Comments on the H_2 Norm of Discrete-Time Stochastic Jump Parameter Linear Systems

Junior R. Ribeiro, Luiz H. Romero, Eduardo F. Costa, and Marcos G. Todorov

***Abstract*—In this note, we consider a general class of stochastic, jump parameter systems and show that the so-called H_2 norm is the Frobenius norm of the impulse response matrix of the system. We present a necessary and sufficient condition for the H_2 norm to be the induced norm from the input to the output for a particular type of deterministic input signal. As expected in the context of time-varying systems, we cannot extend this result to more general deterministic input signals. Moreover, finite H_2 norm does not prevent infinite norm output in response to a finite norm input, unless additional conditions are imposed on the system. Here, we give relatively mild conditions - mean square stability and jumps driven by a finite-dimensional Markov chain starting in equilibrium - and we show that finite H_2 norm implies a bounded ratio between the norms of the output and the input, for certain suitable norms.**

***Index Terms*—Stochastic systems, linear systems, hybrid systems, jump parameter systems, H_2 norm.**

I. Introduction

Several control and filtering results addressing the H_2 norm have been developed in the literature of discrete-time, jump parameter linear systems (JPLS), which is a broad class of systems that comprise, e.g., Markov jump linear systems (MJLS), semi-MJLS, and MJLS with uncertain probability transitions. See [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24] for the H_2 norm definition and results on H_2 stabilization, control, and filtering of JPLS.

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Junior R. Ribeiro, Luiz H. Romero, and Eduardo F. Costa are with the Instituto de Ciências Matemáticas e de Computação, University of São Paulo, São Paulo 13560970, Brazil (e-mail: j5rodrib@gmail.com; neoluizz@gmail.com; efcosta@icmc.usp.br).

Marcos G. Todorov is with the Department of Mathematical and Computational Methods, National Laboratory for Scientific Computing (LNCC), Petrópolis 25651070, Brazil (e-mail: todorov@lncc.br).

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However, to the extent of our knowledge, there is no explanation/proof that the H_2 norm is a norm indeed for such a general class of systems. More specifically, if we denote the JPLS by Φ and its norm by $\|\Phi\|_2$ (following a common notation, as in [3]), in what sense does $\|\Phi\|_2$ satisfy the axioms of norm? It is also not clear whether the H_2 norm is induced in the sense that it satisfies the equivalence:

$$\|\Phi\|_2 = \sup_{w \neq 0} \frac{\|y\|_p}{\|w\|_q}, \tag{1}$$

for $p, q \in [1, \infty)$, where $\|y\|_p$ and $\|w\|_q$ are the norms of the output process $y = \{y(k), k \geq 0\}$, and the deterministic input signal $w = \{w(k), k \geq 0\}$, respectively. Or, at least, if for systems with bounded norm $\|\Phi\|_2 \leq \gamma$,

$$\exists\, p, q, M : \frac{\|y\|_p}{\|w\|_q} \leq M\gamma, \ \forall w \neq 0 : \|w\|_p < \infty, \tag{2}$$

where $M > 0$ and $p, q \in [1, \infty)$, in such a manner that finite H_2 norm would imply "input-to-output stability". As JPLS are time-varying (for a given signal $s(k)$, $k \geq 0$, if we denote $y(k, \tau)$ the output due to $w(k) = s(k - \tau)$, then, e.g., $\mathbb{E}(\|y(k+\tau, \tau)\|^2) \neq \mathbb{E}(\|y(k, 0)\|^2)$, $\tau \neq 0$, in general), one should not expect (2) to hold, and even less (1).

In this note, in Section II we show that $\|\Phi\|_2$ is the Frobenius norm of the impulse response matrix, thus extending to JPLS a well-known property of deterministic linear systems. In Section III, we present a necessary and sufficient condition for (1) to be valid in view of a special type of deterministic input signal (linear combination of impulses at time $k = 0$). As we shall see, the condition is quite restrictive, making (1) valid, e.g., for single input systems and that type of input signals. This is compatible with the fact that the Frobenius matrix norm is not an induced norm [25].

In the context of general deterministic inputs and MJLS (which are also time-varying), we show that $\|y\|_2 \leq \|\Phi\|_2 \|w\|_1$ (hence (2) is valid with $p = 2$ and $M = q = 1$) under some conditions: the system is mean square stable (MSS) and the Markov chain is finite-dimensional with initial invariant distribution. The result is given in Corollary 2, which is based on Theorem 3, where we show how to compute the induced norm $\sup_{\{w \neq 0 : \|w\|_1 < \infty\}} \|y\|_2 \|w\|_1^{-1}$. For a better understanding of the conditions of Corollary 2 in the validity of (2), we relax them in some examples. When MSS and invariance of the initial distribution are dismissed, then (2) is violated, even if we allow M to depend on the parameters of the MJLS;

in fact, in Examples 1 and 2, we obtain relatively small norms $\|\Phi\|_2$, but $\|y\|_2 = \infty$ in response to a finite norm input. In Example 3 we keep MSS valid and relax invariance, and we present a system parameterized by ρ such that $\forall p, q \geq 1$ and $M > 0$, $\exists \rho \in (0, 0.5] : \|y\|_p \|w\|_q^{-1} > M\|\Phi\|_2$ with $w \neq 0$. In this example, $\|\Phi\|_2 = 1$ and (2) is valid, but M can be made arbitrarily large by adjusting ρ, indicating that some care has to be taken when subjecting an MJLS to general inputs if the Markov chain does not start in equilibrium.

