Assignment 1: Adjoint of Elliptic Equation

ID:4873

March 19, 2018

Abstract

1 Code summary

1.1 Development

For this assignment, the following functionalities have been implemented:

- evaluation of the Jacobian matrix of type **SparseMatrixCSC** using a coloring complex-step approach;
- Newton iteration to solve the nonlinear system;
- evaluation of the partial derivative of functionals with respect to the area using the complex-step approach;
- evaluation of the partial derivative of residual with respect to the area using the complex-step approach;
- adjoint solve;
- evaluation of total derivative of functionals.

1.2 How to run the code

For convergence study of functional, run

`julia convergence_study.jl`,

and the results are under directory *results/functional*. For adjoint solution, run

`julia solve_adjoint.jl`,

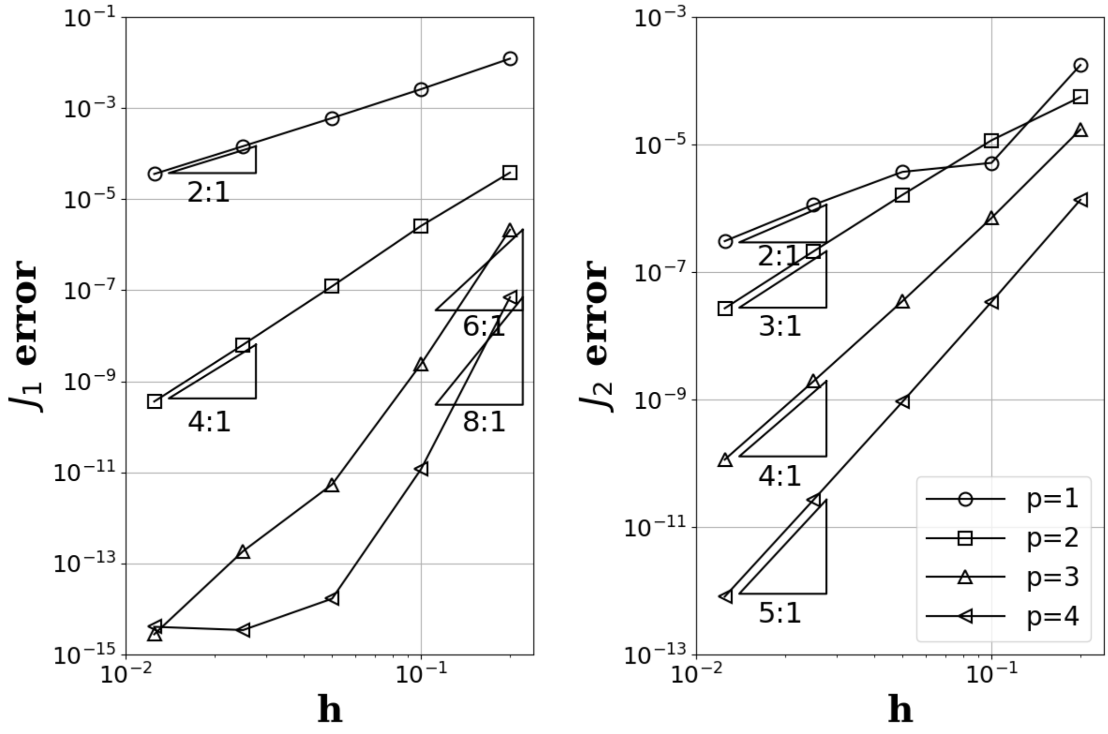
and results are under directory *results/adjoints*.

2 Convergence study of functional

The convergence study of two functionals, $J_{1,h}$ and $J_{2,h}$, is carried out, with $p = 1 \sim 4$. This requires solving the nonlinear system first. Here both explicit and implicit methods are tried. The results are quite the same as long as the convergence tolerance of Runge-Kutta is set to very small, say, 10^{-13}.

The results are shown in Figure 1. As seen, the error in $J_{1,h}$ sees superconvergence while the error in $J_{2,h}$ does not. Specifically, for $J_{1,h}$ the observed convergence rates with $p = 1, 2$ are $2p$ while with $p = 3, 4$ the convergence rates are even higher than $2p$ before the functional error reaches machine zero. On the contrary, the convergence rates for $J_{2,h}$ are uniformly $p + 1$. These results indicate that,

- the discretization is probably adjoint consistent;
- the adjoint solution of functional $J_{1,h}$ is smooth while that of $J_{2,h}$ is not.



