# 15

# The Statistics of the Maximum Drawdown in Financial Time Series

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## 15.1 Introduction

The maximum drawdown (MDD) in financial time series plays an important role in investment management and has been widely studied in the literature. MDD is associated with standards of performance measures such as Calmar or Sterling ratios. Various forms of portfolio optimization based on MDD have been considered (see for example A. Chekhlov and Zabarankin (2005)). In addition, Leal and de Melo Mendes (2005) have proposed a coherent risk measure possessing the properties required by Artzner et al., (1999) similar to the conditional value-at-risk: the maximum drawdown-at-risk $\mathrm{MDaR}_{\alpha}$, which is just a quantile with exceedance probability $\alpha$ of the distribution of the maximum drawdown. Despite the widespread use of maximum drawdown among practitioners, financial economists have not paid much attention to this concept. It provides an alternative or complement to the other commonly used risk measures such as value-at-risk, which is still used extensively by the industry and regulatory standards for the calculation of risk capital in banking and insurance despite its well-known shortcomings. The evaluation of both MDD's expectation value and its probability density function (pdf) is of importance for various practical applications, especially when building a robust framework for risk management and capital allocation. This chapter is motivated by the need to gain insights into the statistical properties of the MDD for stochastic processes that, set aside from the academic example of the Brownian motion, are possibly closer to the stylized facts that characterize the real financial time series (see Rosario N. Mantegna, 2000; Cont, 2001; Bouchaud and Potters, 2003).

To our knowledge, analytical expressions of the distribution function of the MDD and of its expectation value have been derived in the limit of a Brownian motion with drift in Ismail et al. (2004a,b); however, these formula involve an infinite sum of integrals that requires a numerical estimate. Alternatively, a method based on the numerical solution of a PDE for



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computing the expected MDD in the Black-Scholes framework has been proposed in Pospisil and Vecer (2009). Here, a Monte Carlo code is proposed as a versatile tool to explore the higher-order statistics available through the pdf of the MDD in cases that are hardly analytically treatable. In particular, some relevant deviations with respect to the Brownian motion model are analyzed in order to quantify their impact on the statistics of the MDD: the increments of the underlying stochastic process (returns/log-returns) are not independent and present excess kurtosis (heavy tailed) and/or non zero skewness. The chapter is organized as follows. Section 15.2 examines arithmetic processes driven by independent and identically distributed (iid) random variables. The case of Brownian motion with drift provides the framework where the Monte Carlo simulations are presented and validated against the analytical predictions. Section 15.3 analyzes the MDD statistics for stochastic processes following a geometric model with iid increments. Section 15.4 removes the iid ansatz: a general autoregressive model is introduced for the conditional mean and volatility of the innovations. The statistics of the MDD are numerically evaluated when varying the parameters that control the correlation, the symmetry and the tails of the increments distribution. In Section 15.5, a parametric study is performed on the expectation value of the MDD as a function of a number of parameters of practical interest. Section 15.6 presents a comparison between the predictions of the MDD statistics and the historical time series for different asset classes. Finally, Section 15.7 reports the conclusions of this analysis.

## 15.2 Brownian motion with drift

The value of a portfolio $S_t$ is here assumed to follow the SDE

$$dS_t = \mu dt + \sigma dW_t \tag{15.1}$$

where $W_t$ is a Wiener process, $0 \le t \le T$ and both the mean $\mu$ and the volatility $\sigma$ are finite and constant. The MDD is defined as

$$\mathrm{MDD}_T = \max_{0\le t\le T} D_t \qquad \text{with} \qquad D_t = \max_{0\le x\le t} S_x - S_t, \tag{15.2}$$

$D_t$ is the drawdown from the previous maximum value at time $t$. The behavior of the statistics of the MDD in this framework has been studied in Ismail et al. (2004a). An infinite series representation of its distribution function is derived in the case of zero, negative and positive drift, and formulas for the expectation values for the MDD are given. In the limit of $T \to \infty$, it is found that the $T$ dependence of $E(\mathrm{MDD})$ is logarithmic for $\mu > 0$, square root for $\mu = 0$ and linear for $\mu < 0$. Nevertheless, explicit analytical relations cannot be immediately derived: it is required to numerically evaluate special integral functions in the case of $E(\text{MDD})$, while the infinite series representation of the MDD distribution function involves the solution of particular eigenvalue conditions.

Here we prefer to employ a numerical simulation in order to compute the pdf of the MDD distribution function using a Monte Carlo scheme. One of the main advantages of this approach is the possibility to extend this method to different models for the underlying stochastic process. The Monte Carlo code simulates the discretized version of equation (15.1) using the Euler scheme

$$S_{i+1} = S_i + \mu\Delta t + \sigma\sqrt{\Delta t}X_i, \tag{15.3}$$

where $X_i$ are iid pseudo-random variables with normal distribution $\mathcal{N}(0,1)$, that is satisfying $E(X_i) = 0$ and $Var(X_i) = 1$; $\Delta t = T/N$, where $T$ is the time in years and $N$ is the number of time steps in the series. Here and in the remainder of this chapter the average return $\mu$ and volatility $\sigma$ are expressed as annualized values. The code is employed to compute the MDD defined by equation (15.2). In order to verify the goodness of this numerical approach, the code is tested against the analytical predictions of the expectation value of the MDD derived in Ismail et al. (2004a). One of the goals of this verification is to reproduce the transition from the logarithmic to the square root and the linear scaling for $T \to \infty$ when changing the drift from $\mu > 0$ to $\mu = 0$ and $\mu < 0$ respectively.

