

# COSMOS Cluster 5

Jay Paek

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# 1 Reflection and Refraction

**Objective:** Understand the behavior of light as it pass through different mediums.

## 1.1 Ray Approximation

Learn about the mathematical equations that describe reflection and refraction.

Light travels in a straight line in a homogeneous medium until it encounters a boundary between two different media. A **ray** of light is an imaginary line drawn along the direction of propagation of the light. A **wavefront** is a surface passing through points of a wave that have the same phase. The rays, corresponding to the direction of the wave motion, are perpendicular to the wavefronts

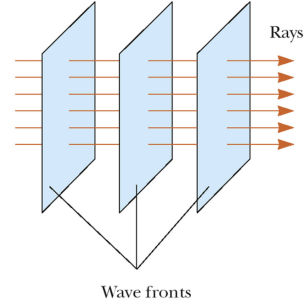
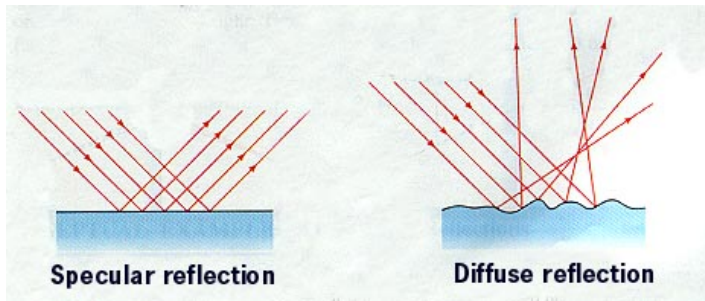


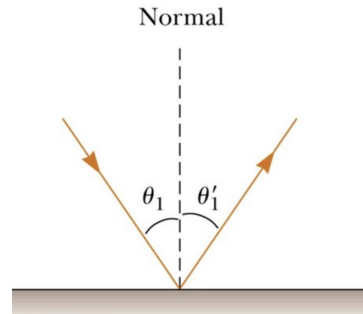
Figure 1: Ray approximations and wavefronts.

## 1.2 Reflection

When light encounters a boundary with a second medium, part of this incident light is directed back into the first medium, and this is known as **reflection**. If the boundary is a smooth surface, the reflection is known as specular reflection. This means all the reflected rays will be parallel to one another. Otherwise if the boundary is not smooth, then the reflection is called diffuse reflection (See Figure 2a). The **normal** to a surface is a



(a) Specular/Diffuse reflection



(b) Reflection

Figure 2: Depictions of reflection.

line which is perpendicular to the surface at given point. The **incident ray** makes an angle  $\theta_1$  with the normal. The **reflected ray** makes an angle  $\theta'_1$  with the normal. The angle of reflection is equal to the angle of incidence:  $\theta_1 = \theta'_1$  (See Figure 2b).

It is important to note that reflection always occurs when light interacts with different mediums.

## 1.3 Refraction

When light, propagating through a transparent medium, encounters a boundary leading into another transparent medium, part of the light is reflected, and part of the light passes into the second medium. The ray that enters the second medium is bent at the boundary. This bending of the ray is called **refraction**. The incident ray, the reflected ray, and the refracted ray (and the normal too) all lie in the same plane. The angle of refraction  $\theta_2$  depends on the refractive indices of the both media.

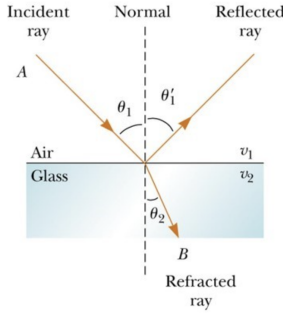


Figure 3: Refraction

The **angle of refraction** ( $\theta_2$  in Figure 3) depends on the material and the angle of incidence

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

The index of refraction, or **refractive index**,  $n$ , of a medium is a unitless value defined as

$$n = \frac{\text{speed of light in a vacuum}}{\text{speed of light in the medium}} = \frac{c}{v}$$

Note that for a vacuum,  $v = c$ , so  $n = 1$ . Otherwise,  $n > 1$ . We can derive **Snell's law** at this point.

Let  $n_1$  and  $n_2$  be the refractive index of the first and second medium, respectively.

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1} \implies n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Consider the two cases in Figure 5. In Figure 5a, observe that the angle of refraction,  $\theta_2$ , is less than the angle of incidence,  $\theta_1$ . On the other hand, in Figure 5b, has the inequality relation to be flipped.

Simply put, in terms of index of refraction, we have that the medium with the greater index of refraction has a lower angle of refraction. It is important to understand the relationships between all of these values. Consider the following example. Let  $\theta_1$  and  $n_1$  be constant. Then we have

$$\underbrace{n_1 \sin \theta_1}_{\text{constant}} = n_2 \sin \theta_2$$

Furthermore  $\sin \theta \approx \theta$  for  $0 \leq \theta \leq \frac{\pi}{2}$ . Then we get that approximately  $n_2 \theta_2 = \eta$ , for some constant  $\eta$  determined by  $n_1$  and  $\theta_1$ . What does this mean mathematically though? We can that given a fixed first medium and angle of incidence, the angle of refraction is inversely proportion to the refractive index.

