A numerical solution for non-Darcian flow to a well in a confined aquifer using the power law function

Article *in* Journal of Hydrology · January 2009

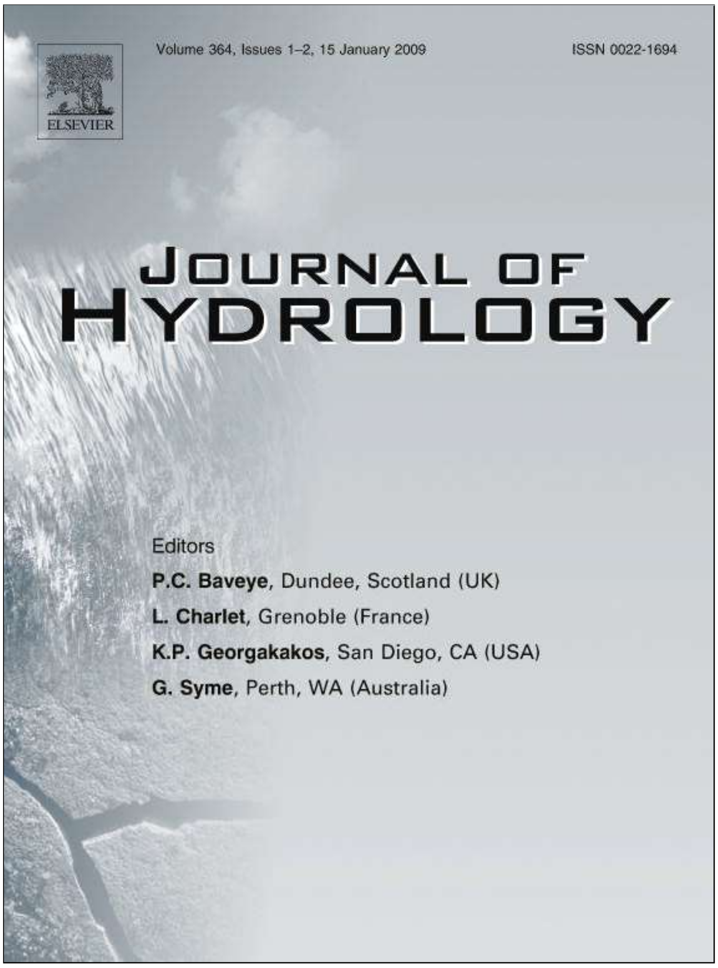
DOI: 10.1016/j.jhydrol.2008.10.009

3 authors:

Zhang Wen
China University of Mining Technology

Guanhua Huang
China Agricultural University

HB Zhan
Texas A&M University


Volume 364, Issues 1–2, 15 January 2009
ISSN 0022-1694
ELSEVIER
JOURNAL OF HYDROLOGY
Editors
P.C. Baveye, Dundee, Scotland (UK)
L. Charlet, Grenoble (France)
K.P. Georgakakos, San Diego, CA (USA)
G. Syme, Perth, WA (Australia)

Journal of Hydrology 364 (2009) 99–106

Contents lists available at ScienceDirect

Journal of Hydrology

journal homepage: www.elsevier.com/locate/jhydrol

A numerical solution for non-Darcian flow to a well in a confined aquifer using the power law function

Zhang Wen [a,b], Guanhua Huang [a,b,*], Hongbin Zhan [c]

[a] Department of Irrigation and Drainage, College of Water Conservancy and Civil Engineering, China Agricultural University, Beijing 100083, PR China
[b] Chinese-Israeli International Center for Research and Training in Agriculture, China Agricultural University, No. 17 Qinghua Donglu, Beijing 100083, PR China
[c] Department of Geology and Geophysics, Texas A&M University, College Station, TX 77843-3115, USA

ARTICLE INFO

Article history:
Received 30 March 2008
Received in revised form 28 August 2008
Accepted 20 October 2008

Keywords:
Non-Darcian flow
Power law
Finite difference method
Laplace transform
Linearization method

SUMMARY

In this study, we have obtained numerical solutions for non-Darcian flow to a well with the finite difference method on the basis of the Izbash equation, which states that the hydraulic gradient is a power function of the specific discharge. The comparisons between the numerical solutions and the Boltzmann solutions and linearization solutions have also been done in this study. The results indicated that the linearization solutions for both the infinitesimal-diameter well and the finite-diameter well agree very well with the numerical solution at late times, while the linearization method underestimates the dimensionless drawdown at early and moderate times. The Boltzmann method works well as an approximate analytical solution for the infinitesimal-diameter well. Significant differences have been found between the Boltzmann solution for a finite-diameter well and the numerical solution during the entire pumping period. The analysis of the numerical solution implies that all the type curves inside the well for different dimensionless non-Darcian conductivity k_D values approach the same asymptotic value at early times, while a larger k_D leads to a smaller drawdown inside the well at late times. A larger k_D results in a larger drawdown in the aquifer at early times and a smaller drawdown in the aquifer at late times. Flow approaches steady-state earlier when k_D is larger.

Introduction

Darcy's law has been used to simulate groundwater flow for more than 100 years. However, when the groundwater flow velocity becomes sufficiently high or sufficiently low, flow can be non-Darcian (e.g. Polubariava-Kochina, 1962; Wright, 1964; Basak and Madhav, 1979; Boast and Baveye, 1989; Venkataraman and Rao, 2000; Moutsopoulos and Tsihrintzis, 2005; Chen et al., 2003; Kohl et al., 1997; Qian et al., 2005, 2007). There are two general types of non-Darcian flow, i.e. pre-linear flow and post-linear flow. The pre-linear flow often occurs at very low Reynolds numbers (Firdaouss et al., 1997) such as in clay-rich aquitards (Teh and Nie, 2002) and in some petroleum reservoirs (Wattenbarger and Ramey, 1969), while the post-linear flow often occurs at very high Reynolds number (Zeng and Grigg, 2006) such as near the pumping wells (Sen, 1987, 1988a,b, 1989, 1990; Wu, 2002a,b; Wen et al., 2006, 2008a,b,c). In this paper, we only consider the post-linear flow for the high velocities near the pumping wells.

A key issue in non-Darcian flow is to quantify the relationship between the specific discharge and hydraulic gradient. Two formulae have been commonly used. The first one is the Forchheimer equation (Forchheimer, 1901), which states that the hydraulic gradient is a second-order polynomial function of the specific discharge. It should be pointed out that there are some alternative ways to present the Forchheimer equation (e.g. Thiruvengadam and Kumar, 1997; Nield, 2002; Moutsopoulos, 2007). For instance, Nield (2002) stated that a local time derivative inertial term and an advective inertial term should be added with the hydraulic gradient on the left hand side of the equation, while the so-called "Brinkman viscous term" should be added with the specific discharge on the right hand side of the equation. Meanwhile, Moutsopoulos (2007) pointed out that these extra terms are non-negligible only for very short times. The second one is the Izbash equation (Izbash, 1931), which states that the hydraulic gradient is a power function of the specific discharge. Many experimental data indicated that both these two functions can describe non-Darcian flow very well (Bordier and Zimmer, 2000; Yamada et al., 2005).

Up to now, many analytical solutions for non-Darcian flow have been presented. For instance, Sen (1987, 1988a,b, 1989, 1990) have obtained analytical solutions for non-Darcian flow to a well using the Boltzmann transform method, a special form of the so-called

* Corresponding author. Address: Department of Imagination and Drainage, College of Water Conservancy and Civil Engineering, China Agricultural University, Beijing 100083, PR China. Tel.: +86 10 62737144; fax: +86 10 62737138.
E-mail addresses: wenzhangcau@gmail.com (Z. Wen), ghuang@cau.edu.cn (G. Huang), zhan@geo.tamu.edu (H. Zhan).

