightarrow du=6x^2dx$	$\frac{du}{dx}=6x^2$ $\Rightarrow du=6x^2dx$	$\frac{du}{dx}=6x^2$ $\Rightarrow du=6x^2dx$
Step 3	Substitute the values of u and du into the original integral.	$\int 6x^2\left(2x^3-3\right)^7 dx$ $=\int\left(2x^3-3\right)^7\cdot 6x^2dx$ $=\int u^7du$	$\int \frac{6x^2}{2x^3-3}\,dx$ $=\int \frac{1}{2x^3-3}\cdot 6x^2dx$ $=\int \frac{1}{u}\,du$	$\int 6x^2e^{2x^3-3}dx$ $=\int e^{2x^3-3}\cdot 6x^2dx$ $=\int e^u du$
Step 4	Integrate with respect to u (using standard formulae).	$=\frac{u^8}{8}+C$	$=\ln\lvert u\rvert+C$	$=e^u+C$
Step 5	Write the answer in terms of x.	$=\frac{\left(2x^3-3\right)^8}{8}+C$	$=\ln\left\lvert 2x^3-3\right\rvert+C$	$=e^{2x^3-3}+C$

The steps summarised above can also be applied to integration of trigonometric functions.

***Example 19*:** Find $\int x^2\cos\left(2x^3-3\right)dx$.

2.6.6 Integration by Substitution & the Definite Integral

When evaluating a definite integral involving substitution (i.e. change of variable from x to u), it is necessary to change the limits for x to the corresponding values of u.

***Example 20*:** Evaluate $\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

Tutorial 2.1

Section A: Basic Integration

1. Find the following integrals:

(a) $\int \left(x - \frac{1}{x^2} \right) dx$ (b) $\int \left(e^{5x} + \frac{3}{e^{3x}} \right) dx$ (c) $\int \frac{x}{3}(2x + \sqrt{x})\, dx$

(d) $\int \left(x^2+2\right)\left(4x-3\right) dx$ (e) $\int e^x \left(2e^x + \frac{1}{e^{3x}} \right) dx$ (f) $\int 2 \tan 3x \, dx$

(g) $\int \cot 6x \, dx$ (h) $\int \frac{2}{9+x^2}\, dx$

2. Evaluate the following definite integrals:

(a) $\int_1^3 x^3 \, dx$ (b) $\int_2^5 dx$ (c) $\int_1^{10} \frac{1}{2x} dx$

(d) $\int_0^1 e^{2x}\, dx$ (e) $\int_{\pi/3}^{\pi} \cos 2x \, dx$ (f) $\int_1^4 \left(x^2+3x\right) dx$

(g) $\int_1^2 \left(x^2 + \frac{1}{x} - 3 \right) dx$ (h) $\int_{-2}^{-1} \left(4e^{-2x} + \frac{3}{x} \right) dx$ (i) $\int_0^1 (5x - \sin 3x)\, dx$

(j) $\int_{\pi/6}^{\pi/3} \left(\sin 3x - \cos 4x\right) dx$ (k) $\int_2^4 \left(5\sin 3x + \frac{2}{x} \right) dx$

Section B: Integration of Functions of a Linear Function

1. Find the following integrals:

(a) $\int (3x+2)^4 \, dx$ (b) $\int (1-2x)^2 \, dx$ (c) $\int \sqrt{4-3x} \, dx$

(d) $\int \frac{1}{(2x-3)^5}\, dx$ (e) $\int \sin(2x+1)\, dx$ (f) $\int \cos\left(3x - \frac{\pi}{6}\right) dx$

(g) $\int e^{\frac{x}{2}+5}\, dx$ (h) $\int 5 e^{3x-2}\, dx$ (i) $\int \frac{1}{8x+3}\, dx$

(j) $\int \frac{3}{2x-25}\, dx$ (k) $\int \frac{1}{2-x} dx$ (l) $\int \frac{4}{25-4x} dx$

2. Evaluate the following definite integrals:

(a) $\int_{-1}^{1} (4x-3)^2 \, dx$ (b) $\int_{4.5}^{10.5} \frac{2}{\sqrt{2x-5}}\, dx$ (c) $\int_{-2/3}^{0} \frac{1}{e^{3x+2}}\, dx$

<u>Section C</u>: Integration using Trigonometric Identities, Application: RMS Value

1. Find the following integrals:

(a) $\int 2\sin x\cos x\, dx$ (b) $\int \frac{1}{\cos^2(2x)}\, dx$ (c) $\int 2\tan^2 2x\, dx$

(d) $\int 2\sin 3x\cos 5x\, dx$ (e) $\int 3\sin\frac{3t}{2}\sin\frac{5t}{2}\, dt$ (f) $\int \sin^2\theta\cos 3\theta\, d\theta$

*(g) $\int \cos^4 x\, dx$ *(h) $\int_0^{\pi/2}\sin^4 x\, dx$

2. Find the root-mean-square (rms) value of:

(a) $f(t) = 1 + 3e^{-t}$ from $t = 0$ to $t = 2$

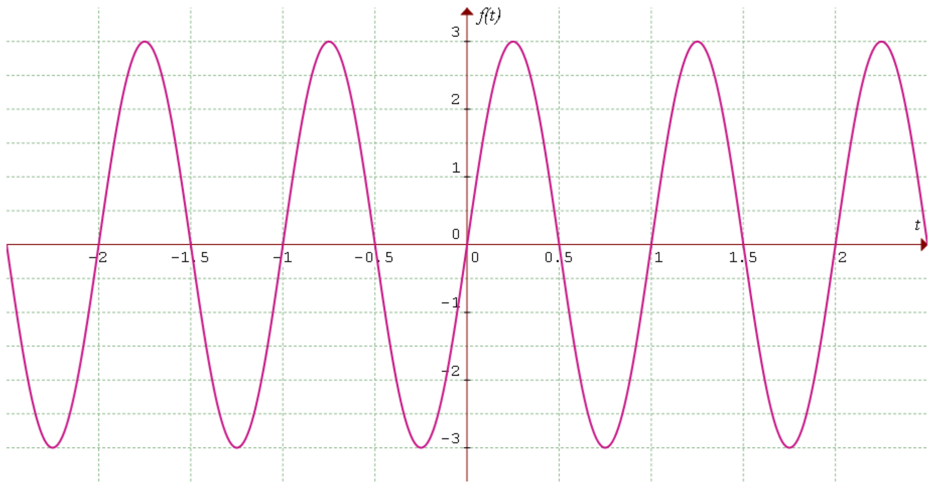
(b) $y = 2(\sin x + \cos x)$ from $x = 0$ to $x = \pi$

3. In an electrical circuit, the voltage across a capacitor at time t seconds (s) is given by

$$V = 1 - e^{-2t} \text{ volts.}$$

Find the root-mean-square (RMS) value of V from $t = 1$ s to $t = 2$ s.

*4. Using integration, derive the root-mean-square (RMS) value of the following sine form:



*5. If a and b are integers, find the integrals for each of the following 3 cases:

(i) $a \neq b$, (ii) $a = b \neq 0$, (iii) $a = b = 0$

(a) $\int \cos ax\cos bx\, dx$

(b) $\int \sin ax\sin bx\, dx$

(c) $\int \sin ax\cos bx\, dx$

*6. If m and n are integers, use the results of question 5 to show that:

(a) $\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}$

(b) $\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases}$ (c) $\int_0^{2\pi} \cos mx \sin nx \, dx = 0$

Miscellaneous Exercises

*1. Find the results of the following integrals:

(a) $\int \frac{(x-2)^2}{x} dx$ (b) $\int \left(2e^{-x} + e^{x}\right)^2 dx$ (c) $\int \frac{e^{2x} + 2e^{x}}{e^{2x}} dx$

(d) $\int \left(1 - e^{-x}\right)^2 dx$ (e) $\int \frac{x}{x-1} dx$ (f) $\int \cot^2 \pi x \, dx$

*2. If the current in an electric circuit is given by $i = I_p \sin \omega t$ Amperes, where I_p is the peak current, show that the rms value of the current from $t = 0$ to $t = \frac{2\pi}{\omega}$ is $\frac{I_p}{\sqrt{2}}$ Amperes.

Multiple Choice Questions

1. Given that $\frac{d}{dx}\left(\sin^3 x \tan x\right) = \sin x \tan^2 x + 3\sin^3 x$, which of the following is equivalent to $\int 3\sin^3 x \, dx$?

(a) $\int \sin^3 x \tan x \, dx$ (b) $\sin^3 x \tan x - \int \sin x \tan^2 x \, dx$

(c) $\int \left(\sin^3 x \tan x - \sin x \tan^2 x\right) dx$ (d) $\sin^3 x \tan x - \sin x \tan^2 x + C$

2. Given the expression $\int f(x) dx = \left[f(x)\right]^2 + C$, where C is an arbitrary constant, which of the following could be $f(x)$?

(a) $f(x) = x + 1$ (b) $f(x) = 2x + 1$

(c) $f(x) = \frac{1}{2}x + 1$ (d) $f(x) = x + \frac{1}{2}$

Tutorial 2.2

Section A: Integration of Rational Functions

1. Find the following integrals:

(a) $\displaystyle\int \frac{-x+7}{(x+3)(3x-1)}\,dx$ (b) $\displaystyle\int \frac{x}{x^2-x-2}\,dx$ (c) $\displaystyle\int_2^3 \frac{1}{(x^2+x)(x-1)}\,dx$

Section B: Integration by Substitution

1. Find the following integrals, by using the given substitutions:

(a) $\displaystyle\int 2x\left(x^2+1\right)^5 dx$, let $u = x^2+1$ (b) $\displaystyle\int \frac{dx}{x\ln x}$, let $u = \ln x$

(c) $\displaystyle\int e^{\sin 2x}\cos 2x\,dx$, let $u = \sin 2x$ (d) $\displaystyle\int \frac{\sec^2\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$, let $u = \sqrt{x}$

2. Find the following integrals, by using the suitable substitutions.

(a) $\displaystyle\int x\left(x^2-3\right)^4 dx$ (b) $\displaystyle\int \frac{3x^2-1}{x^3-x}\,dx$ (c) $\displaystyle\int t\,e^{3-2t^2}\,dt$

(d) $\displaystyle\int \frac{x}{1-2x^2}\,dx$ (e) $\displaystyle\int \frac{x}{(4-x^2)^2}\,dx$ (f) $\displaystyle\int \sin^2\theta\cos\theta\,d\theta$

(g) $\displaystyle\int \frac{5e^{2x}}{\sqrt{1-e^{2x}}}\,dx$ (h) $\displaystyle\int t^3\sin\left(t^4\right)dt$ *(i) $\displaystyle\int \frac{x+1}{\sqrt{x+2}}\,dx$

3. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^2 x\,e^{x^2}\,dx$ (b) $\displaystyle\int_0^{1/2} y\sqrt{\frac{1}{4}-y^2}\,dy$ (c) $\displaystyle\int_1^2 \frac{e^{1/t}}{t^2}\,dt$

(d) $\displaystyle\int_0^4 \frac{4x}{\sqrt{2x+1}}\,dx$

4. A 1.25 F capacitor, that has an initial voltage of 25.0 V, is charged with a current i that varies with time t according to the equation $i = t\sqrt{t^2+6.83}$ Amperes. The formula for the voltage across a capacitor is $V_c = \dfrac{1}{C}\displaystyle\int i\,dt$ Volts .

(a) Show that the general equation of the voltage across the capacitor is given by $V_c = 0.267\left(t^2+6.83\right)^{\frac{3}{2}} + k$, where k is a constant .

(b) Find the value of k .

(c) Hence, find the voltage across the capacitor at 1.00 s.

5. If a circular disk of radius r carries a uniform electrical charge, then the electric potential on the axis of the disk at a point a from its centre is given by the equation:

$$V = k\int_0^r \frac{x}{\sqrt{x^2+a^2}}\,dx$$

where k is a constant depending on the charge density. Integrate to find V as a function of r and a.

6. Find the root-mean-square (rms) value of $i = t^{\frac{1}{2}}e^{-t^2}$ Amperes from $t = 1$ s to $t = 2$ s.

Miscellaneous Exercises

*1. Find the results of the following integrals:

(a) $\int \sin^3 x\,dx$ (Hint: use $\sin^2 x = 1-\cos^2 x$ and let $u = \cos x$.)

(b) $\int \left(27e^{9x}+e^{12x}\right)^{1/3} dx$ (c) $\int \frac{3}{x\ln x}\,dx$ (d) $\int x\sqrt{4-x}\,dx$

(e) $\int e^{2x}\sqrt{1+4e^x}\,dx$ (f) $\int \tan^3 x\,dx$ (g) $\int \sec^6 t\,dt$

(h) $\int \tan^3 x \sec x\,dx$ (i) $\int \sin^2 x\cos^4 x\,dx$ (j) $\int \cos^4 2x\sin^3 2x\,dx$

(k) $\int \sin^3\theta\cos^3\theta\,d\theta$

*2. Integrate the following:

(a) $\int \frac{1}{\sqrt{x}+x}\,dx$ (b) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$ (c) $\int x(2x-5)^3\,dx$

(d) $\int t^3\sqrt{1-t^2}\,dt$ (e) $\int \frac{dx}{3+\sqrt{x+2}}$ (f) $\int \frac{2x+1}{(x-3)^6}\,dx$

(g) $\int \frac{\sqrt{x^3-4}}{x}\,dx$

*3. Evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+\sin^2 2x}\,dx$.

*4. By using the substitution $x = \tan\theta$, or otherwise, find $\int \frac{1}{\left(1+x^2\right)^2}\,dx$.

*5. By using the substitution $t-1 = \sin\theta$, or otherwise, find $\int \sqrt{1-(t-1)^2}\,dt$.

Multiple Choice Questions

1. To find the integral $\int \frac{x-2}{\sqrt{x^2-4x+1}}\,dx$ by substitution method, we should let

   (a) $u = x - 2$ (b) $u = x^2 - 4x + 1$

   (c) $u = 2x - 4$ (d) $u = x$

2. Which of the following integrals **cannot** be found using the substitution method?

   (a) $\int \frac{1}{1+x^2}\,dx$ (b) $\int \frac{x}{1+x^2}\,dx$

   (c) $\int x^2 e^{x^3}\,dx$ (d) $\int 4\cos^2 x \sin x\,dx$

3. To find $\int x\sqrt{x^2+1}\,dx$,

   (a) let $u = x$ (b) let $u = \sqrt{x}$

   (c) let $u = x + 1$ (d) let $u = x^2 + 1$

Answers to Selected Lecture Examples

Example 8: $-\dfrac{\cos 5x}{10}+\dfrac{\cos x}{2}+C$

Example 9: $\dfrac{5}{2}\left(x-\dfrac{\sin 6x}{6}\right)+C$

Example 10: $\dfrac{1}{8}x-\dfrac{1}{32}\sin 4x+C$

Example 12: $\dfrac{3}{\sqrt{2}}$

Example 13: (a) $\ln|x-2|-\ln|x+1|+C$ (b) $6\ln|x-3|-\ln|x|+C$

Example 15: $\dfrac{1}{6}\left(x^2+3\right)^6+C$

Example 16: (a) $-\dfrac{1}{2}\left(1-2x^2\right)^{3/2}+C$ (b) $\dfrac{1}{4}\left(e^x+1\right)^4+C$

Example 17: (a) $\dfrac{2}{3}\ln|x^3-4|+C$ (b) $\dfrac{1}{2}\ln|e^{2x}+1|+C$

Example 18: $e^{x^3}+C$

Example 20: $2\left(1-\dfrac{\pi}{4}\right)$ or 0.43

Answers to Tutorial 2.1

Section A

1. (a) $\dfrac{x^2}{2}+\dfrac{1}{x}+C$ (b) $\dfrac{1}{5}e^{5x}-\dfrac{1}{e^{3x}}+C$ (c) $\dfrac{2}{9}x^3+\dfrac{2}{15}x^{\frac{5}{2}}+C$

   (d) $x^4-x^3+4x^2-6x+C$ (e) $e^{2x}-\dfrac{1}{2}e^{-2x}+C$ (f) $-\dfrac{2}{3}\ln|\cos 3x|+C$

   (g) $\dfrac{1}{6}\ln|\sin 6x|+C$ (h) $\dfrac{2}{3}\tan^{-1}\dfrac{x}{3}+C$

2. (a) 20 (b) 3 (c) 1.15 (d) 3.195

   (e) $-\dfrac{\sqrt{3}}{4}$ (f) 43.5 (g) 0.026 (h) 92.34

   (i) 1.84 (j) 0.766 (k) 1.58

Section B

1. (a) $\dfrac{1}{15}(3x+2)^5+C$ (b) $-\dfrac{1}{6}(1-2x)^3+C$ (c) $-\dfrac{2}{9}(4-3x)^{3/2}+C$

   (d) $-\dfrac{1}{8(2x-3)^4}+C$ (e) $-\dfrac{1}{2}\cos(2x+1)+C$ (f) $\dfrac{1}{3}\sin\left(3x-\dfrac{\pi}{6}\right)+C$

   (g) $2e^{\frac{x}{2}+5}+C$ (h) $\dfrac{5}{3}e^{3x-2}+C$ (i) $\dfrac{1}{8}\ln|8x+3|+C$

   (j) $\dfrac{3}{2}\ln|2x-25|+C$ (k) $-\ln|2-x|+C$ (l) $-\ln|25-4x|+C$

2. (a) 86/3 (b) 4 (c) 0.2882

Section C

1. (a) $-\frac{1}{2}\cos 2x + C$ (b) $\frac{1}{2}\tan 2x + C$ (c) $\tan 2x - 2x + C$

   (d) $\frac{\cos 2x}{2} - \frac{\cos 8x}{8} + C$ (e) $\frac{3}{2}\left(\sin t - \frac{1}{4}\sin 4t\right) + C$ (f) $\frac{\sin 3\theta}{6} - \frac{\sin 5\theta}{20} - \frac{\sin\theta}{4} + C$

   (g) $\frac{1}{8}\left(3x + 2\sin 2x + \frac{1}{4}\sin 4x\right) + C$ (h) $\frac{3\pi}{16}$

2. (a) 2.41 (b) 2 3. 0.942 volts 4. $f(t) = 3\sin(2\pi t)$, $\frac{3}{\sqrt{2}}$

5. (a) (i) $\frac{1}{2}\left[\frac{\sin[(a-b)x]}{a-b} + \frac{\sin[(a+b)x]}{a+b}\right] + C$ (ii) $\frac{1}{2}\left(x + \frac{\sin 2ax}{2a}\right) + C$ (iii) $x + C$

   (b) (i) $\frac{1}{2}\left[\frac{\sin[(a-b)x]}{a-b} - \frac{\sin[(a+b)x]}{a+b}\right] + C$ (ii) $\frac{1}{2}\left(x - \frac{\sin 2ax}{2a}\right) + C$ (iii) C

   (c) (i) $-\frac{1}{2}\left[\frac{\cos[(a-b)x]}{a-b} + \frac{\cos[(a+b)x]}{a+b}\right] + C$ (ii) $-\frac{\cos 2ax}{4a} + C$ (iii) C

Miscellaneous Exercises

1. (a) $\frac{x^2}{2} - 4x + 4\ln|x| + C$ (b) $-2e^{-2x} + 4x + \frac{e^{2x}}{2} + C$ (c) $x - 2e^{-x} + C$

   (d) $x + 2e^{-x} - \frac{1}{2}e^{-2x} + C$ (e) $x + \ln|x-1| + C$

   (f) $-\frac{1}{\pi}\cot \pi x - x + C$

MCQ

1. (b) 2. (c)

Answers to Tutorial 2.2

Section A

1. (a) $-\ln|x+3|+\dfrac{2}{3}\ln|3x-1|+C$ (b) $\dfrac{2}{3}\ln|x-2|+\dfrac{1}{3}\ln|x+1|+C$ (c) 0.0849

Section B

1. (a) $\dfrac{(x^2+1)^6}{6}+C$ (b) $\ln|\ln x|+C$ (c) $\dfrac{1}{2}e^{\sin 2x}+C$ (d) $2\tan\sqrt{x}+C$

2. (a) $\dfrac{1}{10}\left(x^2-3\right)^5+C$ (b) $\ln\left|x^3-x\right|+C$ (c) $-\dfrac{1}{4}\mathrm{e}^{3-2t^2}+\mathrm{C}$

   (d) $-\dfrac{1}{4}\ln\left|1-2x^2\right|+C$ (e) $\dfrac{1}{2\left(4-x^2\right)}+C$ (f) $\dfrac{1}{3}\sin^3\theta+C$

   (g) $-5\sqrt{1-e^{2x}}+C$ (h) $-\dfrac{1}{4}\cos\left(t^4\right)+C$ (i) $\dfrac{2}{3}(x+2)^{3/2}-2(x+2)^{1/2}+C$

3. (a) 26.80 (b) 1/24 (c) 1.07 (d) 13.33
4. (b) 20.2 (c) 26.0 V
5. $V=k\left(\sqrt{r^2+a^2}-a\right)$ Volts
6. 0.18 A

Miscellaneous Exercises

1. (a) $-\cos x+\dfrac{\cos^3 x}{3}+C$ (b) $\dfrac{1}{4}\left(27+e^{3x}\right)^{\frac{4}{3}}+C$

   (c) $3\ln|\ln x|+C$ (d) $\dfrac{2}{5}(4-x)^{5/2}-\dfrac{8}{3}(4-x)^{3/2}+C$

   (e) $\dfrac{1}{40}(1+4e^x)^{5/2}-\dfrac{1}{24}(1+4e^x)^{3/2}+C$ (f) $\dfrac{1}{2}\tan^2 x+\ln|\cos x|+C$

   (g) $\dfrac{1}{5}\tan^5 t+\dfrac{2}{3}\tan^3 t+\tan t+C$ (h) $\dfrac{1}{3}\sec^3 x-\sec x+C$

   (i) $\dfrac{x}{16}-\dfrac{\sin 4x}{64}+\dfrac{\sin^3 2x}{48}+C$ (j) $-\dfrac{1}{10}\cos^5 2x+\dfrac{1}{14}\cos^7 2x+C$

   (k) $\dfrac{1}{6}\cos^6\theta-\dfrac{1}{4}\cos^4\theta+C$ or $\dfrac{1}{4}\sin^4\theta-\dfrac{1}{6}\sin^6\theta+C$

2. (a) $2\ln\left(1+\sqrt{\mathrm{x}}\right)+\mathrm{C}$ (b) $\dfrac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}-\left(1-x^2\right)^{\frac{1}{2}}+C$

   (c) $\dfrac{1}{20}(2x-5)^5+\dfrac{5}{16}(2x-5)^4+C$ or $\dfrac{1}{80}(2x-5)^4(8x+5)+C_1$

   (d) $\dfrac{1}{5}\left(1-t^2\right)^{\frac{5}{2}}-\dfrac{1}{3}\left(1-t^2\right)^{\frac{3}{2}}+C$

(e) $2\left(3+\sqrt{x+2}\right)-6\ln\left(3+\sqrt{x+2}\right)+C$ or $2\sqrt{x+2}-6\ln\left(3+\sqrt{x+2}\right)+C_1$

(f) $\frac{-1}{2}(x-3)^{-4}-\frac{7}{5}(x-3)^{-5}+C$ or $\frac{1-5x}{10(x-3)^5}+C_1$

(g) $\frac{2}{3}\left(x^3-4\right)^{\frac{1}{2}}-\frac{4}{3}\tan^{-1}\frac{\sqrt{x^3-4}}{2}+C$

3. $\frac{\pi}{8}$

4. $\frac{1}{2}\left[\tan^{-1}x+\frac{x}{1+x^2}\right]+C$

5. $\frac{1}{2}\left[\sin^{-1}(t-1)+(t-1)\sqrt{1-(t-1)^2}\right]+C$

<u>MCQ</u>

1. (b) 2. (a) 3. (d)

Chapter 3: First Order Differential Equations and Its Applications

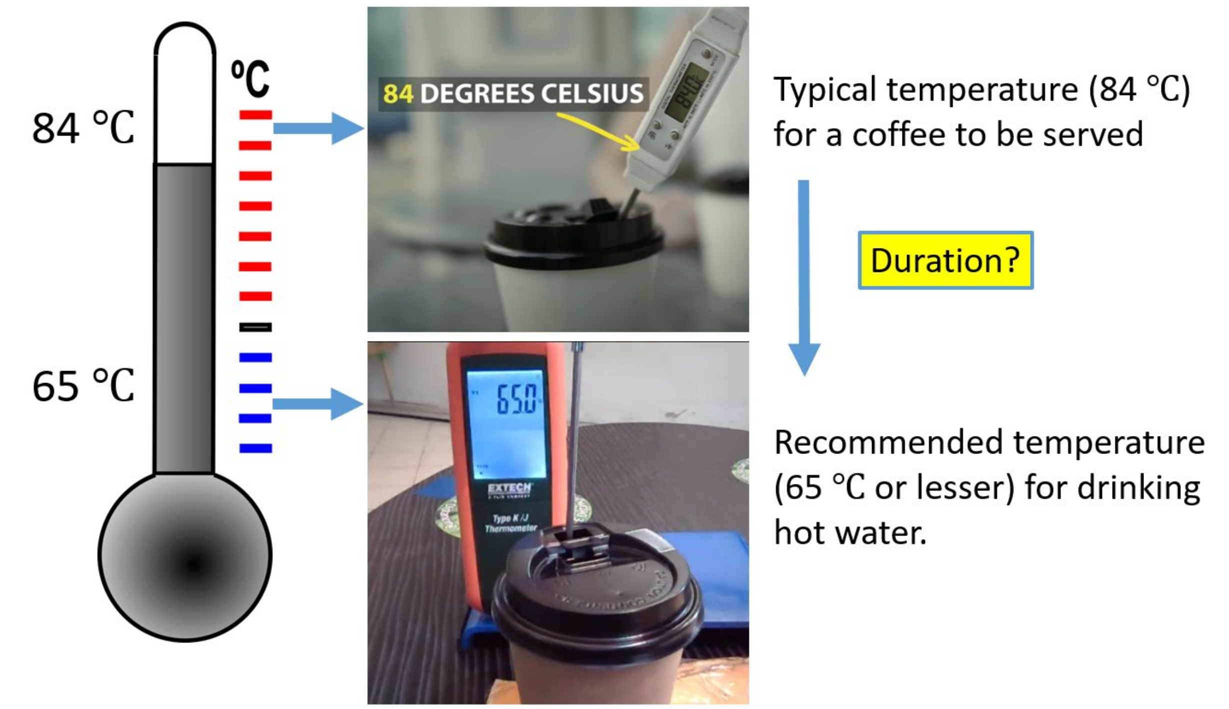
Objectives :

1. *Define differential equation (DE) and state the order of a differential equation.*
2. *Understand the solutions to differential equations.*
3. *Solve first order differential equations by:*
   - *direct integration,*
   - *separation of variables, or*
   - *using the method of integrating factor.*
4. *Apply Newton's Law of Cooling.*
5. *Apply first order DE to other engineeering applications.*

Content

3.1 Application: The Hidden Health Risk of Drinking Hot Coffee

Could your hot drink be putting you at risk of oesophageal cancer?



