Information Theoretic Optimality with Greedy Sensor Scheduling

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***Abstract*—This paper addresses the problem of optimal sensor scheduling in a finite-horizon setting. We formulate the task as an optimal control problem where the objective is to minimize a combination of cumulative sensor usage cost and the terminal entropy of the state belief. We analyze the properties of the belief update dynamics, demonstrating commutativity of observations and monotonicity of expected posterior entropy with respect to sensor noise. Furthermore, we establish that the expected entropy exhibits diminishing returns with respect to sensor precision. Based on these theoretical insights, we propose a greedy algorithm that efficiently determines the sensor selection policy. Experimental results validate the proposed approach, showing intuitive and effective scheduling policies under various cost and noise configurations.**

***Index Terms*—sensor scheduling, optimal control, information theory, Bayesian inference, entropy, greedy algorithms.**

I. Nomenclature

For the equations below, let X, Y be random variables (r.v.s) with pdf/pmf $p(x), p(y)$ and supports $\mathcal{X} := \text{supp } X, \mathcal{Y} := \text{supp } Y$.

Notation	Description
$\mathcal{X}$	State space
$\mathcal{U}$	Control space
$q : \mathcal{X} \to [0, 1]$	pmf (belief) of the state estimate
$p(\cdot \mid s)$	Observation model conditioned on state s

For the equations below, let X, Y be r.v.s with pdf/pmf $p(x), p(y)$ and supports $\mathcal{X} := \text{supp } X, \mathcal{Y} := \text{supp } Y$.

- Differential entropy of r.v. X with pmf $p(.)$:

$$H(p) := -\sum_{\mathcal{X}} p(x) \log p(x)$$

- Conditional entropy:

$$H(Y \mid X) = -\sum\sum_{\mathcal{X}\times\mathcal{Y}} p(x, y) \, \log\big(p(y \mid x)\big)$$

- Mutual information:

$$I(X; Y) = \sum\sum_{\mathcal{X}\times\mathcal{Y}} p(x, y) \, \log\Big(\frac{p(x, y)}{p(x)p(y)}\Big)$$

II. Problem Formulation

Let the robot state $\mathbf{x} \in \mathcal{X} = \{0, \ldots, N-1\}$. Represent the estimate of $\mathbf{x}$ as an r.v. X. Let the initial estimate be a uniform distribution $q_0(s) = 1/N$ for all $s \in \mathcal{X}$.

We have N sensors; sensor i returns an observation $z_t \sim p_i(\cdot \mid \mathbf{x})$. For the toy problem, let $p_i(\cdot \mid \mathbf{x}) = \mathcal{N}(\mathbf{x}, \sigma_i^2)$, thus the tth observation can be represented as

$$Z_n := X_n + N_i$$

where X is our belief prior at the tth step and N_i is the noise from the ith sensor.

At each time t, we sample an observation and update the belief posterior via Bayes' rule:

$$f(q_t, z_t, u_t) := q_{t+1}(s) = \frac{p(z_t \mid s; u_t)\, q_t(s)}{p(z_t; u_t)},$$

which is the belief "motion model," where u_t selects a sensor. We formulate the finite-horizon optimal control problem:

$$\begin{aligned} \min_{\pi_{0:T-1}} \quad & \sum_{t=0}^{T-1} c(u_t) \; + \; \alpha \, H(q_T) \\ \text{s.t.} \quad & q_{t+1} = f(q_t, z_t, u_t), \\ & u_t = \pi_t(q_t). \end{aligned} \tag{1}$$

where the terminal cost $H(q_T)$ is the entropy of the final belief.

III. Commutativity of Observations

A. A Little Lemma

Lemma 1. *Let $\mathcal{V}$ be a vector space on a field F equipped with a linear functional $\phi : \mathcal{V} \to F$. Let L be the set of commutative linear maps on $\mathcal{V}$ such that $\phi(T\boldsymbol{v}) \neq 0$. Define the normalized composition $. \boxtimes . : L \times \mathcal{V} \to \mathcal{V}$ where $T \boxtimes \boldsymbol{v} \mapsto \dfrac{T\boldsymbol{v}}{\phi(Tv)}$. Then,*

$$T_1 \boxtimes (T_2 \boxtimes \boldsymbol{v}) = T_2 \boxtimes (T_1 \boxtimes \boldsymbol{v})$$

Proof. By definition of the operator $\boxtimes$:

$$\begin{aligned} T_1 \boxtimes (T_2 \boxtimes \mathbf{v}) &= \frac{T_1 \frac{T_2 \mathbf{v}}{\phi(T_2 \mathbf{v})}}{\phi\left(T_1 \frac{T_2 \mathbf{v}}{\phi(T_2 \mathbf{v})}\right)} = \frac{\frac{1}{\phi(T_2 \mathbf{v})} T_1 T_2 \mathbf{v}}{\frac{1}{\phi(T_2 \mathbf{v})} \phi(T_1 T_2 \mathbf{v})} \\ &= \frac{T_1 T_2 \mathbf{v}}{\phi(T_1 T_2 \mathbf{v})} \end{aligned}$$