II. The H_2 Norm Is the Frobenius Norm of the Impulse Response Matrix

Consider the stochastic process $\theta = \{\theta(k) \in \mathcal{N}, k \geq 0\}$, where $\mathcal{N} = \{1, 2, \ldots\}$, and the JPLS defined by

$$\Phi\colon \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}w(k), \\ y(k) = C_{\theta(k)}x(k), \\ x(0) = 0, k \geq 0, \end{cases} \tag{3}$$

where $y(k)$ is the output, and $w(k)$ is a deterministic input signal; matrices A_i, B_i, C_i are of dimensions $n \times n$, $n \times n_w$, $n_y \times n$ respectively, for all $i \in \mathcal{N}$; note that θ is not necessarily a Markov chain. Consider $\{\Omega, \mathfrak{F}, P\}$ the fundamental probability space, and denote by $\mathbb{E}(\cdot)$ the expected value, by $\mathbb{E}(\cdot|\cdot)$ the conditional expected value; the H_2 norm is defined as usual in JPLS and MJLS [3],

$$\|\Phi\|_2^2 = \sum_{k=0}^{\infty}\sum_{j=1}^{n_w}\mathbb{E}\Big(\|y^j(k)\|_2^2\Big),$$

where $y^j(k)$ is the output due to the disturbance in the form $w(k) = 0$, $k \geq 1$, $w(0) = e_j$, where e_j stands for the j-th vector of the canonical basis in $\Re^{n_w}$. We define $G = \{G(k), -\infty < k < \infty\}$ by $G(1) = C_{\theta(1)}B_{\theta(0)}$, $G(k+1) = C_{\theta(k+1)}A_{\theta(k)}A_{\theta(k-1)}\cdots A_{\theta(1)}B_{\theta(0)}$, $k \geq 1$, and $G(k) = 0$, $k \leq 0$ (JPLS are causal systems). We refer to the stochastic process G as the impulse response matrix of Φ.

Let us introduce some notation. For a real matrix V, we write its Frobenius norm as $\|V\|_F = (\mathrm{tr}(VV'))^{1/2}$, where the operator $\mathrm{tr}(\cdot)$ is the trace of a matrix. For a vector v, we write its Euclidean norm as $\|v\|_2 = (\mathrm{tr}(vv'))^{1/2}$. The largest eigenvalue of a positive semidefinite matrix $R = R'$ is denoted $\lambda_{\max}(R)$. Regarding the processes G, y and the deterministic signal w, we write,

$$\begin{aligned} \|G\|_F &= \sqrt{\sum_{k=-\infty}^{\infty}\mathbb{E}\Big(\|G(k)\|_F^2\Big)}, \\ \|y\|_{p,r} &= \left[\sum_{k=0}^{\infty}\mathbb{E}\Big(\|y(k)\|_r^p\Big)\right]^{1/p}, \\ \|w\|_{p,r} &= \left[\sum_{k=0}^{\infty}\|w(k)\|_r^p\right]^{1/p}, \quad p, r \in [1, \infty), \end{aligned} \tag{4}$$

where the norms of vectors are the usual ones, e.g., $\|w(k)\|_r = (\sum_{i=1}^{n_w}|w_i(k)|^r)^{1/r}$. When $r = 2$ we dismiss the index r in $\|w\|_{p,r}$ and write it as $\|w\|_p$. We define $\mathcal{L}_{p,r}^{n_w} = \{w;\ w(k) \in \Re^{n_w}, \|w\|_{p,r} < \infty\}$ and $\mathcal{L}_{p,r}^{n_y} = \{y;\ y(k) \in \Re^{n_y}, \|y\|_{p,r} < \infty\}$; similarly as above, we dismiss the index r when $r = 2$. Throughout this note, when dealing with the norm $\|w\|_{p,r}$ of an input signal w, we assume that w belongs to $\mathcal{L}_{p,r}^{n_w}$.

Theorem 1: $\|\Phi\|_2 = \|G\|_F$.

Proof:

$$\begin{aligned} \|\Phi\|_2^2 &= \sum_{k=0}^{\infty}\sum_{j=1}^{n_w}\mathbb{E}\Big(\mathrm{tr}(y^j(k)y^j(k)')\Big) \\ &= \sum_{k=-\infty}^{\infty}\sum_{j=1}^{n_w}\mathbb{E}\Big(\mathrm{tr}\big(G(k)e_je_j'G(k)'\big)\Big) \\ &= \sum_{k=-\infty}^{\infty}\mathbb{E}\Big(\mathrm{tr}\big(G(k)(\sum_{j=1}^{n_w}e_je_j')G(k)'\big)\Big) \\ &= \sum_{k=-\infty}^{\infty}\mathbb{E}\Big(\mathrm{tr}\big(G(k)G(k)'\big)\Big) = \sum_{k=-\infty}^{\infty}\mathbb{E}\Big(\|G(k)\|_F^2\Big) \end{aligned}$$

■

Lemma 1: $\|\cdot\|_F$ defined in (4) is a norm.

For the proof of Lemma 1, see the Appendix.

III. $\|\Phi\|_2$ as Induced Norm and the Input-Output Relation

In this section, we investigate the validity of (1) and (2), including some conditions for them to hold true, and examples of violations.

A. A Specific Deterministic Input Signal

Consider the following type of input signals:

$$\mathcal{S} = \{w : w(k) = 0, k \geq 1, w(0) = v \text{ for } v \in \Re^{n_w}\}. \tag{5}$$

Any $v \in \Re^{n_w}$ can be written as a linear combination of $[e_1, \ldots, e_{n_w}]$, hence, we can think of any $w \in \mathcal{S}$ as a linear combination of impulses at time $k = 0$.

Let us define, for a given system Φ and a nontrivial vector $v \in \Re^{n_w}$, $v \neq 0$, the function

$$\phi(v) = \sum_{\ell=1}^{n_w-1}\sum_{k=-\infty}^{\infty}\mathbb{E}\Big(\mathrm{tr}\big(G(k)z_\ell z_\ell'G(k)'\big)\Big) \tag{6}$$

where z_ℓ, $\ell = 1, \ldots, n_w - 1$, are such that the set of vectors $\{z_1, \ldots, z_{n_w-1}, v\|v\|_2^{-1}\}$ form an orthonormal basis for $\Re^{n_w}$; in particular, if $n_w = 1$ then $v\|v\|_2^{-1}$ works as a basis for $\Re^1$ and $\phi(v) = 0$. Note that $\phi(v)$ is well defined even though the vectors $z_1, \ldots, z_{n_w-1}$ are not unique; in fact, if we denote by O the orthogonal matrix whose columns are formed by the vectors $z_1, \ldots, z_{n_w-1}, v\|v\|_2^{-1}$, we have

$$I = OO' = \frac{vv'}{\|v\|_2^2} + \sum_{\ell=1}^{n_w-1}z_\ell z_\ell', \tag{7}$$

leading to $\sum_{\ell=1}^{n_w-1}\mathrm{tr}(G(k)z_\ell z_\ell'G(k)') = \mathrm{tr}\big(G(k)\big(I - \frac{vv'}{\|v\|_2^2}\big)G(k)'\big)$, irrespective of the choice of $z_1, \ldots, z_{n_w-1}$.

