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Chapter 1

Section 1

Notes

Definition It is a **set** if we can unambiguously say whether an object belongs to it or not. An object a that belongs to a set S is called an **element** of S, $a \in S$.

Significance: Used in almost everything we build on in the class from properties of sets to definition of rational and irrational numbers to Dedekind cuts

Fact: A set with no elements is called "the empty set"- ϕ.

Significance: Definition of the meaning of a non-empty set is important because a lot of theorems require that a set in $\mathbb{R}$ be nonempty

Definition Inclusion: For sets A, B we say $A \subset B$ if every element of A is also an element of B.

Significance: Idea of inclusion is the basis of equivalence of two sets and is the idea used in defining a subset of another set

Fact: Any set is a subset of itself.

Fact: The empty set is a subset of any set.

Fact: For a set S, the power set $P(S)$ of S is the set of all subsets of S.

Definition For two sets A, B we say they are **equivalent** $(A = B)$ if every element of $A \subset B$ and simultaneously $B \subset A$.

Definition The **Union** $(A \cup B)$ of two sets A, B is a set $\{x: x \in A \ or \ x \in B\}$.

Definition The **Intersection** $(A \cap B)$ of two sets A, B is a set $\{x: x \in A \ and \ x \in B\}$.

Definition The **Set Difference** $(A \setminus B)$ of two sets A, B is a set $\{x: x \in A \ and \ x \notin B\}$.

Definition A **Direct Proof** shows that $Q \Rightarrow R$ or $Q \Leftrightarrow R$ and shows that Q is true.

Significance: Definition provides a way of going about showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually axioms, existing lemmas and theorems, without making any further assumptions

Definition Proof of Contradiction: Let Q be the opposite of what you want to prove. Note that $R \wedge \sim R$ is always false. Show that $Q \Rightarrow (R \wedge \sim R)$ is true.

Significance: The definition of proof by contradiction is used to in many proofs when proceeding by trying to prove that every case holds is a nontrivial task such as in Theorem 1.1.1

Definition A **De Morgan's Law** shows for any two sets A, B

$$(A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c.$$

Theorem (Area of a right triangle) If a, b, c are the lengths of the legs of a $right\ triangle$ Δ, then the area of this triangle is $\frac{ab}{2}$.

Definition We say a number is **rational** if (and only if) it can be expressed as a quotient of two integers M/N, with $N \neq 0$. Otherwise, we say it is **irrational**.

Significance: We define rational and irrational numbers because they are both subsets of the real numbers, and each reserve unique properties that are useful in computation. Both rational and irrational numbers will be referenced throughout the rest of these definitions and theorems such as proving the density of rational and irrational numbers and the definition of a Dedekind cut, which in and of itself defines the real number system

Theorem (Area of a right triangle: case when base is the longest side) Let h be the length of the height of the triangle. Then the area of the triangle equals $\frac{bh}{2}$.

Book

Theorem 1.1.1 *There are no integers M and N such that* $\sqrt{2} = {}^{M}/_{N}$.

Significance: Used to prove the definition and existence of irrational numbers, which are then referenced later in ideas such as the density of rational numbers and density of irrational numbers

Lemma 1.1.2 *Each Natural number $N > 1$ can be written as a product of prime numbers*.

Significance: Used in the fundamental theorem of arithmetic, partially the idea that prime numbers are the building block of natural numbers since they can be factorized into a unique (find in subsequent theorem) factorization of solely prime numbers

Theorem 1.1.3 (The Fundamental Theorem of Arithmetic) *Each natural number $N > 1$ has a unique factorization into prime numbers. That is, N can be written in one and only one way as*

$$N = p_1 p_2 p_3 \dots p_r,$$

where the p's are all prime and $p_1 \leq p_2 \leq p_3 \leq \dots \leq p_r$.

Significance: Links the idea of prime numbers and their importance as the building blocks of the natural numbers by actually proving the unique factorization (up to order of multiplication) of any number in the reals into prime numbers. A lot can be said for breaking down any number into its building blocks to construct other numbers and components.

Lemma 1.1.4 If a prime p divides M^2, then p divides M.

Theorem 1.1.5 For any prime, p, $\sqrt{p}$ is irrational.

Significance: Establishing that there is in fact a common link between prime numbers, as just defined, and irrational numbers

Theorem 1.1.6 (Infinitude of Primes) There are an infinite number of primes.

Significance: References the fundamental theorem of arithmetic to arrive at a contradiction from the fact that all numbers are amenable to prime factorization

Section 2

Definition 1.2.1 A **Dedekind cut** is a subset α of the rational numbers $\mathbb{Q}$ with the following properties:

i. α is not empty and $\alpha \neq \mathbb{Q}$;
ii. if $p \in \alpha$ and $q < p$, then $q \in \alpha$;
iii. if $p \in \alpha$, then there is some $r \in \alpha$ such that $r > p$.

