Math 487 Homework 5

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Section 6.1

Ch 6.1 Q4 With 38 possible spots to place a bet on, we know that the probability of winning 35 dollars is 1/38, and the probability of losing a dollar is 37/38. Let X be the random variable which denotes your winnings on a 1 dollar bet in Las Vegas roulette. Then, the distribution of X is given by

$$m_X = \begin{pmatrix} -1 & 35 \\ 37/38 & 1/38 \end{pmatrix}.$$

Thus, we see then that $-1 \cdot (37/38) + 35 \cdot (1/38) = -1/19$.

Ch 6.1 Q17 Since X is the first time that a failure occurs in an infinite sequence, let's assume that the failure occurs after k trials. Then, the probability of a failure is $k-1$ multiplications of p and 1 multiplication of $1-p$. This is valid because $k-1+1=k$ for our k trials. Thus, for $q = 1-p$, we can formally see that

$$p_k = p^{k-1}(1-p) = p^{k-1}q.$$

Then, we can show $\sum_k p_k = 1$. Since $k = 1, 2, \ldots$, the limits of the summation are originally $k = 1$ to ∞.

$$\begin{aligned}
\sum_k p_k &= \sum_{k=1}^{\infty} p^{k-1} q \\
&= q \sum_{k=0}^{\infty} p^k \\
&= q \Big(\frac{1}{1-p} \Big) \\
&= q \frac{1}{q} \\
&= 1.
\end{aligned}$$

In this case, $p_k = p^{k-1}q$ for $k = 1, 2, \ldots$. So,

$$\begin{aligned} \mathrm{E}(X) &= 1q + 2pq + 3p^2q + \ldots \\ &= q(1 + 2p + 3p^2 + \ldots). \end{aligned}$$

Then, we can see that the infinite series $1 + p + p^2 + \ldots$, which is equivalent to $\frac{1}{1-p}$, is embedded in the previous equation. By differentiating this, we get $1 + 2p + 3p^2 + \ldots$, which is the term we have above and is equivalent to $\frac{1}{(1-p)^2}$. By substituting into the expanded equation of $\mathrm{E}(X)$, we get

$$\begin{aligned} \mathrm{E}(X) &= q(1 + 2p + 3p^2 + \ldots) \\ &= q\frac{1}{(1-p)^2} \\ &= q\left(\frac{1}{q^2}\right) \\ &= \frac{1}{q}. \end{aligned}$$

Thus, the expected value of the first tail of a fair coin is $\frac{1}{q} = \frac{1}{\frac{1}{2}} = 2$.

Ch 6.1 Q18 To compute the expected value, $\mathrm{E}(X)$ for some random variable X, we can use the definition of $\mathrm{E}(X)$ using a sum and a distribution function,

$$\begin{aligned} \mathrm{E}(X) &= \sum_{k=1}^{6} k \cdot \frac{1}{6} \\ &= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} \\ &= \frac{7}{2}. \end{aligned}$$

Ch 6.1 Q21 To compute the expected value of X, we need to note that the distribution function of X is $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$. Then,

$$\begin{aligned}
\mathrm{E}(X) = \sum_{x=0}^{\infty} x \cdot p(x) &= \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}
\end{aligned}$$

Now, we see that the denominator yields 0 when $x = 1$, so we can set $k = x - 1$ and adjust the summation,

$$\mathrm{E}(X) = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!}$$

Now, we see that $\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = 1 + x + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$, which is equivalent to e^{λ}. By substituting into $\mathrm{E}(X)$, we get

$$\mathrm{E}(X) = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

Section 6.2

Ch 6.2 Q2 A random variable X has the distribution

$$p_x = \begin{pmatrix} 0 & 1 & 2 & 4 \\ 1/3 & 1/3 & 1/6 & 1/6 \end{pmatrix}$$

Find the expected value, variance, and standard deviation of X.