Theorem 2: For any input signal $w \in \mathcal{S}$, we have

$$\|y\|_2^2 = \|w\|_2^2\big(\|\Phi\|_2^2 - \phi(w(0))\big).$$

Proof: If $w(0) = 0$, the result is trivial. If $w(0) \neq 0$, let us denote $v = w(0)$ and from (7) we have $vv' = \|v\|_2^2(I - \sum_{\ell=1}^{n_w-1} z_\ell z_\ell')$. This allows us to write

$$\begin{aligned}\|y\|_2^2 &= \sum_{k=0}^{\infty} \mathbb{E}\Big(\mathrm{tr}(y(k)y(k)')\Big) = \sum_{k=-\infty}^{\infty} \mathbb{E}\Big(\mathrm{tr}\big(G(k)vv'G(k)'\big)\Big)\\ &= \sum_{k=-\infty}^{\infty} \mathbb{E}\Big(\mathrm{tr}\Big(G(k)\|v\|_2^2\Big(I - \sum_{\ell=1}^{n_w-1} z_\ell z_\ell'\Big)G(k)'\Big)\Big)\\ &= \|v\|_2^2\Big(\sum_{k=-\infty}^{\infty} \mathbb{E}\Big(\mathrm{tr}\big(G(k)G(k)'\big)\Big)\\ &\qquad - \sum_{k=-\infty}^{\infty}\sum_{\ell=1}^{n_w-1} \mathbb{E}\Big(\mathrm{tr}\big(G(k)z_\ell z_\ell' G(k)'\big)\Big)\Big)\\ &= \|w\|_2^2\Big(\|\Phi\|_2^2 - \sum_{k=-\infty}^{\infty}\sum_{\ell=1}^{n_w-1} \mathbb{E}\Big(\mathrm{tr}\big(G(k)z_\ell z_\ell' G(k)'\big)\Big)\Big).\end{aligned}$$

■

Corollary 1: For input signals $w \in \mathcal{S}$, we have

i) $\|y\|_2 \leq \|w\|_2\|\Phi\|_2$,

ii) $\|y\|_2 = \|w\|_2\|\Phi\|_2$ if and only if $\phi(w(0)) = 0$.

iii) Given a system Φ and the associated ϕ as defined in (6), the H_2 norm is auto-induced ($\|\Phi\|_2 = \sup_{w\in\mathcal{S},w\neq 0} \|y\|_2/\|w\|_2$) if and only if $\phi(w(0)) = 0$ for some $w(0) \neq 0$.

Proof: Since $\phi(v) \geq 0$ for all $v \in \Re^{n_w}$, Theorem 2 leads to (i). (ii) is immediate from Theorem 2. (iii): ($\Rightarrow$) is straightforward from (i) and (ii); ($\Leftarrow$) if $\phi(w(0)) > 0$ for all $w(0) \in \Re^{n_w}$, $w(0) \neq 0$, then Theorem 2 yields $\|\Phi\|_2 > \|y\|_2\|w\|_2^{-1}$ and the supremum is never attained. ■

Remark 1: $\phi(v) > 0$ for all $v \neq 0$ is quite common so that by statement (iii) of Corollary 1, the H_2 norm is frequently not auto-induced, even in the particular scenario of input signals $w \in \mathcal{S}$. For instance, if there is a positive probability that $G(k)$ is of full rank for some k, then $\mathbb{E}(\mathrm{tr}(G(k)z_\ell z_\ell' G(k)')) > 0$ and $\phi(v) > 0$ for all $v \neq 0$. As an example where there exists v such that $\phi(v) = 0$, consider $G(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ almost surely for all $k \geq 0$, then $\phi([1 \;\; 0]') = 0$. Another example is when $n_w = 1$ and, consequently, $\phi = 0$ by definition. As we see, the H_2 norm is auto-induced in quite restrictive scenarios.

B. Mean Square Stable MJLS

In this subsection, the input signals $w = \{w(k) \in \Re^{n_w};\ k \geq 0\}$ are assumed to be finite-energy signals only, and are no longer restricted to $\mathcal{S}$. Although (2) is violated for general JPLS (as shown in Examples 1–3, later on), it holds true for MJLS under additional assumptions.

Definition 1: System (3) is said to be mean square stable (MSS) if, regardless of the initial conditions $x(0), \theta(0)$, the following holds:

$$w(\cdot) \equiv 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \mathbb{E}\big(\|x(k)\|^2\big) = 0. \tag{8}$$

Assumption 1: $\mathcal{N} = \{1, 2, \ldots, N\}$ and $\theta(k) \in \mathcal{N}$, for $k \geq 0$, is a time-homogeneous Markov chain.

Let us denote $p_{ji} = P(\theta(k+1) = i \mid \theta(k) = j)$.

Assumption 2: The initial distribution of the Markov chain θ is stationary:

$$P(\theta(0) = i) = \sum_{j=1}^{N} P(\theta(0) = j)p_{ji} \quad \forall\, i = 1, \ldots, N. \tag{9}$$

Consider the operator $\mathcal{T}(Y) = (\mathcal{T}_1(Y), \ldots, \mathcal{T}_N(Y))$, where $Y = \{Y_i \in \Re^{n\times n}, i = 1, \ldots, N\}$ is a set of matrices,

$$\mathcal{T}_i(Y) = A_i'\Big(\sum_{j=1}^{N} p_{ij}Y_j\Big)A_i, \qquad i = 1, \ldots, N, \tag{10}$$

and write $\mathcal{T}_i^{(k)}(Y) = \mathcal{T}_i(\mathcal{T}^{(k-1)}(Y))$, for $k \geq 1$, with $\mathcal{T}_i^{(0)}(Y) \equiv Y_i$. The following results are well-known (see [3]):

Lemma 2: Under Assumption 1, the following conditions are equivalent:

(i) System (3) is MSS.

