Math 487 Homework 4

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October 20, 2017

Ch5.2 Q1 Choose a number U from the unit interval [0, 1] with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = U + 2$.

Since U is uniformly distributed on the unit interval [0, 1], we can redefine the limits in terms of Y where $U = Y - 2$,

$$0 \leq Y - 2 \leq 1,$$
$$2 \leq Y \leq 3.$$

We can now find the probability density function, $f_U(x)$ by $\frac{du}{dy}Y - 2$. We then see that $f_U = 1$. The cumulative distribution function $F_U(x)$ is then found by

$$\int f_U \; dx = \int 1 \; dx$$
$$= x + c.$$

To ensure that $F_U(x)$ is in the unit interval [0, 1], we set $c = -2$ and thus see that $F_U(x) = x - 2$.

(b) $Y = U^3$.

Since U is uniformly distributed on the unit interval [0, 1], we can redefine the limits in terms of Y where $U = \sqrt[3]{Y}$,

$$0 \leq \sqrt[3]{Y} \leq 1,$$
$$0 \leq Y \leq 1.$$

We can now find the probability density function, $f_U(x)$ by $\frac{du}{dy}\sqrt[3]{Y}$. We then see that $f_U = \frac{1}{3}y^{\frac{-2}{3}}$. The cumulative distribution function $F_U(x)$ is then found by

$$\begin{aligned}\int f_U \; dx &= \int \frac{1}{3} y^{\frac{-2}{3}} \; dx \\ &= x^{\frac{1}{3}} + c.\end{aligned}$$

To ensure that $F_U(x)$ is in the unit interval [0, 1], we set $c = 0$ and thus see that $F_U(x) = x^{\frac{1}{3}}$.

Ch5.2 Q2 Choose a number U from the interval [0, 1] with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = 1/(U+1)$.

Since U is uniformly distributed on the unit interval [0, 1], we can redefine the limits in terms of Y where $U = \frac{1}{Y} - 1$,

$$\begin{gathered}0 \leq \tfrac{1}{Y} - 1 \leq 1, \\ 1 \leq \tfrac{1}{Y} \leq 2, \\ 1 \leq Y \leq \tfrac{1}{2}.\end{gathered}$$

We can now find the probability density function, $f_U(x)$ by $\frac{du}{dy}\frac{1}{Y} - 1$. We then see that $f_U = -\frac{1}{y^2}$. The cumulative distribution function $F_U(x)$ is then found by

$$\begin{aligned}\int f_U \; dx &= \int_{\frac{1}{2}}^{1} -\frac{1}{y^2} \; dx \\ &= \left[\frac{1}{y} + c\right]_{\frac{1}{2}}^{1} \\ &= 1.\end{aligned}$$

To ensure that $F_U(x)$ is in the unit interval [0, 1], we set $c = 2$ and thus see that $F_U(x) = 2 - \frac{1}{y^2}$.

(b) $Y = \log(U+1)$.

Since U is uniformly distributed on the unit interval [0, 1], we can redefine the limits in terms of Y where $U = e^Y - 1$,

$$\begin{gathered}0 \leq e^Y - 1 \leq 1, \\ 1 \leq e^Y \leq 2, \\ 0 \leq Y \leq \ln(2).\end{gathered}$$

We can now find the probability density function, $f_U(x)$ by $\frac{du}{dy}e^Y - 1$. We then see that $f_U = e^Y$. The cumulative distribution function $F_U(x)$ is then found by

$$\begin{aligned}\int f_U \; dx &= \int e^Y \; dx, \\ &= e^Y + c \; dx.\end{aligned}$$

To ensure that $F_U(x)$ is in the unit interval [0, 1], we set $c = -1$ and thus see that $F_U(x) = e^Y - 1$.

Ch5.2 Q10 Let U, V be random numbers chosen independently from the interval $[0, 1]$. Find the cumulative distribution and density for the random variables

(a) $Y = \max(U, V)$.