$$\begin{aligned}
\mathrm{E}(X) &= 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} \\
&= \frac{1}{3} + \frac{2}{6} + \frac{4}{6} \\
&= \frac{4}{3}.
\end{aligned}$$

$$
\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}(X^2) - \left(\frac{4}{3}\right)^2 \\
&= \left[0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6}\right] - \frac{16}{9} \\
&= \left[\frac{11}{3}\right] - \frac{16}{9} \\
&= \frac{17}{9}.
\end{aligned}
$$

$$
\mathrm{D}(X) = \sqrt{\mathrm{V}(X)} = \sqrt{\frac{17}{9}} = \frac{\sqrt{17}}{3}.
$$

Ch 6.2 Q4 X is a random variable with $\mathrm{E}(X) = 100$ and $\mathrm{V}(X) = 15$. Find

(a) $\mathrm{E}(X^2)$

We can rearrange $\mathrm{V}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$ for $\mathrm{E}(X^2)$, which yields $\mathrm{V}(X) + \mathrm{E}(X)^2$. Plugging in values, we see that $\mathrm{E}(X^2) = 10015$.

(b) $\mathrm{E}(3X + 10)$

We can use the fact that $\mathrm{E}(cX) = c\mathrm{E}(X)$. So, $\mathrm{E}(3X+10)$ is equivalent to $3\mathrm{E}(X)+10$. Thus, $\mathrm{E}(3X + 10) = 310$.

(c) $\mathrm{E}(-X)$

We can use the fact that $\mathrm{E}(cX) = c\mathrm{E}(X)$. So, $\mathrm{E}(-X)$ is equivalent to $-1\mathrm{E}(X)$. Thus, $\mathrm{E}(-X) = -100$.

(d) $\mathrm{V}(-X)$

We can use the fact that $V(cX) = c^2V(X)$. So, $\mathrm{V}(-X) = -1^2\mathrm{V}(X)$. Thus, $\mathrm{V}(-X) = 15$.

(e) $\mathrm{D}(-X)$

We can use the previous result that said $\mathrm{V}(-X) = 15$ since $\mathrm{D}(-X) = \sqrt{\mathrm{V}(-X)}$. Thus, $\mathrm{D}(-X) = \sqrt{15}$.

Ch 6.2 Q5 In a certain manufacturing process, the (Fahrenheit) temperature never varies by more than 2° from 62°. The temperature is, in fact, a random variable F with distribution

$$
P_F = \begin{pmatrix} 60 & 61 & 62 & 63 & 64 \\ 1/10 & 2/10 & 4/10 & 2/10 & 1/10 \end{pmatrix}.
$$

(a) Find E(F) and V(F).

$$\mathrm{E}(F) = 60 \cdot \frac{1}{10} + 61 \cdot \frac{2}{10} + 62 \cdot \frac{4}{10} + 63 \cdot \frac{2}{10} + 64 \cdot \frac{1}{10} = 62.$$

$$\mathrm{V}(F) = \left[60^2 \cdot \frac{1}{10} + 61^2 \cdot \frac{2}{10} + 62^2 \cdot \frac{4}{10} + 63^2 \cdot \frac{2}{10} + 64^2 \cdot \frac{1}{10}\right] - 62^2 = 1.2.$$

(b) Define $T = F - 62$. Find E(T) and V(T), and compare these answers with those in part (a).

We know that $\mathrm{E}(X + a) = \mathrm{E}(X) + a$, so $\mathrm{E}(T) = \mathrm{E}(F) - 62$. Thus, $\mathrm{E}(T) = 0$. Additionally, we know that $\mathrm{V}(X + a) = \mathrm{V}(X)$, so $\mathrm{V}(T) = \mathrm{V}(F)$. Thus, $\mathrm{V}(T) = 1.2$.

(c) It is decided to report the temperature readings on a Celsius scale, that is, $C = (5/9)(F - 32)$. What is the expected value and variance for the readings now?

We know that $\mathrm{E}(X + a) = \mathrm{E}(X) + a$, so $\mathrm{E}(C) = \frac{5}{9}\Big(\mathrm{E}(F) - 32\Big)$. Thus, $\mathrm{E}(C) = \frac{50}{3}$. Additionally, we know that $\mathrm{V}(cX + a) = c^2\mathrm{V}(X)$, so $\mathrm{V}(C) = \frac{5}{9}^2 \cdot \mathrm{V}(F)$. Thus, $\mathrm{V}(C) = \frac{5}{9}^2 \cdot \frac{6}{5} = \frac{10}{27}$.

