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Risk-Constrained Kelly Portfolios Under Alpha-Stable Laws

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Abstract
This paper provides a detailed framework for modeling portfolios, achieving the highest growth rate under risk constraints such as value at risk (VaR) and expected shortfall (ES) in the presence of α-stable laws. Although the maximization of the expected logarithm of wealth induces outperforming any other significantly different strategy, the Kelly criterion implies larger bets than a risk-averse investor would accept. Restricting the Kelly optimization by spectral risk measures, the authors provide a generalized mapping for different measures of growth and risk. Analyzing over 30 years of S&P 500 returns for different sampling frequencies, the authors find evidence for leptokurtic behavior for all respective sampling frequencies. Given that lower sampling frequencies imply a smaller number of data points, this paper argues in favor of α-stable laws and its scaling behavior to model financial market returns for a given horizon in an i.i.d. world. Instead of simulating from the class of elliptically α-stable distributions, a semiparametric scaling approximation, based on hourly NASDAQ data, is proposed. Our paper also uncovers that including long put options into the portfolio optimization, improves portfolio growth for a given level of VaR or ES, leading to a new Kelly portfolio providing the highest geometric mean.

Keywords Growth-optimal · Kelly criterion · Protective put · Portfolio optimization · Stable distribution · Value at risk · Expected shortfall

JEL Classification C13 · C46 · C61 · C73 · G11

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1 Introduction

Given a set of investment opportunities, how should the investment weights be chosen in order to have more wealth than any other investor at the end of the investment period? The Kelly growth-optimum strategy is a betting scheme for an investor, who seeks to asymptotically maximize his growth rate of capital. This strategy outperforms any other significantly different strategy, given knowledge of the true underlying process (Breiman 1961). But, the sole use of the Kelly criterion implies larger bets than a representative, risk-averse investor would accept in terms of risk (Clark and Ziemba 1987; Hausch and Ziemba 1985). Thus, the Kelly optimization needs to be restricted by a risk measure. We use α-stable laws and its scaling behavior in order to model the underlying financial market returns. Upon the Generalized Central Limit Theorem (GCLT), the horizon distribution is modelled in an discrete i.i.d. framework.

The aim is to maximize the geometric portfolio return, i.e. Kelly Criterion and restrict the objective to a subjective risk constraint, formulated as spectral risk measure, including quantile (VaR) or expected shortfall as special cases. The formulated trade-off introduces a mapping over growth and risk in order to evaluate the investment decision. The contribution of this paper is three-fold: The first contribution represents the application of multidimensional α-stable laws, in the form of elliptically α-stable distributions, to the constrained Kelly portfolio. Second, instead of simulating from the class of elliptically α-stable distributions, a semiparametric scaling approximation, based on the data set itself, is proposed. Third, assets with non-linear payoff structure, long put-options, are incorporated into the nonlinear optimization to allow for asymmetric payoffs, which lead to a higher growth criterion, given a fixed risk constraint.

The Kelly criterion originates from Kelly (1956), dealing with, from the point of information theory, an optimal investment strategy in a binary channel. Breiman (1961) formally proves the asymptotic outperformance of the Kelly strategy for arbitrary distributions in an i.i.d. world. For arbitrarily distributed, possibly non-stationary processes, those results have been extended by Algeot and Cover (1988). Incorporating risk measures into the Kelly optimization, MacLean et al. (1992) discuss the growth-risk trade-off in terms of efficiency. Roll (1973) compares the Markowitz arithmetic mean maximization with the Kelly geometric mean maximization. In contrast to Constant Proportion Portfolio Insurance (CPPI), the investment strategy remains fixed fraction, given stationarity. More recently, Busseti et al. (2016) introduce an alternative risk constraint, limiting the probability of a drawdown of wealth to a given undesirable level.

The distribution of financial market returns for a chosen horizon is modelled as the sum of hourly random variables. As the distribution in some horizon is presumed to be non-Gaussian, the classical Central Limit Theorem (CLT) does not apply as second and higher moments may not exist. Thus, the generalized central limit theorem (GCLT) of Gnedenko and Kolmogorov (1954) is applied for the sum of random variables, whose second and higher moments may not be bounded. For the financial application this implies the use of α-stable laws (Fama 1965; Lévy 1925; Mandelbrot 1963). As multidimensional α-stable random variables are difficult to evaluate for larger dimensions, elliptical α-stable distributions are employed, allowing for efficient portfolio estimation for dimensions $k \leq 40$ (Nolan 2013) in the presence of linear dependence.

Price data, both for assets with linear and non-linear payoff structure, were gathered from Lobster and Bloomberg. For computation, Matlab 2016a was utilized. In order to solve the formulated nonlinear optimization problem the sequential quadratic algorithm in *fmincon* was employed.

The paper is organized as follows: In chapter one the portfolio allocation problem is stated. The financial model is formulated by using generalized measures for growth and risk. Chapter two, the estimation, starts with a case for non-Gaussianity of financial log-returns of different sampling frequencies, reasoning the utilization of α-stable laws. For the multidimensional case, elliptically α-stable distributions are introduced in order to have an analytically tractable class of distributions. As the semiparametric scaling approximation is introduced, the estimation of location and scale is illustrated. An application is given in chapter three, the implementation. For a representative investor with a planning horizon of one year, the optimally VaR/ES-constrained Kelly portfolios are found, benefitting from the protective put strategy.

2 Model

2.1 Portfolio Allocation Problem

Given initial wealth of the investor $W_0 \in \mathbb{R}^+$, there are $j = 1, \ldots, k$ investment opportunities with fractions $f_t = \left[f_{1,t}, \ldots, f_{k,t}\right]^\top \in \mathbb{R}^k$ in period $t = 1, 2, \ldots, T$. $T \in \mathbb{N}^+$ represents the planning horizon. Assessing solely self-financing strategies, the budget constraint is given by $\sum_{j=1}^{k} f_{j,t} \leq 1$. Given a statistical model for continuous returns $X_t \in \mathbb{R}^k$, discrete returns are calculated by $\tilde{X}_t = \exp\{X_t\} - 1$. Given outcomes in $t - 1, \ldots, T$ the wealth in T is given by

$$\begin{aligned} W_T(f_t) &= W_0 \prod_{t=1}^{T} \left\{ 1 + \sum_{j=1}^{k} f_{j,t} \tilde{X}_{j,t} \right\} \\ &= W_0 \prod_{t=1}^{T} \left\{ 1 + f_t^\top \tilde{X}_t \right\}. \end{aligned} \tag{1}$$

Given the stochastic wealth process, measures for growth and risk are formulated in order to choose investment fractions f_t, which suit investor preferences.

For a cdf $F_{W_T}(x)$ the spectral risk/growth measure with weight function $\phi(x)$ is defined through the quantile function $F_{W_T}^{-1}(x) \overset{\text{def}}{=} \{x : \mathrm{P}(W_T(f_t) \leq x) = \alpha\}$, $\alpha \in (0, 1)$.

$$M_\phi \{W_T(f_t)\} = \int_0^1 \phi(x) F_{W_T}^{-1}(x) \mathrm{d}x \tag{2}$$

Within the context of spectral risk measures, the measure will be coherent iff the weight function is positive $\phi(x) \geq 0$, increasing $\phi'(x) \geq 0$ and normalized $\int_0^1 \phi(x) =$

1 (Acerbi 2002). For the discrete framework (1) with $n \in \mathbb{N}^+$ wealth trajectories, the measure is defined as

$$M_\phi \{W_T(f_t)\} = \sum_{i=1}^{n} \phi_i W_{T,i}(f_t), \tag{3}$$

where $W_{T,i}$ denotes element i out of n wealth paths with weight ϕ_i.