The parameters of this test case are: $\mu = -0.1,\ 0, +0.1,\ \sigma = 0.2$ and $T = 1$ d, 1 m, 6 m, 1 y, 5y, 10 y, 30 y (1 year is considered as being composed of 260 working days). The series that are simulated contain $N = 2 \cdot 10^4$ time steps and each simulation considers the statistics of $M = 4 \cdot 10^4$ samples. The estimator of the expectation value of the MDD is the mean of the MDD samples generated by the code. Figure 15.1 summarizes the numerical estimates in comparison with the analytical predictions derived in Ismail et al. (2004a). The simulations are able to reproduce the analytical expectations for $\mu < 0$, $\mu = 0$ and $\mu > 0$ very well, with a relative error on $E(\text{MDD})$ which always stays below 1 per cent. Recalling the definition of equation (15.2), the MDD is derived from the drawdown process $D_t$. The latter is itself a Brownian motion reflected in its maximum; a rigorous justification of this statement can be found in Karatzas and Shreve (1997, p. 210) and Graversen and Shiryaev (2000). Hence, if one is interested in the distribution function of the maxima of $D_t$ (the MDD), the extreme value theory (see Reiss and Thomas, 2007; Coles, 2004) provides some powerful insights. Thanks to the Fisher-Tippett theorem, it is expected that the distribution function of the MDD converges to a generalized extreme value distribution (GEV), whose

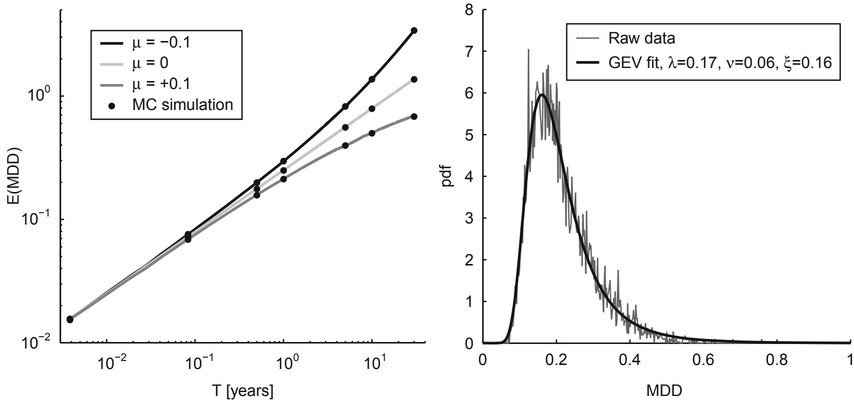


*Figure 15.1* Numerical estimates of MDD

*Notes*: On the left: $E(\text{MDD})$ vs T, analytical expectations compared with the numerical results from the Monte Carlo simulations. On the right: pdf of the MDD obtained by the Monte Carlo simulation fitted to a GEV distribution ($\mu = 0.1$, $\sigma = 0.2$, $T = 1$ y).

density is

$$H_\xi(x) = \begin{cases} \dfrac{1}{\nu}\left(1+\xi\dfrac{x-\lambda}{\nu}\right)^{-\frac{1+\xi}{\xi}} \exp\left[-\left(1+\xi\dfrac{x-\lambda}{\nu}\right)^{-\frac{1}{\xi}}\right] & \xi \neq 0 \\ \dfrac{1}{\nu}e^{-\frac{x-\lambda}{\nu}} \exp\left[-\exp\left(-\dfrac{x-\lambda}{\nu}\right)\right] & \xi = 0 \end{cases} \tag{15.4}$$

where $1+\xi\frac{x-\lambda}{\nu} > 0$; $\xi$ is the shape parameter that controls the tail behavior of the distribution. For completeness, the first two moments of the GEV density are:

$$E(x) = \lambda + \frac{\nu}{\xi}(g_1 - 1) \qquad Var(x) = \frac{\nu^2}{\xi^2}\left(g_2 - g_1^2\right) \tag{15.5}$$

where $g_k = \Gamma(1-k\xi)$, $\Gamma(x)$ being the Gamma function.

The expectation of recovering the GEV density is well verified by the Monte Carlo simulations when computing the pdf of the MDD samples; the raw numerical results can in fact be fitted to the analytical expression (15.4). A nonlinear least squares fitting method is applied: $\lambda, \nu, \xi$ are the free parameters that are computed by the regression, obtaining a typical coefficient of determination $R^2 > 0.96$ for all the cases presented in this chapter. An example of the application of this procedure is shown in Figure 15.1.

A good test for the code is also to sample the iid random variables (rvs) $X_i$ appearing in equation (15.3) from a non normal distribution. We still require that $E(X_i) = 0$ and $Var(X_i) = 1$, but their skewness and kurtosis can take arbitrary values. In the continuous limit of $N \to \infty$ ($t = N\Delta t$) and in the case of zero drift $\mu = 0$, the P. Lévy's martingale characterization of Brownian motion theorem (1997, p. 156) establishes that $S_t$ will still be a Brownian motion. In the more general case of $\mu \neq 0$, as suggested by the central limit theorem, the process $S_t$ will be a semimartingale, that is again a Brownian motion with drift with marginal distribution $\mathcal{N}\left(\mu t, \sigma^2 t\right)$. The skewness and the kurtosis are here defined as:

$$\text{Skewness}(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] \qquad \text{Kurtosis}(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3. \tag{15.6}$$

