MATH190A - HW1

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Problem 1

Part (a)

Let X be a set and let I be an index set. Suppose that for each $i \in I$, we have a topology $\mathcal{T}_i$ on X. Prove that $\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$ is also a topology on X. Prove that $\mathcal{T} \leq \mathcal{T}_i$ for all $i \in I$, and in fact, if there is another topology $\mathcal{T}'$ such that $\mathcal{T}' \leq \mathcal{T}_i$ for all $i \in I$, then $\mathcal{T}' \leq \mathcal{T}$ (i.e., $\mathcal{T}$ is the "greatest lower bound" of all of the $\mathcal{T}_i$).

Solution: First we prove that $\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$ is also a topology on X.

- Clearly, $\emptyset, X \in \mathcal{T}_i, \forall i \in I$, so $\emptyset, X \in \mathcal{T}$.
- Let J be an index set and $A_j \in \mathcal{T}, j \in J$, then $\forall j \in J$ we have $A_j \in \mathcal{T}_i, \forall i \in I$. Let $i \in I$, then $\bigcup\limits_{j\in J} A_j \in \mathcal{T}_i$ since each $\mathcal{T}_i$ is a topology. We picked i arbitrarily, so it follows that $\bigcup\limits_{j\in J} \in \mathcal{T}_i, \forall i \in I$. Hence, $\bigcup\limits_{j\in J} A_j \in \mathcal{T}$
- Let J be a finite index set and $A_j \in \mathcal{T}, j \in J$, then $\forall j \in J$ we have $A_j \in \mathcal{T}_i, \forall i \in I$. Let $i \in I$, then $\bigcap\limits_{j\in J} A_j \in \mathcal{T}_i$ since each $\mathcal{T}_i$ is a topology. We picked i arbitrarily, so it follows that $\bigcap\limits_{j\in J} \in \mathcal{T}_i, \forall i \in I$. Hence, $\bigcap\limits_{j\in J} A_j \in \mathcal{T}$

With the four axioms of a topology, $\mathcal{T}$ is a topology.

Let $i \in I$, then we have that $\mathcal{T} \subseteq \mathcal{T}_i$ by definition. Therefore, $\mathcal{T} \leq \mathcal{T}_i, \forall i \in I$.

Let $\mathcal{T}'$ be a topology on X such that $\mathcal{T}' \leq \mathcal{T}_i, \forall i \in I$. Assume that $\mathcal{T}$ is a strict subset of $\mathcal{T}'$ i.e. $\mathcal{T} < \mathcal{T}'$. Then $\mathcal{T}' - \mathcal{T}$ is nonempty, and we can take any $A \in \mathcal{T}' - \mathcal{T}$. Such A does not belong to $\mathcal{T} = \cap_{i\in I}\mathcal{T}_i$, which means that $\exists j \in I$ such that $A \notin \mathcal{T}_j$, which contradicts that fact that $A \in \mathcal{T}'$. ∎

Part (b)

Let $X = \{1, 2, 3\}$ and find two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X such that $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology.

Solution: $\mathcal{T}_1 = \{\emptyset, \{1\}, X\}, \mathcal{T}_2 = \{\emptyset, \{2\}, X\}$. ∎

Problem 2

Let X be a topological space with topology $\mathcal{T}$, let A be a subset of X, and let B be a subset of A, i.e., $B \subseteq A \subseteq X$. There are two potentially different topologies we can put on B: First, B is a subset of X so we can give it the subspace topology $\mathcal{T}_B$. Second, we can give A the subspace topology $\mathcal{T}_A$ from X, and then give B the subspace topology $(\mathcal{T}_A)_B$ that comes from being a subset of A. Prove that they are actually the same: $\mathcal{T}_B = (\mathcal{T}_A)_B$.

Solution: Let $E \in \mathcal{T}_B$, then $E = B \cap U$ for some $U \in \mathcal{T}$. Since $B \subseteq A$, we have $B = A \cap B$. Then $E = (A \cap B) \cap U = B \cap (A \cap U) \in (\mathcal{T}_A)_B$. Hence, $\mathcal{T}_B \subseteq (\mathcal{T}_A)_B$.