0022-1694/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.jhydrol.2008.10.009

Nomenclature

r	distance from the center of well [L]	Q	pumping discharge [L^3/T]
r_w	effective radius of well screen [L]	$\Gamma(x)$	Gamma function
r_c	radius of well casing [L]	$K_\nu(x)$	the second kind ν-order modified Bessel function
t	pumping time [T]	r_D	dimensionless distance defined in Table 1
$q(r, t)$	specific discharge [L/T]	r_{wD}	dimension radius of well screen defined in Table 1
$s(r, t)$	drawdown [L]	r_{cD}	dimensionless radius of well casing defined in Table 1
$S_w(t)$	drawdown inside well [L]	t_D	dimensionless time defined in Table 1
m	aquifer thickness [L]	q_D	dimensionless specific discharge defined in Table 1
S	storage coefficient of aquifer	s_D	dimensionless drawdown defined in Table 1
n	power index, an empirical constant in the Izbash equation	s_{wD}	dimensionless drawdown inside the well defined in Table 1
k	quasi hydraulic conductivity, an empirical constant in the Izbash equation	k_D	dimensionless non-Darcian flow hydraulic conductivity defined in Table 1
p	Laplace variable		

similarity method. Wen et al. (2006, 2008c) have also used the Boltzmann transform to solve the non-Darcian flow problem to a well in confined aquifers. Meanwhile, Wen et al. (2008c) pointed out that the Boltzmann transform can only be used in an approximate rather than a rigorous mathematical sense. Additionally, Camacho and Vasquez (1992) pointed out that the Boltzmann transform cannot be employed to solve non-Darcian flow problems because of non-linearity of the governing equations. Recently, Wen et al. (2008a,b) have used a linearization method coupling with the Laplace transform to solve the non-Darcian flow toward a well in a confined aquifer. They stated that this linearization procedure might work very well at late times, while at early times it will bring about some errors.

As summarized before, both the Boltzmann transform and the linearization method have some limitations. The primary limitation of the Boltzmann transform is that such a transform requires both the governing equation and the initial and boundary conditions to be transformable by the Boltzmann variable which is the ratio of the radial distance square over time. Such a requirement is often not satisfied for non-Darcian flow. The limitation of the linearization method is that it involves a quasi steady-state approximation which does not work well at early times. Fortunately, many numerical solutions have been developed for non-Darcian flow. Mathias et al. (2008), Choi et al. (1997), Wu (2002a,b), and Ewing and Lin (2001) developed numerical solutions on the basis of the finite difference scheme, whereas Ewing et al. (1999) solved the Forchheimer non-Darcian flow based on the finite element method. Kolditz (2001) has used the finite element scheme to solve non-Darcian flow in fractured rock based on the assumption that the relationship between the specific discharge and hydraulic gradient can be described by a power function. Mathias et al. (2008) developed a numerical solution for the Forchheimer non-Darcian flow to a well and compared their results with those obtained by the Boltzmann transform and linearization methods. They found that both the "Boltzmann solution" (Sen, 1988b) and the linearization solution (Wen et al., 2008a) worked well at late times, while some differences have been found at early and moderate times. Ewing et al. (1999) developed several numerical schemes, e.g. the cell-centered finite difference, the Galerkin finite element, and the mixed finite element models for the Forchheimer non-Darcian flow. So far, most of the numerical solutions for non-Darcian flow are based on the Forchheimer equation (e.g. Mathias et al., 2008; Choi et al., 1997; Wu, 2002b). However, it is equivalently important to study the Izbash non-Darcian flow with the numerical simulations.

The objectives of this paper are to develop a finite difference solution for the non-Darcian flow toward a finite-diameter well in a confined aquifer with the Izbash equation, and to verify the previous solutions obtained by the Boltzmann transform based method (Sen, 1989; Wen et al., 2008c) and the linearization approximation method (Wen et al., 2008a) using the numerical solution.

Continuity equation

Governing equation

The schematic system discussed here is the same as Papadopulos and Cooper (1967). As shown in Fig. 1, flow to a finite-diameter well which fully penetrates a confined aquifer is considered. The aquifer is assumed to be homogenous and horizontally isotropic. The whole system is hydrostatic before the start of pumping, and the pumping rate is supposed to be constant. Under these assumptions, the problem discussed here can be described as (Papadopulos and Cooper, 1967; Wen et al., 2008a,c):

$$\frac{1}{r}\frac{\partial}{\partial r}[rq(r,t)] = \frac{S}{m}\frac{\partial s(r,t)}{\partial t}, \tag{1}$$

$$s(r,0) = 0, \tag{2}$$

$$s(\infty,t) = 0, \tag{3}$$

$$2\pi r_w m q(r,t)|_{r\to r_w} - \pi r_c^2 \frac{ds_w(t)}{dt} = -Q, \tag{4}$$

in which $q(r, t)$ is the specific discharge at radial distance r and time t, $s(r, t)$ is the drawdown, S is the storage coefficient of aquifer, m is the thickness of aquifer, r_w is the effective radius of the well, r_c is the radius of well casing. In most cases, the radius of well casing r_c is larger than, instead of equal to, the effective radius of the well r_w.

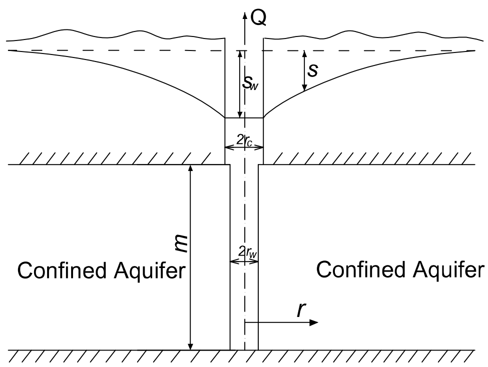


Figure 1. The schematic diagram of the problem.

Table 1
Definition of the dimensionless variables.

$r_D = \frac{r}{m}$	$q_D = -\frac{4\pi m^2}{Q} q$
$r_{cD} = \frac{r_c}{m}$	$s_D = \frac{4\pi k^{1/n} m}{Q} s$
$r_{wD} = \frac{r_w}{m}$	$s_{wD} = \frac{4\pi k^{1/n} m}{Q} s_w$
$t_D = \frac{k^{1/n} t}{Sm}$	$k_D = \left(\frac{4\pi m^2 k^{1/n}}{Q}\right)^{n-1}$

Q is the constant pumping rate which is positive for pumping, and $s_w(t)$ is the drawdown inside the well. In this paper, we employed the following Izbash (1931) equation to describe non-Darcian flow:

$$[q(r,t)]^n = k\frac{\partial s(r,t)}{\partial r}, \tag{5}$$

in which n and k are empirical constants. When n is less than 1, flow is pre-linear. While n is larger than 1 and less than 2, flow is post-linear, which always occurs in high velocity systems such as near the pumping wells. If n is equal to 2, flow becomes fully developed turbulent flow which often occurs at very high velocities such as flow in fractured media (Qian et al., 2005, 2007). If n happens to be equal to 1, flow is Darcian. Then k is the same as the hydraulic conductivity. Thus, k can be regarded as an apparent conductivity which reflects how easy the aquifer can transmit water.

