# On the $H_2$ Control of Hidden Markov Jump Linear Systems

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***Abstract*—This letter studies $H_2$ control of linear systems with parameters driven by a hidden Markov chain, emphasizing a property of invariance of the norm to time shifts in the disturbance signal, which we call $H_2$ time-invariance (H2TI). We show that some systems are not H2TI and that the lack of H2TI may deteriorate the performance of H2 control systems when they are subject to disturbances other than impulses at the time instant $k = 0$ (as in standard $H_2$ control). This motivates us to propose a new variant of the $H_2$-norm, based on a stochastic process having an impulse at a random time. We develop the formula for computing this new variant and apply the result to an LMI optimization problem. Numerical tests are performed, indicating the superiority of the proposed controller for systems subject to different types of disturbances.**

***Index Terms*—Linear systems, stochastic systems, hybrid systems, Markov jump linear systems, hidden Markov chain.**

## I. Introduction

AMONG the main concerns when designing control systems is fulfilling performance and safety requirements in real-world problems that include systems subject to random, abrupt changes in their internal structures. In such a scenario, Markov jump linear systems (MJLS) have been a subject of great attention during the last decades, with several aspects covered in the monographs [1], [2], [3], [4].

This letter presents a new $H_2$-type controller for MJLS, emphasizing the scenario where the observation of the Markov state is indirect – the so-called Markov hidden jump variable or the detector approach for MJLS. Some of the main recent developments in literature are as follows. In [3], the approach consists of minimizing an upper bound related to the $H_2$ control problem whereas the $H_\infty$ control problem seeks for some degree of robustness for hidden Markov jump systems. In [4], the $H_2$ performance considers a $2-D$ hidden Bernoulli jump system. In [5], the authors designed a mode-independent robust controller for cases in which the controller does not have access to the Markov jump variable, common in many applications. In the context where it is challenging to measure the exact value of the transition probabilities, also appealing for real-world applications, [6] presents a solution to a robust $H_2$ problem. In [7], one can find a solution for an impulsive Markov jump linear model with periodic state jumps in the context of sampled-data systems. In [8], [9], [10], [11], the detector approach is developed for discrete-time systems and in [12], [13], we can find a similar approach in continuous-time.

In this letter, we build upon the formulation developed in [8], [9], [10]. We address an issue related to a property of time-invariance of the control system, which may corrupt the performance of the controller designed following the formulations in [8], [9], [10] in some situations. Let us explain this issue in more detail. The $H_2$ norm has the physical interpretation of the energy of the response to an impulse applied at time $k = 0$. This type of signal is not frequently found in real-world problems and the motivations for using it are: (i) the impulsive input "excites all dynamics", in the sense that its z-transform (or Laplace transform in continuous-time problems) is one, flat at all frequencies, and therefore, a good response to the impulse indicates good potential for other signals; (ii) $k = 0$ is simple and, usually, the norm of the output as a response to the input would be the same if we apply the impulse at a generic time instant.

In fact, for many systems, if $x(k)$ denotes the state of the system when subject to an impulse applied at time $k = 0$, then a time-delayed $x(k-\tau)$ is obtained if the impulse is applied at $k = \tau$, $\tau > 0$; we refer to such systems as $H_2$-Time-Invariant (H2TI). It turns out that MJLS are not, in general, H2TI, as shown in Fig. 1 (further details are given in Example 1). Note that the output $y(k)$ goes to zero with different speeds depending on when the impulse is applied. We propose in this letter a formal notion of H2TI relying on an $H_2$ norm which is defined here for the first time, using impulses at time $\tau \geq 0$ and denoted by $\|\Phi\|_{(2,\tau)}$.

For systems that are not H2TI, one could reasonably argue that, in the context of optimal control, perhaps the optimal solution is irrespective of time shifts in the impulse. In other

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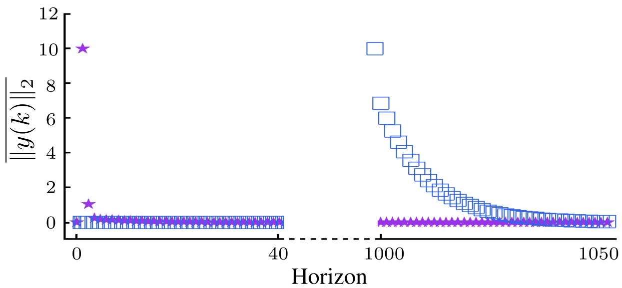


Fig. 1. Mean of ten thousand realizations of the norm of the output, $\|y(k)\|_2$, for the system in Example 1 responding to an impulse at $k = 0$ (marked with ★), and at $k = 1000$ (marked with □).

words, perhaps the $H_2$ norm/objective function changes, but not the optimal controller. Unfortunately, this is not the case for linear systems with hidden Markov jump parameters. The Examples 3 and 4 show that the controller proposed in this letter outperforms the standard $H_2$ controller.

We tackle the issue by defining a version of the $H_2$ norm based on a stochastic process having an impulse at a random time, and denote it by $\|\cdot\|_{2\text{rti}}$. Theorems 1 and 2 present formulas for computing $\|\cdot\|_{(2,\tau)}$ and $\|\cdot\|_{2\text{rti}}$. As an application, in Eq. (22) and Theorem 3, we adapt the convex formulation found in [8, Th. 3] to the proposed norms. Controllers are obtained via LMIs that provides an upper bound on the norm; numerical tests in Section VI illustrate the superiority of the proposed control over the standard $H_2$ control when the system is subject to impulses at random time instants and square wave disturbances.

The rest of this letter is organized as follows. We define the basic notation and the system under study in Section II. In Section III, we define the norm $\|\cdot\|_{(2,\tau)}$, H2TI and give an example showing that MJLS are not, in general, H2TI. In Section IV, we propose the semi-norm $\|\cdot\|_{2\text{rti}}$ and show how to compute it, as well as $\|\cdot\|_{(2,\tau)}$. In Section V, we present convex optimization problems that give gains that minimize upper bounds on the proposed norms. Two numerical applications, one in a direct-current motor and other in a system of coupled tanks via a faulty valve are presented in Section VI.

