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A flowing partially penetrating well in a finite-thickness aquifer: a mixed-type initial boundary value problem

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Abstract

An analytical approach using integral transform techniques is developed to deal with a well hydraulics model involving a mixed boundary of a flowing partially penetrating well, where constant drawdown is stipulated along the well screen and no-flux condition along the remaining unscreened part. The aquifer is confined of finite thickness. First, the mixed boundary is changed into a homogeneous Neumann boundary by discretizing the well screen into a finite number of segments, each of which at constant drawdown is subject to unknown a priori well bore flux. Then, the Laplace and the finite Fourier transforms are used to solve this modified model. Finally, the prescribed constant drawdown condition is reinstated to uniquely determine the well bore flux function, and to restore the relation between the solution and the original model. The transient and the steady-state solutions for infinite aquifer thickness can be derived from the semi-analytical solution, complementing the currently available dual integral solution. If the distance from the edge of the well screen to the bottom/top of the aquifer is 100 times greater than the well screen length, aquifer thickness can be assumed infinite for times of practical significance, and groundwater flow can reach a steady-state condition, where the well will continuously supply water under a constant discharge. However, if aquifer thickness is smaller, the well discharge decreases with time. The partial penetration effect is most pronounced in the vicinity of the flowing well, decreases with increasing horizontal distance, and vanishes at distances larger than 1–2 times the aquifer thickness divided by the square root of aquifer anisotropy. The horizontal hydraulic conductivity and the specific storage coefficient can be determined from vertically averaged drawdown as measured by fully penetrating observation wells. The vertical hydraulic conductivity can be determined from the well discharge under two particular partial penetration conditions.

Keywords: Partial penetration; Mixed boundary; Constant drawdown; Groundwater flow; Confined aquifers

1. Introduction

1.1. Mixed boundary

The constant head test is particularly useful for the determination of hydraulic properties in a low-permeability aquifer (e.g. Jones et al., 1992; Chen and Chang, 2002). Also, constant drawdown pumping is a common means of controlling off-site migration of contaminated groundwater (Hiller and Levy, 1994), or recovering light non-aqueous phase liquids (Murdoch and Franco, 1994). Rice (1998) discussed some advantages associated with the constant drawdown test. Under field conditions, wells usually only partially penetrate aquifers,

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Nomenclature

a_{ij}	matrix element defined by Eq. (29)
b	aquifer thickness, [L]
C_k	dimensionless pseudo-skin factor
C_p	dimensionless coefficient in Eqs. (45) or (46)
d_d	depth to top of the well screen, [L]
d_l	depth to bottom of the well screen, [L]
$h_1(\rho,\tau)$	Laplace inversion of $H_1(\rho,p)$
$h_2(\rho,\xi,\tau)$	Laplace inversion of $H_2(\rho,\xi,p)$
h_w	constant drawdown prescribed at the well, [L]
$h_w(t)$	well bore drawdown as unknown a priori, [L]
$h(r,z,t)$	transient drawdown, [L]
$h(\rho,\xi,\tau)$	dimensionless drawdown equal to $h(r,z,t)/h_w$
$\bar{H}(\rho,n,p)$	dimensionless drawdown in the Laplace and Fourier domain
$H(\rho,\xi,p)$	dimensionless drawdown in the Laplace-domain
K_r	horizontal (radial) hydraulic conductivity, [L/t]
K_z	vertical hydraulic conductivity, [L/t]
$K_0(x)$	modified Bessel function of the second kind of zero order
$K_1(x)$	modified Bessel function of the second kind of first order
l	screen length, [L]
M	total number of the well segments
n	finite Fourier cosine transform parameter of ξ
p	Laplace transform parameter of τ
$q_w(z,t)$	well bore flux as unknown a priori, [L/t]
$\bar{q}_w(\xi,\tau)$	dimensionless well bore flux equal to $\frac{q_w(z,t)r_w}{K_r h_w}$
$\bar{q}_w(\xi,p)$	Laplace transform of $\bar{q}_w(\xi,\tau)$
$q_i(t)$	well bore flux of the ith segment
$q_i(\tau)$	dimensionless well bore flux of the ith segment equal to $\frac{q_i(t)r_w}{K_r h_w}$
$q_i(p)$	Laplace transform of $q_i(\tau)$
Q	constant pumping rate, [L^3/t]
$Q_w(t)$	well discharge, [L^3/t]
$Q_w(\tau)$	dimensionless well discharge equal to $Q_w(t)/2\pi h_w K_r \ell$
$Q_w(p)$	Laplace transform of $Q_w(\tau)$
$Q_z(\xi_d)$	total vertical flow rate across the plane at ξ_d
$Q_z(\xi_l)$	total vertical flow rate across the plane at ξ_l
r	radial distance, [L]
r_w	well bore radius, [L]
S_s	specific storage coefficient, [1/L]
t	time, [t]
z	vertical distance, [L]
$\beta = b/r_w$	dimensionless aquifer thickness
$\rho = r/r_w$	dimensionless radial distance
$\tau = (K_r t)/S_s r_w^2)$	dimensionless time
$\kappa = K_z/K_r$	anisotropy ratio
$\chi_n = \sqrt{p + (n^2\pi^2\kappa)/\beta^2}$	
$\chi_n' = \sqrt{p + u^2\kappa}$	
$\lambda = l/r_w$	dimensionless screen length
$\omega = \lambda/\beta = l/b$	partial penetration ratio
$\xi = z/r_w$	dimensionless vertical distance
$\xi_d = d_d/r_w$	
$\xi_l = d_l/r_w$	
ξ_i	depth to the top of the ith well segment
$\bar{\xi}_i$	depth to the center of the ith well segment
$\Delta\xi_i$	length of the ith well segment

especially when aquifer thickness is large. Therefore, well hydraulics concerning a partially penetrating well under a constant drawdown condition is of practical importance.

As shown in Fig. 1, a partially penetrating well withdraws groundwater through the well screen extending from d_d to d_l under a prescribed constant drawdown h_w. The boundary involving such a flowing partially penetrating well is under a mixed boundary condition because a Dirichlet condition of constant drawdown is prescribed along the well screen, while a Neumann condition of

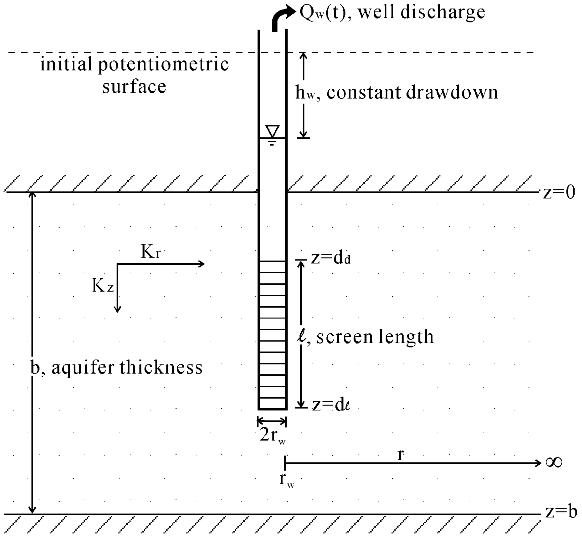

Fig. 1. A well partially penetrates a confined aquifer of finite thickness, and a constant drawdown is imposed on the well resulting in a temporal varying well discharge.

no flux is stipulated over the remaining unscreened part, or

$$h(r_w, z, t) = h_w \qquad d_d \le z \le d_l \tag{1a}$$

$$\left.\frac{\partial h}{\partial r}\right|_{r_w} = 0 \qquad 0 \le z \le d_d, \ \ d_l \le z \le b \tag{1b}$$

1.2. Background information

Methods of integral transforms, or separation of variables, are not suitable for dealing with the mixed boundary, and hence different analytical solution techniques are required (e.g. see Sneddon, 1966; Noble, 1958; Fabrikant, 1991). A few analytical solutions are available for well hydraulics models involving a flowing partially penetrating well. Selim and Kirkham (1974) used the Gram–Schmidt orthonormalization method to find a steady-state solution for a confined aquifer of finite horizontal extent. Wilkinson and Hammond (1990) used a perturbation method to give an approximate solution for drawdown changes at the well in a confined aquifer of semi-infinite thickness (i.e. $b \to \infty$). Using the dual integral equation method, Cassiani et al. (1999) developed semi-analytical solutions (the Laplace-domain solutions) with infinitesimal skin for a confined aquifer of semi-infinite thickness. Moreover, similar problems involving a mixed boundary arise in the field of heat conduction. Among others, Huang (1985) used the Weiner–Hopf technique to find the solution in a semi-infinite slab, and Huang and Chang (1984) combined the Green function approach with conformal mapping to determine the solution in an elliptic disk.

Attempting to analyze groundwater movement toward a flowing partially penetrating well with finite-thickness skin, Novakowski (1993) used the point-source function approach to derive a semi-analytical solution. This solution satisfies a homogeneous Dirichlet boundary condition where constant drawdown prevails along the well screen, but zero drawdown, instead of zero flux, is invoked over the remaining unscreened part. Cassiani et al. (1999) pointed out that this approach led to physically inconsistent semi-infinite total flow to the well when skin effect was excluded.

