Flow to a Well in a Water-Table Aquifer: An Improved Laplace Transform Solution

by Allen F. Moench[a]

Abstract

An alternative Laplace transform solution for the problem, originally solved by Neuman, of constant discharge from a partially penetrating well in a water-table aquifer was obtained. The solution differs from existing solutions in that it is simpler in form and can be numerically inverted without the need for time-consuming numerical integration. The derivation involves the use of the Laplace transform and a finite Fourier cosine series and avoids the Hankel transform used in prior derivations. The solution allows for water in the overlying unsaturated zone to be released either instantaneously in response to a declining water table as assumed by Neuman, or gradually as approximated by Boulton's convolution integral. Numerical evaluation yields results identical with results obtained by previously published methods with the advantage, under most well-aquifer configurations, of much reduced computation time.

Introduction

The analytical solutions for flow to a well in an unconfined aquifer developed by Neuman (1972, 1974) are considered by many hydrogeologists to represent the state-of-the-art for analysis of pumping tests conducted in water-table aquifers. Neuman's time-domain solutions contain considerable complexity and, for certain combinations of parameters, are difficult to evaluate numerically. Even with the use of present-day computers a significant amount of computation time may be needed to obtain accurate results. This is because the time-domain solution involves the evaluation of a semi-infinite integral, the roots of equations for each value of the integrand, and an infinite-series summation for each value of the integrand. An alternative approach that reduces computation time and improves accuracy is described by Moench (1993). The approach involves numerical inversion of the Laplace transform solutions provided by Neuman.

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In this paper an alternative Laplace transform solution to the boundary-value problem formulated and solved by Neuman (1974) is proposed that provides greater simplicity and ease of numerical evaluation. By avoiding use of the Hankel transform in its derivation, the Laplace transform solution does not contain a term involving integration. Consequently, evaluation of the alternative solution by numerical inversion requires less computation time, for most practical values of parameters, than the numerical inversion approach described by Moench (1993). As derived in this paper the proposed solution also approximately accommodates effects of noninstantaneous drainage from the zone above the water table, discussed by Moench (1995).

Mathematical Solution

The governing equation for flow to a fully or partially penetrating well in a compressible, unconfined aquifer is

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{K_z}{K_r}\frac{\partial^2 h}{\partial z^2} = \frac{S}{bK_r}\frac{\partial h}{\partial t} \qquad (1)$$

The symbols are defined in the Notation. The initial and boundary conditions are as follows:

$$h(r, z, 0) = 0 \qquad (2)$$

$$h(\infty, z, t) = 0 \qquad (3)$$

$$\lim_{r \to 0} r \frac{\partial h}{\partial r} = -\frac{Q}{2\pi K_r} \qquad b - \ell \le z \le b - d \tag{4a}$$

$$\lim_{r \to 0} r \frac{\partial h}{\partial r} = 0 \qquad z < b - \ell;\ z > b - d \tag{4b}$$

$$\frac{\partial h}{\partial z}(r, 0, t) = 0 \tag{5}$$

$$K_z \frac{\partial h}{\partial z}(r, b, t) = -\alpha_1 S_y \int_0^t \frac{\partial h}{\partial t'} e^{-\alpha_1(t-t')} dt' \tag{6}$$

For a discussion of the background leading to the above boundary-value problem, the reader is referred to Moench (1993, 1995). The assumptions, excluding those required for the water-table condition, are identical to those of Neuman (1972, 1974) and the assumptions required for the water-table condition (6) are described in detail by Moench (1995). Figure 1 is a diagrammatic cross section through a part of an idealized water-table aquifer with a partially penetrating pumped well, an observation well, and an observation piezometer.