On the other hand, the drawdown process $D_n = \max_{0 \le j \le n} S_j - S_n$ is a reflected random walk whose marginal distribution is the same as that of $-\min_{0 \le j \le n} S_j$. This result, as well other theorems for this kind of reflected process, is formally treated in Doney et al. (2009); Karatzas and Shreve (1997, p. 210, 418) discusses in detail the reflected Brownian motion. As $S_j$ is a discrete Brownian motion with drift, its marginal distribution is $\mathcal{N}\left(\mu j \Delta t, \sigma^2 j \Delta t\right)$. Therefore, provided that $X_i$ are iid rvs with finite mean and variance and regardless of their moments higher than the second, the MDD for the arithmetic process described by equation (15.3) on the time horizon $T$ converges in the continuous limit of $N \to \infty$ to the statistics obtained in the case of the Brownian motion with drift, in terms of both density and expectation value.

Within the Monte Carlo approach already introduced, the iid pseudo-random variables $X_i$ of equation (15.3) can be sampled from a distribution whose third and fourth moments are non vanishing; the coefficients $\mu$ and $\sigma$ and the first two moments are instead not modified. The code generates the iid sequence $X_i$ from a Pearson type IV distribution satisfying the prescriptions $E\left(X_i\right) = 0$, $Var\left(X_i\right) = 1$, Skewness $\left(X_i\right) = s$ and Kurtosis $\left(X_i\right) = k$, where $s$ and $k$ can be arbitrarily prescribed. Some brief details around the Pearson family of distributions are reported in Appendix.

As expected, despite varying the skewness and the kurtosis of the underlying iid random variables, the statistics of the MDD computed by the Monte Carlo simulations systematically recovers the case of the Brownian motion with drift for a given set of $\mu$, $\sigma$ and $T$.

## 15.3 Geometric process driven by iid increments

It is well known that, instead of the arithmetic model treated in the previous section, geometric processes are more appropriate to describe the financial time series. In terms of SDE, equation (15.1) will then be modified in the following way:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{15.7}$$

The latter equation describes a geometric Brownian motion. The natural extension of the definition of the maximum drawdown for a geometric model takes the form:

$$\mathrm{MDD}_T = \max_{0\leq t\leq T} D_t \qquad \text{with} \qquad D_t = \frac{1}{\max_{0\leq x\leq t} S_x}\left(\max_{0\leq x\leq t} S_x - S_t\right). \tag{15.8}$$

The MDD is then the maximum loss relative to the previous peak; consequently, both the relations $0 \leq D_t < 1$ and $0 \leq \mathrm{MDD}_T < 1$ hold.

Itô's calculus is a powerful instrument to handle the dynamics of equation (15.7); it is in fact possible to write $S_t = \exp(\tilde{X}_t)$, along with the SDE:

$$d\tilde{X}_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \tag{15.9}$$

Thanks to this result, the geometric Brownian motion (15.7) can be effectively treated through an arithmetic model for the process $\tilde{X}_t$ with an adjusted drift $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2$. For the same reason, the analytical results of Ismail et al. (2004a) concerning the expectation value and the distribution function of the MDD, can be immediately extended to the case of a geometric Brownian motion according to the new definition of equation (15.8).

At this point it is interesting to consider

$$\frac{dS_t}{S_t} = \mu dt + \sigma dP_t \tag{15.10}$$

where $P_t$ is a process formed by the sum of iid random variables sampled from a non normal distribution, with the assumptions $E(P_t) = 0$ and $Var(P_t) = 1$, but with arbitrary skewness and kurtosis. In the light of the previously cited Lévy's martingale characterization theorem, $P_t$ is a Brownian motion (it is however important to note that it is not possible to derive a direct generalization of Itô's formula for any arbitrary distribution of the log-increments; this point is treated in detail in Kleinert (2009), where it is shown that only a weaker formula for the expectation value of such a process can be obtained). Itô's formula applies here if we can still write $S_t = f(\tilde{X}_t)$, where $f$ is a $C^2$ function and the process $\tilde{X}_t$ is a semimartingale (Karatzas and Shreve (1997, p. 149). It is easy to prove that the choice $f(x) = \exp(x)$ and

$$d\tilde{X}_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dP_t \tag{15.11}$$

fulfills these requirements; $\exp(x)$ is in fact a $C^2$ function and the process $\tilde{X}_t$ is a semimartingale, since it can be decomposed into a local martingale and a drift term of finite variation. Therefore, Itô's calculus prescribes that $S_t$ is a semimartingale too and that the statistical dynamics of equation (15.10) is equivalent to the SDE (15.11), with $S_t = \exp(\tilde{X}_t)$. Consequently if the process $P_t$ is driven by iid increments sampled from a non normal distribution satisfying the previous requirements, we can expect that the Euler discrete version of equation (15.10) will recover the discretization of equation (15.11) with $S_t = \exp(\tilde{X}_t)$ when using a big enough number of discretization points $N$.

The latter statement has been numerically verified, evaluating the relative error between the two different approaches as a function of $N$. In particular, the simulations of a geometric process have been performed in two different ways: $S_i^{\text{direct}}$ through the direct discretization of (15.10) and $S_i^{\text{Ito}}$ obtained by discretizing equation (15.11) using Itô's formula, where $i = 1 \dots N$. In both cases the iid random variables are sampled from a Pearson type IV distribution with $\mu = 0.1$, $\sigma = 0.2$, Skewness $= -1.5$ and Kurtosis $= 7$. The normalized relative error defined as $\text{Error} = \sqrt{\sum_{i=1}^{N} \left(S_i^{\text{Ito}} - S_i^{\text{direct}}\right)^2} / \sqrt{\sum_{i=1}^{N} \left(S_i^{\text{direct}}\right)^2}$ is finally found to follow the law: $\text{Error} \propto 1/\sqrt{N}$.