Clearly, for the CDF of X, we know $F_X(x) = \mathbb{P}(X \leq x)$. Additionally, since U and V are independent and uniformly distributed, we know that $F_U(u) = \mathbb{P}(U \leq x)$ and $F_V(v) = \mathbb{P}(V \leq x)$. So,

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(U \leq x, V \leq x) = \mathbb{P}(U \leq x) \cdot \mathbb{P}(V \leq x) = x^2.$$

Thus, considering the [0,1] range for normally distributed random variables, we can define the cumulative distribution functions and probability density functions as follows,

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Similarly, the probability density functions is defined as follows after differentiating the cumulative distribution function,

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(b) $Y = \min(U, V)$.

Clearly, for the CDF of Y, we know $F_Y(y) = \mathbb{P}(Y \leq y)$. Additionally, since U and V are independent and uniformly distributed, we know that $F_U(u) = 1 - \mathbb{P}(U \leq y)$ and $F_V(v) = 1 - \mathbb{P}(V \leq y)$. So,

$$F_Y(y) = 1 - \mathbb{P}(XY \leq y) = 1 - \mathbb{P}(U \leq y, V \leq y) = 1 - \mathbb{P}(U \leq y) \cdot \mathbb{P}(V \leq y) = 1 - (1-y)^2.$$

Thus, considering the [0,1] range for normally distributed random variables, we can define the cumulative distribution functions and probability density functions as follows,

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - (1-y)^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Similarly, the probability density functions is defined as follows after differentiating the cumulative distribution function,

$$f_X(x) = \begin{cases} 2(1-y) & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Ch5.2 Q16 Let X be a random variable with density function

$$f_X(x) = \begin{cases} cx(1-x) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of c?

To find c, we will integrate the $f_X(x)$ function with respect to x and evaluate the integral from 0 to 1,

$$\begin{aligned} \int cx(1-x)\ dx &= c\int x - x^2\ dx \\ &= c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\ &= 1 \end{aligned}$$

Thus, we see that $c = 6$.

(b) What is the cumulative distribution function F_X for X?

Based on the results from the previous part, we know $F_X = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) = 3x^2 - 2x^3$.

(c) What is the probability that $X < 1/4$?

To find $\mathbb{P}(X < 1/4)$, we can just plug 1/4 into our cumulative distribution function, $F_X(1/4) = 3(1/4)^2 - 2(1/4)^3 = .15625$.

Ch5.2 Q17 Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \sin^2(\pi x/2) & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 < x. \end{cases}$$

(a) What is the density function f_X for X?

The density function f_X can be calculated by

$$\frac{d}{dy} F_X(x) = f_X(x) = \begin{cases} \pi \sin(\pi x/2) \cos(\pi x/2) & \text{if } 0 \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$
$$= \begin{cases} \frac{\pi}{2} \sin(\pi x) & \text{if } 0 \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(b) What is the probability that $X < 1/4$? To find $\mathbb{P}(X < 1/4)$, we can just plug $1/4$ into our cumulative distribution function, $F_X(1/4) = \sin^2(\pi/8) = .146447$.

Ch5.2 Q18 Let X be a random variable with cumulative distribution function F_X, and let $Y = X + b$, $Z = aX$, and $W = aX + b$, where a and b are any constants. Find the cumulative distribution functions F_Y, F_Z, and F_W. *Hint*: The cases $a > 0$, $a = 0$, and $a < 0$ require different arguments.

By the properties of cumulative distribution functions, we know that they are non-decreasing. That leads us to the following three cases: $a > 0$, $a = 0$, and $a < 0$. We can also define $F_Y(y)$, $F_Z(z)$, and $F_W(w)$ in terms of $F_X(x)$,

(i) $F_Y(y)$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X + b \le y) \\ &= F_X(y - b) \end{aligned}$$

Since there are no restrictions on a here, $F_Y(y) = F_X(y - b)$.

