Math 325-001 - Fall 2016. Homework 1

Due: Solutions to the "individual" problems.

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1. Prove the second De Morgan's law for two sets $A, B \subset X$:

$$(A \cap B)^c = A^c \cup B^c$$

Solution.

You can prove it with truth tables, by starting with all the combinations of the truth-values of statements $x \in A$ and $x \in B$. Here's another proof without truth tables:

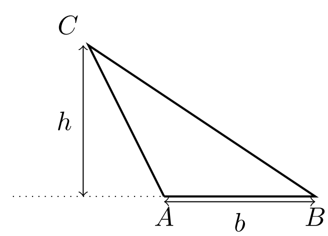
The statement $x \in (A \cap B)^c$ is equivalent to $\sim (x \in A \cap B)$. The latter is, in turn, equivalent to $\sim (x \in A \wedge x \in B)$. The AND clause is negated when at least one of the statements is false. Summarizing, we end up with

$$x \in (A \cap B)^c \quad \Leftrightarrow \quad x \notin A \vee x \notin B$$

And the right-most statement is equivalent to stating that $x \in A^c \vee x \in B^c$, or, equivalently, $x \in A^c \cup B^c$.

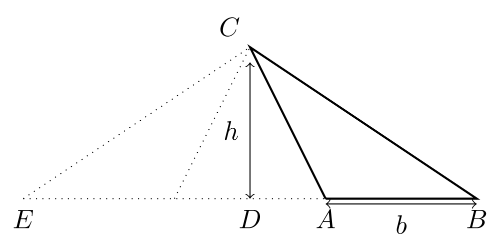
Note that **each** step in this derivation is equivalent to the next one, hence this argument says not only that $(A \cap B)^c \subset A^c \cup B^c$ but, in fact, that $x \in (A \cap B)^c$ if and only if $x \in A^c \cup B^c$. It follows that these sets are included into each other, hence are identical. $\square$

2. We proved that the area of a triangle with height h and base b is $\frac{1}{2}bh$. Only we didn't really; we only proved it for right and acute triangles. Now prove it for obtuse triangles (when the base is not the longest side), as shown on the figure:



Solution.

There are several ways to approach this problem. Here's one. Let "D" denote the point where the height falls on the dotted line. Then augment this picture by its reflection in the line CD:



Let d denote the length of DA. The length of the base of the big triangle is then twice $b+d$. By the case proved in class, when the height comes out from a vertex with obtuse angle, the area of the resulting triangle is

$$\text{Area } \Delta BCE = \frac{1}{2} \cdot [2(d+b)] \cdot h$$

and the area of the right triangle DAC is given by

$$\text{Area } \Delta DAC = \frac{1}{2} \cdot d \cdot h\,.$$

At the same time, the area that we are looking for can be expressed as:

$$\text{Area } \Delta ABC = \text{Area } \Delta BCE \cdot \frac{1}{2} - \text{Area } \Delta DAC.$$

Substitute into the last identity the two preceding ones to get:

$$\text{Area } \Delta ABC = \frac{1}{2} \cdot [2(d+b)] \cdot h \cdot \frac{1}{2} - \frac{1}{2} \cdot d \cdot h = \frac{1}{2}dh + \frac{1}{2}bh - \frac{1}{2}dh = \boxed{\frac{1}{2}bh}$$

which is what we wanted to prove. □

3. Do Problem 1 in Section 1.1, on page 9 of the textbook. (Pythagorean theorem).

   Solution.

   Following the illustration in that problem: the quadrilateral in the middle has all its sides equal to c, hence it is a square. Likewise, the largest quadrilateral in the figure is a square with side length $(a+b)$. Also note that the area of each of the four triangles on the picture is $ab/2$. Using this information we can express the area A of this entire figure in two different ways:

   $$A = (a+b)^2 \quad \text{and} \quad A = c^2 + 4 \cdot (ab/2)$$

   Hence,

   $$(a+b)^2 = c^2 + 2ab$$

   which is equivalent to

   $$a^2 + 2ab + b^2 = c^2 + 2ab\,.$$

   This last identity is in turn equivalent to

   $$a^2 + b^2 = c^2\,.$$

   □

4. Let $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ be the set of all integers. For any $n \in \mathbb{Z}$ define the set

   $$n\mathbb{Z} = \{n \cdot m : m \in \mathbb{Z}\}.$$

   Describe, as compactly as you can, the following:

   a) $2\mathbb{Z} \cap 7\mathbb{Z}$
      Answer. This is the intersection of even numbers with the numbers divisible by 7. Hence the resulting set consists of all the integers that are divisible by 14, which we could write as $14\mathbb{Z}$.

   b) $3\mathbb{Z} \cup 6\mathbb{Z}$
      Answer.
      This is the union of integers divisible by 3 and those divisible by 6. Note that if an integer is divisible by $6 = 2 \cdot 3$, then it is necessarily divisible by 3. Consequently $6\mathbb{Z} \subset 3\mathbb{Z}$. And when we take the union of these two sets we just end up with the bigger one:

      $$6\mathbb{Z}$$

   c) $6\mathbb{Z} \cap 15\mathbb{Z}$
      The numbers in $6\mathbb{Z}$ are divisible by 6. So they are divisible by each of the prime factors in 6, namely, by 2 and 3. Likewise, the numbers in $15\mathbb{Z}$ are precisely the integers divisible by 5 and 3. The common elements of these two sets are the integers divisible by $2, 3$ and 5. They are relatively prime and their product is 30. Thus, the set of integers we obtain is

      $$30\mathbb{Z}$$