Maximum Flows and Parametric Shortest Paths in Planar Graphs [3]

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Maximum flow problem

Given a

- Directed graph (V, E) (simple)
- Edge capacities (weights): $c : E \to \mathbb{R}^+$ (maximum available edge flow)

and $s, t \in V$
Find a feasible flow $f : E \to \mathbb{R}^+$

- $f(e) \leq c(e)$ Satisfies capacity constraints
- Conserves flow:

$$\forall v \notin \{s, t\}. \sum_{u \to v} f(u \to v) = \sum_{v \to w} f(v \to w)$$

Find one with

$$max(\sum_{s \to v} f(s \to v))$$

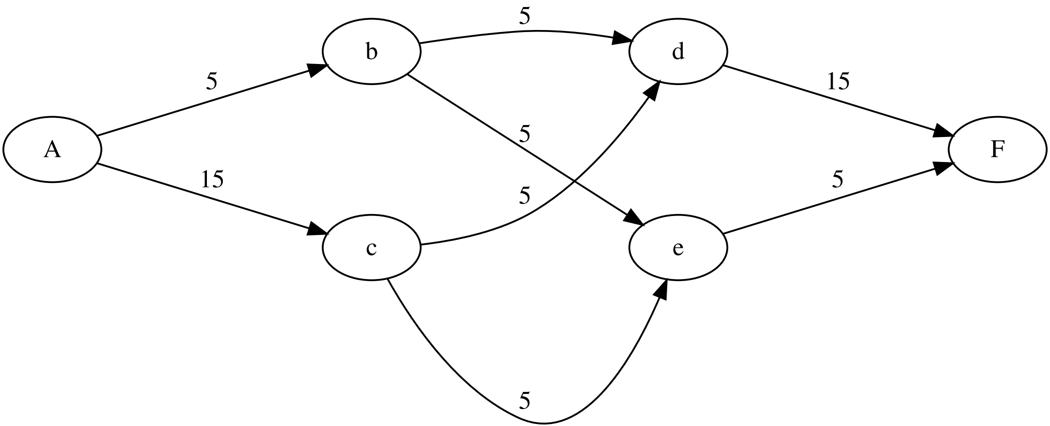


Figure: Input graph

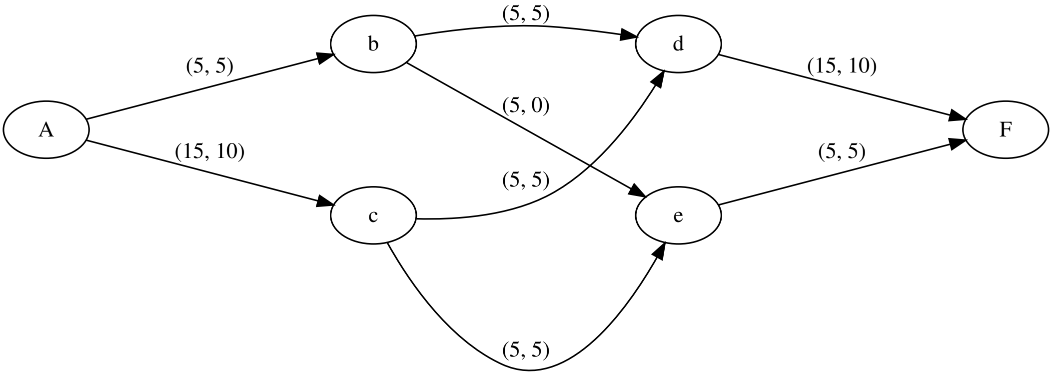


Figure: Possible solution: 15

Applications

Direct practical applications

- Max flow (routing problems)
- Huge impact in data centers
- Various aspects of resource management and scheduling ("baseball elimination" problem, project resource allocation)

First studied by Soviets before WW2, used by USA military planers during the cold war to asses military capabilities [6].
The solution for East European railway (10 stations, ∼70 connections) was produced by hand.

(General) Solutions

- Classical Ford-Fulkerson algorithm [4] ($O(Ef)$) and Edomnds-Karp ($O(VE)$), Dinic ($O(VE \log E)$ using dynamic trees).
- Max-flow min-cut theorem (Max-flow and min-cut are a reformulation as a primal-dual linear programs)

Most direct applications of maximum flow problems can be modeled as planar graphs, so the question is, whether better solutions exists in such cases.