The collection of all Dedekind cuts is called the set of **real numbers.** Two real numbers α and β are **equal** if and only if both cuts are the same subset of $\mathbb{Q}$.

Significance: References the definition of a rational number; references the idea of equivalence of two cuts, which are actually sets, meaning that it is built on the meaning of equivalence of two sets; Used in the definition of a (least) upper and (greatest) lower bounds, to define numbers like π, the theorem of the Archimedean property

Definition 1.2.2 Given two real numbers (cuts) α and β we say that $\alpha < \mathbb{Q}$ ($\alpha \leq \beta$) whenever $\alpha \subset \beta$ ($\alpha \subseteq \beta$).

Significance: Building on the definition of inclusion, we can define what it means for two real numbers (essentially sets) to involve inequality

Definition 1.2.3 Let α, λ, and γ be real numbers (cuts). A nonempty set S of real numbers has an **upper bound** γ, if for any number $\alpha \in S$ we have $\alpha \leq \gamma$. An upper bound λ is called the **least upper bound** for the set S if for any other upper bound γ we have $\lambda \leq \gamma$. λ is denoted by lub(S).

Significance: Involves the definition of inequality of two real numbers, the requires definition of an empty set, applies the validity of an upper bound which will help to establish the basis of what it means for an interval to be bounded

Theorem 1.2.4 (The Least Upper Bound Property) Every nonempty set of real numbers that is bounded above has a least upper bound.

Significance: Building on the definition of inclusion, we can define what it means for two real numbers (essentially sets) to involve inequality

Definition 1.2.5 Let $\alpha \subset \beta$ and $\alpha \subset \beta$ be two real numbers (i.e., Dedekind cuts). Then the **sum** of α and β is defined by

$$\alpha + \beta \equiv \{p + r \mid p \in \alpha \; and \; r \in \beta\}$$

Theorem 1.2.6 If α and β are Dedekind cuts, then $\alpha + \beta$ is a Dedekind cut.

Significance: References the definition of a Dedekind cut to establish that the combination of two Dedekind cuts results in a Dedekind cut (also a real number) – the construction of a complete, ordered field (e.g. the real number line) relies on this property

Theorem 1.2.7 If α, β, and γ are cuts, then

a) $\alpha + \beta = \beta + \alpha$
b) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

Significance: Building on the definition of addition for cuts to establish properties of adding cuts together

Theorem 1.2.8 (The Archimedean Property for the Real Numbers) If α and β are positive real numbers, then there is some positive integer n such that $n\alpha > \beta$.

Significance: Used in the proof that $\mathbb{N}$ is not bounded above, referenced in the proof of the least upper bound property of a non-empty set,

Theorem 1.2.9 (Density of Rational Numbers) If α and β are real numbers with $\alpha < \beta$, then there is a rational number r such that $\alpha < r < \beta$.

Significance: Even though rational numbers are rather sparse in the real number system compared to the irrational numbers, uses the Archimedean property to show there is always a rational between two numbers of the real numbers

Theorem 1.2.10 (Density of Irrational Numbers) If α and β are real numbers with $\alpha < \beta$, then there is a irrational number γsuch that $\alpha < \gamma < \beta$.

Significance: Complement of the Density of Rational Numbers, saying that there will always be an irrational number between any two real numbers, which can be used in subsequent proofs when dealing with real numbers

Section 3

Definition 1.3.1 A **Field** is a set together with two binary operations γ and γ, called **addition** and **multiplication**, that satisfies the following eleven axioms.

Significance: The rational numbers, which are the integers and the fractions allow you to perform addition and subtraction. Fields allow us to generalize properties of certain sets of numbers for computation and categorization. For example, we work to derive axioms that create a complete, ordered field (also known as the real number system)

Theorem 1.3.2 (Uniqueness of Identities and Inverses) the additive and multiplicative identities are unique.

a) If $a + b = a$, then $b = 0$.
b) If $a \cdot b = a$ and $a \neq 0$, then $b = 1$.
c) If $a + b = 0$, then $b = -a$.
d) If $a \cdot b = 1$, then $b = a^{-1}$.

Significance: Building on top of the additive and multiplicative properties of a very basic field, helping to define the behavior of a major operation on a field

Corollary 1.3.3 If $\alpha \in \mathbb{R}$, then $-(-\alpha) = a$. If $\alpha \neq 0$, then $(\alpha^{-1})^{-1} = a$.

Theorem 1.3.4 If $\alpha \in \mathbb{R}$, then $\alpha \cdot 0 = 0$.

Significance: Establishes the multiplicative property of zero, which varies from the multiplicative identity saying that a number multiplied by zero is always zero

Theorem 1.3.5 If $\alpha \cdot b = 0$, then either $a = 0$ or $b = 0$.