## II. Notation and the System Definition

Consider the following sets: $\mathbf{M} = \{1, \dots, N\}$, $\mathbf{D} = \{1, \dots, N_d\}$, and, for each $i \in \mathbf{M}$, $\mathbf{D}_i$ such that $\bigcup_{i\in\mathbf{M}} \mathbf{D}_i = \mathbf{D}$; also, $\mathbf{N} = \{(i, \ell); i \in \mathbf{M}, \ell \in \mathbf{D}_i\}$. Given a set $H$ whose elements are scalars or pairs of scalars, we define a vector space $\mathcal{M}^{n,m}(H)$ comprised of sequences of real matrices $X = (X_{i_1}, \dots, X_{i_r})$ where $X_{i_k} \in \Re^{n\times m}$ for all $i_k \in H$. For brevity, when $n = m$, we write $\mathcal{M}^n(H)$, and when $H = \mathbf{M}$, we write $\mathcal{M}^{n,m}$ (instead of $\mathcal{M}^{n,m}(\mathbf{M})$). The subspace $\mathcal{M}^{n0}(H) \subset \mathcal{M}^n(H)$ is composed of positive semi-definite matrices only. We denote by $I_n$ the identity matrix of order $n$, by $0_{n\times m}$ the null matrix of dimensions $n \times m$, $\text{tr}(\cdot)$ the trace of a matrix and by $X'$ the transposed of $X$.

Denote by $\{\Omega, \mathfrak{F}, \mathcal{P}\}$ the fundamental probability space, by $\mathbb{E}(\cdot)$ the expected value and by $\mathbb{E}(\cdot|\cdot)$ the conditional expected value. Consider a time-homogeneous Markov chain $\{\theta(k) \in \mathbf{M},\ k \geq 0\}$, with transition probability matrix $\mathbb{P} = [p_{ij}] \in [0, 1]^{N\times N}$ such that

$$\mathcal{P}(\theta(k+1) = j|\theta(k) = i) = p_{ij}, \tag{1}$$

with $\sum_{j\in\mathbf{M}} p_{ij} = 1$, $i \in \mathbf{M}$. Also, let $\pi(k) \in [0, 1]^{1\times N}$ denote the distribution of $\theta(k)$ such that $\pi_i(k) = \mathcal{P}(\theta(k) = i)$ and the Cesàro limit distribution $\pi_{\text{ces}} = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \pi(t)$.

The MJLS under study is the system $\Phi$ below

$$\Phi : \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + E_{\theta(k)}w(k), \\ y(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k), \\ \theta(0) \sim \pi(0),\ x(0) = 0,\ k \geq 0, \end{cases} \tag{2}$$

where the processes $\{u(k)\}$, $\{y(k)\}$ and $\{w(k)\}$ are the control, output and disturbance of the system $\Phi$, respectively. Also, it is assumed that $C_i'D_i = 0$ and $D_i'D_i > 0$ for all $i \in \mathbf{M}$. The parameters are such that $A \in \mathcal{M}^n$, $B \in \mathcal{M}^{n,n_u}$, $E \in \mathcal{M}^{n,n_w}$, $C \in \mathcal{M}^{n_y,n}$ and $D \in \mathcal{M}^{n_y,n_u}$.

Regarding the structure of information, a key assumption is that $\theta(k)$ *is indirectly observed* via the process $\{\eta(k), k \geq 0\}$, where $\eta(k)$ takes values in $\mathbf{D}$. For each $\ell \in \mathbf{D}$, $i \in \mathbf{M}$,

$$\mathcal{P}(\eta(k) = \ell \mid \theta(k) = i) = q_{i\ell}. \tag{3}$$

We assume that each $i \in \mathbf{M}$ has a set of outputs $\mathbf{D}_i = \{\ell \in \mathbf{D} : q_{i\ell} > 0\}$. We denote $\mathbb{Q} = [q_{i\ell}] \in [0, 1]^{N\times N_d}$; $x(k)$ is observed.

This allows us to use a feedback law in the form

$$u(k) = K_{\eta(k)}x(k), \tag{4}$$

where $K \in \mathcal{M}^{n_u,n}(\mathbf{D})$ is called gain of the control.

Linked with (4), we introduce the closed loop matrices for a given gain $K$ as $\hat{A}_{i\ell} = A_i + B_iK_\ell$, $\hat{C}_{i\ell} = C_i + D_iK_\ell$, for $i \in \mathbf{M}, \ell \in \mathbf{D}_i$, and the operators $\mathcal{D} : \mathcal{M}^{n0}(\mathbf{N}) \to \mathcal{M}^{n0}$ and $\mathcal{T} : \mathcal{M}^{n0} \to \mathcal{M}^{n0}$ such that, for each $j \in \mathbf{M}$,

$$\mathcal{D}_j(U) = \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} p_{ij}\, q_{i\ell}\, U_{i\ell},\ \ \mathcal{T}_j(U) = \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} p_{ij}\, q_{i\ell}\, \hat{A}_{i\ell} U_i \hat{A}_{i\ell}'. \tag{5}$$

Throughout this letter, $e_j \in \Re^{n_w}$ is the vector whose $j$-th element is one and the remaining ones are zero. Also, $\mathbf{e} = \{\mathbf{e}(k), k \geq 0\}$ is a stochastic process satisfying $\mathcal{P}(\mathbf{e}(0) = e_j) = 1/n_w$ for each $j = 1, \dots, n_w$, and $\mathbf{e}(k) = 0$ almost surely (a.s.), $k > 0$. For a given $\tau \geq 0$, the process $\mathbf{e}_{(\tau)} = \{\mathbf{e}_{(\tau)}(k), k \geq 0\}$ is a time-shifted version of $\mathbf{e}$, satisfying $\mathcal{P}(\mathbf{e}_{(\tau)}(\tau) = e_j) = 1/n_w$ for each $j = 1, \dots, n_w$, and $\mathbf{e}_{(\tau)}(k) = 0$ a.s. for $k \neq \tau$. When $\mathbf{e}_{(\tau)}$ is applied in the exogenous input $w$, $\tau$ can be regarded as the time instant when the system is subject to an impulse.