Growth measures
Following Roll (1973), there are two main strands dealing with the accumulation of wealth and thus, the allocation of wealth into a portfolio. On the one hand, the Markowitz optimization aims to maximize the expected portfolio return (Lintner 1965; Markowitz 1952; Sharpe 1964; Tobin 1958). On the other hand, the Kelly growth-optimum approach by Kelly (1956), Breiman (1961) and Thorp (1971), aims to maximize the expected logarithm of wealth, which is equivalent to maximizing the geometric portfolio return. Within the framework of spectral growth/risk measures, the growth measures for the Markowitz and the Kelly optimization are evaluated:

- G_1 : For the expected wealth, the growth criterion from the Markowitz optimization, the weight function is

$$\phi_E(x) = 1,$$

giving

$$G_{\phi_E} \{W_T(f_t)\} = \int_0^1 F_{W_T}^{-1}(x)\mathrm{d}x = \mathsf{E}\{W_T(f_t)\}. \tag{4}$$

- G_2 : The expected logarithm of wealth, representing the optimization criterion for the Kelly strategy, is obtained for the weight function

$$\phi_{\mathrm{Elog}}(x) = \log(x),$$

giving

$$G_{\phi_{\mathrm{Elog}}} \{W_T(f_t)\} = \int_0^1 \log F_{W_T}^{-1}(x)\mathrm{d}x = \mathsf{E}\{\log W_T(f_t)\}. \tag{5}$$

The growth measure will be denoted by $G_\phi \{W_T(f_t)\}$ and the optimization for horizon T without risk constraints is formulated as

$$\max_{f_t \in \mathbb{R}^k} \left[G_\phi \{W_T(f_t)\} \,\middle|\, \sum_{j=1}^{k} f_{j,t} \leq 1 \right]. \tag{6}$$

This paper focusses on the Kelly growth criterion as it represents a betting scheme for an investor, who seeks to asymptotically maximize his growth rate of capital. The betting strategy outperforms any other significantly different strategy asymptotically and minimizes the expected time to reach a goal (Algeot and Cover 1988; Breiman 1961). For a comprehensive treatment of the Kelly criterion, see MacLean et al. (2011). Whereas the maximization of the expected wealth in the Markowitz optimization, given favorable investment possibilities, always implies betting the entire fortune, the maximization of the expected logarithm of wealth leads to one growth-optimal portfolio, which is not necessarily optimal in terms of the Markowitz portfolio (Thorp 1971). Accordingly Markowitz (1976) considers the Kelly portfolio to be the upper limit for a a conservative investor. Furthermore, the log-optimal strategy is fixed fraction, independent of time (MacLean et al. 1992).

Risk measures

The sole use of the Kelly criterion implies larger bets than a representative, risk-averse investor would accept in terms of risk (Clark and Ziemba 1987; Hausch and Ziemba 1985). In order to formulate individual risk measures for different investors, the spectral risk measure from (2), denoted by $S_\phi\{W_T(f_t)\}$, will be used. Two specific risk measures to include the degree of risk-aversion into the portfolio optimization are quantile (Value at Risk) and conditional tail expectation (expected shortfall) constraints:

- S_1 : The quantile constraint (VaR) is a special case of the spectral risk measure from (2)
$$\phi_{\mathrm{Q}_\alpha}(x) = \delta(x = \alpha),\ \alpha \in (0, 1), \tag{7}$$
where $\delta(x = \alpha)$ is the Dirac delta function, well known to be a non-coherent risk measure. Further drawbacks of the quantile constraint are treated in Basak and Shapiro (2001). However, the quantile restriction allows to ask the investor specifically to name a fraction of his wealth he can accept to lose with probability $1 - \alpha$.
- S_2 : In contrast, Conditional Tail Expectation (ES) is a coherent risk measure representing the average loss beyond a given quantile constraint. Being a special case of the spectral measure, the weight function is given as
$$\phi_{\mathrm{CTE}_\alpha}(x) = \alpha^{-1}\mathbf{1}(x < \alpha). \tag{8}$$

Growth-risk frontier

Following MacLean et al. (1992), the possible combinations of growth and risk measures are given by the set

$$U = \left[G_\phi\{W_T(f_t)\}, S_\phi\{W_T(f_t)\}\right], f_t \text{ feasible}. \tag{9}$$

The growth-risk frontier is accordingly formulated as

$$U_t^* = \left[G_\phi\left\{W_T(f_t^*)\right\}, S_\phi\left\{W_T(f_t^*)\right\}\right], f_t^* \text{ feasible}, \tag{10}$$

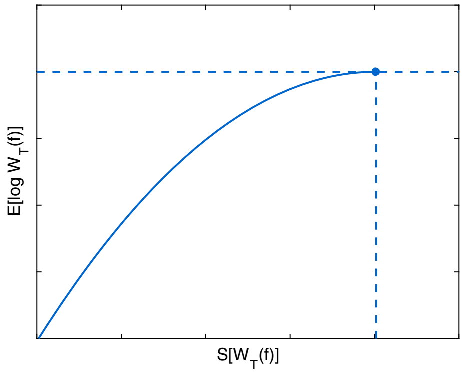

Fig. 1 Kelly-risk frontier with unconstrained Kelly portfolio exhibiting the highest geometrical mean

where the $f_t^* \in \mathbb{R}^k$ is the investment fraction maximizing the growth measure under risk restriction.

$$
\begin{aligned}
f_t^* = \underset{f_t^* \in \mathbb{R}^k}{\arg\max}\ & G_\phi \left\{ W_T(f_t) \right\} \\
\text{s.t. } & S_\phi \left\{ W_T(f_t) \right\} \leq b,\ \ b \in \mathbb{R}, \\
& \sum_{j=1}^{k} f_{j,t} \leq 1
\end{aligned} \tag{11}
$$

For the Kelly criterion with a risk constraint as proposed, the frontier is illustratively visualized in Fig. 1. In contrast to the Markowitz maximization, implying a steady tradeoff between mean and risk, the geometric mean maximization implies one specific portfolio—the Kelly portfolio—exhibiting the highest geometric mean possible (horizontal dotted line). From this viewpoint, portfolios exhibiting a larger risk constraint than the Kelly portfolio (to the right of the vertical dotted line) are not efficient. If the investor prefers a smaller risk constraint than the full Kelly investor, restricted Kelly portfolios (solid line) constitute the Kelly-risk frontier. These are portfolio strategies with the highest growth criterion given risk constraint.

2.2 Tail Constraints and Non-linear Instruments

The introduction of assets as nonlinear functions of the underlyings, derivatives, allows for controlling the asymmetry of the wealth distribution in such a way, that it will be skewed to the left. Albeit the distribution of the risk measure, the loss of the portfolio is limited by construction for high confidence levels. The instruments to achieve the asymmetric payoff profile are long put options. By construction, corridor options, as argued in the context of quantile constraints, are circumvented (Basak and Shapiro 2001). A simplified representation of the protective put strategy is given in Fig. 2, consisting of one stock (blue) and one long put option (green) with chosen strike (dotted black). The result is the protective put strategy (red). The difference in payoff above the strike level is due to the put price, which the option holder has to pay. For

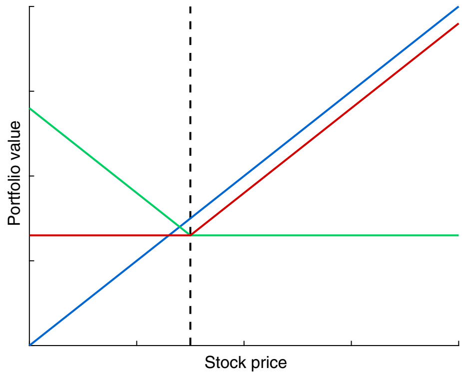

Fig. 2 Protective put strategy (red) consisting of long stock (blue) and long put (green) with chosen strike (dotted black). (Color figure online)

Table 1 Log-return descriptives for the different sampling frequencies, S&P 500 1985–2015 Frequency weekly* omits one week in the financial crisis 2009

Descriptives	S&P (daily)	S&P (weekly)	S&P (weekly*)	S&P (monthly)	S&P (yearly)
Data points	7564	1513	1512	360	30
Mean (p.a. in %)	8.37	8.37	10.41	8.37	8.37
Std (p.a. in %)	18.35	20.93	17.96	16	16.61
Skewness	− 1.29	− 6.7	− 1.27	− 1.98	− 1.78
Kurtosis	31.26	131.09	13.67	12.48	7.13

$k \in \mathbb{N}^+$ linear assets with multiple put options each, given a pre-specified horizon, the choice of the fraction of linear and nonlinear assets is not obvious.