Significance: Building on top of the previous theorem (multiplicative property of zero) to say identify why a product would be zero between two numbers – defines behavior of a major operation on a field

Theorem 1.3.6 Let a and b be real numbers. Then

a) $(-\alpha) \cdot b = -(\alpha \cdot b)$ and, in particular, $(-1) \cdot b = -b$;
b) $(-\alpha) \cdot (-b) = \alpha \cdot b$.

Significance: Expands very basic multiplicative properties of a field to include negative numbers – defines behavior of a major operation on a field

Definition 1.3.7 The element a is **greater than** b (denoted γ) if and only if $a - b > 0$. We say a is **less than** b ($a < b$) if and only if $b > a$. (Similar definitions hold for ≤ and ≥).

Significance:

Theorem 1.3.8 Let a, b, c, and d be real numbers.

a) If $a > b$ and $b > c$, then $a > c$ (transitivity).
b) If $a > b$ and $c \geq d$, then $a + c > b + d$.

Theorem 1.3.9 Let a, b, c, and d be real numbers.

a) If $a > b$ and $c > 0$, then $a \cdot c > b \cdot c$.
b) If $a > b$ and $c < 0$, then $a \cdot c < b \cdot c$.

Definition 1.3.10 The **absolute value** of a real number a is denoted by a and is defined as

$$|a| = \begin{cases} a, & if\ a \geq 0 \\ -a, & if\ a < 0. \end{cases}$$

Significance: Used in theorems such as the triangle inequality; and proofs of theorems such as Bolzano-Weierstrass Theorem, Limit Theorem,

Definition 1.3.11 The **distance** from a to b is $|b - a|$.

Significance: Imperative in proofs of theorems such as Bolzano-Weierstrass Theorem, Limit Theorem, and the Mean Value Theorem where proofs reference the distance between two distance metric points

Theorem 1.3.12 The statement $|a| \leq b$ is equivalent to $-b \leq a \leq b$.

Significance: Defines a property of the absolute value that is utilized in the epsilon-delta definition of the limit

Theorem 1.3.13 (The Triangle Inequality) For any real numbers a and b,

$$|a| + |b| \geq |a + b|.$$

Significance: The triangle inequality concerns *distance* between points and says that the "straight line" distance between *A* and B is less than or equal to the sum of the distances from *A* to C and from *C* to B. It is an intuitive result but is useful in the future of inequality use.

Definition 1.3.14 A system of elements with two operations obeying Axioms 1 through 15 is called a **complete ordered field**.

Significance: A complete, ordered field has the Archimedean Property for the Real Numbers and is foundational to arithmetic and number theory – many theorems, such as the extreme value theorem, requires a continuous function from a set to the real number system.

Section 4

Definition 1.4.1 Let S be a set of real numbers. An **open cover** for S is a collection of open intervals, $\{\mathcal{O}_\alpha \mid \alpha \in A\}$, such that each point in S is contained in at least one of these intervals. That is, for each $s \in S$ we have $s \in \mathcal{O}_\beta$ for some $\beta \in A$. A **finite subcover** for an open cover $\{\mathcal{O}_\alpha\}$ consists of a finite number of the open intervals $\mathcal{O}_\alpha$, which together form an open cover for S.

Significance: the definition of an open cover of a set, arises in the statement of the Heine-Borel theorem, which is in turn used to prove the Bolzano-Weierstrass theorem about accumulation points, or the uniform continuity of continuous functions on closed intervals [a, b].

Theorem 1.4.2 (The Heine-Borel Theorem) Given any open cover $\{\mathcal{O}_\alpha\}$ for the closed and bounded interval $[a, b]$, there is a finite subcover (i.e., a finite subcollection of the sets in $\{\mathcal{O}_\alpha\}$ will include all points of the interval $[a, b]$).

Significance: References the Least Upper Bound Property and the definition of a finite subcover of an open interval to establish the basis of a closed and bounded interval

Definition 1.4.3 Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an **accumulation point** or **cluster point** of S if for every $\epsilon > 0$ there is a point $s \in S$ with $s \neq x$ such that $0 < |s - x| < \epsilon$.

Significance: Uses the Least Upper Bound property to establish a closed and bounded interval, the Bolzano-Weierstrass Theorem is built on the basis of the definition of an accumulation point

Theorem 1.4.4 (The Bolzano-Weierstrass Theorem) Every bounded infinite set of real numbers has an accumulation point.

Significance: References the bounded property of a set in $\mathbb{R}$ and the definition of an accumulation point to say that there is a significant point such that the set is dense with at least one other set member at that point. It can be built on to ensure that functions defined and continuous on intervals reach extremum points and values.

Lemma 1.4.5 If x is an accumulation point of S and $\epsilon > 0$, then there are an infinite number of points of S within ϵ of x.

