MATH/COSC 303

Assignment 3

Due: Feb 25th, assignments are due at the end of lab.

Hand Written Questions:

1. Suppose that $f(b) - f(a) \neq 0$. Prove that
$$a - \frac{f(a)(b-a)}{f(b)-f(a)} = b - \frac{f(b)(b-a)}{f(b)-f(a)}.$$

2. Consider the function $f(x) = e^x - x - 2$.
   a) Prove that $f(x) = 0$ has a solution on $[1, 2]$.
   b) Prove that $f(x) = 0$ has exactly one solution on $[1, 2]$.

3. Consider the function
$$g(x) = -4 + 4x - \frac{1}{2}x^2.$$
   a) Prove that $\bar{x}_1 = 2$ and $\bar{x}_2 = 4$ are the only fixed points of g.
   b) Compute $g'(\bar{x}_1)$ and $g'(\bar{x}_2)$. Use this information to determine which fixed point should be an attractor and which fixed point should be a repeller.

4. Three fixed point iterations are proposed to solve the equation $0 = 2x^2 - 6x + 1$ starting at $x^0 = 3$ (solution at $\bar{x} \approx 2.8229$). These are
$$\begin{array}{lll} x^{k+1} = r(x^k) & \text{where} & r(x) = \frac{1}{3}x^2 + \frac{1}{6} \\ x^{k+1} = s(x^k) & \text{where} & s(x) = 3 - \frac{1}{2x} \\ x^{k+1} = t(x^k) & \text{where} & t(x) = 2x^2 - 5x + 1. \end{array}$$
   Which method is more likely to succeed? (Justify your answer.)

5. (Newton's Root Finding Method) Assume $A > 0$. Newton proposed the following iteration to approximate the square root of A
$$x^{k+1} = \frac{x^k + \frac{A}{x^k}}{2}.$$
   Assuming x^0 is sufficiently close to $\sqrt{A}$, explain why this method works. (Hint, begin by considering the function $f(x) = x^2 - A$, then notice that it was NEWTON's idea.)

Computer Assisted Questions:

6. Consider the function $f(x) = e^x - x - 2$. In question 2, you showed that this function had exactly one solution ($\bar{x}$ such that $f(\bar{x}) = 0$) in $[1, 2]$.
   a) Apply the Bisection Methods starting at $a^0 = 1$ and $b^0 = 2$ to approximate $\bar{x}$ to 4 decimal places. (I.e., iterate until MATLAB reports the first 4 decimals remain unchanged.)
   b) Apply the Method of False Position starting at $a^0 = 1$ and $b^0 = 2$ to approximate $\bar{x}$ to 4 decimal places. (I.e., iterate until MATLAB reports the first 4 decimals remain unchanged.)

7. Consider the function

$$g(x) = -4 + 4x - \frac{1}{2}x^2.$$

a) Starting at $x^0 = 1.9$, apply a fixed point iterative methods to find x^1, x^2, x^3, x^4, x^5. Computer the relative errors R_k between x^k and the closest fixed point to x^k for each iteration $k = 0, 1, 2, ...5$.

b) Starting at $x^0 = 2.1$, apply a fixed point iterative methods to find x^1, x^2, x^3, x^4, x^5. Computer the relative errors R_k between x^k and the closest fixed point to x^k for each iteration $k = 0, 1, 2, ...5$.

c) Starting at $x^0 = 3.8$, apply a fixed point iterative methods to find x^1, x^2, x^3, x^4, x^5. Computer the relative errors R_k between x^k and the closest fixed point to x^k for each iteration $k = 0, 1, 2, ...5$

d) Starting at $x^0 = 4.2$, apply a fixed point iterative methods to find x^1, x^2, x^3, x^4, x^5. Computer the relative errors R_k between x^k and the closest fixed point to x^k for each iteration $k = 0, 1, 2, ...5$

e) Compare your results to the Summarize your observations.

8. Use Netwon's Root Finding Method (see question 5) to approximate $\sqrt{27}$. For each iteration compute the number of significant digits obtained.

9. Use any numerical method you desire to find $x \in (0, 1)$ that solves $x^{\cos(x)} = x^x$ to at least 12 significant digits.