Other direction follows similarly. Let $E \in (\mathcal{T}_A)_B$, then $E = (U \cap A) \cap B$ for some $U \in \mathcal{T}$. But $E = (U \cap A) \cap B = U \cap (A \cap B) = U \cap B \in \mathcal{T}_B$. So, $(\mathcal{T}_A)_B \subseteq \mathcal{T}_B$. ■

Problem 3

Let X be a topological space and let A be a subspace. Prove that if U is open in A, then for any other subset B of X, $U \cap B$ is open in the subspace $A \cap B$.

Solution: U is open in A, so $U = A \cap V$ for some $V \in \mathcal{T}$ (the topology on X). It is clear that $U \cap B = (A \cap B) \cap V \in \mathcal{T}_{A \cap B}$. $U \cap B$ is open. ■

Problem 4

Let X be a topological space and let A, B be subsets of X.

Part (a)

Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution: By definition, $\overline{A} = \bigcap_{i \in I} E_i, \overline{B} = \bigcap_{j \in J} F_j$, where each I, J indexes over all of the closed sets such that $A \subseteq E_i, B \subseteq F_j$, respectively.

$$\overline{A} \cup \overline{B} = \Big(\bigcap_{i \in I} E_i\Big) \cup \Big(\bigcap_{j \in J} F_j\Big) = \bigcap_{(i,j) \in I \times J} E_i \cup F_j$$

For any choice of (i, j), $E_i \cup F_j$ is a closed set that contains $A \cup B$. Therefore $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

The other direction follows nicely. We have $A, B \subseteq A \cup B$, so $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$. ■

Part (b)

Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Solution: Proceed similarly as in (a). We have

$$\overline{A} \cap \overline{B} = \Big(\bigcap_{i \in I} E_i\Big) \cap \Big(\bigcap_{j \in J} F_j\Big) = \bigcap_{(i,j) \in I \times J} E_i \cap F_j$$

For any choice of (i, j), $E_i \cap F_j$ is a closed set that contains $A \cap B$. Therefore $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. ■

Part (c)

Give an example where $\overline{A \cap B}$ is not equal to $\overline{A} \cap \overline{B}$.
[Hint: There is an example where $X = \mathbb{R}$ and A, B are open intervals.]

Solution: With $X = \mathbb{R}$, consider $A = (-1, 0), B = (0, 1)$. Then $\overline{A \cap B} = \emptyset$ and $\overline{A} \cap \overline{B} = \{0\}$. ■

Problem 5

Let X be a topological space and let A be a subset of X. Prove the identities

$$X - \overline{A} = (X - A)^\circ, \quad X - A^\circ = \overline{X - A}.$$

Solution: The solution is trivial for $A = \emptyset, X$. Consider the when it is not.

For the first statement, we have by definition that $(X - A)^\circ = \bigcup_{i \in I} E_i$, where I indexes over every open set E_i, such that $E_i \subseteq X - A$. $X - \overline{A}$ is an open set that is contained in $X - A$, so $X - \overline{A} \subseteq (X - A)^\circ$.

Now, assume that $X - \overline{A} \subset E_i$ for some $i \in I$. Then $E_i - (X - \overline{A}) = E_i \cap \overline{A}$ is nonempty. $E_i \cap \overline{A} = E_i \cap (A \cup A') = (E_i \cap A) \cup (E_i \cap A')$ is nonempty. Since $E_i \subseteq X - A$, we have that $E_i \cap A$ is empty and that $E_i \cap A'$ must be nonempty. Take any $p \in E_i \cap A'$ and E_i is a neighborhood of p, which is a limit point. p is a limit point if and only if every neighborhood of p intersects A. E_i must intersect A. We have a contradiction.

For the second statement, we have by definition that $\overline{X - A} = \bigcap_{i \in I} E_i$ where I indexes over every closed set E_i such that $X - A \subseteq E_i$. $X - A^\circ$ is a closed set that contains $X - A$, so $\overline{X - A} \subseteq X - A^\circ$.