Significance: Uses Bolzano-Weierstrass Theorem to extend the nature of a single accumulation point, which will help in the future of the definition of functions on intervals and defining extremum points.

Definition An **ordered pair** (a, b) is characterized by a set of two $\{a, b\}$ and an ordering. The a element is called "the first element" and the other is "the second."

Significance: Useful in representing input and output function values $(x, f(x))$. The (a, b) notation has otherwise been useful in the context of representing open intervals.

Chapter 2

Section 1

2.1.1 A **function** f is a collection of ordered pairs of elements such that if (a, b) and (a, c) are both in the collection, then $b = c$. That is, the collection cannot contain two different ordered pairs with the same first element.

Significance: References the idea of equality between two real numbers (i.e. Dedekind cuts) in a definition of a function, functions are used to also visually relate the mapping of an input to an output and are used throughout theoretical and applied mathematics

Definition The **domain** D_f of f is the set $\{x: (x, y) \in f$ for some y$\}$.
Definition The **range** R_f of f is the set $\{y: (x, y) \in f$ for some x$\}$.
Definition A function **sum** is the function $(f + g)(x) = \{(x, f(x) + g(x)): x \in D_f \cap D_g\}$.
Definition A function **product** is the function $(f \cdot g)(x) = f(x)g(x)$ with $D_{f \cdot g} = D_f \cap D_g$.
Definition A function **composition** is the function $(f \circ g)(x) = f(g(x))$ with $D_{f \circ g} = D_f \cap D_g$.

Section 2

2.2.1 Let f be a function defined at each point in some open interval containing a, except possibly at a itself. Then a number L is the **limit of f at a** if for ever number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

This is denoted by writing $\lim_{x \to a} f(x) = L$.

Significance: Utilizes the definition of absolute value and inequality, the limit is broken down into one-sided limits for the overall limit to exist, used in the definition of point continuity and extended to continuity on an open interval

Section 3

2.3.1 If $\lim_{x \to a} f(x)$ exists, then this limit is unique. That is, L and M are both limits of f at a, then $L = M$.

Significance: Important to establish so that in continuity, when one must assert that the value of a limit of a number equals the value of the function at that point. If the value wasn't unique, the limit could then be evaluated to equal multiple values, and determining continuity and differentiability would be that much harder.

2.3.2 (Sum Theorem) If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

a) $\lim_{x \to a} (f + g)(x) = L + M$;
b) $\lim_{x \to a} (f - g)(x) = L - M$.

Significance: Instead of working with epsilon delta proofs for combining limits from scratch every time, the sum theorem allows us to easily evaluate combinations.

2.3.3 (Constant Multiple Theorem) If $\lim_{x \to a} f(x) = L$, then for any constant c, $\lim_{x \to a} cf(x) = cL$.

Significance: Instead of working with epsilon delta proofs for combining limits from scratch every time, the sum theorem allows us to easily evaluate combinations

Theorem 2.3.4 (Product Theorem) Assume that $\lim_{x \to a} f(x) = L$ and that $\lim_{x \to a} g(x) = M$. Then $\lim_{x \to a} (f \cdot g)(x) = L\,M$.

Significance: Instead of working with epsilon delta proofs for combining limits from scratch every time, the sum theorem allows us to easily evaluate combinations

Corollary 2.3.5 If $p(x)$ is a polynomial, then $\lim_{x \to a} p(x) = p(a)$. That is, a polynomial has a limit at every point in a and this limit is just the value of the polynomial at that point.

Significance: Easily defines the value of a limit for any polynomial, eliminating the need to prove from scratch via epsilon-delta every time

Theorem 2.3.6 (Quotient Theorem) Assume that $\lim_{x \to a} f(x) = L$ and that $\lim_{x \to a} g(x) = M$ and $M \neq 0$. Then $\lim_{x \to a} (\frac{f}{g})(x) = \frac{L}{M}$.

Significance: Instead of working with epsilon delta proofs for combining limits from scratch every time, the sum theorem allows us to easily evaluate combinations

Section 4

Definition 2.4.1 Let f be defined on the open interval (a, c). The number L is the **limit of** $f(x)$ **as** x **approaches** a **from above** (or **from the right**) if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } 0 < x - a < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Significance: Introducing the idea of the one-sided limit from the right

Definition 2.4.2 Let f be defined on the open interval (b, a). The number L is the **limit of** $f(x)$ **as** x **approaches** a **from below** (or **from the left**) if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } -\delta < x - a < 0, \text{ then } |f(x) - L| < \epsilon.$$

Significance: Introducing the idea of the one-sided limit from the left

Theorem 2.4.3 Both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal if and only if $\lim_{x \to a} f(x)$ exists.

Significance: Defining the requirement of the existence of a limit based on the existence and equivalence of one sided limits at a point. Thus, a limit can be rendered illegitimate if both one sided limits DNE or are not equivalent.