(ii) The coupled Lyapunov equations $Y_i \equiv \mathcal{T}_i(Y) + Q_i$ have a unique solution for each set of matrices $Q = \{Q_i = Q_i' \geq 0, i = 1, \ldots, N\}$, which is $Y_i \equiv \sum_{k=0}^{\infty} \mathcal{T}_i^{(k)}(Q)$.

Let us write $G(k+1, \ell) = C_{\theta(k+1)}\Pi(k, \ell)$, where

$$\Pi(k, \ell) := \begin{cases} A_{\theta(k)} \cdots A_{\theta(\ell+1)}B_{\theta(\ell)}, & k > \ell,\\ B_{\theta(\ell)}, & k = \ell,\\ 0, & k < \ell.\end{cases}$$

Lemma 3: Consider system (3) and Assumption 1. If the system is MSS, then:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \mathbb{E}\Big(G(k, \ell)'G(k, \ell)\Big) &= \mathbb{E}\Big(\sum_{k=-\infty}^{\infty} G(k, \ell)'G(k, \ell)\Big)\\ &= \sum_{i=1}^{N} P(\theta(\ell) = i)B_i'Y_iB_i \end{aligned}\tag{11}$$

where, for each $i = 1, \ldots, N$:

$$Y_i = \mathcal{T}_i(Y) + \sum_{j=1}^{N} p_{ij}C_j'C_j. \tag{12}$$

Proof: Let us denote $\mathfrak{F}_k = \{\theta(0), \ldots, \theta(k)\}$, and $Q_i \equiv \sum_{j=1}^{N} p_{ij}C_j'C_j$ for $i = 1, \ldots, N$. We therefore have, from the Tower Rule,

$$\begin{aligned}&\mathbb{E}\Big(G(k+1, \ell)'G(k+1, \ell)\Big)\\ &\quad= \mathbb{E}\Big\{\Pi(k, \ell)'\mathbb{E}\big(C_{\theta(k+1)}'C_{\theta(k+1)} \mid \mathfrak{F}_k\big)\Pi(k, \ell)\Big\}\\ &\quad= \mathbb{E}\Big(\Pi(k, \ell)'Q_{\theta(k)}\Pi(k, \ell)\Big).\end{aligned}\tag{13}$$

In addition, also from the Tower rule:

$$\begin{aligned}&\mathbb{E}\Big\{\Pi(k-t, \ell)'\mathcal{T}_{\theta(k-t)}^{(t)}(Q)\ \Pi(k-t, \ell)\Big\}\\ &= \mathbb{E}\Big\{\Pi(k-t-1, \ell)'\\ &\qquad \times \mathbb{E}\Big[A_{\theta(k-t)}'\mathcal{T}_{\theta(k-t)}^{(t)}(Q)\ A_{\theta(k-t)}\,\Big|\,\mathfrak{F}_{k-t-1}\Big]\\ &\qquad \times \Pi(k-t-1, \ell)\Big\}\\ &= \mathbb{E}\Big\{\Pi(k-t-1, \ell)'\mathcal{T}_{\theta(k-t-1)}^{(t+1)}(Q)\ \Pi(k-t-1, \ell)\Big\},\end{aligned}$$

so, using induction on (13), we obtain

$$\mathbb{E}\big(G(k+1,\ell)'G(k+1,\ell)\big) = \mathbb{E}\Big(B'_{\theta(\ell)}\mathcal{T}^{(k-\ell)}_{\theta(\ell)}(Q)\ B_{\theta(\ell)}\Big) = \sum_{i=1}^{N}\pi_i(\ell)B'_i\mathcal{T}^{(k-\ell)}_i(Q)\ B_i$$

where $\pi_i(\ell) := P(\theta(\ell) = i)$, and therefore

$$\sum_{k=-\infty}^{\infty}\mathbb{E}\big(G(k+1,\ell)'G(k+1,\ell)\big) = \sum_{k=\ell}^{\infty}\sum_{i=1}^{N}\pi_i(\ell)B'_i\mathcal{T}^{(k-\ell)}_i(Q)\ B_i = \sum_{i=1}^{N}\pi_i(\ell)B'_iY_iB_i$$

where, from the MSS hypothesis, $Y = \sum_{k=0}^{\infty}\mathcal{T}^{(k)}(Q)$ is the only solution to (12). As for the interchangeability of the expectation with the infinite sum in (11), for any vector $v \in \Re^{n_w}$ we have that the random variable $v'G(k,\ell)'G(k,\ell)v$ is nonnegative, hence [26, Corollary 5.3.1] yields

$$\sum_{k=-\infty}^{\infty}\mathbb{E}\Big(v'G(k,\ell)'G(k,\ell)v\Big) = \mathbb{E}\Big(\sum_{k=-\infty}^{\infty}v'G(k,\ell)'G(k,\ell)v\Big)$$

which is valid for all $v \in \Re^{n_w}$, leading to the result. ■

Remark 2: In view of Theorem 1, if we set $\ell = 1$ in (11), and take the trace, we have $\|\Phi\|_2^2 = \sum_{i=1}^{N}\mathrm{tr}(P(\theta(\ell) = i)B'_iY_iB_i)$. This also means that, under Assumption 1, if system (3) is MSS, then it has finite H_2 norm.

Proceeding further, we show that an induced system norm from $\mathcal{L}_1^{n_w}$ to $\mathcal{L}_2^{n_y}$ can be precisely characterized in terms of the solution of (12).