Ch 6.2 Q9 A die is loaded so that the probability of a face coming up is proportional to the number on that face. The die is rolled with outcome X. Find V(X) and D(T).

The total of all faces on a die is 21. Thus, the probability of rolling a 1 is 1/21, rolling a 2 is 2/21, and so on. Thus, we can compute expected value using the summation definition.

$$\sum_{x=1}^{6} x \cdot \frac{x}{21} = \frac{13}{3}.$$

Then, $\mathrm{V}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$. So,

$$\begin{aligned}\mathrm{V}(X) &= \left(\sum_{x=1}^{6} x^2 \cdot \frac{x}{21}\right) - \frac{13}{3}^2 \\ &= 21 - \frac{13}{3}^2 = \frac{20}{9}.\end{aligned}$$

Lastly, $\mathrm{D}(X) = \sqrt{\mathrm{V}(X)} = \sqrt{\frac{20}{9}} = \frac{\sqrt{20}}{3}$.

Ch 6.2 Q28 In Example 5.3, assume that the book in question has 1000 pages. Let X be the number of pages with no mistakes. Show that $\mathrm{E}(X) = 905$ and $\mathrm{V}(X) = 86$. Using these results, show that the probability is $\leq .05$ that there will be more than 924 pages without errors or fewer than 866 pages without errors.

The likelihood that there are no mistakes on a page can is given by $\frac{e^{-0.1}0.1^0}{0!} = e^{-.1} = 0.9048374$. Since the likelihood of a mistake (or lack thereof) on a page is independent of other pages, we can multiply this probability by 1000, which represents the number of pages in the book, to find the expected value of pages without mistakes. So, $\mathrm{E}(X) = 1000 \cdot e^{-.1} = 905$ pages.

Per the usual definition, we know $\mathrm{V}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$. However, since we know that a mistake on the i^{th} page does not affect a mistake on the $i + 1^{\text{th}}$ page, we can reduce the equation to

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}(X) - \mathrm{E}(X)^2 \\
&= \left[\frac{e^{-0.1}0.1^0}{0!} - \left(\frac{e^{-0.1}0.1^0}{0!}\right)^2\right] \\
&= \frac{e^{-0.1}0.1^0}{0!}\left(1 - \frac{e^{-0.1}0.1^0}{0!}\right) \\
&= e^{-0.1}\left(1 - e^{-0.1}\right) \\
&\sim 0.086107
\end{aligned}$$

So, for total variance see that $\mathrm{V}(X) = 1000 \cdot 0.086107 = 86$.

We can use the cumulative distribution function for a binomial distribution to show $\mathbb{P}(X > 924) \leq .05$ and $\mathbb{P}(X < 866) \leq .05$. We see that $F(924; 0.9048374, 1000) \sim 0.985$, so the probability of more than 924 pages is approximately .015. Additionally, $F(866; 0.9048374, 1000) \sim 4.209053 \times 10^{-5}$, and this is obviously less than .05.

Ch 6.2 Q29 Let X be Poisson distributed with parameter λ. Show that $\mathrm{V}(X) = \lambda$.

To find $\mathrm{V}(X)$ of the Poisson distributed random variable X with parameter λ, we can use the equation

$$\mathrm{V}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2.$$

To find $\mathrm{E}(X^2)$, we need a second order value of X, but there is no i, j where $i \neq j$ such that $X = x_i = x_j$. So, $\mathrm{E}(X^2) = \mathrm{E}(X(X-1) + X)$.

$$
\begin{aligned}
\mathrm{E}(X(X-1)+X) &= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} \\
&= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \\
&= \lambda^2 e^{-\lambda} e^{-\lambda} = \lambda^2
\end{aligned}
$$

Thus, we can substitute values and we see that $\mathrm{V}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.