## III. The Notion of H2TI and an Example Showing That Not All MJLS Are H2TI

*Definition 1:* The $H_2$ norm associated with system $\Phi$ with $w = \mathbf{e}$ is defined as

$$\|\Phi\|_2^2 = n_w \sum_{k=0}^{\infty} \mathbb{E}(\|y(k)\|^2). \tag{6}$$

We generalize this norm as follows. Given a $\tau \geq 0$, we write $y_\tau(k)$ the output of system $\Phi$ when the disturbance is $w = \mathbf{e}_{(\tau)}$ (that is, $w(k) = \mathbf{e}_{(\tau)}(k), k \geq 0$).

$$\|\Phi\|^2_{(2,\tau)} = n_w \sum_{k=0}^{\infty} \mathbb{E}(\|y_\tau(k)\|^2). \tag{7}$$

*Remark 1:* By the total probability law,

$$\|\Phi\|_2^2 = n_w \sum_{k=0}^{\infty} \sum_{j=1}^{n_w} \mathbb{E}(\|y(k)\|^2 \mid w(0) = e_j, \\ w(t) = 0, t \geq 1)\, \mathcal{P}(\mathbf{e}(0) = e_j) \\ = \sum_{k=0}^{\infty} \sum_{j=1}^{n_w} \mathbb{E}(\|y(k)\|^2 \mid w(0) = e_j, w(t) = 0, t \geq 1),$$

which is the "classic" norm as defined, e.g., in [1]; we use (6) because it relies on the process $\mathbf{e}$, which makes several passages in this letter more compact and clearer when dealing with a general $\tau \geq 0$.

*Remark 2:* This letter addresses the closed-loop system, i.e., both (2) and (4). The open-loop case is obtained by setting $K_\ell = 0$ for all $\ell \in \mathbf{D}$, and we suggest $\mathbf{D} = \mathbf{D}_i = \{1\}$ for all $i \in \mathbf{M}$. The matrix $\mathbb{Q}$ plays no role anymore once $\hat{A}_{i\ell} = A_i$ and $\hat{C}_{i\ell} = C_i$ for all $i \in \mathbf{M}, \ell \in \mathbf{D}_i$. Also, $\sum_{\ell \in \mathbf{D}_i} q_{i\ell} = 1$ vanishes in both operators $\mathcal{D}$ and $\mathcal{T}$ in (5).

*Definition 2:* We say that $\Phi$ is $H_2$ time-invariant (H2TI) if, for any $\ell \geq 1$, we have

$$\|\Phi\|_{(2,\tau=0)} = \|\Phi\|_{(2,\tau=\ell)}. \tag{8}$$

Next, we present an MJLS that is not H2TI. It involves the simplest possible MJLS in terms of dimensions (only two Markov states and "scalar modes", $n = 1$), suggesting that the lack of H2TI may be commonplace among MJLS. More details are found in Remark 4, in Section IV.

*Example 1:* Consider the system $\Phi$ with the following parameters: $A_1 = -0.101$, $A_2 = 0.878$, and for $i \in \{1, 2\}$, $B_i = 0$, $C_i = [10,\ 0]'$, $D_i = [0,\ 10]'$, $E_i = 1$;

$$\mathbb{P} = \begin{bmatrix} 0.999 & 0.001 \\ 0.0001 & 0.9999 \end{bmatrix}, \quad \mathbb{Q} = [1,\ 1]', \quad \pi(0) = [1,\ 0].$$

In order to compute $\|\Phi\|_2 = \|\Phi\|_{(2,\tau=0)}$ we set $w = \mathbf{e}_{(\tau=0)}$. Fig. 1 illustrates the estimated norm of the output along time. We have performed $10^4$ repetitions, yielding

$$\|\Phi\|_{(2,0)} \approx 10.06.$$

In a second experiment we set $w = \mathbf{e}_{(\tau=1000)}$, i.e., we apply the impulse at time $k = 1000$ instead of $k = 0$, and once again performed $10^4$ repetitions. The estimated norm of the output is also illustrated in Fig. 1, yielding

$$\|\Phi\|_{(2,1000)} \approx 17.53.$$

As we see, there is a significant change in the norm of the output when the system is subject to impulses at different time instants.

*Example 2:* Consider $\Phi$ as in Example 1 with $\mathbb{P}$ replaced by a $2 \times 2$ matrix with $p_{12} = p_{21} = 1$, i.e., a periodic Markov chain. We have performed $10^4$ repetitions and estimated the norm of the output, yielding $\|\Phi\|_{(2,\tau)} \approx 10.42$ for $\tau$ odd and $\|\Phi\|_{(2,\tau)} \approx 13.18$ for $\tau$ even.

## IV. An $H_2$ Semi-Norm Based on Random-Time Impulses and Computation of $\|\cdot\|_{(2,\tau)}$ and $\|\cdot\|_{2\text{rti}}$

Motivated by the facts that MJLS are not H2TI in general and the norm $\|\Phi\|_{(2,\tau)}$ does not necessarily converge when $\tau \to \infty$, as shown in Examples 1 and 2, we define an H2TI semi-norm based on $\|\Phi\|_{(2,\tau)}$.