Since the operators T_1, T_2 commute, $T_1 T_2 = T_2 T_1$, the expression is symmetric in T_1 and T_2, proving the identity. $\square$

B. Application to Bayesian Updates

The lemma shows that the posterior distribution dynamics commute as long as the selected sensor makes the same observation. This is because the Bayesian update is an instance of the normalized composition defined above, where the operator T_i is multiplication by the likelihood $p(z_i|s)$ and ϕ is the normalization constant (marginal likelihood).

Therefore, for two different processes with sensor-observation pairs $(u_1, \mathbf{z}_1)$ and $(u_2, \mathbf{z}_2)$, the final belief state is the same: $q_{t+2} = q'_{t+2}$. The policy space can thus be reduced to combinations of sensor uses, rather than permutations, which has size $\binom{T+N-1}{N-1}$.

IV. Monotonicity

Proposition 1. ***(Expected posterior entropy is monotone in sensor variance)*** *Let X be any random variable, and let a measurement from sensor i be $Z_i = X + N_i$, where $N_i \sim \mathcal{N}(0, \sigma_i^2)$ is independent of X. Let q_t be the prior for X and q_{t+1} be the posterior of X given Z_i. Then*

$$\frac{d}{d\sigma_i^2}\,\mathbb{E}\big[H(q_{t+1})\big] \;\geq\; 0,$$

with equality iff X is almost surely constant.

Proof. Let σ_i^2 be denoted by τ. First, we show that the expected posterior entropy is the conditional entropy, i.e., $\mathbb{E}[H(q_{t+1})] = H(X \mid Z_i)$.

$$\begin{aligned}
\mathbb{E}[H(q_{t+1})] &= -\mathbb{E}\Big[\sum_{\mathcal{X}} \log(q_{t+1}(s \mid z_t))q_{t+1}(s \mid z_t)ds\Big] \\
&= -\mathbb{E}\Big[\sum_{\mathcal{X}} \log(q_{t+1}(s \mid z_t))\frac{q_{t+1}(s, z_t)}{p(z_t)}ds\Big] \\
&= -\int_{\mathbb{R}}\Big[\sum_{\mathcal{X}} \log(q_{t+1}(s \mid z_t))\frac{q_{t+1}(s, z_t)}{p(z_t)}ds\Big]p(z_t)dz_t \\
&= -\int_{\mathbb{R}}\Big[\sum_{\mathcal{X}} \log(q_{t+1}(s \mid z_t))q_{t+1}(s, z_t)ds\Big]dz_t \\
&= H(X \mid Z_t)
\end{aligned}$$

By the identity $H(X \mid Z_i) = H(X) - I(X; Z_i)$ [1], we have

$$\frac{d}{d\tau}\,H(X \mid Z_i) = -\frac{d}{d\tau}I(X; Z_i).$$

We know $I(X; Z_i) = H(Z_i) - H(Z_i \mid X) = H(X + N_i) - \frac{1}{2}\log(2\pi e\tau)$. By de Bruijn's identity [2],

$$\frac{d}{d\tau}H(X + N_i) \;=\; \tfrac{1}{2}\,J(X + N_i),$$

where $J(\cdot)$ is the Fisher information w.r.t. the expected value estimator. Therefore

$$\frac{d}{d\tau}I(X; Z_i) \;=\; \tfrac{1}{2}J(X + N_i) \;-\; \frac{1}{2\tau}.$$

By Stam's inequality in [2],

$$\frac{1}{J(X + N_t)} \geq \frac{1}{J(X)} + \frac{1}{J(N_t)} = \frac{1}{J(X)} + \tau \geq \tau$$

Hence $\frac{d}{d\tau}I(X; Z_i) \leq 0$, which implies

$$\frac{d}{d\tau}\,\mathbb{E}\big[H(q_{t+1})\big] \;=\; -\frac{d}{d\tau}I(X; Z_i) \;\geq\; 0.$$

The continuous case follows similarly. ☐

V. Diminishing Return in Precision

Let X be any real-valued r.v. and for a precision parameter $\lambda = \sigma^{-2} \geq 0$, consider the AWGN observation model $Z_\lambda = \sqrt{\lambda}\,X + N$, where $N \sim \mathcal{N}(0, 1)$ is independent of X. Let q_λ be the posterior of X given Z_λ. The expected posterior entropy is

$$\mathbb{E}\big[H(q_\lambda)\big] = H(X \mid Z_\lambda) = H(X) - I(X; Z_\lambda) \tag{2}$$

a) I–MMSE relation: Define the minimum mean-squared error as $\mathrm{mmse}(\lambda) := \mathbb{E}\big[(X - \mathbb{E}[X \mid Z_\lambda])^2\big]$. The I–MMSE identity [3] gives

$$\frac{d}{d\lambda}\,I(X; Z_\lambda) \;=\; \frac{1}{2}\,\mathrm{mmse}(\lambda), \quad \lambda \geq 0. \tag{3}$$

Moreover, $\mathrm{mmse}(\lambda)$ is nonincreasing in λ.

