Suppose that A, B, C are sets satisfying

$$(1)\ A \cap B = A \cap C \text{ and } (2)\ A \cup B = A \cup C$$

Prove that $B = C$

PROOF

(1) For every $x \in A \cap B$, $x \in A$ and $x \in B$ simultaneously. Since $A \cap B = A \cap C$, every $x \in A$ must also be contained in C simultaneously. There is no such x where $x \in A \cap B$ and $x \notin A \cap C$ and vice versa. By rule of intersection, no such x exists such that $x \in B$ and $x \notin C$ and vice versa.

(2) Assume for every $x \in A \cup B$ that $x \notin A$. x must therefore explicitly be contained in B. Since $A \cup B = A \cup C$ and $x \notin A$, x must also explicitly be contained in C. Therefore no such x exists that $x \in B$ and $x \notin C$ and vice versa.

CONSLUSION

Statement (1) showed that every $x \in A \cap B$ and $x \in A \cap C$ must mean that every $x \in B$ and $x \in C$ at the same time. Additionally, statement (2) showed that every $x \in A \cup B$ and $x \in A \cup C$ simultaneously. By considering both statements to be true, there can not exist an $x \in B$ where $x \notin C$. Hence, the definition of subsets and the equality of sets B and C.

Show that assuming only one of the conditions in the part I is not sufficient to prove $B = C$

(1) Only assuming that $A \cap B = A \cap C$ is not sufficient because there could exist an $x \in B$ and $x \notin C$ because $A \cap B$ and $A \cap C$ are not said to be equal.

(2) Only assuming that $A \cup B = A \cup C$ is not sufficient because there could exist an $x \in A \cap B$ and $x \notin A \cap C$. $A \cup B$ and $A \cup C$ do not elimate this possibility independently.