Dimensionless transformation

Defining the dimensionless variables in Table 1, Eqs. (1)–(5) can be transformed accordingly as follows:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}[r_D q_D] = -\frac{\partial s_D}{\partial t_D}, \tag{6}$$

$$s_D(r_D, 0) = 0, \tag{7}$$

$$s_D(\infty, t_D) = 0, \tag{8}$$

$$\frac{1}{2}(r_D q_D)|_{r_D \to r_{wD}} + \frac{r_{cD}^2}{4S}\frac{ds_{wD}(t_D)}{dt_D} = 1, \tag{9}$$

where the subscript D denotes the dimensionless terms. The Izbash equation Eq. (5) can also be expressed in dimensionless format as

$$(q_D)^n = -k_D\frac{\partial s_D}{\partial r_D} \quad \text{or} \quad q_D = k_D^{1/n}\left(-\frac{\partial s_D}{\partial r_D}\right)^{1/n}, \tag{10}$$

where k_D can be regarded as a dimensionless apparent conductivity. Note that the minus sign must be included in the bracket to make q_D positive.

Numerical modeling

We employed the finite difference method to solve the governing equation (i.e. Eq. (6)). First, the dimensionless spatial domain $[r_{wD}, r_{eD}]$ is discretized into $N+1$ number of sub-domains with $N+2$ number of nodes, where r_{eD} is a very large dimensionless distance which is used to approximate the outer boundary condition of $r_D \to \infty$. The value of r_D at ith node can be referred as r_i, where the subscript i represents the node of interest. For any node of r_i, $r_{wD} < r_i < r_{eD}$, $i = 1, 2, \ldots, N$, and $r_0 = r_{wD}$ and $r_{N+1} = r_{eD}$. For the case of convenience, r_i is expressed as (Wu, 2002a; Mathias et al., 2008):

$$r_i = (r_{i-1/2} + r_{i+1/2})/2, \quad i = 1, 2, \ldots, N, \tag{11}$$

in which

$$\log_{10}(r_{i+1/2}) = \log_{10}(r_{wD}) + i\left[\frac{\log_{10}(r_{eD}) - \log_{10}(r_{wD})}{N}\right], \quad i = 0, 1, \ldots, N. \tag{12}$$

Notice that i starts from one in Eq. (11) whereas i starts from zero in Eq. (12). With the spatial discretization, the governing equation (Eq. (6)) reduces to the following differential equation with respect to dimensionless time t_D:

$$\frac{ds_i}{dt_D} \approx \frac{r_{i-1/2}q_{i-1/2} - r_{i+1/2}q_{i+1/2}}{r_i(r_{i+1/2} - r_{i-1/2})}, \quad i = 1, 2, \ldots, N, \tag{13}$$

where s_i and q_i are the dimensionless drawdown s_D and dimensionless specific discharge q_D at node i, respectively. With the Izbash equation Eq. (10), one has

$$q_{i-1/2} \approx k_D^{1/n}\left(\frac{s_{i-1} - s_i}{r_i - r_{i-1}}\right)^{1/n}, \quad i = 2, 3, \ldots, N, \tag{14}$$

$$q_{i+1/2} \approx k_D^{1/n}\left(\frac{s_i - s_{i+1}}{r_{i+1} - r_i}\right)^{1/n}, \quad i = 1, 2, \ldots, N-1. \tag{15}$$

At the boundaries, one can obtain:

$$q_{1-1/2} \approx k_D^{1/n}\left(\frac{s_{wD} - s_1}{r_1 - r_{wD}}\right)^{1/n}, \tag{16}$$

$$q_{N+1/2} \approx k_D^{1/n}\left(\frac{s_N}{r_{eD} - r_N}\right)^{1/n}, \tag{17}$$

where s_{wD} is the dimensionless drawdown inside the well, which can be approximated as follows when considering Eq. (9):

$$\frac{ds_{wD}}{dt_D} \approx \frac{4S}{r_{cD}^2}\left(1 - \frac{1}{2}r_{wD}q_{1-1/2}\right). \tag{18}$$

With these preparations, the above problem can be solved by using the stiff integrator ODE15s in MATLAB 6.5 (Shampine and Reichelt, 1997; Shampine et al., 1999; Mathias et al., 2008), which is available in any version of MATLAB. ODE15s is a variable order method which employs an adaptive time steps that are continuously adjusted such that the numerical error associated with each step is always below some specified tolerance. It is a stiff method (as opposed to a non-stiff method) was used because of the non-linearity of the problem (Mathias et al., 2006). We have developed a MATLAB program to do this calculation, which is free of charge upon request from the authors. We have tried several values for N and r_{eD} and found that when N is larger than 2000 and r_{eD} is larger than 10^6, the results become very stationary. In this study, we set $N = 3000$ nodes and $r_{eD} = 10^8$ for all the simulations. These values are large enough to ensure sufficient accuracy of the numerical calculation. It should be pointed out that the time step is not needed here because the integrator ODE15s will use an adaptive time grid automatically. What we discussed before is for a finite-diameter well. The methodology can be equally applied to solve non-Darcian flow to an infinitesimal-diameter well by setting very small values for the dimensionless radius of the well screen r_{wD} and the well casing r_{cD}. It was found that the dimensionless drawdown approached the Theis type curve for the Darcian flow case ($n = 1$ in our study) when the dimensionless radius r_{cD} and r_{wD} were set to 10^{-5}.

Linearization solutions

Wen et al. (2008a) have used the linearization procedure to solve non-Darcian flow toward a well and have obtained some approximate analytical solutions. In that study, Wen et al. (2008a) used the dimensional format to analyze the results instead of the dimensionless formats. In order to compare our numerical results with the approximate analytical solutions obtained by the linearization method, we solved the problem with dimensionless terms defined here by using the linearization procedure. Similar to Wen et al. (2008a), with the substitution of Eq. (10) into Eq. (6), one has

$$\frac{\partial^2 s_D}{\partial r_D^2}+\frac{n}{r_D}\frac{\partial s_D}{\partial r_D}=\frac{n}{k_D^{1/n}}\left(-\frac{\partial s_D}{\partial r_D}\right)^{\frac{n-1}{n}}\frac{\partial s_D}{\partial t_D}. \tag{19}$$

The term $\left(-\frac{\partial s_D}{\partial r_D}\right)^{\frac{n-1}{n}}$ in Eq. (19) is a non-linear term, which can be eliminated by the linearization procedure. At the steady-state, flow rates passing through any closed interfaces surrounding the well are equal to the pumping rate Q. Using this to approximate the specific discharge at any time, then one has

$$\frac{1}{2}r_D q_D \approx 1. \tag{20}$$

With Eqs. (10) and (20), the non-linear term can be approximated as

$$\left(-\frac{\partial s_D}{\partial r_D}\right)^{\frac{n-1}{n}}\approx\left(\frac{2}{r_D k_D^{1/n}}\right)^{n-1}. \tag{21}$$

Substituting Eq. (21) to Eq. (19) results in

$$\frac{\partial^2 s_D}{\partial r_D^2}+\frac{n}{r_D}\frac{\partial s_D}{\partial r_D}\approx\frac{n r_D^{1-n}}{2^{1-n}k_D}\frac{\partial s_D}{\partial t_D}. \tag{22}$$

