MATH/COSC 303

4.1 Interpolation Errors

Theorem 4.1.1 *Let $f \in \mathcal{C}^\infty$ and suppose $x_0 < x_1 < ...x_n$.*

Let $p(x) = a_0 + a_1 x + a_2 x^2 + ...a_n x^n$ be the unique interpolation polynomial of degree n for f over $\{x_0, x_1, ...x_n\}$. (I.e., $p(x_i) = f(x_i)$ for $i = 0, 1, ...n$.)

Let $M = \max\{|f^{(n+1)}(x) : x \in [x_0, x_n]\}$ and $\Delta = |x_n - x_0|$.

Then

$$|f(x) - p(x)| \le \frac{1}{(n+1)!} M |\prod_{i=0}^{n} (x - x_i)| \le \frac{1}{(n+1)!} M \Delta^{n+1}, \tag{1}$$

and

$$|f'(x) - p'(x)| \le \frac{1}{(n+1)!} M |\sum_{j=0}^{n} \prod_{i \ne j} (x - x_i)| \le \frac{1}{n!} M \Delta^n. \tag{2}$$

PROOF: Let $x \in [x_0, x_n]$ be such that $x \ne x_i$ for all i. (We will deal with the case $x = x_i$ later by using limits.) Define (note that x is now fixed)

$$w(t) := \prod_{i=0}^{n} (t - x_i), \tag{3}$$

$$\lambda := \frac{f(x) - p(x)}{w(x)}, \tag{4}$$

and

$$\phi(t) := f(t) - p(t) - \lambda w(t). \tag{5}$$

Notice that $\lambda w(x) = f(x) - p(x)$, so $\phi(x) = 0$. Also

$$\begin{aligned} \phi(x_j) &= f(x_j) - p(x_j) - \lambda w(x_j) \\ &= 0 - \lambda \textstyle\prod_{i=0}^{n} (x_j - x_i) \\ &= 0, \end{aligned}$$

as $(x_i - x_i) = 0$. Finally, we note that $\phi \in \mathcal{C}^\infty$ (as f, p, and w are all $\mathcal{C}^\infty$).

In summary, we now have $\phi \in \mathcal{C}^\infty$ with (at least) $n+2$ roots in $[x_0, x_n]$. Applying the MVT (or Rolle's Theorem) this implies,

- $\phi' \in \mathcal{C}^\infty$ with (at least) $n+1$ roots in (x_0, x_n),
- $\phi'' \in \mathcal{C}^\infty$ with (at least) n roots in (x_0, x_n),
- ...
- $\phi^{(n)} \in \mathcal{C}^\infty$ with (at least) 2 roots in (x_0, x_n), and
- $\phi^{(n+1)} \in \mathcal{C}^\infty$ with (at least) 1 root in (x_0, x_n).

Let $c \in (x_0, x_n)$ be a root of $\phi^{(n+1)}$ (i.e., $\phi^{(n+1)}(c) = 0$).

Now, notice that

$$\begin{aligned} \phi^{(n+1)}(t) &= f^{(n+1)}(t) - p^{(n+1)}(t) - \lambda w^{(n+1)}(t) \\ &= f^{(n+1)}(t) - \lambda w^{(n+1)}(t) && \text{(as p is a polynomial of degree n)} \\ &= f^{(n+1)}(t) - \lambda \tfrac{d^{n+1}}{dx^{n+1}} \textstyle\prod_{i=0}^{n} (x - x_i) \\ &= f^{(n+1)}(t) - \lambda (n+1)! \end{aligned}$$

as $\prod_{i=0}^{n} (x - x_i)$ is a polynomial of degree $n+1$ with the first term x^{n+1}. So

$$f^{(n+1)}(c) - \lambda (n+1)! = 0.$$

Recalling the definition of λ (equation (4)), this yields

$$\frac{f(x) - p(x)}{w(x)}(n+1)! = f^{(n+1)}(c),$$

so

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \prod_{i=0}^{n} (x - x_i). \tag{6}$$

Noting $|f^{(n+1)}(c)| \leq M$, we have

$$|f(x) - p(x)| = \frac{1}{(n+1)!} M |\prod_{i=0}^{n} (x - x_i)|, \qquad \text{for all } x \neq x_i,\ x \in [x_0, x_n].$$

As f, p, and $\prod_{i=0}^{n}(x - x_i)$ are all continuous in x, this bound will hold at $x = x_i$ as well, which proves the first inequality in (1). The second inequality follows from $|x - x_i| \leq \Delta$.

To see (2), take the derivative on each side of equation (6) to yield

$$f'(x) - p'(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \sum_{i=0}^{n} \prod_{i \neq j}^{n} (x - x_i),$$

(by the product rule). Using $|f^{(n+1)}(c)| \leq M$ yields the first inequality in (2), and the second inequality follows from $|x - x_i| \leq \Delta$.