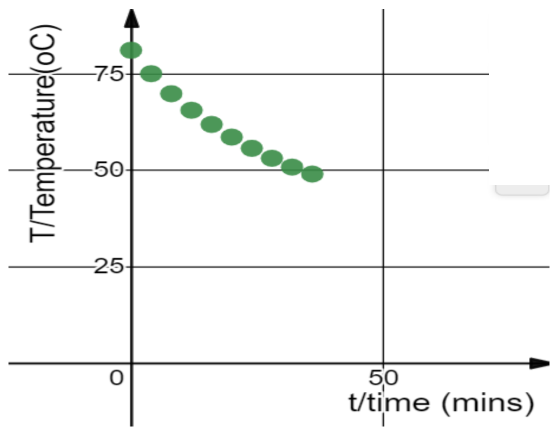
According to the international agency for research on cancer, drinking any beverage above 65 °C may increase the risk of oesophageal cancer.
(*Retrieved from*: https://www.youtube.com/watch?v=XmjrCKjgVkg).

The oesophagus is the long, hollow tube that runs from your throat to your stomach, which transports food to the stomach to be digested.

From the moment the hot beverage in SP food court 3 (FC3) is served, how long do you need to wait for the beverage to cool down to $65\,°C$ before drinking it?

To find out the answer, we can conduct an experiment by measuring the temperature of the coffee.

t	T
0	81.3
4	75.2
8	70
12	65.7
16	62
20	58.7



Another way to solve this problem is to **model** the temperature of the hot beverage by using Newton's law of Cooling.

Newton's Law of Cooling states that:

The rate of cooling (how fast is the decrease in temperature) is proportional to the temperature difference between the object and the object's surroundings.

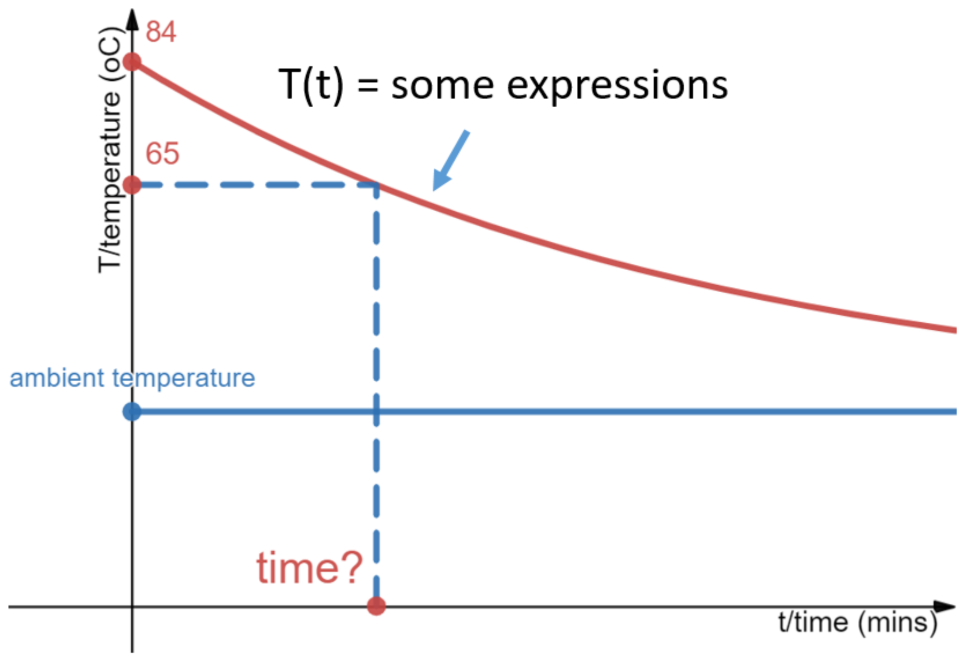
Let $T(t)$ = temperature of beverage at time t, T_s = temperature of the surrounding, then,

$$\frac{dT}{dt} = -k(T - T_s) \quad \ldots\ldots(1)$$

where k is a positive constant determined by other factors, e.g., shape of the glass, the material it is made from, whether it is open or closed and so on.

The negative sign in front of k indicates that the temperature T decreases with time t.

Equation (1) is known as a **Differential Equation** since it contains derivative.
Here we need to find a formula for $T(t)$, which is the temperature of beverage at time t.
Once the formula of $T(t)$ is obtained, we can make use of the formula to find the time t, at any temperature, depicted below:



In the subsequent sections, we will learn how to solve differential equations. After that, we will revisit this problem again.

3.2 Definitions

3.2.1 Differential Equations

A differential equation is an equation that contains derivative(s). It is used to model various activities/happenings around us. It is used to examine the motion of a particle in Physics, to calculate compound interest in Economics, to model electrical circuit in engineering and so on.

3.2.2 Ordinary Differential Equation

An ***ordinary differential equation*** is a differential equation involving only **one** independent variable. Examples of ordinary differential equations are:

- $2\dfrac{dy}{dx}+6y=x+1$
- $3\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-8y=\sin x$

3.2.3 Order of a Differential Equation

The ***order*** of a differential equation is the order of the highest derivative in the differential equation.

Example 1: Identify the order of each of the following differential equations:

(a) $\dfrac{dy}{dx}+3y=x$ (b) $\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3=0$ (c) $\left(\dfrac{d^3y}{dx^3}\right)^2+6\dfrac{dy}{dx}=0$

3.2.4 Methods to Solve Differential Equation

There are 3 methods to solve a differential equation:

- by direct integration
- by separating variables
- by using an integrating factor

We will go through each method in the ensuing sections.

3.3 Solutions of a Differential Equation

Any relation between the dependent and independent variables of a differential equation, not containing any derivatives, which *satisfies* the differential equation is called a **solution** of the equation.

Example 2: Show that $y = 5e^{2x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 2y$.

3.3.1 General Solution

The *general solution* of a differential equation of order n is the solution containing n arbitrary constants.

3.3.2 Particular Solution

A *particular solution* of a differential equation is one in which the arbitrary constants in the general solution have assigned values.

3.4 Method 1: Solve DE by Direct Integration

A first order differential equation of the form $\dfrac{dy}{dx} = f(x)$ can be solved by direct integration.

Thus, if $\dfrac{dy}{dx} = f(x)$, then $y = \int f(x)\,dx$.

Example 3**:** Find the general solution of the differential equation $x^2 \dfrac{dy}{dx} = 2x^4 + \dfrac{1}{2}$.

Find also the particular solution for which $y = 2$ when $x = 1$.

3.5 Method 2: Solve DE by Separation of Variables

Variable separable technique is commonly used to solve first order differential equations.

Equations of the form:

$$\frac{dy}{dx} = f(x)g(y) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

can be re-written as:

$$\frac{1}{g(y)}\,dy = f(x)\,dx \qquad \text{or} \qquad \mathrm{g}(y)\,dy = f(x)\,dx$$

All the terms in x can be collected on one side of the equation and all the terms in y on the other side.

The general solution is then obtained by integrating each side of the equation separately:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx \qquad \text{or} \qquad \int \mathrm{g}(y)\,dy = \int f(x)\,dx$$

Example 4: Solve the differential equation $\dfrac{dy}{dx} = 2y$ for y in terms of x.

Solution: First rewrite the differential equation in differential form:

$$dy =$$

Then separating the variables, we have:

$$\frac{dy}{y} =$$

Integrate each side of the equation:

$$\int \frac{1}{y}\,dy \ =$$

***Example 5*:** Solve $y\,e^{x+y^2}\dfrac{dy}{dx}=1$.

Example 6: Solve $2xy\dfrac{dy}{dx}+3(y^2+1)=0$, given that $y(1)=2$.

3.6 Method 3: Solve DE by using an Integrating Factor

A **standard linear** first order differential equation is in the form:

$$\frac{dy}{dx}+P(x)\cdot y=Q(x) \qquad \text{----- (2)}$$

where $P(x)$ and $Q(x)$ are functions of x, and $P(x)$ is not zero.

An ordinary differential equation may not always be given in standard linear form, we then need to re-arrange the terms to put it into the desired linear form.

For example, $\frac{dy}{dx}=-xy+x^2$ is not linear, but it can be rewritten as $\frac{dy}{dx}+xy=x^2$, which is a linear form, where $P(x)=x$ and $Q(x)=x^2$.

Integrating Factor

Equation (2) can be solved by using an integrating factor (I.F.) : $\boxed{\mu(x)=e^{\int P(x)dx}}$

Multiplying both sides of equation (2) by this integrating factor, we have:

$$e^{\int P(x)\,dx}\cdot\frac{dy}{dx} + e^{\int P(x)\,dx}\cdot P(x)\,y = e^{\int P(x)\,dx}\cdot Q(x)$$

Due to product rule, the two terms on the left-hand side can be written as $\frac{d}{dx}\left(y\,e^{\int P(x)\,dx}\right)$. Hence,

$$\frac{d}{dx}\left(y\,e^{\int P(x)\,dx}\right) = e^{\int P(x)\,dx}Q(x)$$

Notice that, now, this differential equation is of the type $\frac{dy}{dx}=f(x)$ which can be solved by direct integration:

$$y\,e^{\int P(x)\,dx}=\int e^{\int P(x)\,dx}Q(x)\,dx$$

The general solution will be: $y\cdot\mu(x)=\int\mu(x)Q(x)\;dx$

In summary, we can solve a linear DE with the help of an integrating factor as follow:

Step 1: Rewrite the first order linear differential equation in the standard form, $$\frac{dy}{dx} + P(x)y = Q(x)$$
Step 2: Find the integrating factor, $\mu(x) = e^{\int P(x)\,dx}$.
Step 3: Obtain the general solution, $y \cdot \mu(x) = \int \mu(x)Q(x)\,dx$.

In dealing with linear equations, it is important to remember that $\boxed{e^{\ln f(x)} = f(x)}$ which is useful for simplifying certain integrating factors.

Example 7: Find the general solution of the differential equation $\frac{dy}{dx} + 2y = e^x$.

Solution: Comparing the equation $\frac{dy}{dx} + 2y = e^x$

with the standard form $\frac{dy}{dx} + P(x)y = Q(x)$

we see that: $P(x) = 2$, $Q(x) = e^x$

The integrating factor is:
$$\mu(x) = e^{\int P(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

The general solution is given by: $y \cdot \mu(x) = \int \mu(x)Q(x)\,dx$

That is:
$$y e^{2x} = \int e^{2x} \cdot e^x\,dx = \int e^{3x}\,dx = \frac{e^{3x}}{3} + C$$

Hence the general solution is: $y e^{2x} = \frac{e^{3x}}{3} + C$

Example 8: Solve $\frac{dy}{dx} + 2xy = x$.

Example 9: Solve $x\frac{dy}{dx} - 5y = x^6 \cos^2 x$ for y in terms of x , given that $y(\pi) = 0$.

3.7 Application of Differential Equations: Newton's Law of Cooling

Back to The Hidden Health Risk of Drinking Hot Coffee

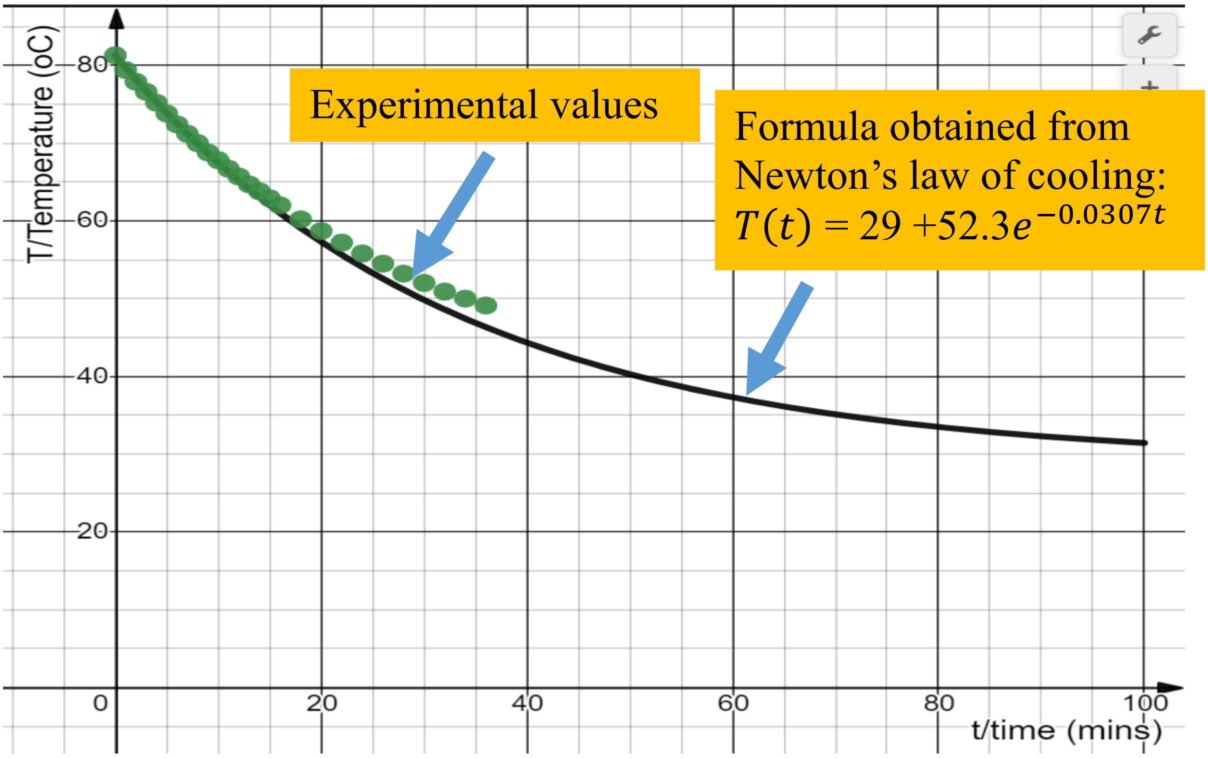
To reduce the risk of getting oesophageal cancer, a hot drink should be allowed to cool down to $65°\text{C}$ before drinking it. Thomas wants to find out when is the appropriate time to drink the hot coffee from Food Court 3 after it is served. He models the temperature of the coffee using Newton's law of cooling.

He found out that temperature of the coffee is $81.3°\text{C}$ when it is delivered to him. The cup of hot coffee is allowed to cool with the surrounding at $29°\text{C}$. After 3 minutes, the temperature of the coffee has dropped to $76.7°\text{C}$.

(a) Set up the differential equation that depicts the cooling process of the coffee.

(b) Find the particular solution of the differential equation in part (a).

(c) How long should Thomas wait for the coffee to cool down to $65°\text{C}$?



Note: Since k is positive, as $t \to \infty$, $e^{-kt} \to 0$ and hence $T \to T_s$. The limiting value of T is called the **ultimate temperature** of the body. The body will cool down to its surrounding temperature T_s after a very long time. **Ultimate** temperature is also known as **Final** temperature, **Limiting** temperature or T**erminal** temperature. Condition for ultimate temperature is $\frac{dT}{dt} = 0$.

3.8 Extended Learning: Application on RLC Circuits

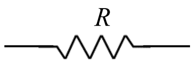
Below are some electrical formulae we will be using to form DE in electrical circuits.

3.8.1 Current and Charge in a Circuit

The current i (in amperes) is the rate of change of the charge q (in coulombs) flowing through a circuit at any time t (in seconds), that is, $\boxed{i = \dfrac{dq}{dt}}$.

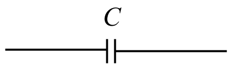
The charge q can be expressed as $\boxed{q = \int i\,dt}$.

3.8.2 Voltage across a Resistor



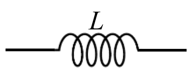
The potential difference v_R (in volts) across a resistor of resistance R (in ohms) is given by $\boxed{v_R = R\,i}$.

3.8.3 Voltage across a Capacitor



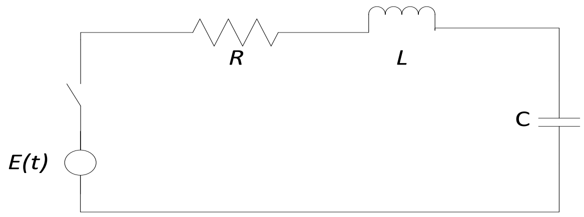
The voltage v_C (in volts) across the plates of a capacitor with capacitance C (in farads), is given by $\boxed{v_C = \dfrac{q}{C}}$.

3.8.4 Voltage across an Inductor



A voltage v_L (in volts) across the inductor with inductance L (in henries) is given by $\boxed{v_L = L\dfrac{di}{dt}}$.

3.8.5 *RLC* Series Circuit



For a RLC series circuit, when the switch is closed at time $t = 0$, by Kirchhoff 's voltage law,

$$v_R + v_L + v_C = E(t)$$

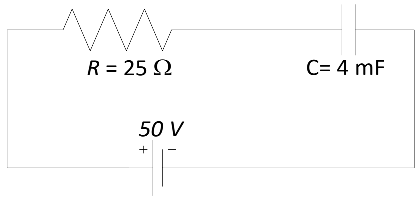
that is, $$Ri + L\frac{di}{dt} + \frac{q}{C} = E(t)$$

***Example 10*:** A circuit has, in series, a voltage source of 30 V, a resistor of 60 Ω, and an inductor of 3 H. If the initial current is 0 A, find the current at time $t > 0$. Hence, find the steady-state current.

Example 11: Assuming that the charge on the capacitor is zero at $t=0$, find:

(a) the charge and current at any time t;

(b) the steady-state charge.



Solution:

(a) $v_R + v_C = V$

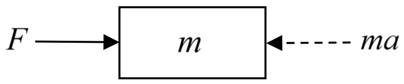
3.9 Extended Learning: Application on Resisted Motion

When a force F acts on an object, it will accelerate. The acceleration a is directly proportional to the force, that is, $F \propto a$ or $F = ma$.

The constant of proportionality, m, is commonly known as the **mass** of the object which is a measure of its inertia.

$F = ma$ is commonly known as Newton's second law. It is often useful to rewrite it as $F - ma = 0$, and consider ma as a force known as the inertia force, which is acting in a direction opposite to the motion (this is known as the D'Alembert Principle). Then, the equation says that, at all time, the sum of the force and the inertia force equal to zero.

The free body diagram is as shown below:

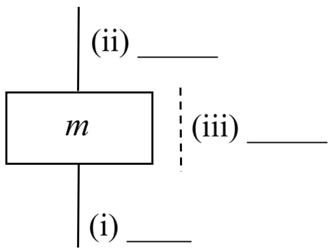


Example 12:

An object of mass 1 kg is projected upward from the level ground with an initial velocity of 45m/s. The air resistance is $2v$, where v is the instantaneous velocity of the object. Model the upward motion of the mass by setting up the differential equation that describe it. [Approximate $g = 10\ m/s^2$.]

(a) When the object is moving upward, there are two external forces acting on it due to:
- (i) gravity,
- (ii) air resistance, and
- (iii) the inertia force.

Indicate the magnitude and direction of each of these forces in the free body diagram:



(b) Set up the differential equation of motion by adding up all the three forces in the free body diagram and equate to zero. Also, state the initial condition.

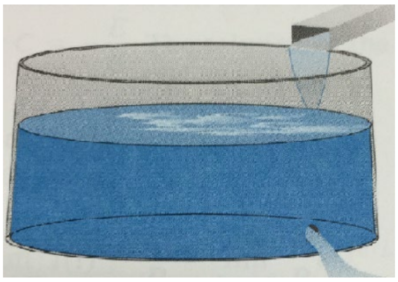
(c) Determine if the variables in the differential equation are separable.

You can try Tutorial 3.2 Miscellaneous Exercises Q1.

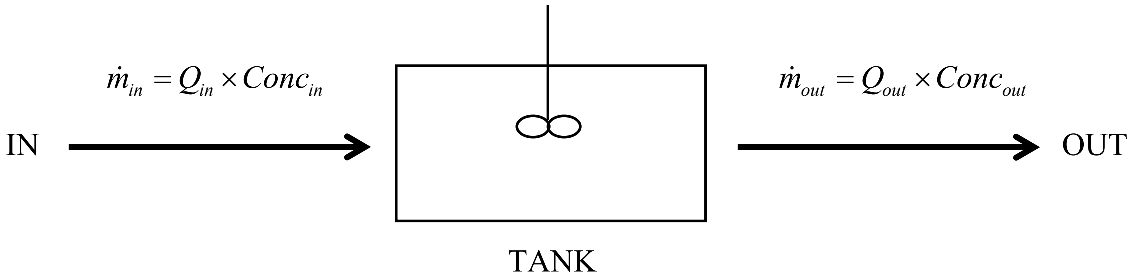
3.10 Extended Learning: Application on Mixture Problem

A problem that is modelled by a 1st order differential equation involves the process of mixing liquids that contain dissolved substances such as salt or dye.

A typical situation involves chemical in a liquid solution flowing into a container which contains the liquid with a specified amount of the dissolved chemical. The mixture is kept uniform by stirring and flows out of the container at a known rate.



Consider the following schematic diagram:



Suppose we are mixing salt in the liquid. Let x be the mass of salt in the tank at any time t.

Rate of change of mass of salt in tank,

$$\frac{dx}{dt} = \text{Mass Flow rate}_{in} - \text{Mass Flow rate}_{out} = \dot{m}_{in} - \dot{m}_{out}$$

where mass flow rate $\dot{m}$ is the mass of the salt which passes per unit of time.

Mass flow rate into the tank $\dot{m}_{in}$ is given by

$$\dot{m}_{in} = Q_{in} \times Conc_{in}$$

where volumetric flow rate Q is the volume of liquid which passes per unit of time and $Conc$ is the concentration of salt in the liquid.

Similarly, we can find the mass flow rate out of the tank $\dot{m}_{out}$.