Theorem 3: Consider Assumptions 1 and 2, and assume system (3) is MSS. Then, letting $\nu_i \equiv P(\theta(0) = i)$:

$$\sup_{w\in\mathcal{L}_1^{n_w}}\frac{\|y\|_2}{\|w\|_1} = \sqrt{\lambda_{\max}\left(\sum_{i=1}^{N}\nu_iB'_iY_iB_i\right)}. \tag{14}$$

Proof: Due to MSS, we get from [3, Th. 3.9] that $\|y\|_2 < \infty$. This yields the following (all the inequalities $\le$ are due to the Cauchy-Schwarz inequality in different setups, and $\|\cdot\|$ stands for the Euclidean norm)

$$\begin{aligned}\|y\|_2^2 &= \sum_{k=0}^{\infty}\mathbb{E}\left[\left(\sum_{\ell=0}^{\infty}G(k,\ell)w(\ell)\right)'y(k)\right]\\ &\le \sum_{\ell=0}^{\infty}\mathbb{E}\left(\sum_{k=0}^{\infty}\|G(k,\ell)w(\ell)\|\|y(k)\|\right)\\ &\le \sum_{\ell=0}^{\infty}\mathbb{E}\left[\left(\sum_{k=0}^{\infty}\|G(k,\ell)w(\ell)\|^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\|y(k)\|^2\right)^{\frac{1}{2}}\right]\\ &\le \sum_{\ell=0}^{\infty}\left[\mathbb{E}\left(\sum_{k=0}^{\infty}\|G(k,\ell)w(\ell)\|^2\right)\mathbb{E}\left(\sum_{k=0}^{\infty}\|y(k)\|^2\right)\right]^{\frac{1}{2}}\\ &= \sum_{\ell=0}^{\infty}\left[\mathbb{E}\left(\sum_{k=0}^{\infty}\|G(k,\ell)w(\ell)\|^2\right)\right]^{\frac{1}{2}}\|y\|_2.\end{aligned}$$

We therefore obtain for all deterministic w that (trivially, if $\|y\|_2 = 0$)

$$\begin{aligned}\|y\|_2 &\le \sum_{\ell=0}^{\infty}\left[\mathbb{E}\left(\sum_{k=0}^{\infty}\|G(k,\ell)w(\ell)\|^2\right)\right]^{\frac{1}{2}}\\ &= \sum_{\ell=0}^{\infty}\left[w(\ell)'\mathbb{E}\left(\sum_{k=0}^{\infty}\big[G(k,\ell)'G(k,\ell)\big]\right)w(\ell)\right]^{\frac{1}{2}}\\ &\overset{*}{=} \sum_{\ell=0}^{\infty}\left[w(\ell)'\left(\sum_{i=1}^{N}\nu_iB'_iY_iB_i\right)w(\ell)\right]^{\frac{1}{2}}\\ &\le \sqrt{\lambda_{\max}(R)}\ \|w\|_1\end{aligned} \tag{15}$$

where $R := \sum_{i=1}^{N}\nu_iB'_iY_iB_i$ (and $\overset{*}{=}$ is due to Lemma 3). Therefore, if we show that $\|y\|_2^2 = \lambda_{\max}(R)\ \|\bar{w}\|_1^2$ is attained for some $\bar{w} \in \mathcal{L}_1^{n_w}$, then (14) will follow. To this end, consider the following input: $\bar{w}(\ell) = 0$, $\ell \ge 1$, and $\bar{w}(0) = \sigma$ with $\sigma \in \mathbb{R}^{n_w}$ standing for an eigenvector of R, with unit Euclidean norm, which corresponds to its maximal eigenvalue,[1] i.e., such that $R\sigma = \lambda_{\max}(R)\sigma$ and $\sigma'\sigma = 1$, yielding $\|\bar{w}\|_1 = 1$. We therefore have, in this case, the corresponding output $y(k) = \sum_{\ell=0}^{\infty}G(k,\ell)\bar{w}(\ell) = G(k,0)\sigma$, so, indeed,

$$\begin{aligned}\|y\|_2^2 &= \sigma'\left(\sum_{k=0}^{\infty}\mathbb{E}\big[G(k,0)'G(k,0)\big]\right)\sigma\\ &= \sigma'R\sigma = \lambda_{\max}(R)\sigma'\sigma = \lambda_{\max}(R).\end{aligned}$$

■

Remark 3: There is no formula similar to (14) for JPLS in general as the Gramian in (12) is specific for MJLS. For unstable MJLS, the Gramian may have one solution, multiple solutions, or no solutions, making the situation far more complex. In addition, if the initial distribution of the Markov chain is not invariant, then, instead of (15), we would have

$$\|y\|_2 \le \sum_{\ell=0}^{\infty}\left[w(\ell)'\left(\sum_{i=1}^{N}P(\theta(\ell)=i)B'_iY_iB_i\right)w(\ell)\right]^{\frac{1}{2}},$$

which is also more complex, making (14) void.

One useful consequence of the preceding theorem is that (2) holds true with suitable norms and $M = 1$ for the class of JPLS considered in this sub-section. This yields the following submultiplicative property for the H_2 norm.

Corollary 2: Under the same assumptions of Theorem 3, we have, for all inputs $w \in \mathcal{L}_1^{n_w}$, that

$$\|y\|_2 \le \|\Phi\|_2\|w\|_1. \tag{16}$$

Proof: Due to Theorem 3, together with the fact that $R = \sum_{i=1}^{N}\nu_iB'_iY_iB_i \ge 0$, we obtain for all such w that

$$\|y\|_2 \le \sqrt{\lambda_{\max}(R)}\|w\|_1 \le \sqrt{\mathrm{tr}(R)}\|w\|_1 = \|\Phi\|_2\|w\|_1,$$

where $\|\Phi\|_2 = \sqrt{\mathrm{tr}(R)}$ follows from Theorem 1 and Lemma 3 with $\ell = 0$. ■

Next, we present examples where we relax the assumptions of Theorem 3 and (16) is no longer valid. In the first two

[1] This follows from the fact that $R = R' \ge 0$, so spectrum$(R) \subset [0,\infty)$.

examples, $\|\Phi\|_2$ is relatively small but $\|y\|_2 = \infty$ in response to a finite norm input, hence there is no M satisfying (2).

