Math 325-001 - Fall 2016. Homework 6

Due: Friday, Nov 18, in class. [Solution to 6B provided below]

Do not start the group problem until you've received an email from me. If you don't have a partner, let me know immediately.

- *STAPLE!*
- *NO fuzzy edges on pages improperly torn from spiral binder notebooks!*
- *Typing is not mandatory, but highly encouraged if you have a suitable typesetting system at your disposal.*
- *If it's not legible, it's not there.*

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1. Suppose f is continuous at x_0 and $f(x_0) < M$. Define $d = |M - f(x_0)|$ (same as $M - f(x_0)$ in this case). Prove that there is $\delta > 0$, such that for any $x \in (x_0 - \delta, x_0 + \delta)$ we have $f(x) < M - \frac{3}{4}d$.
2. Prove that if a function f is **uniformly continuous** on an open interval (a, b), then f is **continuous** on (a, b). *Hint: let p denote an arbitrary point in (a, b) and proceed to show that f is continuous at p by checking the requirements of the definition of continuity.*
3. Prove that for any numbers $m, b \in \mathbb{R}$, the function given by $f(x) = mx + b$ is uniformly continuous on $\mathbb{R}$. *Hint: it may help to consider the case $m = 0$ separately.*
4. Prove that $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$ (recall the problem 1.3.17d on page 36).
5. The following facts about trigonometric functions are well known (and can be established by geometric arguments): for any $\alpha, \beta \in \mathbb{R}$
$$|\sin(\beta)| \leq |\beta|$$
and
$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right) \cdot \sin\left(\frac{\alpha - \beta}{2}\right)$$
   (a) Use these results to show that
$$|\sin(\alpha) - \sin(\beta)| \leq |\alpha - \beta|$$
   (b) Prove that $\sin(x)$ is uniformly continuous on $(-\infty, \infty)$.

> This group assignment, question #5, has two problems. **I will provide you via email a complete solution to ONE of these problems. And you will be writing a solution to the OTHER one.**
> Give the solution that you wrote to your partner. Get feedback from them and prepare a revision. Turn in all those as you did in previous assignments.
> Please **DO NOT SHARE THE SOLUTION THAT I PROVIDE** with your partner and **DO NOT GIVE THEM HINTS till the revision**. My solution is meant to help you write the feedback. Your **FEEDBACK MUST BE DETAILED.** Your job now is to point out all errors and omissions, if any, in your partner's draft solution.

6(A) Suppose functions f and g are uniformly continuous on $\mathbb{R}$. Prove that the composition $f \circ g$ is uniformly continuous on $\mathbb{R}$.

Hint: to make $|f(g(x)) - f(g(y))|$ small, you need to ensure that $g(x)$ and $g(y)$ are sufficiently close. . .

6(B) Prove that if functions f and g are uniformly continuous on $\mathbb{R}$ and bounded, then $f \cdot g$ is uniformly continuous on $\mathbb{R}$.

Hint: look back at how we proved that the limit of a product amounts to the product of the limits (if they exist). Start with the difference $|f(x)g(x) - f(y)g(y)|$, then add and subtract the term $f(x)g(y)$ inside it. Also recall that if a function f, for example, is bounded then we can find a number M_f (that we can choose to be positive) such that $|f(x)| < M_f$ for any x in the function's domain.

Proof. Because f and g are bounded on $\mathbb{R}$, then we know there exist positive numbers M_f and M_g such that $|f(x)| \leq M_f$ and $|g(x)| \leq M_g$ for all $x \in \mathbb{R}$. Let's set $M = \max\{M_f, M_g\}$. Take note of the following estimate which holds for any $x, y \in \mathbb{R}$:

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| =& |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\ \leq& |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \\ =& |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \\ \leq& M|f(x) - f(y)| + M|g(x) - g(y)| \end{aligned} \tag{1}$$

Now let $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2M}$ (or any two positive numbers whose sum is less than ε/M). Then by the uniform continuity of f and g on $\mathbb{R}$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for any two real numbers x, y satisfying $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon_1$, and $|x - y| < \delta_2$ implies $|g(x) - g(y)| < \varepsilon_2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$ (note that δ is positive, since each of δ_1 and δ_2 is). Then according to estimate (1), for any $x, y \in \mathbb{R}$, the condition $|x - y| < \delta$ implies that

$$|f(x)g(x) - f(y)g(y)| < M\frac{\varepsilon_1}{2} + M\frac{\varepsilon_2}{2} = M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since such a $\delta > 0$ can be found for any given $\varepsilon > 0$, then the product function $f \cdot g$ is uniformly continuous on $\mathbb{R}$.