3 Estimation

3.1 A Case for Non-Gaussianity

Although Fama (1965) finds evidence for α-stable characteristics for all returns of the Dow Jones Index, it can be observed that financial (log-)returns tend to the Gaussian distribution as the sampling frequency decreases, see also McFarland et al. (1982), Boothe and Glassman (1987), and Dacorogna et al. (2001). The subsequent textbook example for the Standard and Poor's 500 reads as Table 1. Due to the 2009 financial crisis, an outlier week of −60% increases (decreases) the sample kurtosis (skewness) for the weekly frequency significantly from 13.67 (− 1.27) to 131.09 (− 6.7). If the outlier week is omitted, see column S&P (weekly*) of Table 1, the general observation of decreasing kurtosis and increasing negative skewness is supported for *different* sample sizes. Still, including the outlier week of 2009, erratic behavior of sample moments definitely appears for this reference data series.

The empirical observation of Gaussian convergence for lowering sampling frequencies cannot be shown explicitly by existing data, as data-records capture only 7564

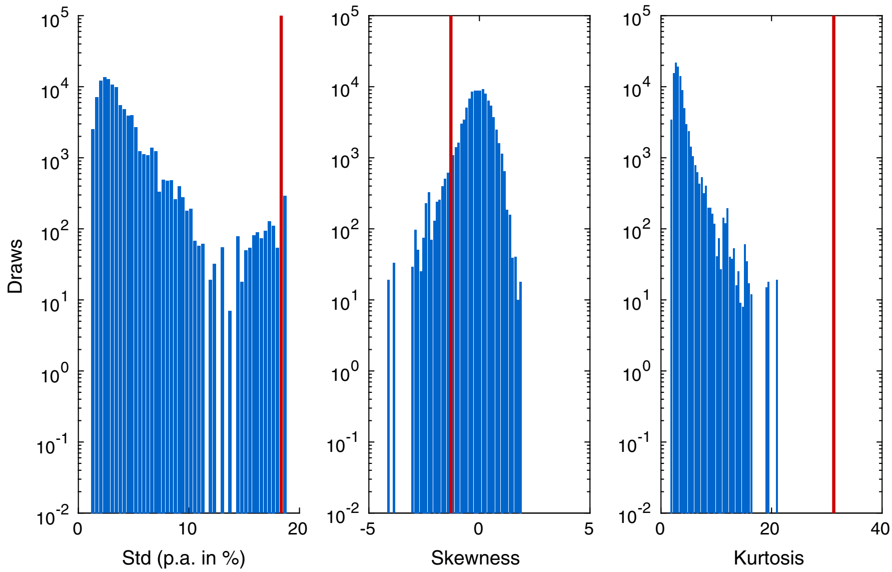

Fig. 3 Whole sample (red) and block-bootstrapped standard deviations (p.a.), skewness and kurtosis for 10^5 draws of 30 subsequent daily returns (blue) from the S&P 500, 1985 to 2015. (Color figure online)

trading days, representing 30 years of data. The empirical verification would require an appropriately large number of weeks, months and years.

In order to show that the annual return distribution, consisting of 30 data points, is with large probability not Gaussian, we randomly sample 10^5 30 blocks of daily returns from the S&P 500 and calculate second, third and fourth moments in order to evaluate dispersion, skewness and leptokurtic behavior (Bootstrap). Hence, for the three moments, the block-bootstrap estimators are plotted as histogram in Fig. 3. The vertical red lines represent the moment estimators for the whole daily data series. In essence, dispersion, skewness and especially leptokurtic behavior of the bootstrap estimators are significantly biased, compared to the estimator of the whole series. Fewer sampled data-points imply less probability of sampling data in the tails of the return distribution. There was not one out of 10^5 30 day sub-samples, which resulted in a comparable kurtosis of the complete data-series. The result holds for sampling 30 separate days randomly under the i.i.d. assumption. Moments of order larger than one behave erratically over an increasing data sample, as first analyzed for commodity prices in Mandelbrot (1963). Figure 4 plots standard deviation (in %), skewness and kurtosis as function of the used data-points of the series. The red lines represents the empirical moment behavior with increasing daily data points. The blue lines represent 100 trajectories of Gaussian moments with increasing data points. The observation of erratic moment behavior stands in contrast to Gaussian behavior. The observation holds over sampling frequencies daily, weekly, monthly and annually. This specific sample-size problem is crucial in risk management, especially for estimating quantiles of high confidence of the wealth distribution as in the constrained portfolio optimization in (11). As the confidence level tends to one, having only a limited amount of data,

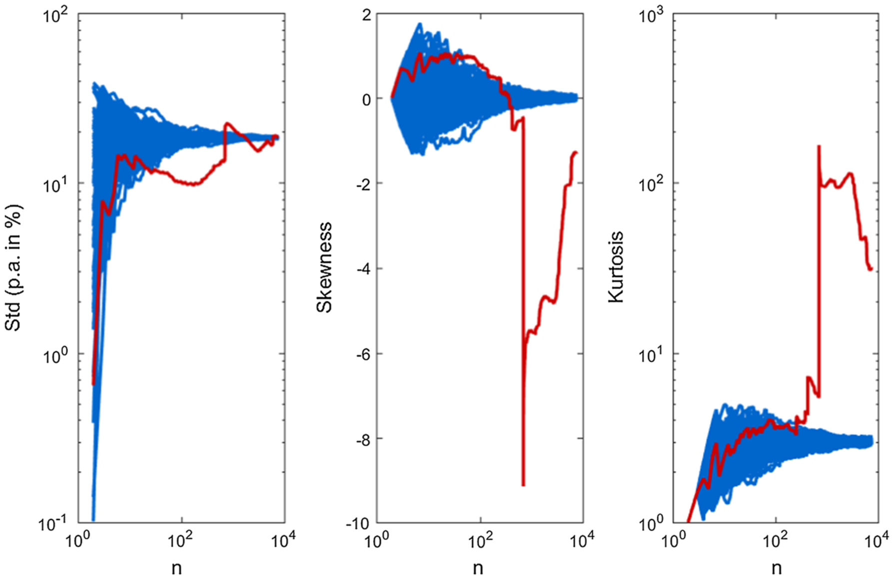

Fig. 4 (Log-)Log plots of standard deviation (in %), skewness and kurtosis with increasing data points, S&P 500 from 1985 to 2015 (red) and 100 Gaussian simulations with S&P 500 moments (blue). (Color figure online)

the quantile estimate is systematically biased as the quantile is overestimated. The portfolio analyst has to evaluate if the estimated quantile given the chosen confidence level still has an acceptable distribution.

Consequently, for investors with longer investment horizons, such as a year, the sum of daily random variables, constituting the yearly distribution, should not converge to the Gaussian, but to a heavy-tailed distribution, which will turn out to be the class of α-stable distributions. For financial markets, this assumption will imply infinite variance, skewness and kurtosis, leading to non-converging moments, i.e. the observed erratic behavior. The model of Sect. 2 will be estimated within a stationary framework for elliptically α-stable distributions, striving for scale invariance. Although daily and higher frequency returns exhibit non-stationary characteristics, the horizon distribution, i.e. yearly, cannot be shown to exhibit significant volatility clustering.

