HW 4

1. a) $Y_i|\theta \sim \text{Poisson}(\theta)$

$$p(y|\theta) = \frac{\theta^{x} e^{-\theta}}{x!} \cdot \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} = \frac{e^{-n\theta}\,\theta^{\sum y_i}}{\prod y_i!}$$

$$\ln(p(y|\theta)) = x\ln(\theta) + (-\theta)\ln e - \ln(x_i!) + a\ln b - \ln\Gamma(a) + (a-1)\ln\theta + (-b\theta)\ln e = -n\theta + \sum y_i \frac{1}{\theta} - \sum \ln(y_i)$$

$$\frac{d}{d\theta} = \frac{x}{\theta} - 1 + \frac{a-1}{\theta} - b = -n + \frac{1}{\theta}\sum y_i$$

$$\frac{d^2}{d\theta^2} = -x\theta^{-2} + (a-1)(-1)\theta^{-2} = \frac{-1}{\theta^2}\sum y_i$$

$$\Rightarrow -E[\ldots] = x\,E[\theta]^{-2} + (a-1)E[\theta]^{-2}$$

$$= \frac{\cancel{x}}{\cancel{x^2}} + \frac{\cancel{a-1}}{\cancel{E^2}} \frac{-1}{E[\sum y_i]\,\theta^2} \Rightarrow f(\lambda) \propto \frac{\sqrt{\cancel{x(a-1)}}\,1}{\cancel{x}\sqrt{\theta^2}} \propto \boxed{\frac{1}{\theta}}$$

b) $Y_i|\theta \sim \text{Exponential}(\theta) \rightarrow p(y|\theta) = \theta^n e^{-\theta\sum y_i}$

$$\frac{d^2}{d\theta^2}\ln(p(y|\theta)) = \frac{d^2}{d\theta^2}(n\ln\theta - \theta n\bar{y}) = \frac{-n}{\theta^2}$$

$$\Rightarrow -E[\ldots] = \frac{n}{\theta^2} \Rightarrow f(\theta) \propto \boxed{\frac{1}{\theta}}$$

c) $Y_i|\theta \sim \text{Normal}(\theta, s^2)$ known. $p(y|\theta) = \left(\frac{1}{\sqrt{2\pi}s}\right)^n e^{-\frac{1}{2s^2}\sum(y_i-\theta)^2}$

$$\Rightarrow \ln(p(y|\theta)) = n\ln\left(\frac{1}{\sqrt{2\pi}s}\right) + \frac{-1}{2s^2}(\ln e)\sum(y_i-\theta)^2$$

$$= n\ln(\ldots) - \frac{1}{2s^2}\sum(y_i-\theta)^2 = n\ln(.) - \frac{1}{2s^2}n(\bar{y}-\theta)^2$$

$$\frac{d^2}{d\theta^2}\ln(\ldots) = \left(\frac{-1}{2s^2}\right)n\,\bar{y}^2 - 2\theta\bar{y}\left(\frac{-n}{2s^2}\right) + \theta^2\left(\frac{-n}{2s^2}\right) \frac{d^2}{d\theta^2} = \frac{-n}{s^2}$$

$\Rightarrow -E[\ldots] = -E\left[\frac{n}{s^2}\right] = \boxed{\frac{-n}{s^2}} \ (?) = \boxed{\frac{1}{s}}$

d) $\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(y-m)^2}{2\sigma^2}\right)}\right)$

$= \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \left(\frac{-(y-m)^2}{2s^2}\right)\ln e$

$\frac{d}{d\theta} = \sqrt{2\pi\sigma^2} \cdot \frac{1}{2}(2\pi\sigma^2)^{-\frac{1}{2}} \cdot 2\pi\sigma$

$\Rightarrow \frac{d^2}{d\theta^2} = \sqrt{2\pi} \cdot (-\pi)\sigma^{-2} \cdot 2\pi$

$\Rightarrow -E[\ldots] = -\pi \, E[(\sigma)]^{-2} = -\pi \, m^{-2}$

$I(\theta) = \boxed{\frac{1}{m}}$

2. a and b $\rightarrow$ improper Gamma $(0, 0)$
c $\rightarrow$ no because it doesn't integrate to a finite value
d $\rightarrow$ $\frac{1}{m} = \frac{1}{\sigma^2}$ $\rightarrow$ proper because the integral has a finite range

