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An analytical solution for non-Darcian flow in a confined aquifer using the power law function

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Abstract

We have developed a new method to analyze the power law based non-Darcian flow toward a well in a confined aquifer with and without wellbore storage. This method is based on a combination of the linearization approximation of the non-Darcian flow equation and the Laplace transform. Analytical solutions of steady-state and late time drawdowns are obtained. Semi-analytical solutions of the drawdowns at any distance and time are computed by using the Stehfest numerical inverse Laplace transform. The results of this study agree perfectly with previous Theis solution for an infinitesimal well and with the Papadopulos and Cooper's solution for a finite-diameter well under the special case of Darcian flow. The Boltzmann transform, which is commonly employed for solving non-Darcian flow problems before, is problematic for studying radial non-Darcian flow. Comparison of drawdowns obtained by our proposed method and the Boltzmann transform method suggests that the Boltzmann transform method differs from the linearization method at early and moderate times, and it yields similar results as the linearization method at late times. If the power index n and the quasi hydraulic conductivity k get larger, drawdowns at late times will become less, regardless of the wellbore storage. When n is larger, flow approaches steady state earlier. The drawdown at steady state is approximately proportional to r^{1-n}, where r is the radial distance from the pumping well. The late time drawdown is a superposition of the steady-state solution and a negative time-dependent term that is proportional to $t^{(1-n)/(3-n)}$, where t is the time.

Keywords: Non-Darcian flow; Power law; Laplace transform; Linearization method

1. Introduction

Scientists and engineers have been fascinated with non-Darcian flow for more than one hundred years [8] and there are still plenty of unanswered questions. Among many difficulties encountered in dealing with non-Darcian flow, two of them are notable. One of them is to adequately characterize non-Darcian flow using physically measurable parameters. The other is to find a robust mathematical tool to solve the non-Darcian flow governing equation.

Non-Darcian flow can arise from a number of different ways. For flow at low rates in fine-grained media such as clay and silt aquitards, the non-Darcian flow may be attributed to the electro-chemical surface effect between the fluid and the solid, and is named pre-linear flow [32,36]. The pre-linear flow has also been extensively investigated in the petroleum engineering [35,5]. For flow at high rates in coarse grained and fractured media, or near pumping wells, the non-Darcian flow may be caused by the inertial effect and the onset of turbulent flow, and is subsequently named post-linear flow [1,14,24,36,39].

Many formulas have been proposed to quantify the relationship between the hydraulic gradient and the specific

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Nomenclature

$A = \frac{S}{m}\frac{n}{k}\left(\frac{Q}{2\pi m}\right)^{n-1}$, a parameter used in Eq. (10)
$B = 2\pi r_{\mathrm{w}} mk\left[\frac{Q}{2\pi m r_{\mathrm{w}}}\right]^{(1-n)}$, a parameter used in Eq. (22)
C a parameter defined in Appendix Eq. (A10) and used in Eq. (27)
C_0, C_1, C_2 integration constants
k quasi hydraulic conductivity, an empirical constant in the Izbash equation [$(\mathrm{L/T})^n$]
m aquifer thickness [L]
n power index, an empirical constant in the Izbash equation
p Laplace variable
$q(r,t)$ specific discharge [L/T]
Q pumping discharge [L^3/T]
r distance from the center of the well [L]
r_{c} radius of well casing [L]
r_{w} effective radius of well screen [L]
R radius of influence of the pumping well [L]
$s(r,t)$ drawdown [L]
$s_w(t)$ drawdown inside well [L]
S Aquifer storage coefficient
t pumping time [T]
Γ Gamma function
$I_v(x)$ the first kind order v modified Bessel function
$K_v(x)$ the second kind order v modified Bessel function

discharge for non-Darcian flow. The most commonly used ones are the Forchheimer equation [8] and the Izbash equation [13]. The Forchheimer equation states that the hydraulic gradient is a second-order polynomial function of the specific discharge; while the Izbash equation states that the hydraulic gradient is a power function of the specific discharge. Many investigators preferred to choose the Forchheimer equation to describe non-Darcian flow based on the argument that the Forchheimer equation includes both the viscous and inertial effects and is an expansion of the linear Darcy's law e.g. [25,37,38,9,20]. The choice of the Izbash equation is based on a different philosophical argument: many post-linear non-Darcian flows are caused predominately by turbulent effects, which are conveniently modeled by power law functions [17,6,24]. In some cases, the Forchheimer equation is favored; in other cases, the Izbash equation is a better choice; and in certain circumstances, both equations can describe non-Darcian flow equally well [2,27].

In terms of the mathematical tool to study the non-Darcian flow, the Boltzmann transform has been regarded as an appropriate analytical method to solve the non-Darcian flow governing equation e.g. [25–28]. The Boltzmann transform is a special member of a family named "similarity method" that has been broadly used in solving diffusion type of problems [19,36]. The idea of the Boltzmann transform is to combine the spatial coordinate (denoted as r) and the temporal coordinate (denoted as t) into a new Boltzmann variable $\eta = r/\sqrt{t}$ and to change the partial differential equation of flow into an ordinary differential equation. Although theoretically appealing, the Boltzmann transform is rather restrictive because it requires that all the initial and boundary conditions can also be simultaneously transformed into forms only containing the Boltzmann variable η. Unfortunately, such a requirement has often not be rigorously checked or even ignored in many studies. This has caused considerable confusion in the literatures on non-Darcian flow. It is unclear how much error will be introduced when such a requirement is not satisfied.

A careful review of previous studies of non-Darcian flow indicates that the Boltzmann transform works very well when flow field is planar. For instance, Sen [25,28] and Wen et al. [36] have successfully applied the Boltzmann transform to study non-Darcian flow to a single fracture. However, the Boltzmann transform seems to be problematic when radial non-Darcian flow is considered, such as those near the pumping wells. When the groundwater flow partial differential governing equation is converted into an ordinary differential equation after the Boltzmann transform, integration is often the next step to yield the solution with an unknown integration "constant" that must be determined via the boundary condition. For a radial non-Darcian flow problem, the boundary condition near the pumping well can not be transformed into a form only depending on the Boltzmann variable. This will make the integration "constant" to be temporal or spatially dependent. For instance, the integration "constant" in Eq. (7) of Sen [26] is actually a function of time (see Eq. (9) of Sen [26]). Several scientists including Camacho-V and Vasquez-C [3] have noticed this problem before.

If the Boltzmann transform is found to be inappropriate for a particular non-Darcian flow problem, then one has to find an alternative mathematical tool to solve the non-Darcian flow problem. Fortunately, we have noticed that many non-Newtonian flow problems resemble similar non-linearity characteristics as the non-Darcian flow problems. The linearization approximation method has been proven to be a successful tool to deal with the non-linearity of the non-Newtonian flow e.g. [12,18,34]. To our knowledge, the linearization approximation method has rarely been used to investigate the non-Darcian flow, particularly for the Izbash type of non-Darcian flow. The purpose of this paper is to develop a new method for solving the power law based radial non-Darcian flow near a pumping well combining the linearization approximation with the Laplace transform. The results of the non-Darcian flow will be compared with that of the Darcian flow. The importance of different non-Darcian flow parameters will be assessed. We will also compare this method with the Boltzmann transform method used before.