3.2 Scale Invariance

Let $X_t \in \mathbb{R}^k$ be a multidimensional, i.i.d. random variable from distribution P^t, where t indicates the scale e.g. days. Given the investment horizon of the investor, T days, the wealth equation of (1)

$$W_T(f_t) = W_0 \prod_{t=1}^{T} \left\{ 1 + f_t^\top \tilde{X}_t \right\} = W_0 \prod_{t=1}^{T} \left\{ f_t^\top \exp(X_t) \right\} \tag{12}$$

can be simplified, given $f_t = f \ \forall t = 0, \ldots, T$.

$$W_T(f) = W_0 \left\{ f^\top \exp\left(\sum_{t=1}^{T} X_t \right) \right\} = W_0 \left\{ f^\top \exp(X) \right\}, \quad X \stackrel{\text{def}}{=} \sum_{t=1}^{T} X_t \tag{13}$$

As the horizon T grows, the sum of the random variables X_t tends to the Gaussian as long as the first two moments of the underlying distribution are finite. Formally, let random variable X_t have expectation vector $\mu_t = \mathsf{E}(X_t)$ and covariance matrix $\sigma_t = \mathsf{E}\left[\{X_t - \mathsf{E}(X_t)\}\{X_t - \mathsf{E}(X_t)\}^\top\right]$. Then

$$\begin{aligned} \sum_{t=1}^{T} X_t &\xrightarrow{\mathcal{L}} \mathrm{N}\left(\sum_{t=1}^{T} \mu_t, \sum_{t=1}^{T} \sigma_t \right) \\ X &\xrightarrow{\mathcal{L}} \mathrm{N}(\mu, \sigma) \\ T^{-\frac{1}{2}} \sum_{t=1}^{T} (X_t - \mu_t) &\xrightarrow{\mathcal{L}} \mathrm{N}(0, \sigma). \end{aligned} \tag{14}$$

If the distribution in horizon T is modelled as the sum of higher frequency distributions, the multidimensional process of returns, which may not be Gaussian, but of finite variance, converges to the Gaussian. In contrast, as argued in Sect. 3.1, returns of horizons beyond the sampled frequency, are presumed to be heavy-tailed. Hence, the standard Central Limit Theorem (CLT) does not apply.

Except for the Gaussian itself, finite variance distributions change their shape under aggregation. In contrast, the class of α-stable distributions is scale invariant (Mandelbrot 1963). Scale invariance of distribution P is defined via a continuous function g, such that for all x

$$g(\lambda)P(x) = P(\lambda x), \tag{15}$$

with $\lambda x \geq x_0$ and $x_0 > 0$. Equivalently, distribution P has a power-law tail, implying that for $x \geq x_0 \geq 0$, $c \geq 0$ and $\alpha > 0$

$$P(x) = cx^{-\alpha}. \tag{16}$$

In that respect, a one-dimensional random variable $X \sim S(\alpha, \beta, \gamma, \delta)$ will be α-stable distributed with parameters $0 < \alpha \leq 2,\ -1 \leq \beta \leq 1,\ \gamma \geq 0$ and $\delta \in \mathbb{R}$ (Cizek et al. 2011; Nolan 2017), if

$$X \stackrel{\mathcal{L}}{=} \begin{cases} \gamma Z + \delta, & \alpha \neq 1 \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \log \gamma), & \alpha = 1. \end{cases} \tag{17}$$

$S(Z \mid \alpha, \beta, 1, 0)$ represents the standard α-stable form. Only special cases of α-stable distribution are available as real-valued densities (e.g. Gaussian, Cauchy and Lévy).

Scale invariance under addition implies that for the sum of α-stable variables $X_t \sim S(\alpha, \beta, \gamma, \delta_t),\ t = 1, \ldots, T$

$$X_1 + X_2 + \ldots + X_T = \sum_{t=1}^{T} X_t = X \sim S\left(\alpha, \beta, T^{\frac{1}{\alpha}}\gamma, \delta\right), \tag{18}$$

where $\delta = T\delta_t$.

According to Gnedenko and Kolmogorov (1954), the limiting distribution of T i.i.d. α-stable random variables, $0 < \alpha \leq 2$ is

$$a_T \left(\sum_{t=1}^{T} X_t\right) - b_T \xrightarrow{\mathcal{L}} S(\alpha, \beta, 1, 0), \tag{19}$$

where $a_T > 0$ and $b_T \in \mathbb{R}$. The special case of the Generalized Central Limit Theorem (GCLT) is the CLT of Eq. (14) for $\alpha = 2$, $\beta = 0$, $\gamma = \frac{\sigma}{\sqrt{2}}$ and $\delta_t = \mu_t$, given $a_T = \frac{1}{\sigma\sqrt{T}}$ and $b_T = \frac{\sqrt{T}\mu}{\sigma}$. In general, for $0 < \alpha \leq 2$,

$$T^{-\frac{1}{\alpha}} \sum_{t=1}^{T} (X_t - \delta_t) \xrightarrow{\mathcal{L}} S(\alpha, 0, \gamma, 0). \tag{20}$$

3.3 Elliptically Contoured α-Stable Distributions

For the multidimensional estimation, α-stable laws are not extensively accessible as closed-form densities are only available for special cases. One computationally tractable exception are elliptically contoured α-stable laws, which can be efficiently estimated for dimensions $k \leq 40$ (Nolan 2013). This class of distributions enables the modeling of heavy tails while preserving its shape under aggregation in the presence of linear dependence.

Random vector $Y = [Y_1, \ldots, Y_k]^\top$ has a spherical distribution iff the characteristic function $\varphi_Y(u)$ satisfies for all $u \in \mathbb{R}^k$

$$\varphi_Y(u) = \mathsf{E}\left\{\exp\left(iu^\top Y\right)\right\} = \psi(u^\top u) = \psi(u_1^2 + \ldots + u_k^2), \tag{21}$$

where ψ is the characteristic generator of the spherical distribution.

Random vector $X \sim S_k(\delta, \Gamma, \psi)$ is elliptically distributed with positive definite scaling matrix $\Gamma = AA^\top,\ A \in \mathbb{R}^{k \times k}$ and location vector $\delta \in \mathbb{R}^k$ when

$$X \stackrel{\mathcal{L}}{=} \delta + AY, \tag{22}$$

where Y is spherical with characteristic generator ψ. The characteristic function is given by

$$\varphi_X(u) = \mathsf{E}\left\{\exp\left(iu^\top X\right)\right\} = \exp\left(iu^\top \delta\right) \psi\left(u^\top \Gamma u\right). \tag{23}$$

A subclass of elliptical distributions are normal variance mixtures $X = [X_1, \ldots, X_k]^\top$ for

$$X \stackrel{\mathcal{L}}{=} W^{1/2} A Z + \delta, \tag{24}$$

with $Z \sim \mathrm{N}(0, I_k)$ and $W \geq 0$ being a non-negative one-dimensional random variable, independent of Z (Kring et al. 2009).

