Adjoints for Scientists and Engineers
Spring 2018
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Rensselaer

Assignment #3 **Due Date: Apr 6, 2018**

This assignment is used to explore output-based error estimation using the adjoint-weighted residual method.

1 Boundary Value Problem and Functionals

1.1 The Quasi-1D Euler Equations

We once again consider the quasi-1D Euler equations to model the flow in a converging-diverging nozzle. Refer to the previous assignment for details on the boundary-value problem and its discretization.

One notable addition from last assignment is that we will consider both subsonic and transonic flows. The transonic case is interesting, because it exhibits a shock at $x = 0.7$.

As in Assignment 2, we have an exact solution available to compare against — this is true for both the subsonic and transonic cases. Both cases use a stagnation temperature of 300 K and stagnation pressure of 100 kPa. However, the subsonic flow has a critical nozzle area of $A^* = 0.8$ while the transonic flow has a critical nozzle area of $A^* = 1.0$. Since the nozzle has a throat area of $A(0.5) = 1.0 = A^*$, the flow becomes supersonic at the throat in the transonic flow; it then returns to subsonic again after the shock at $x = 0.7$.

1.2 Functionals

We will once again consider the integrated momentum source,

$$J_1(q) = \int_{x=0}^{1} p \frac{dA}{dx}\, dx.$$

In addition, we will consider a weighted momentum source, where the weight function is a Gaussian bump centered at $x = 0.25$:

$$J_2(q) = \int_{x=0}^{1} G(x) p \frac{dA}{dx}\, dx,$$

where $G(x) = \exp(-(x - \frac{1}{4})^2/(0.05)^2)$.

2 Discretization

As before, the quasi-1D Euler equations are discretized using a nodal discontinuous Galerkin method [1]. To capture the shock in the transonic flow, the discretization uses the artificial viscosity approach of Persson and Peraire [2].

Note that, due to the sonic point at $x = 0.5$ in the transonic flow, a small amount of artificial viscosity is also added at the throat of the nozzle.

You do not need to understand all the details of the shock-capturing scheme, but it is important to appreciate that the transonic problem is significantly more difficult to solve; use the provided `solveHomotopy!` method to solve for q in this case.

3 Provided Code

Please download the new supplementary zip file for this assignment, which has code that has been updated from Assignment 2. Here is a list of some of the functions you will want to make use of.

- `solveNewton!` uses Newton's method, without globalization, to solve the flow equations. This is useful for the subsonic flow.
- `solveHomotopy!` is a more robust method (than Newton's method) for solving nonlinear equations. Use it for the transonic flow.
- `calcIntegratedSource` and `calcWeightedSource` can be used to evaluate the functionals J_1 and J_2, respectively.
- `interpSolution!` can be used to interpolate a coarse-space solution to a fine-space solution based on p enrichment. This function requires that you create a new solver object for the fine space.
- `solveAdjoint!` will solve the adjoint variables for you based on a given solver object and solution array `q`. You have to provide the array corresponding to $\partial J_h/\partial q_h$, which you can compute using...
- `calcIntegratedSourcedJdq!` and `calcWeightedSourcedJdq!` evaluate the partial derivative of J_1 and J_2 with respect to q_h, respectively.

The file `shock_example.jl` is provided to show you how to run the transonic problem. **Be sure to set the keyword to `shocks=false` in the solver construction when solving the subsonic flow, and `shocks=true` when solving the transonic flow**. Also, the arguments for the function for computing the exact solution need to change for the subsonic flow.

4 Questions

1. Implement the adjoint-weighted residual (AWR) method using p enrichment. Please describe the details of your implementation in your report.

   Bonus: 2 bonus points if you implement the AWR without requiring state and adjoint solves on the fine space.