(a) $J_{1,h}$ error

(b) $J_{2,h}$ error

Figure 1: Functional errors

3 Adjoints of $J_{1,h}$ and $J_{2,h}$

The adjoint solve requires the evaluation of the Jacobian matrix of the nonlinear system. As aforementioned, a coloring complex-step method is implemented. For this specific problem, elements $i, i+3, i+6, \ldots,$ are grouped together, resulting three colors totally. With number of degrees of freedom per element being $N_{dof,elem}$, the evaluation of Jacobian matrix requires only $3N_{dof,elem}$ residual evaluations.

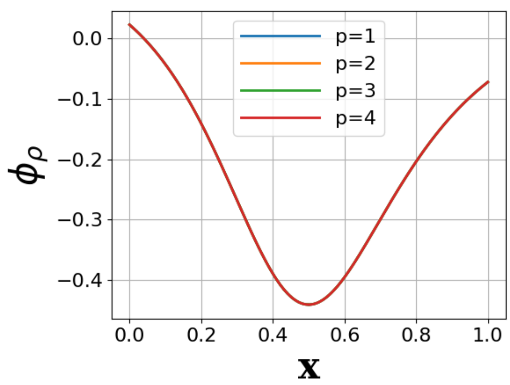
With the Jacobian matrix available, a raw Newton iteration is implemented to solve the nonlinear system, as an alternative to the explicit Runge-Kutta. Although no effort is spend to improve the robustness, no convergence problem has been encountered for all the test cases shown in this assignment. Compared to the explicit methods, the implicit method is usually dozens times faster, especially for large cases.

Similar to the evaluation of Jacobian matrix, the derivative of functionals with respect to the solution q_h, $\partial J_{i,h}/\partial q_h$ is also evaluated using complex-step. Finally, the linear adjoint system is solved using a direct solver.

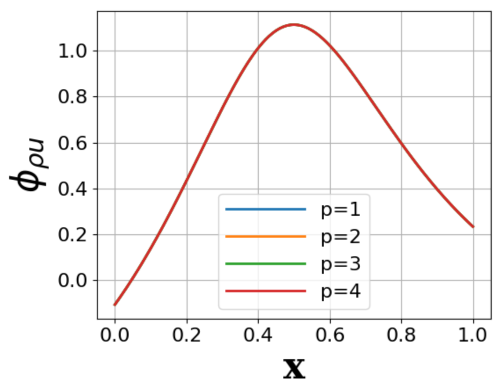
Although several difference grids are used to ensure the solution is grid independent, only results on `numelem=100` are displayed here.

The adjoints of $J_{1,h}$ and $J_{2,h}$ are plotted in Figure 2 and 3, respectively. Obviously, with `numelem=100`, all the adjoint solutions coincide with each other, which indicates that the mesh is sufficiently refined. The only exception occurs near $x = 1$ in Figure 3.

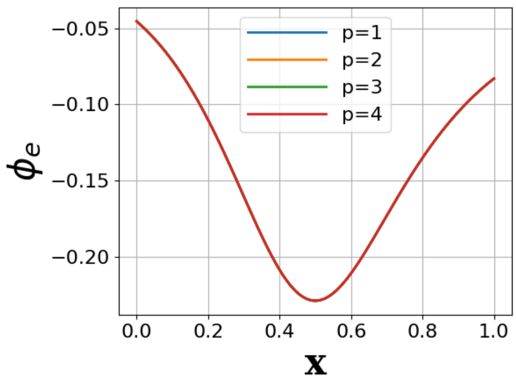
The adjoints of $J_{1,h}$ are all smooth, while the adjoints of $J_{2,h}$ see severe oscillations near $x = 1$. These oscillations destroy the smoothness of the adjoint solution, which is one of the prerequisites to obtain super-convergence in evaluation of functionals. This is why in Figure 1b the convergence rates of $J_{2,h}$ are $p + 1$ rather than $2p$.