A further subclass of normal variance mixtures are α-stable sub-Gaussian $X = [X_1, \ldots, X_k]^\top$ for $W \sim S(\alpha/2, (\cos \pi\alpha/4)^{2/\alpha}, 1, 0)$, $0 < \alpha < 2$, being one-dimensionally α-stable distributed, parameterized following Nolan (2017). $G \sim \mathrm{N}(0, \Gamma)$ is multidimensional Gaussian with scaling matrix $\Gamma = AA^\top$. Then $X \sim S_k(\alpha, \beta, \Gamma, \delta, \psi)$, $\beta = 0$ is α-stable sub-Gaussian if

$$\begin{aligned} X &\stackrel{\mathcal{L}}{=} W^{1/2} G + \delta \\ &\stackrel{\mathcal{L}}{=} W^{1/2} A Z + \delta, \ Z \sim \mathrm{N}(0, I_k) \\ &\stackrel{\text{def}}{=} AY + \delta, \end{aligned} \tag{25}$$

while $Y \sim S_k(\alpha, 0, I_k, 0)$ is radially symmetric α-stable. The according characteristic function of X is

$$\begin{aligned} \varphi_X(u) &= \int_{-\infty}^{\infty} f_X(x) \exp(iu^\top X) \mathrm{d}x = \mathsf{E}\left(iu^\top X\right) \\ &= \exp\left\{-\left(\frac{1}{2} u^\top \Gamma u\right)^{\alpha/2} + iu^\top \delta\right\}, \end{aligned} \tag{26}$$

$f_X(x)$ as probability density function. $\Gamma \in \mathbb{R}^{k \times k}$ is the positive definite scale matrix and $\delta \in \mathbb{R}^k$ the location vector. The characteristic generator is therefore given by

$$\psi(s, \alpha) = \exp\left\{-\left(\frac{1}{2} s\right)^{2/\alpha}\right\}. \tag{27}$$

This implies that α-stable sub-Gaussian distributions are scale mixtures of multivariate normal distributions (Samorodnitsky and Taqqu 1994). Note, for $\alpha = 2$, the characteristic function collapses to the Gaussian. For $G \sim \mathrm{N}(0, I_k)$, the characteristic function of Y in Eq. (25) simplifies to

$$\varphi_Y(u) = \mathsf{E}\left(iu^\top Y\right) = \exp\left(-\gamma^\alpha |u|^\alpha\right). \tag{28}$$

For the horizon of the investor, T, the estimated higher sampling frequency log-returns are summed to the chosen frequency:

$$\tilde{X} = TX \ \sim S_k(\alpha, 0, T\Gamma, T\delta, \psi) \tag{29}$$

For subsequent estimation, stability parameter $0 < \alpha \leq 2$, scale Γ and location δ need to be estimated, given that $-1 \leq \beta \leq 1$ can be assumed to be not significantly different from zero.

3.4 Parameter Estimation

The utilization of α-stable laws implies that fractional moments of random variable X

$$\mathsf{E}|X|^p = \int_{-\infty}^{+\infty} |X|^p f(X) \mathrm{d}X \tag{30}$$

are finite for $0 < p < \alpha,\ p \in \mathbb{R}$ and infinite for $p \geq \alpha$. This implies that for the α-stable Paretian case, representing a slower decay than under the Gaussian, $0 < \alpha < 2$, the second moment $\mathsf{E}|X|^2 = \infty$ and higher moments such as skewness and kurtosis are infinite. For the empirical financial market returns $1 < \alpha < 2$ (see Sect. 4), the first moment remains finite. For elliptically α-stable random variable $X \sim S_k(\alpha, 0, \Gamma, \delta, \psi))$ the expectation is

$$\mathsf{E}\, X = \delta < \infty. \tag{31}$$

In general, for univariate α-stable laws the mean is undefined for $\alpha \leq 1$ and $\mathsf{E}\, X = \delta - \beta\gamma \tan\left(\frac{\pi\alpha}{2}\right) < \infty$ for $\alpha > 1$. From the perspective of a data scientist, analyzing the sample, empirical moments are always finite. But under the assumptions of being α-stable distributed, fractional moments with $p \geq \alpha$ have no intrinsic meaning. As shown in Sect. 3.1 higher moments behave erratic with increasing data points, contrary to moment convergence under Gaussianity.

For portfolio allocation the estimation of location and scale are crucial. Founding on the analysis of Chopra and Ziemba (1993), the mean represents the largest source of error for estimating the portfolio fraction. Their final implication is straightforward: "[...] the bulk of resources should be spent on obtaining the best estimates of expected returns of the asset classes under consideration".

Simulating from the class of elliptically α-stable distributions implies to estimate the stability parameter α, scaling matrix Γ and location δ, given that the skewness parameter β is zero. For the characteristic exponent α the method of Rachev and Mittnik (2000) is used:

i. Simulate $U_1, \dots, U_n$ uniformly i.i.d. random variables on the unit hypersphere S^{k-1}.
ii. Estimate the MLE for the index of stability $\hat{\alpha}_i$ (Nolan 2001) for each i from 1 to n, $U_i^\top X_1, \dots, U_i^\top X_n$.
iii. Calculate the index of stability by $\hat{\alpha} = n^{-1} \sum_i^n \hat{\alpha}_i$.

By utilizing the MLE for the characteristic exponent α, severe estimation biases from e.g. the Hill estimator (Hill 1975) are circumvented, see also McCulloch (1997) and Kearns and Pagan (1997). For the proposed semiparametric scaling approximation in Sect. 3.5, the estimation of stability α will not be necessary.

Estimating the location vector $\delta \in \mathbb{R}^k$ of multidimensional variable $X \sim S_k(\alpha, 0, \Gamma, \delta, \psi)$ is of crucial importance for portfolio allocation, representing *the* driver for asset growth.

From the perspective of information theory, we aim to chose the parameter vector, which maximizes the probability of coming from the empirical data-set. From the perspective of decision theory, this method coincides with the minimization of expected loss under the 0–1 loss function:

$$L(\delta, \hat{\delta}) = 1(\delta \neq \hat{\delta}). \tag{32}$$

The according risk function is

$$\mathrm{R}(\delta, \hat{\delta}) = \mathsf{E}\left\{\mathrm{L}(\delta, \hat{\delta})\right\} = \mathsf{E}\left\{1(\delta \neq \hat{\delta})\right\} \tag{33}$$

Consequently the optimization

$$\delta^* = \underset{\delta^* \in \mathbb{R}^k}{\arg\min} \left[\mathsf{E}\left\{1(\delta \neq \hat{\delta})\right\}\right] \tag{34}$$

leads to the common Maximum Likelihood Estimate (MLE). If the loss function is not presumed to be 0–1 loss, e.g. quadratic, the usual ML estimator may not be suitable. The inadmissability of the sample mean under the Gaussian for dimensions $k > 2$ has been first shown by Stein (1955), leading to the class of shrinkage estimators, starting with James and Stein (1961). An overview over the class of shrinkage estimators is given in Hansen (2015). To our knowledge, those results have not been extended to α-stable laws.

Following Nolan (2013), there are two methods to estimate the scale matrix Γ:

i. Given that X is elliptically α-stable,

$$\forall u,\ u^\top X \sim S_k\left(\alpha, 0, (u^\top \Gamma u)^{\frac{1}{2}}, u^\top \delta, \psi\right). \tag{35}$$

The $k(k+1)/2$ parameters of the scale matrix Γ are estimated by

$$\begin{aligned} \hat{\Gamma}_{j,j} &= \hat{\gamma}_j^2 \\ \hat{\Gamma}_{j,i} &= \tfrac{1}{2}\left\{\hat{\gamma}^2(1,1) - \hat{\gamma}_i^2 - \hat{\gamma}_j^2\right\}, \end{aligned} \tag{36}$$

where $\hat{\gamma}^2(1,1) = (1,1)^\top (X_j, X_i) = X_j + X_i$ and $\hat{\gamma}_j$ is the univariate scale ML estimate of asset j. Note that $\hat{\Gamma}_{j,i}$ depends solely on directions $(1,1)$,$(1,0)$ and $(0,1)$.

ii. As $\mathsf{E}\left\{\exp(iu^\top X)\right\} = \exp\left\{-\gamma(u)^\alpha\right\}$

$$\left\{-\log \mathsf{E} \exp(iu^\top X)\right\}^{\frac{2}{\alpha}} = u^\top \Gamma u = \sum_i u_i^2 \Gamma_{i,i} + 2\sum_{i<j} u_i u_j \Gamma_{i,j}, \tag{37}$$

so $\Gamma_{i,j}$ can be estimated as linear function via regression, taking more directions into account than the first method.