Assuming the tank is well-stirred, $Conc_{out}$ will be the same as the concentration of salt in the tank.

$$Conc_{out} = \frac{\text{Mass of salt in tank}}{\text{Volume of liquid in tank}} = \frac{x}{\text{Volume of liquid in tank}}$$

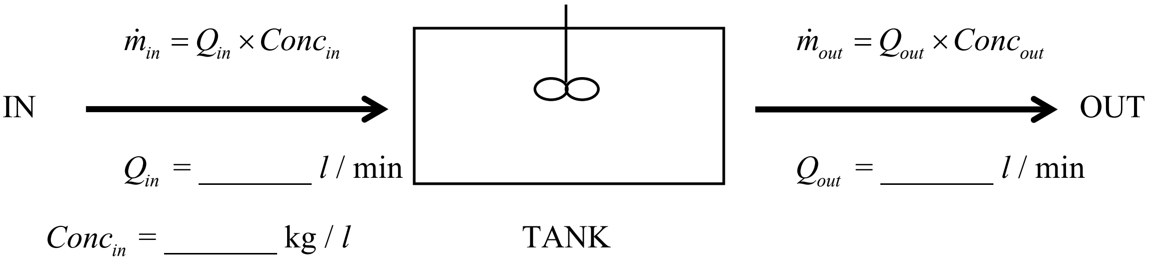
Differential equation: $\dfrac{dx}{dt} = (Q_{in} \times Conc_{in}) - \left(Q_{out} \times \dfrac{x}{\text{Volume of liquid in tank}} \right)$

Example 13:

Fifty litres of sodium chloride (NaCl) solution originally containing 3 kg of NaCl is in a tank. 2 l of water is pumped into the tank per minute, with the same amount of mixture flowing out of the tank each minute. How much NaCl salt is in the tank after 10 min?

Solution:

Volume in tank = ______ l



Let x be the mass of salt in the tank at any time t.

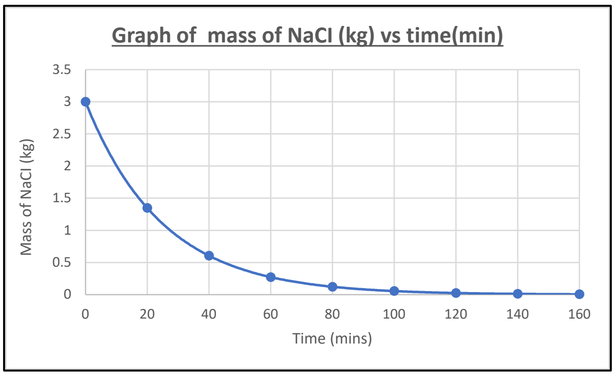
Initial conditions: Initial mass of salt in tank, $x(0) =$ _____________ kg

$$\frac{dx}{dt} = (Q_{in} \times Conc_{in}) - \left(Q_{out} \times \frac{x}{\text{Volume of liquid in tank}} \right)$$

$$\frac{dx}{dt} = (\qquad\qquad \times \qquad\qquad) - \left(\qquad\qquad \times \frac{x}{\qquad\qquad} \right)$$

$$=$$

x is non-negative, so $|x|$ is x.



When $t = 10$ min, $x =$ _____________ kg

You can try Tutorial 3.2 Miscellaneous Exercises Q2.

Tutorial 3.1

Section A: Introduction to Differential Equations (DE)

1. What is the order of each of the following differential equations?

(a) $y\dfrac{d^2y}{dx^2}+\left[\dfrac{dy}{dx}\right]^2=0$ (b) $\left[\dfrac{dy}{dx}\right]^3-y=x$

2. Verify that the function $y=x^2+x$ is a solution of the differential equation:

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}-2=2y-x$$

(*You may refer to Example 2 on page 3-5.*)

Section B: Solve First Order DE by Separation of Variables Method

1. Solve the following differential equations by separating variables:

(a) $\dfrac{dy}{dx}=2xy$ (b) $\dfrac{dy}{dt}=e^{t-y+2}$

(c) $\left(1+x^2\right)\dfrac{dy}{dx}=xy$ (d) $\dfrac{dy}{dx}-x^2+1=0$

2. Find the general solution for each of the following differential equations:

(a) $\dfrac{dy}{dx}-\dfrac{2x}{3y^2+1}=0$ (b) $\dfrac{y+1}{\sin(2x)}\dfrac{dy}{dx}-\cos x=0$

3. Find the particular solution for each of the following differential equations:

(a) $2x^2y\dfrac{dy}{dx}=-(y+1)$, given that $y=0$ when $x=1$.

(b) $\cos y+(1+e^{-x})\sin y\dfrac{dy}{dx}=0$, given that $y=\dfrac{\pi}{4}$ when $x=0$.

Section C: Solve First Order DE by using An Integrating Factor

1. Solve the following linear differential equations:

(a) $\dfrac{dy}{dx}+\dfrac{3y}{x}=6x^2$ (b) $\dfrac{dy}{dx}+3y=3x^2e^{-3x}$

(c) $y'+\dfrac{y}{x}-\dfrac{\sin^2 x}{x}=0$ (d) $\dfrac{dy}{dx}+5x=x-xy$, given that $y=1$ when $x=0$.

Miscellaneous Exercises

1. Find the particular solution for the following differential equation:

$$x^2(y+1)+y^2(x-1)\frac{dy}{dx}=0 \quad , \text{ given that } y=0 \text{ when } x=0 .$$

2. Solve the differential equation $\dfrac{d^2y}{dx^2}=x\left(\dfrac{dy}{dx}\right)^2$, if $y(0)=2$ and $y'(0)=1$.

[*Hint*: Let $z=\dfrac{dy}{dx}$.]

3. Solve the differential equation $dy=(9x+4y+1)^2\,dx$. [*Hint*: Let $z=9x+4y+1$.]

4. By using the substitution $y=\dfrac{1}{v}$, show that the differential equation $-xy^2=y^2+y^3\dfrac{dy}{dx}$ can be reduced to the form $\dfrac{1}{v^3}\dfrac{dv}{dx}=1+x$. Hence, find the particular solution that passes through the origin.

5. The gradient of a curve at the point (x, y) is given by $\dfrac{y}{\ln y}$. Given that the curve passes through the point $(0, e)$, find the equation of the curve.

6. Obtain the general solution of each of the given differential equations:

(a) $\tan x=-y\cos^2 x\dfrac{dy}{dx}$ (b) $\dfrac{dy}{dx}\left(1+\sin^2 x\right)=\sin y\cos x$

(c) $\dfrac{dy}{dx}=\dfrac{y^2\sqrt{1-y^2}}{\sqrt{25-4x^2}}$ [Hint: Let $y=\sin\theta$.]

(d) $y\dfrac{dy}{dx}=\dfrac{\sqrt{y^4+1}}{2+\sin x}$ [Hint: Let $y^2=\tan\theta$ and $u=\tan\dfrac{x}{2}$.]

7. Solve the differential equation $2xy\dfrac{dy}{dx}=4x^2+3y^2$. [*Hint*: Let $u=y^2$.]

8. Show that the following equations are linear. Hence, solve them.

(a) $xy'=e^x(1-xy)-y$ (b) $\left(1+y^2\right)^2=2y\left[1+x\left(1+y^2\right)\right]\dfrac{dy}{dx}$

(c) $\cos^2 y + \left[\dfrac{2x}{\tan y} - \dfrac{\cos^2 y}{\tan y - y} \right] \dfrac{dy}{dx} = 0$, given that $y(0) = \dfrac{\pi}{4}$

<u>Multiple Choice Questions</u>

1. Which of the following differential equations <u>cannot</u> be solved by separating the variables?

(a) $\dfrac{dy}{dx} = \dfrac{y}{x}$ (b) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(c) $\dfrac{dy}{dx} = xy$ (d) $\dfrac{dy}{dx} = x + y$

2. Which of the following differential equations can be solved by separating the variables?

(a) $\dfrac{dy}{dx} = \dfrac{xe^{x^2} \sin y}{\cos y}$ (b) $\dfrac{dy}{dx} = \dfrac{x^2 + x - 1}{xe^y - \sin y}$

(c) $\dfrac{dy}{dx} = \dfrac{e^{x^2}}{\tan y}$ (d) $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

3. Which of the following is NOT a solution to the differential equation $\dfrac{dy}{dx} = ky$, where k is a constant?

(a) $\ln y = kx + c$ (b) $y = ce^x + k$

(c) $\ln(cy) = kx$ (d) $y = ce^{kx}$

4. The expression $e^{\frac{1}{2}\ln(1+x)}$ can be simplified as

(a) $\dfrac{1}{2}(1+x)$ (b) $\sqrt{1+x}$ (c) $e^{\frac{1}{2}}(1+x)$ (d) $\dfrac{1}{\sqrt{1+x}}$

5. Reduce $x\dfrac{dy}{dx} - \dfrac{y}{x^2} = \ln x$ to linear form and identify $P(x)$ and $Q(x)$.

(a) $P(x) = -\dfrac{1}{x^2}$, $Q(x) = \ln x$ (b) $P(x) = -\dfrac{1}{x^3}$, $Q(x) = \ln x$

(c) $P(x) = -\dfrac{1}{x^2}$, $Q(x) = \dfrac{\ln x}{x}$ (d) $P(x) = -\dfrac{1}{x^3}$, $Q(x) = \dfrac{\ln x}{x}$

Tutorial 3.2

Section A: Application of First Order DE – Newton's Law of cooling

1. A cup of boiling coffee is allowed to cool in a room where the temperature is maintained constant at $25\,°C$. The cooling process follows Newton's law of cooling. After 2 minutes, the coffee temperature dropped to $80\,°C$.
   (a) Set up the differential equation that depicts the cooling process of the coffee.
   (b) Find the particular solution of the differential equation in part (a).
   (c) Find the coffee temperature after 8 minutes.

2. A cup of boiling water is allowed to cool in a room where the temperature is maintained constant at $30\,°C$. The cooling process follows Newton's law of cooling whereby the rate of change of temperature of the water is proportional to the difference between the water and its surrounding temperature. After 3 minutes, the water temperature dropped to $78\,°C$.
   (a) Set up the differential equation that depicts the cooling process of the water.
   (b) Find the particular solution of the differential equation in part (a).
   (c) Find the water temperature after 10 minutes.

3. A body cools from $100\,°C$ to $80\,°C$ in 10 minutes in air, which is maintained at $20\,°C$. The cooling process follows Newton's law of cooling.
   (a) Set up the differential equation that depicts the cooling process of the body.
   (b) Solve the equation in part (a) using given conditions.
   (c) How long will it takes the body to cool down from $80\,°C$ to $60\,°C$?

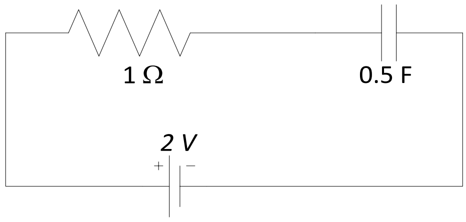
*4. A cup of hot coffee is served at $84\,°C$. The room temperature is modelled by $T_s = 30e^{-0.1t}$, where t is the time elapsed in minutes after the coffee is served. The cooling process follows Newton's law of cooling and it can be modelled by

$$\frac{dT}{dt} = -0.4\left(T - T_s\right) ,$$

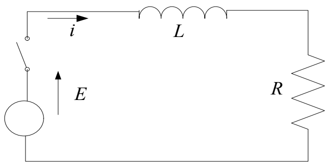
where T is the temperature of the coffee.
   (a) Solve the differential equation.
   (b) Find the temperature of the coffee after 10 minutes.

Section B: Application of First Order DE – *RLC* Series Circuit

1. For the circuit shown in the figure on the right, find the charge $q(t)$ on the capacitor, given that $q(0)=0$ coulomb.



2. Consider the circuit in the figure below with inductance 1 H, resistance 5000 Ω and a voltage source of 12 V. Assume that no current flows in the circuit when the switch is closed at $t = 0$.
   (a) Find the current in the circuit at any time t.
   (b) Find the steady-state current (i.e. when t tends to infinity.).



3. The current i (amperes) flowing through an *RL* series circuit at time t (seconds) satisfies the differential equation:

$$L\frac{di}{dt}+R\,i=V$$

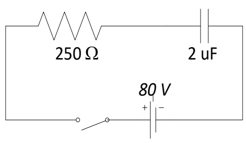
where R, L and V are constants. Given that $i=0$ at $t=0$, find:
   (a) the current i, at any time t, in terms of R, L and V;
   (b) an expression for the steady-state current.

*4. A *RL* series circuit has a 35 volt supply connected to an inductor with inductance L (henry) and a resistor with resistance R (ohms). The current i (amperes) flowing through the circuit at time t (seconds) satisfies the differential equation:

$$L\frac{di}{dt}+Ri=V$$

where V is the voltage supply.
   (a) Find the current i, at any time t, in terms of R and L given that $i = 0$ A when $t = 0$.
   (b) If the circuit is to have an instantaneous current of 5 mA at a time of 22 ms from switch-on and the resistance $R = 4.7$ kΩ, determine the required value of the inductance L.

*5. For the circuit on the right, the capacitor is initially fully discharged. How long after the switch is closed will the capacitor voltage be 76 volts? Determine the current in the resistor at that time, using $i=\dfrac{dq}{dt}$.



Miscellaneous Exercises

1. A parachutist is falling with speed 50 m/s when the parachute opens. The air resistance is $\frac{mgv^2}{81}$, where m is the total mass of the man and the parachute. Model the downward motion of the parachutist after the parachute has opened by setting up the differential equation that describes it.

   (a) Indicate all the external forces and the inertia force (both magnitude and direction) in a free body diagram.

   (b) Set up the differential equation of motion by adding up all the forces in the free body diagram and equate to zero. Also, state the initial condition.

   (c) Show if the variables in the differential equation are separable.

2. A tank contains 20 kg of salt dissolved in 4000 litres of water. Brine containing 0.02 kg of salt per litre of water enters the tank at a rate of 15 *l*/min. The solution is kept thoroughly mixed and drained from the tank at the same rate.

   Find an expression for the amount of salt in the tank at any time, *t*. Hence, calculate how much salt is in the tank after 1 hour.

Answers to Selected Lecture Examples

Example 5: $\frac{1}{2}e^{y^2} = -e^{-x} + C$

Example 6: $\ln\left|y^2+1\right| = -3\ln|x| + \ln 5$

Example 8: $y = \frac{1}{2} + Ce^{-x^2}$

Example 9: $y = \frac{x^5}{4}\left[2x + \sin(2x) - 2\pi\right]$

Example 10: $i(t) = \frac{1}{2} - \frac{1}{2}e^{-20t}$ A ; 0.5 A

Example 11: (a) $q(t) = \frac{1}{5} - \frac{1}{5}e^{-10t}$ C , $i(t) = 2e^{-10t}$ A (b) 0.2 C

Answers to Tutorial 3.1

Section A

1. (a) 2 (b) 1

Section B

1. (a) $\ln|y| = x^2 + C$ or $y(x) = Ae^{x^2}$ (b) $e^y = e^{t+2} + C$ or $y(t) = \ln\left(e^{t+2} + C\right)$

 (c) $\ln|y| = \frac{1}{2}\ln\left|1+x^2\right| + C$ or $y(x) = A\sqrt{1+x^2}$ (d) $y(x) = \frac{x^3}{3} - x + C$

2. (a) $y^3 + y = x^2 + C$ (b) $\frac{y^2}{2} + y = -\frac{1}{2}\left[\frac{\cos(3x)}{3} + \cos x\right] + C$

3. (a) $2\left(y - \ln|y+1|\right) = \frac{1}{x} - 1$ (b) $(1+e^x)\sec y = 2\sqrt{2}$ or $(1+e^x) = 2\sqrt{2}\cos y$

Section C

1. (a) $y(x) = x^3 + Cx^{-3}$ (b) $y(x) = (x^3 + C)e^{-3x}$

 (c) $xy = \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$ (d) $y(x) = 5e^{-\frac{x^2}{2}} - 4$

Miscellaneous Exercises

1. $x^2 + y^2 + 2x - 2y + 2\ln|x-1| + 2\ln|y+1| = 0$

2. $y(x) = 2 + \frac{1}{\sqrt{2}}\ln\left|\frac{x+\sqrt{2}}{x-\sqrt{2}}\right|$

3. $3\tan(6x+c) = 2(9x+4y+1)$

4. $x^2 + 2x + y^2 = 0$

5. $(\ln y)^2 = 2x + 1$

6. (a) $y^2 + \sec^2 x = C$ (b) $\tan^{-1}(\sin x) + \ln|\csc y + \cot y| = C$

(c) $\frac{1}{2}\sin^{-1}\frac{2x}{5} + \frac{\sqrt{1-y^2}}{y} = C$ (d) $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(\tan\frac{x}{2} + \frac{1}{2}\right)\right] = \frac{1}{2}\ln\left(\sqrt{y^4+1} + y^2\right) + C$

7. $y^2 = -4x^2 + Cx^3$

8. (a) $xy = 1 + Ce^{-e^x}$ (b) $x(y) = -\frac{1}{2(1+y^2)} + C(1+y^2)$ (c) $x\tan^2 y = \ln\left|\frac{\tan y - y}{1 - \frac{\pi}{4}}\right|$

MCQ

1. (d) 2. (a) 3. (b) 4. (b) 5. (d)

Answers to Tutorial 3.2

Section A

1. (b) $T(t) = 25 + 75e^{-0.155t}$ $(°C)$ (c) $46.7°C$

2. (b) $T(t) = 30 + 70e^{-0.125t}$ $(°C)$ (c) $50°C$

3. (b) $T(t) = 20 + 80e^{-0.029t}$ $(°C)$ (c) 14.1 min

4. (a) $T(t) = 40e^{-0.1t} + 44e^{-0.4t}$ $(°C)$ (b) $T(10) = 15.52$ $(°C)$

Section B

1. $q(t) = 1 - e^{-2t}$ coulombs

2. (a) $i(t) = \frac{3}{1250}\left(1 - e^{-5000t}\right)$ amperes (b) $\frac{3}{1250}$ amperes

3. (a) $i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ amperes (b) $\frac{V}{R}$ amperes

4. (a) $i(t) = \frac{35}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ amperes (b) 92.91 henries

5. 1.50 ms , 16 mA

Miscellaneous Exercises

1. (b) $m\frac{dv}{dt} + \frac{mg}{81}v^2 - mg = 0$, $v(0) = 50$ m/s (c) $\frac{dv}{81 - v^2} = \frac{g}{81}dt$

2. $x = 80 - 60e^{-0.00375t}$; 32.1kg

Chapter 4: Integration by Parts & Simpson's Rule

Objectives:

1. *Derive the formula for integration by parts.*
2. *State the guideline (LIATE) and apply the integration by parts formula.*
3. *Use Simpson's rule to obtain approximate values of definite integrals.*

4.1 Integration by Parts

If u and v are both functions of x,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\text{i.e. } uv = \int u\,dv + \int v\,du$$

Thus,

$$\boxed{\int u\,dv = u\cdot v - \int v\,du} \quad \cdots\cdots\cdots\cdots(1)$$

The method of "***integration by parts***" is used to integrate:

- product of two different types of functions,
  such as $x^2 \sin x$, $x e^x$, $x \ln x$, $e^x \sin x$, $x \tan^{-1} x, \ldots$ etc.
- single functions that cannot be directly integrated with standard formulae,
  such as $\ln x$, $\sin^{-1}x$, $\tan^{-1}x$, $\cos^{-1}x$, $\ldots$etc.

In this method, we make use of formula (1) to change the original integral $\int u\,dv$ to a different integral $\int v\,du$, which can be integrated easily.

One important step in carrying out the above formula is to select the appropriate u and dv.
To select u, we use the acronym **L I A T E** to guide us.

This acronym spells out the priority of u according to the following order:

Logarithm expression

Inverse Trigonometric expression

Algebraic expression

Trigonometric expression

Exponential expression

4.1.1 Integrals of the Type: $\int x^n e^{ax}\, dx$, $\int x^n \sin ax\, dx$, $\int x^n \cos ax\, dx$, where n is a positive integer and a is a constant

For such integrals, we let $u = x^n$ and $dv = e^{ax}dx$ or $dv = \sin ax\,dx$ or $dv = \cos ax\,dx$.

Example 1: Find $\int x\,e^{2x}\,dx$.

Solution 1: $\int x\,e^{2x}\,dx$

$u =$ $\qquad$ $dv =$

$du =$ $\xrightarrow[-\int]{}$ $v =$

Solution 2: $\int xe^{2x}\,dx$

$\underline{u}$ $\qquad$ $\underline{dv}$

Example 2: Find: (a) $\int x \sin 2x\,dx$ (b) $\int x^2 e^x dx$

4.1.2 Integrals of the Type: $\int x^n \ln(ax+b)\,dx$ or $\int x^n \tan^{-1}(ax+b)\,dx$, where n is a non-negative integer and a, b are constants

For such integrals, we let $u = \ln(ax+b)$ or $u = \tan^{-1}(ax+b)$ and $dv = x^n dx$.

***Example 3*:** Find $\int \ln(2x+1)\,dx$.

Solution: $\int \ln(2x+1)\,dx$

$\underline{u}$	$\underline{dv}$

Integration by parts formula for definite integrals:

$$\boxed{\int_a^b u\,dv = [u \cdot v]_a^b - \int_a^b v\,du}$$

Example 4: Evaluate $\int_1^{\mathrm{e}} x \ln x\,dx$.

Example 5: Find $\int \tan^{-1} x\,dx$.

4.2 Numerical Integration

We have learnt how to deal with various types of integrals. But there are still some integrals, that seem simple enough, yet they cannot be integrated by any of the standard method we have studied.

For examples, $\int_0^1 e^{-x^2}\,dx$, $\int_1^2 \ln\sqrt{1+x^3}\,dx$.

When a definite integral $\int_a^b f(x)\,dx$ does not fit into the 'normal patterns' of the standard integral, a numerical approximation of the integral is often sought after. One such approximation method is called **Simpson's rule**.

Numerical methods of integration are used to approximate definite integrals $\int_a^b f(x)\,dx$ when:

(i) The integrand $f(x)$ cannot be integrated by any standard integrals or integration methods that we have encountered so far.

For example: Evaluate $\int_2^4 \sqrt{1+x^3}\,dx$ $\left[\text{Here } f(x)=\sqrt{1+x^3}\right]$

(ii) The integrand $f(x)$ is not known explicitly but only as a set of points.

For example: Given the corresponding data in x and y in the table below:

x	1	2	3	4	5	6	7
$y=f(x)$	0.25	0.78	1.28	1.43	0.96	0.32	0.18

Since $y=f(x)$ is not defined, $\int_1^7 y\,dx$ cannot be integrated. Hence, the value of the integral $\int_1^7 y\,dx$ can only be approximated by numerical method.

4.3 Simpson's Rule

The two most popular numerical methods of integration are Trapezoid rule and Simpson's rule. However, we will learn only Simpson's rule in our module.

The definite integral $\int_a^b f(x)\,dx$ represents the area between the curve $y=f(x)$ and the x-axis from $x=a$ to $x=b$. So, even if $\int f(x)\,dx$ cannot be found in terms of elementary functions, an approximate value for $\int_a^b f(x)\,dx$ can be obtained by evaluating the appropriate area.

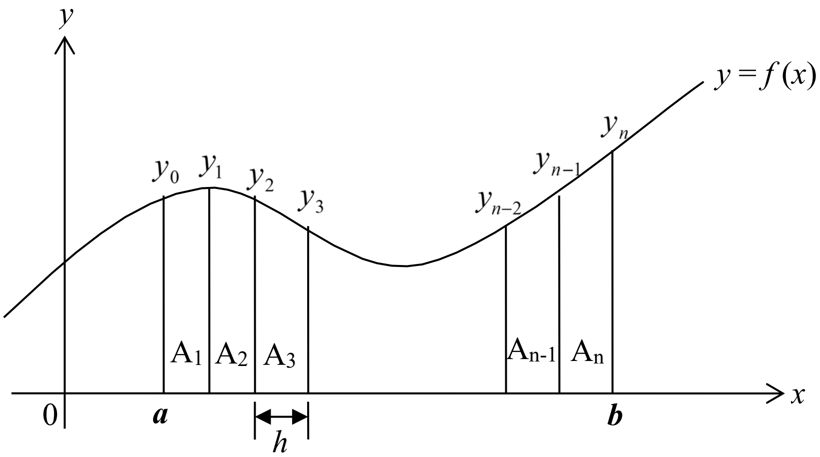
In Simpson's rule, this area is considered to be made up of areas of n vertical strips of equal width h, where n is an even integer.