(ii) $F_Z(z)$

$$\begin{aligned} F_Z(z) &= \mathbb{P}(aX \le z) \\ &= F_X\left(\frac{z}{a}\right) \end{aligned}$$

So, we see the form of $F_Z(z)$ but we need to consider the cases when $a > 0$, $a = 0$, and $a < 0$. Furthermore, we know that $F_X(x)$ is strictly increasing when $a > 0$ and is strictly decreasing when $a < 0$.

$$F_Z(z) = \begin{cases} F_X\left(\frac{z}{a}\right) & \text{if } a > 0 \\ 1 - F_X\left(\frac{z}{a}\right) & \text{if } a < 0, \\ 1 & \text{if } a = 0. \end{cases}$$

(iii) $F_W(w)$

$$\begin{aligned} F_W(w) &= \mathbb{P}(aX + b) \\ &= F_X\left(\frac{x+b}{a}\right) \end{aligned}$$

So, we see the form of $F_W(w)$ but we need to consider the cases when $a > 0$, $a = 0$, and $a < 0$. Furthermore, we know that $F_X(x)$ is strictly increasing when $a > 0$ and is strictly decreasing when $a < 0$.

$$F_W(w) = \begin{cases} F_X\left(\frac{x+b}{a}\right) & \text{if } a > 0 \\ 1 - F_X\left(\frac{x+b}{a}\right) & \text{if } a < 0, \\ \begin{cases} 1 & \text{if } w \geq b, \\ 0 & \text{elsewhere.} \end{cases} & \text{if } a = 0. \end{cases}$$

Ch5.2 Q19 Let X be a random variable with density function f_X, and let $Y = X + b$, $Z = aX$, and $W = aX + b$, where $a \neq 0$. Find the density functions f_Y , f_Z, and f_W. (See Exercise 18.)

(i) $f_Y(y)$

Since Y is X with a positive shift by b without any influence of the value of a, we know the density of Y at y is that of X at $x - b$,

$$f_Y(y) = f_X(y - b).$$

(ii) $f_Z(z)$

Since Z is X with scaled by a, we know the density of Z at z is $|\frac{1}{a}|$ that of X at $\frac{z}{a}$,

$$f_Z(z) = \begin{cases} \frac{1}{|a|} f_X\left(\frac{z}{a}\right) & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

(iii) $f_W(w)$

Since W is X with scaled by a and shifted by b, we know the density of Z at z is $|\frac{1}{a}|$ that of X at $aX+b$,

$$f_W(w) = \begin{cases} \frac{1}{|a|} f_X\left(\frac{w-b}{a}\right) & \text{if } a \neq 0, \\ 0 & \text{if } a = 0 \text{ and } w \neq 0. \end{cases}$$

Ch5.2 Q23 Let X be a random variable with density function f_X. The mean of X is the value $\mu = \int x f_x(x)dx$. Then μ gives an average value for X (see Section 6.3). Find μ if X is distributed uniformly, normally, or exponentially, as in Exercise 22.

Based on Exercise 22, we will find the value of $\mu = \int x f_x(x)dx$ with the following distributions:

(i) uniformly distributed over the interval [a, b].

$$\begin{aligned} \mu = \int x f_x(x)dx &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b \\ &= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] \\ &= \frac{b+a}{2} \end{aligned}$$

(ii) normally distributed with parameters μ and ρ.

$$\begin{aligned} \mu = \int x f_x(x)dx &= \int_{-\infty}^{\infty} x \frac{e^{\frac{-(x-\mu)^2}{(2\sigma)^2}}}{\sigma\sqrt{2\pi}} dx \\ &= \mu \end{aligned}$$

(iii) exponentially distributed with parameter λ.

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\begin{aligned}
\mu = \int x f_x(x) dx &= \lambda \int_0^\infty x e^{-\lambda x} dx \\
&= \lambda \Big[(\frac{-x}{\lambda} - \frac{1}{\lambda^2}) e^{-\lambda x} \Big]_0^\infty \\
&= \lambda \Big[\frac{1}{\lambda^2} \Big] \\
&= \frac{1}{\lambda}
\end{aligned}
$$