If the wellbore storage is excluded, the inner boundary condition at the center of the well in dimensionless format can be expressed as

$$\frac{1}{2}r_D q_D|_{r_D\to 0}=1. \tag{23}$$

Eq. (22) can be solved by the Laplace transform. The details of solving Eq. (22) will not be reported here, which can be found in our previous studies (Wen et al., 2008a,b). With the boundary Eq. (23), the solution for Eq. (22) in the Laplace domain can be expressed as

$$\overline{s}_D(r_D,p)=\frac{2^{n+1}\left(\frac{1}{3-n}\sqrt{\frac{np}{2^{1-n}k_D}}\right)^{\frac{2}{3-n}}}{k_D p\sqrt{\frac{np}{2^{1-n}k_D}}\,\Gamma\left(\frac{2}{3-n}\right)}r_D^{\frac{1-n}{2}}K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r_D^{\frac{3-n}{2}}\sqrt{\frac{np}{2^{1-n}k_D}}\right), \tag{24}$$

where p is the Laplace variable respect to the dimensionless time t_D, $\overline{s}_D(r_D,p)$ is the dimensionless drawdown in the Laplace domain, $K_\nu(x)$ is the second kind ν-order modified Bessel functions, where $\nu=(1-n)/(3-n)$. $\Gamma(x)$ is the gamma function. When the wellbore storage is considered, the boundary at the face of the wellbore should be replaced by Eq. (9). Applying the Izbash equation, Eq. (9) is also a non-linear boundary. Similarly, Eq. (9) can also be transformed with the linearization procedure as

$$k_D\left(\frac{r_D}{2}\right)^n\frac{\partial s_D}{\partial r_D}|_{r_D\to r_{wD}}-\frac{r_{cD}^2}{4S}\frac{ds_{wD}(t_D)}{dt_D}\approx -1. \tag{25}$$

The solution with wellbore storage in dimensionless format in the Laplace domain can be expressed as

$$\overline{s_D}(r_D,p)=\frac{r_D^{\frac{1-n}{2}}K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r_D^{\frac{3-n}{2}}\sqrt{\frac{n}{2^{1-n}k_D}p}\right)}{p\left\{k_D\frac{r_{wD}}{2^n}\sqrt{\frac{n}{2^{1-n}k_D}p}K_{\frac{2}{3-n}}\left(\frac{2}{3-n}r_{wD}^{\frac{3-n}{2}}\sqrt{\frac{n}{2^{1-n}k_D}p}\right)+\frac{r_{cD}^2}{4S}p r_{wD}^{\frac{1-n}{2}}K_{\frac{1-n}{3-n}}\left(\frac{2}{3-n}r_{wD}^{\frac{3-n}{2}}\sqrt{\frac{n}{2^{1-n}k_D}p}\right)\right\}}. \tag{26}$$

The detailed derivation is similar to those in Wen et al. (2008a), and will not be repeated here. Eqs. (24) and (26) can be inverted numerically using the method proposed by Stehfest (1970a,b).

Results and discussion

In the following discussion, we only considered the case for $n > 1$ because the pre-linear flow is unlikely to occur near the pumping wells. We have used the same values for r_{wD} and r_{cD} in the analysis. The results corresponding to different r_{wD} and r_{cD} values can also be easily obtained.

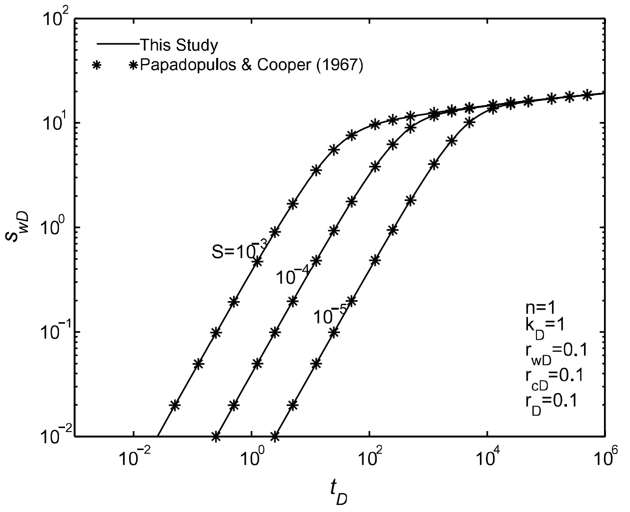


Figure 2. Comparison of the numerical solution with the result of Papadopulos and Cooper (1967) for the Darcian flow case.

Comparison with the solution of Papadopulos and Cooper (1967)

In order to test the finite difference solutions, first we compared the numerical solution with Papadopulos and Cooper (1967) for the Darcian flow case by setting the parameter $n = 1$. As shown in Fig. 2, our finite difference solution agrees well with the solution obtained by Papadopulos and Cooper (1967) for the whole pumping period. The general agreement indicates that our numerical solution and MATLAB program are validated. Here we have compared the numerical results with the solution of Papadopulos and Cooper (1967) for Darcian flow case. It should be pointed out that it might be better to check the numerical solution with the analytical solution for non-Darcian flow case. However, to our knowledge, there is no analytical solution for such a radial non-Darcian flow case. Moutsopoulos and Tsihrintzis (2005) derived an analytical solution for one-dimensional non-Darcian flow based on the Forchheimer equation, and Song and Tao (2007) used their analytical solution to validate the numerical solutions. Although Moutsopoulos and Tsihrintzis (2005) derived an analytical solution for non-Darcian flow by using a similarity method, the continuity equation in that study is different from that for radial flow in this study. Moutsopoulos and Tsihrintzis (2005) dealt with a uniform flow type of problem and an exact analytical solution indeed can be derived for that problem. However, we have tried every possible ways for the radial non-Darcian flow problem and still cannot find an exact analytical solution for an arbitrary n value except the special case of $n = 1$. We suspect that such an exact analytical solution for the radial flow case may not exist at all. Due to these reasons, we only compare our numerical solution with that of Papadopulos and Cooper (1967) for Darcian flow case. It is evident to see the numerical solution is sufficiently accurate as the perfect agreement reflected in Fig. 2.

Comparison with the linearization solution and Boltzmann solution

We have also used the numerical solution to test the approximate analytical solutions obtained by the linearization method and the Boltzmann transform, as shown in Fig. 3. Fig. 3a is about the dimensionless drawdowns for an infinitesimal-diameter well with $n = 1.5$, $k_D = 10$ and $r_D = 10$, 100, respectively. From Sen (1989), the solution of the drawdown with the Boltzmann transform can be expressed as

$$s(r,t) = \frac{1}{k}\left(\frac{Q}{2\pi m}\right)^n \int_r^\infty \frac{1}{\tau^n}\left[-2\frac{n-1}{n-3}\frac{\tau^2 S}{4mk^{1/n}t}\left(\frac{2\pi m\tau}{Q}\right)^{1-n}+1\right]^{n/(1-n)} d\tau, \tag{27}$$

in which τ is a dummy variable. Eq. (27) can be expressed in dimensionless format as

$$s_D = \frac{2^n}{k_D}\int_{r_D}^\infty \frac{1}{\tau_D}\left[-2\frac{n-1}{n-3}\frac{\tau_D^{3-n}}{2^{3-n}k_D t_D}+1\right]^{n/(1-n)} d\tau_D, \tag{28}$$