*Definition 3:* Consider system $\Phi$. The $H_2$ semi-norm with random-time impulse is defined by

$$\|\Phi\|_{2\text{rti}} = \lim_{\tau\to\infty} \mathbb{E}\big(\|\Phi\|_{(2,T_\tau)}\big), \tag{9}$$

where $T_\tau$ is a random variable with discrete uniform distribution in the interval $[0,\ \ \tau - 1]$.

*Remark 3 ($\|\cdot\|_{(2,\tau)}$ Is a Norm and $\|\cdot\|_{2rti}$ Is a Semi-Norm):* Consider the time-shifted impulse response matrix $G_\tau = \{G_\tau(k), -\infty < k < \infty\}$ defined as $G_\tau(k) = 0$, $k \leq \tau$, $G_\tau(\tau + 1) = C_{\theta(\tau+1)}E_{\theta(\tau)}$, and $G_\tau(k + \tau + 1) = C_{\theta(k+\tau+1)}A_{\theta(k+\tau)}A_{\theta(k+\tau-1)}, \ldots A_{\theta(\tau+1)}E_{\theta(\tau)}$, $k \geq 1$. We have

$$\|\Phi\|^2_{(2,\tau)} = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{n_w} \mathbb{E}\Big(\mathrm{tr}\big(G_\tau(k)e_je_j'G_\tau(k)'\big)\Big) \\ = \sum_{k=-\infty}^{\infty} \mathbb{E}\Big(\mathrm{tr}\big(G_\tau(k)G_\tau(k)'\big)\Big) = \|G_\tau\|_F^2.$$

It can be shown that $\|\cdot\|_F$ is a norm and, via (9); $\|\cdot\|_{2\text{rti}}$ inherits the axioms of non-negativeness, absolute homogeneity and the triangle inequality, hence $\|\cdot\|_{2\text{rti}}$ is a semi-norm.

We devote the remainder of this section to obtaining conditions for existence and formulas for $\|\Phi\|_{(2,\tau)}$ and $\|\Phi\|_{2\text{rti}}$. We start with an auxiliary result.

*Lemma 1:* Consider the closed-loop system (2) and (4) with $w = \mathbf{e}_{(\tau)}$. Let $X(k) \in \mathcal{M}^{n0}$ be such that $X_i(k) = \mathbb{E}(x(k)x(k)'\mathbb{1}_{\{\theta(k)=i\}})$. Then,

$$\begin{cases} X_i(k) = 0, & \text{for } 0 \leq k \leq \tau, \\ X_i(\tau+1) = \mathcal{D}_i\big(\frac{1}{n_w}\pi(\tau)EE'\big), & \\ X_i(k+1) = \mathcal{T}_i(X(k)), & \text{for } k \geq \tau + 1. \end{cases} \tag{10}$$

*Proof:* We have that $x(t) = 0$ for $0 \leq t \leq \tau$, and consequently $X(t) = 0$. Then, from (2) and by total probability rule, we get

$$X_i(\tau+1) = \mathbb{E}\Big(x(\tau+1)x(\tau+1)'\mathbb{1}_{\{\theta(\tau+1)=i\}}\Big) \\ = \mathbb{E}\Big(E_{\theta(\tau)}w(\tau)w(\tau)'E'_{\theta(\tau)}\mathbb{1}_{\{\theta(\tau+1)=i\}}\Big) \\ = \sum_{j=1}^{n_w}\sum_{r\in\mathbf{M}} \mathbb{E}\Big(E_{\theta(\tau)}w(\tau)w(\tau)'E'_{\theta(\tau)}\Big|\theta(\tau+1) = i, \\ \theta(\tau) = r, w(\tau) = e_j\Big)\mathcal{P}(\theta(\tau+1) = i|\theta(\tau) = r, \\ w(\tau) = e_j)\mathcal{P}(\theta(\tau) = r|w(\tau) = e_j)\mathcal{P}(w(\tau) = e_j) \\ = \sum_{r\in\mathbf{M}} p_{ri}\Big(\sum_{\ell\in\mathbf{D}_r} q_{r\ell}\Big)\pi_r(\tau)E_r\Big(\sum_{j=1}^{n_w}\tfrac{1}{n_w}e_je_j'\Big)E_r' \\ = \mathcal{D}_i\big(\tfrac{1}{n_w}\pi(\tau)EE'\big).$$

The third equality in (10) can be found in [8, Proposition 1], and its proof is omitted for brevity. ■

For any gain $K \in \mathcal{M}^{n_u,n}(\mathbf{D})$, the associated operator $\mathcal{T}$ and $\rho \in \Re^{1\times N}$, $\rho_i \geq 0$, $i \in \mathbf{M}$, define

$$V(k,\rho) = \sum_{r=0}^{k} \mathcal{T}^r(\mathcal{D}(\rho EE')),\ k \geq 0. \quad (11)$$

*Theorem 1:* Assume that there exists $\bar{V} \in \mathcal{M}^{n0}$ such that, for all $k \geq 0$, $V(k,\pi(\tau)) \leq \bar{V}$. Then, $V_\infty(\pi(\tau)) = \lim_{k\to\infty} V(k,\pi(\tau))$ exists and

$$\|\Phi\|^2_{(2,\tau)} = \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\big(\hat{C}_{i\ell}\, V_{\infty,i}(\pi(\tau))\, \hat{C}'_{i\ell}\big). \quad (12)$$

