Zadanie 1

Podpunkt a.

In[ ]:= `Limit[x^2 - x, x → 1]`

Out[ ]=

0

Podpunkt b.

In[ ]:= `Limit[Sin[x]/x, x → 0]`

Out[ ]=

1

Podpunkt c.

In[ ]:= `Limit[(1 + x/n)^n, n → ∞]`

Out[ ]=

e^x

Podpunkt d.

In[ ]:= `Limit[x/(x - 1), x → 1, Direction → "FromAbove"]`

`Limit[x/(x - 1), x → 1, Direction → "FromBelow"]`

Out[ ]=

∞

Out[ ]=

$-\infty$

Podpunkt e.

In[ ]:= `Limit[ArcTan[x], x → ∞]`

Out[ ]=

$\frac{\pi}{2}$

Podpunkt f.

In[ ]:= `Limit[Sin[x], x → ∞]`

Out[ ]=

Indeterminate

Zadanie 2

Podpunkt a.

```
f[x_] := x Sin[x] + Cos[x];

f'[x]
f''[x]
f'''[x]
```

Out[◦]=

x Cos[x]

Out[◦]=

Cos[x] - x Sin[x]

Out[◦]=

-x Cos[x] - 2 Sin[x]

Podpunkt b.

In[◦]:= $f[x\_] := \frac{x}{x + 2};$

```
f'[x] // Simplify
f''[x] // Simplify
f'''[x] // Simplify
```

Out[◦]=

$$\frac{2}{(2 + x)^2}$$

Out[◦]=

$$-\frac{4}{(2 + x)^3}$$

Out[◦]=

$$\frac{12}{(2 + x)^4}$$

Podpunkt c.

In[ ]:=
```
f[x_] := (x^2 - x)/(x + 2);

f'[x] // Simplify
f''[x] // Simplify
f'''[x] // Simplify
```

Out[ ]=

$$\frac{-2 + 4 x + x^2}{(2 + x)^2}$$

Out[ ]=

$$\frac{12}{(2 + x)^3}$$

Out[ ]=

$$-\frac{36}{(2 + x)^4}$$

Zadanie 3

Podpunkt a.

In[∘]:=
```
f[x_, y_] := (x^2 - y - x)/(x^2 + y^2);

Print["Pochodne cząstkowe 1. rzędu:"]
Print["fₓ = ", Simplify[∂ₓ f[x, y]]]
Print["f_y = ", Simplify[∂_y f[x, y]]]

Print["\nPochodne cząstkowe 2. rzędu:"]
Print["f_xx = ", Simplify[∂_xx f[x, y]]]
Print["f_xy = f_yx = ", Simplify[∂_xy f[x, y]]]
Print["f_y = ", Simplify[∂_yy f[x, y]]]

Print["\nPochodne cząstkowe 3. rzędu:"]
Print["f_xxx = ", Simplify[∂_xxx f[x, y]]]
Print["f_xxy = f_xyx = f_yxx = ", Simplify[∂_xxy f[x, y]]]
Print["f_yyx = f_yxy = f_xyy = ", Simplify[∂_yyx f[x, y]]]
Print["f_yyy = ", Simplify[∂_yyy f[x, y]]]

Print["\nGradient:"]
Print["∇f = ", MatrixForm[Simplify[∇_{x,y} f[x, y]]]]

Print["\nHesjan:"]
Print["H = ",
 MatrixForm[
  {{Simplify[∂_xx f[x, y]], Simplify[∂_xy f[x, y]]},
   {Simplify[∂_yx f[x, y]], Simplify[∂_yy f[x, y]]}}
 ]
]
```

Pochodne cząstkowe 1. rzędu:

$$f_x = \frac{x^2 - y^2 + 2\,x\,y\,(1 + y)}{\left(x^2 + y^2\right)^2}$$

$$f_y = \frac{2\,x\,y + y^2 - x^2\,(1 + 2\,y)}{\left(x^2 + y^2\right)^2}$$

Pochodne cząstkowe 2. rzędu:

$$f_{xx} = \frac{-2\,x^3 + 6\,x\,y^2 - 6\,x^2\,y\,(1 + y) + 2\,y^3\,(1 + y)}{\left(x^2 + y^2\right)^3}$$

$$f_{xy} = f_{yx} = \frac{2\left(-3\,x^2\,y + y^3 + x^3\,(1 + 2\,y) - x\,y^2\,(3 + 2\,y)\right)}{\left(x^2 + y^2\right)^3}$$

$$f_y = -\frac{2\left(-x^3 + x^4 + 3\,x\,y^2 + y^3 - 3\,x^2\,y\,(1 + y)\right)}{\left(x^2 + y^2\right)^3}$$