2. Conceptual and mathematical models

2.1. Problem statement and solutions

Considering a vertical pumping well in a confined aquifer, the thickness of the aquifer is m. The aquifer is homogeneous and isotropic. The physical system is the same as that of Sen [26], as shown in Fig. 1a. The continuity equation can be expressed as:

$$\frac{\partial q(r,t)}{\partial r} + \frac{q(r,t)}{r} = \frac{S}{m}\frac{\partial s(r,t)}{\partial t}, \tag{1}$$

where $q(r,t)$ is the specific discharge at radial distance r and time t, $s(r,t)$ is the drawdown, and S is the storage coefficient of the aquifer. With the assumptions used in Sen [26], the initial and boundary conditions can be written as:

$$s(r,0) = 0, \tag{2}$$

$$s(\infty,t) = 0, \tag{3}$$

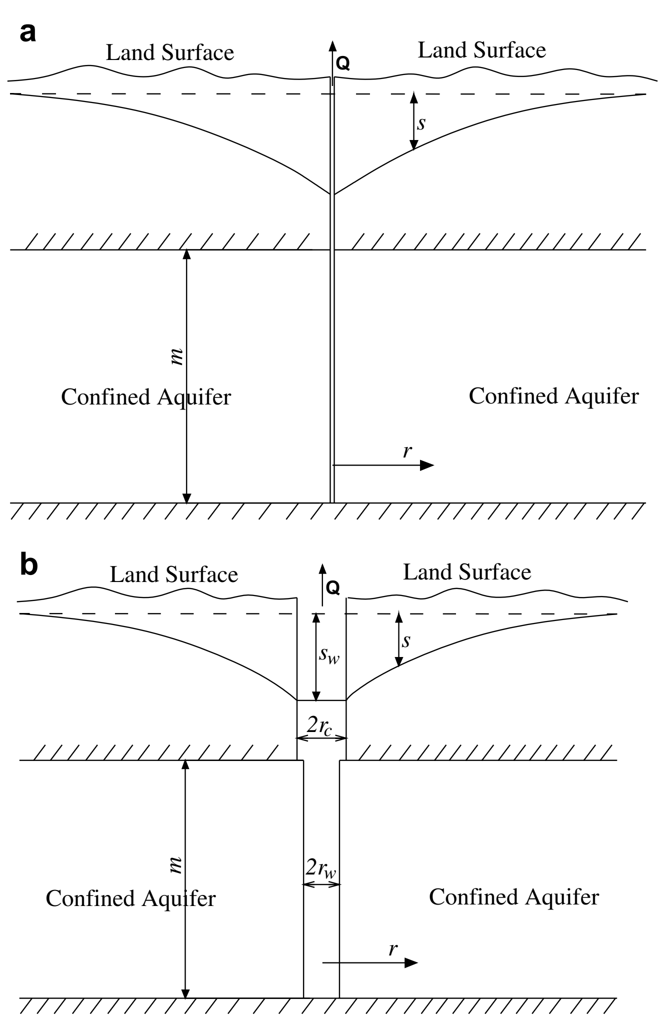

Fig. 1. The schematic diagrams of the problem (a) without wellbore storage and (b) with wellbore storage.

$$\lim_{r\to 0}[2\pi m r q(r,t)] = -Q. \tag{4}$$

where Q is the pumping rate of the well, which is assumed to be constant during the pumping period. Eq. (4) indicates that the wellbore storage is not considered here, all the pumping rate comes from the aquifer. We employ the power law to describe the non-Darcian flow:

$$[q(r,t)]^n = k\frac{\partial s(r,t)}{\partial r}, \tag{5}$$

in which k and n are constants and n is between one and two. Further discussion of Eq. (5) and the parameters k and n are given by several studies including Wen et al. [36]. When n goes to one, flow is Darcian, and k becomes the hydraulic conductivity. Thus, k can be regarded as a quasi hydraulic conductivity term [36], which reflects how easy the aquifer can transmit water. When n is less than one, flow is called pre-linear flow. When n is between one and two, flow is post-linear flow. If n approaches two, flow is fully developed turbulent, meaning that the hydraulic gradient of flow is independent of the Reynolds number [6,17]. The fully developed turbulent flow is likely to occur under certain cases such as very close to the pumping well, near a dam, preferential flow, etc. In some cases, k and n are not real constants, and depend on the property of the media and the velocities. Moutsopoulos and Tsihrintzis [16] suggested that the assumption of constant n and k was more likely not justified if a boundary condition of known piezometric head was considered, whereas it was probably justified if a known flux was considered, such as in this study. If there is strong evidence that k and/or n are not constant for the flow regime, alternative approaches other than Eq. (5) are needed. Nevertheless, k and n are treated as constants in this study. Eq. (5) can be expressed in an alternative way as:

$$[q(r,t)] = k^{1/n}\left[\frac{\partial s(r,t)}{\partial r}\right]^{1/n}. \tag{6}$$

Substituting Eq. (6) to Eq. (1) will yield:

$$\frac{\partial^2 s}{\partial r^2} + \frac{n}{r}\frac{\partial s}{\partial r} = \frac{S}{m}\frac{n}{k^{1/n}}\left(\frac{\partial s}{\partial r}\right)^{\frac{n-1}{n}}\frac{\partial s}{\partial t}. \tag{7}$$

Now one must deal with the $\left(\frac{\partial s}{\partial r}\right)^{\frac{n-1}{n}}$ term on the right hand side of Eq. (7) which is a non-linear governing equation. One possible way to linearize Eq. (7) is to replace $\partial s/\partial r$ on the right hand side of Eq. (7) by a time-independent approximation term as follows:

$$\frac{\partial s(r,t)}{\partial r} = \frac{[q(r,t)]^n}{k} \approx -\frac{\left(\frac{Q}{2\pi rm}\right)^n}{k}. \tag{8}$$

Eq. (8) is referred to as the linearization approximation. This approximation means that the amount of water passed through any radial face per unit time is treated as Q regardless of how far away from the center of the well. In a rigorous sense, the flow rate is equal to Q only at the pumping wellbore. Similar linearization approximations have been commonly used in studying non-

Newtonian flow e.g. [12,18,34]. Finite difference solutions in non-Newtonian flow studies indicated that the errors caused by the approximation are generally small [12,18,34]. This approximation works the best at places near the pumping well and/or at late times. It will be less accurate at early times when drawdown changes rapidly with time. With this approximation, Eq. (7) can be changed to:

$$\frac{\partial^2 s}{\partial r^2}+\frac{n}{r}\frac{\partial s}{\partial r}=\frac{S}{m}\frac{n}{k}\left(\frac{Q}{2\pi m}\right)^{n-1}r^{1-n}\frac{\partial s}{\partial t}. \tag{9}$$

Eq. (9) is a linear partial differential equation, which can be solved by Laplace transform. With the Laplace transform for time t, one has:

$$\frac{\mathrm{d}^2\bar{s}}{\mathrm{d}r^2}+\frac{n}{r}\frac{\mathrm{d}\bar{s}}{\mathrm{d}r}=Ar^{1-n}p\bar{s} \tag{10}$$

where $A=\frac{S}{m}\frac{n}{k}\left(\frac{Q}{2\pi m}\right)^{n-1}$, p is the Laplace variable, over bar of variable s means the Laplace transform of s. The initial condition has been used in the Laplace transform. With this transform, the boundary conditions can be changed to:

$$\bar{s}(\infty,p)=0, \tag{11}$$

$$\lim_{r\to 0} r^n\left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}r}\right)=-\frac{\left(\frac{Q}{2\pi m}\right)^n}{kp}. \tag{12}$$

Eq. (10) is a form of Bessel's equation. Its ordinary solution has been given in previous studies on non-Newtonian flow [12]. Carslaw and Jaeger p. 414, case V [4] have also provided a solution for this problem, which is

$$\bar{s}(r,p)=r^{\frac{1-n}{2}}\left[C_1 I_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)+C_2 K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)\right], \tag{13}$$

in which $I_{\frac{1-n}{3-n}}(x)$ and $K_{\frac{1-n}{3-n}}(x)$ are the first and second kind modified Bessel function with the order $\frac{1-n}{3-n}$, respectively. C_1 and C_2 are the integration constants depending on the boundary conditions. In terms of the boundary condition of Eq. (11), $C_1=0$. Then, Eq. (13) becomes:

$$\bar{s}(r,p)=r^{\frac{1-n}{2}}\left[C_2 K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)\right]. \tag{14}$$