*Proof:* By (11), we have $V(k+1,\pi(\tau)) \geq V(k,\pi(\tau))$ and, since $\bar{V} \geq V(k,\pi(\tau))$ by hypothesis, the sequence $V(k,\pi(\tau))$ converges monotonically to $V_\infty(\pi(\tau))$. Now, using the Lemma 1, we can rewrite (11) as $V(k,\pi(\tau)) = \sum_{r=0}^{k} n_w X(\tau+1+r)$, and taking limits,

$$V_\infty(\pi(\tau)) = \sum_{r=\tau+1}^{\infty} n_w X(r). \quad (13)$$

On the other hand, notice that $y_\tau(k) = \hat{C}_{\theta(k)\eta(k)}x(k)$, and $y_\tau(t) = 0$ for $0 \leq t \leq \tau$ and, for $k \geq \tau+1$, we have, by total probability rule:

$$\begin{aligned}\mathbb{E}(\|y_\tau(k)\|_2^2) &= \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} \mathbb{E}\Big(\mathrm{tr}\big(\hat{C}_{\theta(k)\eta(k)}x(k)x(k)'\hat{C}'_{\theta(k)\eta(k)}\big)\Big| \\ &\qquad \theta(k)=i,\eta(k)=\ell\Big)\mathcal{P}(\eta(k)=\ell|\theta(k)=i)\mathcal{P}(\theta(k)=i) \\ &= \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\big(\hat{C}'_{i\ell}\hat{C}_{i\ell}\mathbb{E}(x(k)x(k)'\mathbb{1}_{\{\theta(k)=i\}})\big) \\ &= \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\big(\hat{C}'_{i\ell}\hat{C}_{i\ell}X_i(k)\big).\end{aligned}$$

Substituting this in (7) leads to

$$\begin{aligned}\|\Phi\|^2_{(2,\tau)} &= n_w \sum_{\substack{k=\tau+1\\ i\in\mathbf{M},\ell\in\mathbf{D}_i}}^{\infty} q_{i\ell}\,\mathrm{tr}\big(\hat{C}_{i\ell}X_i(k)\hat{C}'_{i\ell}\big) \\ &= \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\Big(\hat{C}_{i\ell}\Big(\sum_{k=\tau+1}^{\infty} n_w X_i(k)\Big)\hat{C}'_{i\ell}\Big). \quad (14)\end{aligned}$$

Finally, (13) and (14) yield to (12). ∎

*Remark 4:* Theorem 1 shows how $\|\Phi\|_{(2,\tau)}$ depends on $\pi(\tau)$. As we see, $\|\Phi\|_{(2,\tau=0)} = \|\Phi\|_{(2,\tau=\ell)}$ implies $\pi(0) = \pi(\ell)$, $\ell \geq 0$, except in particular cases where $\pi(\ell)$ is irrelevant in (12). This makes the lack of H2TI quite usual in MJLS. For example, among $10^3$ examples of MJLS created with random parameters, we found no H2TI one.

*Theorem 2:* Assume that there exists $\bar{V} \in \mathcal{M}^{n0}$ such that $V(k,\nu) \leq \bar{V}$, $k \geq 0$, where $\nu_i = \mathbb{1}_{\{[\sum_{k=0}^{N}\mathcal{P}(\theta(k)=i)]>0\}}$. Then, $V_\infty(\pi_{\mathrm{ces}}) = \lim_{k\to\infty} V(k,\pi_{\mathrm{ces}})$ exists and

$$\|\Phi\|^2_{2\mathrm{rti}} = \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\big(\hat{C}_{i\ell}V_{\infty,i}(\pi_{\mathrm{ces}})\hat{C}'_{i\ell}\big). \quad (15)$$

*Proof:* We start with the existence of the limit. Define $\mu = \sum_{k=0}^{N}\pi(0)\mathbb{P}^k$ so that $\nu_i = \mathbb{1}_{\{\mu_i>0\}}$. If $\mu_i = 0$, then $\pi_i(r) = 0$, $0 \leq r \leq N$ is immediate; for $r > N$, Cayley-Hamilton theorem applied in matrix $\mathbb{P}$ in $\pi(k) = \pi(0)\mathbb{P}^k$ yields $\pi_i(r) = 0$, $r \geq 0$ whenever $\mu_i = 0$. Then, it is simple to conclude that: $\pi_i(r) \leq \nu_i$, $i \in \mathbf{M}$, $r \geq 0$ and $\pi_{\mathrm{ces},i} \leq \nu_i$, $i \in \mathbf{M}$; these lead to

$$\begin{aligned}&V(k,\pi(r)) \leq V(k,\nu) \leq \bar{V}, \\ &V(k,\pi_{\mathrm{ces}}) \leq V(k,\nu) \leq \bar{V},\ k \geq 0. \quad (16)\end{aligned}$$

Convergence of both $V(k,\pi(r))$ and $V(k,\pi_{\mathrm{ces}})$ as $k\to\infty$ follows from (16) and the facts that $V(k,\pi(r)) \leq V(k+1,\pi(r))$ and $V(k,\pi_{\mathrm{ces}}) \leq V(k+1,\pi_{\mathrm{ces}})$.

Now we show (15). From (9) and (12) we have

$$\|\Phi\|^2_{2\mathrm{rti}} = \lim_{\tau\to\infty}\sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\Big(\hat{C}_{i\ell}\Big[\mathbb{E}\big(\lim_{k\to\infty} V(k,\pi(T_\tau))\big)\Big]\hat{C}'_{i\ell}\Big).$$

Since the limit inside the expectation exists,

$$\|\Phi\|^2_{2\mathrm{rti}} = \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\Big(\hat{C}_{i\ell}\Big[\lim_{k\to\infty}\lim_{\tau\to\infty}\mathbb{E}\big(V(k,\pi(T_\tau))\big)\Big]\hat{C}'_{i\ell}\Big). \quad (17)$$

Using the linearity of $\mathcal{D}$ and $\mathcal{T}$ and the total probability rule,

$$\begin{aligned}\mathbb{E}\Big(V(k,\pi(T_\tau))\Big) &= \sum_{r=0}^{\tau-1} V(k,\pi(r))\mathcal{P}(T_\tau = r) \\ &= \sum_{r=0}^{\tau-1}\frac{1}{\tau}V(k,\pi(r)) = \frac{1}{\tau}\sum_{r=0}^{\tau-1}\sum_{s=0}^{k}\mathcal{T}^s\big(\mathcal{D}(\pi(r)EE')\big) \\ &= \sum_{s=0}^{k}\mathcal{T}^s\Big(\mathcal{D}\Big(\sum_{r=0}^{\tau-1}\frac{1}{\tau}\pi(r)EE'\Big)\Big).\end{aligned}$$