(a) ϕ_ρ of $J_{1,h}$



(b) $\phi_{\rho u}$ of $J_{1,h}$



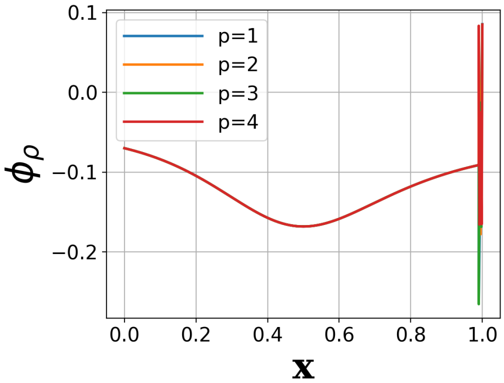
(c) ϕ_e of $J_{1,h}$

Figure 2: Adjoints of $J_{1,h}$

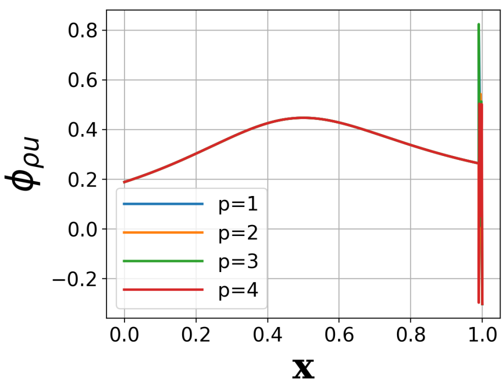
4 Evaluation of total derivatives of functionals

Based on the adoint solution, the total derivatives of $J_{1,h}$ and $J_{2,h}$ with respect to the area, $\frac{DJ_{i,h}}{DA}$, are computed. The results are plotted in Figure 4 and 5. As seen, for both cases, the derivatives are close to zero except inside the first and last elements. For $J_{1,h}$, this can be easy explain as follows: By integration by parts,

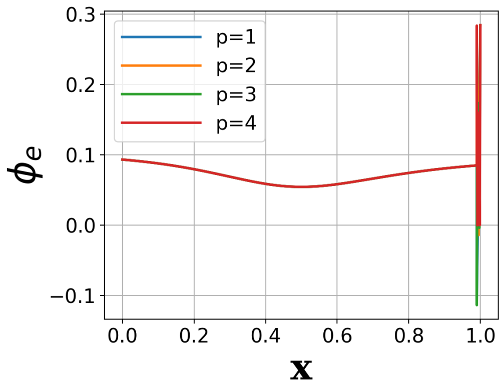
$$J_1 = (pA)|_0^1 - \int_0^1 \frac{\partial p}{\partial x} A \mathrm{d}x$$