To find the area under the curve $y = f(x)$ between $x = a$ and $x = b$:

1. Divide the area into an *even* number (n) of strips each of width $h = \dfrac{b-a}{n}$.
2. Number and evaluate each ordinate: $y_0,\ y_1,\ y_2,\ \cdots,\ y_n$.
3. The area is then given by:

$$\int_a^b f(x)\,dx \approx \frac{1}{3}\,h\left[y_0 + y_n + 4\left(y_1 + y_3 + y_5 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right)\right],$$

where $h = \dfrac{b-a}{n}$ and n is an even integer.



Example 6: By using Simpson's rule with 6 strips, determine the approximate value of the definite integral $\int_0^{\pi} \frac{1}{1+\sin x}\,dx$, correct to 2 decimal places.

Solution: Let $f(x) = \dfrac{1}{1+\sin x}$ and $h = \dfrac{b-a}{n} =$ [Note: x must be in radians]

x	0	$1\cdot\left(\frac{}{}\right)$	$2\cdot\left(\frac{}{}\right)$	$3\cdot\left(\frac{\pi}{6}\right)$	$4\cdot\left(\frac{\pi}{6}\right)$	$5\cdot\left(\frac{\pi}{6}\right)$	π
$\frac{1}{1+\sin x}$	1	0.6667			0.5359		

By Simpson's rule,

$$\int_0^{\pi} \frac{1}{1+\sin x}\,dx \approx \frac{h}{3}\left[y_0 + y_6 + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{1}{3}\cdot\left(\frac{\pi}{6}\right)[2 + 4(\qquad + \qquad + \qquad) + 2(\qquad + \qquad)]$$

$$= \frac{\pi}{18}(\qquad\qquad) =$$

Example 7: Evaluate $\int_1^2 \frac{\sin x}{x}\,dx$ using Simpson's rule with 4 strips, correct to 3 decimal places.

Solution:

***Example 8*:** The voltage (in mV) of a supply at regular intervals of 0.01 s over a half cycle is found to be:

0, 19.5, 35, 45, 40.5, 25, 20.5, 29, 27, 12.5 and 0

By Simpson's rule, find the r.m.s. value of the voltage over the half cycle, correct to 2 decimal places.

Solution: The r.m.s. value of the voltage is, $V_{rms} = \sqrt{\frac{1}{b-a}\int_a^b V^2 dt}$.

t						
V	0	19.5	35	45	40.5	25

t					
V	20.5	29	27	12.5	0

Tutorial 4

<u>Section A</u>: Integration by Parts

Find the following integrals:

1. $\int x \cos x \, dx$

2. $\int (x^2 + x) e^{2x} \, dx$

3. $\int x^2 \sin 3x \, dx$

4. $\int_0^1 x e^{-5x} \, dx$

5. $\int_1^e x^2 \ln x \, dx$

6. $\int \theta \sin^2 \theta \, d\theta$

7. $\int \ln(1-4x) \, dx$

<u>Section B</u>: Simpson's Rule

(Give all your final answers correct to 2 decimal places.)

1. Approximate the following integrals by Simpson's rule, using the number of intervals indicated:

   (a) $\int_0^1 \sqrt{1+x^3} \, dx \quad (n = 8)$

   (b) $\int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta \quad (n = 6)$

   (c) $\displaystyle\int_{\pi/2}^{3\pi/2} \frac{\sin x}{x} \, dx \quad (n = 6)$

   (d) $\int_0^2 e^{x^2} dx \quad (n = 4)$

2. Given that $V = 2\pi \int_a^b rh \, dr$, where the values of r and h are given in the table below:

r	0	1	2	3	4	5	6
h	0.599	1.072	1.415	1.588	1.579	1.428	1.003

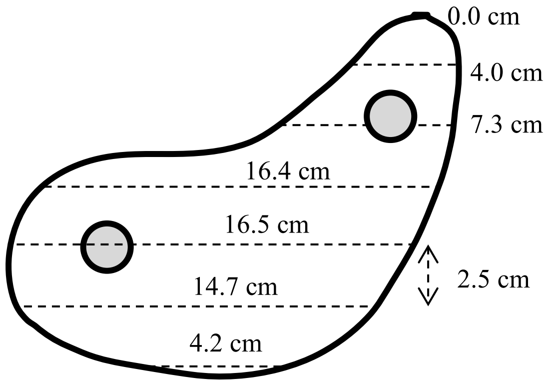
   Use Simpson's rule to find the approximate value of V.

3. When a battery is applied to the sending end of a long telegraph line, the growth of received current i (mA) at 5 ms intervals is given by:

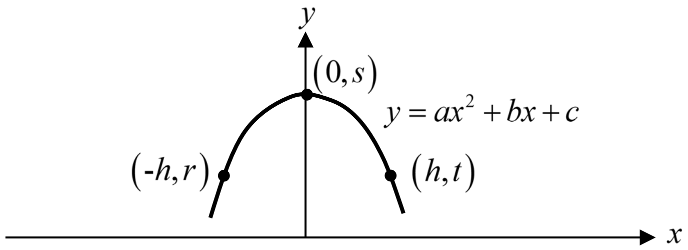
Time t	0	5	10	15	20	25	30	35	40
Current i	0	1.5	7	13	16	18	19	19.5	20

   Use Simpson's rule to evaluate the **root-mean-square (RMS)** value of the current in the complete 40 ms interval.

4. The widths of a bell crank are measured at 2.5 cm intervals as shown in the diagram on the right. Find the approximate area, accurate to two decimal places, of the bell crank if the two connector holes are each 1.5 cm in diameter.

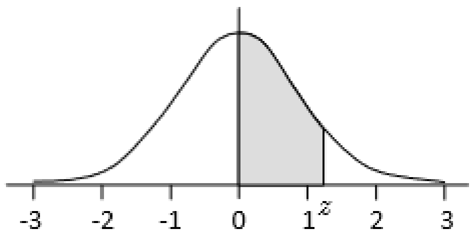


* 5.



Show that the area under a quadratic curve $y = ax^2 + bx + c$ in the interval $-h \le x \le h$ is $\frac{h}{3}(r + 4s + t)$, where r , s and t are the y-coordinates of the three points shown in the graph above. Hence or otherwise, prove the Simpson's rule formula.

*6. A very large number of random variables observed in nature follow a frequency distribution that is approximately bell-shaped. Statistician called this a normal probability distribution.



The function that describes the standardized normal curve is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

The probability of z taking on values from a to b, denoted by $P(a < z < b)$, is given by the area under the curve between $z = a$ and $z = b$. Hence,

$$P(a < z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Unfortunately, $f(z)$ does not have an antiderivative in terms of elementary functions. Hence, we have to use numerical integration method like the Simpson's Rule to evaluate this integral.

Use Simpson's Rule to evaluate $P(0 < z < 1.2) = \int_0^{1.2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$, using $n = 6$.

Miscellaneous Exercises

*1. Given that $\dfrac{d^2y}{dx^2}=\dfrac{1}{x}$, find y in terms of x.

*2. Find the following:

(a) $\displaystyle\int \frac{\ln(x)}{(2x+1)^3}\,dx$ (b) $\displaystyle\int \frac{x\sin^{-1}(2x)}{\sqrt{1-4x^2}}\,dx$

*3. Evaluate $\displaystyle\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$.

*4. Show that $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$. Hence, find $\displaystyle\int x^3 e^x\,dx$.

*5. $\displaystyle\int e^{5x}\cos 2x\,dx$. Learn from this video: https://youtu.be/c_PGa3DAFno

*6. By writing $\sin^n x = \sin^{n-1}x\,\sin x$, derive the reduction formula for

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\,\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

Hence, determine $\displaystyle\int \sin^5 x\,dx$.

Multiple Choice Questions

1. To find $\displaystyle\int x\sec^2(5x)\,dx$ using integration by parts, we choose

(a) $u = x$ and $dv = \sec^2(5x)dx$ (b) $u=\sec^2(5x)$ and $dv = x\,dx$

(c) $u = x\,dx$ and $dv=\sec^2(5x)$ (d) $u=\sec^2(5x)\,dx$ and $dv = x$

2. The number of panels or strips to be considered in Simpson's rule must be ________.

(a) Odd (b) Even

3. The exact solution of a definite integral can be obtained using the Simpson's rule.

(a) True (b) False

4. A definite integral $\displaystyle\int_0^3 \sqrt{1-x^2}\,dx$ is evaluated using the Simpson's rule with 8 strips. Which of the following could be used to increase the accuracy of the final answer?

(a) Evaluate the definite integral by integrating the function $\sqrt{1-x^2}$ and substituting the limits of integration.

(b) Use the trapezoid method instead of Simpson's rule using the same number of strips.

(c) Reduce the number of strips from 8 to 4.

(d) Increase the number of strips from 8 to 16.

Answers to Selected Lecture Examples

Example 1: $\dfrac{xe^{2x}}{2}-\dfrac{e^{2x}}{4}+C$

Example 2: (a) $-\dfrac{1}{2}x\cos 2x+\dfrac{1}{4}\sin 2x+C$ (b) $x^2e^x-2xe^x+2e^x+C$

Example 3: $x\left[\ln(2x+1)\right]-x+\dfrac{1}{2}\ln\left|2x+1\right|+C$ Example 4: 2.097

Example 5: $x\tan^{-1}x-\dfrac{1}{2}\ln\left|1+x^2\right|+C$

Example 6: 2.00 Example 7: 0.659 Example 8: 28.37 mV

Answers to Tutorial 4

Section A

1. $x\sin x+\cos x+C$
2. $\dfrac{1}{2}x^2e^{2x}+C$
3. $-\dfrac{x^2}{3}\cos 3x+\dfrac{2}{9}x\sin 3x+\dfrac{2}{27}\cos 3x+C$
4. 0.0384
5. 4.575
6. $\dfrac{\theta^2}{4}-\dfrac{\theta}{4}\sin 2\theta-\dfrac{1}{8}\cos 2\theta+C$
7. $x\,\ell n(1-4x)-x-\dfrac{1}{4}\ell n(1-4x)+C$

Section B

1. (a) 1.11 (b) 1.19 (c) 0.24 (d) 17.35
2. 159.62
3. 14.77 mA
4. 156.63 cm^2
6. 0.3849 or 38.49%

Miscellaneous Exercises

1. $y(x)=x\ln x+Ax+B$
2. (a) $\dfrac{1}{4}\left[\ln|x|-\ln|2x+1|+\dfrac{1}{2x+1}\right]-\dfrac{\ln|x|}{4(2x+1)^2}+C$ (b) $\dfrac{1}{4}\left[-\sin^{-1}(2x)\sqrt{1-4x^2}+2x\right]+C$
3. $\dfrac{1}{3}\left(2\sqrt{3}\,\pi-\dfrac{\pi}{2}-2+\ln 2\right)$ or 2.668
4. $x^3e^x-3x^2e^x+6xe^x-6e^x+C$
5. $\dfrac{1}{29}e^{5x}(2\sin 2x+5\cos 2x)+C$
6. $-\dfrac{1}{5}\sin^4 x\cos x-\dfrac{4}{15}\sin^2 x\cos x-\dfrac{8}{15}\cos x+C$

MCQ

1. (a) 2. (b) 3. (b) 4. (d)

Chapter 5: Fourier Series

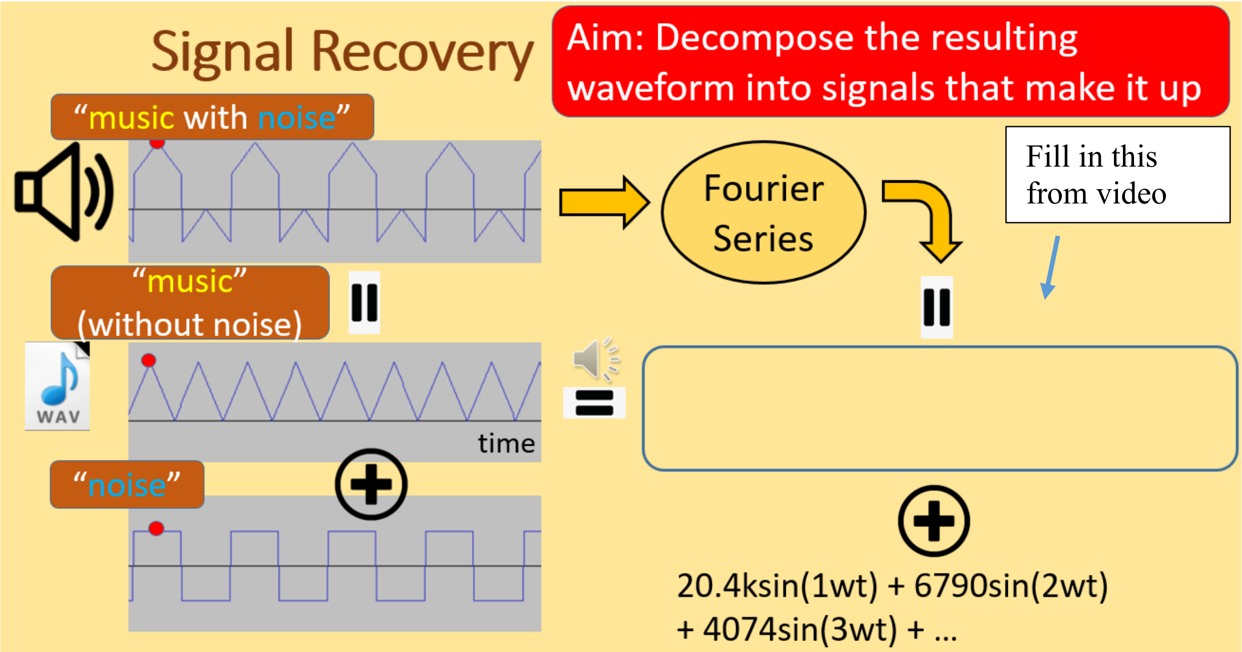
Objectives:

1. *Application: Signal Recovery Using Fourier Series*
2. *Define periodic functions.*
3. *Obtain the Fourier Series of a periodic function.*
4. *Distinguish odd and even functions.*
5. *Obtain the Fourier series of an odd or even periodic function by using the properties of odd and even functions.*

Content

5.1 Application: Signal Recovery using Fourier Series

A music producer is listening to a piece of music and comments that the background noise is too loud, he asks if we can reduce the background noise. To do that, we need an algorithm to separate the noise and the music, the below picture illustrated the process.



Signal recovery process

With reference to the picture above, the steps below outline the algorithm for signal recovery:

1. The desired signal (aka "music" without noise) and noise can usually be distinguished by certain properties.
2. To separate them, we decompose the resulting waveform (aka "music with noise") into the sum of sine and cosine terms using the Fourier series.
3. With the terms from the Fourier series, we can recover the desired signal by extracting the terms that make up that signal. In the above picture, in our example, that is the sum of the cosine terms.

The video in Brightspace explains the above algorithm in detail.

Complete the following sentence from the video:

In conclusion, a Fourier series ____

____

In the subsequent sections, we will learn how to find the Fourier series of periodic functions. Let's start by learning how to sketch a periodic function.

5.2 Periodic Function

A **periodic function** is a function which repeats itself at regular intervals.
A function $f(t)$ is said to be periodic with period T if $f(t+T) = f(t)$.

Example 1: Sketch two cycles of the following periodic functions:

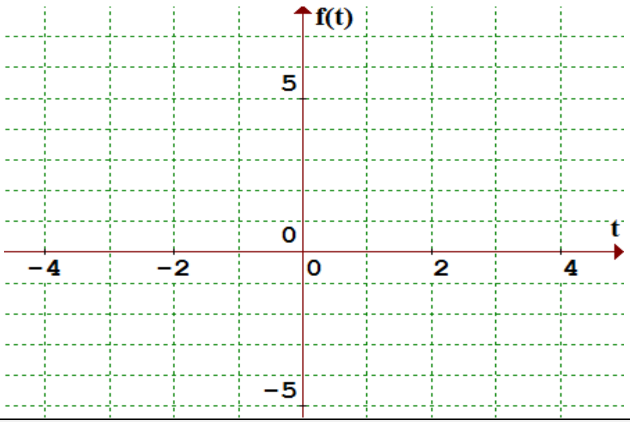
(a) $f(t) = \begin{cases} 2 & , \quad -1 < t < 0 \\ 4 & , \quad 0 < t < 1 \end{cases} \qquad f(t+2) = f(t)$

(b) $f(t) = \begin{cases} t+3 & , \quad 0 < t < 2 \\ t-7 & , \quad 2 < t < 4 \end{cases} \qquad f(t+4) = f(t)$

Solution:

(a)

(b)



5.3 Fourier Series

Any periodic function of period T can be expressed by an infinite series of the form:

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,\omega_0 t + b_n \sin n\,\omega_0 t) \\ &= a_0 + a_1 \cos \omega_0 t \;+\; a_2 \cos 2\omega_0 t \;+\; a_3 \cos 3\omega_0 t \;+ \cdots \\ &\qquad + b_1 \sin \omega_0 t \;+\; b_2 \sin 2\omega_0 t \;+\; b_3 \sin 3\omega_0 t \;+ \cdots \end{aligned}$$

where a_0, a_n, and b_n are constants for $n = 1, 2, 3, \ldots$ and $\omega_0 = \dfrac{2\pi}{T}$.

This series is called the **trigonometric form of a Fourier series**.

Terminology: a_0 is referred to as the *d-c component*.
a_1 is referred to as the *amplitude of the fundamental cosine component*.
b_1 is referred to as the *amplitude of the fundamental sine component*.
a_n is referred to as the *amplitude of the* n^{th} *cosine component*.
b_n is referred to as the *amplitude of the* n^{th} *sine component*.

The Fourier series exists if the following conditions (called the *Dirichlet* conditions) are satisfied:

1. it is a single-valued function,
2. if it is discontinuous, then there is a finite number of discontinuous in the period of T,
3. it has a finite number of positive and negative maxima and minima in any one period,
4. it has a finite average value for the period T.

The series will converge to the value $f(a)$ if $f(t)$ is continuous at $t = a$, and converges to $\frac{1}{2}\left[f(a+)+f(a-)\right]$ if $f(t)$ is not continuous at $t = a$. Note the use of $a+$ to denote approaching a from the right and $a-$ to denote approaching a from the left.

5.4 Some Special Integrals

The following integrals are used throughout the theory of Fourier series:

(i) $\displaystyle\int_K^{K+T} \cos n\omega_0 t \, dt = 0$

(ii) $\displaystyle\int_K^{K+T} \sin n\omega_0 t \, dt = 0$

(iii) $\displaystyle\int_K^{K+T} \cos m\omega_0 t \cdot \cos n\omega_0 t \, dt = \begin{cases} 0 & \text{, if } m \neq n \\ T/2 & \text{, if } m = n \neq 0 \\ T & \text{, if } m = n = 0 \end{cases}$

(iv) $\displaystyle\int_K^{K+T} \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = \begin{cases} 0 & \text{, if } m \neq n \\ T/2 & \text{, if } m = n \end{cases}$

(v) $\displaystyle\int_K^{K+T} \cos m\omega_0 t \cdot \sin n\omega_0 t \, dt = 0$

where m, n are positive integers, K is a constant, and $\omega_0 = \dfrac{2\pi}{T}$.

Proof of (iii), when $m \neq n$:

$$\int_K^{K+T} \cos\, m\,\omega_0 t \cdot cos\ n\,\omega_0 t\ dt$$

$$= \frac{1}{2}\int_K^{K+T}\left[\cos(m+n)\omega_0 t + \cos(m-n)\omega_0 t\right] dt \qquad \text{(applied Product to Sum Identities)}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\omega_0 t}{(m+n)\omega_0} + \frac{\sin(m-n)\omega_0 t}{(m-n)\omega_0}\right]_K^{K+T} \qquad \text{(where } T = \frac{2\pi}{\omega_0}\text{)}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)(\omega_0 K + 2\pi)}{(m+n)\omega_0} + \frac{\sin(m-n)(\omega_0 K + 2\pi)}{(m-n)\omega_0}\right] - \frac{1}{2}\left[\frac{\sin(m+n)\omega_0 K}{(m+n)\omega_0} + \frac{\sin(m-n)\omega_0 K}{(m-n)\omega_0}\right]$$

$$= 0 \text{, if } m \neq n$$

Note: $\sin\left[2\pi(m+n) + \omega_0 K(m+n)\right] = \sin(m+n)\omega_0 K$

5.5 Determination of the Fourier Coefficients

$$\begin{aligned} f(t) = a_0 &+ a_1\cos\omega_0 t + a_2\cos 2\omega_0 t + a_3\cos 3\omega_0 t + \cdots \\ &+ b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + b_3\sin 3\omega_0 t + \cdots \end{aligned}$$

To find a_0, integrate both sides of the above equation over one period:

$$\begin{aligned} \int_K^{K+T} f(t)\,dt &= \int_K^{K+T} a_0\,dt + a_1\int_K^{K+T}\cos\,\omega_0 t\,dt + a_2\int_K^{K+T}\cos\,2\omega_0 t\,dt + \cdots \\ &\quad + b_1\int_K^{K+T}\sin\omega_0 t\,dt + b_2\int_K^{K+T}\sin 2\omega_0 t\,dt + \cdots \\ &= a_0\left[\,t\,\right]_K^{K+T} \\ &= a_0 T \end{aligned}$$

Hence, $$\boxed{a_0 = \frac{1}{T}\int_K^{K+T} f(t)\,dt}$$

To find a_n, multiply both sides by $\cos\, n\,\omega_0 t$ and integrate over one period:

$$\begin{aligned} \int_K^{K+T} f(t)\cos\, n\,\omega_0 t\,dt &= a_0\int_K^{K+T}\cos n\,\omega_0 t\,dt \\ &\quad + a_1\int_K^{K+T}\cos\omega_0 t\cos n\,\omega_0 t\,dt + \cdots + a_n\int_K^{K+T}\cos^2 n\,\omega_0 t\,dt \\ &\quad + b_1\int_K^{K+T}\sin\omega_0 t\cos n\omega_0 t\,dt + \cdots + b_n\int_K^{K+T}\sin n\,\omega_0 t\cos n\,\omega_0 t\,dt \\ &= a_n\left(\frac{T}{2}\right) \end{aligned}$$

Hence,
$$\boxed{a_n = \frac{2}{T}\int_K^{K+T} f(t)\cos n\,\omega_0 t\ dt}$$

Similarly,
$$\boxed{b_n = \frac{2}{T}\int_K^{K+T} f(t)\sin n\,\omega_0 t\ dt}$$

Example 2: A periodic function $f(t)$ is defined by:

$$f(t)=\begin{cases}1 & , \quad 0<t<\pi \\ 0 & , \quad \pi<t<2\pi\end{cases} \qquad \text{and} \qquad f(t+2\pi)=f(t)\,.$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonics.

Example 3: A periodic function $f(t)$ is defined by:

$$f(t) = \begin{cases} t & , \quad 0 < t < 1 \\ 1 & , \quad 1 < t < 2 \end{cases} \quad \text{and} \quad f(t+2) = f(t)\,.$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonics.

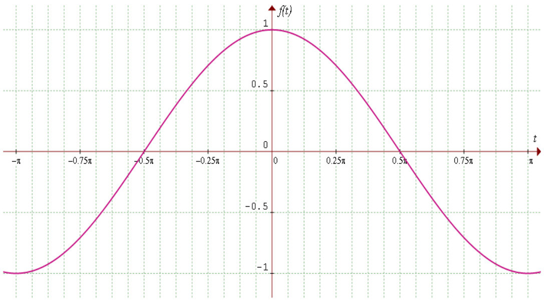
5.6 Odd and Even Functions

5.6.1 Even Function

A function $f(t)$, defined in the interval $-\frac{T}{2} < t < \frac{T}{2}$, is said to be **even** if $f(-t) = f(t)$ for every value of t in the interval.

The graph of an even function is symmetrical about the vertical axis.

For example, $f(t) = \cos t$ is an even function.

