

Engineering Statistics Lectures XV and XVI

Notes by Jonathan Bender

October 24, 2019; October 29, 2019

Abstract

Bell Labs – Subsidiary of AT&T, company invented many things ranging from the BJT to the general transistor to the telephone (Namesake – Alexander Graham Bell) to the IC. Basically died at the late 70's AT&T split.

1 Reprisal of Joint Probability

For three variables,

$$P(X = x, Y = y) = \frac{J_x(x)J_y(y)J_z(x, y)}{\# \text{ of possible cases}}$$

Where $J_x(x)$ refers to a joint probability element which concerns x. and $J_z(x, y)$ refers to a joint probability element which concerns z, a dependent variable upon x and y. Generally, these are nC_r functions such that the sum of all n and the sum of all k in the numerator match that of the denominator.

2 Marginal probabilities

What if we want the chance that, regardless of y, x is fixed? Well, good! We have a way of dealing with it!

$$P(x) = \sum_{j \in Y} P(x, j)$$

Where x is some fixed value and j iterates over the set of possible values for Y. For three independent variables,

$$P(x) = \sum_{j \in Y} \sum_{k \in Z} P(x, j, k)$$

Where, as before, j and k iterate over Y and Z. Think of P(x) to be a lesser-dimensional slice of the relevant space (plane corresponds to line, space corresponds to plane, hyperspace corresponds to space, etc.).

3 Means in multidimensional sets

$$\mu_X = \frac{P(x)}{\|X\|} = \frac{\sum_{j \in Y} P(x, j)}{\# \text{ of terms in } X} = \text{avg}(P(x))$$

Where $P(x)$ is the marginal probability of x occurring.

4 Marathon I guess?

Spoz that the fraction of guys that complete a marathon is X , and the fraction of women is Y . Historically-speaking, $f(X = x, Y = y) = Axy$ s.t. $0 \leq y \leq x \leq 1$.

To get the probability between two values of x , we take the integral of the PDF $f(x)$ between those two values.

To get the probability within two ranges of x and y respectively, we take the integral of a PDF $f(x,y)$ within those ranges; the area under the surface.

4.1 A. find A.

$$\begin{aligned} \int \int Axy \, dx dy &= 1 \quad \forall (x, y) \text{ s.t. } 0 \leq y \leq x \leq 1 \\ \rightarrow \int_{y=0}^{x=1} \int_{x=y}^{x=1} Axy \, dx dy &= 1 \\ \rightarrow \int_{x=0}^{x=1} \int_{y=0}^{y=x} Axy \, dy dx &= 1 \\ \rightarrow \int_{x=0}^{x=1} \frac{A}{2} xy^2 \Big|_{y=0}^{y=x} dy &= 1 \\ \rightarrow \int_{x=0}^{x=1} \frac{A}{2} x^3 dx &= 1 \\ \rightarrow \frac{A}{8} x^4 \Big|_{x=0}^{x=1} &= 1 \\ \rightarrow \frac{A}{8} &= 1 \\ \rightarrow A &= 8 \\ \rightarrow f(x, y) &= 8xy \end{aligned}$$

4.2 B. Find $g(x)$, $h(y)$

Find $g(x)$:

$$\begin{aligned}g(x) &= \int_{y=0}^{y=x} f(x, y) dy \\ \rightarrow g(x) &= 8 \int_{y=0}^{y=x} xy \, dy \\ &= 4xy^2 \Big|_{y=0}^{y=x} \\ \rightarrow g(x) &= 4x^3\end{aligned}$$

Find $h(y)$:

$$\begin{aligned}h(y) &= \int_{x=y}^{x=1} [f(x, y)] dx \\ \rightarrow h(y) &= \int_{x=y}^{x=1} [8xy] dx \\ &= 4x^2y \Big|_{x=y}^{x=1} \\ &= 4y(1 - y^2) \\ \rightarrow h(y) &= 4y(1 + y)(1 - y)\end{aligned}$$