Pochodne cząstkowe 3. rzędu:

$$f_{xxx} = \frac{6\left(x^4 - 6\,x^2\,y^2 + y^4 + 4\,x^3\,y\,(1 + y) - 4\,x\,y^3\,(1 + y)\right)}{\left(x^2 + y^2\right)^4}$$

$$f_{xxy} = f_{xyx} = f_{yxx} = -\frac{2\left(-12\,x^3\,y + 12\,x\,y^3 + y^4\,(3 + 2\,y) + x^4\,(3 + 6\,y) - 2\,x^2\,y^2\,(9 + 8\,y)\right)}{\left(x^2 + y^2\right)^4}$$

$$f_{yyx} = f_{yxy} = f_{xyy} = \frac{-6\,x^4 + 4\,x^5 + 36\,x^2\,y^2 - 6\,y^4 + 12\,x\,y^3\,(2 + y) - 8\,x^3\,y\,(3 + 4\,y)}{\left(x^2 + y^2\right)^4}$$

$$f_{yyy} = \frac{6\left(-4\,x^3\,y + 4\,x\,y^3 + y^4 - 2\,x^2\,y^2\,(3 + 2\,y) + x^4\,(1 + 4\,y)\right)}{\left(x^2 + y^2\right)^4}$$

Gradient:

$$\nabla f = \begin{pmatrix} \frac{x^2-y^2+2\,x\,y\,(1+y)}{\left(x^2+y^2\right)^2} \\ \frac{2\,x\,y+y^2-x^2\,(1+2\,y)}{\left(x^2+y^2\right)^2} \end{pmatrix}$$

Hesjan:

$$H = \begin{pmatrix} \frac{-2\,x^3+6\,x\,y^2-6\,x^2\,y\,(1+y)+2\,y^3\,(1+y)}{\left(x^2+y^2\right)^3} & \frac{2\left(-3\,x^2\,y+y^3+x^3\,(1+2\,y)-x\,y^2\,(3+2\,y)\right)}{\left(x^2+y^2\right)^3} \\ \frac{2\left(-3\,x^2\,y+y^3+x^3\,(1+2\,y)-x\,y^2\,(3+2\,y)\right)}{\left(x^2+y^2\right)^3} & -\frac{2\left(-x^3+x^4+3\,x\,y^2+y^3-3\,x^2\,y\,(1+y)\right)}{\left(x^2+y^2\right)^3} \end{pmatrix}$$

Podpunkt b.

```
In[ ]:= f[x_, y_] := Sin[x + y];

Print["Pochodne cząstkowe 1. rzędu:"]
Print["fx = fy = ", Simplify[∂x f[x, y]]]

Print["\nPochodne cząstkowe 2. rzędu:"]
Print["fxx = fxy = fyx = fy = ", Simplify[∂xx f[x, y]]]

Print["\nPochodne cząstkowe 3. rzędu:"]
Print["fxxx = fxxy = fxyx = fyxx = fyyx = fyxy = fxyy = fyyy = ", Simplify[∂xxx f[x, y]]]

Print["\nGradient:"]
Print["∇f = ", MatrixForm[Simplify[∇{x,y} f[x, y]]]]

Print["\nHesjan:"]
Print["H = ",
 MatrixForm[
  {{Simplify[∂xx f[x, y]], Simplify[∂xy f[x, y]]},
   {Simplify[∂yx f[x, y]], Simplify[∂yy f[x, y]]}}
 ]
]
```

Pochodne cząstkowe 1. rzędu:

$f_x = f_y = \mathrm{Cos}[x + y]$

Pochodne cząstkowe 2. rzędu:

$f_{xx} = f_{xy} = f_{yx} = f_y = -\mathrm{Sin}[x + y]$

Pochodne cząstkowe 3. rzędu:

$f_{xxx} = f_{xxy} = f_{xyx} = f_{yxx} = f_{yyx} = f_{yxy} = f_{xyy} = f_{yyy} = -\mathrm{Cos}[x + y]$