Planar graphs

- Computing flow with a fixed value is equivalent to computing a SSSP in a particular dual Graph G^* [5].
- In 2009 An $O(n \log n)$ algorithm was described for arbitrary directed planar graphs [1] (using clockwise/counterclockwise cycling to saturate network).
- Problem is intricately connected to the shortest paths in dual graph -> most of the solutions rely on fast SSSP algorithms
- SSSP can be computed in linear time (with no negative weights), when s and t are on the same face, the problem is linear.

A simpler derivation of $O(n \log n)$ algorithm [1]

- Reformulate max-flow as a parametric SSSP in dual graph
- Solve SSSP in the dual
- Devise an upper bound of number of changes in the dual
- Construct a higher genus graph, that achieves upper bound

Dual graph

WLOG: Assume that all edges have a reverse pair

$$(u \to v)^* := u^* \uparrow v^* \qquad\qquad u^* \uparrow v^* := (u \to v)^*$$

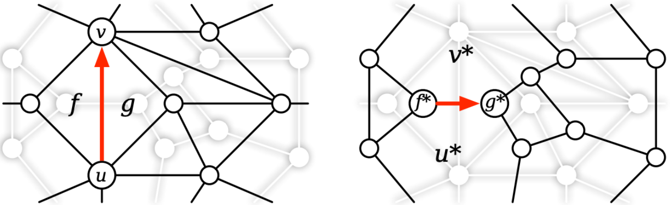


Figure: Dual graph [3]

Flow

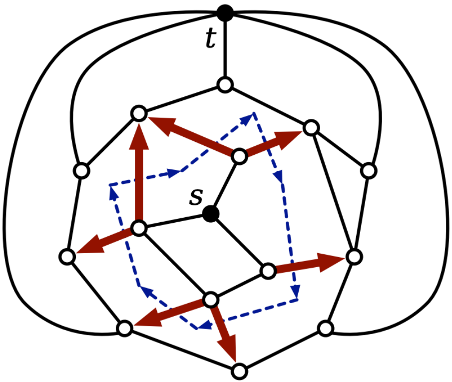
$$\phi : E \to \mathbb{R} \qquad \phi(e) = 0, e \notin E$$

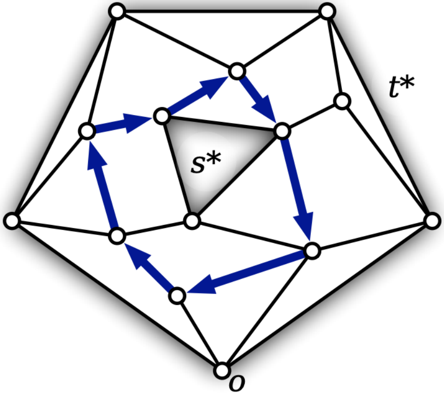
$$\phi(e) = -\phi(rev(e)) \qquad \sum_{w} \phi(v \to w) = 0$$

Feasible if $\phi(e) \leq c(e)$. Flow saturates an edge if $c(e) = \phi(e)$. For a cut $C \subseteq E$: $\phi(C) = \sum_{e \in C} \phi(e)$.
By the max-flow min cut, if C is saturated, then $\phi(C)$ is max flow.

Reformulation

Cut is called a *Cocycle* if the dual produces a simple cycle





Crossing number

For a arbitrary (fixed) path P in G.

$$\pi(e) := \begin{cases} 1 & e \in P \\ -1 & e \notin P \\ 0 & \text{otherwise} \end{cases}$$

For a cycle C in dual, this counts the number of left to right crosses minus the number of right to left crosses.
$\pi(C) \in \{-1, 0, 1\}$ for any cocycle C and $\pi(C) = 1 \iff C$ is a (s, t)-cut.

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$\pi(C) \in \{-1, 0, 1\}$ for any cocycle C and $\pi(C) = 1 \iff C$ is a (s, t)-cut. Apply Jordan curve, and do case analysis on side location of t, s. Cycle is polygon.

Residual flow

For $\lambda \in \mathbb{R}$ consider $\lambda\pi$.
Take the residual graph with $c(\lambda, e) = c(e) - \lambda\pi(e)$.
Flow is feasible $\iff c(\lambda, e) \leq 0$.
Consider the residual dual network $c(\lambda, e^*) := c(\lambda, e)$.
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$$c(\lambda, C^*) = \sum_{e \in C} c(\lambda, e) = \sum_{e \in C} c(e) - \lambda\pi(C) < 0$$

$\pi(C) = 1$ and C is a (s, t)-cut with a capacity smaller than λ.