As there is a formal proof that in the continuous limit and under the previous hypotheses the geometric process (15.10) is equivalent to the arithmetic one (15.11), the same argument as in the previous section can be applied for inferring the statistics of the MDD. In particular, the moments higher than the second of the iid rvs that compose the Brownian motion $P_t$ do not affect the statistics of the MDD defined in (15.8) for the geometric process (15.10).

This statement can also be recovered by the Monte Carlo simulations. The discrete version of equation (15.10) used here is the Euler scheme

$$S_{i+1} = S_i \left(1 + \mu \Delta t + \sigma \sqrt{\Delta t} X_i\right). \tag{15.12}$$

As before, $X_i$ is a sequence of iid pseudo-random variables from a Pearson type IV distribution whose first two moments satisfy $E(X_i) = 0$ and $Var(X_i) = 1$ ($\Delta t = T/N$). The case of sampling from the standard normal distribution $\mathcal{N}(0,1)$ is recovered when all the higher moments are vanishing.

The pdf of the MDD for the geometric model (15.12) has been computed exploring the following parametric space: $\mu = [-0.1, 0, +0.1]$, $\sigma = [0.1, 0.2]$, $s = [-1.3, 0]$, $k = [0, 5]$, $T = [1\text{ d}, 1\text{ y}, 10\text{ y}]$. As expected, for any set of $\mu$, $\sigma$ and $T$, no difference has been found when changing the skewness and/or the kurtosis of the underlying $X_i$. Obviously, as the whole pdf of the MDD does not vary, $E(\text{MDD})$ remains the same.

## 15.4 Autoregressive process

In order to get a deviation from the MDD statistics of the Brownian motion with drift it is necessary to relax the strong hypothesis of iid rvs for the increments (log-returns). In this section this is done using an autoregressive model Bollerslev (1986) for the time series. In particular we consider the following AR(1)-GARCH(1,1) model:

$$Y_i = \mu_i + \epsilon_i \tag{15.13}$$

$$\epsilon_i = \sigma_i Z_i \tag{15.14}$$

$$Z_i \equiv \text{iid rvs} \qquad \text{with} \qquad E\left(Z_i\right) = 0 \qquad Var\left(Z_i\right) = 1 \tag{15.15}$$

for $i = 0, 1, 2, \ldots, N$ $(\Delta t = T/N)$, together with the prescriptions for the conditional mean and volatility

$$\mu_i = c + \phi\left(Y_{i-1} - c\right) \tag{15.16}$$

$$\sigma_i^2 = \alpha_0 + \alpha_1 \epsilon_{i-1}^2 + \beta_1 \sigma_{i-1}^2 \tag{15.17}$$

with $\alpha_0, \alpha_1, \beta_1 > 0$, $\alpha_1 + \beta_1 < 1$ and $|\phi| < 1$. The moments of $Z_i$ higher than the second are explicitly not fixed here, since the skewness and the kurtosis will be varied in the following. The model is defined by the five parameters $c, \phi, \alpha_0, \alpha_1, \beta_1$.

The AR(1)-GARCH(1,1) series $Y_i$ is considered as a sequence of log-returns; therefore, to reconstruct the portfolio value trajectory we use the discrete geometric model

$$S_{i+1} = S_i\left(1 + Y_i\right). \tag{15.18}$$

There is a clear analogy between the latter formulation and the previous geometric model (15.12) driven by iid innovations. To make a direct parallelism between them it is useful to preserve the unconditional mean $\mu$ and volatility $\sigma$ over the time interval $T = N\Delta t$; this is done by fixing the parameters $\alpha_0$ and $c$ of the AR(1)-GARCH(1,1) series according to

$$\alpha_0 = \sigma^2 \Delta t \left(1 - \alpha_1 - \beta_1\right) \tag{15.19}$$

$$c = \mu \Delta t. \tag{15.20}$$

Therefore, for a given set of unconditional $\mu$ and $\sigma$ on the time horizon $T$, the prescriptions (15.19)–(15.20) guarantee that the autoregressive model (equations (15.13)–(15.18)) is completely specified by the three coefficients $\alpha_1, \beta_1, \phi$ and comparable with the iid innovations driven model (15.12). The goal is in fact to evaluate how the MDD distribution is affected passing from one model to the other one. Ultimately, if $Z_i$ has the same distribution of $X_i$