In addition to constant drawdown, another practical condition that can be imposed on a partially penetrating well is a constant pumping rate, Q. A constant pumping rate through the screen length, l, can be stipulated as the Neumann condition of $\partial h/\partial r = -Q/(2\pi r_w K_r l)$; and the unscreened part remains the same as Eq. (1b). As such, the pertinent boundary condition is a homogeneous Neumann condition and the associated problems can be dealt with by using integral transform methods (e.g. Hantush, 1961; Javandel, 1982; Dougherty and Babu, 1984; Moench, 1985; Streltsova, 1988). However, associated with the homogeneous Neumann boundary condition is variable drawdown along the well screen (Hantush, 1964; Muskat, 1946). When considering that pressure variations inside the well are negligibly small compared to pressure drops in the surrounding porous media, drawdown should be uniformly distributed along the well screen (Muskat, 1946; Hantush, 1964; Gringarten and Ramey, 1975). That is, the partially penetrating well of a constant pumping rate can also be specified as a mixed boundary (Gringarten and Ramey, 1975; Dagan, 1978; Cassiani and Kabala, 1998); namely,

$$h(r_w, z, t) = h_w(t) \qquad d_d \le z \le d_l \tag{2a}$$

$$\left.\frac{\partial h}{\partial r}\right|_{r_w} = 0 \qquad 0 \le z \le d_d, \qquad d_l \le z \le b \tag{2b}$$

The mixed boundaries set forth by Eqs. (1a)–(2b) are different in that the constant h_w in Eq. (1a) is known as prescribed while the time-dependent $h_w(t)$ in Eq. (2a) is unknown a priori. In addition, the absence of Q makes Eqs. (2a and 2b) independent of the constant pumping rate condition. These shortcomings of Eqs. (2a and 2b) can be overcome by invoking the physically appropriate constraint across the well screen such as

$$2\pi r_w \int_{d_d}^{d_l} q_w(z,t)\mathrm{d}z = Q \qquad d_d \le z \le d_l \tag{3a}$$

where $q_w(z,t)$ is well bore flux equal to the flow rate per unit area of the well screen defined as

$$q_w(z,t) = -K_r \left.\frac{\partial h}{\partial r}\right|_{r_w} \tag{3b}$$

Although well bore flux $q_w(z,t)$ is also unknown a priori, it can be determined through Eqs. (3a and 3b); detailed procedures for this practice are discussed in Dagan (1978), and Gringarten and Ramey (1975). For an unconfined aquifer of finite extent Dagan (1978), employed the Green function approach to establish a steady-state solution valid for $l/r_w \ge 50$. For a confined aquifer of semi-infinite thickness Gringarten and Ramey (1975), gave a line-sink solution using the source function approach. Using dual integral equations Cassiani and Kabala (1998), determined semi-analytical solutions based on the assumption of $b \to \infty$, where infinitesimal skin thickness and well bore storage were taken into account. For a stratified aquifer, numerical solutions were determined by Hemker (1999) with the hybrid analytical–numerical technique. In this technique, the radial flow component was treated analytically, but the finite difference method was used to deal with the vertical flow component wherein a finite aquifer thickness is required.

Ruud and Kabala (1997) investigated the difference between the aforementioned homogeneous Neumann boundary and the mixed boundary for the constant pumping rate condition. They concluded that the use of the homogeneous Neumann boundary as the substitute for the mixed boundary should be exercised with extreme caution. Whether the associated boundary condition should be a mixed or a homogeneous Neumann boundary is subject to debate for Hantush (1964) pointed out 'neither a uniform flux nor a uniform drawdown is really conceived along the face of the well, because of several involved field and operational conditions.'

1.3. Purpose

However, the boundary involving a flowing partially penetrating well is clearly a mixed boundary. As discussed above, transient analytical or semi-analytical solutions associated with such a mixed boundary condition assume that aquifer thickness is infinity. Since aquifer thickness is finite under field conditions, these solutions may be appropriate for the early-time stage in which pressure change caused by the constant drawdown pumping has not reached the bottom of the aquifer, or for the special condition where the screen length is significantly shorter than aquifer thickness. These limitations, however, can be removed if aquifer thickness is assumed to be finite. For this reason, we are interested in finding the semi-analytical solution for the problem dealing with a flowing partially penetrating well in a confined aquifer of finite thickness. To do so, an analytical solution technique using the Laplace and the finite Fourier cosine transforms is developed.

Assuming that the aquifer is homogeneous and anisotropic, the mathematical model for the problem of interest is:

$$K_r\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\right) + K_z\frac{\partial^2 h}{\partial z^2} = S_s\frac{\partial h}{\partial t} \tag{4}$$

$$r_w \le r < \infty, \quad 0 \le z \le b$$

$$h(r,z,0) = 0 \tag{5}$$

$$h(r \to \infty, z, t) = 0 \tag{6}$$

$$\frac{\partial h}{\partial z} = 0 \qquad z = 0, \quad z = b \tag{7}$$

$$\left.\frac{\partial h}{\partial r}\right|_{r_w} = 0, \qquad 0 \le z \le d_d, \quad d_l \le z \le b \tag{8a}$$

$$h(r_w, z, t) = h_w \qquad d_d \le z \le d_l \tag{8b}$$

The above model cannot be directly solved by the integral transform technique because of the mixed boundary stated by Eqs. (8a and 8b). Considering that the hydraulic gradient across the interface between the well screen and the surrounding aquifer serves as the driving force that moves groundwater into the well, well bore flux $q_w(z,t)$ plays a pivotal role in maintaining a mass balance between groundwater entering and leaving the well to sustain a constant water volume inside the well bore (constant drawdown). Based on this fact, the first step of the analytical approach is to transform Eqs. (8a and 8b) by replacing Eq. (8b) with Eq. (3b) thereby forming a homogeneous Neumann condition. Then, the semi-analytical solution of the modified model can be determined by using the Laplace and the finite Fourier cosine transforms. As a result, this solution involves the unknown well bore flux $q_w(z,t)$ and is independent of the prescribed h_w. By reinstating Eq. (8b), not only is $q_w(z,t)$ uniquely determined but also the relation between the solution and the original problem is restored.

In summary, the purposes of this paper are: (1) to determine the semi-analytical solution for the model described by Eqs. (4–8b) with the approach discussed above, (2) to analyze the well discharge $Q_w(t)$, well bore flux $q_w(z,t)$, and drawdown distributions, and (3) to investigate the significance of finite or semi-infinite aquifer thickness.

2. Development of semi-analytical solution

2.1. Solution to the modified model

The dimensionless forms of Eqs. (3a–8b) are

$$\frac{\partial^2 h}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial h}{\partial \rho} + \kappa\frac{\partial^2 h}{\partial \xi^2} = \frac{\partial h}{\partial \tau} \tag{9}$$

$1 \le \rho \le \infty, \quad 0 \le \xi \le \beta$

$$h(\rho, \xi, \tau = 0) = 0 \tag{10}$$

$$h(\rho \to \infty, \xi, \tau) = 0 \tag{11}$$

$$\frac{\partial h}{\partial \xi} = 0 \qquad \xi = 0, \quad \xi = \beta \tag{12}$$

$$\left.\frac{\partial h}{\partial \rho}\right|_{\rho=1} = 0 \qquad 0 \le \xi \le \xi_d, \quad \xi_l \le \xi \le \beta \tag{13}$$

$$h(\rho = 1, \xi, \tau) = 1 \qquad \xi_d \le \xi \le \xi_l \tag{14}$$

$$\left.\frac{\partial h}{\partial \rho}\right|_{\rho=1} = -\bar{q}_w(\xi, \tau) \qquad \xi_d \le \xi \le \xi_l \tag{15}$$

All the dimensionless parameters are defined in Nomenclature, unless otherwise noted. The modified model consists of Eqs. (9–13), and Eq. (15); Eq. (14) is temporarily left out and will be reinstated upon determination of $\bar{q}_w(\xi,p)$. The integral transform techniques used here are the Laplace transform with respect to τ and the finite Fourier cosine transform with respect to ξ. Successive application of these two transforms to the modified model results in

$$\frac{d^2\bar{H}}{d\rho^2} + \frac{1}{\rho}\frac{d\bar{H}}{d\rho} - \left(p + \frac{n^2\pi^2\kappa}{\beta^2}\right)\bar{H} = 0 \tag{16}$$

$$\bar{H}(\rho \to \infty, n, p) = 0 \tag{17}$$

$$\left.\frac{d\bar{H}}{d\rho}\right|_{\rho=1} = -\int_{\xi_d}^{\xi_l} \bar{q}_w(\xi, p) cos\left(\frac{n\pi\xi}{\beta}\right) d\xi \tag{18}$$