where τ_D is a dummy variable. The Boltzmann solution in Fig. 3a is calculated with Eq. (28). The solution for the linearization method in Fig. 3a is calculated with Eq. (24). Fig. 3b is the dimensionless drawdowns for non-Darcian flow to a finite-diameter well with $n = 1.5$, $k_D = 10$, $r_{wD} = r_{cD} = 0.1$, $S = 0.001$ and $r_D = 0.1$, 1, and 10, respectively. It can be seen that the linearization solution agrees very well with the numerical solution in this study at late times both for the infinitesimal-diameter well and the finite-diameter well. As shown in Fig. 3a, when the wellbore storage is excluded, both the Boltzmann solution and the linearization solution agree very well with the numerical solution at late times. The Boltzmann solution seems to be more accurate than the linearization solution at early times, and the linearization solution underestimates the dimensionless drawdown at early times when the wellbore storage is excluded. Similar results have also been found in Mathias et al. (2008). When the wellbore storage is considered, as shown in Fig. 3b, the linearization solution agrees very well with the numerical solution at late times and underestimates the dimensionless drawdown at early times, which is similar to that without wellbore storage. A relatively large difference has been found between the Boltzmann solution and the numerical solution during the entire pumping period when the wellbore storage is considered. Actually, the main problem of the Boltzmann transform is that the governing equation and the initial and boundary conditions must be transformed in terms of the Boltzmann variable, which is not satisfied in such a non-Darcian flow case. This leads to the integration constant, which depends on the boundary conditions, becomes a function of time t rather than independent of the Boltzmann variable. When the wellbore storage is considered for a finite-diameter well, the integration constant is a further complicated function of time t (see Eq. (A6) in Wen et al., 2008c), which might result in greater differences than those for an infinitesimal-diameter well. In order to quantify the error of the Boltzmann solution and the linearization solution, we have plotted the relative error versus dimensionless time t_D in semi-log scale, as shown in Fig. 4. The relative error, ε, is defined as the ratio of the difference between the Boltzmann solution or the linearization solution and the numerical solution over the numerical solution. Fig. 4a is about the errors when the wellbore storage is excluded for $n = 1.5$, $k_D = 10$, and $r_D = 1$. While Fig. 4b reflects the errors with wellbore storage for $n = 1.5$, $k_D = 10$, $r_{wD} = r_{cD} = r_D = 0.1$, and $S = 0.001$. As shown in Fig. 4a, it is difficult to tell which approximate analytical solution is better. However, it is obvious to see that the linearization method is better when the wellbore storage is considered, as shown in Fig. 4b.

Notice that if the dimensionless drawdown s_D for non-Darcian flow can be solved by the Boltzmann transform, the solution must be a function of the Boltzmann variable, manifested as a function of r_D^2/t_D. This means that the value of the dimensionless drawdown s_D should be the same for different dimensionless distances r_D (or different dimensionless times t_D) as long as the values of r_D^2/t_D are the same. We plotted the dimensionless drawdown s_D versus r_D^2/t_D for different dimensionless distances r_D in Fig. 5a–c, where Fig. 5a refers to Darcian flow to an infinitesimal-diameter well; Fig. 5b is non-Darcian flow to an infinitesimal-diameter well; and Fig. 5c regards non-Darcian flow to a finite-diameter well. If the dimensionless drawdown s_D is a function of r_D^2/t_D, all the curves should converge to a single curve. However, the curves for different dimensionless distance r_D are quite different except for Darcian flow to an infinitesimal-diameter well. In fact, the curves in Fig. 5a reflect the Theis model. It is well-known that the Theis

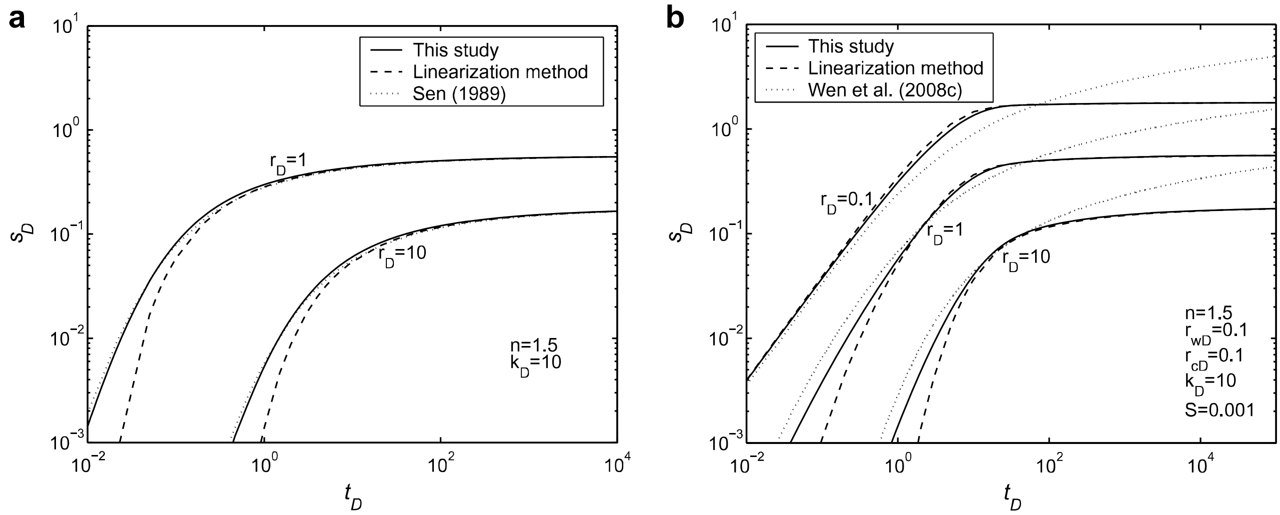


Figure 3. Comparison of the numerical solution with the Boltzmann solution and linearization solution: (a) non-Darcian flow to an infinitesimal-diameter well with $n = 1.5$, $k_D = 10$, and $r_D = 1$, 10, respectively, the Boltzmann solution is calculated from Eq. (28); (b) non-Darcian flow to an finite-diameter well with $n = 1.5$, $k_D = 10$, $r_{wD} = r_{cD} = 0.1$, $S = 0.001$ and $r_D = 0.1$, 1, and 10, respectively.

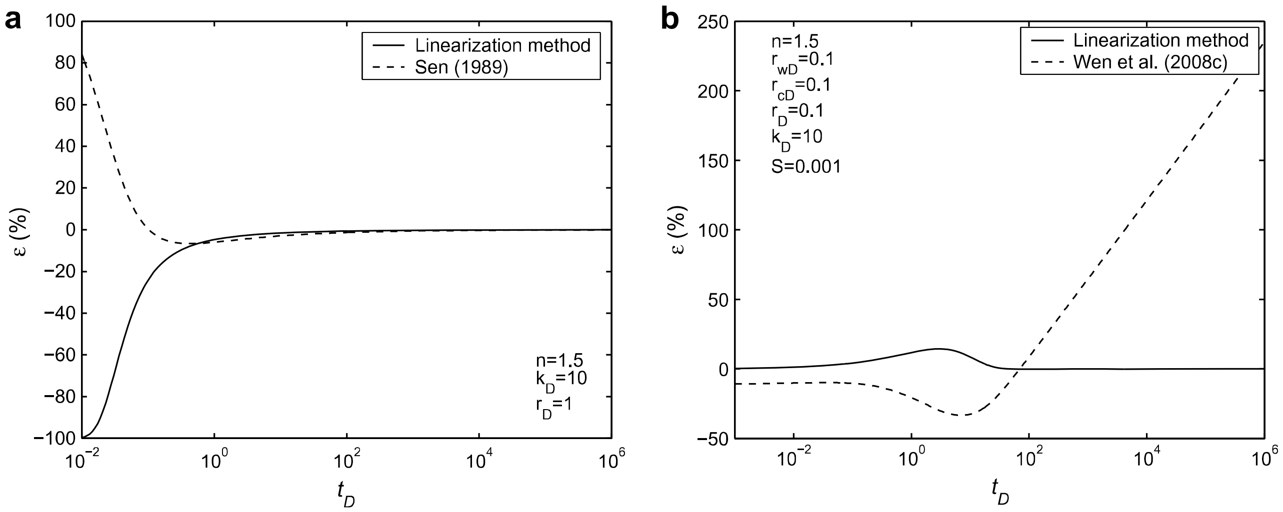


Figure 4. Error comparison of the linearization solution and the Boltzmann solution: (a) for an infinitesimal-diameter well with $n = 1.5$, $k_D = 10$, and $r_D = 1$; (b) for a finite-diameter well with $n = 1.5$, $k_D = 10$, $r_{wD} = r_{cD} = r_D = 0.1$ and $S = 0.001$.