Now, assume that $E_i \subset X - A^\circ$ for some $i \in I$. Then $E_i - (X - A^\circ) = E_i \cap A^\circ$ is empty. However, $E_i \cap A^\circ$ is should be nonempty $\forall i \in I$ since $A^\circ \cap (X - A)$ is nonempty and $X - A \subset E_i$. We have a contradiction. ■

Problem 6

Let I be an index set and suppose we have a topological space X_i for each $i \in I$. Let X be the disjoint union of all of the X_i:

$$X = \bigsqcup_{i \in I} X_i.$$

Formally, this is the set of pairs $\{(i, x) \mid i \in I, x \in X_i\}$. Let $\mathcal{T}$ be the collection of subsets U of X such that for all $i \in I$, the set $U_i = \{x \in X_i \mid (i, x) \in U\}$ is open in X_i. Prove that $\mathcal{T}$ is a topology for X.

Solution: If $U = \emptyset$, then $\forall i \in I$ we have $U_i = \emptyset$, which makes U_i open in X_i. $\emptyset \in \mathcal{T}$.

If $U = X$, then $\forall i \in I$ we have $U_i = X_i$, which makes U_i open in X_i. $X \in \mathcal{T}$.

Let J be an index set, and U_j open $\forall j \in J$. By definition, we have that $(U_j)_i$ is open in $X_i, \forall i \in I$. Let $i \in I$ and consider $\left(\bigcup_{j \in J} U_j\right)_i$, which is just $\bigcup_{j \in J} (U_j)_i$, a union of sets that are open in X_i. Thus $\left(\bigcup_{j \in J} U_j\right)_i$ is open in $X_i, \forall i \in I$, making it open in X.

Proceed with a setup similar to the previous section, but this time let J be a finite index set. Let $i \in I$ and consider $\left(\bigcap_{j \in J} U_j\right)_i$, which is just $\bigcap_{j \in J} (U_j)_i$, a finite intersection of sets that are open in X_i. Thus $\left(\bigcap_{j \in J} U_j\right)_i$ is open in $X_i, \forall i \in I$, making it open in X. ■

Problem 7

Let $X = \mathbb{Z}$ be the set of integers. For each pair of integers m, n such that $m \neq 0$, define the subset

$$b_{m,n} = \{mx + n \mid x \in \mathbb{Z}\}.$$

Part (a)

Prove that the collection of $b_{m,n}$ (with $m \neq 0$ but no restriction on n) form a basis for a topology, which we will just call $\mathcal{T}$.
[Remark: Since each $b_{m,n}$ is infinite, all non-empty open sets in $\mathcal{T}$ are infinite.]

Solution: Let $x \in \mathbb{Z}$ and $x \in b_{m,n} \cap b_{p,q}$. Then $x = my + n = pz + q$ for some $y, z \in \mathbb{Z}$. Then we have $x \in b_{mp,x}$ and $b_{mp,x} \subseteq b_{m,n} \cap b_{p,q}$. ■

Part (b)

Prove that each $b_{m,n}$ is both open and closed in $\mathcal{T}$.

Solution: $b_{m,n}$ is vacuously open by construction. Furthermore, $\mathbb{Z} - b_{m,n} = \bigcup_{i=1, i\neq n}^{m} b_{m,i}$, which open. $b_{m,n}$ is also closed. ■

Part (c)

Prove that

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p} b_{p,0}$$

where the union is over all prime numbers p.

Solution: Proof is trivial if x is prime or composite or 0. If $x = 1, -1$, then assume $mp = x \implies m = \dfrac{x}{p}$ for some m, where p is prime. But $p > 1, \forall p$, so $|m| < 1$, which can not be true. ■

Part (d)

Using the above facts, conclude that there must be infinitely many primes.
[Hint: use proof by contradiction.]

Solution: Assume there are finitely many primes. Then $\mathbb{Z} - \{-1, 1\}$ is nonempty and clopen. $\mathbb{Z} - \{-1, 1\}$ is open, but can't be closed since $\{-1, 1\}$ is not open. A contradiction. ■

Problem 8

Part (a)

Let X be a topological space. Given a subset A, define $f(A) = \overline{A}$, so that we have a function $f : 2^X \to 2^X$ which we call closure. Prove that f satisfies these 4 properties:

(i) $f(\emptyset) = \emptyset$.

(ii) For all $A \subseteq X$, we have $A \subseteq f(A)$.

(iii) For all $A \subseteq X$, we have $f(A) = f(f(A))$.

(iv) For all $A, B \subseteq X$, we have $f(A) \cup f(B) = f(A \cup B)$.