Applying Eq. (12) in Eq. (14), one has:

$$\begin{aligned}C_2 r^n\Bigg[&\frac{1-n}{2}r^{\frac{-1-n}{2}}K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)\\&+r^{\frac{1-n}{2}}K'_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)r^{\frac{1-n}{2}}\sqrt{Ap}\Bigg]\\&=-\frac{\left(\frac{Q}{2\pi m}\right)^n}{kp},\quad \text{for}\quad r\to 0.\end{aligned} \tag{15}$$

With the properties of the modified Bessel functions: $x\mathrm{d}K_v(x)/\mathrm{d}x+vK_v(x)=-xK_{v-1}(x)$ and $K_v(x)=K_{-v}(x)$ p. 505 [29], Eq. (15) can be reduced to:

$$C_2 r^n\left[-r^{1-n}\sqrt{Ap}K_{\frac{2}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)\right]=-\frac{\left(\frac{Q}{2\pi m}\right)^n}{kp},\ \text{for}\ r\to 0. \tag{16}$$

According to the property of the second kind modified Bessel function $K_v(x)\approx\frac{\Gamma(v)}{2}\left(\frac{x}{2}\right)^{-v}$, $v>0$ when x goes to zero p. 505 [29], in which $\Gamma()$ is the gamma function, one has:

$$C_2=\frac{2\left(\frac{Q}{2\pi m}\right)^n\left(\frac{\sqrt{Ap}}{3-n}\right)^{\frac{2}{3-n}}}{kp\sqrt{Ap}\Gamma\left(\frac{2}{3-n}\right)}. \tag{17}$$

Therefore, one can determine the solution of the drawdown in Laplace domain accordingly:

$$\bar{s}(r,p)=\frac{2\left(\frac{Q}{2\pi m}\right)^n\left(\frac{\sqrt{Ap}}{3-n}\right)^{\frac{2}{3-n}}}{kp\sqrt{Ap}\Gamma\left(\frac{2}{3-n}\right)}r^{\frac{1-n}{2}}K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right). \tag{18}$$

Eq. (18) can be solved by using either the analytical or the numerical Laplace inversion methods. Luan [15] has used an analytical Laplace inversion method to investigate non-Newtonian flow in a double-porosity formation. However, the analytical inversion is often complex and not always available. It is proven that the numerical methods are very efficient and useful for such inversion problems e.g. [30,31,7]. We have evaluated Eq. (18) by using the Stehfest method [30,31], the drawdowns at any given distance r and time t can be obtained when the parameters of the aquifer are known.

We have developed a MATLAB program to compute the drawdown by using this inversion method. The Stehfest method requires a choice of N which is the number of terms used in the inversion. Our numerical exercise shows that $N=18$ gives the most accurate and stable solution, thus will be used in all the following calculation. The MATLAB program is simple and straightforward with the Stehfest method as the primary component. The program is free of charge from the authors upon request.

2.2. Solutions with wellbore storage

When the well radius is relatively large, the wellbore storage can not be ignored. Large diameter wells are commonly used all over the world, especially in developing countries such as India and China e.g. [11,10]. Similar to the physical system of Papadopulos and Cooper [21], the schematic diagram is shown in Fig. 1b. If the wellbore storage is considered, the boundary condition will be expressed as:

$$2\pi r_{\mathrm{w}}mk^{1/n}\left[\left.\frac{\partial s(r,t)}{\partial r}\right|_{r\to r_{\mathrm{w}}}\right]^{1/n}-\pi r_{\mathrm{c}}^2\frac{\partial s_{\mathrm{w}}(t)}{\partial t}=-Q,\quad t>0. \tag{19}$$

in which r_w is the effective radius of the well screen and r_c is the radius of the well casing. In most cases, r_c is not equal to and often much larger than r_w. $s_w(t)$ is the drawdown in the well, which is equal to the drawdown at the face of wellbore in the aquifer $s(r_w,t)$. With the Laplace transform, Eq. (19) becomes

$$2\pi r_{\mathrm{w}} m k^{1/n}\left[\left.\frac{\mathrm{d}\bar{s}(r,t)}{\mathrm{d}r}\right|_{r\to r_{\mathrm{w}}}\right]^{1/n} - \pi r_{\mathrm{c}}^2 p \bar{s}_{\mathrm{w}}(t) = -\frac{Q}{p}. \quad (20)$$

Eq. (20) is a non-linear boundary condition. The solution is still in the form of Eq. (13) with $C_1 = 0$. However, it seems difficult to obtain the integration constant C_2 of Eq. (13) directly from the non-linear boundary condition Eq. (20). In light of the linearization method as shown in Eq. (8), then Eq. (19) can be simplified to

$$2\pi r_{\mathrm{w}} m k\left[\frac{Q}{2\pi m r_{\mathrm{w}}}\right]^{(1-n)}\left.\frac{\partial s(r,t)}{\partial r}\right|_{r\to r_{\mathrm{w}}} - \pi r_{\mathrm{c}}^2\frac{\partial s_{\mathrm{w}}(t)}{\partial t} \approx -Q. \quad (21)$$

Taking the Laplace transform of Eq. (21), one has

$$B\left.\frac{\mathrm{d}\bar{s}(r,t)}{\mathrm{d}r}\right|_{r\to r_{\mathrm{w}}} - \pi r_{\mathrm{c}}^2 p\bar{s}(r_{\mathrm{w}},p) = -\frac{Q}{p}, \quad (22)$$

in which $B = 2\pi r_{\mathrm{w}} m k\left[\frac{Q}{2\pi m r_{\mathrm{w}}}\right]^{(1-n)}$. Substituting Eq. (22) into Eq. (13), one has:

$$C_2 = \frac{Q}{p\left\{B r_{\mathrm{w}}^{1-n}\sqrt{Ap}K_{\frac{2}{3-n}}\left(\frac{2}{3-n}r_{\mathrm{w}}^{\frac{3-n}{2}}\sqrt{Ap}\right) + \pi r_{\mathrm{c}}^2 p r_{\mathrm{w}}^{\frac{1-n}{2}} K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r_{\mathrm{w}}^{\frac{3-n}{2}}\sqrt{Ap}\right)\right\}}. \quad (23)$$

After obtaining C_2, the solution with wellbore storage in Laplace domain can be written as

$$\bar{s}(r,p) = \frac{Q r^{\frac{1-n}{2}} K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r^{\frac{3-n}{2}}\sqrt{Ap}\right)}{p\left\{B r_{\mathrm{w}}^{1-n}\sqrt{Ap}K_{\frac{2}{3-n}}\left(\frac{2}{3-n}r_{\mathrm{w}}^{\frac{3-n}{2}}\sqrt{Ap}\right) + \pi r_{\mathrm{c}}^2 p r_{\mathrm{w}}^{\frac{1-n}{2}} K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r_{\mathrm{w}}^{\frac{3-n}{2}}\sqrt{Ap}\right)\right\}}. \quad (24)$$

When n approaches 1, flow becomes Darcian. Eq. (24) can be approximated by

$$\bar{s}(r,p) = \frac{Q K_0\left(r\sqrt{\frac{S}{mk}p}\right)}{p\left\{2\pi r_{\mathrm{w}} m k\sqrt{\frac{S}{mk}p}K_1\left(r_{\mathrm{w}}\sqrt{\frac{S}{mk}p}\right) + \pi r_{\mathrm{c}}^2 p K_0\left(r_{\mathrm{w}}\sqrt{\frac{S}{mk}p}\right)\right\}}. \quad (25)$$

Eq. (25) is the same as the solution of Papadopulos and Cooper [21] in Laplace domain. Eq. (24) can also be evaluated by the numerical inversion which is included in the MATLAB program.