Substituting the above in (17) leads to

$$\|\Phi\|^2_{2\mathrm{rti}} = \sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}\Big(\hat{C}_{i\ell}\lim_{k\to\infty}V(k,\pi_{\mathrm{ces}})\hat{C}'_{i\ell}\Big)$$

and the result is immediate, since we have already shown that $V(k,\pi_{\mathrm{ces}})$ converges as $k\to\infty$. ∎

*Remark 5:* The hypotheses in the above theorems are less conservative than the usual condition of mean square (MS) stability. In fact, MS-stability is equivalent to the spectral radius of $\mathcal{T}$ being smaller than one, which guarantees that $V(k,\rho)$ converges to some $V_\infty(\rho)$ as $k\to\infty$; it is simple to see from (11) that $V(k,\rho) \leq V(k+1,\rho)$, therefore $V(k,\rho) \leq V_\infty(\rho)$; hence, the hypotheses in Theorems 1 and 2 are fulfilled with $\rho = \pi(\tau)$ and $\rho = \nu$, respectively.

## V. Control Synthesis via LMIS

As an application of the result given in Theorem 2, we present an LMI formulation for minimizing an upper bound on the semi-norm $\|\Phi\|_{2\mathrm{rti}}$. Consider the objective function

$$fo = \sqrt{\sum_{i\in\mathbf{M},\ell\in\mathbf{D}_i} q_{i\ell}\,\mathrm{tr}(W_{i\ell})}, \quad (18)$$

and the following constraints for each $i \in \mathbf{M}$, $\ell \in \mathbf{D}_i$,

$$\begin{bmatrix} R_{i\ell} - \pi_{\mathrm{ces},i}E_iE_i' & A_iG_\ell + B_iF_\ell \\ (A_iG_\ell + B_iF_\ell)' & G_\ell + G_\ell' - \mathcal{D}_i(R)\end{bmatrix} > 0, \quad (19)$$

$$\begin{bmatrix} W_{i\ell} & C_iG_\ell + D_iF_\ell \\ (C_iG_\ell + D_iF_\ell)' & G_\ell + G_\ell' - \mathcal{D}_i(R)\end{bmatrix} \geq 0, \quad (20)$$

where $R_{i\ell}$ and $W_{i\ell}$ are positive semidefinite matrices of dimension $n \times n$ and $F_\ell$ and $G_\ell$ are matrices of appropriate dimensions.

The relation between $f_O$ and $\|\Phi\|_{2\text{rti}}$ is given in the next theorem.

*Theorem 3:* If $F_\ell$ and $G_\ell$ is in the feasible set of (19)–(20), then the gain $K_\ell = F_\ell G_\ell^{-1}$ is such that $f_O \geq \|\Phi\|_{2\text{rti}}$.

*Proof:* Let $\tilde{R}_{i\ell} = R_{i\ell} - \pi_{\text{ces},i} E_i E_i'$, $\tilde{A}_{i\ell} = A_i G_\ell + B_i F_\ell$ and $\tilde{C}_{i\ell} = C_i G_\ell + D_i F_\ell$. We use the inequality $Y' X^{-1} Y \geq Y + Y' - X$ and Schur complement to write (19) as

$$\begin{aligned} 0 &< \tilde{R}_{i\ell} - \tilde{A}_{i\ell}[G_\ell + G_\ell' - \mathcal{D}_i(R)]^{-1}\tilde{A}_{i\ell}' \\ &\leq \tilde{R}_{i\ell} - \tilde{A}_{i\ell}[G_\ell' \mathcal{D}_i(R)^{-1} G_\ell]^{-1}\tilde{A}_{i\ell}' \\ &= \tilde{R}_{i\ell} - (A_i + B_i F_\ell G_\ell^{-1})\mathcal{D}_i(R)(A_i + B_i F_\ell G_\ell^{-1})' \\ &= R_{i\ell} - \pi_{\text{ces},i} E_i E_i' - \hat{A}_{i\ell}\mathcal{D}_i(R)\hat{A}_{i\ell}'. \end{aligned}$$

Now, make $\bar{V} := \mathcal{D}(R)$ and apply $\mathcal{D}$ to the inequality to get $\bar{V} - \mathcal{D}(\hat{A}\bar{V}\hat{A}' + \pi_{\text{ces}} EE') > 0$, which is equivalent to $\bar{V} > \mathcal{T}(\bar{V}) + \mathcal{D}(\pi_{\text{ces}} EE')$. This means that the spectral radius of $\mathcal{T}$ is smaller than one, by turn implying that $V(k, \rho)$ defined in (11) converges as $k \to \infty$ (irrespectively of $\rho$), which guarantees that the hypothesis of Theorem 2 is fulfilled. This enables us to use (15). Similarly as above, using the Schur complement we write (20) as

$$0 \leq W_{i\ell} - \tilde{C}_{i\ell}[G_\ell + G_\ell' - \mathcal{D}_i(R)]^{-1}\tilde{C}_{i\ell}' \leq W_{i\ell} - \hat{C}_{i\ell}\mathcal{D}_i(R)\hat{C}_{i\ell}',$$

hence $W_{i\ell} \geq \hat{C}_{i\ell}\bar{V}_i\hat{C}_{i\ell}$. By taking the trace, multiplying by $q_{i\ell}$, summing over the indexes, and using Theorem 2, we obtain $f_O \geq \|\Phi\|_{2\text{rti}}$. ∎

The optimization problem on the variables $R_{i\ell}$, $W_{i\ell}$, $F_\ell$ and $G_\ell$, for $i \in \mathbf{M}$ and $\ell \in \mathbf{D}_i$, is formulated as:

$$\inf_{F_\ell, G_\ell, R_{i\ell}, W_{i\ell}} f_O, \quad \text{subject to} \quad (19), (20). \tag{21}$$