$\xi_d \le \xi \le \xi_l$

where p is the Laplace transform variable, and n is the finite Fourier cosine transform variable. Eq. (16) is a modified Bessel equation, of which the solution subject to Eqs. (17) and (18) is

$$\bar{H}(\rho, n, p) = \frac{K_0(\chi_n\rho)}{\chi_n K_1(\chi_n)} \int_{\xi_d}^{\xi_l} \bar{q}_w(\xi, p) cos\left(\frac{n\pi\xi}{\beta}\right) d\xi \tag{19}$$

where

$$\chi_n = \left(p + \frac{n^2\pi^2\kappa}{\beta^2}\right)^{1/2} \tag{20}$$

Application of the inversion theorem for the finite Fourier cosine transform (Sneddon, 1972) to Eq. (19) yields

$$H(\rho,\xi,p)=\frac{1}{\beta}\frac{K_0(\rho\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})}\int_{\xi_d}^{\xi_l}\bar{q}_w(\xi,p)\mathrm{d}\xi+\frac{2}{\beta}\sum_{n=1}^{\infty}\left[\frac{K_0(\chi_n\rho)}{\chi_n K_1(\chi_n)}\int_{\xi_d}^{\xi_l}\bar{q}_w(\xi,p)cos\left(\frac{n\pi\xi}{\beta}\right)\mathrm{d}\xi\right]cos\left(\frac{n\pi\xi}{\beta}\right) \tag{21}$$

which is in the Laplace domain. Eq. (21) is not ready for Laplace inversion because $\bar{q}_w(\xi,p)$ is unknown. Determination of $\bar{q}_w(\xi,p)$ requires the reinstatement of Eq. (14) as discussed below.

2.2. Reinstatement of the well bore boundary

For the reinstatement of Eq. (14), the dimensionless screen length, λ, is divided into M segments, each having a length of $\Delta\xi_i, i=1,2,\dots M$; such a discretization practice was also employed by others to deal with different problems (e.g. Dagan, 1978; Selim and Kirkham, 1974; Gringarten and Ramey, 1975; Lee and Damiata, 1995). As shown in Fig. 2, $\Delta\xi_i$ can be a constant equal to λ/M if λ is uniformly divided, or a variable if λ is non-uniformly discretized. Corresponding to the M segments, the unknown function of $\bar{q}_w(\xi,p)$ is replaced by $q_i(p)$, $i=1,2,\dots M$. For segment i, $q_i(p)$ is constant over $\Delta\xi_i$ such that

$$\int_{\xi_d}^{\xi_l}\bar{q}_w(\xi,p)\mathrm{d}\xi=\sum_{i=1}^{M}q_i(p)\Delta\xi_i \tag{22}$$

Rigorously speaking, only when M approaches infinity can the right hand side be exactly equal to the left hand side in Eq. (22). Nevertheless, the equality of Eq. (22) can be accurately approximated if the well screen is properly discretized and M is properly chosen, as discussed below. Introduction of Eq. (22) to Eq. (21) yields

$$H(\rho,\xi,p)=H_1(\rho,p)+H_2(\rho,\xi,p) \tag{23}$$

where

$$H_1(\rho,p)=\left\{\frac{K_0(\rho\sqrt{p})}{\beta\sqrt{p}K_1(\sqrt{p})}\right\}\sum_{i=1}^{M}q_i(p)\Delta\xi_i \tag{24}$$

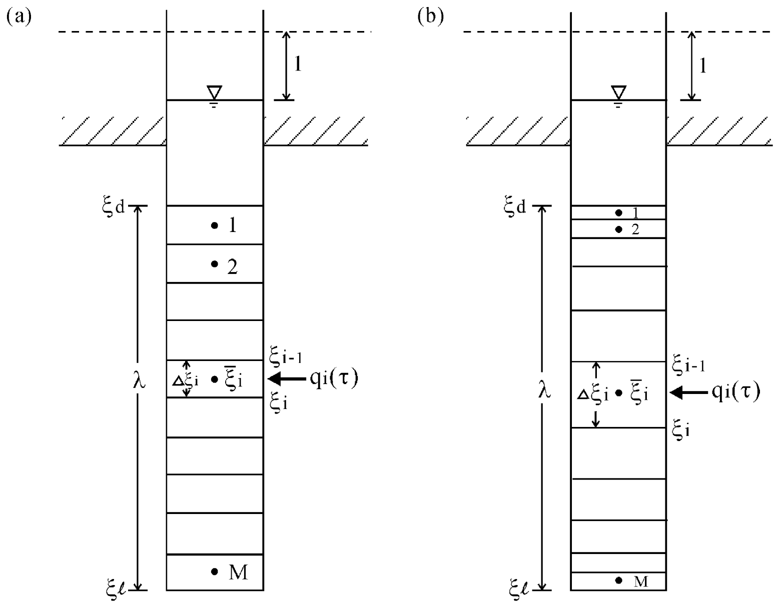

Fig. 2. The screen length is discretized into M segments, and each segment is subject to a constant dimensionless drawdown equal to unity and a specific dimensionless well bore flux $q_i(\tau)$. (a) Uniform discretization condition where $\Delta\xi_i$ is constant. (b) Non-uniform discretization condition where $\Delta\xi_i$ varies.

$$H_2(\rho,\xi,p) = 2\sum_{i=1}^{M} q_i(p)\left\{\sum_{n=1}^{\infty}\frac{K_0(\chi_n\rho)}{\chi_n K_1(\chi_n)}\frac{1}{n\pi}\left[sin\left(\frac{n\pi\xi_i}{\beta}\right) - sin\left(\frac{n\pi\xi_{i-1}}{\beta}\right)\right]cos\frac{n\pi\xi}{\beta}\right\} \tag{25}$$

Since H_1 is independent of ξ, and the integration of $cos(n\pi\xi/\beta)$ from 0 to β in H_2 is zero, H_1 signifies the vertically averaged drawdown as can be measured by fully penetrating observation wells, and H_2 represents the partial penetration effect. However, if β approaches infinity, H_1 is zero, and thus vertically averaged drawdown is unavailable for aquifer thickness being semi-infinitely large. Now Eq. (14) is transformed to the Laplace domain,

$$H(\rho=1,\xi,p)=1/p \quad \xi_d \le \xi \le \xi_l \tag{26}$$

Corresponding to the discretized well screen, Eq. (26) is applied to the center of each well segment as

$$H(\rho=1,\bar{\xi}_i,p)=1/p, \quad i=1,2\ldots M, \quad \xi_d \le \bar{\xi}_i \le \xi_l \tag{27}$$

where $\bar{\xi}_i=(\xi_i+\xi_{i-1})/2$ represents the center of the ith well segment between ξ_i and ξ_{i-1}, with $\xi_0=\xi_d$ and $\xi_M=\xi_l$. For each well segment, there is one equation resulting from the application of Eq. (27) to Eq. (23). For M segments, there are such M equations as

$$[a_{ij}]\{q_i(p)\}=\{1/p\} \quad i,j=1,2,\ldots M \tag{28}$$

where

$$a_{ij}=\frac{\Delta\xi_j}{\beta}\frac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})}+4\sum_{n=1}^{\infty}\frac{K_0(\chi_n)}{\chi_n K_1(\chi_n)}\frac{1}{n\pi} \times sin\left(\frac{n\pi\Delta\xi_j}{2\beta}\right)cos\left(\frac{n\pi\bar{\xi}_j}{\beta}\right)cos\left(\frac{n\pi\bar{\xi}_i}{\beta}\right) \tag{29}$$

Eq. (29) indicates that a_{ij} is equal to a_{ji} for $\Delta\xi_j$ being constant; that is, $[a_{ij}]$ is symmetric when $\Delta\xi_j$ is constant; otherwise $[a_{ij}]$ is asymmetric. After $[a_{ij}]$ is inverted, $\{q_i(p)\}$ can be obtained. Introduction of $\{q_i(p)\}$ to Eq. (23) completes the restatement of Eq. (14) in the Laplace domain and restores the relation between Eq. (23) and the original model. By direct substitution, it can be shown that Eq. (23) satisfies the modified model of Eqs. (9–12) in the Laplace domain. The satisfaction of the remaining boundary conditions of Eq. (13) and Eq. (15) is verified in Appendix.