Gradient:

$$\nabla f = \begin{pmatrix} \mathrm{Cos}[x + y] \\ \mathrm{Cos}[x + y] \end{pmatrix}$$

Hesjan:

$$H = \begin{pmatrix} -\mathrm{Sin}[x + y] & -\mathrm{Sin}[x + y] \\ -\mathrm{Sin}[x + y] & -\mathrm{Sin}[x + y] \end{pmatrix}$$

Zadanie 4

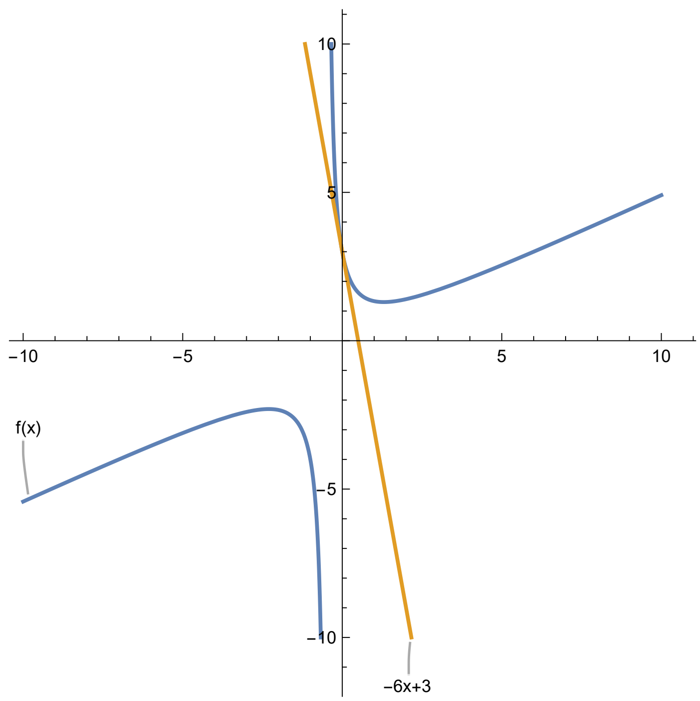
Podpunkt a.

In[∘]:=
```
f[x_] := (x^2 + 3)/(2 x + 1);

x0 = 0;

y = f'[x0] (x - x0) + f[x0];

Print["Styczna do wykresu funkcji f(x) w punkcie (0,3):"]
Print["y = ", y]

Plot[
 {Callout[f[x], "f(x)", Left],
  Callout[y, "-6x+3", Below]},
 {x, -10, 10},
 PlotRange → {-10, 10},
 AspectRatio → 1
]
```

Styczna do wykresu funkcji f(x) w punkcie (0,3):

y = 3 - 6 x

Out[∘]=



Podpunkt b.

```
In[●]:= Print[
  "Mianownik zeruje się w punkcie\n",
  Solve[Denominator[f[x]] == 0, x]
 ]

Print["\nSprawdzamy zatem granice funkcji f(x) w tym punkcie\n"]
Print["lim_{x→(-1/2)^+} f(x)  = ", Limit[f[x], x → -1/2, Direction → "FromAbove"]]
Print["\nlim_{x→(-1/2)^-} f(x)  = ", Limit[f[x], x → -1/2, Direction → "FromBelow"]]
Print["Więc istnieje asymptota pionowa w x = -1/2"]

Print["\nSprawdzamy granice funkcji w ±∞"]
Print["lim_{x→+∞} f(x)  = ", Limit[f[x], x → ∞]]
Print["lim_{x→-∞} f(x)  = ", Limit[f[x], x → -∞]]
Print["Granice są nieskończone, zatem nie istnieją asymptoty poziome"]

Print["\nSprawdźmy zatem istnienie asymptot ukośnych"]
a = Limit[f[x]/x, x → ∞];
b = Limit[f[x] - a x, x → ∞];
Print["a = lim_{x→±∞} f (x)/x  = ", a]
Print["b = lim_{x→±∞} (f(x)-ax)  = ", b]
Print["\nIstnieje więc jedna asymptota ukośna określona wzorem"]
Print["y = ", a x + b]

Plot[
 {Callout[f[x], "f(x)"],
  Callout[a x + b, "y = 1/2 x - 1/4", Left]},
 {x, -10, 10},
 PlotRange → {-10, 10},
 AspectRatio → 1,
 PlotStyle → {{}, {Dashed, Orange}},
 ExclusionsStyle → Directive[Thick, Dashed, Orange]
]
```