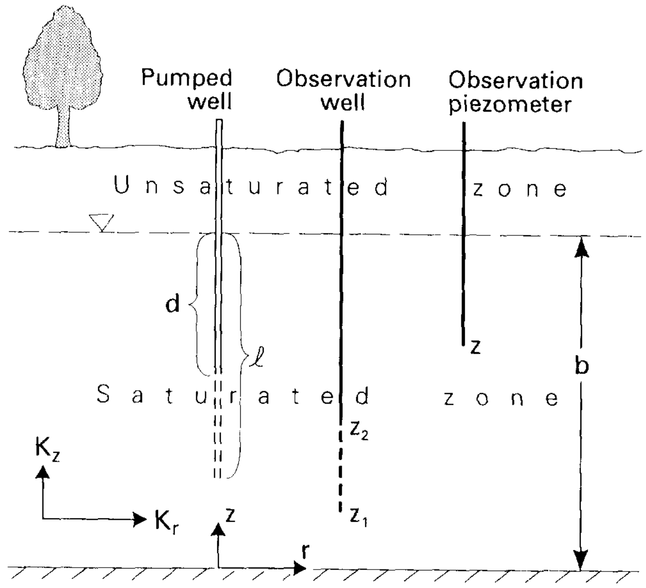

Fig. 1. Schematic diagram of partially penetrating wells in a homogeneous, anisotropic, water-table aquifer.

The dimensional boundary-value problem described by equations (1)-(6) is nondimensionalized by substituting the expressions given in Table 1. Application of the method of Laplace transformation leads to the following subsidiary equations:

$$\frac{\partial^2 \bar{h}_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \bar{h}_D}{\partial r_D} + K_D \frac{\partial^2 \bar{h}_D}{\partial z_D^2} = \frac{p\bar{h}_D}{r_D^2} \tag{7}$$

Table 1. Dimensionless Expressions

t_D	Tt/r^2S
t_{Dy}	Tt/r^2S_y
h_D	$4\pi T(h_i - h)/Q$
r_D	r/b
z_D	z/b
K_D	K_z/K_r
ℓ_D	ℓ/b
d_D	d/b
β	$K_z r_D^2/K_r$
σ	S/S_y
γ	$\alpha_1 b S_y/K_z$

$$\bar{h}_D(\infty, z_D, p) = 0 \tag{8}$$

$$\lim_{r_D \to 0} r_D \frac{\partial \bar{h}_D}{\partial r_D} = -\frac{2}{p(\ell_D - d_D)} \qquad 1 - \ell_D \le z_D \le 1 - d_D \quad \ldots. (9a)$$

$$\lim_{r_D \to 0} r_D \frac{\partial \bar{h}_D}{\partial r_D} = 0 \qquad z_D \le 1 - \ell_D;\ z_D \ge 1 - d_D \quad \ldots. (9b)$$

$$\frac{\partial \bar{h}_D}{\partial z_D}(r_D, 0, p) = 0 \tag{10}$$

$$\frac{\partial \bar{h}_D}{\partial z_D}(r_D, 1, p) = -\frac{\bar{h}_D p}{\sigma\beta + p/\gamma} \tag{11}$$

The Laplace transform variable, p, is inversely related to t_D. The bar over the dependent variable represents the Laplace transform.

The derivation without the Hankel transform of the solution is provided in the Appendix. The Laplace transform solution for dimensionless drawdown at a point in the aquifer due to constant pumpage from a partially penetrating well is written as:

$$\bar{h}_D(\gamma, \beta, \sigma, z_D, p) = \sum_{n=0}^{\infty} \frac{2K_0(x_n)\{\sin[\epsilon_n(1-d_D)] - \sin[\epsilon_n(1-\ell_D)]\}\cos(\epsilon_n z_D)}{p(\ell_D - d_D)[0.5\epsilon_n + 0.25\sin(2\epsilon_n)]} \tag{12}$$

where K_0 is the modified Bessel function of the second kind and order zero, $x_n = [\beta\epsilon_n^2 + p]^{1/2}$, and ϵ_n are the roots of

$$\epsilon_n \tan(\epsilon_n) = \frac{p}{(\sigma\beta + p/\gamma)} \tag{13}$$

For a partially penetrating observation well screened over the interval from z_{D1} to z_{D2}