2.3. Numerical evaluation of solutions

Whether $[a_{ij}]$ is symmetric or asymmetric, appropriate numerical methods can be used for the matrix inversion of $[a_{ij}]$. In the calculation of a_{ij}, the semi-infinite series in Eq. (29) can be effectively computed with the aid of the Euler transformation (Press et al., 1992; pp. 160–163). In general, the convergence of the oscillatory function of Eq. (29) requires 200–1000 terms in calculation, depending on the size of $\Delta\xi_j$. Then $\{q_i(p)\}$ is substituted into Eqs. (24) and (25) to close Eq. (23). For any specific τ, transient solutions of $h(\rho,\xi,\tau)$ are obtained by numerically inverting Eq. (23) with the Talbot (1979) method. In processing this Laplace inversion, $\{q_i(\tau)\}$, $h_1(\rho,\tau)$, and $h_2(\rho,\xi,\tau)$ are determined. From $\{q_i(\tau)\}$, the well discharge $Q_w(\tau)$ of the partially penetrating well can be determined as

$$Q_w(\tau) = \sum_{i=1}^{M} q_i(\tau)\Delta\xi_i/\lambda \tag{30}$$

of which the Laplace trasform is

$$Q_w(p) = \sum_{i=1}^{M} q_i(p)\Delta\xi_i/\lambda \tag{31}$$

The validity of solutions can be verified by inspecting whether Eqs. (14) and (15) are satisfied. The primary reason why Eqs. (14) and (15) fail to be satisfied is due to improper discretization of the screen length. If the screen length is uniformly divided by too large an M, each segment is so small that the well bore flux associated with Eq. (15) becomes oscillatory for small τ. However, if M is too small, each segment becomes too coarse, and well bore drawdown loses accuracy, particularly in the vicinity of either end of the screen. To deal with this dilemma, the screen length should be non-uniformly divided as demonstrated in Fig. 2(b), where finer grids exist near the ends while coarser grids in the middle portion.

To illustrate these points, well bore drawdown and well bore flux resulting from two uniform grids and

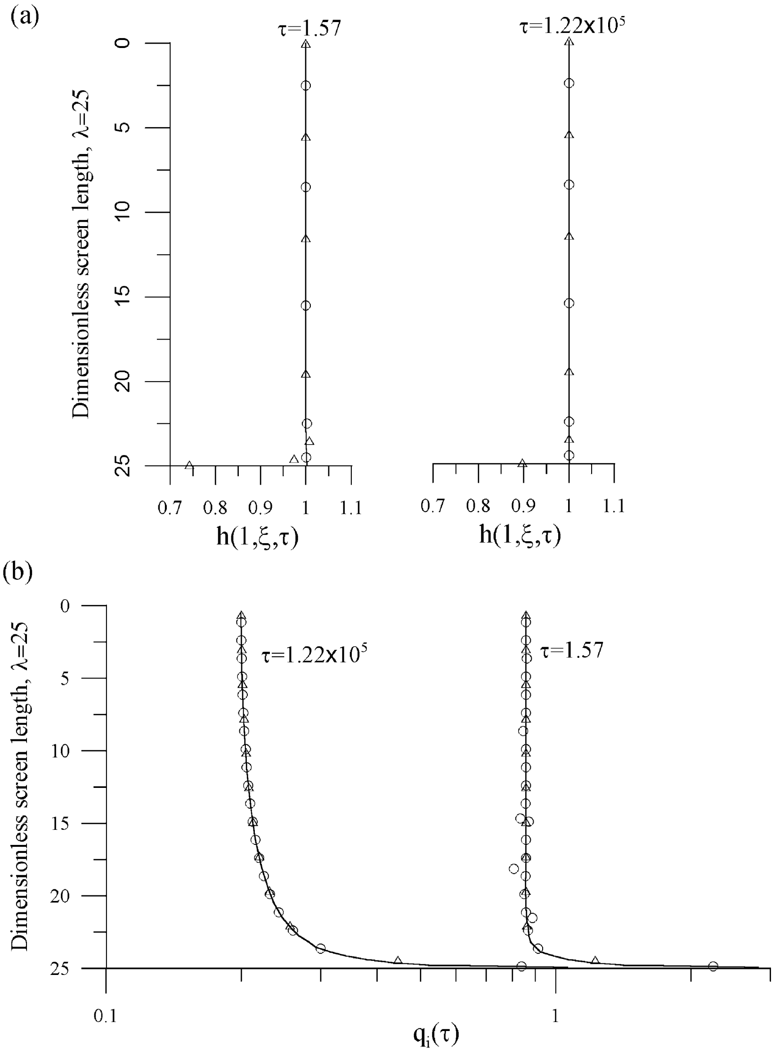

Fig. 3. Influence of well screen discretization on the boundary conditions at the well bore for $\beta = 50$ and $\kappa = 1$; the solid line represents the nonuniform grid of $M = 20$, Δ represents the uniform grid of $M = 20$, and ○ represents the uniform grid of $M = 100$. (a) Dimensionless well bore drawdown $h(1,\xi,\tau)$ along the well screen for the three grid systems. (b) Dimensionless well bore flux $q_i(\tau)$ along the well screen for the three grid systems.

one non-uniform grid are compared in Fig. 3. In the two uniform grids, M is 20 and 100; respectively, and the non-uniform grid has $M = 20$. Extending from the top of the aquifer, the screen length is assumed to be 25 (i.e. $\lambda = 25$ from $\xi_d = 0$ to $\xi_l = 25$). For $M = 20$, well bore drawdown of the non-uniform grid is equal to unity along the full screen length, while well bore drawdown of the uniform grid is less than unity near the bottom screen at $\lambda = 25$. To eliminate the imprecision near $\lambda = 25$, M needs to be increased from 20 to 100 in the uniform grid. This indicates that the same accuracy of well bore drawdown of a uniform grid with $M = 100$ can be achieved by using a non-uniform grid with $M = 20$. Moreover, accompanying this increased M is oscillatory well bore flux at $\tau = 1.57$. However, oscillatory well bore flux is not present for the non-uniform grid at small or large τ. Therefore, it is recommended that the screen length be divided non-uniformly as illustrated in Fig. 2(b) and M be chosen around 20.

3. Analysis and discussion

3.1. Semi-analytical solutions for semi-infinite aquifer thickness

Since aquifer thickness is assumed finite in Eq. (23), it is feasible to derive the semi-analytical solution for aquifer thickness being semi-infinite from Eq. (23) by letting β approach infinity. In this event, H_1 of Eq. (24) is zero, and H_2 forms the complete solution for $H(\rho,\xi,p)$. However, the infinite series in Eq. (25) becomes an infinite integral when β approaches ∞, as discussed in Appendix. As a result, the semi-analytical drawdown solution for aquifer thickness being semi-infinite is

$$H(\rho,\xi,p) = \sum_{i=1}^{M} q_i(p)\left\{\frac{2}{\pi}\int_0^{\infty}\frac{K_0(\rho\chi'_n)}{\chi'_n K_1(\chi'_n)} \times [sin(u\xi_i) - sin(u\xi_{i-1})]cos(u\xi)\frac{du}{u}\right\} \quad (32)$$

where $\chi'_n = (p + u^2\kappa)^{1/2}$. In this case, a_{ij} is

$$a_{ij} = \frac{4}{\pi}\int_0^{\infty}\frac{K_0(\chi'_n)}{\chi'_n K_1(\chi'_n)} \times sin\left(\frac{u\Delta\xi_j}{2}\right)cos(u\bar{\xi}_i)cos(u\bar{\xi}_j)\frac{du}{u} \quad (33)$$

Since a function at large τ corresponds to its Laplace transform counterpart at small p, the Laplace inversion of $H(\rho,\xi,p)$ for small p gives $h(\rho,\xi,\tau)$ for large τ. As suggested by Chen and Stone (1993), such an asymptotic inversion for the constant-drawdown condition is better accomplished using the Tauberian theorem (e.g. see Sneddon, 1972), which can be written as

$$\lim_{p\to 0} pH(\rho,\xi,p) = h(\rho,\xi,\tau\to\infty) \quad (34)$$

The steady-state solution occurs as τ approaches infinity. In accordance with Eq. (34), the Laplace inversion of Eq. (23) as p approaches zero does not exist, indicating there is no steady-state solution available for Eq. (23). However, if Eq. (34) is applied to Eq. (32), the result is

$$h(\rho,\xi) = \sum_{i=1}^{M} q_i\left\{\frac{2}{\pi\sqrt{\kappa}}\int_0^{\infty}\frac{K_0(\rho\sqrt{\kappa}u)}{K_1(\sqrt{\kappa}u)} \times [sin(u\xi_i) - sin(u\xi_{i-1})]cos(u\xi)\frac{du}{u^2}\right\} \quad (35a)$$

where well bore flux is no longer dependent on τ and is defined by

$$\{q_i\} = [a_{ij}]^{-1}\{1\} \quad (35b)$$

with a_{ij} being defined by

$$a_{ij} = \frac{4}{\pi\sqrt{\kappa}}\int_0^{\infty}\frac{K_0(u\sqrt{\kappa})}{K_1(u\sqrt{\kappa})} \times sin\left(\frac{u\Delta\xi_j}{2}\right)cos(u\bar{\xi}_i)cos(u\bar{\xi}_j)\frac{du}{u^2} \quad (35c)$$

Therefore, if aquifer thickness is assumed to be semi-infinite, a steady-state condition can be established for drawdown distribution, and the associated well discharge and well bore flux are independent of time as well. Since Eq. (35a) decreases rapidly as ρ increases, drawdown changes due to the partially penetrating well are limited to finite horizontal extent. The vertical hydraulic gradient at any specific location within the steady-state groundwater flow field is

$$\frac{\partial h}{\partial \xi} = -\sum_{i=1}^{M} q_i\frac{2}{\pi\sqrt{\kappa}}\int_0^{\infty}\frac{K_0(\rho\sqrt{\kappa}u)}{K_1(\sqrt{\kappa}u)}[sin(\xi_i u) - sin(\xi_{i-1}u)]sin(\xi u)\frac{du}{u} \quad (36)$$