Mianownik zeruje się w punkcie

$\left\{\left\{x \to -\frac{1}{2}\right\}\right\}$

Sprawdzamy zatem granice funkcji f(x) w tym punkcie

$\lim_{x\to\left(-\frac{1}{2}\right)^+} f(x) = \infty$

$\lim_{x\to\left(-\frac{1}{2}\right)^-} f(x) = -\infty$

Więc istnieje asymptota pionowa w $x = -\frac{1}{2}$

Sprawdzamy granice funkcji w $\pm\infty$

$\lim_{x\to+\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

Granice są nieskończone, zatem nie istnieją asymptoty poziome

Sprawdźmy zatem istnienie asymptot ukośnych

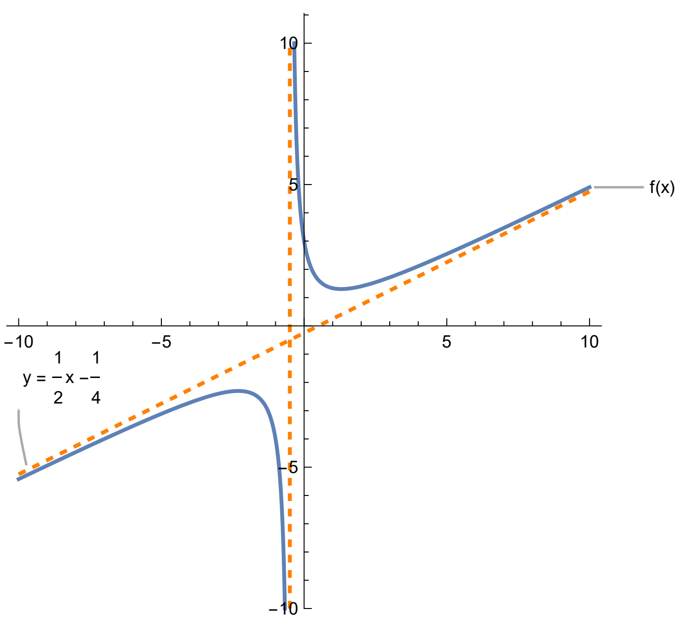
$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \frac{1}{2}$

$b = \lim_{x\to\pm\infty} (f(x) - ax) = -\frac{1}{4}$

Istnieje więc jedna asymptota ukośna określona wzorem

$y = -\frac{1}{4} + \frac{x}{2}$

Out[ ]=



Podpunkt c.

In[∘]:=
```
f''[x] /. Solve[f'[x] == 0] // Simplify

{x, f[x]} /. Solve[f'[x] == 0] // Simplify;
Grid[Prepend[%, {"x", "f(x)"}], Frame → All]
```

Out[∘]=

$$\left\{-\frac{2}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right\}$$

Out[∘]=

x	f(x)
$\frac{1}{2}\left(-1-\sqrt{13}\right)$	$\frac{1}{2}\left(-1-\sqrt{13}\right)$
$\frac{1}{2}\left(-1+\sqrt{13}\right)$	$\frac{1}{2}\left(-1+\sqrt{13}\right)$

Zadanie 5

Podpunkt a.

In[∘]:=

$$\int \left(x^4 - x^2 + 3x + 1\right)^4 dx$$

Out[∘]=

$$x + 6x^2 + \frac{50x^3}{3} + 18x^4 - \frac{17x^5}{5} - 6x^6 + \frac{146x^7}{7} + 3x^8 - \frac{89x^9}{9} + \frac{36x^{10}}{5} + \frac{38x^{11}}{11} - 3x^{12} + \frac{10x^{13}}{13} + \frac{6x^{14}}{7} - \frac{4x^{15}}{15} + \frac{x^{17}}{17}$$

Podpunkt b.

In[∘]:=

$$f[x\_] := \begin{cases} x & x < 1 \\ 2 & x == 1; \\ x^2 - 1 & x > 1 \end{cases}$$

$$\int f[x]\, dx$$

Out[∘]=

$$\begin{cases} \frac{x^2}{2} & x \leq 1 \\ \frac{7}{6} - x + \frac{x^3}{3} & \text{True} \end{cases}$$

Podpunkt c.

