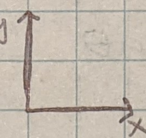
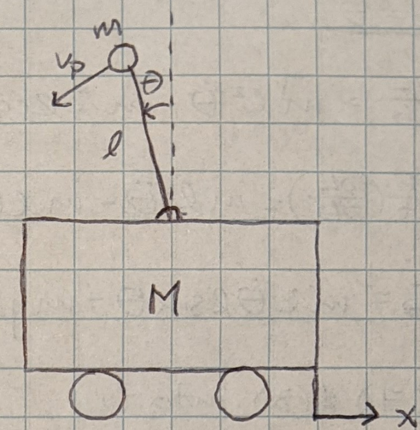
Inverted Pendulum on a cart





M: mass of cart
m: mass of pendulum
θ: Angle of pendulum
l: length of pendulum
v_p: Velocity of the pendulum

Lagrangian Approach:

(1) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

(2) $$L = T - V$$

(3) $$V = mgl\cos\theta$$

$$\begin{aligned}
T &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m v_p^2 \\
&= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(v_{p/x}^2 + v_{p/y}^2\right) \\
&= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x} - l\dot{\theta}\cos\theta)^2 + \tfrac{1}{2}m(-l\dot{\theta}\sin\theta)^2 \\
&= \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}m\left(-2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1})\right)
\end{aligned}$$

(4) $$T = \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 - m\dot{x}\dot{\theta}l\cos\theta$$

Plug (3) & (4) into (2):

(5) $$L = \tfrac{1}{2}(M+m)\dot{x}^2 - m\dot{x}l\dot{\theta}\cos\theta + \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

Use (5) to evaluate (1) where $q_1 = x$ & $q_2 = \theta$:

$\boxed{q_1 = x}$:

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - ml\dot{\theta}\cos\theta$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} - ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$\frac{\partial L}{\partial x} = 0$$

(6) $$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = u(t)$$; $u(t)$: input in x direction

$\boxed{q_2 = \theta}$

$$\frac{\partial L}{\partial \dot\theta} = m\ell^2\dot\theta - m\dot x\ell\cos\theta$$

(7) $$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m\ell^2\ddot\theta - m\ell(\ddot x\cos\theta - \dot x\dot\theta\sin\theta)$$

(8) $$\frac{\partial L}{\partial \theta} = m\dot x\dot\theta\ell\sin\theta + mg\ell\sin\theta$$

Plug (7) & (8) into (1)

$$m\ell^2\ddot\theta - m\ell\ddot x\cos\theta + m\ell\cancel{\dot x\dot\theta}\sin\theta - m\ell\cancel{\dot x\dot\theta}\sin\theta - mg\ell\sin\theta = 0$$

(9) $$\rightarrow m\ell\ddot\theta - m\ddot x\cos\theta - mg\sin\theta = 0$$

(6) & (9) are our EoMs: $$\boxed{\begin{aligned}&(M+m)\ddot x - m\ell\ddot\theta\cos\theta + m\ell\dot\theta^2\sin\theta = u(t)\\ &m\ell\ddot\theta - m\ddot x\cos\theta - mg\sin\theta = 0\end{aligned}}$$

For the purposes of control, we linearize about the vertical pendulum position, or $\theta \approx 0$. This implies $\cos\theta \approx 1$ and $\sin\theta \approx \theta$

(6) becomes:

$$(M+m)\ddot x - m\ell\ddot\theta + m\ell\dot\theta^2\theta = u(t) \qquad \dot\theta = 0$$

(10) $$\rightarrow (M+m)\ddot x - m\ell\ddot\theta = u(t)$$

(9) becomes:

(11) $$m\ell\ddot\theta - m\ddot x - mg\theta = 0$$

Using (10) & (11) solve for $\ddot x$ & $\ddot\theta$

From (10):

$$\ddot x = \frac{m\ell\ddot\theta + u(t)}{(M+m)}$$

Plugging into (11):

$$m\ell\ddot\theta - m\left(\frac{m\ell\ddot\theta + u(t)}{(M+m)}\right) = 0$$

From (11)

$$\ddot{\theta} = \frac{\cancel{m}\ddot{x} + \cancel{m}g\theta}{\cancel{m}\ell} = \frac{\ddot{x} + g\theta}{\ell}$$

Plugging into (10)

$$(M+m)\ddot{x} - m\ell\left(\frac{\ddot{x} + g\theta}{\ell}\right) = u(t)$$

(12) $$\ddot{x} = \left(\frac{mg}{M}\right)\theta + \left(\frac{1}{M}\right)u(t)$$

$$\rightarrow \ddot{\theta} = \frac{\left(\frac{u(t) + mg\theta}{M}\right) + g\theta}{\ell}$$

(13) $$\ddot{\theta} = \left(\frac{1}{M\ell}\right)u(t) + \left(\frac{mg}{M\ell} + \frac{g}{\ell}\right)\theta$$

Using (12) & (13) we can convert our system into state-space form which we can use to develop a controller:

Let $$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

So our system Becomes:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\mu_c & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \left(\frac{mg}{M\ell} + \frac{g}{\ell}\right) & -\mu_\theta \end{bmatrix}$$

(μ_θ & μ_c are friction coefficients for the cart & hinge)

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M\ell} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note: C implies we are keeping track of the carts position x and the angular position θ