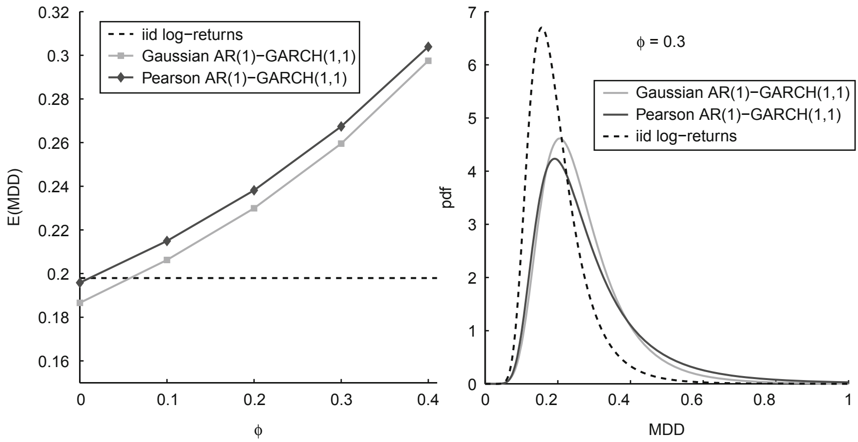


*Figure 15.2* Unconditional $\mu = 0.1$, $\sigma = 0.2$, $T = 1$ y

*Notes*: Left plot: $E(\text{MDD})$ vs. $\phi$ computed by the Monte Carlo code considering the AR(1)-GARCH(1,1) model and sampling the iid $Z_i$ from $\mathcal{N}(0,1)$ or from a Pearson type IV distribution with $s = -1$ and $k = 0$. Right plot: comparison of the MDD pdf fitted with a GEV at fixed $\phi = 0.3$.

appearing in equation (15.12) and $\alpha_1 = \beta_1 = \phi = 0$, the two models describe exactly the same statistical dynamics.

This autocorrelated geometric process is used to study the statistics of the MDD when changing the coefficient $\phi$ that controls the serial correlation of the log-returns. In this case we have fixed the following parameters: $\alpha_1 = 0.0897$ and $\beta_1 = 0.9061$ (these values come from the best fit of the daily log-returns of the S&P 500 between 16 May 1995 and 29 April 2003 to a GARCH(1,1) model Starica (2004)), the unconditional mean $\mu = 0.1$, the unconditional standard deviation $\sigma = 0.2$ and the time horizon $T = 1$ y. The Monte Carlo simulations are run sampling the iid $Z_i$ either from $\mathcal{N}(0,1)$ or from a Pearson type IV distribution with $s = -1$ and $k = 0$. The kurtosis of the Pearson distribution has not been increased since the previous parameters of the AR(1)-GARCH(1,1) model already imply a high kurtosis; in particular the skewness implied by the model for the series of log-returns is 0 in the first case and $\approx -1.7$ in the second, while the kurtosis is $\approx 12$ in both cases. These results are summarized in Figure 15.2. The serial correlation has quite a strong impact on the expected MDD, while it appears that the skewness added in the series of log-returns increases the $E(\text{MDD})$ at maximum by 5 percent with respect to the Gaussian AR(1)-GARCH(1,1) expectation. The analysis of this kind of autoregressive model still deserves further attention in the future.

## 15.5 Parametric studies

Another possible application of the Monte Carlo simulations presented in the previous sections is to study how the statistics of the MDD are related to a number of parameters of interest. Here the following dependencies are explored when dealing either with iid or autocorrelated log-returns:

1. $\frac{\mathrm{E(MDD)}}{\sigma}$ vs $T$;
2. $\frac{\mathrm{E(MDD)}}{\sigma}$ vs $\frac{\mu}{\sigma}$, $\mu/\sigma$ being the Sharpe ratio where the risk-free rate is set to 0;
3. $\frac{\mu T}{\mathrm{E(MDD)}}$ vs $\frac{\mu}{\sigma}$, (the Calmar ratio vs. the Sharpe ratio).

The first case is summarized in the left plot of Figure 15.3, considering the time horizons $T = 1$ d, 1 w, 1 m, 6 m, 1 y. We use the unconditional mean and volatility $\mu = 0.1$ and $\sigma = 0.2$. The AR(1)-GARCH(1,1) model of the previous section is applied to generate skewed log-returns ($Z_i$ are sampled from a Pearson distribution with $s = -1$); $\alpha_1 = 0.0897$ and $\beta_1 = 0.9061$ as before, and $\phi = 0.3$.

The results of the dependence of $E(\mathrm{MDD})/\sigma$ on the Sharpe ratio are reported in the central plot of Figure 15.3. The time horizon is $T = 1$ y and the change in the Sharpe ratio is obtained fixing the unconditional $\sigma = 0.15$ while varying the unconditional mean of the log-returns. The parameters for the AR(1)-GARCH(1,1) are the same as before. As is clear from the figure, the Monte Carlo simulations find that the expected MDD for the autocorrelated process decays more slowly with $\mu/\sigma$ with respect to the case of iid innovations.

Finally, the right plot of Figure 15.3, refers to the dependence of the Calmar ratio on the Sharpe ratio for the same parameters of the previous figure. The picture highlights how the risk adjusted returns are lower when the underlying series of log-returns presents positive autocorrelation and negative skewness.

Coming back to the basic hypothesis of iid log-returns with finite and constant mean and volatility described in Section 15.3, despite the simplicity of such a framework it is still of practical interest to derive a heuristic approximation for the behavior of $E(\mathrm{MDD})$ as a function of $\mu$, $\sigma$ and $T$. A simple approach could be to assume that this function can be written in terms of a power law,

$$E(\mathrm{MDD}) = f(\mu, \sigma, T) \approx C_0 \mu^{C_1} \sigma^{C_2} T^{C_3}. \tag{15.21}$$

In this framework the function $f(\mu, \sigma, T)$ has an analytical expression Ismail et al. (2004a), but the actual value is obtained through numerically computed functions. On the other hand it is possible to apply a multivariate linear regression to fit the expected MDD to the power law equation (15.21).