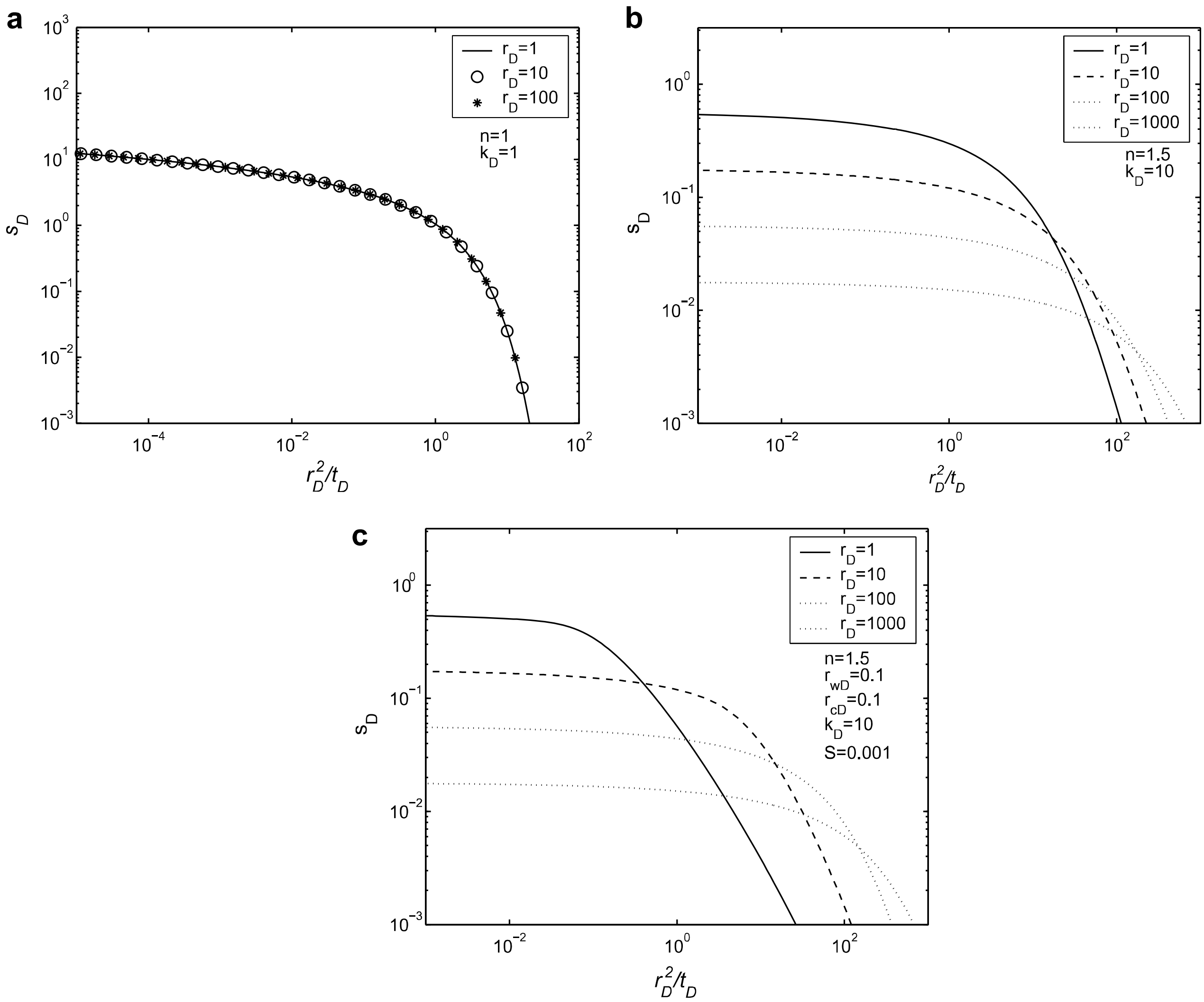


Figure 5. Dimensionless drawdown s_D versus r_D^2/t_D: (a) Darcian flow to an infinitesimal-diameter well with $n = 1$, $k_D = 1$ and $r_D = 1$, 10 and 100, respectively; (b) non-Darcian flow to an infinitesimal-diameter well with $n = 1.5$, $k_D = 10$ and $r_D = 1$, 10, 100 and 1000, respectively; (c) non-Darcian flow to a finite-diameter well with $n = 1.5$, $k_D = 10$, $r_{wD} = r_{cD} = 0.1$ and $r_D = 1$, 10, 100 and 1000, respectively.

curve is a function of r_D^2/t_D. Thus, it is no wonder all the curves for different dimensionless distance r_D converge to a single curve as shown in Fig. 5a. However, the curves for non-Darcian flow to a well are quite different for different dimensionless distance r_D, as can be seen from Fig. 5b and c. This indicates that the dimensionless drawdown for non-Darcian flow is no longer a function of r_D^2/t_D. That is why curves for different r_D should not converge to a single curve. Therefore, we can conclude that the Boltzmann transform cannot be used to solve the non-Darcian flow problems in a rigorous sense.

Effect of the dimensionless non-Darcian conductivity k_D

The sensitivity analysis of the dimensionless non-Darcian conductivity k_D, which is an important parameter to describe non-Darcian flow, is shown in Fig. 6. Fig. 6a is the dimensionless drawdown versus the dimensionless time inside the well, while Fig. 6b is the dimensionless drawdown versus the dimensionless time in the aquifer. As shown in Fig. 6a, all the curves approach the same asymptotic value at early times, and they are straight lines in log–log scales. This is an obvious wellbore storage effect (Park and Zhan, 2002, 2003), meaning that the pumping rate only comes from the wellbore at early times. As the pumping time increases, the curves start to deviate from the straight line. At moderate times, the pumping rate partially comes from the wellbore storage and partially comes from the aquifer. While at late times, a larger k_D leads to a smaller drawdown inside the well. This finding can be understood physically. A larger k_D represents a larger "hydraulic conductivity", thus, it is easier for the medium to transmit water, and a smaller drawdown inside the well will be seen. Similar results have also been found by Wen et al. (2008b) in the two-region non-Darcian flow model. The impact of k_D on the drawdowns in the aquifer is shown in Fig. 6b. A larger k_D leads to larger aquifer drawdowns at early times. This is because at early times, q_D is not a constant; rather, it gradually increases from zero to its steady-state value. However, the slope of q_D versus t_D is different for different k_D. For the extreme case of an extremely large k_D, q_D increases to its steady-state value within an extremely small time interval. Mathematically, it appears like a step function in the $q_D \sim t_D$ diagram. On the other hand, for an extremely small k_D, q_D is going to take an extremely long time to reach its steady-state value. Mathematically, it is manifested by a slowly increase curve over a very long time in the $q_D \sim t_D$ diagram. For a k_D value within those two extremes, q_D increases with t_D faster to its steady-state value for a larger k_D value. Therefore, for a given t_D during the early time, a greater q_D will be resulted for the case of a larger k_D. Then from Eq. (10), it is possible to see that a greater s_D will be resulted for a larger k_D. Where at late times, the whole system approaches a quasi steady-state, a larger dimensionless hydraulic conductivity k_D leads to a smaller aquifer drawdown. This is the same as the drawdown inside the well. It can also be found that flow approaches steady-state earlier when k_D is larger.