For the remainder of the paper, the first method is utilized due to its analytical tractability.

3.5 Semiparametric scaling approximation

Instead of simulating from the estimated elliptically contoured α-stable distribution, a semiparametric scaling approximation based on higher sampling frequency data is proposed. Simulating from the elliptically α-stable distribution implies that

$$\begin{aligned} &\forall j = 1, \ldots, k \qquad \alpha_j = \alpha, \ 0 < \alpha < 2, \\ &\forall j = 1, \ldots, k \qquad \beta_j = 0. \end{aligned} \tag{38}$$

We deal with this drawback by using α-stable properties of the empirical data-set. Assume that the higher sampling frequency data-set $X_t \sim S_k(\alpha, 0, \Gamma_t, \delta_t, \psi)$ is elliptically α-stable distributed. Then,

i. estimate location δ_t and scale $\Gamma_t = A_t A_t^\top$ of higher frequency returns X_t as proposed in section 3.4.
ii. Normalize X_t to radially symmetric $Y \sim S_k(\alpha, 0, I_k, 0, \psi)$

$$Y = A_t^{-1} X_t - \delta_t. \tag{39}$$

iii. Rescale radially symmetric Y to distribution $X \sim S_k(\alpha, 0, \Gamma, \delta, \psi), \ \Gamma = AA^\top$ with investment horizon T,

$$X = AY + \delta \tag{40}$$

with $\Gamma = T\Gamma_t$ and $\delta = T\delta_t$.

The resulting distribution for horizon T, represented by convoluted higher frequency distributions, is simply an affine transformation of its radially symmetric analogue, given its scaling nature. Given that $\beta = 0$, we can use the potentially different stabilities α_j of the marginals, having no effect on location δ and scale Γ.

As the horizon distribution represents a limited number of data points (see section 3.1), empirical quantiles $Q_\alpha, \ \alpha < 0.02$ are overestimated, implying that risk measures for large confidence levels are underestimated. Vice versa, quantiles $Q_\alpha, \ \alpha > 0.98$ are consequently underestimated, see Fig. 5. By using empirical higher sampling frequency data, we can scale high-frequency events to a manifold of large scale events, which never happened in the original data history of the lower sampling frequency, enriching the tails of the horizon distribution.

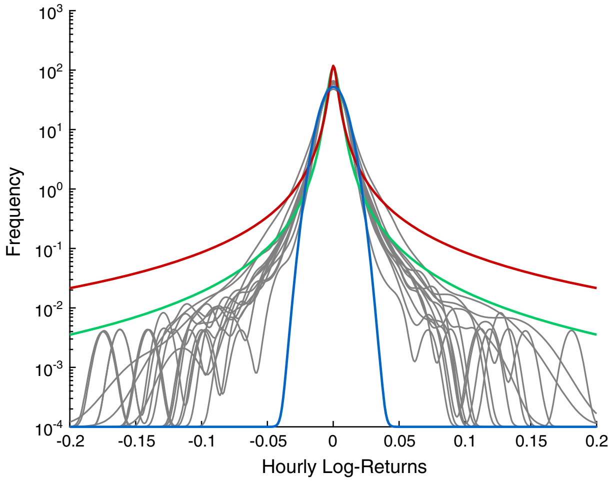

Fig. 5 Semi-log densities for hourly Apple log-returns for Gaussian, $\alpha = 2$ (blue), Stable, $\alpha = 1.33$ (green), Cauchy, $\alpha = 1$ (red) and financial assets (gray). (Color figure online)

4 Implementation

4.1 Data

The hourly financial stock prices come from Lobster and cover the time span from 2007-06-27 to 2018-05-25, representing 17862 hourly prices per asset. The $k = 14$ assets with a linear payoff structure (stocks) are the stocks with the biggest market capitalization in the NASDAQ 100, representing a technology driven portfolio. A risk-free asset, which can be bought with annual rate $r = 0.01$ is included into the optimization. Relevant asset statistics including Maximum Likelihood Estimates (MLE) under α-stability (Nolan 2001) are given in Table 2.

The assets with a non-linear payoff structure are represented as long put options, written on the stock market index NASDAQ 100. As will be assumed for the representative investor in section 4.4, the maturity, and hence the investment horizon T, is chosen to be one year. The prices coming from the ask implied volatilities of the long put options determine the price of the hedge and accordingly the reduction in wealth if the stocks close above the chosen strike levels. For the distribution of wealth in T, the put option price O_T at maturity is given by the inner value

$$O_T = \max\{0; K - S_T\}. \tag{41}$$

Solely for evaluating the price of the non-linear assets between $t = 1$ and horizon T, a pricing model is needed.

Table 2 Log-return descriptive statistics with Maximum Likelihood Estimates under α-stability, 2007-06-27 to 2018-05-25

	$\mu \times T$	$\sigma \times T^{1/2}$	Skewness	Kurtosis	α	β	$\gamma \times T^{1/\alpha}$	$\delta \times T$
Apple	0.22	0.31	−0.75	41.09	1.33	0.03	0.69	0.08
Adobe	0.16	0.32	−0.97	68.52	1.42	−0.05	0.55	0.33
Amgen	0.11	0.26	1.12	44.33	1.49	0.00	0.39	0.06
Amazon	0.29	0.38	1.79	62.00	1.40	0.03	0.67	0.14
Comcast	0.08	0.30	−0.16	29.31	1.43	−0.02	0.51	0.14
Costco	0.11	0.23	−0.64	32.68	1.44	0.03	0.38	0.04
Cisco	0.04	0.29	−0.94	62.65	1.44	−0.02	0.47	0.09
Gilead	0.11	0.30	−0.94	48.60	1.47	0.02	0.47	−0.01
Intel	0.08	0.28	−0.24	25.55	1.43	−0.01	0.51	0.07
Microsoft	0.11	0.27	−0.05	37.09	1.42	−0.01	0.47	0.09
Nvidia	0.16	0.47	−2.95	134.48	1.40	−0.01	0.86	0.20
Pepsi	0.04	0.18	−0.74	38.56	1.44	0.00	0.30	0.08
Qualcomm	0.03	0.31	−0.31	59.96	1.39	−0.01	0.58	0.07
Texas Instruments	0.10	0.28	−0.71	28.31	1.44	−0.04	0.49	0.25

4.2 Stable tests

In order to verify if the class of elliptically α-stable distributions is suitable for the financial assets, the following prerequisites have to be met:

- heavy tails beyond the Gaussian (Leptokurtic behavior),
- linear dependence structure between the margins,
- comparable range of α_j (for simulation),
- skewness parameter β not coherently different from zero.

As examined descriptively in Table 2, empirical financial market returns are significantly non-Gaussian. In Fig. 5, the densities of the normalized log-returns on log-scale are plotted for Gaussian ($\alpha = 2$), Stable ($\alpha = 1.33$), Cauchy ($\alpha = 1$) and the individual assets using Kernel Density Estimates. Within the α-stable framework all examined assets lie between Gaussian and Cauchy, $1 < \alpha < 2$. The α-stable fit for $\alpha = 1.33$ captures the tails adequately, although events are captured, which never took place in the data history. The range of characteristic exponents stands in line with results of Westerfield (1977), McCulloch (1997) or Nolan (2013). The elliptical behavior is assessed by using two dimensional scatter matrices of the empirical log-returns. The significance of the skewness parameters β_j is verified by the utilization of the Fisher information from the MLE. The respective confidence intervals for the individual parameters show that β_j are not consistently different from zero, given a confidence level of 99%. For larger dimensions, Nolan (2013) reaches the same conclusion for the Dow Jones constituents.