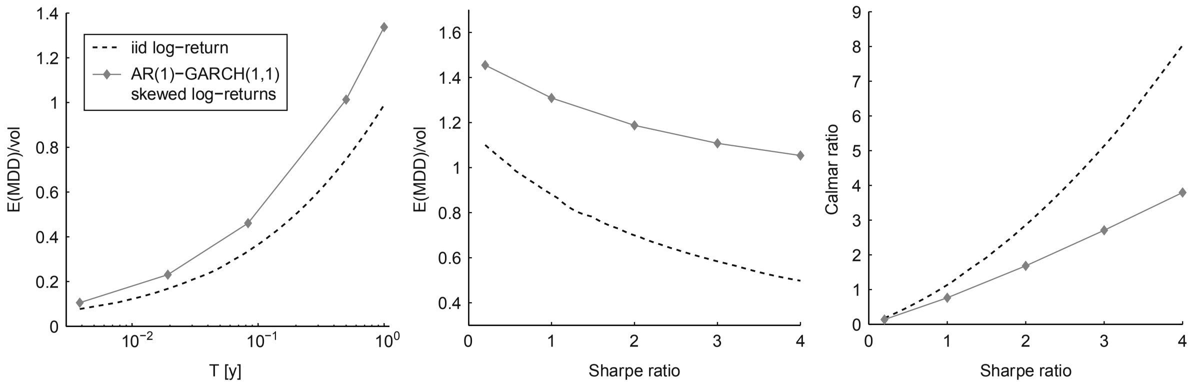


*Figure 15.3* MDD related performance figures obtained when considering the underlying log-returns either as iid or as a AR(1)-GARCH(1,1) series with non zero skewness.

*Notes:* The unconditional $\mu = 0.1$ and $\sigma(\text{vol}) = 0.2$ are kept fixed in both cases; the AR(1)-GARCH(1,1) parameters are $\alpha_1 = 0.0897$, $\beta_1 = 0.9061$, $\phi = 0.3$. On the left: $E(\text{MDD})/\sigma$ vs. $T$. In the center: $E(\text{MDD})/\sigma$ vs. the Sharpe ratio $\mu/\sigma$ at $T = 1$ y. On the right: Calmar ratio $\mu T/E(\text{MDD})$ vs. the Sharpe ratio $\mu/\sigma$ at $T = 1$ y.

Here we focus the attention on the regime of $\mu > 0$. Using the parametric space $[0.02 < \mu < 0.1]$, $[0.05 < \sigma < 0.2]$, $[1\ \mathrm{d} < T < 6\ \mathrm{m}]$ a linear regression has been performed on the analytical predictions, obtaining the following estimates for the coefficients of equation (15.21):

$$C_0 = 1.0305 \qquad c_1 = -0.0431 \qquad c_2 = 1.0416 \qquad c_3 = 0.4704. \tag{15.22}$$

The mean of the error between the power law with coefficients (15.22) and the analytical results is $3.75 \cdot 10^{-4}$ while the error standard deviation is 0.028. In principle a similar approach can be applied for the case of autoregressive processes, but this would require the creation of a large database of results computed by the Monte Carlo simulations.

## 15.6 Comparison against historical financial time series

For the purposes of the comparison against the historical financial data, the model for the underlying time series $S_t$ is the following GARCH(1,1) model:

$$S_{t+1} = S_t(1 + Y_t) \tag{15.23}$$

$$Y_t = k + \epsilon_t \tag{15.24}$$

$$\epsilon_t = \sigma_t Z_t \tag{15.25}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{15.26}$$

where $Z_t$ are iid random variables sampled from a Student's T distribution with $\nu$ degrees of freedom and unitary variance; the drift $k$ is a constant, $\alpha_0, \alpha_1, \beta_1 > 0$ and $\alpha_1 + \beta_1 < 1$. With respect to the model previously described in Section 15.4 by equations (15.13)–(15.18), here we consider a vanishing autocorrelation of the increments and a less generic distribution for the innovations (Student's T instead of Pearson type IV); despite this simplification the latter model is able to fit the historical series of daily returns analyzed here reasonably well, as will be shown in the following.

The Monte Carlo code is used to evaluate the distribution function of the maximum drawdown $\mathrm{MDD_T}$ on the time horizon $T$. The simulations presented in this section consider the statistics of $M = 1 \cdot 10^6$ MDD samples, each one obtained from a time series composed by $N = 50 \cdot 10^3$ steps. The numerical predictions are compared against six historical financial time series (the data source is Yahoo!Finance), namely the S&P 500 index, CAC40 index, Credit Suisse stock, IBM stock, US Dollar/Japanese Yen spot, US treasury 30-years yield. The analysis is composed by the following steps.

1. The historical data at daily frequency have been considered on a eight year period, namely from 1 January 2002 to 31 December 2009 (nearly 2000 points). The log-returns of the closing prices are used to perform

*Table 15.1* Parameters obtained from a maximum likelihood fit to the GARCH(1,1) model with student's T innovations described by equations (15.23)–(15.26).

| Instrument | $k$ | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\nu$ | LLK |
|---|---|---|---|---|---|---|
| S&P500 | 4.89E-4 | 5.46E-7 | 0.0676 | 0.9298 | 9.90 | 6365.83 |
| CAC40 | 6.49E-4 | 1.33E-6 | 0.0858 | 0.9091 | 11.97 | 6110.21 |
| CS | 6.95E-4 | 3.17E-6 | 0.0833 | 0.9151 | 6.85 | 4896.05 |
| IBM | 2.76E-4 | 1.65E-6 | 0.0591 | 0.9341 | 5.96 | 5828.22 |
| USD/JPY | -1.29E-5 | 3.83E-7 | 0.0389 | 0.9534 | 6.14 | 7543.11 |
| US 30y | -1.86E-4 | 5.83E-7 | 0.0404 | 0.9561 | 12.79 | 6204.93 |

*Notes*: LLK is the log-likelihood value obtained from the fit.

a maximum likelihood fit to a GARCH(1,1) model with Student's T distributed innovations (see the above equations).