Theorem 1 (Monotonicity and convexity in precision)**.** *The map $\lambda \mapsto \mathbb{E}[H(q_\lambda)]$ is* monotone decreasing *and* convex *on* $[0, \infty)$.

Proof. Differentiate (2) with respect to λ and apply (3):

$$\frac{d}{d\lambda}\,\mathbb{E}[H(q_\lambda)] = -\frac{d}{d\lambda}I(X; Z_\lambda) = -\tfrac{1}{2}\,\mathrm{mmse}(\lambda) \leq 0,$$

which proves monotonicity. Differentiating again yields

$$\frac{d^2}{d\lambda^2}\,\mathbb{E}[H(q_\lambda)] = -\tfrac{1}{2}\,\frac{d}{d\lambda}\,\mathrm{mmse}(\lambda) \geq 0,$$

which proves convexity, as $\mathrm{mmse}(\lambda)$ is nonincreasing. ☐

This implies that the incremental entropy reduction per unit increase in precision exhibits diminishing marginal returns.

VI. Information Additivity

Proposition 2 (Information depends only on total precision)**.** *Assume the AWGN model with independent measurements $Z_t = X + N_t$, where $N_t \sim \mathcal{N}(0, \sigma_t^2)$. Define precisions $\lambda_t := \sigma_t^{-2}$ and total precision $\Lambda := \sum_{t=0}^{T-1} \lambda_t$. Then*

$$I(X; Z_{0:T-1}) \;=\; I(X; \sqrt{\Lambda}\,X + N_0),$$

where $N_0 \sim \mathcal{N}(0, 1)$ is independent of X. Hence

$$\mathbb{E}[H(q_T)] \;=\; H(q_0) - I(\Lambda).$$

Proof. The set of observations $\{Z_t\}_{t=0}^{T-1}$ can be shown to be equivalent to a single observation from a channel with precision equal to the sum of the individual precisions, Λ.

Stack $Z = (Z_1, \ldots, Z_T)^\top$ and let $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_T)$. Define $Y := D^{-1}Z = v\,X + W$, where $v = (1/\sigma_1, \ldots, 1/\sigma_T)^\top$ and $W \sim \mathcal{N}(0, I_T)$. Choose an orthogonal U with $Uv = \|v\|e_1$, where $\|v\| = \sqrt{\sum_t \lambda_t} = \sqrt{\Lambda}$. Then $\tilde{Y} := UY = \|v\|X\,e_1 + \tilde{W}$, with $\tilde{W} \sim \mathcal{N}(0, I_T)$. Thus $\tilde{Y}_2, \ldots, \tilde{Y}_T$ are pure noise independent of X, so

$$I(X; Z) = I(X; Y) = I(X; \tilde{Y}) = I\big(X; \tilde{Y}_1\big) = I\big(X; \sqrt{\Lambda}\,X + N_0\big)$$

because mutual information is invariant under bijective transforms □

VII. A Greedy Algorithm

The optimal control problem is equivalent to

$$\max_{u_{0:T-1}} \sum_{t=0}^{T-1} \left(\Delta I_t \; - \; \frac{c(u_t)}{\alpha} \right),$$

where $\Delta I_t := I(\Lambda_{t-1} + \lambda_{u_t}) - I(\Lambda_{t-1})$. The term inside the summation is the marginal "net gain" from choosing sensor u_t. Due to the convexity of $\mathbb{E}[H(q_T)]$ (and thus concavity of $I(\Lambda)$), a greedy approach is optimal. At each step, we choose the sensor that provides the largest marginal net gain.

The mutual information $I(\Lambda)$ for a discrete prior can be approximated using Gauss-Hermite quadrature. The resulting greedy selection policy is detailed in Algorithm 1.