2. Apply the AWR to both J_1 and J_2 for the subsonic nozzle flow ($A^* = 0.8$). For the subsonic flow, you can assume that the "exact" value of the integrated-source functional is

   $$J_1(q) \approx -0.35194635479522557,$$

   and that the "exact" value of the weighted-source functional is

   $$J_2(q) \approx -0.11000142657405404.$$

(a) Provide a grid convergence study that plots the true functional error, the estimated functional error, and the corrected functional error on the same graph. Provide a separate plot for the error-estimate effectivity, η. The studies should include discretizations of degree $p = 1$, $p = 2$, $p = 3$, and $p = 4$, but each of these discretizations should have its own plot.

(b) Choose a particular mesh size, say `numelem=80`, and plot the elementwise localized error, that is, plot the elementwise quantities

$$\epsilon_{k_H} = \left| \sum_{j=1}^{3} \psi_{h,j}^T R_{h,j}(q_H^h) \right|,$$

where $\psi_{h,j}$ is the fine-space approximation to the jth fine-space adjoint variable, q_H^h is the coarse-space solution interpolated to the fine space, and $R_{h,j}$ is the fine-space residual of the jth equation, $j = 1, 2, 3$. Use these plots to describe where you would refine or coarsen the mesh to improve the accuracy of J_1 and J_2.

3. Repeat the above study for the transonic flow. That is, apply the AWR to both J_1 and J_2 when $A^* = 1.0$. For the transonic flow, you can assume that the "exact" value of the integrated-source functional is

$$J_1(q) \approx -0.38482425832590694,$$

and that the "exact" value of the weighted-source functional is

$$J_2(q) \approx -0.10202215753949603.$$

(a) Provide a grid convergence study that plots the true functional error, the estimated functional error, and the corrected functional error on the same graph. Provide a separate plot for the error-estimate effectivity, η. **Unlike the subsonic case**, it is sufficient to consider just degrees $p = 1$ and $p = 2$.

(b) Choose a particular mesh size (i.e. fix the number of elements), and plot the elementwise localized error, that is, plot the elementwise quantities

$$\epsilon_{k_H} = \left| \sum_{j=1}^{3} \psi_{h,j}^T R_{h,j}(q_H^h) \right|,$$

where $\psi_{h,j}$ is the fine-space approximation to the jth fine-space adjoint variable, q_H^h is the coarse-space solution interpolated to the fine space, and $R_{h,j}$ is the fine-space residual of the jth equation, $j = 1, 2, 3$. Use these plots to describe where you would refine or coarsen the mesh to improve the accuracy of J_1 and J_2.

Please highlight the differences between the transonic and subsonic cases. Use physical reasoning to explain any differences that you observe.

5 Report and Code Expectations

Your report should be typeset in LaTeX. As before, the report should be fewer than 10 pages if you use a 12 point font. This is a "soft" page limit, but please do not be overly verbose.

The following should be included in the report.

- An abstract that summarizes your results.
- Provide a separate section or subsection in which you answer each question in the above list of questions.

You do not need to include an introduction, since your audience (me!) knows the background. You can include the results of additional studies that you performed out of interest.

The report will include several figures as requested in the questions. Please use an adequate font size on the figures! It should be legible if printed.

Please provide the modified code for your project as a separate zip file. You do not need to include in the report itself. All code should have adequate commenting.

Collaboration

You are permitted and encouraged to discuss the assignment with each other, provided each of you writes your own code and report. A good policy to follow in order to avoid academic misconduct is to not take notes or exchange files with one another; i.e. exchange information verbally and you should be fine.

References

[1] Hesthaven, J. S. and Warburton, T., *Nodal discontinuous Galerkin methods: algorithms, analysis, and applications*, Springer-Verlag, New York, 2008.

[2] Persson, P.-O. and Peraire, J., "Sub-Cell Shock Capturing for Discontinuous Galerkin Methods," *44th AIAA Aerospace Sciences Meeting and Exhibit*, American Institute of Aeronautics and Astronautics, Jan. 2006.