2. The parameters computed by the best fit are used to run the Monte Carlo code. The simulation evaluates the statistics of the $MDD_T$, providing in particular the expectation value, 95 percent and 99 percent confidence intervals for the maximum drawdown.
3. *In-sample* test: The code results are compared to the historical $MDD_T$ (the latter is calculated using high and low intra-day prices) observed within the time interval used for the fit of point 1. Four reference time horizons are considered: 10 days, 40 days, 1 year and 8 years.
4. *Out-of-sample* test: Using the same fit parameters obtained at point 1, the code is tested against the historical MDD in the period between 1 Jan 2010 and 15 Mar 2010. The time horizons considered here are: 10, 20 and 40 days.

Table 15.1 summarizes the results from the maximum likelihood fit to a GARCH(1,1) model with Student's T innovations on the previous time series.

The plots of Figure 15.4 firstly report two examples of the *in-sample* comparison between the numerical expectations and the historical maximum drawdown observed between 1 Jan 2002 and 31 Dec 2009. Secondly, using the parameters of Table 15.1 obtained on the training dataset, the *out-of-sample* estimates of the Monte Carlo code are presented for the period between 1 Jan 2010 and 15 Mar 2010.

In conclusion, the Monte Carlo code has been used to evaluate a total number of 42 MDD observations (24 in the 1 Jan 2002 to 31 Dec 2009 period and 18 in the 1 Jan 2010 to 15 Mar 2010 period). Hence the expected number of violations from the model is 2.1 when working with 95 percent CI, and 0.42 when working with 99 percent CI. The present analysis reported three violations from the 95 percent CI and no violations from the 99 per cent CI.

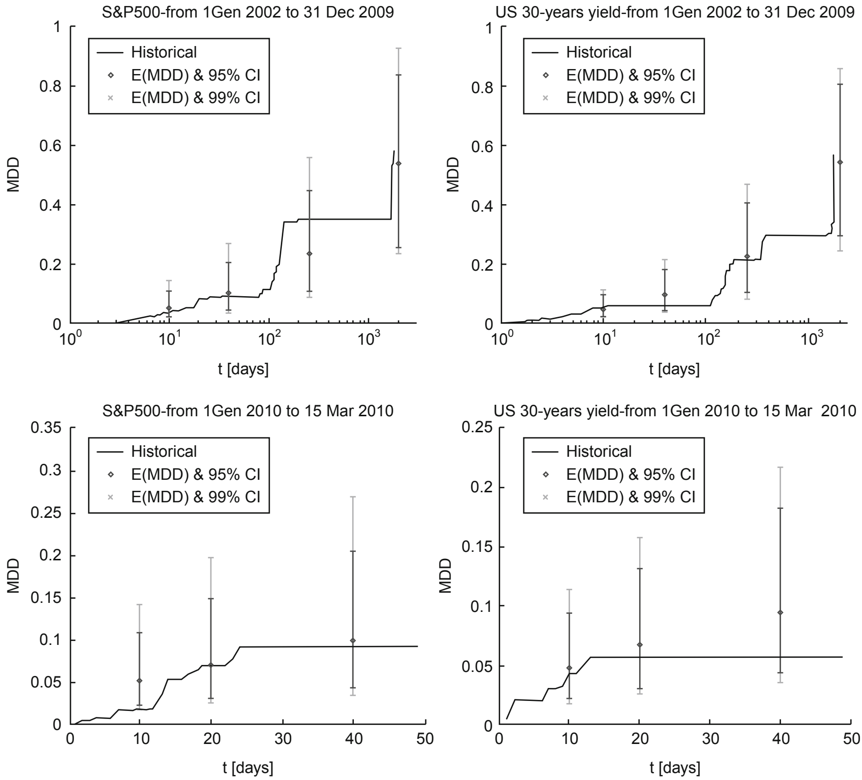


*Figure 15.4* On the top: time evolution of the MDD of the historical S&P 500 and US 30-years yield series between 1 Jan 2002 and 31 Dec 2009 compared with the *in-sample* expectation values and their confidence intervals computed by the Monte Carlo code. On the bottom: time evolution of the MDD of the historical S&P 500 and US 30-years yield series between 1 Jan 2010 and 15 Mar 2010 compared with the *out-of-sample* expectation value and their confidence intervals computed by the Monte Carlo code

## 15.7 Conclusions

The analytical predictions for the expectation value of the maximum drawdown in the case of a Brownian motion with drift are very well recovered by the Monte Carlo code presented here. The numerical simulations also allow us to consistently recover the MDD distribution function in terms of a generalized extreme value distribution.

In the continuous limit of the discretized version of a geometric stochastic process, the MDD statistics are shown to be unaffected by the moments higher than the second that characterize the distribution of the underlying increments, provided that they are iid. This result is consistent with the expectations from the central limit theorem and Itô's formula, and it is recovered by the Monte Carlo code.

When relaxing the iid constraint for the increments of the process and considering an autoregressive AR(1)-GARCH(1,1) model, the MDD statistics are strongly affected; in the cases examined here, the expectation value of the MDD increases roughly linearly with the serial correlation coefficient. In the case of both the geometric Brownian motion and the autoregressive stochastic process, some parametric studies have been performed in order to derive the heuristic behavior of the MDD expected value as a function of time, mean return, volatility and Sharpe ratio.