Although our focus is on $\|\Phi\|_{2\text{rti}}$, it is interesting to compare results with the controllers that minimize an upper bound on $\|\Phi\|_{(2,\tau)}$ with various $\tau \geq 0$; this is given by the following version of (21), for $i \in \mathbf{M}$, $\ell \in \mathbf{D}_i$:

$$\begin{gathered} \inf_{F_\ell, G_\ell, R_{i\ell}, W_{i\ell}} f_O, \quad \text{subject to} \quad (20), \text{ and} \\ \begin{bmatrix} R_{i\ell} - \pi_i(\tau) E_i E_i' & A_i G_\ell + B_i F_\ell \\ (A_i G_\ell + B_i F_\ell)' & G_\ell + G_\ell' - \mathcal{D}_i(R) \end{bmatrix} > 0. \end{gathered} \tag{22}$$

This is a generalization of the convex optimization problem presented in [8, Th. 3] (which is specific for system $\Phi$ responding to impulses in the form $w = \mathbf{e}_{(\tau=0)}$). Solving (22) for a specific $\tau$ yields the gain that minimizes an upper bound for $\|\Phi\|_{(2,\tau)}$, which provides a suitable response to an impulse applied at time $k = \tau$.

## VI. Numerical Examples

Throughout this section, we denote by $K_\text{p}$ the gain proposed in this letter, calculated via (21), and by $K_\text{s}$ the "standard" gain found by solving (22) with $\tau = 0$ (which corresponds to the "standard $H_2$ control" as in [8]). We apply three different disturbance signals to the following Examples 3 and 4 to compare the performance of the controllers.

*Example 3 (Direct-Current Motor):* Consider the system given in [14] with the parameters in [14, Table 1] and assume that the inertial constant of the motor can change from the nominal value $I = 3.2 \times 10^{-5}\text{kg} \cdot \text{m}^2$ to $2I$ or $10I$, due to the coupling of different loads into the motor axis. This leads to three different system matrices $A_1$, $A_2$, $A_3$. We have performed a time discretization using a sampling time $10^{-4}$s and a zero-order holder in the input, as usual, and considered the associated MJLS with

$$\mathbb{P} = \begin{bmatrix} 0.999 & 0.001 & 0 \\ 0.001 & 0.998 & 0.001 \\ 0 & 0.001 & 0.999 \end{bmatrix}$$

and $\mathbb{Q} = [1,\ 1,\ 1]'$, so that the observed process $\eta$ has only one output (i.e., $\theta$ is not observed). For $i \in \{1, 2, 3\}$, the matrices $A_i \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 1}$ can be obtained from the above; the matrices $C_i \in \mathbb{R}^{3\times 2}$ are formed by null entries except the elements $[C_i]_{11} = 1$ and $[C_i]_{22} = 0.1$; the matrices $D_i \in \mathbb{R}^{3\times 1}$ are also formed by null entries except the element $[D_i]_{31} = 0.1$. The results with three different input signals are as follows:

1) *Impulse at time $\tau = 5 \times 10^4$.* We obtained $\|\Phi\|_{(2,\tau)} \approx 1.71$ using $K_\text{p}$ against $\|\Phi\|_{(2,\tau)} \approx 2.66$ with $K_\text{s}$.
2) *Noise after $k = 5 \times 10^4$.* We set $w(k) = [S_1(k),\ S_2(k)]'$ for $5\times 10^4 \leq k \leq 10^5$, where $S_a(k) \sim \mathcal{N}(0, 1)$, $a = 1, 2$; $w(k) = 0$, otherwise. We obtained $\sum_{k=0}^{\infty} \mathbb{E}(\|y(k)\|_2^2) \approx 3.83 \times 10^2$ with using $K_\text{p}$ against $\sum_{k=0}^{\infty} \mathbb{E}(\|y(k)\|_2^2) \approx 5.87 \times 10^2$ using $K_\text{s}$.
3) *Square wave after $k = 5 \times 10^4$.* Consider the square wave signal $s(k) = \text{sign}(\sin(\pi k/(2 \times 10^4)))$, $k \geq 0$. The results are illustrated in Fig. 2.

*Example 4 (Faulty Valve Coupling Two Tanks):* The system is adapted from [12], [15], and consists of two coupled tanks $T_1$ and $T_2$, connected by two valves, $V_1$ and $V_2$, and controlled by the signals $u_2(k)$ and $u_3(k)$, respectively.

The tank $T_1$ is filled via pump $P_1$. The continuous state vector is $x = [v_1,\ v_2,\ h_1,\ h_2,\ s_1,\ s_2]'$, where $v_1$ and $v_2$ are the flows, $h_1$ and $h_2$ are the heights of the fluids (in a frame of reference where zero corresponds to the desired operating point), and $s_1$ and $s_2$ are sensor readings; more details can be found in [15]. The input vector is given by $u = [u_1,\ u_2,\ u_3]'$, where $u_1$ is the control signal for the pump, and $u_2$ and $u_3$ are the control signals for the two valves. The control aims to regulate the system, i.e., to keep $x(k)$ close to the operating point $x = 0$.