Solution: For (i), $\emptyset$ is closed, so the closure is itself.
For (ii), we have $A \subseteq \overline{A}$.
For (iii), $f(A) = \overline{A}$, which is closed. The closure of a closed set is itself, so $\overline{A} = f(\overline{A})$.
For (iv), see Problem 4 Part (a). ■

Part (b)

Conversely, suppose that X is a set and we are given a function $g : 2^X \to 2^X$ satisfying the 4 conditions above. Prove that there is a unique topology on X so that f is the closure function of this topology. In particular, this says that we could define topologies in terms of functions satisfying (i)–(iv) instead of with open sets.

Solution: From (iv), we get monotonicity of f. Let $A \subset B$, then $A \cup B = B$. And thus

$$f(A) \subseteq f(A) \cup f(B) = f(A \cup B) = f(B)$$

Let $\mathcal{T} = \{A \subseteq X : X - A = f(X - A)\}$. $f(\emptyset) = \emptyset \implies f(X - X) = X - X \implies X \in \mathcal{T}$. $X - \emptyset = X \subseteq f(X)$ and $f(X) \in 2^X \iff f(X) \subseteq X$. $f(X) = X$, so $\emptyset \in \mathcal{T}$.

Let I be an index set and $A_i \in \mathcal{T}, \forall i \in I$. From (ii), $X - \bigcup_{i \in I} A_i \subseteq f\Big(X - \bigcup_{i \in I} A_i\Big)$. Then we have $X - \bigcup_{i \in I} A_i = \bigcap_{i \in I}(X - A_i) = \bigcap_{i \in I} f(X - A_i)$ since $A_i \in \mathcal{T}, \forall i \in I$. Observe that

$$f\Big(\bigcap_{i \in I}(X - A_i)\Big) \subseteq f(X - A_j), \forall j \in I \implies f\Big(\bigcap_{i \in I}(X - A_i)\Big) \subseteq \bigcap_{i \in I} f(X - A_i) = X - \bigcup_{i \in I} A_i$$

And thus, $X - \bigcup_{i \in I} A_i \subseteq f\Big(X - \bigcup_{i \in I} A_i\Big)$, making $\mathcal{T}$ closed under arbitrary union.

Let I be a finite index set and $A_i \in \mathcal{T}, \forall i \in I$. Then $f\Big(X - \bigcap_{i \in I} A_i\Big) = f\Big(\bigcup_{i \in I}(X - A_i)\Big) = \bigcup_{i \in I} f(X - A_i) = \bigcup_{i \in I}(X - A_i) = X - \bigcap_{i \in I} A_i$. $\mathcal{T}$ is closed under finite intersection. Therefore, $\mathcal{T}$ is a topology.

Now we show that $f(A) = \overline{A}, \forall A \subseteq X$. Let $A \subseteq X$. The solution is trivial if $A = \emptyset, X$. If not, then we have $f(A) = f(f(A))$, so $A = X - B$ for some $B \in \mathcal{T}$. This makes $f(A)$ closed and it contains A. $\overline{A} \subseteq f(A)$. But then $\overline{A}$ is the intersection of all closed sets E_i such that $A \subseteq E_i$. By monotonicity, $f(A) \subseteq f(E_i) = E_i, \forall i \in I \implies f(A) \subseteq \overline{A}$. Therefore $f(A) = \overline{A}$, and f is the closure function of $\mathcal{T}$.

Uniqueness follows from construction where there exists unique A such that $A = f(A) = \overline{A}$, which defined a unique collection of open sets. Essentially, we have defined a topology by first defining the closed sets. ■

Part (c)

Find and prove the analogous statement for the function that takes a subset to its interior.

Solution: Let X be a topological space. If we have a function $g : 2^X \to 2^X$ such that it satisfies:

(i) $g(\emptyset) = \emptyset$.

(ii) For all $A \subseteq X$, we have $g(A) \subseteq A$.

(iii) For all $A \subseteq X$, we have $g(g(A)) = g(A)$.

(iv) For all $A, B \subseteq X$, we have $g(A) \cap g(B) = g(A \cap B)$.

then g is the interior function of the topology on X.

This result follows from part (b). Let $g(A) = X - f(X - A)$, where f is the closure function. ■