In[∘]:=

$$\int \frac{\text{Sin}[x]}{x} dx$$

Out[∘]=

SinIntegral[x]

Podpunkt d.

In[∘]:= $\int \sqrt{\sqrt{r}\,(r-3)}\, dr$

Out[∘]=

$\frac{4}{45} r^{5/4} (-27 + 5\, r)$

Podpunkt e.

In[∘]:= $\int \text{Log}[\text{Log}[x]]\, dx$

Out[∘]=

x Log[Log[x]] - LogIntegral[x]

Podpunkt f.

In[∘]:= $\int \frac{\text{Sin}[x]}{\text{Log}[x]}\, dx$

Out[∘]=

$\int \frac{\text{Sin}[x]}{\text{Log}[x]}\, dx$

Zadanie 6

Podpunkt a.

In[∘]:=
```
ℛ = ImplicitRegion[(0 < x < 3) && (1 < y < x - 1), {x, y}];
```

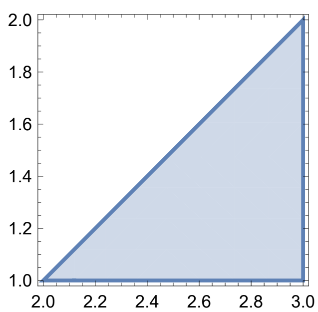
$\int_{\{x,y\}\in\mathcal{R}} (x+y)$

```
RegionPlot[ℛ]
```

Out[∘]=

2

Out[∘]=



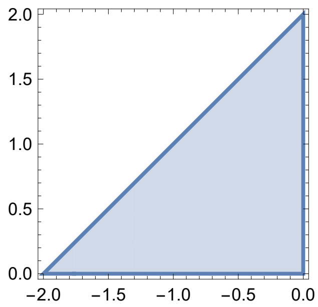
Podpunkt b.

In[∘]:=
```
ℛ = ImplicitRegion[(-2 < x < 0) && (0 < y < x + 2), {x, y}];
```
$$\int_{\{x,y\}\in\mathcal{R}} (x + y)$$
```
RegionPlot[ℛ]
```

Out[∘]=

0

Out[∘]=



Podpunkt c.

In[∘]:=
```
ℛ = ImplicitRegion[(0 < x < 1) && (c < y < d), {x, y}];
```
$$\int_{\{x,y\}\in\mathcal{R}} (x^2 + 3\,y)$$

Out[∘]=

$$\begin{cases} \frac{1}{6}\left(-2\,c - 9\,c^2 + 2\,d + 9\,d^2\right) & d > c \\ 0 & \text{True} \end{cases}$$

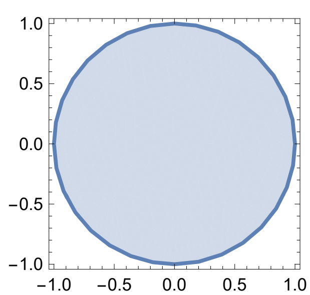
Podpunkt d.

In[∘]:=
```
ℛ = ImplicitRegion[x² + y² ≤ 1, {x, y}];
```
$$\int_{\{x,y\}\in\mathcal{R}} (x^2 + y^2)$$
```
RegionPlot[ℛ]
```

Out[∘]=

$$\frac{\pi}{2}$$

Out[∘]=



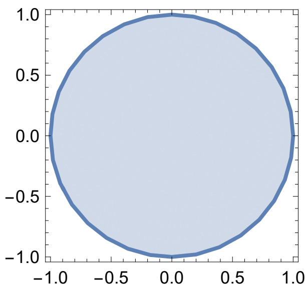
Podpunkt e.

In[ ]:=
```
ℛ = ImplicitRegion[x^2 + y^2 ≤ 1, {x, y}];
∫_{{x,y}∈ℛ} ((x - 1)^2 + y^2)
RegionPlot[ℛ]
```