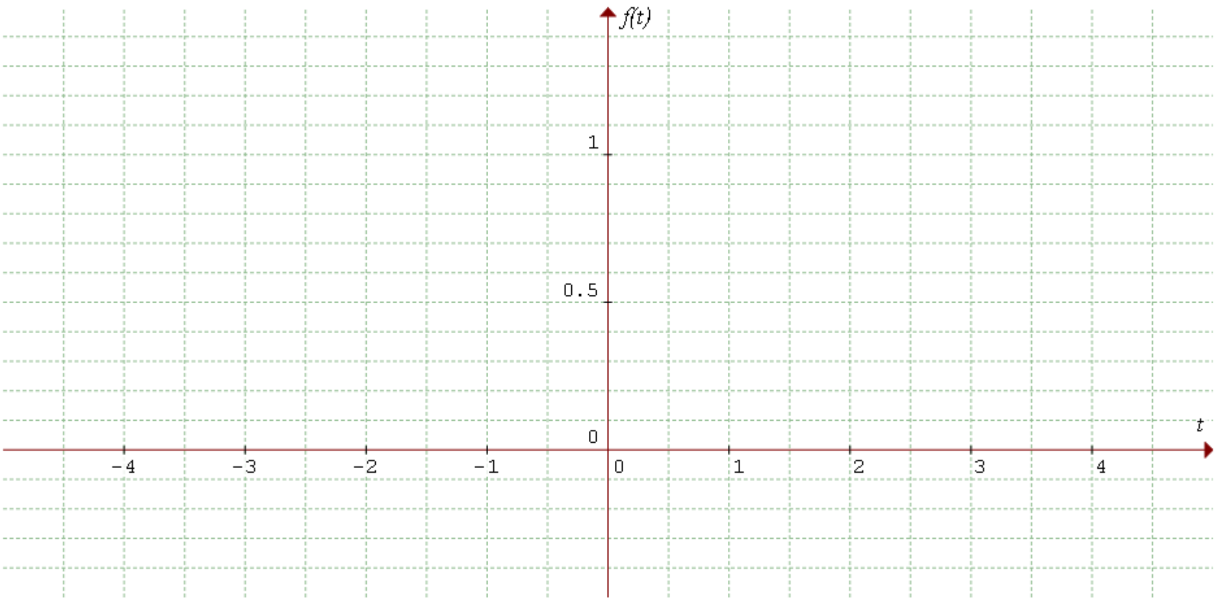


Example 4 Sketch one cycle of the function:

$$f(t) = \begin{cases} 0 & , \quad -2 < t < -1 \\ 1 & , \quad -1 < t < 1 \\ 0 & , \quad 1 < t < 2 \end{cases} \qquad \text{and} \qquad f(t+4) = f(t).$$

Is $f(t)$ an even function?

Solution:

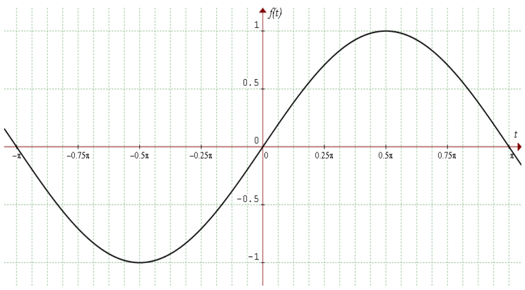


5.6.2 Odd Function

A function $f(t)$, defined in the interval $-\frac{T}{2} < t < \frac{T}{2}$, is said to be **odd** if $f(-t) = -f(t)$ for every value of t in the interval. Note that for this definition to be consistent, $f(0) = 0$.

The graph of an odd function is symmetrical about the origin.

For example, $f(t) = \sin t$ is an odd function.

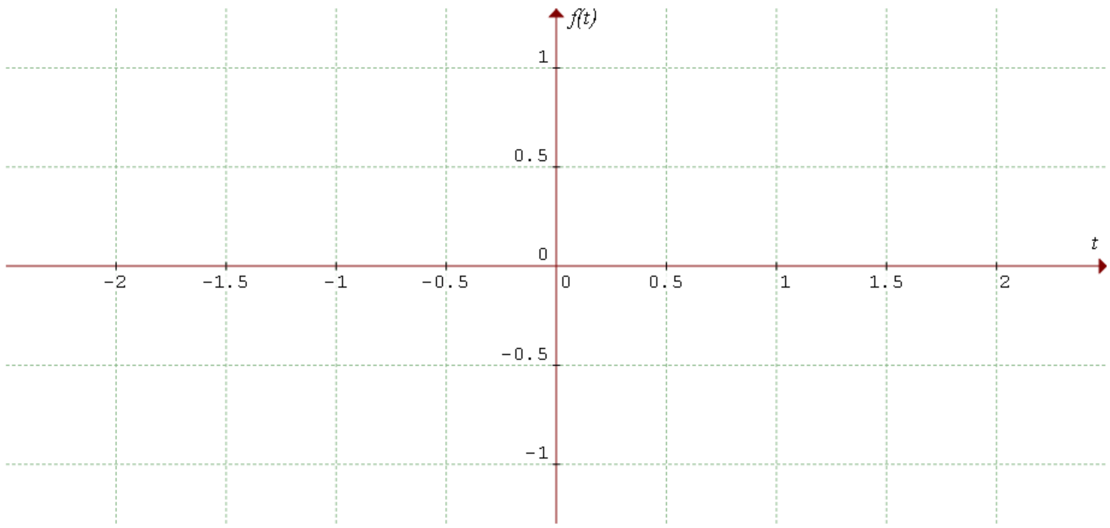


Example 5 Sketch one cycle of the function:

$$f(t) = \begin{cases} -t-1 & , \quad -1 < t < 0 \\ 0 & , \quad t = 0 \\ -t+1 & , \quad 0 < t < 1 \end{cases} \qquad \text{and} \qquad f(t+2) = f(t).$$

Is $f(t)$ an odd function?

Solution:



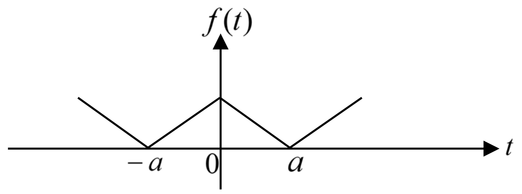
5.6.3 Properties of Odd and Even Functions

There are some special properties of odd and even functions when we take sum or product of them:

1. The product of two odd functions is an *even* function.
2. The product of two even functions is an *even* function.
3. The sum of two odd functions is an *odd* function.
4. The sum of two even functions is an *even* function.
5. The product of an odd function and an even function is an *odd* function.
6. The sum of an odd function and an even function is *neither even nor odd*.

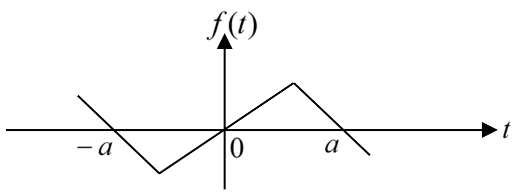
Two useful facts emerge from odd and even functions, in terms of area under their graphs.

- Area under the graph of an **even** function: $\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$



Since the graph of an even function is symmetrical about the y-axis, then the area under the graph from $-a$ to a will be symmetrical too. Thus, we only need to integrate half the interval from 0 to a, but multiply the integral by a factor of two.

- Area under the graph of an **odd** function: $\int_{-a}^{a} f(t)\,dt = 0$



Due to its special symmetry, the area under the graph from $-a$ to 0 will be equal to the area under the graph from 0 to a, but opposite in signs. Thus, the areas will cancel out.

5.7 Fourier Series for Odd and Even Functions

Let $f(t)$ be a periodic function of period T. The Fourier series of $f(t)$ is given by:

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega_0 t + b_n \sin n\,\omega_0 t)$$

$$= a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots$$
$$+ b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

where $\omega_0 = \dfrac{2\pi}{T}$

$$a_0 = \frac{1}{T}\int_K^{K+T} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_K^{K+T} f(t)\cdot\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_K^{K+T} f(t)\cdot\sin n\omega_0 t\,dt$$

By making use of the properties of odd and even functions, the formulae for finding the Fourier series coefficients can be simplified.

- If $f(t)$ is an **even** periodic function, then

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\cos\, n\omega_0 t\,dt$$

$$b_n = 0$$

- If $f(t)$ is an **odd** periodic function, then

$$a_0 = 0$$

$$a_n = 0$$

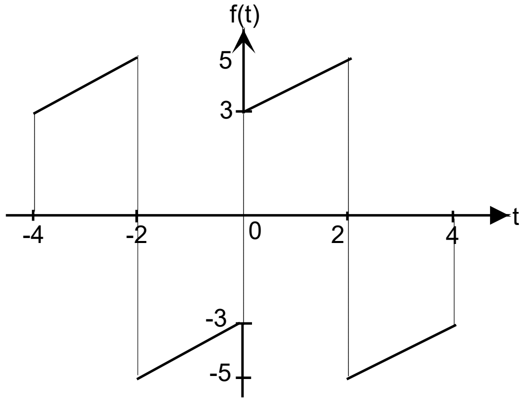
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\sin n\omega_0 t\,dt$$

Note: The Fourier coefficients formulae for odd and even periodic functions will <u>not</u> be provided during examination. Students are required to derive these formulae from the standard ones in the formulae card.

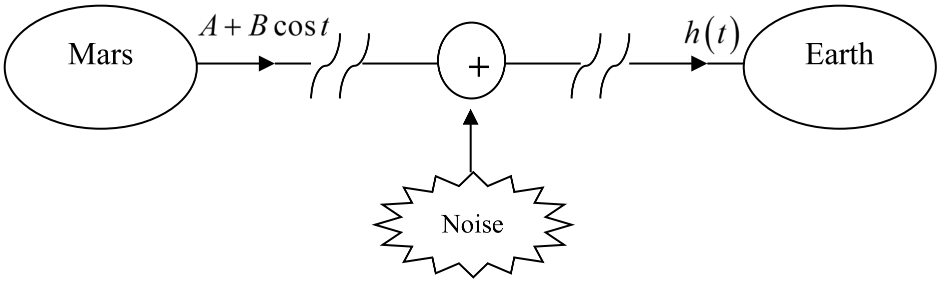
Example 6: The graph of the function $f(t)=\begin{cases} t+3 & 0<t<2 \\ t-7 & 2<t<4 \end{cases}$ and $f(t+4)=f(t)$ for $-4<t<4$ is shown below.

(a) Indicate whether $f(t)$ is even or odd or neither.

(b) Determine the Fourier series of $f(t)$ up to the fourth harmonics.



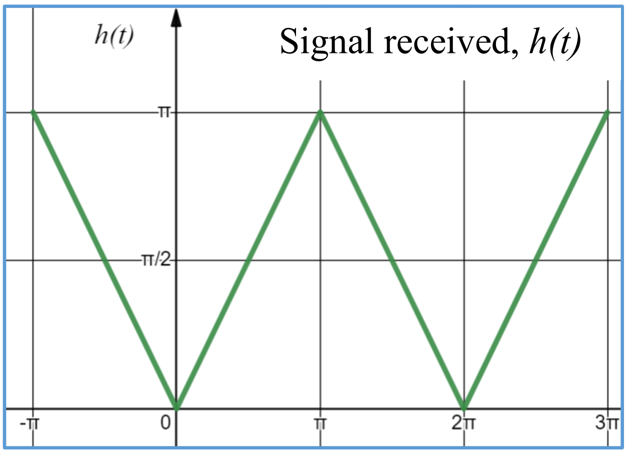
Example 7:



Suppose that a spacecraft from Mars has measured a quantity B and sent it to earth in the form of a periodic signal $A + B\cos t$ of amplitude B. On its way to earth, the signal picks up periodic noise, containing only second and higher harmonics. Suppose that the signal

$$h(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$

of period 2π actually received on earth is graphed as on the right.



(i) State whether the function $h(t)$ is even or odd.

(ii) Find the Fourier series of $h(t)$ as far as the second harmonic. Given that $a_0 = \dfrac{\pi}{2}$.

(iii) Determine the signal that the spacecraft originally sent and hence the value B of the measurement.

Tutorial 5.1

1. Sketch the waveforms of the following periodic functions.

(a) $f(x)=\begin{cases}0 & , & 0<x<2\\ 1 & , & 2<x<6\\ 2 & , & 6<x<10\end{cases}$ and $f(x+10)=f(x)$

(b) $f(t)=\begin{cases}t & , & 0<t<2\\ 0 & , & 2<t<4\end{cases}$ and $f(t+4)=f(t)$

2. A periodic function $f(t)$ of period 4 is defined as:

$$f(t)=\begin{cases}t-1 & , & -1<t<1\\ 2 & , & 1<t<3\end{cases} \quad \text{and} \quad f(t+4)=f(t)$$

Find:

(a) the d.c. component (i.e. a_0),

(b) the second sine harmonic (i.e. $b_2\sin(2\omega t)$), and

(c) the third cosine harmonic (i.e. $a_3\cos(3\omega t)$) of the Fourier series of $f(t)$.

3. A periodic function $f(t)$ is defined by:

$$f(t)=\begin{cases}3 & , & 0<t<\pi\\ -1 & , & \pi<t<2\pi\end{cases} \quad \text{and} \quad f(t+2\pi)=f(t)$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonics.

4. $f(t)$ is a periodic function defined over one period as follows:

$$f(t)=\begin{cases}2 & , & 0<t<\pi\\ -1 & , & \pi<t<3\pi/2\\ 0 & , & 3\pi/2<t<2\pi\end{cases} \quad \text{and} \quad f(t+2\pi)=f(t)$$

Find the Fourier series of $f(t)$ as far as the third harmonics.

5. A function $f(t)$ is defined by:

$$f(t)=\begin{cases}t & , & 0<t<1\\ 0 & , & 1<t<2\end{cases} \quad \text{and} \quad f(t+2)=f(t)$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonics.

Tutorial 5.2

1. Sketch the following periodic functions and state whether the functions are even, odd or "neither even nor odd".

   (a) $f(t)=\begin{cases} t+1 & , \quad -1<t<0 \\ -t+1 & , \quad 0<t<1 \end{cases}$ and $f(t+2)=f(t)$

   (b) $f(t)=t \quad , \quad -\pi<t<\pi$ and $f(t+2\pi)=f(t)$

   (c) $f(t)=t^2 \quad , \quad 0<t<2$ and $f(t+2)=f(t)$

2. A function $f(t)$ is defined by:

$$f(t)=\begin{cases} -0.5 & , \quad -2<t<-1 \\ 0.5 & , \quad -1<t<1 \\ -0.5 & , \quad 1<t<2 \end{cases} \qquad \text{and} \qquad f(t+4)=f(t)$$

   (a) Sketch the graph and indicate whether $f(t)$ is even or odd.

   (b) Find the Fourier series of $f(t)$ as far as the third harmonics.

3. A periodic function $f(t)$ is defined as:

$$f(t)=\begin{cases} -1 & , \quad -2<t<0 \\ 0 & , \quad t=0 \\ 1 & , \quad 0<t<2 \end{cases} \qquad \text{and} \qquad f(t+4)=f(t)$$

   Find the Fourier series of $f(t)$ as far as the third harmonics.
   (Hint: First find out whether the function is odd or even, then make full use of the advantages of this classification.)

*4. A periodic function $f(t)$ of period 2 is defined over <u>half</u> a period as follows:

$$f(t)=t^2 \quad , \quad 0<t<1$$

   If $f(t)$ is an even function,

   (i) sketch the graph of $f(t)$ for $-1<t<1$.

   (ii) Find the Fourier Series of $f(t)$ as far as the third harmonics.

*5. A periodic function $f(t)$ of period 2π is defined over <u>half</u> a period as follows:

$$f(t) = 2t \quad , \quad 0 < t < \pi$$

If $f(t)$ is an odd function,

(i) sketch the graph of $f(t)$ for $-\pi < t < \pi$.

(ii) Find the Fourier Series of $f(t)$ as far as the third harmonics.

Miscellaneous Exercises

*1. Show that the Fourier series of $f(t) = t$, $-\pi < t < \pi$, and $f(t+2\pi) = f(t)$ is

$f(t) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nt)}{n}$.

Hence, deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.

*2. Show that the Fourier series of

$$f(t) = \begin{cases} 0 & , \quad -1 < t < 0 \\ t+1 & , \quad 0 < t < 1 \end{cases} \qquad \text{and} \qquad f(t+2) = f(t)$$

is $f(t) = \frac{3}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t}{(2n-1)^2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi t}{2n-1} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi t}{2n}$.

*3. A periodic function $f(t)$ of period 4 is defined as:

$$f(t) = 4 - t^2 \quad , \quad -2 \le t \le 2 \quad \text{and} \quad f(t+4) = f(t)$$

(a) Sketch the graph of $f(t)$ for the interval $-2 \le t \le 2$.

(b) Show that the Fourier series of $f(t)$ is

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2}\left(\cos\frac{\pi t}{2} - \frac{1}{4}\cos\pi t + \frac{1}{9}\cos\frac{3\pi t}{2} + \cdots\right) .$$

(c) Choose a suitable value of t in the Fourier series of $f(t)$ above to show that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{12} .$$

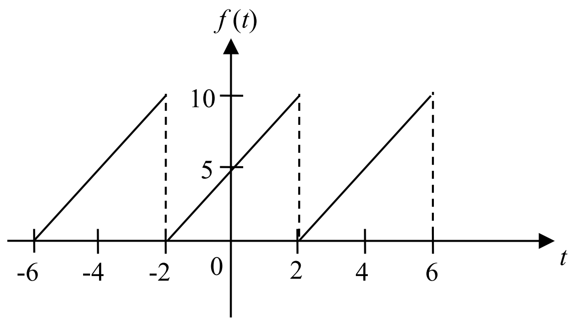
*4. Sketch the graphs of the following periodic functions for two periods and classify them as even, odd or "neither even nor odd". Find their Fourier series.

(a) $f(t) = |t|$, $-1 < t < 1$ and $f(t+2) = f(t)$

(b) $f(t) = t - t^3$, $-1 < t < 1$ and $f(t+2) = f(t)$

Multiple Choice Questions

1.



In the figure above, $f(t)$ is a periodic function. The period of $f(t)$ is

(a) 2 (b) 4

(c) 6 (d) 10

2. The d.c. component a_0 of the Fourier series of $f(t)$ (as shown in the figure in MCQ 1) is

(a) 0 (b) 2

(c) 5 (d) 10

3. The trigonometric Fourier series representation of the periodic function $f(t)$ of period 2π is given by $f(t) = \frac{4}{\pi^2}\left(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \cdots\right) + \frac{1}{\pi}\left(\sin t - 2\sin 3t + 3\sin 5t + \cdots\right) + \cdots$.

Then $f(t)$ is

(a) an even function (b) an odd function

(c) an odd function plus constant (d) a function with no symmetry

4. The d.c. component a_0 of the Fourier series of $f(t)$ (as given in MCQ 3) is

(a) 0 (b) 1

(c) 2 (d) $\frac{4}{\pi^2}$

*5. If the Fourier series of a periodic function $f(t)$ of period π is given by

$$f(t) = \frac{2}{\pi} + \frac{4}{3\pi}\cos 2t - \frac{4}{15\pi}\cos 4t + \frac{4}{35\pi}\cos 6t + \cdots ,$$

then the value of $\int_0^{\pi} f(t)\cos 2t \, dt$ is given by

(a) $\frac{4}{3\pi}$ (b) $\frac{4}{3}$

(c) $\frac{2}{3}$ (d) none of the above

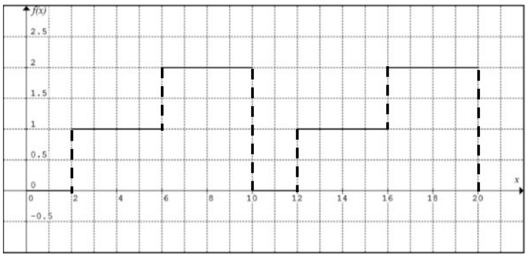
Answers to Selected Lecture Examples

Example 2: $f(t)=\frac{1}{2}+\frac{2}{\pi}\sin t+\frac{2}{3\pi}\sin 3t+\cdots$

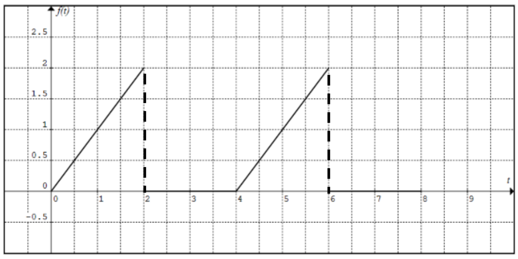
Example 6: $f(t)=\frac{16}{\pi}\sin\frac{\pi}{2}t-\frac{2}{\pi}\sin \pi t+\frac{16}{3\pi}\sin\frac{3\pi}{2}t-\frac{1}{\pi}\sin 2\pi t+......$

Answers to Tutorial 5.1

1. (a)



(b)



2. (a) $\frac{1}{2}$ (b) $\frac{1}{\pi}\sin(\pi t)$ (c) $\frac{2}{\pi}\cos\left(\frac{3\pi}{2}t\right)$

3. $a_0=1$, $a_n=0$, $b_n=\frac{4}{n\pi}\left[1-\cos\left(n\pi\right)\right]$; $f(t)=1+\frac{8}{\pi}\sin t+\frac{8}{3\pi}\sin 3t+\cdots$

4. $a_0=\frac{3}{4}$, $a_n=-\frac{1}{n\pi}\sin\left(\frac{3n\pi}{2}\right)$, $b_n=\frac{1}{\pi}\left[-\frac{3}{n}\cos\left(n\pi\right)+\frac{2}{n}+\frac{1}{n}\cos\left(\frac{3n\pi}{2}\right)\right]$;

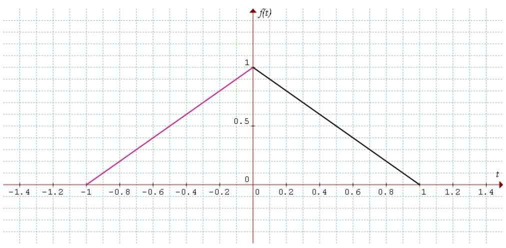
$f(t)=\frac{3}{4}+\frac{1}{\pi}\cos t-\frac{1}{3\pi}\cos 3t+\cdots+\frac{5}{\pi}\sin t-\frac{1}{\pi}\sin 2t+\frac{5}{3\pi}\sin 3t+\cdots$

5. $a_0=\frac{1}{4}$, $a_n=\frac{1}{(n\pi)^2}\left[\cos\left(n\pi\right)-1\right]$, $b_n=-\frac{1}{n\pi}\cos\left(n\pi\right)$;

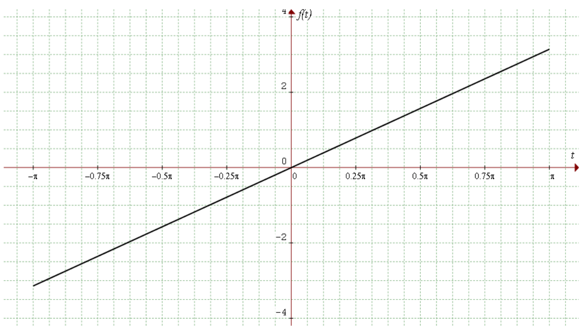
$f(t)=\frac{1}{4}-\frac{2}{\pi^2}\cos \pi t-\frac{2}{9\pi^2}\cos 3\pi t+\ldots+\frac{1}{\pi}\sin \pi t-\frac{1}{2\pi}\sin 2\pi t+\frac{1}{3\pi}\sin 3\pi t+\ldots$

Answers to Tutorial 5.2

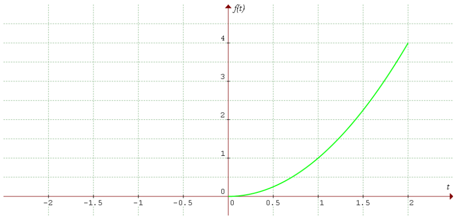
1. (a) even



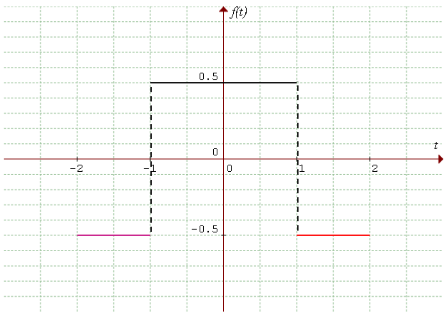
(b) odd



(c) neither odd nor even



2. (a) even

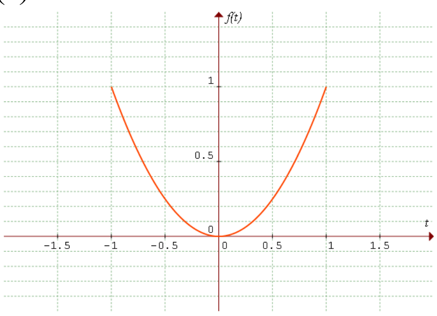


(b) $a_0 = 0$, $a_n = \dfrac{2}{n\pi}\sin\left(\dfrac{n\pi}{2}\right)$;

$$f(t) = \frac{2}{\pi}\cos\frac{\pi t}{2} - \frac{2}{3\pi}\cos\frac{3\pi t}{2} + \ldots$$

3. $b_n = -\dfrac{2}{n\pi}\left[\cos(n\pi) - 1\right]$; $f(t) = \dfrac{4}{\pi}\sin\dfrac{\pi t}{2} + \dfrac{4}{3\pi}\sin\dfrac{3\pi t}{2} + \cdots$

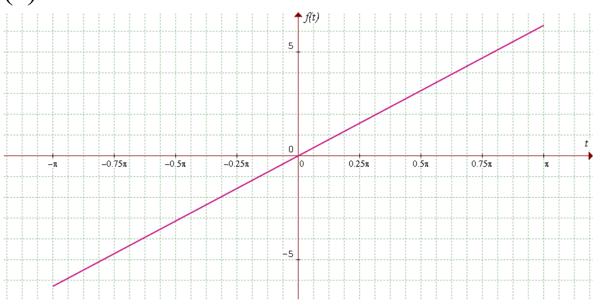
4. (a)



(b) $a_0 = \dfrac{1}{3}$, $a_n = \dfrac{4}{(n\pi)^2}\cos(n\pi)$, $b_n = 0$;

$$f(t) = \frac{1}{3} - \frac{4}{\pi^2}\cos\pi t + \frac{1}{\pi^2}\cos 2\pi t - \frac{4}{9\pi^2}\cos 3\pi t + \ldots$$

5. (a)



(b) $a_0 = 0$, $a_n = 0$, $b_n = -\dfrac{4}{n}\cos(n\pi)$;

$$f(t) = 4\sin t - 2\sin 2t + \frac{4}{3}\sin 3t + \cdots$$

Miscellaneous Exercises

3. (c) $t = 0$

4. (a) even , $f(t) = \dfrac{1}{2} - \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)\pi t}{(2n-1)^2}$ (b) odd , $f(t) = \dfrac{12}{\pi^3}\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n^3}\sin(n\pi t)$

MCQ

1. (b) 2. (c) 3. (d) 4. (a) 5. (c)

Chapter 6: Laplace Transform

Objectives:

1. *Application: Optimizing Design of Bungee Jumping Systems*
2. *Understand the definition of Laplace transform.*
3. *Know the Laplace transforms of common functions.*
4. *Use standard results and apply basic theorems.*

Content

6.1 Application: Optimizing Design of Bungee Jumping Systems

Figure 1: Bungee Jump

As shown in ***Figure 1***, a person is doing bungee jump. A bungee jump consists of three phases: free-fall of jumper when the rope is still slack, the stretch phase until the rope reaches its maximum length and the rebound phase. It is an example of **damped oscillatory motion** where the amplitude of oscillation reduces with time due to external forces like friction, air resistance and other resistive forces.