By Eq. (36), the total vertical flow rate as normalized by $2\pi h_w K_r l$ across the plane at the depth of ξ_1 is

$$Q_z(\xi_1) = -\frac{\kappa}{\lambda}\int_1^{\infty}\rho\frac{\partial h}{\partial\xi}d\rho = \frac{2}{\pi}\frac{\sqrt{\kappa}}{\lambda}\sum_{i=1}^{M} q_i \times \int_0^{\infty}\frac{[sin(\xi_i u) - sin(\xi_{i-1}u)]sin(\xi_1 u)}{uK_1(\sqrt{\kappa}u)} \times \int_1^{\infty}\rho K_0(\rho u\sqrt{\kappa})d\rho du \quad (37)$$

In Eq. (37), ρ in the integration is permitted to vary from the well bore face (i.e. $\rho = 1$) to infinity, rather than to finite horizontal extent. Using the equations presented by Gradshteyn and Ryzhik (1980; Equation 6.561–8) and Abramowitz and Stegun (1970;

Equation 11.4.22), the following relation can be established for the inner integral of Eq. (37);

$$\int_1^\infty \rho K_0(\rho u\sqrt{\kappa})\mathrm{d}\rho = \frac{K_1(u\sqrt{\kappa})}{u\sqrt{\kappa}} \tag{38}$$

Substitution of Eq. (38) in Eq. (37) gives

$$Q_z(\xi_l) = \frac{2}{\pi\lambda}\sum_{i=1}^{M} q_i \int_0^\infty [sin(\xi_i u) - sin(\xi_{i-1}u)] \times sin(\xi_l u)\frac{\mathrm{d}u}{u^2} \tag{39}$$

The following relation given by Gradshteyn and Ryzhik (1980, Equation 3.741–3) is useful in carrying out the infinite integral in Eq. (39)

$$\int_0^\infty sin(a_1 x)sin(a_2 x)\frac{\mathrm{d}x}{x^2} = \frac{a_1\pi}{2} \qquad 0 < a_1 \le a_2 \tag{40}$$

Since ξ_l is greater than both ξ_i and ξ_{i-1}, Eq. (39) is simplified in accordance with Eq. (40) as

$$Q_z(\xi_l) = \frac{2}{\lambda\pi}\sum_{i=1}^{M} q_i\left[\frac{\pi}{2}\xi_i - \frac{\pi}{2}\xi_{i-1}\right] = \frac{1}{\lambda}\sum_{i=1}^{M} q_i\Delta\xi_i = Q_w \tag{41}$$

Therefore, the steady-state well discharge is completely supported by the lower portion of the aquifer ($\xi_l \le \xi \le \infty$), where groundwater is sufficiently available to sustain the steady-state water supply. Furthermore, the total vertical flow rate across the plane at the depth of ξ_d can be determined by replacing ξ_l with ξ_d in Eq. (39). Then, because both ξ_i and ξ_{i-1} are greater than ξ_d, $Q_z(\xi_d)$ is

$$Q_z(\xi_d) = \frac{2}{\lambda\pi}\sum_{i=1}^{M} q_i\left[\frac{\pi}{2}\xi_d - \frac{\pi}{2}\xi_d\right] = 0 \tag{42}$$

Consequently, under the steady-state condition, the total vertical flow rate is zero across the plane of ξ_d. However, Eq. (42) should not be misconstrued as having no vertical flow in the upper portion. As illustrated in Fig. 4, the drawdown contours as determined with Eqs. (35a)–(35c) are not strictly vertical in the upper portion, indicating vertical flow. However, the schematically added flow vectors indicate that the vertical flow components across the plane of ξ_d are downward in the near field, and upward in the far field. Accordingly, the upper portion of the aquifer ($0 \le \xi \le \xi_d$) merely serves as a flow path on which distant groundwater from the lower portion of the aquifer ($\xi_l \le \xi \le \infty$) migrates to the well. Thus, it is the 'net' vertical flow rate across the plane at ξ_d that is zero, as indicated by Eq. (42). Across the plane at ξ_l, the vertical flow components are all upward and their summation supplies the well discharge as shown by Eq. (41).

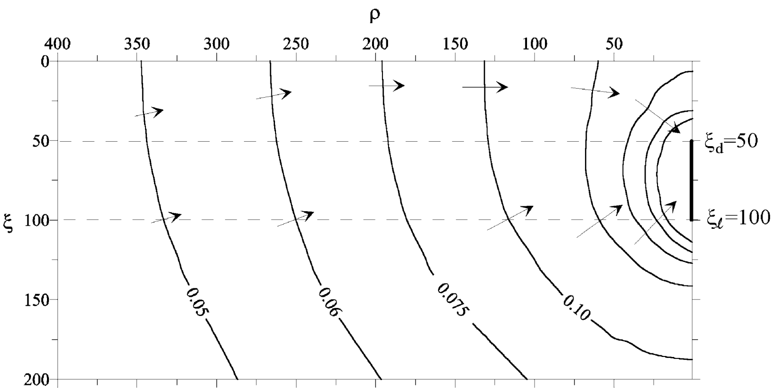

Fig. 4. Steady-state drawdown contours determined by Eq. (35a) and schematically added flow vectors show that the vertical flow components across the plane at ξ_d are downward in the near field and upward in the far field. The vertical flow components across the plane at ξ_l are all upward.

3.2. Well discharge variation

Assuming semi-infinite aquifer thickness, Cassiani et al. (1999) presented the well discharges for $\lambda = 10$, 20 and 50 with $\xi_d = 0$. For this current study, the condition of semi-infinite aquifer thickness can be properly handled in Eq. (23) by setting β significantly larger than λ. For the three cases, if the penetration ratio $\omega = \lambda/\beta$ is smaller than 0.01, $Q_w(\tau)$ of the current study matches well with the published results as verified in Fig. 5. Therefore, if aquifer thickness is greater than 100 times the screen length, it can be assumed semi-infinite for times of practical significance, provided the well screen extends from the top of the aquifer (i.e. $\xi_d = 0$). For the more general situations where ξ_d is greater than zero as illustrated in Figs. 1 or 2, aquifer thickness can be assumed semi-infinite if $(\beta - \xi_d)$ or $(\beta - \xi_l)$ is 100 times greater than the screen length. That is, dependent on the well screen position, if $\lambda/(\beta - \xi_d)$ or $\lambda/(\beta - \xi_l)$ is less than 0.01, then aquifer thickness can be assumed semi-infinite.

As ω increases to unity, the fully penetrating condition is invoked. Associated with such a condition, the well discharge in the Laplace domain was given by Chen and Stone (1993) as

$$Q_w(p) = K_1(\sqrt{p})/\sqrt{p}K_0(\sqrt{p}) \tag{43}$$

where $Q_w(p)$ is the Laplace transform of $Q_w(\tau)$. As shown in Fig. 5, $Q_w(\tau)$ for the current study of $\omega = 1$ is in excellent agreement with $Q_w(\tau)$ determined by Eq. (43) with the Stehfest (1970) method, supporting the accuracy of the solutions obtained herein. As a side note, the Stehfest method gives spurious results of $\{q_i(\tau)\}$ at small τ, and thus it is not used to invert Eq. (23). In Appendix, it is formally proven that $Q_w(p)$ and the semi-analytical drawdown solution of $\omega = 1$ pertains to the fully penetrating conditions.

3.3. Well bore flux variation

Under the isotropic condition ($\kappa = 1$), influence of aquifer thickness on well bore flux $q_i(\tau)$ is illustrated in Fig. 6, where ω is assumed to be 0.2 and 0.01. At

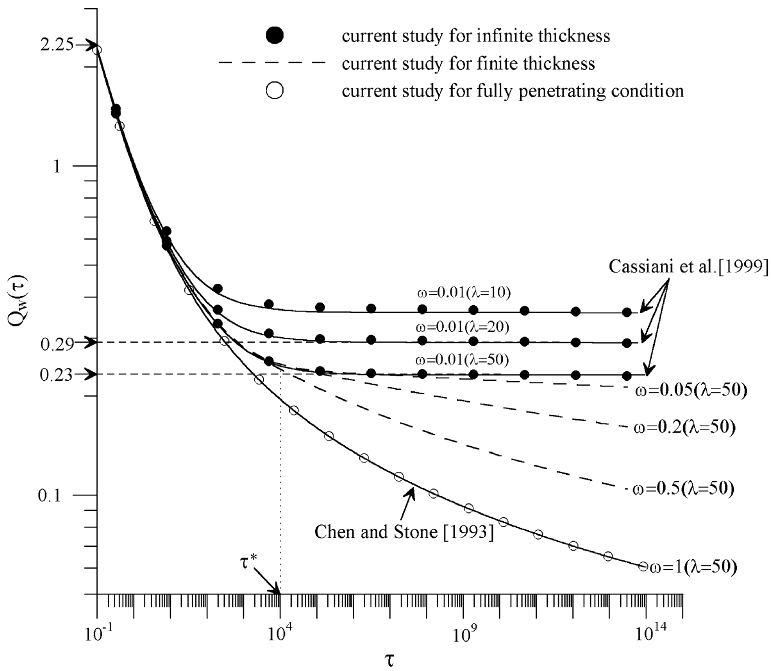

Fig. 5. Influence of aquifer thickness on the dimensionless well discharge $Q_w(\omega)$. Results of the current study for $\omega = 0.01$ coincide with the published results of infinite aquifer thickness. As $\omega = 1$, the fully penetrating condition is invoked and results of the current study coincide with the results of a pertinent model.