Out[ ]=

$$\frac{3\pi}{2}$$

Out[ ]=



Zadanie 7

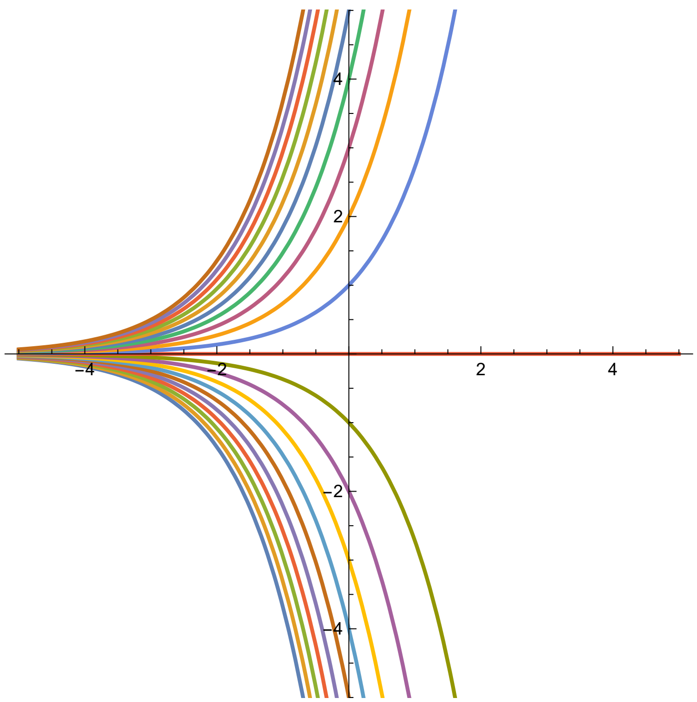
Podpunkt a.

In[ ]:=
```
DSolve[y'[x] == y[x]]

solution = DSolve[y'[x] == y[x] && y[0] == a];
Plot[
 Evaluate[y[x] /. solution /. {a → Range[-10, 10]}],
 {x, -5, 5},
 PlotRange → {-5, 5},
 AspectRatio → 1
]
```

Out[ ]=

$\{\{y[x] \to e^{x}\, c_1\}\}$

Out[ ]=



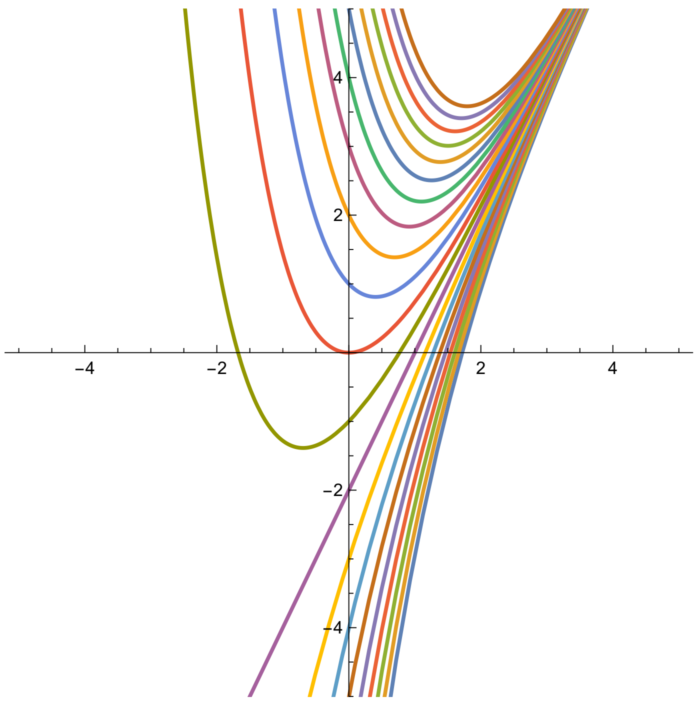
Podpunkt b.

In[∘]:=
```
DSolve[y'[x] + y[x] == 2 x]

solution = DSolve[y'[x] + y[x] == 2 x && y[0] == a];
Plot[
 Evaluate[y[x] /. solution /. {a → Range[-10, 10]}],
 {x, -5, 5},
 PlotRange → {-5, 5},
 AspectRatio → 1
]
```

Out[∘]=

$\{\{y[x] \to 2\,(-1 + x) + e^{-x}\,\mathbb{c}_1\}\}$

Out[∘]=



Zadanie 8

In[•]:=
```
solution = DSolve[x'[t] == x[t] && x[0] == 1]

Plot[
 Evaluate[x[t] /. solution], {t, -5, 5},
 PlotRange → {-5, 5},
 AspectRatio → 1
]
```

Out[•]=

$\{\{x[t] \to e^t\}\}$

Out[•]=