3. a) $\phi = \ln\left(\frac{\theta}{1-\theta}\right)$

$\Rightarrow \frac{\theta}{1-\theta} = e^{\phi} \Rightarrow \theta = e^{\phi} - e^{\phi}\theta$

$\Rightarrow \theta(1+e^{\phi}) = e^{\phi} \Rightarrow \theta = \frac{e^{\phi}}{1+\phi}$

$\Rightarrow f_{\phi}(\phi)$ PDF $= \left(\frac{e^{\phi}}{1+\phi}\right)^{\frac{1}{2}} \left(1 - \frac{e^{\phi}}{1+\phi}\right)^{-\frac{1}{2}} \cdot \left|\frac{d}{d\phi} \cdot \frac{e^{\phi}}{1+\phi}\right|$

$\frac{\phi e^{\phi}}{(\phi+1)^2} \Rightarrow \boxed{f_{\phi}(\phi) = \left(\frac{e^{\phi}}{1+\phi}\right)^{-\frac{1}{2}} \left(1 - \frac{e^{\phi}}{1+\phi}\right)^{-\frac{1}{2}} \left(\frac{\phi e^{\phi}}{(\phi+1)^2}\right)}$

b) $p(y|\theta) = \left(\frac{e^{\phi}}{1+\phi}\right)^{\frac{n}{2}} \left(1 - \frac{e^{\phi}}{1+\phi}\right)^{\frac{n}{2}} \left(\frac{\phi^n e^{n\phi}}{(\phi+1)^{2n}}\right)$

$$\Rightarrow \ln(y|\theta) = \frac{-n}{2}\ln\left(\frac{e^{\phi}}{1+\phi}\right) + \left(\frac{-n}{2}\right)\ln\left(1-\frac{e^{\phi}}{1+\phi}\right) + \ln \dots$$

$(\ln 1 = 0)$
$$= \frac{-n}{2}\phi\ln e - \frac{-n}{2}\ln(1+\phi) + \left(\frac{-n}{2}\right)\frac{\ln 1}{\phi\ln e - \ln(1+\phi)} + \ln$$

$$= \frac{-n}{2}\phi + \cancel{\frac{n}{2}}\, n\ln\phi + n\phi\ln e - 2n\ln(\theta+1)$$

$$= \frac{-n}{2}\phi + n\ln\phi + n\phi$$

$$\frac{d^2}{d\theta^2}\ln(p(y|\theta)) = -n\phi^{-2} \Rightarrow I(\theta) = \boxed{\phi^{-1}}$$

c) I have a feeling that I should have gotten ϕ instead of ϕ^{-1}. But aside from that, the fact that ϕ survived tells me that, yes, Jeffreys Prior is invariant of parametrization, as we already knew.

4. a) $$BF(H_0 : H_1) = \frac{p(D \mid \theta = 3)}{\int p(D \mid \theta)\, p(\theta \mid H_1)\, d\theta}$$

$$\Rightarrow p(D \mid H_0) = \prod^{n} \frac{3^{y_i} \cdot e^{-3}}{y_i!}$$

$$p(D \mid H_1) = \int \prod_{i=1}^{n} \frac{\theta^{y_i} e^{-\theta}}{y_i!} \cdot \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}\, d\theta$$

jammed this into an integral calculator
$$\Rightarrow = \frac{b^a}{\Gamma(a)} \cdot \frac{\Gamma(\Sigma(y_i + a))}{(b+n)^{\Sigma(y_i + a)}}$$

Consider again $p(D \mid H_0)$ with the data:

$$p(D \mid H_0) = \frac{3^2 e^{-3}}{2!} \cdot \frac{3^3 e^{-3}}{3!} \cdot \frac{3^3 e^{-3}}{3!} \cdot \frac{3\, e^{-3}}{1} \cdot \frac{3^4 e^{-3}}{4!}$$

$$= \frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!}$$

$$p(D \mid H_1) = \frac{b^a\, \Gamma(13+a)}{\Gamma(a)(b+5)^{13+a}}$$ (with data).