2.4.4 Let f be defined on the interval $(a, +\infty)$. The number L is the **limit of** $f(x)$ **at** $+\infty$ if for every $\epsilon > 0$, there is a number N such that

$$\text{if } x > N, \text{ then } |f(x) - L| < \epsilon.$$

When this limit exists, we will write $\lim_{x \to +\infty} f(x) = L$.

2.4.5 Let f be defined on the interval $(-\infty, \boldsymbol{b})$. The number L is the **limit of** $f(x)$ **at** $-\infty$ if for every $\epsilon > 0$, there is a number N such that

$$\text{if } x < N, \text{ then } |f(x) - L| < \epsilon.$$

When this limit exists, we will write $\lim_{x \to -\infty} f(x) = L$.

Significance: This and the previous theorem (2.4.4) extend the definition of a limit to dealing with functions that approach infinite values, which becomes important when dealing with the definition of differentiability and limit combinations (i.e. quotient combination)

2.4.6 Assume that f is a function defined on the interval of the form $(a, +\infty)$. Let $y = {}^{1}/_{x}$. Then

$$\lim_{x \to +\infty} f(x) = L \iff \lim_{y \to 0^+} f\left({}^{1}/_{y}\right) = L.$$

The analogous result holds for limits as $x \to -\infty$.

Section 5

2.5.1 We say a function f is continuous at a point $p \in \mathbb{R}$ if

a) f is defined at p
b) $\lim_{x \to p} f(x) = f(p)$

Significance: References what it means for a function to be defined at a point, the definition of both the limit and one-sided limits existing and equaling the same value of $f(p)$. Defining continuity of a function at a single point is integral for then defining continuity on an interval. Continuity is then a required parameter for a large number of theorems and definitions.

2.5.2 Assume that f and g are both continuous at a. Then

a) $f + g$ is continuous at a;
b) cf is continuous at a for any constant c;
c) $f \cdot g$ is continuous at a;
d) if $g(a) \neq 0$, then f/g is continuous at a.

Significance: Developing simple rules for the combination of continuous functions saves a lot of trouble later when trying to prove the continuity of combined functions from scratch every time

Lemma A function f is said to be continuous on (a, b) if and only if f is continuous at each point in the open interval (a, b).

2.5.3 If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Significance: The continuity of function composition may seem simple, but it can be really helpful in breaking down complex functions – similar to the extreme usefulness of the chain rule with differentiation.

Definition A function f is said to be **right-continuous** at $p \in \mathbb{R}$ if $f(p)$ is defined and $\lim_{x \to p^+} f(x) = f(p)$.

Definition A function f is said to be **left-continuous** at $p \in \mathbb{R}$ if $f(p)$ is defined and $\lim_{x \to p^-} f(x) = f(p)$.

Significance: The one-sided continuity of a function at a specific point is similar to the usefulness of the one-sided limits. It allows one to break down the nature of the function as it approaches a specific point from a single direction. Both left and right continuity must hold for overall continuity to hold. Will be used in establishing continuity on a closed interval.

2.5.4 A function f is **continuous on the closed interval** $[a, b]$ if the following two conditions are satisfied:

i. f is continuous on the open interval (a, b);
ii. $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

or

iii. f is right-continuous at a and left-continuous at b.

Significance: Up until this point, continuity has only been defined on an open interval, but that becomes an issue when ideas arise such as intermediate value theorem, min-max theorem, and mean value theorem. Said theorems require the continuity of a function on a closed interval.

Proposition Suppose that function f is continuous on an **open** interval $\mathcal{O}$. If a **closed** interval $[a, b]$, with $a < b$, is contained in $\mathcal{O}$, then f is continuous on $[a, b]$.

Section 6

Theorem 2.6.1 (The Intermediate Value Theorem) Assume that f is continuous on the closed interval $[a, c]$ and let y be any real number such that

$$f(a) < y < f(c)$$

Then there exists at least one number b in the interval $[a, c]$ such that $f(b) = y$. (A similar theorem is obtained by replacing the hypothesis with $f(a) > y > f(c)$.)

Significance: Relies on the definition of continuity on a closed interval, inequality of numbers that can be traced back to inequalities of Dedekind cuts, combines with Bolzano-Weierstrass Theorem to say that if a continuous function has values of opposite signs that it has a root in that interval

Corollary 2.6.2 If n is an odd positive integer, then the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has a root.

Definition 2.6.3 A function f is **uniformly continuous on an interval I** if for every $\epsilon > 0$, there is a $\delta > 0$ such that if x and y are in I and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Significance: A stricter form of continuity that doesn't require a specific point $p \in I$, used in a variety of theorems and definitions such as the proof that every continuous function of a closed interval is Riemann integrable.

Theorem 2.6.4 If f is continuous on the closed, bounded interval $[a, b]$, then f is uniformly continuous there.