Figure 2: Maximum Displacement of Jumper

To design and optimize bungee jumping systems, we may want to know the **maximum displacement** of a jumper i.e. how far from ground level the jumper will be at the **lowest point** as shown in above ***Figure 2***.

The displacement of the jumper is the distance between the initial position and the position at a given time. This information can be used to study the stretching of the bungee cord during the jump and to assess the safety of the system. The maximum displacement can be obtained by finding the maximum value of the position function. This information is important as it tells us how much the bungee cord stretches during the jump, which can have **safety implications**.

Ackowledgement: Above screenshots taken from YouTube video. https://www.youtube.com/watch?v=dwv3jos4Ipc

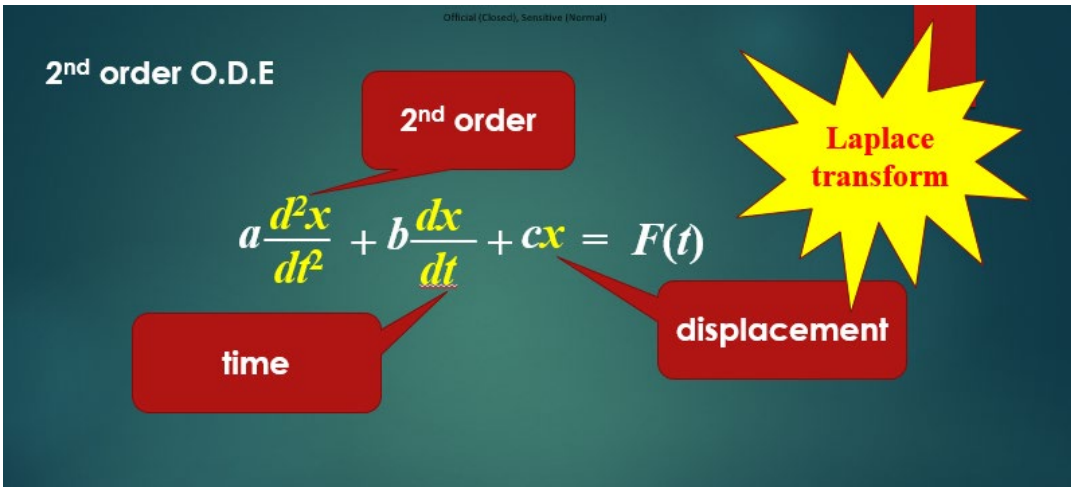


Figure 3: Modeling Bungee Jump using 2nd order ordinary differential equations

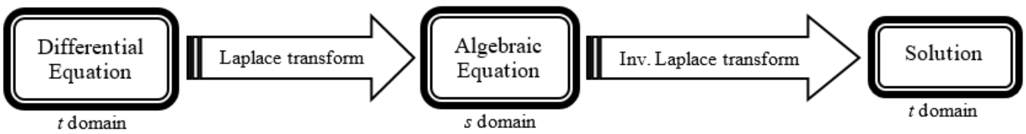
The bungee jump can be described by a **2nd order ordinary differential equation**, where x is the vertical displacement of the jumper at time t, as shown in ***Figure 3***. The solution to the differential equation gives the position and velocity of the jumper as a function of time, allowing us to model the motion of the bungee jumper during a jump, which can then be used to make informed decisions about safety and design.

There are many methods to solve this differential equation. These include the auxillary equation method and the Laplace transform method. This week, we will define **Laplace transform**, and we will revisit the bungee jump problem in Chapter 9.

6.2 Introduction

The Laplace transform has a key role to play in the modern approach to the analysis and design of engineering systems. It is an example of a class called integral transforms, in which Fourier transform is another widely used example.

In Laplace transform, a function $f(t)$ of one variable t (time) is changed into a function $F(s)$ of another variable s. In doing so, it transforms a *differential* equation in the t (time) domain into an *algebraic* equation in the s (frequency) domain, denoted as $F(s)$, which makes solving easier. After obtaining the solution of the algebraic equation in s, the solution of the original differential equation in t can be obtained by using the inverse Laplace transform. See diagram below.



Another advantage of using the Laplace transform to solve differential equations is that initial conditions play an important role in the transformation process, so they are automatically incorporated into the solution. Hence, it is an ideal tool for solving initial-value problems such as those occurring in the investigation of electrical circuits and mechanical vibrations.

A further important advantage is that the method of Laplace transform enables us to deal with situations where the function is discontinuous or periodic, or even impulsive.

6.3 Definition and Notation of the Laplace Transform

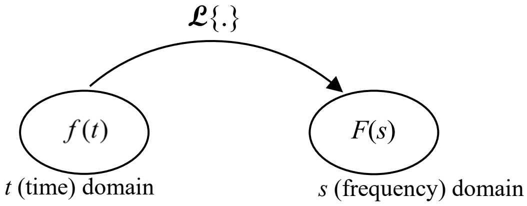
Let $f(t)$ be a function defined for $t > 0$.
The Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is <u>defined</u> by:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \text{, where } s > 0$$

The Laplace transform of $f(t)$ is said to exist if the above integral converges for some value of s, otherwise it does not exist.

It is usual to represent the Laplace transform of a function by the corresponding capital letter. That is, $\mathcal{L}\{f(t)\} = F(s)$. We may similarly write $\mathcal{L}\{q(t)\} = Q(s)$, $\mathcal{L}\{i(t)\} = I(s)$, $\mathcal{L}\{v(t)\} = V(s)$, etc.

The symbol $\mathcal{L}$ denotes the Laplace transform operator.
The relationship between $f(t)$ and $F(s)$ is depicted graphically on the right:



6.4 Laplace Transforms of Simple Functions

Let us use the <u>definition</u> stated in section 6.2 to derive the results of Laplace transform of some common functions.

Result 1: $\mathcal{L}\{1\} = \dfrac{1}{s}$

Proof:
$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\, dt && [\text{ taking } f(t) \text{ as the value } 1\ ] \\
&= -\frac{1}{s}\left[e^{-st}\right]_0^\infty && \left[\text{ Let } [\ ]_0^\infty \text{ denote } \lim_{b\to\infty}[\ ]_0^b\ \right] \\
&= -\frac{1}{s}\lim_{b\to\infty}\left[e^{-st}\right]_0^b = -\frac{1}{s}\lim_{b\to\infty}\left(e^{-sb}-1\right) \\
&= -\frac{1}{s}(0-1) = \frac{1}{s} && \text{since } e^{-sb} \to 0 \text{ as } b \to \infty \text{ if } s > 0.
\end{aligned}$$

Result 2: $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}, s > a$

Proof:
$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st}e^{at}\, dt \\
&= \int_0^\infty e^{-(s-a)t}\, dt = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty
\end{aligned}$$

Since $e^{-(s-a)b} \to 0$ as $b \to \infty$ if $(s-a) > 0$ or $s > a$,

$$\mathcal{L}\left\{e^{at}\right\} = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}.$$

Result 3: $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{s^{n+1}}$ for positive integer n.

Proof: $\mathcal{L}\left\{t^n\right\} = \int_0^\infty t^n e^{-st}\, dt$

$$= -\frac{1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\, dt, \text{ using integration by parts.}$$

As $b \to \infty$, $b^n e^{-sb} = \dfrac{b^n}{e^{sb}} \to 0$ since e^{sb} for $s > 0$ grow at a greater rate than b^n, so

$$\mathcal{L}\left\{t^n\right\} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\, dt = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}.$$

This is a recurrence relation that can be used to find an explicit formula for $\mathcal{L}\left\{t^n\right\}$.

$$\mathcal{L}\{t\} = \frac{1}{s}\mathcal{L}\left\{t^0\right\} = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2} \qquad \mathcal{L}\left\{t^2\right\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\frac{1}{s^2} = \frac{2!}{s^3}$$

$$\mathcal{L}\left\{t^3\right\} = \frac{3}{s}\mathcal{L}\left\{t^2\right\} = \frac{3}{s}\frac{2!}{s^3} = \frac{3!}{s^4} \qquad \therefore \mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

In addition to the above functions, the Laplace transforms of other functions can be determined in much the same way. The final results have been collated and presented into a **formulae table** which can be found at the end of this book as well as in the Maths Formulae Card. We shall make references to these **standard results** from this point onwards.

Formula $\boxed{2}$

t^n (n is a positive integer)	$\dfrac{n!}{s^{n+1}}$

***Example 1*:** (a) $\mathcal{L}\left\{t^2\right\}$ (b) $\mathcal{L}\left\{t^5\right\}$

Formula $\boxed{3}$	e^{at}	$\dfrac{1}{s-a} \quad (s>a)$

***Example 2*:** (a) $\mathcal{L}\left\{e^{3t}\right\}$ (b) $\mathcal{L}\left\{e^{-2t}\right\}$ (c) $\mathcal{L}\left\{e^{\frac{t}{2}}\right\}$

Formula $\boxed{4}$	$\sin at$	$\dfrac{a}{s^2+a^2}$
Formula $\boxed{5}$	$\cos at$	$\dfrac{s}{s^2+a^2}$

***Example 3*:** (a) $\mathcal{L}\left\{\sin 3t\right\}$ (b) $\mathcal{L}\left\{\cos t\right\}$

Formula $\boxed{6}$	$t\sin at$	$\dfrac{2as}{\left(s^2+a^2\right)^2}$
Formula $\boxed{7}$	$t\cos at$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$

***Example 4*:** (a) $\mathcal{L}\left\{t\sin 3t\right\}$ (b) $\mathcal{L}\left\{t\cos 4t\right\}$

6.5 Linearity Property

Theorem: Linearity Property

If $f_1(t)$ and $f_2(t)$ are functions of t and, a and b are constants, then

$$\mathcal{L}\left\{a\, f_1(t) + b\, f_2(t)\right\} = a\,F_1(s) + b\,F_2(s)$$

where $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively.

***Example 5*:** (a) $\mathcal{L}\left\{5\sin 3t\right\}$

(b) $\mathcal{L}\left\{e^{3t} + \cos t\right\}$

***Example 6*:**

(a) $\mathcal{L}\left\{\pi - 2t^3 + e^{4t}\right\}$

(b) $\mathcal{L}\left\{6e^{-t} - 3 + 7\,t\sin \pi t\right\}$

Sometimes, we may need to apply algebraic manipulations, law of indices or trigonometric identities/formulae before Laplace transform can be done.

***Example 7*:**

(a) $\mathcal{L}\left\{3t\left(t-2\right)\right\}$

(b) $\mathcal{L}\left\{e^{t+2}\right\}$

(c) $\mathcal{L}\left\{\sin\left(3t+\dfrac{\pi}{4}\right)\right\}$

(d) $\mathcal{L}\left\{\sin^2 3t\right\}$

(e) $\mathcal{L}\left\{t\sin 2t\cos 2t\right\}$

6.6 First Shift Theorem

<u>First Shift Theorem (Formula $\boxed{8}$)</u>

If $\mathcal{L}\left\{f(t)\right\} = F(s)$, then

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a), \quad s > a$$

The transform $\mathcal{L}\left\{e^{at}f(t)\right\}$ is the same as $\mathcal{L}\left\{f(t)\right\}$ but with 's' everywhere in the result replaced by ($s - a$).

Hence, we can also write: $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s)\big|_{s\to s-a} = F(s-a)$

For example: Since $\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$, then to obtain $\mathcal{L}\left\{e^{2t}\,t^3\right\}$,

we replace 's' with (s – 2) and get $\mathcal{L}\left\{e^{2t}t^3\right\} =$

Alternatively, we can write: $\mathcal{L}\left\{e^{2t}t^3\right\} = \left.\frac{6}{s^4}\right|_{s \to s-2} =$

***Example 8*:** Find, by first shift theorem:

(a) $\mathcal{L}\left\{e^t \sin 3t\right\}$

(b) $\mathcal{L}\left\{t\,e^t \sin 3t\right\}$

(c) $\mathcal{L}\left\{2t\,e^{5t}\right\}$

(d) $\mathcal{L}\left\{5e^{-2t} \cos 3t\right\}$

(e) $\mathcal{L}\left\{\left(e^{2t} - e^{-2t}\right)\cos 2t\right\}$

6.7 Laplace Transforms of Derivatives

If we are to use Laplace transform methods to solve differential equations, we need to find the Laplace transforms of derivatives such as $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$. Note that $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$ can also be written as $y'(t)$ and $y''(t)$ respectively.

By definition, $\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st} y'(t)\, dt$

$$= \left[e^{-st} y(t)\right]_0^\infty + s\int_0^\infty e^{-st} y(t)\, dt \qquad \text{(using integration by parts)}$$

As $b \to \infty$, $e^{-sb} y(b) = \dfrac{y(b)}{e^{sb}} \to 0$. This is because e^{sb} for $s > 0$ grow at a faster rate than $|y(b)|$ in most of case. So,

$$\mathcal{L}\{y'(t)\} \;=\; 0 - y(0) + s\int_0^\infty e^{-st} y(t)\, dt \;=\; s\mathcal{L}\{y(t)\} - y(0)$$

We can further use this result to find the Laplace transform of $y''(t)$:

$$\mathcal{L}\{y''(t)\} = \mathcal{L}\left\{\left(y'(t)\right)'\right\} = s\mathcal{L}\{y'(t)\} - y'(0)$$

$$= s\left(s\mathcal{L}\{y(t)\} - y(0)\right) - y'(0) = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$$

These results are summarized in the **formulae table**:

Theorems: Derivatives (Formulae $\boxed{9}$ & $\boxed{10}$)

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\mathcal{L}\{y\} - y(0) \quad \text{and} \quad \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)$$

where $y(0)$ and $y'(0)$ are values of y and y' respectively when $t = 0$.

Example 9: (a) Given $y' = 6t$ and $y(0) = 5$, find $\mathcal{L}\{y\}$.

(b) Given $y(0) = 3$ and $y'(0) = 7$, find $\mathcal{L}\left\{\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 13y\right\}$.

6.8 Partial Fractions

Partial fraction expansion is performed whenever we want to represent a complicated fraction as a sum of simpler fractions. This occurs when working with the inverse Laplace Transform in which we have methods of efficiently processing simpler fractions.

A **proper fraction** $\dfrac{f(x)}{g(x)}$ can be expressed as a sum of simpler fractions if the denominator $g(x)$ can be factorised. These simpler fractions are called **partial fractions**. Each partial fraction corresponds to a factor of $g(x)$.

The rules of partial fractions are as follows:

Rule 1 The fraction $\dfrac{f(x)}{g(x)}$ must be a proper fraction. (If it is not, then first divide out by long division.)

Rule 2 Factorise the denominator $g(x)$ into its prime factors. This is important since the factors obtained determine the form of the partial fractions.

Rule 3 Corresponding to each **linear factor** $\boxed{ax+b}$ in the denominator, there is a partial fraction of the form $\boxed{\dfrac{A}{ax+b}}$.

For example, $\dfrac{3x-2}{(x+1)(2x-1)} = \dfrac{A}{x+1} + \dfrac{B}{2x-1}$.

Rule 4 Corresponding to a **repeated linear factor** $\boxed{(ax+b)^n}$ in the denominator, there will be n partial fractions of the form $\boxed{\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}}$.

For example, $\dfrac{3x-2}{(x+3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2}$.

Rule 5 Corresponding to an irreducible **quadratic factor** $\boxed{ax^2+bx+c}$ in the denominator, there will be a partial fraction of the form $\boxed{\dfrac{Ax+B}{ax^2+bx+c}}$.

For example, $\dfrac{3x-2}{(x+1)(x^2+x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+x+1}$.

Note: All the constants A's and B's can then be determined by "Cover-up" method, substitution method, and/or equating coefficients of like terms.

Example 10: Write down the form of partial fractions, without evaluating the constants.

Original Fraction	**Form of the Partial Fractions**	**Unknown coefficients that can be solved by cover-up method**
$\dfrac{s}{(s+1)(s+3)}$		
$\dfrac{s}{(s+1)(s+3)^2}$		
$\dfrac{s}{(s+1)(s^2+3)}$		

Example 11: Resolve $\dfrac{9}{(s-1)(s^2+2)}$ into partial fractions.

Tutorial 6

Find the following:

1. (a) $\mathcal{L}\left\{4-9e^{-4t}\right\}$ (b) $\mathcal{L}\{7+\pi\}$ (c) $\mathcal{L}\{\cos \pi t\}$

   (d) $\mathcal{L}\left\{5t^3+3\sin 2t\right\}$ (e) $\mathcal{L}\{2\sin 4t - 9\cos 6t\}$

   (f) $\mathcal{L}\{A\sin \omega t\}$, where A and ω are constants.

2. (a) $\mathcal{L}\{(t+1)(t+2)\}$ (b) $\mathcal{L}\left\{e^{2t+3}\right\}$ (c) $\mathcal{L}\left\{\sin\left(t+\frac{\pi}{6}\right)\right\}$

   (d) $\mathcal{L}\left\{\cos 2\left(t-\frac{\pi}{4}\right)\right\}$ (e) $\mathcal{L}\left\{2\sin^2 t\right\}$ (f) $\mathcal{L}\left\{(\sin t-\cos t)^2\right\}$

3. (a) $\mathcal{L}\{t\sin 2t\}$ (b) $\mathcal{L}\left\{t\cos^2 3t\right\}$ (c) $\mathcal{L}\{t\sin 2t \sin 5t\}$

4. (a) $\mathcal{L}\left\{e^{3t}\sin t\right\}$ (b) $\mathcal{L}\left\{e^{-2t}\cos\sqrt{3}\,t - t^2e^{-2t}\right\}$

   (c) $\mathcal{L}\left\{t\,e^{2t}\cos 5t\right\}$ (d) $\mathcal{L}\left\{t\,e^{3t}\sin 2t\right\}$ (e) $\mathcal{L}\left\{e^{4t}\sin 3t\cos 2t\right\}$

5. (a) Given $y' = t^2$ and $y(0)=1$, find $\mathcal{L}\{y\}$.

   (b) Find $\mathcal{L}\left\{\frac{dv}{dt}+3v-13\sin 2t\right\}$, if $v(0)=6$.

   (c) Given $i(0)=0$, find $\mathcal{L}\left\{\frac{di}{dt}+5i+6e^{-2t}\right\}$.

   (d) If $y(0)=1$ and $y'(0)=-2$, find $\mathcal{L}\left\{\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y-e^{-2t}\cos 3t\right\}$.

*6. Use $\mathcal{L}\{t\cos 3t\}$ to evaluate $\int_0^\infty te^{-2t}\cos 3t\,dt$.

Multiple Choice Questions

1. $\mathcal{L}\left\{e^{-3t-5}\right\}$ is equal to

   (a) $\frac{1}{e^3(s+5)}$ (b) $\frac{e^3}{s+5}$ (c) $\frac{1}{e^5(s+3)}$ (d) $\frac{e^5}{s+3}$

2. $\mathcal{L}\left\{\left(1-e^{-t}\right)\cos 2t\right\}$ is equal to

(a) $\left(\dfrac{1}{s}-\dfrac{1}{s+1}\right)\left(\dfrac{s}{s^2+4}\right)$ (b) $\dfrac{s}{s^2+4}-\dfrac{s}{(s+1)(s^2+4)}$

(c) $\dfrac{s}{s^2+4}-\dfrac{s}{(s+1)^2+4}$ (d) $\dfrac{s}{s^2+4}-\dfrac{s+1}{(s+1)^2+4}$

Answers to Selected Lecture Examples

Example 9: (a) $Y(s)=\dfrac{6}{s^3}+\dfrac{5}{s}$ (b) $\left(s^2+6s+13\right)Y(s)-3s-25$

Example 11: $\dfrac{9}{(s-1)(s^2+2)}=\dfrac{3}{(s-1)}-\dfrac{3(s+1)}{(s^2+2)}$

Answers to Tutorial 6

1. (a) $\dfrac{4}{s}-\dfrac{9}{s+4}$ (b) $\dfrac{7+\pi}{s}$ (c) $\dfrac{s}{s^2+\pi^2}$

   (d) $\dfrac{30}{s^4}+\dfrac{6}{s^2+4}$ (e) $\dfrac{8}{s^2+16}-\dfrac{9s}{s^2+36}$ (f) $\dfrac{A\omega}{s^2+\omega^2}$

2. (a) $\dfrac{2}{s^3}+\dfrac{3}{s^2}+\dfrac{2}{s}$ (b) $\dfrac{e^3}{s-2}$ (c) $\dfrac{\sqrt{3}+s}{2(s^2+1)}$ or $\dfrac{0.866+0.5s}{s^2+1}$

   (d) $\dfrac{2}{s^2+4}$ (e) $\dfrac{1}{s}-\dfrac{s}{s^2+4}$ (f) $\dfrac{1}{s}-\dfrac{2}{s^2+4}$

3. (a) $\dfrac{4s}{(s^2+4)^2}$ (b) $\dfrac{1}{2}\left[\dfrac{1}{s^2}+\dfrac{s^2-36}{(s^2+36)^2}\right]$ (c) $\dfrac{1}{2}\left[\dfrac{s^2-9}{(s^2+9)^2}-\dfrac{s^2-49}{(s^2+49)^2}\right]$

4. (a) $\dfrac{1}{(s-3)^2+1}$ (b) $\dfrac{s+2}{(s+2)^2+3}-\dfrac{2}{(s+2)^3}$ (c) $\dfrac{(s-2)^2-25}{\left[(s-2)^2+25\right]^2}$

   (d) $\dfrac{4(s-3)}{\left[(s-3)^2+4\right]^2}$ (e) $\dfrac{1}{2}\left[\dfrac{5}{(s-4)^2+25}+\dfrac{1}{(s-4)^2+1}\right]$

5. (a) $\mathcal{L}\{y\}=\dfrac{2}{s^4}+\dfrac{1}{s}$ (b) $(s+3)\mathcal{L}\{v\}-6-\dfrac{26}{s^2+4}$

   (c) $(s+5)\mathcal{L}\{i\}+\dfrac{6}{s+2}$ (d) $(s^2+2s+5)\mathcal{L}\{y\}-s-\dfrac{s+2}{(s+2)^2+9}$

6. $-\dfrac{5}{169}$

MCQ

1. (c) 2. (d)

Chapter 7: Inverse Laplace Transform

Objectives:

1. *Find inverse Laplace transforms using standard results.*
2. *Use techniques like linearity property, completing the square and partial fractions to evaluate inverse Laplace transforms.*

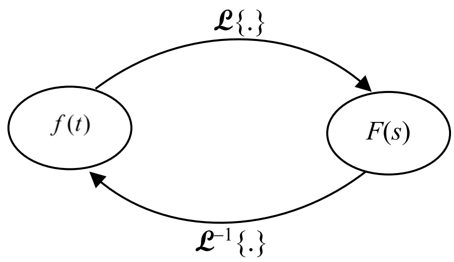
7.1 Definition

If the Laplace transform of a function $f(t)$ is $F(s)$, then $f(t)$ is called an **inverse Laplace transform** of $F(s)$. That is:

$$\text{if } \mathcal{L}\{f(t)\} = F(s) \ , \text{ then } f(t) = \mathcal{L}^{-1}\{F(s)\}$$

where $\mathcal{L}^{-1}$ is called the **inverse Laplace transformation operator**.