With a similar strategy, we can show that  $\theta_1$  is directly proportional to  $\theta_2$  if  $n_1, n_2$  are kept constant i.e. increasing  $\theta_1$  will increase  $\theta_2$  and decreasing  $\theta_1$  will decrease  $\theta_2$ .

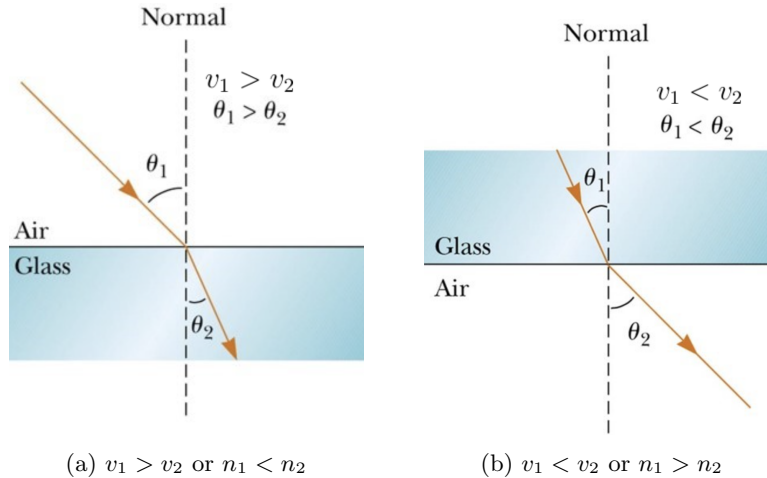


Figure 4: Two cases of refraction.

Figure 4 shows how the angle made with the normal is smaller in the medium with a higher refractive index. We will prove the result for Figure 4a. The proof for 4b is similar.

**Proposition:** If  $n_1 < n_2$ , then  $\theta_1 > \theta_2$ .

*Proof:* Recall the following proportions:

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1}$$

Since  $n_2 > n_1 > 0$ , so  $\frac{n_2}{n_1} > 1$ . Then  $\frac{\sin \theta_1}{\sin \theta_2} > 1 \implies \sin \theta_1 > \sin \theta_2$ .  $\sin$  is a monotone increasing function in the domain  $(0, \frac{\pi}{2})$ . This means that for any  $x, y$  between 0 and  $\frac{\pi}{2}$ , if  $\sin x > \sin y$  then  $x > y$ . We can now conclude that  $\theta_1 > \theta_2$ . ■

Consider the physical example in Figure 5, where we shine a laser through a prism. Ray 1 is the incident ray. Ray 2 is the reflected ray. Ray 3 is refracted into the crystal. Ray 4 is internally reflected in the crystal. Ray 5 is refracted as it enters the air from the crystal.

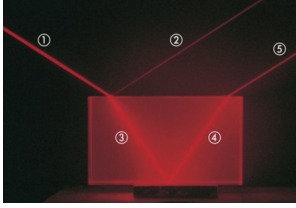


Figure 5: Light path of laser through a prism.

Another interesting example is to observe that Ray 2 and Ray 5 look parallel. It is left as an exercise to the reader to prove that these two rays are indeed parallel given that the prism is a perfect rectangular prism.

Just kidding, I will prove it here.

**Proposition:** Ray 2 and Ray 5 are parallel.

*Proof:* Let  $\theta_1$  denote the angle of incidence of the laser (Ray 1) and  $\theta_2$  denote the angle of the first reflection, Ray 2.

Let  $\theta_3$  be the refracted angle of Ray 3. Given that the top and bottom surfaces are parallel, this must mean that all lines perpendicular to these parallel lines are parallel to each other. We can draw all of the normals between Rays (1,2), (3,4), and (4,5), which is depicted with the magenta arrows.

Ray 3 is a transversal of the left and middle vertical lines. By the alternate interior angles theorem,  $\theta_3 = \theta_r$  and  $\theta_r' = \theta_4$ . Furthermore,  $\theta_r = \theta_r'$  by the properties of reflection of a smooth surface. By the transitive property and the congruent set of angles, we have  $\theta_3 = \theta_4$ .

We know that  $(\theta_1, \theta_3)$  and  $(\theta_4, \theta_5)$  satisfy Snell's law. But with what we had earlier,  $n_2 \sin \theta_3 = n_2 \sin \theta_4$ . So

$$n_1 \sin \theta_1 = n_2 \sin \theta_3 = n_2 \sin \theta_4 = n_1 \sin \theta_5 \implies \sin \theta_1 = \sin \theta_5$$

Since  $\theta_1$  and  $\theta_5$  can only be values between 0 and  $\frac{\pi}{2}$ , there must be unique values that satisfy this equality (in other words, we can say that the  $\sin$  function is **bijective** in  $(0, \frac{\pi}{2})$ ). Necessarily, we must have  $\theta_1 = \theta_5$ . With reflection,  $\theta_1 = \theta_2 = \theta_5$ .