Discussion

In this study, we used a finite difference scheme to solve the radial non-Darcian flow to a well in a confined aquifer. The flow system was assumed to be axi-symmetrical. This assumption might be true at very early times because the depletion cone has a very small radius at this stage. However, because of the heterogeneity or anisotropy of the aquifer, such axi-symmetrical flow is not valid for many real-world problems. Two-dimensional flow or three-dimensional flow might occur. In that case, new mathematical model should be established and further research is needed in the future. Nevertheless, the numerical scheme developed in this study might also be useful to solve some other problems such as one-dimensional flow after some changes in the governing equation and the boundary equations.

Because of the non-linearity nature of the non-Darcian flow problems, it seems very difficult to solve the non-Darcian flow problems analytically. Analytical methods based on the linearization approximation and the Boltzmann transform have been presented in literatures. However, they have their specific limitations and can only be used in an approximate sense. The numerical methods might be a promising tool to solve the non-Darcian flow problems. The finite difference scheme developed in this study only can be used for radial non-Darcian flow without considering sophisticated situations, such as multi-phase, multi-dimensional non-Darcian flow, contaminant transport with non-Darcian flow, and flow problems with different boundary conditions. All these issues are deserved to be investigated in the future.

Another interesting issue is to estimate the parameters of the aquifer by using the derived type curves associated with a pumping test. We usually used the matching method to carry out such works. However, there are more variables, e.g. the power index n,

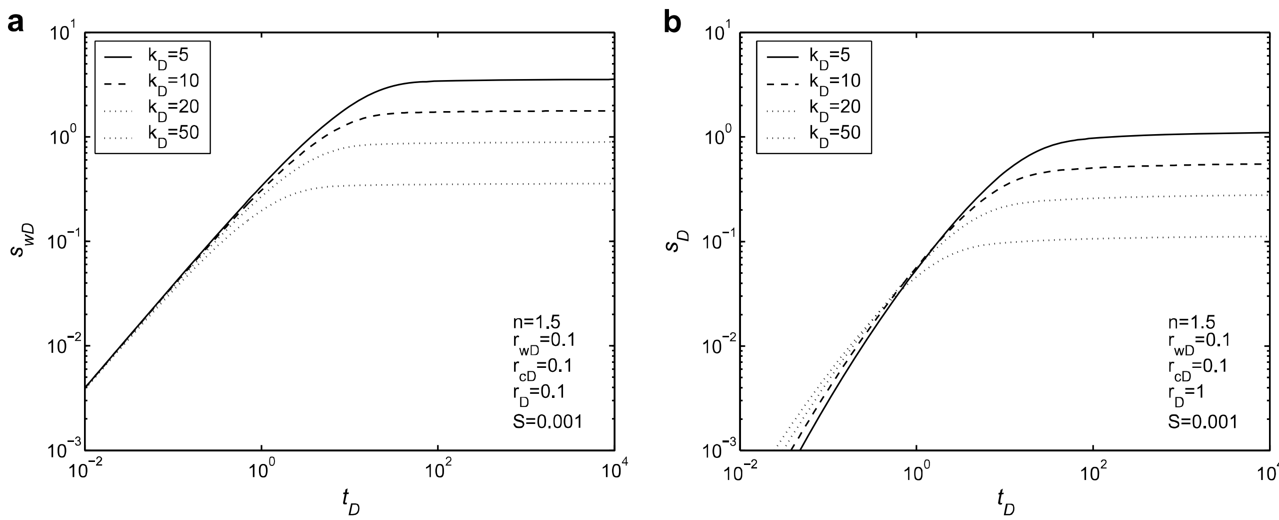


Figure 6. Type curves for different dimensionless non-Darcian conductivity k_D = 5, 10, 20, 50 with n = 1.5, $r_{wD} = r_{cD}$ = 0.1, S = 0.001: (a) inside the well, r_D = 0.1 and (b) in the aquifer, r_D = 1.

the dimensionless non-Darcian hydraulic conductivity k_D, needed to be evaluated. The nature of the non-linearity of the non-Darcian flow makes the inverse problem a little complicated. To obtain the parameters of the aquifer, e.g. the storage coefficient S, maybe first one needs to determine the power index n for non-Darcian flow case. This issue is also deserved to be investigated in the future.

Summary and conclusions

We have obtained a finite difference numerical solution for non-Darcian flow to a well in a confined aquifer. The Izbash equation has been used to describe the non-Darcian flow. The wellbore storage has also been included in this study. We have used our numerical solution to compare with previous studies, i.e. the Boltzmann solution (Sen, 1989; Wen et al., 2008c) and the linearization solution (Wen et al., 2008a). Several findings can be drawn from this study:

(1) The linearization solutions for both the infinitesimal-diameter well and the finite-diameter well agree very well with the numerical solutions at late times, while the linearization method underestimates the dimensionless drawdown at early and moderate times.
(2) The Boltzmann solution of the drawdown for the infinitesimal-diameter well agrees well with the numerical solution. Significant differences have been found between the Boltzmann solution for a finite-diameter well and the numerical solution during the entire pumping period.
(3) All the type curves inside the well for different dimensionless non-Darcian conductivity k_D values approach the same asymptotic value at early times, while a larger k_D leads to a smaller drawdown inside the well at late times.
(4) A larger dimensionless non-Darcian conductivity k_D results in a larger drawdown in the aquifer at early times and a smaller drawdown in the aquifer at late times. Flow approaches steady-state earlier when k_D is larger.

Acknowledgments

This research was partly supported by the National Natural Science Foundation of China (grant numbers: 50779067, 50428907), the Elitist Program of the Ministry of Education of China (Grant number: NCET-05-0125) and the Program for Changjiang Scholars and Innovative Research Team in University. We would like to thank Dr. Simon A. Mathias for his great help on developing the MATLAB program. We also would like to thank two anonymous reviewers and the Editor for their constructive comments that help us improve the quality of the manuscript.