The relationship between $f(t)$ and $F(s)$ is depicted graphically here:



When finding the inverse Laplace transform of $F(s)$, the following strategies are normally used, sometimes in combinations, before referring to the Laplace transforms **formulae table**:

- Linearity rule
- Completing the square
- Partial fractions

These methods will be gradually introduced and demonstrated in the latter sections of this chapter.

7.2 Inversion using Standard Results and Linearity Property

Since $\mathcal{L}\left\{e^{3t}\right\} = \dfrac{1}{s-3}$, then inversely, $\mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t}$.

Similarly, since $\mathcal{L}\left\{t^2\right\} = \dfrac{2}{s^3}$, then inversely, $\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = t^2$.

The linearity property works for the inverse Laplace transform operator as well.

Theorem: Linearity Property

If $f_1(t)$ and $f_2(t)$ are functions of t, a and b are constants, $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$\mathcal{L}^{-1}\{a\,F_1(s)+b\,F_2(s)\}=a\,f_1(t)+b\,f_2(t)$$

Formula $\boxed{1}$

1	$\dfrac{1}{s}$

***Example 1*:** (a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{\pi}{s}\right\}$

Formula $\boxed{2}$

t^n (n is a positive integer)	$\dfrac{n!}{s^{n+1}}$

***Example 2*:** (a) $\mathcal{L}^{-1}\left\{\dfrac{24}{s^5}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\}$

Formula $\boxed{3}$

e^{at}	$\dfrac{1}{s-a}$

***Example 3*:** (a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{4}{s-2}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{1}{3s+2}\right\}$

Formula $\boxed{4}$	$\sin at$	$\dfrac{a}{s^2+a^2}$
Formula $\boxed{5}$	$\cos at$	$\dfrac{s}{s^2+a^2}$

***Example 4*:** (a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}$

Formula $\boxed{6}$	$t\sin at$	$\dfrac{2as}{\left(s^2+a^2\right)^2}$
Formula $\boxed{7}$	$t\cos at$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$

***Example 5*:** (a) $\mathcal{L}^{-1}\left\{\dfrac{3s}{\left(s^2+16\right)^2}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{s^2-16}{\left(s^2+16\right)^2}\right\}$

***Example 6*:**

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s}-\dfrac{7}{s-5}+\dfrac{1}{s^4}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^4}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{-\frac{1}{5}s+\frac{4}{5}}{s^2+4}\right\}$

7.3 Inversion using First Shift Theorem

7.3.1 First Shift Theorem

This is first shift theorem expressed in the inverse form:

First Shift Theorem (Formula $\boxed{8}$)

$$\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at} f(t)$$

We can also write: $\mathcal{L}^{-1}\left\{F(s-a)\right\} = \mathcal{L}^{-1}\left\{F(s)\big|_{s\to s-a}\right\} = e^{at} f(t)$

For example: Since we know that $\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} = t^3$,

and that $\dfrac{6}{(s-2)^4}$ is just $\dfrac{6}{s^4}$ with 's' replaced by $(s-2)$,

hence $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} =$

Alternatively, we can write: $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-2)^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\bigg|_{s\to s-2}\right\} =$

***Example 7*:** Find the following by first shift theorem:

(a) $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^5}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{3}{(s+2)^2}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-5)^2+4}\right\}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2+25}\right\}$

***Example 8*:**

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^2+25}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{(s+2)^3}\right\}$

7.3.2 Complete the Squares Method

In rational expressions (i.e. fractions) where the denominator is a quadratic function of the form as^2+bs+c which cannot be factorised, we will perform "completing the square" method for the denominator. Briefly, to complete the square:

$$s^2+ks=\left(s+\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)^2$$

For example: To find $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\}$, we would want to "complete the square" for the quadratic denominator $s^2+2s+10$ which cannot be factorised.

Let $k=$, then $s^2+2s+10=\left(s+\dfrac{\quad}{2}\right)^2-\left(\dfrac{\quad}{2}\right)^2+10=$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\}=\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{\qquad}\right\}$

$=\mathcal{L}^{-1}\left\{\dfrac{\quad}{\qquad}\Bigg|_{s\to s+1}\right\}$

***Example 9*:**

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{\frac{1}{5}s-\frac{2}{5}}{s^2+2s+2}\right\}$

7.4 Inversion by Resolving into Partial Fractions

In inverse Laplace Transform, a proper rational expression of the form $\dfrac{p(s)}{q(s)}$ can be written as the sum of partial fractions having the forms:

Original Fraction	Form of the Partial Fractions	Unknown coefficients that can be solved by cover-up method
$\dfrac{s}{(s+1)(s+3)}$	$\dfrac{A}{(s+1)}+\dfrac{B}{(s+3)}$	A and B (both are unknown coefficients of linear factors)
$\dfrac{s}{(s+1)(s+3)^2}$	$\dfrac{A}{(s+1)}+\dfrac{B}{(s+3)}+\dfrac{C}{(s+3)^2}$	A (linear factor) and C (highest power of the repeated factor)
$\dfrac{s}{(s+1)(s^2+3)}$	$\dfrac{A}{(s+1)}+\dfrac{Bs+C}{(s^2+3)}$	A (linear factor) only

Summary:

(a) Unknown coefficients of linear factors and highest power of the repeated factor can be solved by cover-up method.
(b) Unknown coefficients of quadratic factors cannot be solved by cover-up method.
(c) If unknown coefficients cannot be solved by cover-up method, then use compare coefficients method to solve it.

By finding the inverse Laplace transform for each of the partial fractions, we can then evaluate $\mathcal{L}^{-1}\left\{\dfrac{p(s)}{q(s)}\right\}$.

***Example 11*:** Find $\mathcal{L}^{-1}\left\{\dfrac{9s+14}{(s-2)(s^2+4)}\right\}$.

***Example 12*:** Find $\mathcal{L}^{-1}\left\{\dfrac{7s-6}{(s+2)(s-3)}\right\}$.

***Example 13*:** Find $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)(s+1)^2}\right\}$.

***Example 14*:** Which method should we use to find the following?

(a) $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-8s+15}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{4s^2-4s+5}\right\}$

Tutorial 7

1. Express the following into partial fractions:

(a) $\dfrac{s+1}{(s+2)(s-1)^2}$

(b) $\dfrac{2s-1}{(s+2)\left(s^2+1\right)}$

(c) $\dfrac{6s^2+7s-49}{(s-4)(s+1)(2s-3)}$

(d) $\dfrac{5s+3}{s^3-2s^2-3s}$

[Hint: factorize the denominator first]

2. Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{2}{s}-\dfrac{8}{s^3}+\dfrac{16}{s^5}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s+6}-\dfrac{3s}{s^2+25}+\dfrac{1}{s^2+49}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s^2-100}{\left(s^2+100\right)^2}-\dfrac{4s}{\left(s^2+81\right)^2}\right\}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{1}{2s-3}\right\}$

(e) $\mathcal{L}^{-1}\left\{\dfrac{3(1+s)}{s^5}\right\}$

(f) $\mathcal{L}^{-1}\left\{\dfrac{3s+2}{s^2+36}\right\}$

3. Use First Shift Theorem to find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-1)^3}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-2)^2+9}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+25}\right\}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{2(s-5)}{(s-5)^2+49}\right\}$

4. Use the methods of completing the square or partial fractions to find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+6s+13}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{s-1}{s^2-4s+20}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^2+6s+9}\right\}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^2-2s+5}\right\}$

(e) $\mathcal{L}^{-1}\left\{\dfrac{s-\frac{3}{2}}{2s^2-6s+\frac{13}{2}}\right\}$

(f) $\mathcal{L}^{-1}\left\{\dfrac{s^2-2s+3}{s(s-1)(s-2)}\right\}$

(g) $\mathcal{L}^{-1}\left\{\dfrac{s^2+1}{(s-1)(s^2+2)}\right\}$

(h) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2(s^2+9)}\right\}$

*5. Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{4e^{-3}}{2s-1}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{s-1}{4s^2+60}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{7s-7}{3s^2+5s-2}\right\}$

*6. Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{s-3}{(s-2)^2+2(s-2)+1}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+s+1}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s}{5s^2-10s+9}\right\}$

*7. Find $\mathcal{L}\left\{e^{2t}\cos 3t\right\}$. Hence, given that $\mathcal{L}^{-1}\left\{F(s+3)\right\} = e^{3(1-t)}\cos 3t$, find $F(s-2)$.

Multiple Choice Questions

1. If $\mathcal{L}^{-1}\left\{F(s)\right\} = \sin 2t$, then $\mathcal{L}^{-1}\left\{F(s+\pi)\right\}$ is equal to

(a) $\sin 2t$ (b) $-\sin 2t$

(c) $e^{-\pi t}\sin 2t$ (d) $e^{\pi t}\sin 2t$

2. If $\mathcal{L}^{-1}\left\{F(s+2)\right\} = e^{2(1-t)}t^3$, then $\mathcal{L}^{-1}\left\{F(s)\right\}$ is equal to

(a) t^3 (b) e^2t^3

(c) $e^{2(1+t)}t^3$ (d) $e^{2(1-2t)}t^3$

3. When performing the following transformations, which one does NOT involve First Shift Theorem?

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\}$ (b) $\mathcal{L}\left\{\left(e^{-t}-e^{3t}\right)\sin 2t\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s-1}{\left(s-3\right)^3}\right\}$ (d) $\mathcal{L}\left\{te^{t}\right\}$

Answers to Selected Lecture Examples

Example 8: (a) $e^{-t}\left(\cos 5t - \frac{1}{5}\sin 5t\right)$ (b) $e^{-2t}\left(2t - \frac{3}{2}t^2\right)$

Example 9: (a) $\frac{1}{2}e^{-2t}\sin 2t$ (b) $\frac{1}{5}e^{-t}\left(\cos t - 3\sin t\right)$

Example 12: $4e^{-2t} + 3e^{3t}$

Example 13: $\frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - te^{-t}$

Example 14: (a) $-7e^{3t} + 13e^{5t}$ (b) $\frac{1}{4}e^{t/2}\sin t$

Answers to Tutorial 7

1. (a) $-\frac{1}{9(s+2)} + \frac{1}{9(s-1)} + \frac{2}{3(s-1)^2}$ (b) $-\frac{1}{s+2} + \frac{s}{s^2+1}$

   (c) $\frac{3}{s-4} - \frac{2}{s+1} + \frac{4}{2s-3}$ (d) $-\frac{1}{s} + \frac{3}{2(s-3)} - \frac{1}{2(s+1)}$

2. (a) $2 - 4t^2 + \frac{2}{3}t^4$ (b) $e^{-6t} - 3\cos 5t + \frac{1}{7}\sin 7t$ (c) $t\cos 10t - \frac{2}{9}t\sin 9t$

   (d) $\frac{1}{2}e^{\frac{3}{2}t}$ (e) $\frac{1}{8}t^4 + \frac{1}{2}t^3$ (f) $3\cos 6t + \frac{1}{3}\sin 6t$

3. (a) $3t^2e^t$ (b) $e^{2t}\sin 3t$ (c) $e^{-2t}\cos 5t$ (d) $2e^{5t}\cos 7t$

4. (a) $e^{-3t}\sin 2t$ (b) $e^{2t}\left(\cos 4t + \frac{1}{4}\sin 4t\right)$ (c) $e^{-3t}(1-t)$

   (d) $e^{t}\left(2\cos 2t + \frac{5}{2}\sin 2t\right)$ (e) $\frac{1}{2}e^{\frac{3}{2}t}\cos t$ (f) $\frac{3}{2} - 2e^{t} + \frac{3}{2}e^{2t}$

   (g) $\frac{2}{3}e^{t} + \frac{1}{3}\left(\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}\sin\sqrt{2}\,t\right)$ (h) $\frac{1}{9} + \frac{1}{9}t - \frac{1}{9}\cos 3t - \frac{1}{27}\sin 3t$

5. (a) $2e^{\frac{t}{2}-3}$ (b) $\frac{1}{4}\left(\cos\sqrt{15}\,t - \frac{1}{\sqrt{15}}\sin\sqrt{15}\,t\right)$ (c) $-\frac{2}{3}e^{t/3} + 3e^{-2t}$

6. (a) $e^{t}(1-2t)$ (b) $\frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$ (c) $\frac{1}{5}e^{t}\left(\cos\left[\frac{2}{\sqrt{5}}t\right] + \frac{\sqrt{5}}{2}\sin\left[\frac{2}{\sqrt{5}}t\right]\right)$

7. $e^{3}\left(\frac{s-2}{(s-2)^2+9}\right)$

MCQ

1. (c) 2. (b) 3. (a)

Chapter 8: Solving Second Order Differential Equations

Objectives :

1. *Define homogeneous and non-homogeneous second order ordinary differential equation with constant coefficients.*
2. *Solve differential equations using auxiliary equation.*
3. *Solve differential equations using Laplace transform method.*

8.1 Introduction

A **2nd order linear differential equation with constant coefficients** is an equation of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \text{ , where } a(\neq 0) \text{ , } b \text{ and } c \text{ are constants.}$$

If $f(x)=0$, the equation is said to be **homogeneous**.
If $f(x)\neq 0$, the equation is said to be **nonhomogeneous or inhomogeneous.**

8.2 Solving 2nd order Linear Homogeneous D.E. using Auxiliary Equation

The general form of the differential equation is:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = 0 \quad \text{..................} \quad (1)$$

where a, b and c are constants. Notice that $f(x)=0$, thus equation (1) is a **homogeneous** differential equation.

To solve this equation, we observe that the function $y=e^{\lambda x}$ satisfies equation (1), that is, it is a solution of the differential equation. Substitute $y=e^{\lambda x}$ into equation (1) , we have:

$$e^{\lambda x}(a\lambda^2+b\lambda+c) = 0 \quad \text{..................} \quad (2)$$

Since $e^{\lambda x}$ cannot be zero, equation (2) is satisfied only if λ satisfies the quadratic equation:

$$a\lambda^2+b\lambda+c = 0 \quad \text{......................} \quad (3)$$

Equation (3) is called the **<u>auxiliary equation</u>** or **<u>characteristic equation</u>**. Solving this, we have:

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

If λ_1 and λ_2 are the roots of auxiliary equation (3), then we get two solutions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$.

It can be shown that if y_1 and y_2 are the solutions to the differential equation (1), then $y = Ay_1 + By_2$ (where A and B are constants) is also a solution.

The different forms of general solution of differential equation (1) depending on the nature of the roots of equation (3) are summarised below:

Case	Nature of Root(s)	General Solution
Case 1	Two distinct real roots λ_1 and λ_2 ($b^2 - 4ac > 0$)	$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
Case 2	One repeated real root λ ($b^2 - 4ac = 0$)	$y = e^{\lambda x}(Ax + B)$
Case 3	Complex roots $\lambda = \alpha \pm \beta j$ ($b^2 - 4ac < 0$)	$y = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$

Example 1: Find the general solution to each differential equation:

(a) $9\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + y = 0$ (b) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$

Example 2: Find the particular solution to the differential equation $y''-2y'+5y=0$, given that $y(0)=1$ and $y'(0)=-1$.

8.3 Solving 2nd Order Linear Non-Homogeneous D.E. using Laplace Transform Method

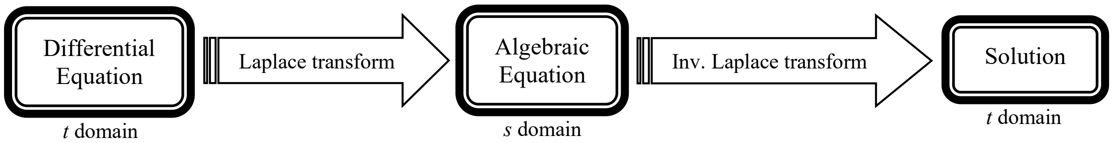
The Laplace transform is useful in solving ordinary linear differential equations with constant coefficients. The general second-order linear differential equation is as shown, subjected to initial conditions:

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t)\,, \quad t\geq 0$$

Notice that $f(t)\neq 0$, thus this equation is a **nonhomogeneous** differential equation. To solve this differential equation by the method of Laplace transforms, four distinct steps are required:

1. Take Laplace transform of each term in the differential equation.
2. Insert the given initial conditions.
3. Re-arrange the algebraic equation to obtain the transform of the solution, i.e. $Y(s)$.
4. Determine the inverse Laplace transform to obtain the solution, i.e. $y(t)$.

The concept is summarized here:



The actual detailed process will be illustrated in Example 3.

Example 3: Solve the differential equation $\dfrac{d^2y}{dt^2}+5\dfrac{dy}{dt}+6y=2e^{-t}$, given that $y(0)=1$ and $y'(0)=0$.

Solution:

Step 1: Take Laplace transform of each term in the differential equation.

Use linearity property.	$\mathcal{L}\left\{\dfrac{d^2y}{dt^2}\right\}+5\mathcal{L}\left\{\dfrac{dy}{dt}\right\}+6\mathcal{L}\{y\}=2\mathcal{L}\{e^{-t}\}$
Recall the Laplace transforms of differentials mentioned in Chapter 6.7.	

Step 2: Insert the given initial conditions.

Given $y(0) = 1$ and $y'(0) = 0$, and recall the convention $\mathcal{L}\{y\} = Y(s)$.	$s^2Y(s)-s+5[sY(s)-1]+6Y(s)=\dfrac{2}{s+1}$

Step 3: Re-arrange the algebraic equation to obtain the transform of the solution, i.e. $Y(s)$.

Make $Y(s)$ the subject of the formula. *Some useful tips:* • *Factorise the quadratic expressions?* • *Combine common denominator?*	$(s^2+5s+6)Y(s)=\dfrac{2}{s+1}+s+5$

Step 4: Determine the inverse Laplace transform to obtain the solution, i.e. $y(t)$.

Inverse Laplace transform $Y(s)$ to get $y(t)$.	$y(t)=\mathcal{L}^{-1}\{Y(s)\}$ $=\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+7}{(s+1)(s+2)(s+3)}\right\}$
Before performing inversion, we need to first resolve into partial fractions.	Let $\dfrac{s^2+6s+7}{(s+1)(s+2)(s+3)}=\dfrac{A}{s+1}+\dfrac{B}{s+2}+\dfrac{C}{s+3}$ Use cover-up rule to find A, B and C: (*do your own working*) $A=1,\ B=1,\ C=-1$
Perform inversion to get solution	$\therefore y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}+\dfrac{1}{s+2}-\dfrac{1}{s+3}\right\} = e^{-t}+e^{-2t}-e^{-3t}$

Example 4: Given the differential equation $\frac{d^2i}{dt^2}+6\frac{di}{dt}+9i=6t^2e^{-3t}$ with initial conditions $i(0)=0$ and $i'(0)=0$, find $i(t)$.

Tutorial 8

Section A

Find the general solution to each differential equation in (1) to (3).

1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ 2. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ 3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Find the particular solution to each differential equation in (4) to (10).

4. $4\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 0$, when $y(0) = -1$ and $y'(0) = 1$.

5. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$, , when $y(0) = 0$ and $y'(0) = 3$.

6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, , when $y(0) = 5$ and $y'(0) = -9$.

7. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$, , when $y(1) = 1 + e^2$ and $y'(1) = 2e^2$.

8. $\frac{d^2y}{dx^2} - 4y = 0$, , when $y(0) = 1$ and $y'(0) = -1$.

9. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$, , when $y(0) = 0$ and $y'(0) = 1$.

10. $2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$, when $y(0) = 1$ and $y'(0) = 1$.

Section B

1. Solve the initial-value problem: $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 4, \quad y(0) = -1, \ y'(0) = 2$.

2. Solve the following differential equation using Laplace transform method:

$$q'' + 9q = 0 \ , \text{ where } q(0) = 0 \text{ and } q'(0) = 2.$$

3. (a) Resolve $\dfrac{8}{(s+2)^2(s^2+4)}$ into partial fractions of the form $\dfrac{A}{s+2}+\dfrac{B}{(s+2)^2}+\dfrac{Cs+D}{s^2+4}$.

   (b) Hence, use the result from part (a) to solve the differential equation for $v(t)$:

$$v'' + 4v' + 4v = 4\sin 2t \ , \quad \text{where } v(0) = 1 \text{ and } v'(0) = 0.$$

4. (a) Express $\dfrac{2}{(s^2+1)(s^2+2s+5)}$ as a sum of partial fractions.

   (b) Show that $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+5}\right\} = e^{-t}\left(\cos 2t - \dfrac{1}{2}\sin 2t\right)$.

   (c) Given the differential equation $\dfrac{d^2y}{dt^2} + y = e^{-t}\sin 2t$, where $y(0) = 0$ and $y'(0) = 1$,

      (i) use the result from part (a) to show that:

$$\mathcal{L}\{y(t)\} = Y(s) = \frac{-s+7}{5(s^2+1)} + \frac{s}{5(s^2+2s+5)}$$

      (ii) Hence, use the result from (b) to solve for $y(t)$.

*5. Use Laplace transform method to solve the following differential equation for $y(t)$:

$$\frac{d^2y}{dt^2} + 9y = \cos 2t \ , \quad \text{where } y(0) = 1 \text{ and } y\left(\frac{\pi}{2}\right) = -1.$$

Multiple Choice Question

1. If the differential equation $4\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + ky = 0$ has a general solution of the form $y(x) = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$, where α , β , A and B are constants, then the value of the constant k is ________.

   (a) < 4 (b) ≤ 4

   (c) > 4 (d) ≥ 4

Answers to Selected Lecture Examples

Example 1: (a) $y(x)=e^{\frac{1}{3}x}\left(Ax+B\right)$ (b) $y(x)=e^{-1.25x}\left[A\cos\left(1.20x\right)+B\sin\left(1.20x\right)\right]$

Example 2: $y(x)=e^{x}\left(\cos 2x-\sin 2x\right)$

Example 4: $i(t)=\frac{1}{2}e^{-3t}t^{4}$

Answers to Tutorial 8

Section A

1. $y(x)=Ae^{3x}+Be^{-2x}$
2. $y(x)=(A+Bx)e^{2x}$
3. $y(x)=e^{-2x}\left(A\cos 3x+B\sin 3x\right)$
4. $y(x)=\frac{4}{9}e^{\frac{9}{4}x}-\frac{13}{9}$
5. $y(x)=3xe^{-3x}$
6. $y(x)=(5-14x)e^{x}$
7. $y(x)=1+e^{2x}$
8. $y(x)=\frac{1}{4}\left(3e^{-2x}+e^{2x}\right)$
9. $y(x)=e^{-x}\sin x$
10. $y(x)=e^{-\frac{1}{2}x}\left(\cos\frac{3}{2}x+\sin\frac{3}{2}x\right)$

Section B

1. $y(t)=-3-2t+2e^{2t}$
2. $q(t)=\frac{2}{3}\sin 3t$
3. (a) $\frac{1}{2(s+2)}+\frac{1}{\left(s+2\right)^{2}}-\frac{s}{2(s^{2}+4)}$ (b) $v(t)=\frac{3}{2}e^{-2t}+3te^{-2t}-\frac{1}{2}\cos 2t$
4. (a) $\frac{-s+2}{5(s^{2}+1)}+\frac{s}{5(s^{2}+2s+5)}$ (c) $y(t)=\frac{7}{5}\sin t-\frac{1}{5}\cos t+\frac{1}{5}e^{-t}\left(\cos 2t-\frac{1}{2}\sin 2t\right)$
5. $y(t)=\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t+\frac{1}{5}\cos 2t$

MCQ

1. (c)

Chapter 9: Applications of Second Order Differential Equations

Objectives :

1. Solve application problems using auxiliary equation and Laplace transform method.

9.1 Modelling a Mass Spring Damper System

The differential equation is given by Newton's second law of motion:

mass $\times$ acceleration = sum of forces acting on the mass

$$m\frac{d^2x}{dt^2} = F_g + F_s + F_d + F(t)$$

where:

m is the mass

$\frac{d^2x}{dt^2}$ is the acceleration (i.e. rate of change of velocity, $\frac{dv}{dt}$)

F_g is the force due to the earth's gravity

F_s is the tension in the spring

F_d is the damping force due to air resistance, etc.

