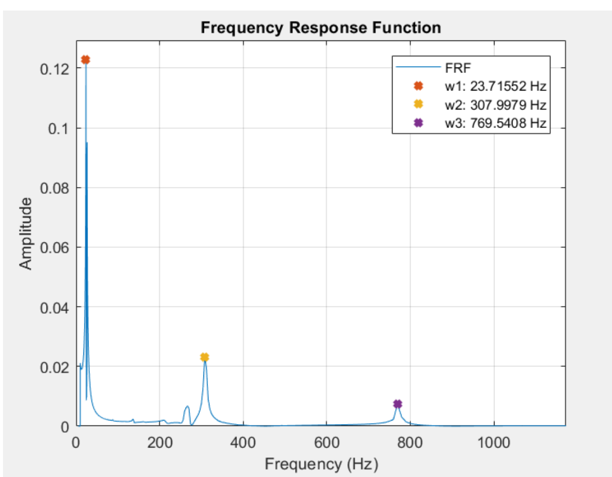
Analysis of a Cantilevered Beam with Tip Mass

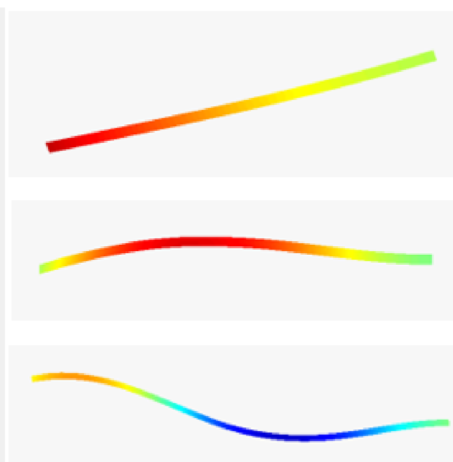
The following is a summarized portion of a final project I completed for the course Vibration and Shock II during my final semester at the University at Buffalo. The purpose is to analyze the modal parameters of a cantilevered beam with a tip mass using three different techniques and compare the results. The techniques used are experimental, distributed parameter analysis using the Euler Bernoulli Equations, and Finite Element Analysis. The finite element analysis code and all plots were created in Matlab.

Experimental:

This experiment was conducted in the University at Buffalo Sound and Vibrations Lab using a shaker and laser vibrometer, and accelerometer. To determine the natural frequencies of the beam it was excited using the shaker with a sine sweep from 5 – 4000 Hz while sampling the tip of the beam at a rate of 2.4 kHz. These values were chosen with the Nyquist Criteria in mind in order to prevent aliasing. Once the natural frequencies were determined the beam was excited at each natural frequency. Using a grid of sample points along the length of the beam the mode shapes were determined. The beam properties, frequency response function, and mode shapes are shown below:

- ρ = 2700 kg/m^2
- E = 69 GPa
- Tip mass = 0.015 kg
- l =0.16m
- b = 0.0092m
- h = 0.00238m





Distributed Parameter Analysis with Euler Bernoulli Equations:

$$c^2 \frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \quad c = \sqrt{\frac{EI}{\rho A}}$$

Using the separation of variables technique, we can separate w into two independent functions, one spatial and one temporal. Plugging into our PDE we can evaluate and rearrange out terms such that we have functions of x equal to functions of t. We can assume each side is equal to a constant since this is the only way we can draw an equivalency from a function of x and a function of t.

$$w(x,t) = X(x)T(t) \implies c^2 \frac{X''''(x)}{X(x)} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2$$

Through this equivalency we can separate them into two separate governing equations and based on these forms we can assume the form of each solution.

$$\ddot{T}(t) + \omega^2 T(t) = 0 \implies T(t) = A\sin(\omega t) + B\cos(\omega t)$$

$$X''''(x) - \sigma^4 X(x) = 0 \implies X(x) = C\sin(\sigma x) + D\cos(\sigma x) + E\sinh(\sigma x) + F\cosh(\sigma x)\ \sigma^4 = \left(\frac{\omega}{c}\right)^2$$

Applying our boundary conditions to the spatial equation we obtain a system of equations which we can assemble in matrix form. In order for our solution to be nontrivial the determinant of this matrix must be equal to zero. Evaluating our determinant we get the characteristic equation from which we can determine values of sigma and therefore omega:

Boundary Conditions:

$$w(0,t) = 0,\ \frac{\partial w(0,t)}{\partial x} = 0,\ \frac{\partial}{\partial x}\left(EI \cdot \frac{\partial^2 w(l,t)}{\partial x^2}\right) = m\frac{\partial^2 w(l,t)}{\partial t^2},\ EI \cdot \frac{\partial^2 w(l,t)}{\partial x^2} = 0$$

Matrix Form:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(\sigma l) & -\cos(\sigma l) & \sinh(\sigma l) & \cosh(\sigma l) \\ -\cos(\sigma l) + \frac{\sigma m}{\rho A}\sin(\sigma l) & D\sin(\sigma l) + \frac{\sigma m}{\rho A}\cos(\sigma l) & \cosh(\sigma l) + \frac{\sigma m}{\rho A}\sinh(\sigma l) & \sinh(\sigma l) + \frac{\sigma m}{\rho A}\cosh(\sigma l) \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Characteristic Equation:

$$1 + \cos(\sigma l)\cosh(\sigma l) + \frac{\sigma m}{\rho A}(\cos(\sigma l)\sinh(\sigma l) - \sin(\sigma l)\cosh(\sigma l)) = 0$$

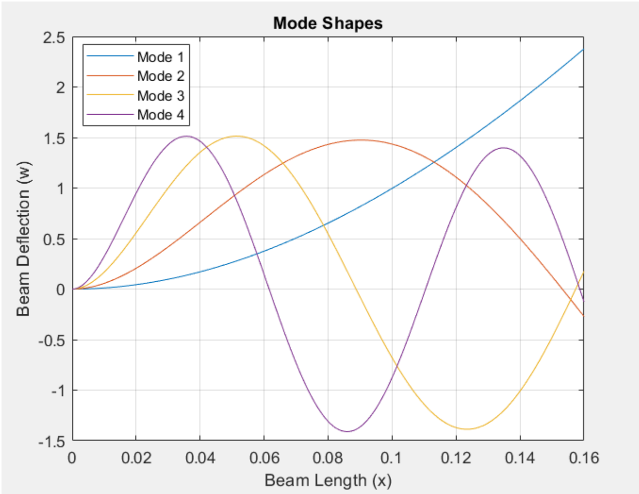
From this we get $\sigma_{1,2,3,4} =$ 7.096, 24.979, 44.4470, 64.002

Therefore, $\omega_{1,2,3,4}$ = 27.834, 344.9048, 1092.0, 2264.3 (Hz)

From our system of equations, we can also relate the coefficients to one another and write our solution in terms of one coefficient. This our spatial solution and therefore determines the mode shapes which are plotted below:

$$C_n[\sin(\sigma_n x) - \sinh(\sigma_n x) + \gamma_n(\cos(\sigma_n x) - \cosh(\sigma_n x))] = X_n(x)$$

$$\gamma_n = \frac{-\cos(\sigma_n l) - \cosh(\sigma_n l) + \frac{\sigma_n m}{\rho A}(\sin(\sigma_n l) - \sinh(\sigma_n l))}{-\sin(\sigma_n l) + \sinh(\sigma_n l) - \frac{\sigma_n m}{\rho A}(\cos(\sigma_n l) - \cosh(\sigma_n l))}$$



Finite Element Model:

To create an accurate Finite Element Model of our system for the first 3 natural frequencies it need to be discretized into at least 6 elements. This will give us 14 degrees of freedom, two of which can be eliminated at the fixed end. To account for the tip mass, the quantity of the mass was added to the transverse component of the last node along the diagonal of our global mass matrix. Using the model developed here the first three natural frequencies were found to be:

$\omega_{1,2,3}$ = 27.84, 344.94, 1093.4 (Hz)

Conclusions:

It is clear that the results found in all three methods are very similar to each other with the exception of the frequency of 769.54 Hz found in the experimental results. This can be attributed to experimental error. The results of these analysis can be applied in the real world in different ways depending on what your goal is. If the goal is to avoid failure of your structure, exciting the beam at these frequencies is something that should be avoided. Data should be collected near the structure site and analyzed to ensure that it will not be excited as such. If the goal is to excite the structure as much as possible, such as in a piezoelectric energy harvesting device, then we will want to excite this structure at these frequencies to maximize the amount of energy harvested.