(a) ϕ_ρ of $J_{2,h}$



(b) $\phi_{\rho u}$ of $J_{2,h}$

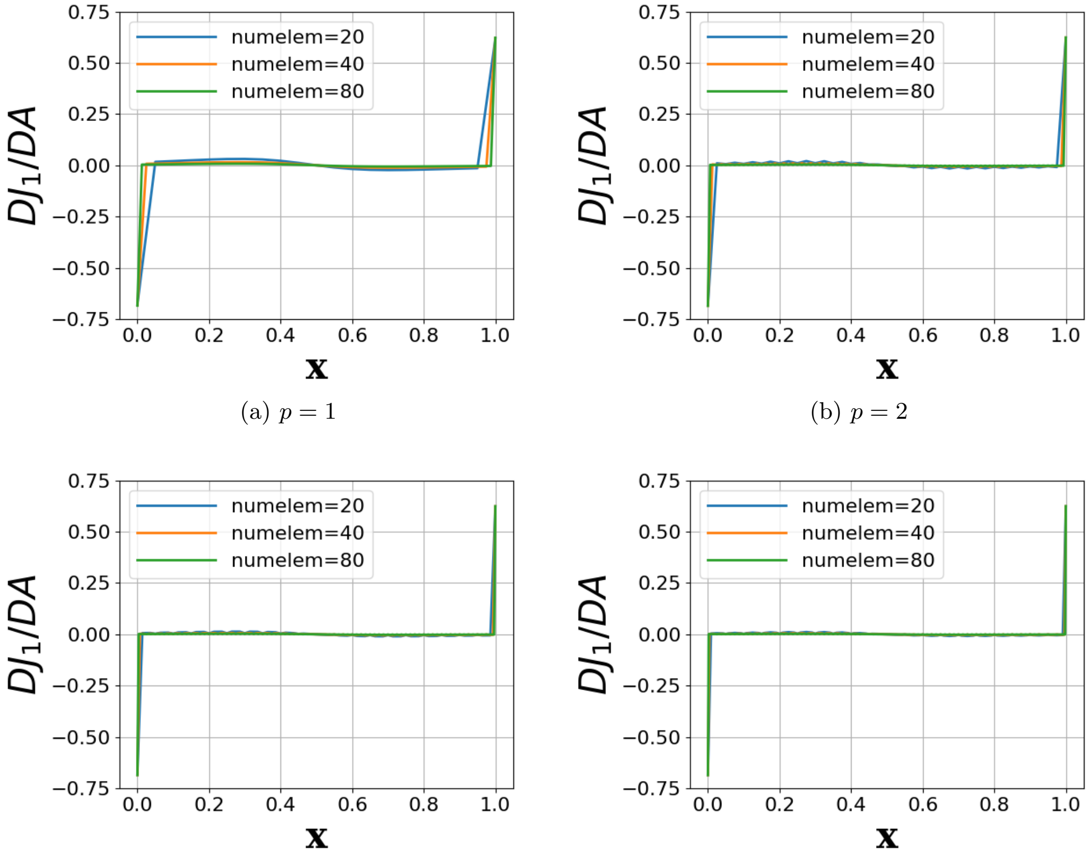


(c) ϕ_e of $J_{2,h}$

Figure 3: Adjoints of $J_{2,h}$

Assuming the area is smooth along x, $\frac{\partial p}{\partial x}$ is small for subsonic flow, then

$$\frac{DJ_1}{DA} \approx \begin{cases} 0, 0 < x < 1 \\ p, x = 0, 1 \end{cases}$$



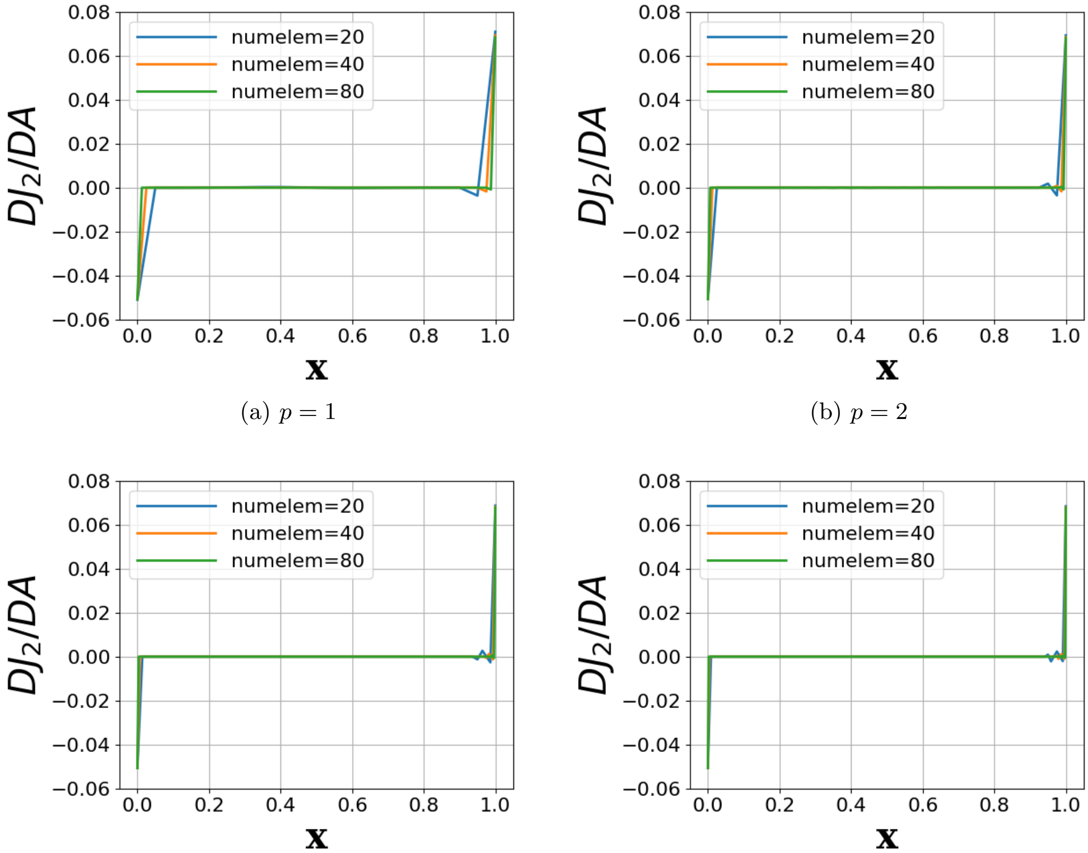
(a) $p = 1$

(b) $p = 2$

(c) $p = 3$

(d) $p = 4$

Figure 4: Value of $\frac{DJ_1}{DA}$



(a) $p = 1$

(b) $p = 2$

(c) $p = 3$

(d) $p = 4$

Figure 5: Value of $\frac{DJ_2}{DA}$