$F\left(t\right)$ is the external applied force

Since

- $F_g = mg$, where g is the acceleration due to gravity,
- $F_s = -k\left(L_0 + x\right)$, where k is the spring constant and L_0 is the extension of the spring at equilibrium position. [Hooke's law states that the tension in the spring is proportional to the extension of the spring],
- $F_d = -c\frac{dx}{dt}$, where c is the damping constant, in which the minus sign indicates that the damping force is always opposite to the direction of motion, and
- $mg = kL_0$ in equilibrium,

the equation becomes:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= mg - k\left(L_0 + x\right) - c\frac{dx}{dt} + F(t) \\ &= mg - kL_0 - kx - c\frac{dx}{dt} + F(t) \\ &= -kx - c\frac{dx}{dt} + F(t) \end{aligned}$$

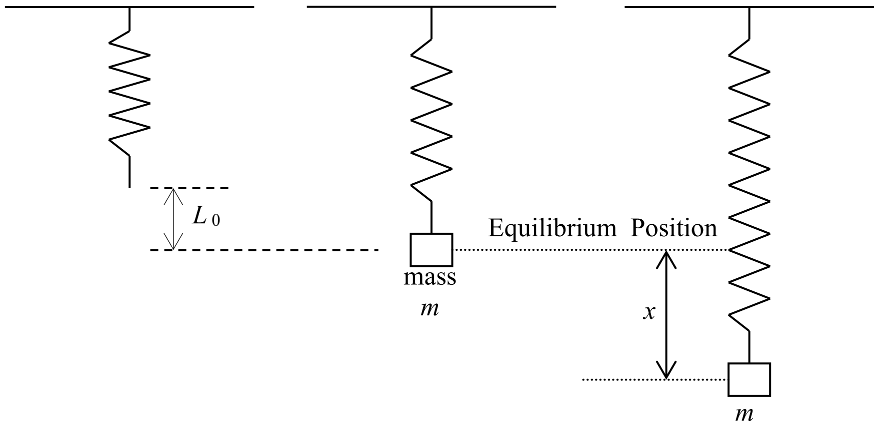
$$\therefore \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Hence, the mechanical vibration of a spring-mass-damper system is described by the 2nd order ODE:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F(t)$$

where:

x (m) is the displacement of the mass m (kg) at any time t (s) from equilibrium position,
c is the damping coefficient,
k is stiffness of spring (N/m),
$F(t)$ is the external force.



The value of x is measured as $\begin{cases} \text{positive, if it is below the equilibrium position.} \\ \text{negative, if it is above the equilibrium position.} \end{cases}$

9.1.1 Mass Spring Damper System: Without External Force, i.e. Free Oscillations

There is no driving force $F(t)$, i.e. $F(t)=0$.
Differential equation of motion becomes:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

Notice that this is a homogeneous differential equation.

(A) No Damping (c = 0)

In un-damped free vibration (which is known as **Simple Harmonic Motion**), there is no damping force, i.e. $c = 0$.
The differential equation becomes:

$$m\frac{d^2x}{dt^2}+kx=0.$$

Example 1: A 1 kg mass is suspended from a spring of stiffness 25 N/m. The mass is pulled below the equilibrium position and released. Assuming no frictional force and air resistance,

(a) set up the differential equation that describes the motion of the mass.

(b) Find the position of the mass at any time t.

(B) With Damping ($c \neq 0$)

With a damping force, that is, $c \neq 0$, the differential equation becomes:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

The characteristic equation is:

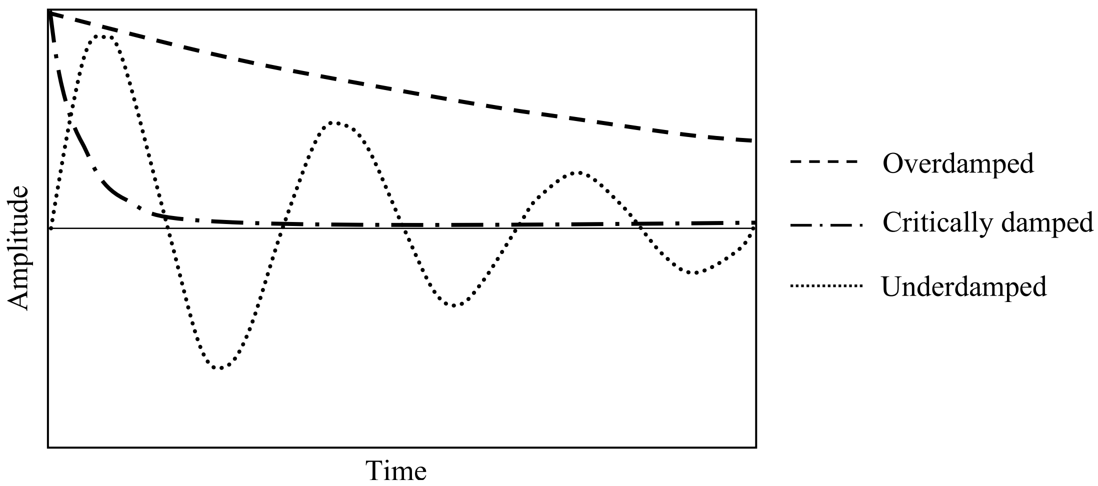
$$m\lambda^2 + c\lambda + k = 0$$

Its roots are:

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

There are 3 possible cases:

Roots	Discriminant	Condition	Solution
2 real roots: λ_1 and λ_2	$c^2 - 4km > 0$	***Overdamped***	$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
2 equal roots: $\lambda_1 = \lambda_2 = \lambda$	$c^2 - 4km = 0$	***Critically damped***	$x(t) = e^{\lambda t}\left(At + B\right)$
2 complex roots: $\lambda = \alpha \pm j\beta$	$c^2 - 4km < 0$	***Underdamped***	$x(t) = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$



Example 2: A 2 kg mass is suspended from a spring of stiffness 20 N/m. The mass is being pulled 1 m below the equilibrium position with no initial velocity and released. If the air resistance is numerically equal to $4V$, where V is the instantaneous velocity in m/s, find the position of the mass at time t and determine whether the motion is one of over damping, critical damping or under damping.

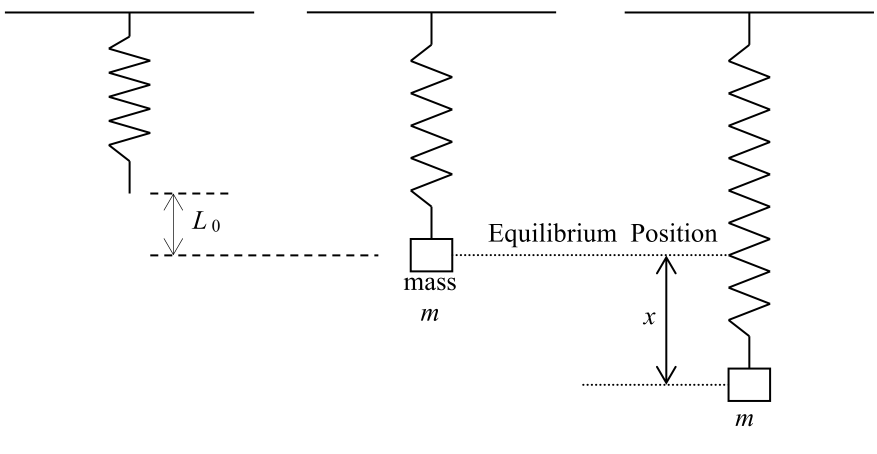
9.1.2 Mass Spring Damper System: With External Force

Recall that the mechanical spring vibration is described by the 2nd order ODE:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F(t)$$

where:

x (m) is the displacement of the mass m (kg) at any time t (s) from equilibrium position,
c is the damping coefficient,
k is stiffness of spring (N/m),
$F(t)$ is the external force.



The value of x is measured as $\begin{cases} \text{positive, if it is below the equilibrium position.} \\ \text{negative, if it is above the equilibrium position.} \end{cases}$

The solution is: $$x = \underbrace{\boxed{\text{Solution of motion due to applied force}}}_{\text{steady state solution}} + \underbrace{\boxed{\text{Solution of motion due to damped oscillation}}}_{\text{transient state solution}}$$

The damped motion in the solution is called the **transient solution** which diminishes with time. The undamped motion in the solution is called the **steady-state solution** which remains throughout.

In Chapter 6: Laplace transform, we introduced the Bungee Jump example. In view of the safety implications on design of the system, we may want to determine the lowest position of the jumper from ground level. As mentioned earlier (in Chapter 6), the Bungee Jump scenario comprises 3 phases. Hence, the actual analysis of the displacement of the jumper can be complex. The following example provides a simplified model of the system to simulate the rebound phase of the jumper.

***Example 3*:** A mass of 5 kg is suspended from a spring of stiffness 5 N/m. Initially, the mass was at its equilibrium position at a velocity of 2 m/s. An external force $F(t) = 50$ N is simultaneously and constantly applied to the mass. If the air resistance is numerically equal to v, where v (m/s) is the velocity of the mass at time t (s),

(a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions.

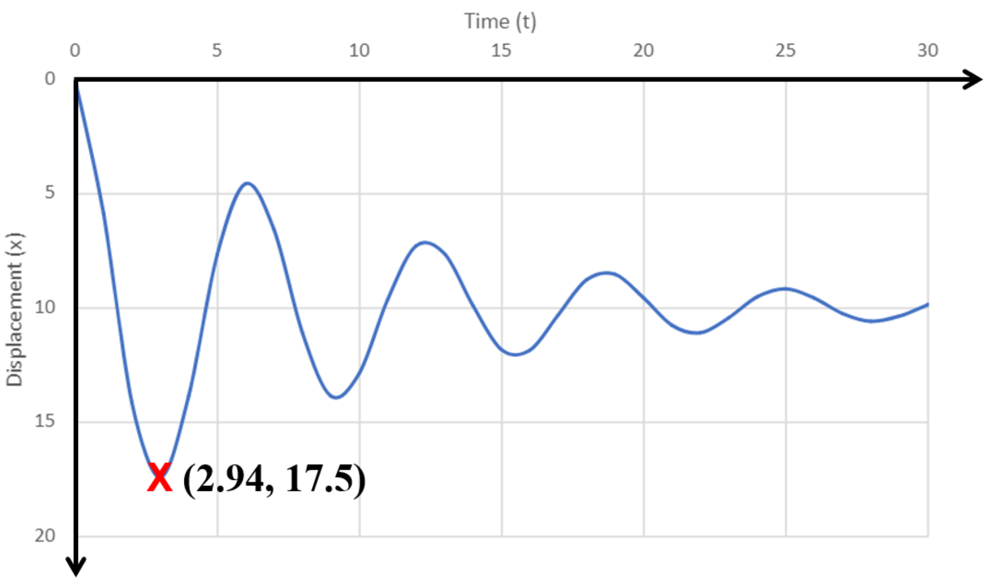
(b) Find the displacement and the velocity of the mass at any time t.

A graph of displacement against time for the motion is plotted. The vertical axis is **inverted** to help you visualise displacement as position below the equilibrium.

(c) Using the graph, comment on the damped motion (underdamped, critically damped or overdamped), and state the maximum displacement of the mass.

(d) If the mass was 20 meters above the ground from its equilibrium position, what would be the lowest point of the mass from the ground?

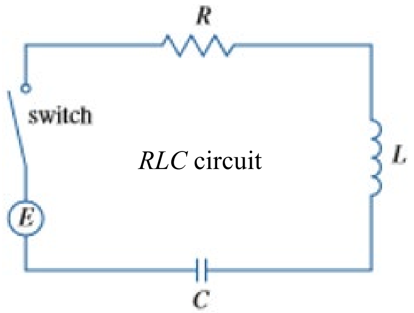
(c)



9.2 Electrical Circuits

In chapter 3, we were able to use first order differential equation to analyze *RL* and *RC* series circuits that contain resistor, inductor and capacitor. Now that we know how to solve second order differential equations, we are able to extend the analysis to *RLC* series electrical circuits.

In the *RLC* circuit below, it contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C, in series.
If the charge on the capacitor at time t is $q = q(t)$, then the current i is the rate of change of q with respect to t, that is, $i = \frac{dq}{dt}$.



It is known from physics that the voltage drops across the resistor, inductor, and capacitor are respectively,

$$v_R = iR \quad , \quad v_L = L\frac{di}{dt} \quad , \quad v_C = \frac{q}{C}$$

Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t) \quad \cdots\cdots\cdots\cdots(1)$$

Since $i = \frac{dq}{dt}$, then equation (1) becomes:

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)} \quad \cdots\cdots\cdots(2)$$

which is a second-order linear differential equation with constant coefficients.

If the charge q_0 and the current i_0 are known at time $t = 0$, then we have the initial conditions:

$$q(0) = q_0 \qquad q'(0) = i(0) = i_0$$

and the initial-value problem can be solved by the Laplace Transform method.

A differential equation for the current can be obtained by differentiating equation (1) with respect to t and remembering that $i = \frac{dq}{dt}$:

$$\boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)} \quad \cdots\cdots\cdots(3)$$

Example 4: Find the **charge** on the capacitor at time t in the *RLC* series circuit (given on pg. 7), if $L = 0.1$ H, $R = 2\ \Omega$, $C = 0.1$ F and $E(t) = 0$V, with initial conditions $q(0) = 0$ and $q'(0) = 1$.

Example 5:

(a) Given $-\dfrac{8}{(s^2+4)(s^2+2s+2)} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+2s+2}$. If $A = \dfrac{4}{5}$ and $B = \dfrac{4}{5}$, find the constants *C* and *D*.

(b) In the *RLC* series circuit (shown on page 7), it is known that $R = 2\Omega$, $L = 1$ H, $C = 0.5$ F, and $E(t) = 2\cos 2t$ V. Initially, there is no current flowing through the circuit, and the rate of change of the current is zero.

i. Set up the differential equation to model the **current** $i(t)$ flowing through the circuit. Indicate clearly the initial conditions.

ii. Find the current at time t in the circuit. Hence, find the transient and steady-state current.

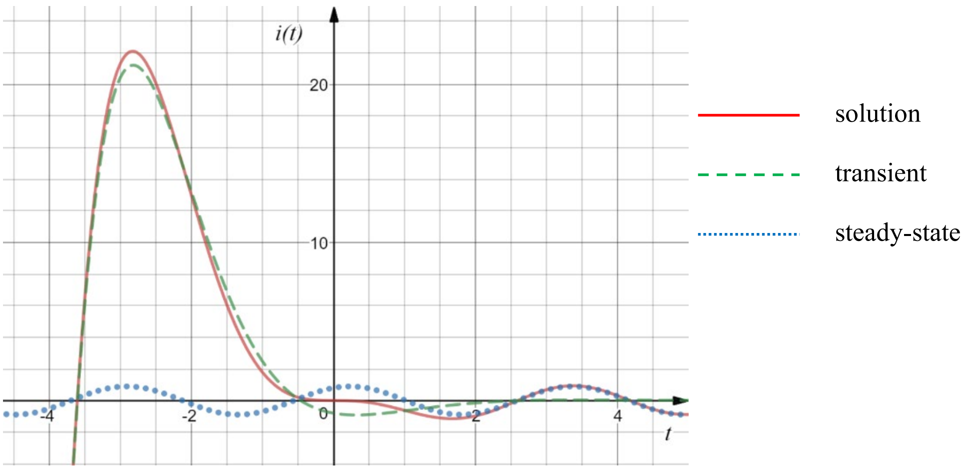
Note that:

- The first term of the solution, $i_S(t) = \frac{2}{5}\sin(2t) + \frac{4}{5}\cos(2t)$, is called the **steady state solution**.
- The second term of the solution, $i_t(t) = -\frac{4}{5}e^{-t}[\cos(t) + 2\sin(t)]$, is called the **transient state solution**.

Remark:

The transient state of the current decays to zero as $t \to \infty$. Damping is caused by the resistance in the circuit. It is the damping force which caused the free vibration to disappear after some time while the steady state vibration remained, which is constantly maintained by the action of the applied external force $2\cos(2t)$.

The graph of the solution is shown here:



Tutorial 9

1. A mass of 0.6 kg is attached to the lower end of a vertical spring of stiffness 200 N/m. The mass is raised 3 cm above the equilibrium position, i.e. $x(0) = -3$ cm, and released from rest, i.e. $v(0) = x'(0) = 0$ cm/s. Assuming no air resistance,
   (a) describe the motion of the mass;
   (b) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions;
   (c) find the position of the mass 5 seconds after it is released; and
   (d) determine the frequency of the motion.

2. A mass of 10 kg is suspended from a spring of spring constant 300 N/m. The mass is pushed up 15 cm above its equilibrium position and released from rest. Assuming there is no damping force,
   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions ;
   (b) find the position of the mass after 1 second ;
   (c) determine the amplitude, period and frequency of the vibration.

3. A 1 kg mass is attached to the lower end of a vertical spring of stiffness 25 N/m. The mass is set into motion from rest at the equilibrium position by an external force $F(t) = \sin(5t)$ (N). The resistance to the motion is numerically equal to $8v$ (N) , where v (m/s) is the velocity of the mass at time t (s).
   (a) Set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions.
   (b) Find the displacement x (m) of the mass at any time t (s) .
   (c) State the amplitude of the steady-state vibration of the mass.
   (d) What is the ratio of the displacement in the steady-state motion to that in the transient-state motion when $t = 0.5$ (s) ?

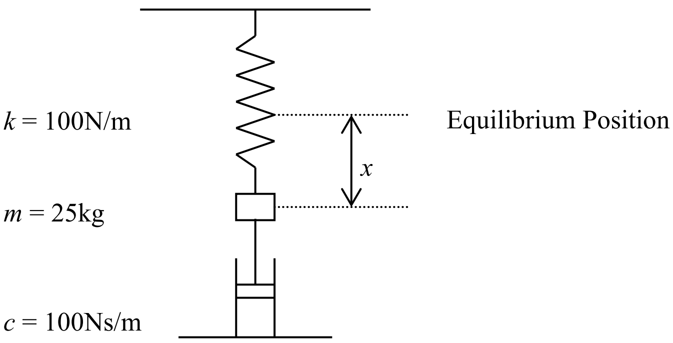
4. A spring has a spring constant of 125 Nm^{-1}. A mass of 5 kg is suspended from the spring and, after it has come to equilibrium, is pulled down 20 cm and released from rest. Assuming that there is a damping force numerically equal to $30v$, where v (m/s) is the instantaneous velocity at time t (s),
   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions.
   (b) Hence, find the position and the velocity of the mass at any time.

5. A 10 kg mass is attached to a spring of stiffness 120 N/m. The mass is set into motion from the equilibrium position with an initial velocity of 1 m/s in the upward direction. If the air resistance is numerically equal to $70v$ N where v is the velocity at time t,

   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions.

   (b) Hence, find the position of the mass at any time t .

6. Determine the differential equation of motion for the damped vibratory system shown. Indicate the initial conditions clearly as well.



Given that the mass is pushed down 1 cm and released from rest, determine the position of the mass at time $t = 1$ s.

7. Find the charge on the capacitor in the *RLC* series circuit when $L = 0.25$ H, $R = 20\ \Omega$, $C = \dfrac{1}{300}$ F, $E(t) = 0$ V, $q(0) = 4$ C and $q'(0) = 0$ A.

8. In a *RLC* circuit, it is known that $R = 10$ ohms, $L = 5/3$ henry, $C = 1/30$ farad, and the electromotive force $E(t) = 300$ volts. If initially, there is no current flowing thru the circuit, and the rate of change of the current is 180 amperes/s,

   (a) set up the differential equation to model the current in the circuit, and indicate clearly the initial conditions.

   (b) Hence, find the current $i(t)$.

9. In a *RLC* circuit, it is known that $R = 10$ ohms, $L = 0.5$ henry, $C = 0.01$ farad, and the electromotive force $E(t) = 150$ volts. If initially, the charge on the capacitor is 1 coulomb and there is no current,

   (a) set up the differential equation to model the charge on the capacitor, and indicate clearly the initial conditions.

   (b) Hence, find the charge $q(t)$.

*10. In a *RLC* circuit, it is known that $R = 2$ ohms, $L = 1$ henry, $C = 0.25$ farad, and the electromotive force $E(t) = 50\cos(t)$ volts.

   (a) Set up the differential equation to model the current in the circuit;

   (b) Hence, find the steady-state current $i_{\text{steady-state}}(t)$.

   [Hint: treat $i(0)$ and $i'(0)$ as constants]

*11. Use Laplace transform method to find the current i given the differential equation:

$$Ri + \frac{1}{C}\int_0^t i\,dt - v_C = 0$$

where v_C, R and C are constants.

Multiple Choice Question

1. If the motion of an engineering system is described by

$$y(t) = \frac{1}{2}[e^{-2t}\cos(t) + 3e^{-2t}\sin(t) - e^{-t}],$$

then the motion is considered ___________.

(a) un-damped (b) under-damped

(c) critically-damped (d) over-damped

Answers to Selected Lecture Examples

Example 1: (b) $x(t) = A\cos 5t + B\sin 5t$

Example 2: $x(t) = e^{-t}\left[\cos(3t) + \frac{1}{3}\sin(3t)\right]$, underdamped

Example 3: (b) $-10e^{-0.1t}\left(\cos t\right) + e^{-0.1t}\left(\sin t\right) + 10$

Example 4: $q(t) = t\,e^{-10t}$

Example 5: (a) $C = -\frac{4}{5}$ and $D = -\frac{12}{5}$

(b)(ii) $i(t) = \frac{1}{5}\left\{2\sin 2t + 4\cos 2t - 4e^{-t}\left(\cos t + 2\sin t\right)\right\}$ A ,

$i_s(t) = \frac{1}{5}\left(2\sin 2t + 4\cos 2t\right)$ A , $i_t(t) = -\frac{4}{5}e^{-t}\left(\cos t + 2\sin t\right)$ A

Answers to Tutorial 9

1. (a) Simple harmonic motion (c) 3 cm below equilibrium position (d) 2.9 Hz
2. (a) 10.4 cm above equilibrium position (b) 0.15 m , 1.15 s , 0.87 Hz
3. (b) $x(t) = -\frac{1}{40}\cos 5t + \frac{1}{40}e^{-4t}[\cos 3t + \frac{4}{3}\sin 3t]$ m (c) $\frac{1}{40}$ m (d) 4.22
4. (a) $x(t) = e^{-3t}[0.2\cos(4t) + 0.15\sin(4t)]$ m (b) $x'(t) = -1.25\,e^{-3t}\sin(4t)$ m/s
5. $x(t) = e^{-4t} - e^{-3t}$ m
6. 0.41 cm
7. $q(t) = 6e^{-20t} - 2e^{-60t}$ C
8. $i(t) = 60e^{-3t}\sin 3t$ amperes
9. $q(t) = \frac{3}{2} - \frac{1}{2}e^{-10t}(\cos 10t + \sin 10t)$ coulombs
10. $i_s(t) = \frac{100}{13}\cos t - \frac{150}{13}\sin t$ amperes
11. $i(t) = \frac{v_C}{R}e^{-\frac{t}{RC}}$ amperes

MCQ

1. (b)