2.3. Approximate analytical solutions at steady state and late times

Before the discussion of the results, it is necessary to obtain the analytical solutions at steady state and late times. For steady-state flow, the drawdown does not depend on time t, thus $\frac{\partial s}{\partial t} = 0$. Appendix shows the detailed derivation of obtaining the steady-state drawdown which is

$$s(r) = \left(\frac{Q}{2\pi m}\right)^n \frac{1}{k(n-1)}\left(\frac{1}{r^{n-1}} - \frac{1}{R^{n-1}}\right), \quad r_{\mathrm{w}} \leqslant r \leqslant R, \quad (26)$$

where r_{w} is the effective radius of the well screen and R is the radius of influence of the well at which the drawdown is negligible. Notice that $1 < n \leqslant 2$, thus the drawdown decreases with distance with a power of $(n-1)$. For the special case of Darcian flow, n is approaching 1, and Eq. (26) can be simplified as follows. Recalling the identity that $\lim_{n\to 1}(1/r^{n-1}) = \lim_{n\to 1}\{\exp[-(n-1)\ln r]\} \cong 1-(n-1)\ln r$, thus Eq. (26) becomes the well known steady-state solution for Darcian flow to a well: $s(r) = \left(\frac{Q}{2\pi mk}\right)\ln(r/R)$. A minor issue to point out is that the radius of influence R must be used when discussing the steady-state Darcian flow in a confined aquifer; otherwise, the drawdown will become infinity when r goes to infinity.

It is notable that van Poollen and Jargon [33] have provided an analytical solution of steady-state non-Newtonian fluid flow. They have also provided a numerical solution of transient non-Newtonian fluid flow. Eq. (26) indicates that the steady-state drawdown will increase with the pumping rate Q, and decrease with the quasi hydraulic conductivity k and the thickness of the aquifer m. Taking the derivative with respect to r on both sides of Eq. (26), it is found that $\frac{\mathrm{d}s(r)}{\mathrm{d}r} = -\frac{(Q/2\pi mr)^n}{k}$, which is consistent with Eq. (8).

In addition to the steady-state solution, it is also interesting to derive the time-dependent late time solution which may be very useful in well testing analysis because the late time solution is normally accurate enough after a certain time of pumping, as demonstrated by Wu [37]. The late time analytical solution can be obtained by conducting the inverse Laplace transform of Eq. (18) by allowing the Laplace transform parameter p to be sufficiently small. The Appendix has given the detailed derivation of the later time analytical solution which is

$$s(r,t) = \left(\frac{Q}{2\pi m}\right)^n \frac{1}{k(n-1)}\left(\frac{1}{r^{n-1}} - \frac{C}{t^{\frac{n-1}{3-n}}}\right), \quad (27)$$

where the constant C is given in Eq. (A10) of the Appendix.

It is interesting to see that the late time drawdown is simply a superposition of the steady-state solution and a negative term that decreases with time with a power of $(n-1)/(3-n)$. Interestingly, the time-dependent decreasing term in Eq. (27) is independent of spatial coordinate. Since the wellbore storage will not affect the steady-state and late time drawdowns. Above solutions Eqs. (26) and (27) are also valid for the case with the wellbore storage.

3. Results and discussion

There are two different ways to present the results. One way is to present the dimensionless drawdown versus dimensionless time in log–log scales, the so-called type curves, as often done in Darcian flow studies. The other way is to analyze the result in dimensional forms. For Darcian flow, one of the advantages of using type curves is based on the consideration that the dimensionless drawdown is often a function of dimensionless time only in a confined aquifer, provided that other effects such as leakage across the aquifer boundaries are not considered. For non-Darcian flow discussed here, the benefits of using type curves are not always obvious because of the non-linear power law relationship between the discharge and the hydraulic gradient. Furthermore, it is straightforward to test the sensitivity of the solutions to two parameters n and k in dimensional forms. Therefore, we prefer a dimensional analysis for most parts of the following analysis. The type curves are only used when the results are compared with previous solutions of Darcian flow under the special case of $n = 1$.

Nevertheless, the type curves for non-Darcian flow can be easily obtained based on the dimensional analysis if the dimensionless terms can be adequately defined. With the developed MATLAB program for the numerical Laplace inversion, we have obtained the drawdown values against time which are presented in log–log scales, as shown in Figs. 2–13. In the following discussion, we only consider the case that n is larger than one, because pre-linear flow is unlikely to occur near the pumping wells.

3.1. Drawdowns without wellbore storage

3.1.1. Comparison of the non-linear type curves with Theis curves

When n approaches one, flow is approaching Darcian. The result of Eq. (18) will approach the classical Theis solution in the Laplace domain. We have compared our results for $n = 1$ with the Theis type curves as shown in Fig. 2, which has the same axes as those defined in Theis type curves, i.e. $u = \frac{r^2 S}{4mkt}$ and $W(u) = \frac{4\pi mk}{Q} s(r, t)$. It is clear to see that our results agree perfectly with the Theis type curves. This indicates that our MATLAB program based

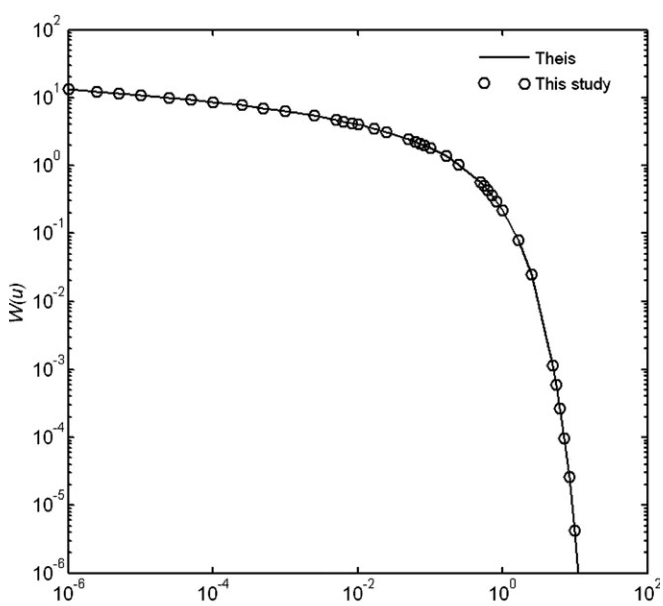

Fig. 2. Comparison of the type curves for non-Darcian flow and Theis type curves.

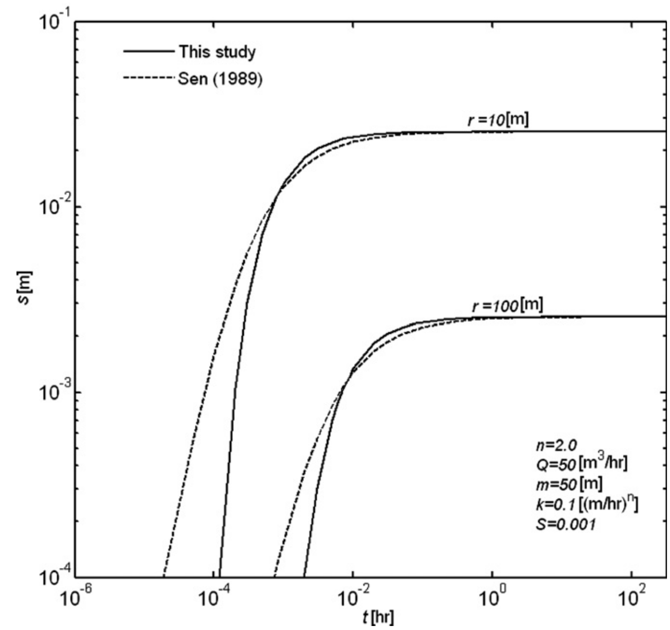

Fig. 3. Comparison of the drawdowns obtained by the proposed linearization and Laplace transform method and the Boltzmann transform method with $n = 2.0$, $Q = 50\ \mathrm{m^3/h}$, $m = 50$ m, $k = 0.1(\mathrm{m/h})^n$, and $S = 0.001$ for the distances $r = 10$ m and 100 m, respectively.