$$\bar{h}_D(\gamma, \beta, \sigma, z_{D1}, z_{D2}, p) = \frac{1}{(z_{D2} - z_{D1})} \int_{z_{D1}}^{z_{D2}} \bar{h}_D(\gamma, \beta, \sigma, z_D, p)\, dz_D =$$

$$\sum_{n=0}^{\infty} \frac{2K_0(x_n)\{\sin[\epsilon_n(1-d_D)] - \sin[\epsilon_n(1-\ell_D)]\}[\sin(\epsilon_n z_{D2}) - \sin(\epsilon_n z_{D1})]}{p(\ell_D - d_D)\,\epsilon_n[0.5\epsilon_n + 0.25\sin(2\epsilon_n)]\,(z_{D2} - z_{D1})} \tag{14}$$

Table 2. Comparison of Computer Runtimes Using WTAQ1 and WTAQ2 for Various Values of β in a Hypothetical Aquifer

β	*WTAQ1 runtime (seconds)*	*WTAQ2 runtime (seconds)*
10^4	29.33	0.55
10^3	28.83	0.54
10^2	29.93	0.66
10^1	32.69	1.10
10^0	38.23	2.30
10^{-1}	33.89	4.72
10^{-2}	32.90	10.27
10^{-3}	33.18	23.18
10^{-4}	33.83	61.02
10^{-5}	38.45	133.96

[Each runtime represents the determination of about 60 values of dimensionless drawdown over a range of dimensionless time from $t_D = 0.1$ to $t_D = 10^7$. A PC with an 80486-33 M Hz microprocessor was used. (The relevant parameters are $\gamma = \infty$, $\sigma = 10^{-2}$, $\ell_D = 0.10$, $d_D = 0.05$, and two observation wells with $z_{D1} = 0.90$ and $z_{D2} = 0.95$ for one, and $z_{D1} = 0.05$ and $z_{D2} = 0.10$ for the other.)]

For a fully penetrating pumped and observation well (14) reduces to

$$\bar{h}_D(\gamma, \beta, \sigma, p) = \sum_{n=0}^{\infty} \frac{2K_0(x_n)\sin^2(\epsilon_n)}{p\,\epsilon_n[0.5\epsilon_n + 0.25\sin(2\epsilon_n)]} \quad (15)$$

It can be seen by inspection that (15) reduces to the Theis equation if the specific yield becomes zero. That is, if $\sigma \rightarrow \infty$, $\epsilon_n \rightarrow 0$, and since, as $u \rightarrow 0$, $\sin(u)/u = 1$, then $\bar{h}_D = 2K_0(p^{1/2})/p$, which is the Theis equation in the Laplace domain.

The solution (12) is a simpler alternative to the solution given by Moench (1995, eq. 20). That the two solutions are equivalent is inferred from the fact that they were both obtained by formal procedures using the same boundary-value problem as a starting point. Evaluation of the two solutions leads to identical results.

Evaluation

Equations (12), (14), or (15) can be inverted using the Stehfest (1970) algorithm. The roots of (13) are readily obtained by the Newton-Raphson method. A computer program named WTAQ2 was written for the purpose of generating type curves and is available from the author upon request. Because of the relative simplicity of the Laplace transform solutions the necessary code is considerably shorter than that of WTAQ1 described by Moench (1993). The computation time required by WTAQ2 is significantly less than that of WTAQ1 for most well-aquifer configurations and combinations of parameters. As β decreases, however, the number of terms required for convergence of the indicated summation increases, resulting in increased computation time. For values of β much less than 10^{-3} (as, for example, if $K_z/K_r = 0.1$ and $r/b \ll 0.1$) the computation time used by WTAQ2 exceeds that of WTAQ1.