Making use of the semiparametric scaling approximation implies that there is no need to estimate one specific α for the elliptical α-stable distribution. As $\forall j \; 1 < \alpha_j < 2$ we can deny the null of Gaussianity coherently for the 99% confidence level, speaking in favour of the α-stable hypothesis. As we are interested in the horizon distribution, constituted by the sum of hourly random variables, the generalized CLT is utilized.

4.3 Stable estimation

Following section 3.4, the parameter estimates for the hourly distribution $X_t \sim E_k(\alpha, \beta, \Gamma_t, \delta_t, \psi)$, $\beta = 0$ are scaled to the chosen horizon of one year. Exemplary. the semi-log densities for yearly Apple log-returns under Gaussian, Cauchy, Stable and the semiparametric scaling are plotted in Fig. 6. Additionally, Gaussian scaling, representing the scaling of the hourly distribution utilizing the square root of time rule under Gaussianity whilst neglecting the CLT, is displayed. In comparison, the semiparametric scaling distribution exhibits heavier tails than under Gaussianity, implying stock market events, which never occurred in the history of the original sampling frequency. The utilized scaling approximation provides the horizon distribution $X_T \sim E_k(\alpha, 0, \Gamma_T, \delta_T, \psi)$ with location vector $\delta_T = T\delta_t$ and scaling matrix $\Gamma_T = T\Gamma_t$, given that $\beta = 0$.

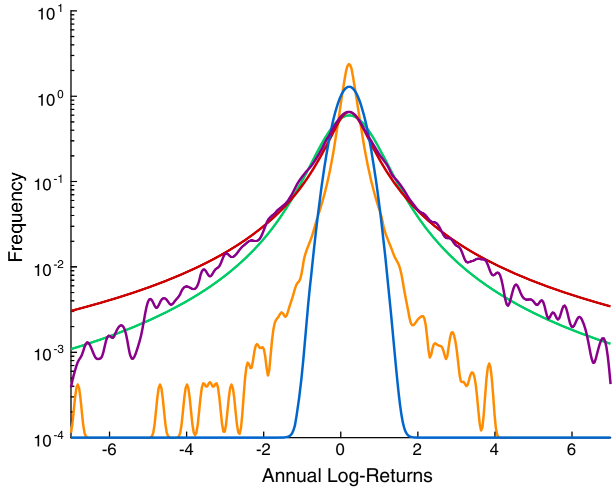

Fig. 6 Semi-log densities for yearly Apple log-returns for Gaussian, $\alpha = 2$ (blue), Stable, $\alpha = 1.33$ (green), Cauchy, $\alpha = 1$ (red), Gaussian scaling (orange) and semiparametric scaling (violet). (Color figure online)

4.4 Portfolio implementation

Exemplary, the representative investor has an investment horizon of one year. According to his client, no more than 20% ($b = 0.20$) of his wealth should be lost given probability $1 - \alpha = 99.5\%$ (Value at Risk). This implies that only $\alpha = 0.5\%$ of the wealth return paths should end below -20%. The investor is able to buy risk-free bonds with risk free rate $r_f = 1\%$ per year, representing the 15th asset. The maximization problems, without ($k = 15$) and with options ($k = 101$), as a special case of optimization in (11), are formulated within the framework of spectral measures. Subject to the $\mathrm{VaR}(1-\alpha)$, $\alpha = 0.5\%$ constraint, the Kelly criterion $G_{\phi_{\text{Elog}}}$ is maximized to achieve the portfolio with the highest growth rate:

$$\begin{aligned} f^* = \underset{f^* \in \mathbb{R}^k}{\arg\max}\ & G_{\phi_{\text{Elog}}}\{W_T(f)\} \\ \text{s.t. } & S_{\phi_{\text{Q}_{0.5\%}}}\left\{1 - \frac{W_T(f)}{W_0}\right\} \leq 0.2, \\ & \sum_{j=1}^{k} f_j \leq 1. \end{aligned} \tag{42}$$

Additionally the client aims to replace the VaR(99.5%) constraint with the expected shortfall restriction $\mathrm{ES}(1-\alpha)$ in order to account for events beyond the VaR level.

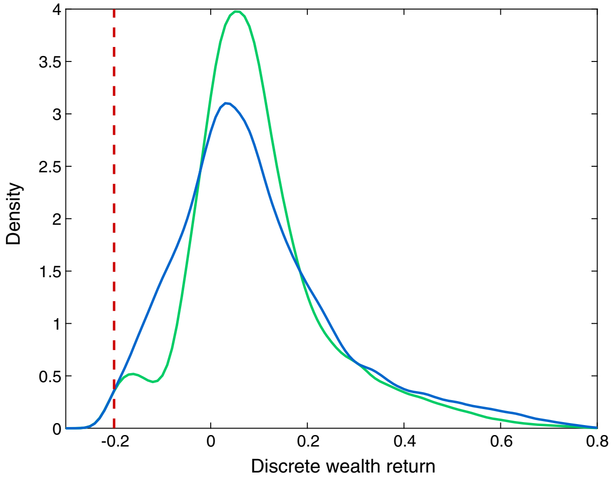

Fig. 7 Wealth return densities for VaR restricted Kelly optimization without (blue) and with (green) put-options, VaR constraint (red). (Color figure online)

$$
\begin{aligned}
f^* = \underset{f^* \in \mathbb{R}^k}{\arg\max}\ & G_{\phi_{\text{Elog}}}\{W_T(f)\} \\
\text{s.t. } & S_{\phi_{\text{CTE}_{0.5\%}}}\left\{1 - \frac{W_T(f)}{W_0}\right\} \leq 0.2, \\
& \sum_{j=1}^{k} f_j \leq 1
\end{aligned}
\tag{43}
$$

The resulting discrete wealth return distributions for the VaR restricted portfolios are given in Fig. 7. Including non-linear instruments into the restricted optimization proves to be beneficial for the Kelly criterion (Geometric mean), whilst preserving the VaR restriction (see Table 3). The protective put strategy allows to reduce probability mass for negative discrete wealth returns. The investment fractions of Table 4 show that the decrease in risk free bond for the portfolio with options is equivalent to the option investment.

Replacing the VaR constraint by the ES constraint, indicates that the stock investment is reduced for both cases with and without options, although not substantially. Enriching the ES restricted Kelly portfolios with put options has the same effect in terms of portfolio fractions as in the VaR restricted case (see Table 4) implying a higher geometric mean for the same ES constraint.

Extending the VaR constraint to the interval $0 \leq b \leq 1$, leads to a series of optimizations for all relevant quantile levels. $b = 0$ represents the risk free portfolio, whereas $b = 1$ implies that the investor can loose all of his fortune, given chosen confidence level.