Example 1: Consider a Markov chain with $\mathcal{N} = \{1, 2, 3\}$, $\theta(0) = 1$, $P(\theta(k+1) = 2 \mid \theta(k) = 1) = P(\theta(k+1) = 3 \mid \theta(k) = 2) = 1$, $P(\theta(k+1) = 3 \mid \theta(k) = 3) = 1$, $A_1 = 0$, $A_2 = 0$, $A_3 = 1.1$, and, for $i \in \mathcal{N}$, $B_i = 1$, $C_i = 1$. The input signal is $w(20) = 1$, $w(\ell) = 0$, $\ell \geq 0$, $\ell \neq 20$, so that $\|w\|_2 = 1$. It is simple to check that $\|\Phi\|_2 = \|G\|_F = 1$ and $\|y(k)\|_2$ diverges as $k \to \infty$.

The system in Example 1 is observable and controllable in the "strong" sense that one can retrieve $x(k)$ from $y(k)$, and any state $x(k)$ can be driven to a desired state $x(k+1) = \bar{x}$ by defining $w(k) = \bar{x} - A_{\theta(k)}x(k)$. In such a scenario, perhaps one would expect the impulses at time $k = 0$ inherent to the H_2 norm to lead to $\|\Phi\|_2 = \infty$, making the instability of the system apparent, but this is not the case.

Still on the previous example, note that all subsystems except A_3 (the unstable one) have finite impulse responses, and the initial Markov state is concentrated at $\theta(0) = 1$. However, these properties are not the key for $\|y\|_2 = \infty$ vs $\|\Phi\|_2 < \infty$, as illustrated in Example 2. The true reason why $\|y\|_2 = \infty$ and $\|\Phi\|_2 < \infty$ is the lack of MSS combined with the fact that the impulse at time $k = 0$ never "reaches" the unstable mode A_3.

Example 2: Consider $\mathcal{N} = \{1, 2, 3\}$, $P(\theta(0) = 1) = P(\theta(0) = 2) = 1/2$, satisfying (9), and

$$\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix},$$

$A_1 = 0.1$, $A_2 = 0$, $A_3 = 2$, $B_1 = 1$, $B_2 = 0$, $B_3 = 1$, and $C_i = 1$, for all $i \in \mathcal{N}$. The input signal w is the same as in Example 1. We have checked that the system is not MSS via [3, Proposition 3.2]. We have computed $\|\Phi\|_2 \approx 0.710$ following [22, Th. 3], and estimated $\|y\|_2 \approx 6.71 \times 10^{18}$ by Monte Carlo simulation with 10.000 repetitions, indicating that $\mathbb{E}(\|y(k)\|^2)$ diverges as $k \to \infty$.

In the next example, the system is MSS and we do not require that the Markov chain starts in equilibrium; (2) is valid but M is arbitrarily large depending on a parameter ρ.

Example 3: Consider the parameter $0 < \rho \leq 1/2$ and the system Φ with $A_1 = A_2 = 0$, $A_3 = 1 - \rho$, $B_i = C_i = 1$, $i \in \mathcal{N} = \{1, 2, 3\}$; θ is a Markov chain with state space $\mathcal{N}$, $\theta(0) = 1$, $P(\theta(k+1) = j \mid \theta(k) = i) = [\mathbb{P}]_{ij}$,

$$\mathbb{P} = \begin{bmatrix} 1-\rho & \rho & 0 \\ \rho & 1-2\rho & \rho \\ 0 & \rho & 1-\rho \end{bmatrix}.$$

The input signal is $w(k_0) = 1$, $w(k) = 0$, $k \geq 0$, $k \neq k_0$, where $k_0 = \lceil 1/\rho^2 \rceil$, and $\lceil \cdot \rceil$ is the ceil function. We shall show that

$$\begin{aligned} &\forall M \geq 0 \text{ and } p, q, r, s \geq 1, \\ &\qquad \exists \rho : \|y\|_{p,r}\|w\|_{q,s}^{-1} > M\|\Phi\|_2. \end{aligned} \tag{17}$$

First, it is simple to check that $\|\Phi\|_2 = 1$, because $G(1) = C_{\theta(1)}B_{\theta(0)} = 1$ and $G(k) = 0, k \neq 1$, almost surely, irrespective of ρ. Also, $\|w\|_{q,s} = 1$ for any $q, s \geq 1$. Now we address the output. Let us define the following random variable linked with the permanence of θ in state 3, where $K_t = \{k_0, \ldots, k_0 + t\}$:

$$T = \begin{cases} 0, & \text{if} \quad \theta(k_0) = 3, \\ \max\{t;\ \theta(k) = 3 \quad \forall k \in K_t\}, & \text{otherwise.} \end{cases}$$

Also, the time instant k_0 has been defined in such a manner that $P(\theta(k_0) = 3) > 1/4$ for any ρ (we have checked it by direct calculation using the Chapman-Kolmogorov equation), allowing us to write

$$\|y\|_{p,r}^p = \mathbb{E}\Big(\textstyle\sum_{k=0}^{\infty} \|y(k)\|_r^p\Big) \geq \frac{1}{4}\mathbb{E}\Big(\textstyle\sum_{k=k_0+1}^{k_0+T+1} \|y(k)\|_r^p \Big| \theta(k_0) = 3\Big).$$

Note that, if $\theta(k_0) = 3$, then $y(k) = (1-\rho)^{k-k_0-1}$, $k_0 + 1 \leq k \leq k_0 + T + 1$; substituting above, after some algebra,

$$\begin{aligned} \|y\|_{p,r}^p &\geq \frac{1}{4}\mathbb{E}\left(\sum_{k=k_0+1}^{k_0+T+1} (1-\rho)^{p(k-k_0-1)} \middle| \theta(k_0) = 3\right) \\ &> \frac{1}{4}\mathbb{E}(T(1-\rho)^{pT} | \theta(k_0) = 3). \end{aligned}$$

Since $T|\theta(k_0) = 3$ is geometric with parameter ρ,

$$\begin{aligned} \|y\|_{p,r} &> \left[\frac{1}{4}\sum_{t=0}^{\infty} t(1-\rho)^{(p+1)t}\rho\right]^{1/p} \\ &= \left[\frac{\rho(1-\rho)^{p+1}}{4[1-(1-\rho)^{p+1}]^2}\right]^{1/p} =: g(\rho) \end{aligned}$$

Gathering the above evaluations involving y, w and $\|\Phi\|_2$, for any $p, q, r, s \geq 1$ we have that

$$\|y\|_{p,r}\|w\|_{q,s}^{-1} > g(\rho)\|\Phi\|_2.$$

This implies (17), because $g(\rho)$ tends to infinity when ρ tends to zero.