4.3 C. Find μ_X, μ_Y

$$\begin{aligned}\mu_X &= \int_{x=0}^{x=1} xg(x)dx \\ &= \int_{x=0}^{x=1} 4x^4dx \\ &= \frac{4}{5}x^5 \Big|_0^1\end{aligned}$$

$$\rightarrow \mu_X = \frac{4}{5}$$

Find μ_Y :

$$\begin{aligned}\mu_Y &= \int_{y=0}^{y=1} yh(y)dy \\ &= \int_0^1 y * 4y(y - y^3)dy \\ &= \int_0^1 4(y^3 - y^5)dy \\ &= [y^4 - \frac{2}{3}y^6]_{y=0}^{y=1}\end{aligned}$$

$$\rightarrow \mu_Y = \frac{1}{3}$$

4.4 D. What's $P(0.1 \leq X \leq 0.3, 0.2 \leq Y \leq 0.4)$

Find the probability by summing up the contents under the curve. However, Y is strictly bounded by X: $Y \leq X$!! However, the same rule follows: $Y \not\leq X$. So, the bound becomes:

$$0.2 \leq X \leq 0.3, 0.2 \leq Y \leq 0.3$$

So, we take the integral as:

$$\begin{aligned} P(etc) &= \int_{x=0.2}^{x=0.3} \int_{y=0.2}^{y=0.3} f(x) dy dx \\ &= \int_{x=0.2}^{x=0.3} \int_{y=0.2}^{y=0.3} 8xy dy dx \\ &= \int_{x=0.2}^{x=0.3} [4xy^2]_{y=0.2}^{y=0.3} dx \\ &= \int_{x=0.2}^{x=0.3} 4\left(\frac{9}{100} - \frac{4}{100}\right) x dx \\ &= \frac{1}{5} * \frac{1}{2} [x^2]_{x=0.2}^{x=0.3} \\ &= \frac{1}{10} \left[\frac{9}{100} - \frac{4}{100} \right] \end{aligned}$$

5 Brian has two pumps?

Spoz that Brian runs a gas station with two pumps.

Let X be the fraction of time that a customer has to wait in line (only has two pumps) for the first pump. Let Y be the fraction of time that a customer has to wait in line for the other pump. Historically-speaking, after much data compiled and taken using python, the PDF $f(x,y)$, given that 10 minutes is the maximum wait time possible:

$$f(x,y) = \begin{cases} A(2x + 3y) & \text{iff } 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ & \\ 0 & \text{elsewhere} \end{cases}$$

5.1 A. Find A

$$\begin{aligned} A &= \frac{1}{F(x,y)} \\ \rightarrow F(x,y) &= \int_{y=0}^{y=1} \int_{x=0}^{x=1} (2x + 3y) dx dy \\ &= \int_{y=0}^{y=1} (x^2 + 3xy) \Big|_{x=0}^{x=1} dy \\ &= \int_{y=0}^{y=1} (1 + 3y) dy \\ &= \left[y + \frac{3}{2}y^2 \right]_{y=0}^{y=1} \\ &= \frac{5}{2} \\ \rightarrow A &= \frac{2}{5} \end{aligned}$$

5.2 B. Find μ_X, μ_Y

Find μ_X by finding $g(x)$:

$$\begin{aligned}g(x) &= \int_{y=0}^{y=1} \frac{2}{5}(2x+3y)dy \\&= \frac{2}{5}[2xy + \frac{3}{2}y^2]_{y=0}^{y=1} \\&= \frac{2}{5}[2x + \frac{3}{2}] \\&= \frac{4x+3}{5}\end{aligned}$$

$$\begin{aligned}\rightarrow \mu_X &= \int_{x=0}^{x=1} \frac{4x^2+3x}{5}dx \\&= \frac{\frac{4}{3}x^3 + \frac{3}{2}x^2}{5} \Big|_{x=0}^{x=1}\end{aligned}$$

$$\rightarrow \mu_X = \frac{17}{30}$$

Find μ_Y by finding $h(y)$:

$$h(y) = \int_{x=0}^{x=1} \frac{2}{5}(2x+3y)dx$$

5.3 Find the probability that the wait time for the first pump is 3-8 minutes and that the time waiting for pump 2 is 3-7 minutes