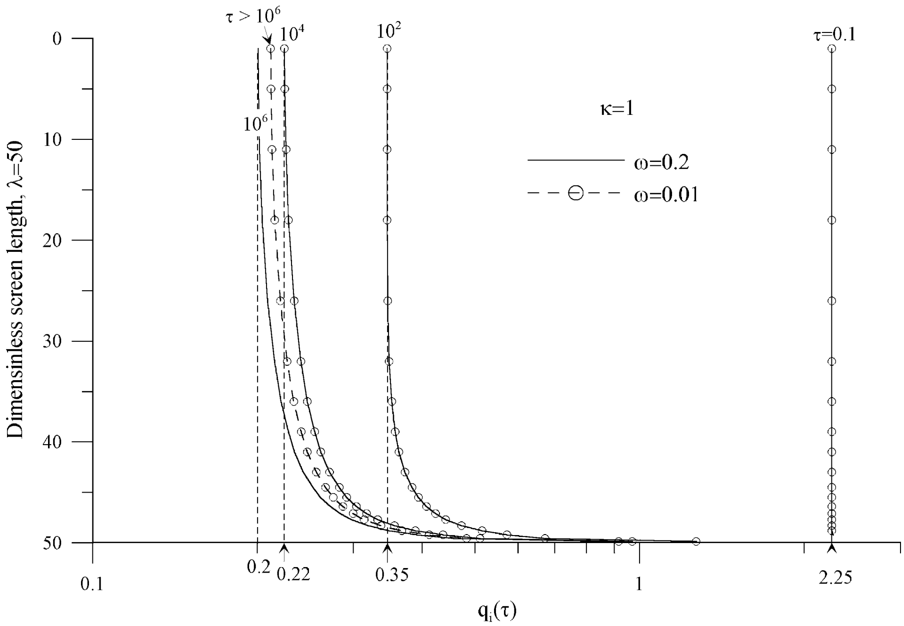

Fig. 6. Temporal well bore flux variation for $\kappa = 1$.

early time of $\tau = 0.1$, well bore flux for both $\omega = 0.2$ and 0.01 is uniformly distributed along the well screen with a constant $q_i(\tau)$ equal to 2.25, except at the end where a tiny difference occurs because of vertical flow influence. Ignoring this tiny difference, the associated $Q_{\mathrm{w}}(\tau)$ determined by Eq. (30) is 2.25, also denoted in Fig. 5. A uniform well bore flux distribution is characteristic of a flowing fully penetrating well, and hence at small τ the flowing partially penetrating well acts as a flowing well that fully penetrates an aquifer of finite thickness equal to λ. This result is also confirmed in Fig. 5, where for τ less than one, all of the $Q_{\mathrm{w}}(\tau)$ curves merges to the fully penetrating condition of $\omega = 1$. This fact is further mathematically validated in Appendix.

When τ increases, well bore flux $q_i(\tau)$ becomes non-uniformly distributed with smaller magnitude. The non-uniform distribution possesses a maximum at the end of the screen due to the strong contribution from vertical groundwater flow induced by the partial penetration effect. Similar non-uniform well bore flux distributions are also noted in Dagan (1978) and Cassiani et al. (1999). For both $\omega = 0.2$ and 0.01, well bore flux variation remains the same until τ is 10^4. After τ exceeds 10^4, then well bore flux variations for both cases become different. This indicates that during the period of τ less than 10^4, pressure change caused by the flowing partial penetrating well has not reached the bottom of the aquifer; it is as if aquifer thickness were semi-infinite. Aquifer thickness begins to have an influence on groundwater flow after τ is greater than 10^4. As τ increases to 10^6, well bore flux $q_i(\tau)$ of $\omega = 0.01$ reaches the steady-state condition, while well bore flux $q_i(\tau)$ of $\omega = 0.2$ continues to decrease. The variation of well bore flux is correspondingly reflected on the well discharge. Referring to Fig. 5, the deviation time of 10^4 is denoted by τ^*, and the well discharge of $\omega = 0.01$ approaches a constant value for τ greater than 10^6, but the well discharge of $\omega = 0.2$ is reduced at large τ.

By assuming $\kappa = 0.5$, the anisotropy effect on well bore flux is relatively minor. At $\tau = 0.1$, well bore flux is identical to that of the isotropic condition. At $\tau = 10^2$, well bore flux is non-uniformly distributed in the same way as shown in Fig. 6, except that well bore flux between $\lambda = 30$ and 50 is slightly smaller. For τ greater than 10^4, well bore flux under the anisotropic condition becomes noticeably smaller than under the isotropic condition, because a smaller K_z tends to impede the vertical flow. Therefore, aquifer anisotropy has little influence on well bore flux for small τ, while it can slightly reduce well bore flux at large τ.

3.4. Drawdown distribution analysis

For a flowing fully penetrating well without well skin, K_r and S_s can be estimated by the late-time linear relation between $h_w/Q_w(t)$ and logarithmic time, as discussed by Jacob and Lohman (1952). For a flowing partially penetrating well without well skin, curves of dimensionless drawdown versus logarithmic τ/ρ^2 are presented in Fig. 7, where vertically averaged drawdown in terms of $h_1(\rho,\tau)/[0.5\omega Q_w(\tau)]$ is used for $\rho > 1$, and constant well bore drawdown in terms of $1/[0.5\omega Q_w(\tau)]$ is used for $\rho = 1$. Also shown is the curve of $1/[Q_w(\tau)]$ for the fully penetrating well condition. At late time, a straight line of a slope equal to 2.303 can represent this curve, demonstrating semi-log radial flow characteristic at late time. This straight line intercepts the abscissa at 1/2.25.

For $\rho = 1$, the curve of $\kappa = 0.001$ shows two straight-line sections, occurring at early and late time, respectively. The slope of the late-time segment is 2.303, and that of the early-time segment is $2.303/\omega$. As discussed earlier, well bore flux at early time is uniformly distributed along the screen length, and thus early-time groundwater flow is radial and confined to the well screen interval in the aquifer. Because κ is so small, vertical flow at early time is insignificant and the early-time radial flow period can last so long that the curve resolves itself into a semi-log straight line. But when κ becomes larger, the early-time radial flow period is reduced and a semi-log linear relation of $1/[Q_w(\tau)]$ cannot be developed. At late time, regardless of the anisotropy ratio and the screen locations, all curves become parallel straight lines of a constant slope of 2.303, indicating the late-time aquifer response over the entire aquifer thickness. In addition, the late-time straight lines are separated by vertical displacement representative of restricted flow entry influence, which is manifested by aquifer anisotropy, the penetration ratio, and the location of the screen length. A dimensionless variable C_k denotes the vertical displacement with respect to the straight line of the fully penetrating condition, as indicated in Fig. 7. Then, the dimensional form of these straight lines of large τ is

$$\frac{h_w}{Q_w(t)} = \frac{2.303}{4\pi K_r b}\left[\log\left(\frac{2.25tK_r}{S_s r_w^2}\right) + C_k\right], \quad \omega < 1 \tag{44}$$

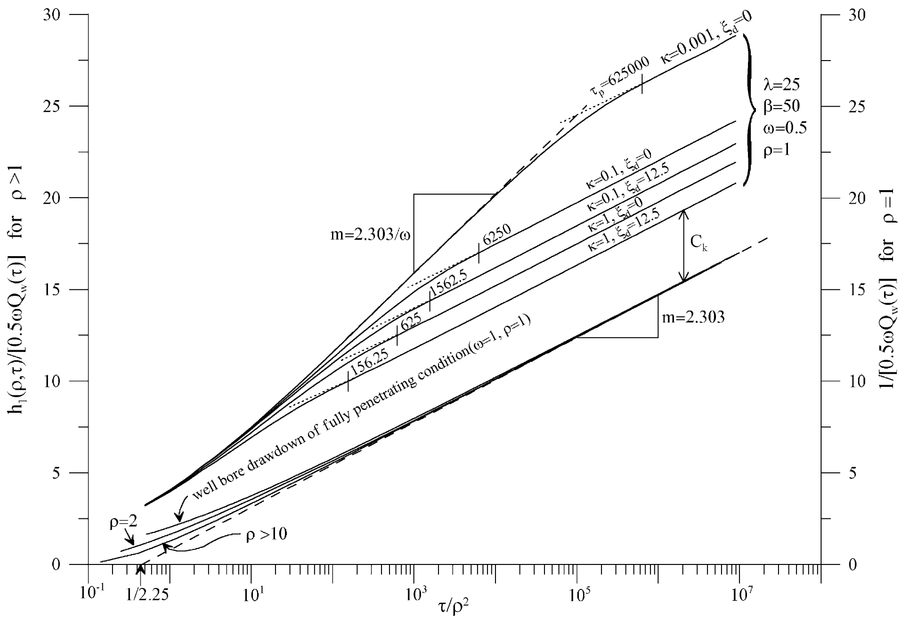

Fig. 7. Well bore drawdown in terms of $1/[0.5\omega Q_w(\tau)]$ and vertically averaged drawdown in terms of $h_1(\rho,\tau)/[0.5wQ_w(\tau)]$ are plotted against logarithmic τ/ρ^2. At late time, all curves become straight lines parallel to the straight line of the fully penetrating condition; τ_p marks the beginning time for the late-time linear relation of well bore drawdown.