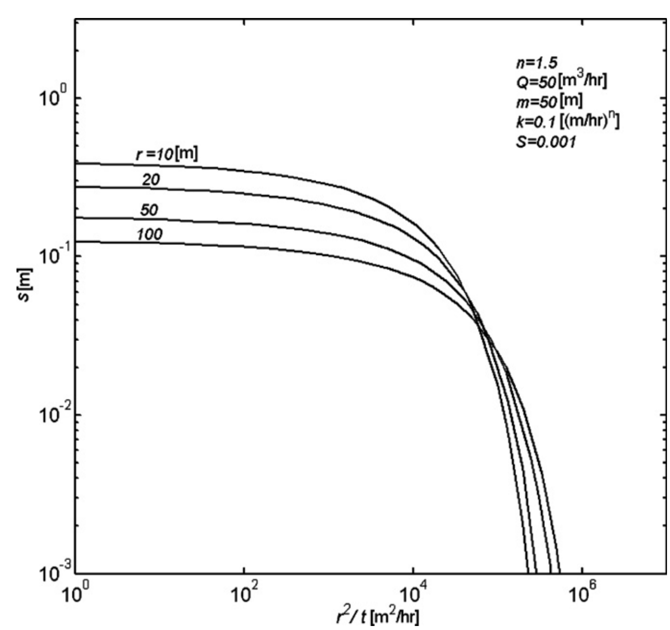

Fig. 4. Drawdowns versus r^2/t with $n = 1.5$, $Q = 50\ \mathrm{m^3/h}$, $m = 50$ m, $k = 0.1(\mathrm{m/h})^n$, and $S = 0.001$ for the distances $r = 10$ m, 20 m, 50 m, and 100 m, respectively.

on the numerical inversion is applicable. On the other hand, it may suggest that the errors of the linearization approximation are negligible at least when n is close to one. In the following analysis, we consider the linearization solutions as ''quasi-exact'' solutions when comparing to the results of the Boltzmann method.

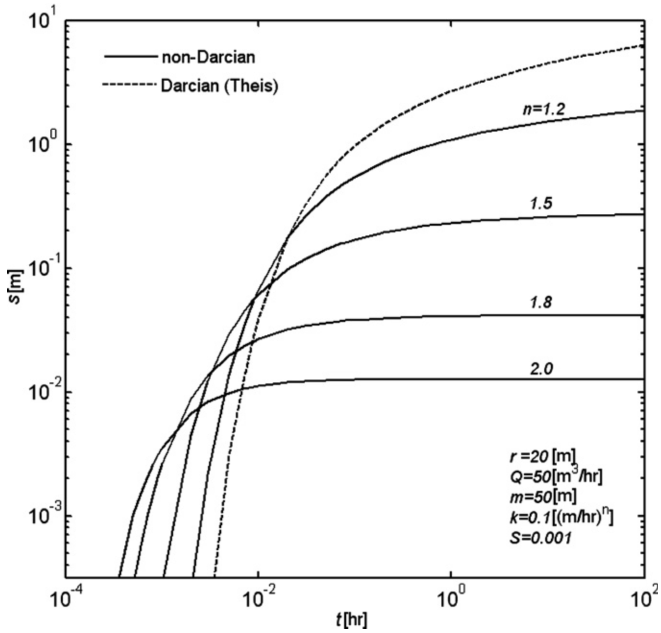

Fig. 5. Drawdowns versus time with $Q = 50\ \mathrm{m^3/h}$, $r = 20$ m, $m = 50$ m, $k = 0.1(\mathrm{m/h})^n$, and $S = 0.001$ for the values of power index $n = 1.2$, 1.5, 1.8, and 2.0, respectively.

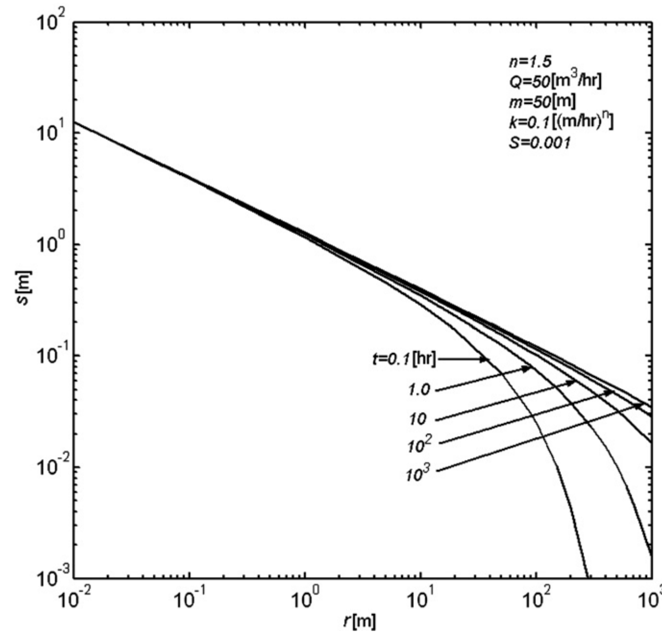

Fig. 7. Drawdowns versus distance with $n = 1.5$, $Q = 50\ \mathrm{m^3/h}$, $m = 50$ m, $k = 0.1(\mathrm{m/h})^n$, and $S = 0.001$ for time $t = 0.1$ h, 1 h, 10 h, 100 h and 1000 h, respectively.

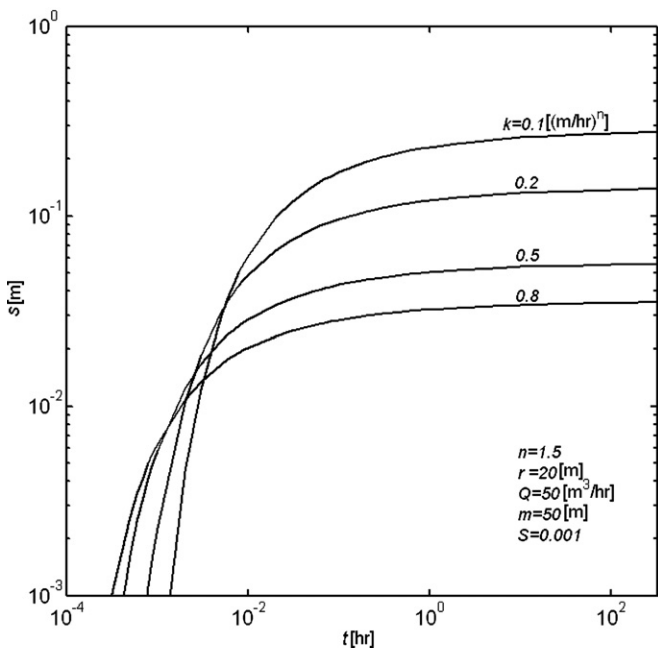

Fig. 6. Drawdowns versus time with $n = 1.5$, $Q = 50\ \mathrm{m^3/h}$, $r = 20$ m, $m = 50$ m, and $S = 0.001$ for the quasi hydraulic conductivity values $k = 0.1(\mathrm{m/h})^n$, $0.2(\mathrm{m/h})^n$, $0.5(\mathrm{m/h})^n$, and $0.8(\mathrm{m/h})^n$, respectively.