Table 2 shows comparative runtimes for the two programs on a PC with an 80486, 33 MHz microprocessor. Dimensionless drawdown was computed for two observation wells located near the top and bottom, respectively, of a hypothetical water-table aquifer in which the pumped well is screened over a small portion of the top part of the saturated thickness. (See the footnote in Table 2 for details of the well-aquifer configuration and values of parameters.) Drainage from the unsaturated zone is assumed to occur instantaneously in response to pumping (representing a large value of α_1, so that γ is effectively infinite [10^9 in these computations]). Both programs provided values of dimensionless drawdown, over a range of dimensionless time from 0.1 to 10^7, that agree with one another with a difference of less than 1 percent. Four points spaced equally on a logarithmic time scale were computed for each decade in time. Type curves drawn from results computed with WTAQ2 are illustrated in Figure 2. They are identical to those of Moench (1993, Figure 5) using WTAQ1. It is perhaps of interest to note that inclusion in the computations realistic values of the dimensionless parameter γ (say 1 to 10) actually leads to reductions in computation time beyond those shown in Table 2.

Summary and Conclusion

An alternative, improved mathematical solution for the problem of flow to a partially penetrating well in a water-table aquifer has been derived. The proposed Laplace transform solution is simpler in form than solutions previously available and generally requires much less computation time to invert than other Laplace transform solutions for the same level of accuracy. Results suggest that the alternative solution may prove to be advantageous for automated, least-squares fitting of theoretical drawdown with measured drawdown.

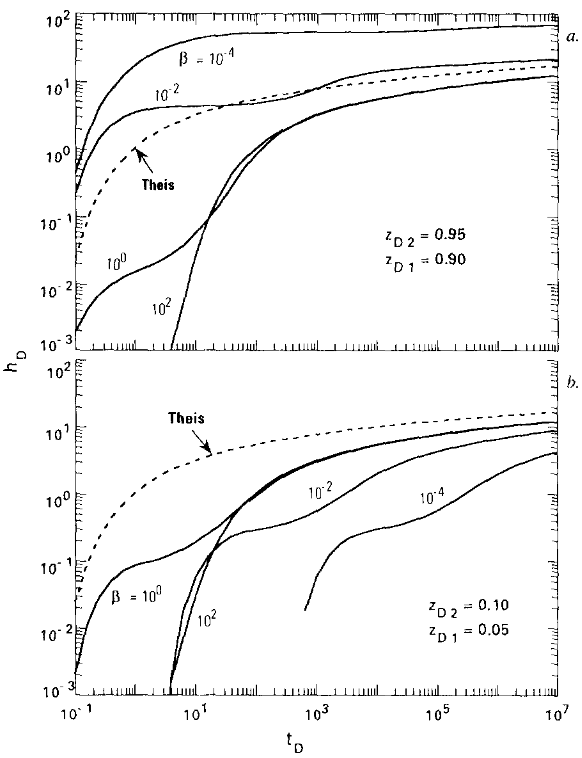

Fig. 2. Type curves for partially penetrating pumping and observation wells for $\gamma = \infty$, $\sigma = 10^{-2}$, $\ell_D = 0.10$, and $d_D = 0.05$ using the indicated values of β, z_{D1}, and z_{D2}: (a) observation well screen located near the water table opposite the pumping well, (b) observation well screen located near the bottom of the aquifer. A Theis type curve is shown for comparison.

Appendix

This appendix provides a derivation of equation (12). The boundary-value problem in Laplace space is given by (7)-(11). A solution to (7) that satisfies (10) is

$$\bar{h}_D = \sum_{n=0}^{\infty} \bar{f}_n(r_D, p) \cos[(n + 1/2)\pi z_D] \tag{A1}$$

Applying (11) and evaluating at $z_D = 1$

$$-[(n + 1/2)\pi] \tan[(n + 1/2)\pi] = -\frac{p}{\sigma\beta + p/\gamma} \tag{A2}$$

where $n = 0, 1, 2, \ldots$

Thus

$$\bar{h}_D = \sum_{n=0}^{\infty} \bar{f}_n(r_D, p) \cos(\epsilon_n z_D) \tag{A3}$$

where ϵ_n are the roots of $\epsilon_n \tan(\epsilon_n) = p/(\sigma\beta + p/\gamma)$.