References

Basak, P., Madhav, M.R., 1979. Analytical solutions to the problems of transient drainage through trapezoidal embankments with Darcian and non-Darcian flow. J. Hydrol. 41, 49–57.
Boast, C.W., Baveye, P., 1989. Solutions of the flow at a corner problem with a stagnation zone. Water Resour. Res. 25 (4), 757–763.
Bordier, C., Zimmer, D., 2000. Drainage equations and non-Darcian modeling in coarse porous media or geosynthetic materials. J. Hydrol. 228, 174–187.
Camacho, R.G., Vasquez, M., 1992. Comment on "analytical solution incorporating nonlinear radial flow in confined aquifers" by Zekai Sen. Water Resour. Res. 28 (12), 3337–3338.
Chen, C., Wan, J., Zhan, H., 2003. Theoretical and experimental studies of coupled seepage pipe flow to a horizontal well. J. Hydrol. 281, 159–171.
Choi, E.S., Cheema, T., Islam, M.R., 1997. A new dual-porosity/dual-permeability model with non-Darcian flow through fractures. J. Petrol. Sci. Eng. 17, 331–344.
Ewing, R.E., Lin, Y., 2001. A mathematical analysis for numerical well models for non-Darcy flows. Appl. Numer. Math. 39 (1), 17–30.
Ewing, R.E., Lazarov, R.D., Lyons, S.L., Papavassiliou, D.V., Pasciak, J., Qin, G., 1999. Numerical well model for non-Darcy flow through isotropic porous media. Comput. Geosci. 3, 185–204.
Firdaouss, M., Guermond, J., Le Quere, P., 1997. Nonlinear corrections to Darcy's law at low Reynolds numbers. J. Fluid Mech. 343 (1), 331–350.
Forchheimer, P.H., 1901. Wasserbewegung durch Boden. Zeitschrift des Vereines Deutscher Ingenieure 49, 1736–1749. 50, 1781–1788.
Izbash, S.V., 1931. O filtracii V Kropnozernstom Materiale. Leningrad, USSR (in Russian).
Kohl, T., Evans, K.F., Hopkirk, R.J., Jung, R., Rybach, L., 1997. Observation and simulation of non-Darcian flow transients in fractured rock. Water Resour. Res. 33 (3), 407–418.
Kolditz, O., 2001. Non-linear flow in fractured rock. Int. J. Numer. Methods. Heat Fluid Flow 11 (6), 547–575.
Mathias, S.A., Butler, A.P., Jackon, B.M., Weather, H.S., 2006. Transient simulations of flow and transport in the Chalk unsaturated zone. J. Hydrol. 330, 10–28.
Mathias, S.A., Butler, A.P., Zhan, H., 2008. Approximate solutions for Forchheimer flow to a well. ASCE J. Hydraul. Eng-ASCE 134, 1318–1325.
Moutsopoulos, K.N., 2007. One-dimensional unsteady inertial flow in phreatic aquifers induced by a sudden change of the boundary head. Trans. Porous Med. 70, 97–125.
Moutsopoulos, K.N., Tsihrintzis, V.A., 2005. Approximate analytical solutions of the Forchheimer equation. J. Hydrol. 309, 93–103.
Nield, D.A., 2002. Modeling fluid flow in saturated porous media and at interfaces, second ed.. Transport Phenomena in Porous Media. Pergamon, Amsterdam. pp. 1–19.
Papadopulos, I.S., Cooper, H.H., 1967. Drawdown in a well of large diameter. Water Resour. Res. 3 (1), 241–244.
Park, E., Zhan, H., 2002. Hydraulics of a finite-diameter horizontal well with wellbore storage and skin effect. Adv. Water Resour. 25, 389–400.
Park, E., Zhan, H., 2003. Hydraulics of horizontal wells in fractured shallow aquifer systems. J. Hydrol. 281, 147–158.
Polubariava-Kochina, P., 1962. Theory of Ground Water Movement. Princeton University Press, Princeton, New Jersey, USA (translated by J.M. De Wiest).
Qian, J., Zhan, H., Zhao, W., Sun, F., 2005. Experimental study of turbulent unconfined groundwater flow in a single fracture. J. Hydrol. 311, 134–142.
Qian, J., Zhan, H., Zhao, W., Sun, F., 2007. Experimental evidence of scale-dependent hydraulic conductivity for fully developed turbulent flow in a single fracture. J. Hydrol. 339, 206–215.
Sen, Z., 1987. Non-Darcian flow in fractured rocks with a linear flow pattern. J. Hydrol. 92, 43–57.
Sen, Z., 1988a. Type curves for two-region well flow. J. Hydraul. Eng-ASCE 114 (12), 1461–1484.
Sen, Z., 1988b. Analytical solution incorporating nonlinear radial flow in confined aquifers. Water Resour. Res. 24 (4), 601–606.
Sen, Z., 1989. Nonlinear flow toward wells. J. Hydraul. Eng-ASCE 115 (2), 193–209.
Sen, Z., 1990. Nonlinear radial flow in confined aquifers toward large-diameter wells. Water Resour. Res. 26 (5), 1103–1109.
Shampine, L.F., Reichelt, M.W., 1997. The MATLAB ODE Suite. SIAM J. Sci. Comput. 18, 1–22.
Shampine, L.F., Reichelt, M.W., Kierzenka, J.A., 1999. Solving Index-1 DAEs in MATLAB and Simulink. SIAM J. Sci. Comput. 41, 538–552.
Song, H., Tao, L., 2007. Homotopy analysis of 1D unsteady, nonlinear groundwater flow through porous media. In: Journal of Coastal Research. Proceedings of the 9th International Coastal Symposium, Gold Coast, Australia SI 50, 292–296. ISSN 0749.0208.
Stehfest, H., 1970a. Algorithm 368 numerical inversion of Laplace transforms. Commun. ACM. 13 (1), 47–49.
Stehfest, H., 1970b. Remark on algorithm 368 numerical inversion of Laplace transforms. Commun. ACM 13 (10), 624–625.
Teh, C.I., Nie, X., 2002. Coupled consolidation theory with non-Darcian flow. Comput. Geotech. 29, 169–209.
Thiruvengadam, M., Kumar, G.N.P., 1997. Validity of Forchheimer equation in radial flow through coarse granular media. J. Eng. Mech. 123 (7), 694–704.
Venkataraman, P., Rao, P.R.M., 2000. Validation of Forchheimer's law for flow through porous media with converging boundaries. J. Hydraul. Eng-ASCE 126 (1), 63–71.
Wattenbarger, R.A., Ramey Jr., H.J., 1969. Well test interpretation of vertically fractures gas wells. J. Petrol. Tech. AIME 246, 625–632.
Wen, Z., Huang, G., Zhan, H., 2006. Non-Darcian flow in a single confined vertical fracture toward a well. J. Hydrol. 330, 698–708.
Wen, Z., Huang, G., Zhan, H., 2008a. An analytical solution for non-Darcian flow in a confined aquifer using the power law function. Adv. Water Resour. 31, 44–55.
Wen, Z., Huang, G., Zhan, H., Li, J., 2008b. Two-region non-Darcian flow toward a well in a confined aquifer. Adv. Water Resour. 31, 818–827.
Wen, Z., Huang, G., Zhan, H., 2008c. Non-Darcian flow toward a finite-diameter vertical well in a confined aquifer. Pedosphere 18, 288–303.
Wright, D.E., 1964. Nonlinear flow through granular media. J. Fluid Mech. 19, 103–112.
Wu, Y.S., 2002a. An approximate analytical solution for non-Darcy flow toward a well in fractured media. Water Resour. Res. 38 (3), 1023. doi:10.1029/2001WR000713.
Wu, Y.S., 2002b. Numerical simulation of single-phase and multiphase non-Darcy flow in porous and fractured reservoirs. Trans. Porous Med. 49, 209–240.
Yamada, H., Nakamura, F., Watanabe, Y., Murakami, M., Nogami, T., 2005. Measuring hydraulic permeability in a streambed using the packer test. Hydrol. Process. 19, 2507–2524.
Zeng, Z., Grigg, R., 2006. A criterion for non-Darcy flow in porous media. Trans. Porous Med. 63, 57–69.