Finally, under the hypothesis of a GARCH(1,1) process driven by Student's T distributed innovations, the MDD statistics predicted by the Monte Carlo code has been compared to the events observed in different historical financial time series. The relevance of this comparison relies not only on the MDD expectation value but most of all on its confidence interval (at 95 per cent and 99 per cent), available through the MDD distribution function derived from the simulation. Both the *in-sample* and the *out-of-sample* tests show a good match between the numerical expectations and the historical data.

## Appendix: Some Details on the Pearson Distributions

The Pearson family of distributions Pearson (1895) is particularly well suited to generating random variables with arbitrary skewness and kurtosis. Seven distribution types are in fact encompassed in the Pearson system; particular members of that family are normal, beta, gamma, Student's T, F, inverse Gaussian and Pareto distributions. The general probability density function $p(x)$ of a Pearson distribution is defined through the differential equation

$$\frac{d\log p(x)}{dx} = \frac{x-\alpha}{a_0 + a_1 x + a_2 x^2}. \tag{A1}$$

The normal distribution is for example recovered for $a_1 = a_2 = 0$.

For our purposes, the Pearson type IV distribution is of particular importance and some of its properties will be briefly recalled here. When $a_1 \neq 0$ and $a_2 \neq 0$, the roots of the denominator of equation (A1) are in general the complex quantities $b \pm ia$. The density of the Pearson type IV distribution can be written as

$$p(x) \propto \left(1 + \frac{x^2}{a^2}\right)^{-m} \exp\left[\delta \arctan\left(\frac{x}{a}\right)\right] \tag{A2}$$

where $m = -\frac{1}{2}a_2$ and $\delta = \left(b - a\right)/aa_2$. Equation (A2) is an asymmetric leptokurtic density function. Denoting the skewness by $s$ and the kurtosis by $k$

(in the context of this appendix only, the kurtosis is defined as the fourth standardized moment, that is $k = 3$ for the normal distribution), the following relations between the parameters of the Pearson type IV distribution (A2) and $s$ and $k$ can be derived:

$$r = \left[ \frac{6\left(k - s^2 - 1\right)}{2k - 3s^2 - 6} \right] \qquad \text{with} \qquad r = 2m - 2 \tag{A3}$$

$$\delta = \frac{r\,(r-2)\,s}{\sqrt{16\,(r-1) - s^2\,(r-2)^2}} \tag{A4}$$

$$a = \sqrt{\frac{\mu_2}{16}\left[16\,(r-1) - s^2\,(r-2)^2\right]} \tag{A5}$$

where $\mu_2$ is the second central moment. The parameters $\alpha, a_0, a_1, a_2$ appearing in the differential equation (A1) can then be determined by fixing the first four moments of the distribution.

The Pearson family of distributions is hence used in this chapter in order to generate a sequence of iid random variables given a set of mean $\mu$, standard deviation $\sigma$, skewness $s$ and kurtosis $k$ for the underlying density.

## References


P. Artzner, F. Delbaen, J. E. and Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance*, 9: 203–28.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31: 307–27.

Bouchaud, J.-P. and Potters, M. (2003). *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management*. (Cambridge University Press: Cambridge, 2nd edition).

A. Chekholov, S. Uryasev, S. U. and Zabarankin, M. (2005). Drawdown Measure in Portfolio Optimization. *International Journal of Theoretical and Applied Finance*, 8: 1–46.

Coles, S. (2004). *An Introduction to Statistical Modeling of Extreme Values* (Springer: London).

Cont, R. (2001). Empirical Properties of Asset Returns: Stylized Facts and Statistical Issues. *Quantitative Financei*, 1: 223–36.

Doney, R., Maller, R., and Savov, M. (2009). Renewal Theorems and Stability for the Reflected Process. *Stochastic Processes and their Applications*, 119: 270–97.

Graversen, S. E. and Shiryaev, A. (2000). An Extension of P. Lévy's Distributional Properties to the Case of a Brownian Motion with Drift. *Bernoulli*, 6: 615–20.

Ismail et al., M.-M. (2004a). On the Maximum Drawdown of a Brownian Motion. *Journal of Applied Probability*, 41: 147–61.

Ismail et al., M.-M. (2004b). Maximum Drawdown. *Risk Magazine*, 17: 99–102.

Karatzas, I. and Shreve, S. E. (1997). *Brownian Motion and Stochastic Calculus*. (Springer: New York, 2nd edition).
Kleinert, H. (2009). Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific: Singapore, 5th edition).
Leal, R. P. C. and de Melo Mendes, B. V. (2005). Maximum Drawdown: Models and Application. *Journal of Alternative Investments*, 7: 83–91.
Pearson, K. (1895). Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material. *Philosophical Transactions of the Royal Society of London*, 186: 343–414.
Pospisil, L. and Vecer, J. (2009). PDE Methods for the Maximum Drawdown. *Journal of Computational Finance*, 12: 59–76.
Reiss, R.-D. and Thomas, M. (2007). *Statistical Analysis of Extreme Values*. (Birkenhauser: Basel 3rd edition).
Rosario N. Mantegna, H. E. S. (2000). *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press: Cambridge).
Starica, C. (2004). Is Garch(1,1) as Good a Model as the Nobel Prize Accolades Would Imply? *Econometrics 0411015*, EconWPA.