Significance: References both the definition of continuity and uniform continuity and links them on the condition of a closed, bounded interval, referenced in the Riemann-integrability of a non-decreasing function

Theorem 2.6.5 (The Boundedness Theorem) If f is continuous on the closed interval $[a, b]$, then f is bounded above, that is, there is some number M such that for all x in $[a, b]$, $f(x) \leq M$.

Significance: Establishes the significance of a bounded interval on a closed interval that is utilized as a precondition for many other theorems such as the least upper bound property and minimum and maximum point theorem of a function

Corollary 2.6.6 If f is continuous on the closed interval $[a, b]$, then f is bounded below; that is, there is some number m such that for all x in $[a, b]$, $m \leq f(x)$.

Significance: Utilizes the boundedness theorem to establish a reciprocal lower bound to an upper bound of a closed interval, later used in combination with the boundedness theorem to talk about what it means for a function to be defined on a closed and bounded interval

Definition 2.6.7 Let f be a function and let S be a subset of the domain of f. A point x in S is a **maximum point** for f on S if $f(x) \geq f(y)$ for every $y \in S$. The number $f(x)$ is called the **maximum value** of f on S. Similarly, a point x in S is a **minimum point** for f on S if $f(x) \leq f(y)$ for every $y \in S$, and $f(x)$ is called the **minimum value** of f on S. More generally, if x is either a maximum point or minimum point, x is called an **extreme point** of f on S and $f(x)$ is called an **extreme value** of f on S.

Significance: Uses Heine-Borel Theorem for the definition of a closed interval, Uses Bolzano-Weierstrass theorem to prove that $\max_{x \in X} f(x)$ exists when X is a closed, bounded subset of $\mathbb{R}$ and f is continuous. Likewise, the min exists.

Theorem 2.6.8 (The Max-Min Theorem) If f is continuous on the closed interval $[a, b]$, then there is a number x in $[a, b]$ such that $f(x_0) \geq f(x)$ for all x in $[a, b]$. That is, f achieves a maximum value on $[a, b]$. Similarly, there exists a number x in $[a, b]$ such that $f(x_1) \leq f(x)$ for all x in $[a, b]$. That is, f achieves a minimum value on $[a, b]$.

Significance: Uses Heine-Borel Theorem for the definition of a closed interval, utilizes the Boundedness Theorem and continuity to state that the theorem is defined at every point on the interval, the Bolzano-Weierstrass says that there is a convergent subsequence that converges to some $x \in I$, the limit to say that the value of the limit at that point x is equal to the function value at x, and the Archimedean property to say that the function actually achieves minimum and maximum values

Chapter 3

Section 1

Definition 3.1.1 Let f be a function defined in an open interval containing a. The function f is **differentiable at $\boldsymbol{a}$** if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

exists. When the limit does exist, it is denoted by $f'(x)$ and is called the **derivative of** f at a. If f is differentiable at every point in its domain, we say that f is **differentiable**.

Significance: Cites the definition of an open interval and the limit to give meaning to the idea of a derivative of a function at a point along with the value of said derivative, used in the definition of a critical points and henceforth extremum values of a function f,

Theorem 3.1.2 If f is differentiable at a, then f is continuous at a. Equivalently, if f is not continuous at a, then f is not differentiable at a.

Significance: Based on the definitions of both continuity and differentiability.

Theorem 3.1.3 Let f be defined on the open interval (a, b). If x is an extreme point of f on (a, b) and if f is differentiable at x, then $f'(x) = 0$. (Note: We do not assume that f is differentiable or even continuous at points of (a, b) other than x.)

Significance: Speaking about the nature of a function at a specific point (and no other point) on an interval, linking all of the previously defined nature of a function (i.e. extremum points and differentiability) and stating requirements so that the linkage does not have to be proved in subsequent theorems such as Mean Value Theorem/Rolle's Theorem and the derivative combinations.

Definition 3.1.4 The number x is a **critical point** of f if $f'(x)$ exists and equals 0. In this case the number $f(x)$ is called a **critical value** of f.

Significance: References the definition of a derivative of a function, utilizes the idea of an extreme point from the previous theorem and defines critical points and values of a function. Used in later in finding roots of functions and Rolle's theorem.

Theorem 3.1.5 If f is continuous on the closed, bounded interval $[a, b]$, then f has maximum and minimum points on $[a, b]$. Further, the only possible extreme points are:

a) critical points of f;
b) points where f' does not exist;
c) the endpoints of $[a, b]$.

Significance: References what it means for a function to be continuous on a closed interval, defines precise points on said interval where f could reach its minimum and maximum points – will be useful in future theorems such as min-max where minimum and maximum points are discussed.

Corollary 3.1.6 If f is continuous on $[a, b]$ and differentiable on (a, b), then f has maximum and minimum points and these occur either at critical points of f or at endpoints of $[a, b]$.