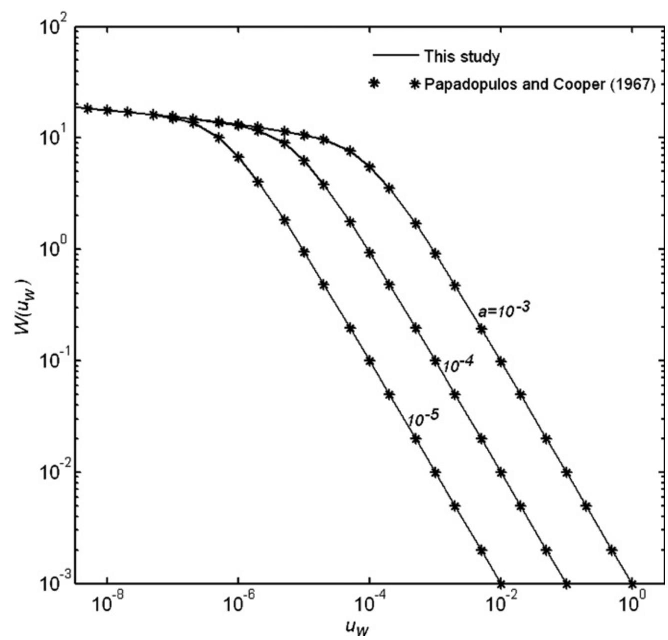

Fig. 8. Comparison of the wellbore well function of this study and that of Papadopulos and Cooper (1967) for $a = 10^{-3}$, 10^{-4}, and 10^{-5}, respectively.

3.1.2. Comparison of this study with those based on the Boltzmann transform

The comparison of our linearization results with the solutions of the Boltzmann transform [26] is shown in Fig. 3. As presented by Sen [26], the drawdown obtained by using the Boltzmann transform can be expressed as $s(r,t) = \frac{1}{k}\left(\frac{Q}{2\pi m}\right)^n \int_r^\infty \frac{1}{r'^n}\left[-2\frac{n-1}{n-3}\frac{r'^2 S}{4mkt}\left(\frac{Q}{2\pi m r'}\right)^{n-1}+1\right]^{n/(1-n)} \mathrm{d}r'$, in which r' is a dummy integration variable. As shown in Fig. 3, considerable differences have been found at early and moderate times for fully turbulent flow ($n = 2$). Whereas the Boltzmann transform method and the linearization method yield nearly the same results at late times at locations that are 10 m and 100 m away from the pumping well, as seen in Fig. 3.

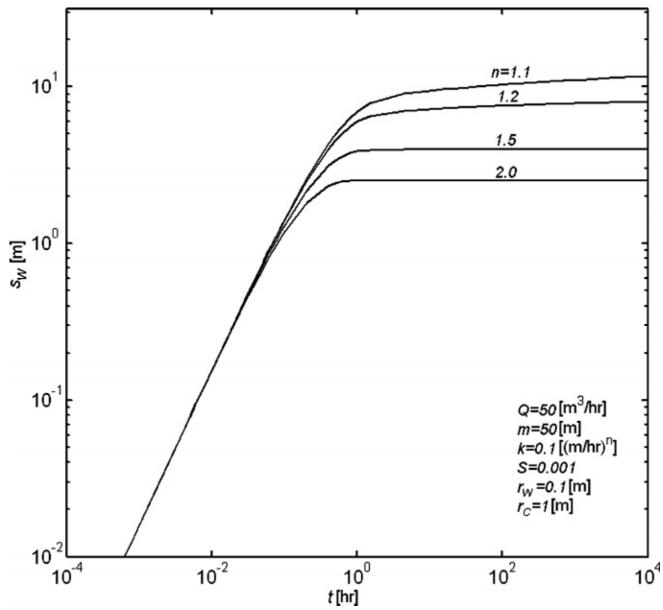

Fig. 9. Drawdowns in the well with wellbore storage with $Q = 50\ \mathrm{m}^3/\mathrm{h}$, $m = 50\ \mathrm{m}$, $k = 0.1(\mathrm{m/h})^n$, $r_w = 0.1\ \mathrm{m}$, $r_c = 1\ \mathrm{m}$, and $S = 0.001$ for the values of power index $n = 1.1$, 1.2, 1.5, and 2.0, respectively.

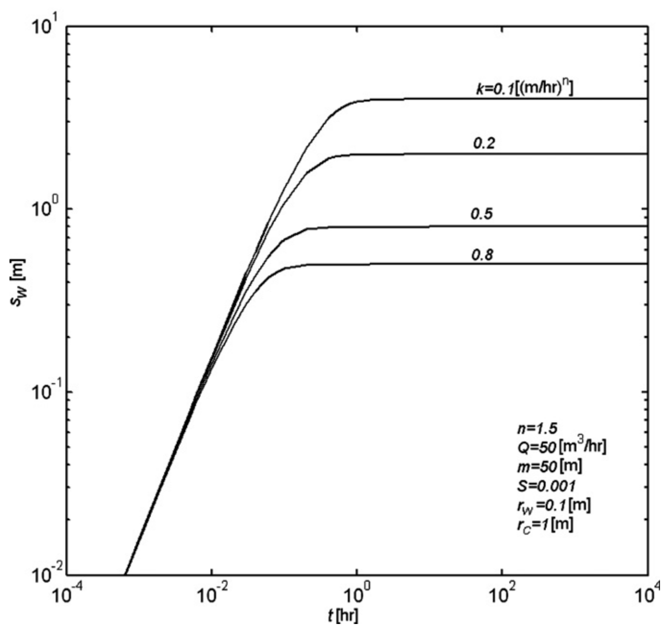

Fig. 10. Drawdowns in the well with wellbore storage with $Q = 50\ \mathrm{m}^3/\mathrm{h}$, $m = 50\ \mathrm{m}$, $n = 1.5$, $r_\mathrm{w} = 0.1\ \mathrm{m}$, $r_\mathrm{c} = 1\ \mathrm{m}$, and $S = 0.001$ for the quasi hydraulic conductivity values $k = 0.1$, 0.2, 0.5 and 0.8, respectively.

A careful check of Sen [26] indicates that the boundary condition can not be transformed into the form in terms of the variable $\eta = r/\sqrt{t}$ only. Therefore, the solutions as shown in Eqs. (9) and (10) of Sen [26] are not only functions of variable η but also functions of time t or r, violating the requirement of using the Boltzmann transform.In order to demonstrate that the drawdown for non-Darcian flow is not only a function of r^2/t, we compute the drawdowns for four different distances $r = 10\ \mathrm{m}$, 20 m, 50 m and 100 m with $n = 1.5$, $Q = 50\ \mathrm{m}^3/\mathrm{s}$, $m = 50\ \mathrm{m}$, $k = 0.1\ (\mathrm{m/hr})^n$ and $S = 0.001$, and present the results in Fig. 4. Differences have been found among the curves of s–r^2/t at the four distances. This proves that the drawdown is indeed not only a function of η.