Table 3 Discrete wealth return statistics (p.a. in %) for (restricted) Kelly portfolios with and without put options and Sharpe maximizing portfolio

Wealth Statistics	Kelly s.t. VaR $\leq$ 20%	Kelly s.t. VaR $\leq$ 20% with puts	Kelly s.t. ES $\leq$ 20%	Kelly s.t. ES $\leq$ 20% with puts	Kelly	Kelly with puts	Sharpe
Geometric mean	8.04	8.44	7.9	8.19	18.25	20.92	$-\infty$
Arithmetic mean	9.64	9.7	9.51	9.45	34.05	32.45	62.08
Standard deviation	17.35	14.45	17.51	14.73	58.42	48.1	124.7
Skewness	1.09	1.06	1.12	1.21	0.65	0.89	1.11
Kurtosis	4.48	5.05	4.53	5.29	3.66	4.77	4.65
Minimum (in %)	− 21.86	−21.93	− 20.66	−20.79	− 95.86	−87.39	− 100.00
$CTE_{0.5\%}$	− 20.9	−20.97	− 20.00	−20.00	− 89.23	−82.41	− 100.00
$Q_{0.5\%}$	− 20.00	−20.00	− 19.37	−19.29	− 84.72	−78.64	− 100.00
$Q_{1\%}$	− 19.11	−19.13	− 18.55	−18.39	− 78.87	−73.43	− 100.00
$Q_{10\%}$	− 9.65	−4.67	− 9.8	−4.54	− 35.1	−9.79	− 86.13
$Q_{50\%}$	6.59	7.39	6.13	6.6	27.17	21.79	40.41
$Q_{90\%}$	33.25	28.97	33.41	29.03	113.15	97.35	229.96
$Q_{99\%}$	64.22	55.94	64.76	57.99	198.52	179.02	453.77
Maximum	94.29	87.05	91.78	84.2	330.45	294.65	849.39

Table 4 Portfolio fractions (in %) for (restricted) Kelly portfolios with and without put options and Sharpe maximizing portfolio

Investment Fractions	Kelly s.t. VaR $\leq$ 20%	Kelly s.t. VaR $\leq$ 20% with puts	Kelly s.t. ES $\leq$ 20%	Kelly s.t. ES $\leq$ 20% with puts	Kelly	Kelly with puts	Sharpe
Apple	4.31	4.65	5.24	4.55	19.57	18.76	72.03
Adobe	1.64	3.73	2.16	2.71	10.91	12.15	30.54
Amgen	0.03	1.66	0.65	0.63	9.35	7.05	21.76
Amazon	12.16	10.57	12.43	11.55	24.84	26.30	61.16
Comcast	0.27	0.17	0	0	1.12	0.43	− 18
Costco	2.75	1.01	0.27	0.73	11.75	7.34	45.61
Cisco	0	0	0	0	0	0	− 53.12
Gilead	0.56	0.39	0.5	0.67	7.97	6.54	14.63
Intel	0.05	0	0	0	0	0	− 24.24
Microsoft	0.16	0	0	0	1.57	1.05	5.49
Nvidia	0.31	0.08	0.14	0.54	4.79	5.9	7
Pepsi	0.22	0	0	0	2.96	0	− 19.98
Qualcomm	0	0	0	0	0	0	− 51.19
Texas Instruments	0.11	0	0.01	0.34	1.05	1.84	8.32
Risk free	77.35	71.96	78.55	73.81	4.06	0	0
Put options	0	5.39	0	4.7	0	12.6	0

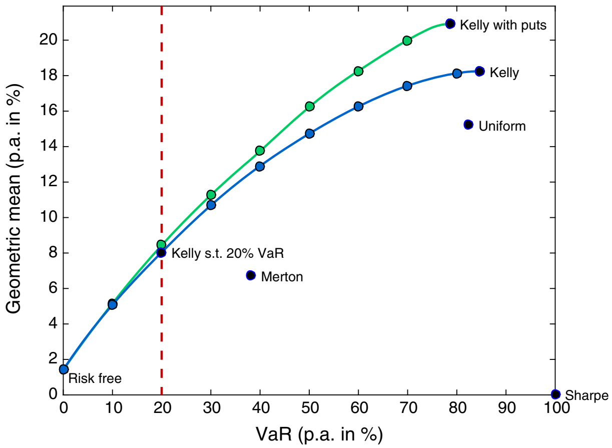

Fig. 8 Kelly-VaR (99.5%) frontier without (blue) and with (green) options, VaR constraint (red), benchmark portfolios for Uniform, Merton, Sharpe and Kelly (black). (Color figure online)

$$
\begin{aligned}
f^* = \underset{f^* \in \mathbb{R}^k}{\arg\max}\ & G_{\phi_{\mathrm{Elog}}}\{W_T(f)\} \\
\text{s.t. } & S_{\phi_{\mathrm{Q}_{0.5\%}}}\left\{1 - \frac{W_T(f)}{W_0}\right\} \le b,\ 0 \le b \le 1, \\
& \sum_{j=1}^{k} f_j \le 1.
\end{aligned} \tag{44}
$$

This series of restricted optimizations constitutes the Kelly-VaR frontier (Fig. 8), in which each point represents a growth-optimal portfolio given quantile (VaR) constraint. The portfolio with (without) options, which can lose at most 20% with 99.5% probability is the portfolio where the green (blue) frontier crosses the quantile constraint (red). Except for the risk-free portfolio, $0 < b \le 1$, every restricted portfolio with put options outperforms the portfolio without options in terms of the geometric mean. The unrestricted Kelly portfolio exhibits the highest geometric mean possible (18.25%), for a given VaR of 84.72%. Including put options into the unrestricted Kelly optimization increases the geometric mean (20.92%) and reduces the VaR to 78.64% at the same time. Relevant benchmark portfolios such as equally distributed (uniform),

$$
f_{\mathrm{Uniform}} = \mathbf{1}\frac{1}{k} \tag{45}
$$

the closed-form Merton solution under log-utility and Gaussianity (Merton 1992),

$$
f_{\mathrm{Merton}} = \Sigma^{-1}(\mu - \mathbf{1}r_f) \tag{46}
$$

and the Sharpe portfolio (Merton 1972),

$$f_{\text{Sharpe}} = \frac{\Sigma^{-1}(\mu - \mathbf{1}r_f)}{\mathbf{1}^\top \Sigma^{-1}(\mu - \mathbf{1}r_f)} \tag{47}$$

are not close to the Kelly-VaR frontier. Specifically the Sharpe investor, who allows for a larger risk constraint b than the unrestricted Kelly solution, should still invest into the growth-optimal portfolio, as the geometric mean of the unrestricted Kelly portfolio cannot be surpassed. Although the one period Sharpe maximizer obtains a larger arithmetic return, the Kelly (geometric mean) optimization rests on a multiperiod investment process. The multiperiod investor cannot sustain substantial draw-downs in one period as for the Sharpe portfolio. Given the α-stable process, see Table 4, the Sharpe investor goes bankrupt every one-hundred years.

5 Conclusion

Whereas the unrestricted Kelly portfolio ensures the asymptotic outperformance of the investor's wealth towards significantly different strategies, the presented model ensures growth-optimal investment subject to personal risk. The constrained optimization is formulated within the framework of spectral measures, inducing quantile (VaR) and Conditional tail expectation (expected shortfall) as special cases. In order to allow for an asymmetric wealth distribution, long put options are included into the optimization.

Financial market returns are with large probability non-Gaussian. Founding on the work of Mandelbrot (1963), it can be observed that the stability parameter α is significantly smaller than two, speaking in favor of the class of α-stable distributions. Given a chosen investment horizon, the distribution of financial market returns is modelled as the sum of hourly random variables. For α-stable laws with $\alpha < 2$, the variance of those random variables is infinite. Hence, the standard CLT does not apply and the generalized CLT of Gnedenko and Kolmogorov (1954) is applied. For the multidimensional estimation elliptical α-stable distributions, implying a linear dependence structure, are used. Instead of simulating from this class of distributions, a semiparametric scaling approximation is proposed. The resulting annual distribution, represented by convoluted hourly distributions, is simply an affine transformation of its normalized hourly analogue, given its scaling nature.

Heavy tails beyond the Gaussian, linear dependence between the marginals and nonsignificant skewness are empirically supported. Correspondingly, the joint distribution of financial market returns for a specified horizon is estimated by elliptical α-stable distributions utilizing a semiparametric scaling approximation. The portfolio model is implemented for a representative investor with quantile (VaR) constraint. The resulting growth-optimum strategy maximizes the geometric mean, given his risk constraint. Including put options into the optimization levers the portfolio by a suitable protective put strategy, leading to an increased geometric mean for the same risk. For the Kelly-quantile frontier, except for the risk-free portfolio, every restricted portfolio with options outperforms the portfolio without options in terms of the geometric mean.

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