450 HW 1

1. - $P(E) \geq 0$ for all $E \subset S$ ($S$ is the sample space)
- $P(S) = 1$
- Let $E_1$ and $E_2$ be mutually exclusive events
$P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

2. a) $A_1 \cup A_2 = A_1 \cup A_2$
b) $A_1 \cap A_2 = \emptyset$
c) $P(A_1 \cup A_2) = P(A_1) + P(A_2)$
d) $P(A_1 \cap A_2) = 0$
e) $\Omega$ is the sample space of flipping a coin
$\rightarrow A_1$ = Heads and $A_2$ = Tails

3. a) $P(A \cap B) = P(A) \cdot P(B)$
b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) \cdot P(B)}{P(B)} = P(A)$

4. a) $p(x, y) = p(x) \cdot p(y)$
b) $P(X > 2 \mid Y < 1)$: these two events don't seem to be dependent on each other so $P(X>2 \mid Y<1) = P(X>2)$.

5. - Discrete: the support consists of a finite number of all possible values of the variable.
- Continuous: the support is a range / interval

6.

| | Type (i) | Support (ii) |
|---|---|---|
| $X \sim Beta(a,b)$ | Continuous | $[0, 1]$ |
| $X \sim Bin(n,p)$ | Discrete | $\{0, 1, 2, \dots, n\}$ |
| $X \sim Gamma(a,b)$ | Continuous | $(0, +\infty)$ |
| $X \sim Normal(\mu, \sigma^2)$ | Continuous | $(-\infty, +\infty)$ |
| $X \sim Poisson(\lambda)$ | Discrete | $\{0, 1, 2, \dots\}$ |
| $X \sim Uniform(a,b)$ | Continuous | $[a, b]$ |

7.

7. PMF:

- $f(x) \geq 0$ for all $x \in S$ → $f(x)$ is defined
- $\sum f(x) = 1$ ($x$ is all elements in $S$)
- $f(x) = P(X = x)$

PDF:

- $f(x) \geq 0$ → $f(x)$ is defined
- The area under the curve totals to 1
- $f(x)$ = the probability of $X$ being in a range.

$P(a \leq X \leq b) = \int_a^b f_x(x)\, dx$

8. a) Let $X$ be the random variable representing the number of defective batteries. $X$ follows a binomial distribution $X \sim Bin(n, \theta)$

a) $E(X) = n \times \theta = 10 \times 0.05 = 0.5$

b) $P(X = x) = \binom{n}{x} \times 0.05^{(x)} \times 0.95^{(n-x)}$

$\Rightarrow P(X=0) = \binom{10}{0} \times 0.05^0 \times 0.95^{10}$

$P(X=1) = \binom{10}{1} \times 0.05^1 \times 0.95^9$

$\Rightarrow P(X=0) + P(X=1) \approx 0.91 = 91\%$

9. $\int_0^\infty \theta^5 e^{-3\theta}\, d\theta$ → $\theta$ is the variable of a PDF $f(\theta)$. Then, $\int_0^\infty \theta^5 e^{-3\theta}\, d\theta$ is the probability $P(\theta \geq 0) \geq 0$

$\Rightarrow \int_0^\infty \theta^5 e^{-3\theta}\, d\theta \geq 0 \Rightarrow \theta^5 e^{-3\theta} \geq 0 \Rightarrow \theta^5 \geq 0 \Rightarrow \theta \geq 0 \Rightarrow \boxed{[0, \infty)} \to$ support

$\boxed{\text{Kernel: } e^{-3}}$ (because it doesn't involve $\theta$)

⇒ This resembles the Gamma distribution:

$[0, \infty)$ is the support so the function

$\int_0^\infty \theta^5 e^{-3\theta}\, d\theta$ is covering the entire range of all possible values of the random variable, so it must equal to 1

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## II. ELISA test

HIV: the person has HIV

TP: test is positive

a) $P(TP) = P(TP | HIV) . P(HIV) + P(TP | HIV^c) . P(HIV^c)$

$= 0.99 \times 1/10000 + (1 - P(TP^c | HIV^c)) \times (1 - P(HIV))$

$\approx 9.95 . 10^{-5}$

b) $P(HIV | TP^c) = \dfrac{P(TP^c | HIV) . P(HIV)}{P(TP^c)}$

$= \dfrac{(1 - P(TP | HIV)) . P(HIV)}{1 - P(TP)}$

$= \dfrac{(1 - 0.01) \times 1/10000}{1 - 0.9949} \approx 9.9 . 10^{-5}$

c) $P(HIV | TP) = \dfrac{P(TP | HIV) . P(HIV)}{P(TP)}$

$\approx \dfrac{0.99 \times 1/10000}{9.9 \times 10^{-5}} \approx 1.$

## III. Genetic Status

a) $P(X = x) = 0.5^{x}(1 - 0.5)^{1-x}$

for $x = \{0, 1\}$. $x$ is equally likely to be either.

b) $P(y_1 = 0, y_2 = 0 | \theta = 1)$

$= P(y_1 = 0 | \theta = 1) . P(y_2 = 0 | \theta = 1)$

$= \dfrac{P(\theta = 1 | y_1 = 0) . P(y_1 = 0)}{P(\theta = 1)} . \dfrac{P(\theta = 1 | y_2 = 0) . P(y_2 = 0)}{P(\theta = 1)}$

$= \dfrac{P(\theta = 1) . P(y_1 = 0)}{P(\theta = 1)} . \dfrac{P(\theta = 1) . P(y_2 = 0)}{P(\theta = 1)}$

$= 0.5 \times 0.5 = 0.25$

$P(y_1 = 0, y_2 = 0 | \theta = 0)$

$= P(y_1 = 0 | \theta = 0) . P(y_2 = 0 | \theta = 0)$

$= 1 \times 1$

$= 1$

fixed thanks to peer reviews

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c) $p(\theta = 1 \mid y_1, y_2) = \dfrac{p(y_1, y_2 \mid \theta = 1) \cdot p(\theta = 1)}{p(y_1, y_2)}$

$$= \frac{p(y_1, y_2 \mid \theta = 1) \cdot p(\theta = 1)}{p(y_1, y_2 \mid \theta = 1) \cdot p(\theta = 1) + p(y_1, y_2 \mid \theta = 0) \cdot p(\theta = 0)}$$

$$= \frac{0.25 \times 0.5}{0.25 \times 0.5 + 1 \times 0.5} = 0.2$$

d) $p(\theta = 1 \mid y_1, y_2, y_3)$

$$= \frac{p(y_1, y_2, y_3 \mid \theta = 1) \cdot p(\theta = 1)}{p(y_3)}$$

OR $$= \frac{p(y_3 \mid \theta = 1) \cdot \overbrace{p(\theta = 1 \mid y_1, y_2)}^{\text{prior}}}{p(y_3 \mid \theta = 1) \cdot p(\theta = 1 \mid y_1, y_2) + p(y_3 \mid \theta = 0) \cdot p(\theta = 0 \mid y_1, y_2)}$$

$$= \frac{0.25 \times 0.5}{0.25 \times 0.5 + 0.25 \times 0.5} = 0.5$$