$$BF(H_0 : H_1) = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \boxed{\frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!} \cdot \frac{\Gamma(a)\,(b+5)^{13+a}}{b^a\,\Gamma(13+a)}}$$

b) $BF(H_0 : H_1) = \frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!} \cdot \frac{\Gamma(1)\,(6)^{14}}{1\,\Gamma(14)}$

$\boxed{\approx 0.00355}$

$p(H_0 \mid y) = \frac{p(y \mid H_0) \cdot p(H_0)}{p(y)}$

$p(y) \overset{LOTP}{=} p(y \mid H_1) \cdot p(H_1) + p(y \mid H_0) \cdot p(H_0)$

$= \frac{\Gamma(14)}{\Gamma(1)\,(6)^{14}} \cdot 0.5 + \frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!} \cdot 0.5$

$= \frac{13!}{6^{14}} \cdot 0.5 + \frac{3^{11} e^{-15}}{2!\,2!\,2!\,4!} \cdot 0.5$

≈ 0.0398

$\Rightarrow p(H_0 \mid y) = 0.5 \cdot \frac{1}{0.0398} \cdot \frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!}$

$\boxed{\approx 0.0035}$

c) $BF(H_0 : H_1) = \frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!} \cdot \frac{\Gamma(4)\,(6)^{17}}{1 \cdot \Gamma(17)}$

$= \frac{3^{13} e^{-15}}{3!\,3!\,2!\,4!} \cdot \frac{3!\,6^{17}}{16!}$

$\boxed{\approx 2.175 \cdot e^{-11}}$

$p(y) = p(y \mid H_1) \cdot p(H_1) + p(y \mid H_0)\,p(H_0)$

$= \frac{1 \cdot \Gamma(17)}{\Gamma(4) \cdot 6^{17}} \cdot 0.5 + \ldots$

$= \frac{16!}{3! \cdot 6^{17}} \cdot 0.5 + \frac{3^{11} e^{-15}}{2!\,2!\,2!\,4!} \cdot 0.5$

≈ 0.103

$\Rightarrow p(H_0 \mid y) \approx 0.5 \cdot \frac{1}{0.103} \cdot 0.00028$

$\boxed{= 0.00135}$

d) The p-values aren't so different, so I would say that what we know about Jeffreys Priors still holds (that they are objective and don't influence the probability). A caveat would be that the p-values are ~~all~~ already low, lower than 0.01, the range where our findings would be statistically significant, so differences within this range wouldn't matter any way.

5. a) $p(y|H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$$p(y|H_1) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\,c} \cdot e^{-\frac{\theta^2}{2c^2}} \, d\theta$$

$$= e^{\frac{cy^2}{2(1+c^2)}} \cdot \sqrt{\frac{2\pi}{1+\frac{1}{c^2}}}$$

$$BF(H_1 : H_0) = \frac{p(y|H_1)}{p(y|H_0)} = e^{\frac{cy^2}{2(1+c^2)}} \cdot \sqrt{\frac{2\pi}{1+\frac{1}{c^2}}} \cdot \sqrt{2\pi} \cdot \frac{1}{e^{-\frac{y^2}{2}}}$$

$$= e^{\frac{cy^2}{2(1+c^2)} + \frac{y^2}{2}} \cdot \frac{2\pi}{\sqrt{1+\frac{1}{c^2}}}$$

$$= \boxed{\underbrace{e^{\frac{y^2(c^2+c+1)}{2(1+c^2)}}}_{A} \cdot \underbrace{\frac{2\pi}{\sqrt{1+\frac{1}{c^2}}}}_{B}}$$

b) The BF slightly increases as y increases. This seems expected; of course it changes but it doesn't change that much (y is only a power of e and e itself is small). In context, this saying that more data shows stronger evidence against the null hypothesis, which makes sense in some cases.

c) If the variance increases and the data is constant, A slightly increases and B increases (at a faster rate). So we have: if variance increases then there is more evidence against the null hypothesis, which sounds counter-intuitive.