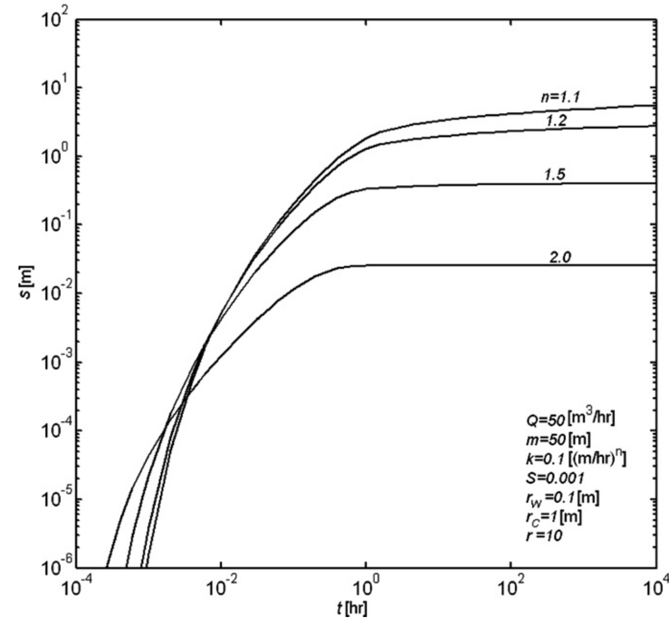

Fig. 11. Drawdowns in the aquifer with wellbore storage with $Q = 50\ \mathrm{m}^3/\mathrm{h}$, $m = 50\ \mathrm{m}$, $k = 0.1(\mathrm{m/h})^n$, $r_\mathrm{w} = 0.1\ \mathrm{m}$, $r_\mathrm{c} = 1\ \mathrm{m}$, $r = 10\ \mathrm{m}$, and $S = 0.001$ for the values of power index $n = 1.1$, 1.2, 1.5 and 2.0, respectively.

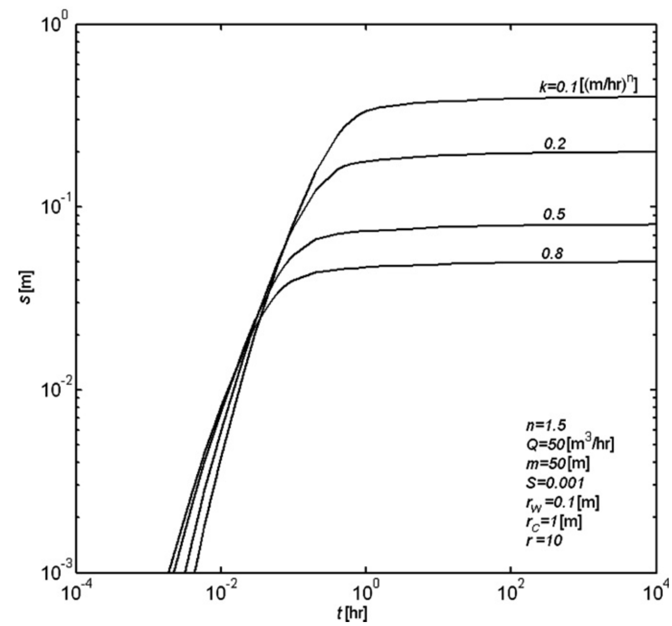

Fig. 12. Drawdowns in the aquifer ($r = 10\ \mathrm{m}$) with wellbore storage with $Q = 50\ \mathrm{m}^3/\mathrm{h}$, $m = 50\ \mathrm{m}$, $n = 1.5$, $r_\mathrm{w} = 0.1\ \mathrm{m}$, $r_\mathrm{c} = 1\ \mathrm{m}$, $r = 10\ \mathrm{m}$, and $S = 0.001$ for the quasi hydraulic conductivity values $k = 0.1$, 0.2, 0.5, and 0.8, respectively.

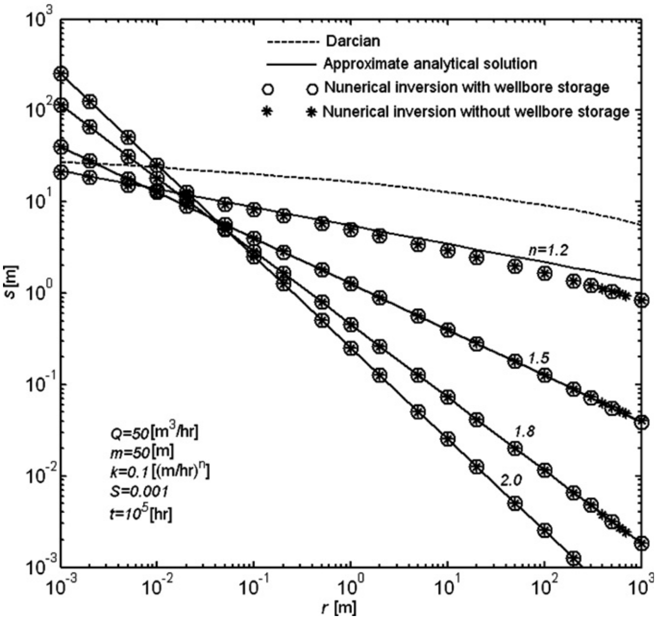

Fig. 13. Drawdowns at $t = 10^5$ h versus distance with $Q = 50\,\mathrm{m}^3/\mathrm{h}$, $m = 50$ m, $k = 0.1(\mathrm{m/h})^n$, and $S = 0.001$ for the values of power index $n = 1.1$, 1.2, 1.5 and 2.0, respectively.

Interestingly, despite of the problems associated with the Boltzmann transform, the results obtained from the linearization method seem to agree reasonably well with those obtained from the Boltzmann transform method at late times, as reflected in Fig. 3. However, when the wellbore storage is included, we have found that the linearization method gives the same drawdown at the pumping wellbore as that of Papadopulos and Cooper [21] for Darcian flow (see detailed discussion in Section 3.2); on the contrary, the Boltzmann transform method can not reproduce the same wellbore drawdown as predicted by Papadopulos and Cooper [21] for Darcian flow.

3.1.3. Effect of the power index n and the quasi hydraulic conductivity k

The sensitivity of the drawdown versus n is shown in Fig. 5 with $n = 1.2$, 1.5, 1.8 and 2.0, respectively. As shown in this figure, the late time drawdown decreases when n increases at any given distance. This finding is consistent with the steady-state solution Eq. (26) and the late time solution Eq. (27). Physically speaking, this is also understandable. A greater n value implies greater deviation from Darcian flow and greater degree of turbulence of flow, indicating that flow near the well will experience greater degree of resistance. As a result, the cone of depression of the pumping well will be shrunk to a smaller volume surrounding the well. In another word, the drawdown will drop to zero faster when moving away from the well. The consequence of this is a smaller drawdown at any given distance r and time t.

Fig. 5 also shows that when n is larger, flow approaches steady state more quickly. This finding is proved by the analytical solution Eq. (27) in which the time-dependent term drops to zero faster when n gets larger.

The impact of the quasi hydraulic conductivity k on the drawdown is also analyzed. The results are shown in Fig. 6. At early times, the drawdown is greater when the value of k is larger; while at late times, the drawdown is less when k is larger. This finding is similar to the influence of the power index n on the drawdown.

3.1.4. Drawdowns versus distance

All the discussion above is about the drawdown versus time t. It is equally important to analyze the drawdown versus distance r. We have analyzed the drawdowns versus distance r at five given times, as shown in Fig. 7. It is interesting to notice that for flow near the pumping well, all the curves approach the same asymptotic value in Fig. 7. This simply reflects the fact that the drawdowns at places very close to the pumping well reach steady state shortly after the start of pumping.

3.2. Drawdowns with wellbore storage

3.2.1. Drawdown in the well

When n approaches one, flow is Darcian, then the flow model with wellbore storage is the same as that of Papadopulos and Cooper [21] for Darcian flow to a large diameter well. We can use the analytical solutions of Papadopulos and Cooper [21] to verify our results. In order to make this comparison feasible, we use the same dimensionless variables as that of Papadopulos and Cooper [21]: the wellbore well function $W(u_\mathrm{w}) = \frac{4\pi mk}{Q} s_\mathrm{w}(t)$, the dimensionless time factor $u_\mathrm{w} = \frac{r_\mathrm{w}^2 S}{4mkt}$, and $a = \frac{r_\mathrm{w}^2 S}{r_\mathrm{c}^2}$. The computed type curves based on both the numerical inversion of Eq. (25) and the analytical solutions of Papadopulos and Cooper [21] are shown in Fig. 8. It is obvious to notice that our numerical inversion results under Darcian flow case ($n = 1$) agree very well with the solutions of Papadopulos and Cooper [21].