In the form of Eq. (44), C_k can be considered as the pseudo-skin factor (Streltsova, 1988), accounting for additional drawdown caused by partial penetration. Since influence of the penetration ratio, aquifer anisotropy, and the location of the well screen is implicitly incorporated into C_k, it is unknown a priori. However, if $\omega = 1$, C_k is zero and Eq. (44) becomes the pertinent solution given by Jacob and Lohman (1952, Eq. (13)).

According to Eq. (44), if all the assumptions made in the current model are satisfied in the field, field data $h_w/Q_w(t)$ versus logarithmic time at late time will be a straight line. Then, by the well-known semi-log method developed by Cooper and Jacob (1946), K_r can be estimated from the slope of the straight line, but S_s cannot be determined from the intercept at the abscissa because it is a function of the unknown C_k. Furthermore, if the aquifer is anisotropic, K_z can be determined with the method for a partially penetrating well subject to a constant pumping rate, as discussed by Streltsova (1988). This method is only valid under two specific conditions: (1) the well screen extends from the top of the aquifer (i.e. $\xi_d = 0$), and (2) the well screen is in the central portion of the aquifer such that the middle line of the well screen aligns with the middle line of the aquifer thickness (i.e. $\xi_d + \xi_l = \beta$). These two conditions can be used as the two extremities for other partial penetration conditions. As noted in Fig. 7, τ_p marks the time at which the late-time straight line begins. In spite of the constant pumping rate conditions, the relation for τ_p given by Streltsova (1988) remains valid for the current study; that is,

$$\tau_p = \beta^2/(C_p\kappa) \tag{45}$$

where C_p is 16 for the condition of $\xi_d + \xi_l = \beta$, and is 4 for the condition of $\xi_d = 0$. The dimensional form of Eq. (45) gives the following relation for K_z

$$K_z = S_s b^2/(C_p t_p) \tag{46}$$

where t_p is the dimensional time at which late-time straight line of field data $h_w/Q_w(t)$ begins. It should be noted that Eqs. (45) or (46) is independent of the partial penetration ratio ($\omega < 1$). The determination of K_z cannot be completed without the knowledge of S_s, which can be obtained from vertically averaged drawdown measured at fully penetrating observation wells as discussed below.

For ρ greater than unity, all the curves of $h_1(\rho,\tau)/[0.5\omega Q_w(\tau)]$ merge to the fully penetrating condition, regardless of aquifer anisotropy, the well screen locations, and the partial penetration ratios. This follows because vertical drawdown variation induced by these factors is averaged out in fully penetrating wells. At large τ/ρ^2, one single straight-line approximation exists for these curves, of which the dimensional form is

$$\frac{h_1(r,t)}{Q_w(t)} = \frac{2.303}{4\pi K_r b}\log\left(\frac{2.25tK_r}{S_s r^2}\right), \qquad r > r_w \tag{47}$$

which is the same as the one given by Mishra and Guyonnet (1992, Equation (20)). Using the method presented by Cooper and Jacob (1946), K_r and S_s can be determined by the slope and the intercept at the abscissa of the straight line described by Eq. (47), respectively. If the aquifer is homogeneous, K_r determined by Eqs. (47) or by (44) should be approximately the same. Now that S_s is known, K_z can be found using Eq. (46).

As discussed above, $H_2(\rho,\xi,p)$ of Eq. (25) represents the partial penetration effect. Its Laplace inversion is denoted by $h_2(\rho,\xi,\tau)$. As shown in Fig. 8, the partial penetration effect in terms of $h_2\rho\xi\tau)/Q_w(\tau)$ is plotted against logarithmic τ for the two extreme partial penetrating conditions, $\xi_d = 0$ and $\xi_d + \xi_l = \beta$. For the condition of $\xi_d = 0$, the maximum penetration effect in terms of positive drawdown occurs at the top of the aquifer, decreases as ξ increases, and reaches a minimum in terms of negative drawdown (pressure buildup) at the bottom of the aquifer. For the condition of $\xi_d + \xi_l = \beta$, the vertical variation of the partial penetration effect is symmetric with respect to the middle line of aquifer thickness at $\xi = 250$. At this middle point, the partial penetration effect in terms of positive drawdown is maximum, smaller than the maximum value of the other condition. Then, the partial penetration effect decreases towards the top and bottom of the aquifer; at both places the effect is minimum and negative. In both conditions, the partial penetration effect is positive within the screen length interval of $\xi_d \le \xi \le \xi_l$, and changes from positive (drawdown) to negative (pressure

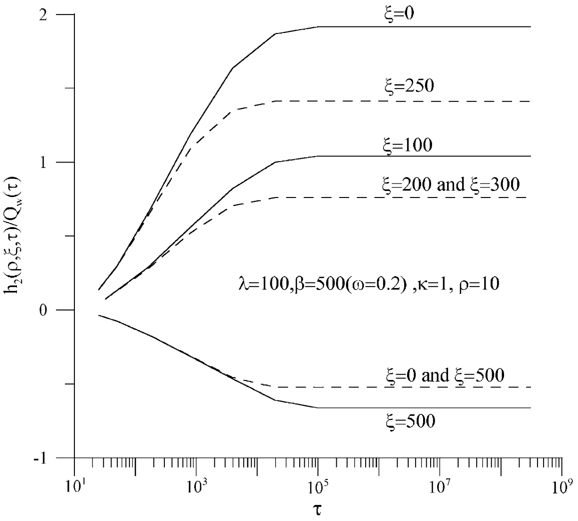

Fig. 8. Partial penetration effect in terms of $h_2(\rho, \xi, \tau)/Q_w(\tau)$ under two partial penetrating conditions: the solid curves represent the partial penetration condition of $\xi_d = 0$, and the dashed curves represent the partial penetration condition of $\xi_d + \xi_l = \beta$. Under both conditions, partial penetration effect at different depths becomes independent of time after a while.

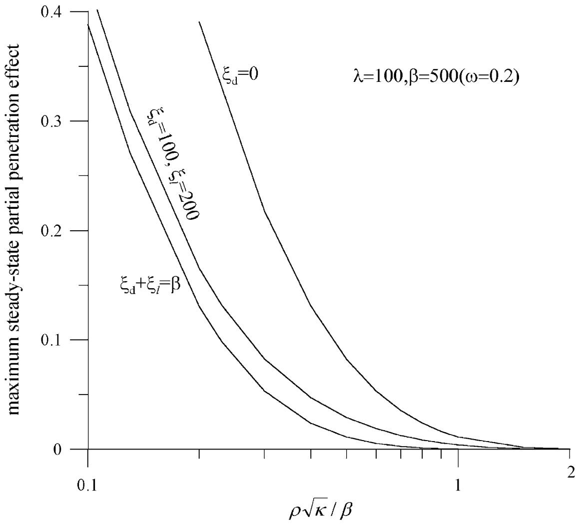

Fig. 9. The steady-state maximum partial penetration effect diminishes as ρ increases, showing that the radius of partial penetration influence ranges from 1 to 2 times $\beta/\sqrt{\kappa}$, depending on the well screen location.

buildup) somewhere in the intervals of $\xi_l \le \xi \le \beta$ and $0 \le \xi \le \xi_d$.

The partial penetration effect of different depths becoming time-independent after a while is also noted in Fig. 8. However, this does not imply the occurrence of steady-state groundwater flow because aquifer thickness is finite. Referring to Eq. (25), $h_2(\rho,\xi,\tau)$ decreases as ρ increases, and thus the partial penetration effect becomes negligible at a certain distance, called the radius of partial-penetration influence, R_p. Beyond R_p, groundwater flow assumes a radial character because of the lack of vertical drawdown variation. Since the penetration effect varies with depth and time, it can be ignored if the associated steady-state maximum value is negligible. As such, R_p can be quantified as shown in Fig. 9, where the steady-state maximum partial penetration effect of different distances are presented for $\xi_d = 0$ and $\xi_d + \xi_l = \beta$, between which lies another penetration effect of $\xi_d = 100$ and $\xi_l = 200$. It is seen that the partial penetration effect becomes negligible at a distance of $2\rho\sqrt{\kappa}/\beta$ for the condition of $\xi_d = 0$, and at a distance of $1\rho\sqrt{\kappa}/\beta$ for the condition of $\xi_d + \xi_l = \beta$. Therefore, the radius of partial-penetration influence ranges from 1 to 2 $\rho\sqrt{\kappa}/\beta$, dependent on the well screen location. This relation is not significantly different from that of the constant pumping rate condition, which is in the range of 0.5–3.18 $\rho\sqrt{\kappa}/\beta$ (Streltsova, 1988).

