MATH 310 Spring 2017 **Homework #1** **Due Wednesday, January 18**

Please indicate how much time you spent on this assignment and give the name(s) of anyone else you worked with.

1. For each of the following statements, figure out what it means and decide whether it is true, false, or neither.

   (a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : y > x$.

   (b) $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z}, y > x$.

   (c) $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z}, y \geq x$.

   (d) $x > 0 \implies x < 1$.

2. Negate the following statements without using any negative words ("no", "not", "neither ... nor", etc.). Try to make your negation sound as much like normal English as possible.

   (a) Every word on this page starts with a consonant or ends with a vowel.

   (b) There is a book on this shelf in which every page has a word that starts and ends with a vowel.

3. Let $A = \{1, 2, \{3, 4\}, \{5\}\}$. Decide whether each of the following statements is true or false: (*Hint:* There are exactly six true statements.)

$$\begin{array}{rrrr} 1 \in A, & \{1,2\} \in A, & \{1,2\} \subseteq A, & \emptyset \in A, \\ 3 \in A, & \{3,4\} \in A, & \{3,4\} \subseteq A, & \emptyset \subseteq A, \\ \{1\} \in A, & \{1\} \subseteq A, & \{5\} \in A, & \{5\} \subseteq A. \end{array}$$

4. Suppose the only set notation we know is $\in$, $\notin$, and $\{x \in U : P(x)\}$. Let A and B be subsets of U. You may also use the mathematical logical symbols for "and" and "or" to combine statements. These symbols are $\wedge$ (and) and $\vee$ (or).

   (a) Define the sets $A \cap B$ and $A \cup B$ using only the available notation.

      - $A \cap B =$
      - $A \cup B =$

   (b) Write down propositions equivalent to $A \subseteq B$ and $A \subsetneq B$ using only the available set and logical notation.

      - $A \subseteq B$ can be expressed:
      - $A \subsetneq B$ can be expressed:

5. Suppose $A = \{x \in U : P(x)\}$ and $B = \{x \in U : Q(x)\}$. Write down the following propositions without referring to A or B at all:

   - $A \subseteq B$ can be expressed:
   - $A \subsetneq B$ can be expressed:
   - $A = B$ can be expressed:
   - $A \neq B$ can be expressed:

6. For each of the following propositions, write down both the proposition and its negation in mathematical notation, using no English words:
   (a) There is no largest natural number.
   (b) Every subset of the natural numbers has a smallest element. (This proposition is called the "well-ordering axiom for the natural numbers.")

7. In this problem, let

$$\begin{gathered} A = \{x \in \mathbb{Z} : x = 2k \text{ for some } k \in \mathbb{Z}\}, \\ B = \{x \in \mathbb{Z} : x = 4k \text{ for some } k \in \mathbb{Z}\}, \\ C = \{x \in \mathbb{Z} : \text{the units digit of } x \text{ is } 0, 2, 4, 6, 8\}, \\ D = \{x \in \mathbb{Z} : x^2 = x\}, \\ E = \{1\}. \end{gathered}$$

   For each of the following statements:
   (i) Determine whether it is true or false.
   (ii) If it is false, determine whether reversing the implication arrow would make the statement true. If it is true, determine whether reversing the implication arrow would make another true statement or if it would make a false statement.

   You do not need to provide a full explanation of your thinking for this problem. It is enough to write a sentence or two capturing the main idea.
   (a) $x \in A \Longrightarrow x \in B$
   (b) $x \in B \Longrightarrow x \in A$
   (c) $x \in C \Longrightarrow x \in B$
   (d) $x \in D \Longrightarrow x \in E$
   (e) $x \in E \Longrightarrow x \in D$

8. For each of the following propositions: (i) Write down the proposition in mathematical notation (no English words); (ii) Write down its negation in mathematical notation, but without using the negation symbol $\neg$; (iii) Prove either the original proposition or its negation (whichever is true). You may assume all the ordinary properties of numbers and arithmetic.
   (a) There is a real number between any two distinct real numbers.
   (b) There is an integer between any two distinct integers.
   (c) There is no smallest positive real number.

9. Write the best proof that you can for the proposition for Bad Proof #2 (see Supplement 1: Bad Proofs).

10. Find examples of a, b, c, $d \in \mathbb{N}$, where a, b, c, $d \neq 0$, such that dividing a into b gives the same quotient and remainder as dividing and d into c, but where $\frac{a}{b} \neq \frac{c}{d}$.

11. Let a, $b \in \mathbb{N}$ and suppose that dividing b into a results in the quotient q with remainder r.
   (a) Let c be a positive integer. Make a conjecture about how to use a, b, c, q, and r to find the quotient and remainder when you divide bc into ac. Support your conjecture with three examples.
   (b) Prove your claim. Communicate your proof in claim/proof format. (Label and state the claim, label the proof.)