Algorithm 1 Greedy Sensor Selection with α-Tradeoff

Input: Precisions $(\lambda_i)_{i=1}^m$, costs $(c_i)_{i=1}^m$, horizon T, tradeoff $\alpha > 0$, oracle $I(\cdot)$, tolerance $\varepsilon \geq 0$.
Initialize: $\Lambda \leftarrow 0$; $n_i \leftarrow 0$ for all i; $C \leftarrow 0$.
for $t = 1$ **to** T **do**
  $b \leftarrow -\infty$; $j^\star \leftarrow \varnothing$.
  for $i = 1$ **to** m **do**
    $\Delta I \leftarrow I(\Lambda + \lambda_i) - I(\Lambda)$
    score $\leftarrow \Delta I - c_i/\alpha$
    if score $> b$ **then**
      $b \leftarrow$ score; $j^\star \leftarrow i$
    end if
  end for
  if $b \leq \varepsilon$ **then**
    break {No nonnegative marginal net gain}
  end if
  $n_{j^\star} \leftarrow n_{j^\star} + 1$; $\Lambda \leftarrow \Lambda + \lambda_{j^\star}$; $C \leftarrow C + c_{j^\star}$.
end for
Output: Counts $(n_i)_{i=1}^m$, total precision Λ, total cost C, term. entropy $H(q_T) = H(q_0) - I(\Lambda)$.

VIII. Results

We test the greedy algorithm with a horizon of $T = 10$ and a cost-entropy tradeoff of $\alpha = 1$. The results show the number of times each sensor is selected by the greedy policy.

It can be seen that the greedy algorithm consistently outperforms the other algorithms by terminating early.

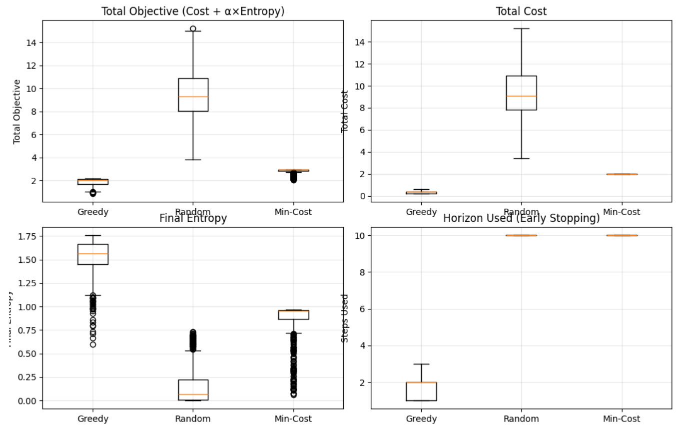

Fig. 1. 4 sensors with costs [2.0, 1.0, 0.5, 0.2] and noise variances [0.5, 1.0, 1.5, 2.0]. The policy balances high precision with low cost.

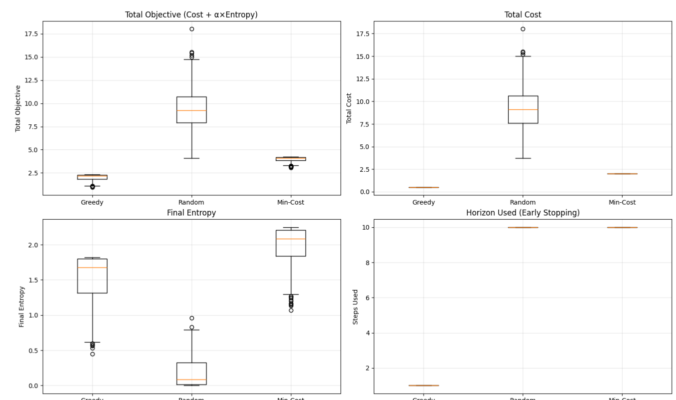

Fig. 2. 4 sensors with costs [2.0, 1.0, 0.5, 0.2] and noise variances [0.5, 1.0, 1.5, 10.0]. The high-noise sensor 4 is correctly ignored.

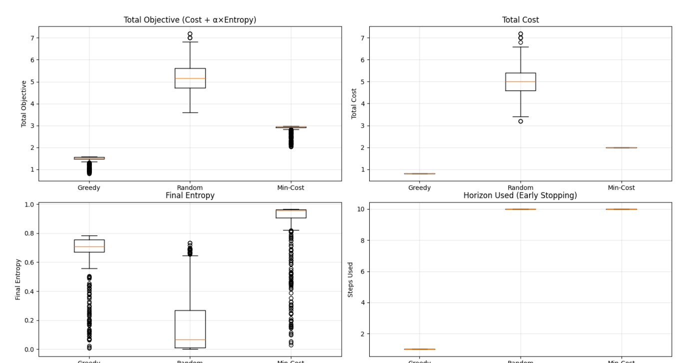

Fig. 3. 4 sensors with costs [0.8, 0.6, 0.4, 0.2] and noise variances [0.5, 1.0, 1.5, 2.0]. With lower costs, the policy heavily favors the most precise sensor.

References

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