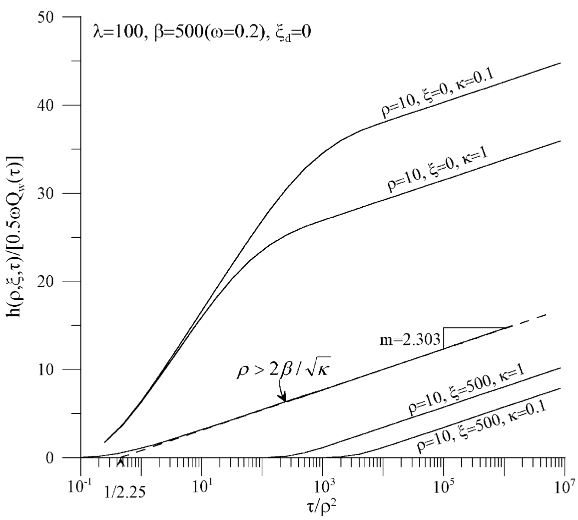

Fig. 10. Depth-specific drawdown in terms of $h(\rho, \xi, \tau)/[0.5\omega Q_w(\tau)]$ against logarithmic τ/ρ^2.

The depth-specific drawdown in terms of $h(\rho,\xi,\tau)/[0.5\omega Q_w(\tau)]$ is plotted against logarithmic τ/ρ^2 in Fig. 10 for the partial penetrating condition of $\xi_d = 0$. As $\rho = 10$, all the curves at late time become parallel straight lines of a constant slope equal to 2.303. They are separated by vertical displacement representative of the partial penetration effect. As ρ is greater than $2\beta/\sqrt{\kappa}$, the partial penetration effect is negligible. Therefore, beyond this distance all the curves merge to one single curve with a late-time approximation of a straight line described by Eq. (47).

4. Conclusion

The analytical solution approach allows for the use of Laplace and finite Fourier transforms to determine the semi-analytical solution of the model involving a flowing partially penetrating well for finite aquifer thickness. Being divided into a finite number of segments, the screen length should be non-uniformly discretized in about 20 segments in calculation to ensure that well bore flux and well bore drawdown conditions are satisfied. The transient and steady-state solutions for aquifer thickness being infinite can be obtained from this semi-analytical solution by asymptotic analysis.

If the distance from the bottom of the well screen to the bottom of the aquifer, or if the distance from the top of the well screen to the top of the aquifer, is 100 times greater than the screen length, then aquifer thickness can be assumed semi-infinite. In this case, steady-state conditions are obtained where the well discharge approaches a constant value, indicating the partially penetrating well will continuously produce water. The steady-state discharge is completely supplied by the portion of the aquifer that has a thickness 100 times greater than the screen length; the other portion merely serves as the flow path for transporting distant groundwater to the well. If aquifer thickness is finite, groundwater flow remains transient and the well discharge decreases with increasing time. Whether aquifer thickness should be assumed finite or semi-infinite warrants careful consideration.

At early time, well bore flux is uniformly distributed along the screen length as if the partially penetrating well were fully penetrating the aquifer having a finite thickness equal to the screen length. Well bore flux decreases and becomes non-uniformly distributed along the well screen as time increases. The aquifer anisotropy has little influence on well bore flux at early time, and tends to reduce it by impeding vertical groundwater flow at late time. The partial penetration effect vanishes at distance large than 1–2 times $\beta/\sqrt{\kappa}$. The horizontal hydraulic conductivity and the specific storage coefficient can be determined from vertically averaged drawdown as measured by fully penetrating observation wells. The vertical hydraulic conductivity can be determined from the well discharge under two partial penetration conditions.

Acknowledgements

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Appendix A

A.1. Satisfaction of boundary conditions of Eqs. (13) and (15)

From Eq. (23), the hydraulic gradient across the interface between the well bore and the surrounding aquifer is

$$\left.\frac{\partial H}{\partial \xi}\right|_{\rho=1} = -\sum_{i=1}^{M} q_i(p)F_i(\xi) \qquad 0 \le \xi \le \beta \tag{A1}$$

where

$$F_i(\xi) = \frac{\Delta\xi_i}{\beta} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{sin\left[\frac{n\pi(\xi_i-\xi)}{\beta}\right] + sin\left[\frac{n\pi(\xi_i+\xi)}{\beta}\right] - sin\left[\frac{n\pi(\xi_{i-1}-\xi)}{\beta}\right] - sin\left[\frac{n\pi(\xi_{i-1}+\xi)}{\beta}\right]\right\} \tag{A2}$$

The following relation is noted (Gradshteyn and Ryzhik, 1980; equation 1.442-1)

$$\sum_{n=1}^{\infty} \frac{sin(n\chi)}{n} = \frac{\pi - \chi}{2} \qquad 0 < \chi < 2\pi \tag{A3}$$

Application of Eq. (A3) to Eq. (A2) gives

$$F_i(\xi) = \begin{cases} 1 & \xi_{i-1} \le \xi \le \xi_i \\ 0 & \text{otherwise} \end{cases} \tag{A4}$$

Finally, substitution of Eq. (A4) to Eq. (A1) gives $\partial H/\partial\xi = 0$ for $0 \le \xi \le \xi_d$ and $\xi_l \le \xi \le \beta$, and $\partial H/\partial\xi = -q_i(p)$ for $\xi_{i-1} \le \xi \le \xi_i$, they are Eq. (13) and Eq. (15) in the Laplace domain, respectively.

A.2. Early-time approximation

When p is large as for small τ, χ_n defined by Eq. (20) approaches $\sqrt{p}$. Then a_{ij} given by Eq. (29) reduces to

$$a_{ij} = \frac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})} F_j(\bar{\xi}_i) \tag{A5}$$

where $F_j(\bar{\xi}_i)$ is given by Eq. (A2). Referring to Eq. (A4), $F_j(\bar{\xi}_i)$ is equal to unity for $i = j$ and is zero for $i \ne j$, and

$$a_{ij} = \begin{cases} \dfrac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})} & i = j \\ 0 & i \ne j \end{cases} \tag{A6}$$

In a straightforward manner, substitution of Eq. (A6) into Eq. (28) yields

$$q_i(p) = \frac{K_1(\sqrt{p})}{\sqrt{p}K_0(\sqrt{p})} \qquad i = 1, 2 \ldots M \tag{A7}$$

which is constant for each well segment. Substitution of Eq. (A7) into Eq. (31) gives Eq. (43), proving that the well discharge of a partially penetrating well is the same as that of fully penetrating wells at early times.

A.3. Fully penetrating conditions

When $\omega = 1$, λ is equal to β, and the summation of a_{ij} for ith equation of Eq. (28) is

$$\sum_{j=1}^{M} a_{ij} = \frac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})} \qquad j = 1, 2, \ldots M \tag{A8}$$

By introduction of Eq. (A8) into Eq. (28), the M's equations for the M segments are

$$\begin{bmatrix} \dfrac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})} & a_{12} & a_{13}\cdots a_{1M} \\ \vdots & \vdots & \vdots \\ \dfrac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})} & a_{M2} & a_{M3}\cdots a_{MM} \end{bmatrix} \begin{bmatrix} q_1(p) \\ \vdots \\ q_M(p) - q_1(p) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{p} \\ \vdots \\ \dfrac{1}{p} \end{bmatrix}. \tag{A9}$$

By Gauss elimination, Eq. (A9) can be easily solved, and the result is

$$q_i(p) = \frac{K_1(\sqrt{p})}{\sqrt{p}K_0(\sqrt{p})}, \qquad i = 1, 2, \ldots, M \tag{A10}$$

which is independent of well screen depth for $\omega = 1$. Replacing Eq. (A10) in Eq. (31) gives $Q_\omega(p)$ as defined by Eq. (43), formally proving that the well discharge for $\omega = 1$ is the same as that for the fully penetrating condition. Since $q_i(p)$ is constant in Eq. (25), the summation of $\sin(n\pi\xi_i/\beta) - \sin(n\pi\xi_{i-1}/\beta)$ for $i = 1, 2 \ldots M$ is zero. Thus H_2 is equal to zero for $\omega = 1$, and H_1 is solely dependent on H_1 defined by Eq. (24). When Eq. (A10) is used in Eq. (24), H_1 becomes identical to the Laplace-domain solution for a fully penetrating well as given by Chen and Stone (1993; Equation 6).

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