The discrete-time model is described as: for $i \in \{1, 2, 3\}$,

$$A_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -0.030 & -0.032 & 0.933 & 0.032 & 0 & 0 \\ 0.028 & 0.030 & 0.063 & 0.836 & 0 & 0 \\ 0.001 & 0.002 & 0.003 & 0.087 & 0.905 & 0 \\ 0 & 0.001 & 0.001 & 0.030 & 0 & 0.967 \end{bmatrix},$$

$C_i = [I_6,\ 0_{6\times 3}]'$, $D_i = [0_{3\times 6},\ I_3]'$, $E_i = I_6$. The matrix $B_1 \in \mathbb{R}^{6\times 3}$ is formed by null entries except $[B_1]_{31} = 0.078$, $[B_1]_{41} = 0.003$, $[B_1]_{23} = 0.002$, and $[B_1]_{12} = 1$. Both $B_2$ and $B_3$ have the same elements of $B_1$ except $[B_2]_{12} = 0.25$ and $[B_3]_{12} = 0$. The different values of $[B_i]_{12}$ are due to failures in valve $V_1$, with three possible modes: (1) $V_1$ is not faulty; (2) the efficiency of $V_1$ is reduced by 75%; (3) $V_1$ is completely irresponsive to control. The system is associated

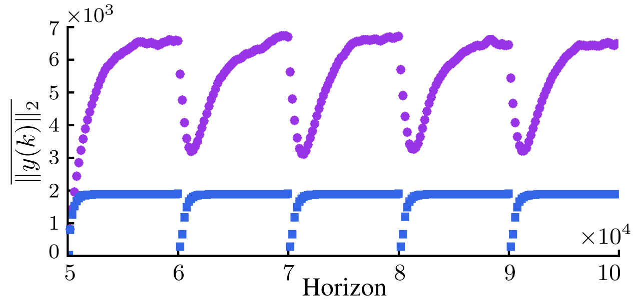


Fig. 2. Mean of ten thousand realizations of the norm of the output of the system in Example 3 with the gains $K_s$ (marked with •) and $K_p$ (marked with ■), responding to a square wave disturbance.

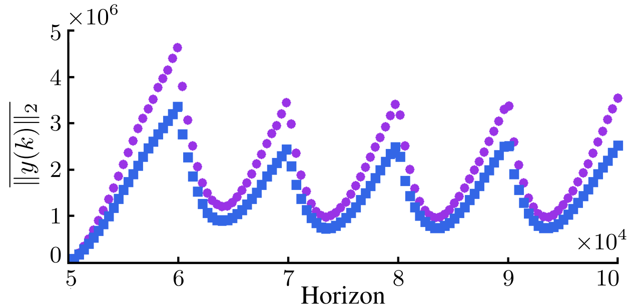


Fig. 3. Mean of ten thousand realizations of the norm of the output of the system in Example 4 with the gains $K_s$ (marked with •) and $K_p$ (marked with ■), responding to a square wave disturbance.

TABLE I
SOME ENTRIES OF THE GAINS $K_S$ AND $K_P$ IN EXAMPLE 4

| Gain \ Entry | 1, 1 | 2, 1 | 2, 4 | 3, 1 | 3, 5 |
|---|---|---|---|---|---|
| $K_s$ | 0.099 | -1.109 | 0.084 | -0.090 | 0.011 |
| $K_p$ | 0.078 | -0.574 | -0.092 | -0.010 | -0.003 |

with the Markov chain

$$\mathbb{P} = \begin{bmatrix} 0.9991 & 0.0007 & 0.0002 \\ 0.0007 & 0.9993 & 0 \\ 0.0002 & 0 & 0.9998 \end{bmatrix},$$

and it is assumed that $\mathbb{Q} = [1,\ 1,\ 1]'$. The results with three different input signals are as follows:

1) *Impulse at time* $\tau = 5 \times 10^4$. We obtained $\|\Phi\|_{(2,\tau)} \approx 55.45$ using $K_p$ against $\|\Phi\|_{(2,\tau)} \approx 66.84$ with $K_s$.
2) *Noise after* $k = 5 \times 10^4$. We set $w(k) = [S_1(k), \ldots, S_6(k)]'$ for $5 \times 10^4 \leq k \leq 10^5$ where $S_a(k) \sim \mathcal{N}(0, 1)$, $a = 1, \ldots, 6$; $w(k) = 0$, otherwise. We obtained $\sum_{k=0}^{\infty} \mathbb{E}(\|y(k)\|_2^2) \approx 1.23 \times 10^4$ using $K_p$ against $\sum_{k=0}^{\infty} \mathbb{E}(\|y(k)\|_2^2) \approx 1.42 \times 10^4$ with $K_s$.
3) *Square wave after* $k = 5 \times 10^4$. Consider the square wave signal $s(k) = \mathrm{sign}(\sin(\pi k/(2 \times 10^4)))$, $k \geq 0$. The results are illustrated in Fig. 3.

The experiments in the Examples above, and in particular the norms of the output seen in Figs. 2, 3, make clear that the control system proposed in this letter benefits from the design based on random-time impulses (which defines the semi-norm $\|\cdot\|_{2\mathrm{rti}}$ when the system is affected by disturbances in long time intervals). We can also see that the gains $K_p$ and $K_s$ obtained for the system in Example 4 are significantly different from each other (see Table I), which indicates that $\|\cdot\|_{2\mathrm{rti}}$ and $\|\cdot\|_2$ do lead to different control solutions when dealing with non-H2TI MJLS without perfect observation of the Markov jumps, as in this application.

## VII. CONCLUSION

This letter introduces the notion of H2TI for MJLS and shows that there exist systems that are not H2TI, with a possible negative reflex on the performance, as illustrated in the examples. We have proposed the semi-norm $\|\cdot\|_{2\mathrm{rti}}$ based on random time instant impulses. We have shown how to compute $\|\cdot\|_{2\mathrm{rti}}$ for the closed-loop system in Theorem 2. The results have been applied to obtain the convex optimization problem (21), leading to a controller that minimizes an upper bound on $\|\cdot\|_{2\mathrm{rti}}$. The numerical examples indicate that the norm of the output is smaller than the one obtained with the standard $H_2$ norm with different input signals, suggesting that the proposed control solution outperforms the standard one for MJLS subject to impulses at time instant other than $k = 0$, as well as other disturbances. Future work will look into the extension to the continuous-time case, as well as the impact of (lack of) H2TI in other results in the literature and how they can benefit from the norm proposed here.

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