IV. Concluding Remarks

This note shows that $\|\Phi\|_2$ is the Frobenius norm of the impulse response matrix of the JPLS in (3), which is a rather general class of stochastic systems comprising well-known systems such as MJLS. In Section III-A, we present a necessary and sufficient condition for $\|\Phi\|_2$ to be the induced norm in the sense of (1), for input signals in $\mathcal{S}$ (linear combinations of impulses at time $k = 0$); the condition is quite restrictive, and probably the most usual situation where it holds is when $n_w = 1$ (single input systems). For general deterministic input signals, (1) is not necessarily valid and, more, (2) may be violated, in such a manner that finite H_2 norm not necessarily implies "input-output stability" (a finite norm input may generate an infinite norm output), as illustrated in Examples 1 and 2. This is not surprising because JPLS are time-varying systems (in a sense explained in the Introduction). Hence, for (2) to be valid, we need more structure: we consider the subclass of MJLS, Assumptions 1 and 2, and MSS, and we show that (2) holds with $p = 2$ and $M = q = 1$. We also explore examples where the assumptions and stability are relaxed, and (2) is violated or M can be made arbitrarily large (depending on the system parameters), giving more insight into the considered conditions and also indicating that they are tight.

APPENDIX
PROOF OF LEMMA 1

We shall show: (i) triangular inequality, (ii) homogeneity $\|\alpha G\|_F = |\alpha|\|G\|_F$, and we split positivity in two parts: (iii) non-negativeness $\|G\|_F \geq 0$, (iv) $\|G\|_F = 0$ if and only if $G = 0$ almost surely. (i):

$$\begin{aligned}\|G_1 + G_2\|_F^2 &= \sum_{k=-\infty}^{\infty} \mathbb{E}\left(\|G_1(k) + G_2(k)\|_F^2\right)\\ &\leq \sum_{k=-\infty}^{\infty} \mathbb{E}\left(\left(\|G_1(k)\|_F + \|G_2(k)\|_F\right)^2\right)\\ &= \sum_{k=-\infty}^{\infty} \mathbb{E}\left(\|G_1(k)\|_F^2 + \|G_2(k)\|_F^2 + 2\|G_1(k)\|_F\|G_2(k)\|_F\right)\\ &= \|G_1\|_F^2 + \|G_2\|_F^2 + 2\sum_{k=-\infty}^{\infty} \mathbb{E}\left(\|G_1(k)\|_F\|G_2(k)\|_F\right)\\ &= \|G_1\|_F^2 + \|G_2\|_F^2 + 2\mathbb{E}\left(\sum_{k=-\infty}^{\infty} \|G_1(k)\|_F\|G_2(k)\|_F\right),\end{aligned}$$

where, in the last passage, the interchangeability of the expectation and the infinite sum follows from [26, Corollary 5.3.1]. By using the Cauchy-Schwarz inequality for summation,

$$\begin{aligned}\|G_1 + G_2\|_F^2 \leq{}& \|G_1\|_F^2 + \|G_2\|_F^2\\ &+ 2\mathbb{E}\left(\sqrt{\sum_{k=-\infty}^{\infty} \|G_1(k)\|_F^2}\sqrt{\sum_{k=-\infty}^{\infty} \|G_2(k)\|_F^2}\right)\end{aligned}$$

Now, the Cauchy-Schwarz inequality for expectation yields

$$\begin{aligned}\|G_1 + G_2\|_F^2 \leq{}& \|G_1\|_F^2 + \|G_2\|_F^2\\ &+ 2\left\{\mathbb{E}\left(\sqrt{\sum_{k=-\infty}^{\infty} \|G_1(k)\|_F^2}^{\,2}\right)\right\}^{1/2}\\ &\times\left\{\mathbb{E}\left(\sqrt{\sum_{k=-\infty}^{\infty} \|G_2(k)\|_F^2}^{\,2}\right)\right\}^{1/2}\\ ={}& \|G_1\|_F^2 + \|G_2\|_F^2\\ &+ 2\sqrt{\sum_{k=-\infty}^{\infty} \mathbb{E}\left(\|G_1(k)\|_F^2\right)}\sqrt{\sum_{k=-\infty}^{\infty} \mathbb{E}\left(\|G_2(k)\|_F^2\right)}\\ ={}& \|G_1\|_F^2 + \|G_2\|_F^2 + 2\|G_1\|_F\|G_2\|_F\\ ={}& \left(\|G_1\|_F + \|G_2\|_F\right)^2.\end{aligned}$$

(ii) and (iii) follow in a straightforward manner from the corresponding properties of the Frobenius norm of matrices: $\|\alpha G(k)\|_F = |\alpha|\|G(k)\|_F$ and $\|G(k)\|_F \geq 0$. (iv) ($\Rightarrow$) It is trivial that $G = 0$ almost surely implies $\|G\|_F = 0$. ($\Leftarrow$) Now, assume $\|G\|_F = 0$, that is, $\sum_{k=-\infty}^{\infty} \mathbb{E}(\|G(k)\|_F^2) = 0$, which implies $\mathbb{E}(\|G(k)\|_F^2) = 0$ for all k. Since the random variable $\|G(k)\|_F^2$ is nonnegative, this yields $\|G(k)\|_F = 0$ almost surely. Now, the fact that the Frobenius norm of a matrix satisfies the positivity axiom leads to $G(k) = 0$, almost surely, $-\infty < k < \infty$, as desired.