Substitution of (A3) into (7) yields

$$\sum_{n=0}^{\infty} \left[\bar{f}_n'' + \frac{1}{r_D}\bar{f}_n' - \left(\epsilon_n^2 K_D + \frac{p}{r_D^2}\right)\bar{f}_n\right] \cos(\epsilon_n z_D) = 0 \quad \ldots \tag{A4}$$

hence,

$$\bar{f}_n'' + \frac{1}{r_D}\bar{f}_n' - \left(\epsilon_n^2 K_D + \frac{p}{r_D^2}\right)\bar{f}_n = 0 \tag{A5}$$

the solution of which can be written as

$$\bar{f}_n = A_n(p) K_0(x_n) + B_n(p) I_0(x_n) \tag{A6}$$

where $x_n = q_n r_D$ and $q_n = (\epsilon_n^2 K_D + p/r_D^2)^{1/2}$. K_0 and I_0 are the modified Bessel functions of the second and first kind and order zero, respectively, and A_n and B_n are coefficients to be determined.

Because of (8), $B_n(p) = 0$ and consequently

$$\bar{f}_n = A_n(p) K_0(x_n) \tag{A7}$$

Substitution of (A7) into (A3) yields

$$\bar{h}_D = \sum_{n=0}^{\infty} A_n(p) K_0(x_n) \cos(\epsilon_n z_D) \tag{A8}$$

Applying (9a)

$$\lim_{r_D \to 0} -A_n(p) r_D q_n K_1(r_D q_n) \cos(\epsilon_n z_D) = -\frac{2}{p(\ell_D - d_D)}$$

$$n = 0, 1, 2, \ldots \tag{A9}$$

where K_1 is the modified Bessel function of the second kind and order unity. The summation sign has been omitted to simplify notation. Because

$$\lim_{r_D \to 0} r_D q_n K_1(r_D q_n) = 1$$

(A9) becomes

$$-A_n(p) \cos(\epsilon_n z_D) = -\frac{2}{p(\ell_D - d_D)} \tag{A10}$$

Multiplying both sides of (A10) by $\cos(\epsilon_n z_D)$ and integrating over z_D to obtain the average drawdown over the screened section of the pumped well, one obtains

$$-A_n(p) \int_0^1 \cos^2(\epsilon_n z_D)\, dz_D = -\frac{2}{p(\ell_D - d_D)} \cdot \int_{1-\ell_D}^{1-d_D} \cos(\epsilon_n z_D)\, dz_D \tag{A11}$$

where the integration limits on the right-hand side are obtained by taking (9b) into consideration. Thus,

$$-A_n(p)\left[\frac{1}{2} + \frac{1}{4\epsilon_n}\sin(2\epsilon_n)\right] = -\frac{2\{\sin[\epsilon_n(1 - d_D)] - \sin[\epsilon_n(1 - \ell_D)]\}}{p(\ell_D - d_D)\,\epsilon_n} \tag{A12}$$

and

$$A_n(p) = \frac{2\{\sin[\epsilon_n(1 - d_D)] - \sin[\epsilon_n(1 - \ell_D)]\}}{p(\ell_D - d_D)[0.5\epsilon_n + 0.25\sin(2\epsilon_n)]} \tag{A13}$$

The solution (12) is obtained by substitution of (A13) into (A8).

Notation

- α_1 Empirical constant for drainage from the unsaturated zone, T^{-1}.
- b Initial saturated thickness of aquifer, L.
- d Vertical distance from initial water table to top of pumped well screen, L.
- h Hydraulic head, L.
- h_i Initial hydraulic head, L.
- K_z Hydraulic conductivity in the vertical direction, LT^{-1}.
- K_r Hydraulic conductivity in the horizontal direction, LT^{-1}.
- ℓ Vertical distance from initial water table to bottom of pumped-well screen, L.
- Q Pumping rate, L^3T^{-1}.
- p Laplace transform variable.
- r Radial distance from axis of pumping well, L.
- S Storativity.
- S_y Specific yield.
- T Transmissivity, L^2T^{-1}.
- t Time since start of pumping, T.
- z Vertical distance above bottom of aquifer, L.

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