Significance: Now that both min-max points and critical points have been defined and discussed slightly, this forms a link between them both and will be used in future theorems such as Rolle's theorem

Theorem (Derivative at extremal points) Let f be defined on an open interval (a, b). If $x_0 \in (a, b)$ is an extreme point of f and if f is differentiable at x_0, then $f'(x_0) = 0$.

Significance: Relating previously defined extremum points and derivative values of a function on an open interval, which again is referenced in Rolle's theorem and the theorem talking about constant functions on an interval

Section 2

Theorem 3.2.1

a) If $f(x) = c$ is a constant function, then f is differentiable and $f'(x) = 0$.
b) If $f(x) = x$, then f is differentiable and $f'(x) = 1$.

Significance: Defining basic properties of the derivative of a function so that future proofs such as the sum, product, chain, and power rules do not have to prove facts (a) and (b) from scratch

Theorem 3.2.2 (The Sum Rule) Let f and g be functions differentiable at $x = a$. Then the sum of $f + g$ is differentiable at $x = a$ and

$$(f + g)'(a) = f'(a) + g'(x).$$

Theorem 3.2.3 (The Product Rule) Let f and g be functions differentiable at $x = a$. Then the product of $f \cdot g$ is differentiable at $x = a$ and

$$(f \cdot g)'(a) = f'(a) \cdot g(x) + f(x) \cdot g'(x).$$

Corollary 3.2.4 If function f is given by $f(x) = x^n$ for $n \in \mathbb{N}$, then for any $p \in \mathbb{R}$

$$f'(p) = np^{n-1}.$$

Lemma 3.2.5 If g is differentiable at a and $g(a) \neq 0$, then the function $r(x) = 1/g(x)$ is differentiable at a and

$$r'(a) = \frac{-g'(a)}{(g(a))^2}.$$

Significance: An extension of the power rule stated in Corollary 3.2.4 that makes division differentiation a little bit easier

Theorem 3.2.6 (The Quotient Rule) Suppose that f and g are both differentiable at a and that $g(a) \neq 0$. then the quotient f/g is differentiable at $x = a$ and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Theorem 3.2.7 (The Chain Rule) If g s differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and

$$(f \circ g)(a) = f'(g(a)) \cdot g'(\text{a})$$

Section 3

Theorem 3.3.1 (Rolle's Theorem) Assume that f is continuous on the closed and bounded interval $[a, b]$, differentiable on (a, b), and that $f(a) = f(\text{b}) = 0$. Then there is a point c strictly between a and b such that $f'(c) = 0$.

Significance: Specific case of the MVT stated first because of its relative simplicity and intuitiveness – set the stage for the more widely applicable MVT since it does not require $f(a) = f(\mathrm{b}) = 0$. We will see after the MVT that the two are equivalent.

Theorem 3.3.2 (The Mean Value Theorem) Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Significance: Uses Heine-Borel Theorem for significance of a closed interval, the Min-Max theorem, definition of continuity on a closed interval, definition of a function on an open interval, extends the usefulness of Rolle's Theorem (even though the two are equivalent)

Corollary 3.3.3 (The Generalized Mean Value Theorem) Suppose f and g are continuous on the closed interval $[a, b]$ and are differentiable on (a, b). Then there is a point c strictly between a and b such that

$$(\mathrm{g(b)} - \mathrm{g(a)})\, f'(c) = (f(b) - f(a))\, g'(c).$$

Significance: Extends the usefulness of the MVT to two functions instead of two points on the same function

Theorem 3.3.4 Suppose that f is continuous on an interval I and that $f'(x) = 0$ are all interior points x of I. Then f is constant on I.

Significance: Seeing as though constant functions are some of the simplest functions, this theorem establishes a simple and somewhat intuitive precedent that allows us to not prove constant functions from scratch every time.

Corollary 3.3.5 Assume f and g are continuous on an interval I and that for each interior point x of I, $f'(x) = g'(x)$. Then there is a constant c such that $f = g + c$.

Significance: Establishing a way to relate functions of different $f(x)$ values but the same derivative. Can be useful when functions appear to be totally different but are linked by derivative values.

Theorem 3.3.6 (L'Hôpital's Rule) Assume that f and g are differentiable on the open interval (a, b) and that

$$\lim_{x \to a^+} f(x) = 0 \text{ and } \lim_{x \to a^+} g(x) = 0$$

Further, assume that $f'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \to a^+} f'(x) / g'(x)$ exists, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Significance: L'Hôpital's Rule makes dealing with limits involving indeterminate forms much more pleasant. Applying L'Hôpital's Rule to a limit in indeterminate form 0/0 results in a limit in determinate form and easy evaluation of said limit through known proof techniques.