Finally, let the top surface be a transversal. The angles of  $\theta_2$  and  $\theta_5$  are congruent, so Ray 2 and Ray 5 are parallel by the converse of the corresponding angles theorem. This concludes the proof. ■

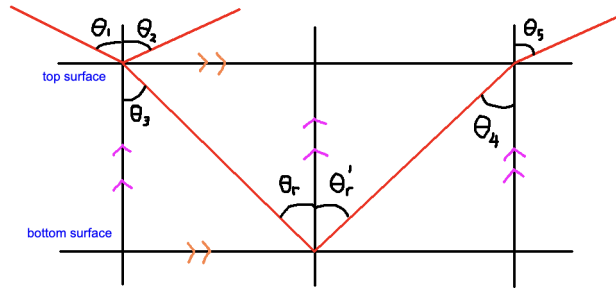


Figure 6: Depiction of laser's path through prism with labeled angles.

## 1.4 Wavelength and Frequency

Recall that the velocity of a wave is the product of the wavelength  $\lambda$  and frequency  $f$  i.e.  $v = \lambda f$ . When light travels through different mediums, what happens to these values?

The equation for the energy of a **photon**, a light particle, is  $E = hf$ , where  $h$  is Planck's constant. From this, we can deduce that the frequency of a wave remains constant as it travels through different mediums since energy must be conserved. With velocity changing and frequency remaining constant, we must have that the wavelength changes!

With this new idea in mind, we can derive another proportional relationship related to Snell's law.

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} = \frac{\lambda_1 f}{\lambda_2 f} = \frac{\lambda_1}{\lambda_2}$$

## 1.5 Total Internal Reflection

**Total internal reflection** can occur when light attempts to move from a medium with a high index of refraction to a medium with a lower index of refraction. A particular angle of incidence will result in angle of refraction of  $90^\circ$ . This angle of incidence is called the critical angle. For angles of incidence greater than the critical angle, the beam is entirely reflected at the boundary. This ray obeys the Law of Reflection. Total internal reflection occurs only when light attempts to move from a medium of higher index of refraction to a medium of lower index of refraction.

## 1.6 Supplementary Materials

- 3blue1brown's video: '4 questions about the refractive index | Optics puzzles 4'

- 2    Diffraction**
- 3    Spectrometer**
- 4    Polarization**
- 5    Solar Cell**

## 6 Quantum Cryptography

### 6.1 Introduction to Cyptography

**Cryptography** is the technique for converting information to an unintelligible form that can only be deciphered by authorized parties. **Encrypting** is altering the data in a way to hide the true message. **Decrypting** is basically undoing the encryption to reveal the true message. A **key** is a piece of information such that it helps encrypt or decrypt the message. The goal is to only let authorized parties have access to the key. Thus, the encrypted message will make no sense to those without the key.

Especially in the digital world, we encode each letter to a specific binary string i.e. a sequence of eight 1's and 0's. Let's take a look at a simple example where we our encoding algorithm is simply adding a binary string to each character's respective binary string.

Before we proceed, let's do an example of adding binary numbers.

**Example:** Let the key be 11010100.

True message	N	O	Encrypted message	N	O
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### 6.2 Quantum Key Distribution

This time, we will make our key by using photons. According to quantum physics, one cannot create a copy of a single photon with the same state (polarization state in this case). This is based on the no-cloning theorem.

### 6.3 Linear Algebra Background

TO-DO

### 6.4 BB84 Protocol

The idea is to generate a secure key that only the sender and receiver have acces to. Single photons transmitted and measured at randomly selected polarization states are used to create a secure key.

Hotizontal and vertical polarizations make an orthogonal basis.  $-45^\circ$  and  $+45^\circ$  polarizations make an orthogonal basis.

In our experiment, a short pulse of laser light is used instead of a single photon. When a  $45^\circ$  light is sent through a polarizing beam splitter, both detectors will record light, In this case, one of the polarizations states will be randomly selectly electronically.

Alice will send a message in either  $-45, 0, 45, 90$  using the waveplate. Bob will receive the message with a waveplate that can be configured to either  $0, 45$ . Bob will use the polarized beam splitter to project the lasers onto two different panels.

### 6.5 Additional Resources

- More cool cryptography videos
  - ‘The Unbreakable Krytos Code’ by LEMMiNO

ASCII Alphabet			
A	1000001	N	1001110
B	1000010	O	1001111
C	1000011	P	1010000
D	1000100	Q	1010001
E	1000101	R	1010010
F	1000110	S	1010011
G	1000111	T	1010100
H	1001000	U	1010101
I	1001001	V	1010110
J	1001010	W	1010111
K	1001011	X	1011000
L	1001100	Y	1011001
M	1001101	Z	1011010

Part of the ASCII alphabet

Figure 7: Light path of laser through a prism.

– ‘Cicada 3301:An Internet Mystery’ by LEMMiNO