The sensitivity analysis of the power index n and the quasi hydraulic conductivity k on the drawdown in the well are shown in Figs. 9 and 10, respectively. In these two figures, all the diagrams in log–log scales can be divided into three portions. For the first portion at early times, all the lines are straight and approach the same asymptotic values, reflecting the fact that the pumped water comes from the wellbore storage entirely during this period. Similar results have been found by Park and Zhan [22,23] in horizontal wells. For the second portion at moderate time, all the curves start to deviate from the straight lines, meaning that the pumping rate is partially from the wellbore storage and partially from the aquifer. While at late times, the third portion, a larger n or k value leads to a smaller drawdown in the well. This finding is consistent with the late time analytical solution of Eq. (27).

3.2.2. Drawdowns in the aquifer

We have also analyzed the drawdown in the aquifer with different n and k values, as shown in Figs. 11 and 12 by using distance $r = 10$ m as an example. It can be found that the drawdown in the aquifer is less when n or k are greater at late times; while the opposite is true at early times. This finding is similar to that of the drawdown in the aquifer without considering wellbore storage.

3.3. Drawdowns versus distances at late times

To demonstrate the late time behavior of the drawdowns, we compute the drawdowns with different n values at $t = 10^5$ h for both Eqs. (14) and (23), the results are shown in Fig. 13. The approximate analytical solution Eq. (26) for steady-state flow is also included in this figure. The subtle difference for the case of $n = 1.2$ might due to that flow is still at unsteady stage even for time as large as 10^5 h. When the time is longer than 10^5 h, the numerical inversion results approach the steady-state analytical results. This indicates when n is smaller, it will take longer time to approach the steady state.

It can also be found that all the drawdown curves are nearly straight with different n values in log–log scales. This again can be explained by the approximate analytical solution of Eq. (26) in which the drawdown is proportional to r^{1-n}. When plotted in log–log scales, the relationship between the drawdown and the distance is a straight line with a slope of $1 - n$.

4. Summary and conclusions

We have developed a method to compute the drawdown of the power law based non-Darcian flow toward a well in a confined aquifer with and without wellbore storage. To use this method, one first has to approximate the non-Darcian flow equation with a linearization equation, then to obtain the solutions of the linearization equation in Laplace domain, and finally to obtain the drawdowns by using a numerical inverse Laplace transform method. The MATALAB based program has been developed to facilitate the numerical computation. Drawdowns obtained by our proposed method have been compared with those obtained by using the Boltzmann transform method. We have also analyzed the sensitivity of the drawdowns both in the well and in the aquifer to a number of parameters such as the power law index n and the quasi hydraulic conductivity k.

Several findings can be drawn from this study. The drawdown for the power law based non-Darcian radial flow can not be expressed as a function of $\eta = r/\sqrt{t}$, this means that the drawdown can not be obtained by directly solving the non-Darcian radial flow equation with the Boltzmann transform. The Boltzmann transform method differs from the linearization method considerably at early and moderate times, but it yields nearly the same results as the linearization method at late times. The results of this new method for the special Darcian flow case ($n = 1$) agree perfectly with that of the Theis solution for an infinitesimally small pumping well, and with that of the Papadopulos and Cooper solution [21] for a finite-diameter pumping well. If the power index n and the quasi hydraulic conductivity k get larger, drawdowns at early times will get greater; whereas drawdowns at late times will become less, regardless of the wellbore storage. When n is larger, flow approaches steady state earlier. And the drawdown is approximately proportional to r^{1-n} at steady state. The late time drawdown is a superposition of the steady-state solution and a negative time-dependent term that is proportional to $t^{(1-n)/(3-n)}$.

Acknowledgements

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Appendix A. Derivation of the analytical solutions at steady state and late times

For steady-state flow, one has $\frac{\partial s}{\partial t} = 0$. Then Eq. (9) can be changed to

$$\frac{\partial^2 s}{\partial r^2} + \frac{n}{r}\frac{\partial s}{\partial r} = 0, \tag{A1}$$

or,

$$\frac{\partial}{\partial r}\left(r^n \frac{\partial s}{\partial r}\right) = 0. \tag{A2}$$

Then one has

$$r^n \frac{\partial s}{\partial r} = C_0, \quad \text{or} \quad \frac{\partial s}{\partial r} = \frac{C_0}{r^n}, \tag{A3}$$

where C_0 is a constant, which can be obtained by the boundary condition Eq. (8). Substituting Eq. (A3) to Eq. (8), one has

$$C_0 = -\frac{\left(\frac{Q}{2\pi m}\right)^n}{k}. \tag{A4}$$

For steady-state flow, the drawdown at a sufficiently far distance R from the well will be essentially be zero, where R is often called the radius of influence of the well. Therefore, integrating Eq. (A3) leads to the final steady-state solution as

$$s(r) = \left(\frac{Q}{2\pi m}\right)^n \frac{1}{k(n-1)}\left(\frac{1}{r^{n-1}} - \frac{1}{R^{n-1}}\right), \quad r_w \leqslant r \leqslant R \tag{A5}$$

The approximate analytical solution at late times can be obtained by allowing the Laplace transform parameter p

to be very small in Eq. (18). Recalling the following two term approximation of $K_v(x)$ p. 505, 51:9:2 [29]:

$$K_v(x) \approx \frac{\Gamma(-v)x^v}{2^{1+v}} + \frac{\Gamma(v)x^{-v}}{2^{1-v}}, \quad 0 < |v| < 1, \quad \text{for small} \quad x \tag{A6}$$

Then

$$K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n} r^{\frac{3-n}{2}} \sqrt{Ap}\right) = \frac{\Gamma\left(\frac{n-1}{3-n}\right)}{2}\left(\frac{r^{\frac{3-n}{2}}\sqrt{Ap}}{3-n}\right)^{\frac{1-n}{3-n}} + \frac{\Gamma\left(\frac{1-n}{3-n}\right)}{2}\left(\frac{r^{\frac{3-n}{2}}\sqrt{Ap}}{3-n}\right)^{\frac{n-1}{3-n}}, \tag{A7}$$

and Eq. (18) becomes:

$$\bar{s}(r,p) = \left(\frac{Q}{2\pi m}\right)^n \frac{1}{k}\left(\frac{1}{p(n-1)r^{n-1}} + \frac{\Gamma\left(\frac{1-n}{3-n}\right)}{\Gamma\left(\frac{2}{3-n}\right)} \times \frac{A^{\frac{n-1}{3-n}}}{(3-n)^{\frac{n+1}{3-n}}} \times \frac{1}{p^{\frac{4-2n}{3-n}}}\right). \tag{A8}$$

Recalling the following inverse Laplace transform identities $L^{-1}[1/p] = 1$, $L^{-1}[1/p^\delta] = t^{\delta-1}/\Gamma(\delta)$, and considering the properties of the Gamma function p. 414 [29], the inverse Laplace transform of Eq. (A8) becomes:

$$s(r,t) = \left(\frac{Q}{2\pi m}\right)^n \frac{1}{k(n-1)}\left(\frac{1}{r^{n-1}} - \frac{C}{t^{\frac{n-1}{3-n}}}\right), \tag{A9}$$

where the constant C equals:

$$C = \frac{A^{\frac{n-1}{3-n}}}{(3-n)^{\frac{2n-2}{3-n}}\Gamma\left(\frac{2}{3-n}\right)}. \tag{A10}$$

Since the wellbore storage will not affect the steady-state and late time drawdowns. Above derived solutions of (A5) and (A10) are also valid for the case with the wellbore storage.

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