Corollary 3.3.7 Assume that f and g are differentiable on the open interval $(a, +\infty)$ and that

$$\lim_{x\to+\infty} f(x) = 0 \text{ and } \lim_{x\to+\infty} g(x) = 0$$

Further, assume that $f'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x\to+\infty} f'(x) / g'(x)$ exists, then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}.$$

An analogous result holds for limits as $x \to -\infty$.

Significance: Extends L'Hôpital's Rule of the previous theorem to indeterminate forms of 0/0 as limits approach infinity. Thus, L'Hôpital's Rule makes dealing with ∞ and limits much more pleasant.

Section 4

Definition 3.4.1 A **partition** P of the closed, bounded interval $[a, b]$ is a finite set of points $\{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

Significance: Definition of a partition used in the lower and upper sums definitions of partitions, which are then referenced in anything involving integralibility of functions

Definition 3.4.2 Let P=$\{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ and let f be a bounded function on $[a, b]$.

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \text{ and } M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

The **lower sum of f relative to the partition P** is

$$L(P, f) = \textstyle\sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

The **upper sum of f relative to the partition P** is

$$U(P, f) = \textstyle\sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Significance: The definition of Lower and Upper sums is referenced later to establish properties involving inequalities between the two, involved with infimum's and supremum's, used in pretty much everything involving integrals

Definition The **supremum** of a set S is another name for the least upper bound of S, denoted by $\sup(S)$.

Significance: References least upper bound, which notes the upper bound property of a non-empty set, related to infimums, utilized in combination of upper and lower sums of partitions to define properties about those partitions and define the Riemann Integrable nature of a function

Definition The **infimum** of a set S is another name for the greatest lower bound of S, denoted by $\inf(S)$.

Significance: References greatest lower bound, which notes the lower bound property of a non-empty set, related to supremums, utilized in combination of upper and lower sums of partitions to define properties about those partitions and define the Riemann Integrable nature of a function

Definition 3.4.3 Let P and Q be partitions of $[a, b]$. Q is said to be finer than P or a **refinement** of P if Q contains all the points of P.

Significance: Defining a refinement of a partition, which is later used in properties of upper and lower sums and infimums versus supremums, and ultimately the basis for a Riemann integral

Theorem 3.4.4 If Q is finer than P, then

$$L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f).$$

Significance: Defining the relationship between the upper and lower sum between a partition and is an integral part of the definition of a what it means for a function to be Riemann Integrable

Corollary 3.4.5 Let f be a bounded function on $[a, b]$ and let P and R be any two partitions of $[a, b]$. Then

$$L(P, f) \leq U(R, f).$$

Significance: Important idea of sums between two different partitions in setting up the definition of a Riemann Integrable function

Theorem 3.4.6 If Q_1 and Q_2 are any partitions of $[a, b]$, then

$$L(Q_1, f) \leq \sup_P\{L(P, f)\} \leq \inf_P\{U(P, f)\} \leq U(Q_2, f)$$

Significance: Identifying quantified relation properties of lower and upper sums and supremums and infimums of said sums to be used in definition of Riemann Integrable functions

Definition 3.4.7 A bounded function f on the interval $[a, b]$ is called **Riemann integrable** if

$$\sup_P\{L(P, f)\} = \inf_P\{U(P, f)\}.$$

Significance: References the bounded property of a finite set (in this case the set $\{f(x): x \in [a, b]\}$, references the supremum and infimum properties and implying that they exist, references the definitions lower and upper sums of partitions, establishes the basis of what it means to be integrable for later use.

Theorem If f is non-decreasing on $[a, b]$, then f is Riemann-integrable.

Definition 3.4.8 Assume that f is nonnegative and Riemann integrable on $[a, b]$. The **area** of the region bounded above by the graph $y = f(x)$ and below by the x-axis and which lies between the vertical lines $x = a$ and $x = b$ is defined to be $\int_a^b f$.

Significance: Extends the idea of a Riemann integrable and the infinitely fine partitions over a closed interval such that the sum of all of the areas of all of those regions represented by partitions has a value and is significant

Theorem 3.4.9 Assume that f is bounded on $[a, b]$. Then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition Q such that $U(Q, f) - L(Q, f) < \epsilon$.

Significance: Establishing the idea of the width of partitions becoming infinitely small that will be useful in integration and finding accurate values of the area under a curve

Section 5

Theorem 3.5.1 If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Significance: Utilizes the definition of continuity on a closed interval (which means that f is also uniformly continuous, f attains min and max points, and f is bounded) to establish a general condition of integrability of functions on closed intervals

Theorem 3.5.2 If f is non-decreasing on $[a, b]$, then f is Riemann integrable on $[a, b]$. (The same result holds for non-increasing functions)

Significance: Utilizes the definition of open intervals, continuity of functions on an interval, and the non-decreasing nature of a function f to establish a general rule for integrability of functions on a closed interval regardless of continuity