No negative cycles

Root a tree of SSSP at arbitrary $o \in G^*_\lambda$

$$\phi(\lambda, e) = d(\lambda, head(e^*)) - d(\lambda, tail(e^*)) + \lambda\pi(e)$$

For an arbitrary v, the edges leaving v define directed (simple) cycle in G^* and the d cancels out -> the net value of sink in 0. This is a valid flow.
$slack(\lambda, e^*) := d(\lambda, tail(e*)) - d(\lambda, head(e^*)) + c(\lambda, e) = c(e) - \phi(\lambda, e)$, which is feasible by [4].

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Since path is arbitrary, for a fixed λ one can calculate the flow in $O(n)$.

Solution

Find the largest value λ that produces no negative cycles.
Find a λ, that induces a zero cycle. This is exactly the parametric shortest path problem.
WLOG: assume, that all the SSSP tree is always unique (standard perturbations, lexicographical...)

For $\lambda \in [0, \lambda_{max}]$ a SSSP tree is well defined (and unique) except for some critical values.

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Of special interest are *tense* edges $slack(\lambda, e^*) = 0$, they always lie on the tree except on critical values of λ (pivots), where they get replaced by some other.
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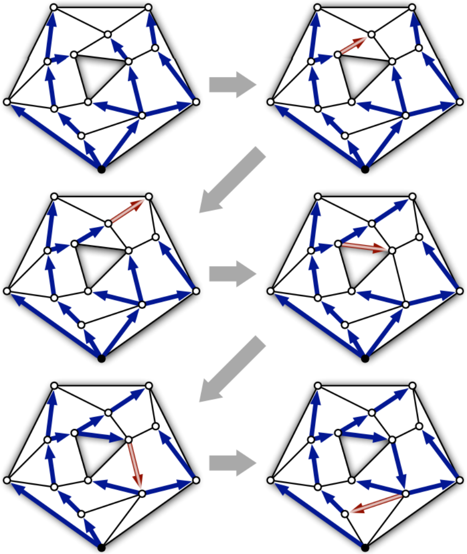

Complexity

Initialize the spanning tree of loose edges L_λ, SSSP predecesors for dual vertices and slack values for dual edges. This can be done in $O(n \log n)$
Crucial part is use of self adjusting top tree to efficiently update spanning tree in $O(\log n)$ time.

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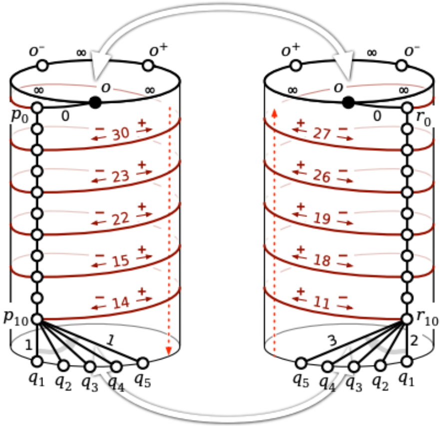
Consider an infinite strip graph with two pivot paths with indices $i < j$ Concatenate $o_{-i} - - > p_0$ and $o_j - - > p_0$ and loop. q_0 is both in and out of the loop.

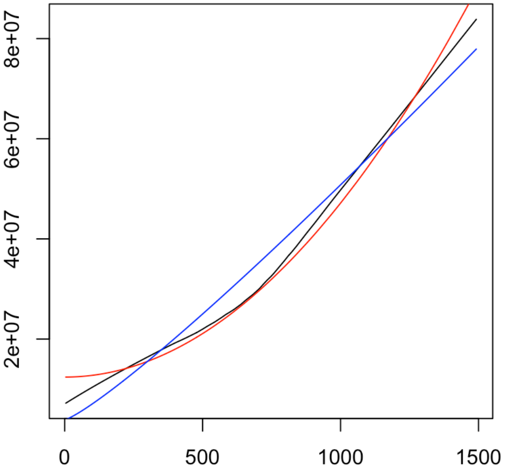
Higher genus

General algorithm given by [2] in $O(g^7 n \log^2 n \log^2 C)$.
In surfaces with higher genus, the decomposition of dual in L_λ is no longer (just) a tree, but might have additional edges.
Self adjusted trees can be modified to preserve the same time complexity.
The time complexity is still $O(N \log n)$, but N is $O(n^2)$.

Dual n^2 case



o^-
∞
o^+
o
p_0
0
30
23
22
15
14
p_{10}
1
q_1 q_2 q_3 q_4 q_5
27
26
19
18
11
r_0
r_{10}
3
2
q_5 q_4 q_3 q_2 q_1



2e+07
4e+07
6e+07
8e+07
0
500
1000
1500

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