REFERENCES

[1] O. L. V. Costa, J. B. R. do Val, and J. C. Geromel, "A convex programming approach to H_2 control of discrete-time Markovian jump linear systems," *Int. J. Control*, vol. 66, no. 4, pp. 557–579, 1997.

[2] J. B. do Val, J. C. Geromel, and A. P. Gonçalves, "The H_2 control for jump linear systems: Cluster observations of the Markov state," *Automatica*, vol. 38, no. 2, pp. 343–349, 2002.

[3] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. London, U.K.: Springer, 2005, doi: 10.1007/b138575.

[4] J. Dong and G.-H. Yang, "Robust H_2 control of continuous-time Markov jump linear systems," *Automatica*, vol. 44, no. 5, pp. 1431–1436, 2008.

[5] A. R. Fioravanti, A. P. de Castro Gonçalves, and J. C. Geromel, "$\mathcal{H}_2$ filtering of discrete-time Markov jump linear systems through linear matrix inequalities," *Int. J. Control*, vol. 81, no. 8, pp. 1221–1231, 2008.

[6] V. Dragan, T. Morozan, and A. M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*. New York, NY, USA: Springer, 2009, doi: 10.1007/978-1-4419-0630-4.

[7] J. Xiong and J. Lam, "Robust H_2 control of Markovian jump systems with uncertain switching probabilities," *Int. J. Syst. Sci.*, vol. 40, no. 3, pp. 255–265, 2009.

[8] M. Shen and G.-H. Yang, "H_2 filter design for discrete-time Markov jump linear systems with partly unknown transition probabilities," *Opt. Control Appl. Methods*, vol. 33, no. 3, pp. 318–337, 2011.

[9] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov, *Continuous-Time Markov Jump Linear Systems*. Berlin, Germany: Springer, 2013, doi: 10.1007/978-3-642-34100-7.

[10] R. C. L. F. Oliveira, A. N. Vargas, J. B. R. d. Val, and P. L. D. Peres, "Mode-independent $\mathcal{H}_2$-control of a DC motor Modeled as a Markov jump linear system," *IEEE Trans. Control Syst. Technol.*, vol. 22, no. 5, pp. 1915–1919, Sep. 2014.

[11] O. L. do Valle Costa, M. D. Fragoso, and M. G. Todorov, "A detector-based approach for the H_2 control of Markov jump linear systems with partial information," *IEEE Trans. Autom. Control*, vol. 60, no. 15, pp. 1219–1234, May 2015.

[12] M. G. Todorov and M. D. Fragoso, "New methods for mode-independent robust control of Markov jump linear systems," *Syst. Control Lett.*, vol. 90, pp. 38–44, Apr. 2016.

[13] L. Qiu, B. Zhang, G. Xu, J. Pan, and F. Yao, "Mixed H_2/H_∞ control of Markovian jump time-delay systems with uncertain transition probabilities," *Inf. Sci.*, vol. 373, pp. 539–556, Dec. 2016.

[14] A. M. de Oliveira and O. L. do Valle Costa, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for Markov jump linear systems," *Int. J. Control*, vol. 49, no. 18, pp. 3023–3036, 2018.

[15] M. Ogura, A. Cetinkaya, T. Hayakawa, and V. M. Preciado, "State feedback control of Markov jump linear systems with hidden-Markov mode observation," *Automatica*, vol. 89, pp. 65–72, Mar. 2018.

[16] Y. Zhang, C. Lim, and F. Liu, "Robust mixed H_2/H_∞ model predictive control for Markov jump systems with partially uncertain transition probabilities," *J. Franklin Inst.*, vol. 355, no. 8, pp. 3423–3437, 2018.

[17] G. W. Gabriel, T. R. Gonçalves, and J. C. Geromel, "Optimal and robust sampled-data control of Markov jump linear systems: A differential LMI approach," *IEEE Trans. Autom. Control*, vol. 63, no. 9, pp. 3054–3060, Sep. 2018.

[18] A. M. de Oliveira and O. L. V. Costa, "Mixed H_2/H_∞ control of hidden Markov jump systems," *Int. J. Robust Nonlinear Control*, vol. 28, no. 4, pp. 1261–1280, 2018.

[19] Z.-G. Wu, Y. Shen, H. Su, R. Lu, and T. Huang, "H_2 performance analysis and applications of $2-D$ hidden Bernoulli jump system," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 10, pp. 2097–2107, Oct. 2019.

[20] A. M. de Oliveira, O. L. V. Costa, and J. Daafouz, "Suboptimal H_2 and H_∞ static output feedback control of hidden Markov jump linear systems," *Eur. J. Control*, vol. 51, pp. 10–18, Jan. 2020.

[21] L. Carvalho, A. M. de Oliveira, and O. L. V. Costa, "H_2/H_∞ simultaneous fault detection and control for Markov jump linear systems with partial observation," *IEEE Access*, vol. 8, pp. 11979–11990, 2020.

[22] L. H. Romero, J. R. Ribeiro, and E. F. Costa, "On the H_2 control of hidden Markov jump linear systems," *IEEE Control Syst. Lett.*, vol. 7, pp. 1315–1320, 2023, doi: 10.1109/LCSYS.2023.3236892.

[23] B. Xin and D. Zhao, "Generalized H_2 control of discrete-time semi-Markov jump systems," *Asian J. Control*, vol. 25, no. 1, pp. 114–127, 2023.

[24] M. Souza, M. D. Almeida, A. R. Fioravanti, and O. L. V. Costa, "H_2 output-feedback cluster control for continuous semi-Markov jump linear systems with Erlang dwell times," *IEEE Control Syst. Lett.*, vol. 7, pp. 109–114, 2023.

[25] V. Chellaboina and W. M. Haddad, "Is the Frobenius matrix norm induced?" *IEEE Trans. Autom. Control*, vol. 40, no. 12, pp. 2137–2139, Dec. 1995.

[26] S. I. Resnick, *A Probability Path*. Boston, MA, USA: Birkhäuser, 2014.