Math 487 Final Review

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Q1 Let S_{100} be the number of heads that turn up in 100 tosses of a fair coin. Use the Central Limit Theorem to estimate

(a) $\mathbb{P}(S_{100} \leq 45)$

$$\begin{aligned}
\mathbb{P}(S_{100} \leq 45) &= \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} \leq \frac{45 - (100 \cdot .5)}{\sqrt{100 \cdot .5 \cdot .5)}}\right) \\
&= \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} \leq -1\right) \\
&= .1587
\end{aligned}$$

(b) $\mathbb{P}(45 < S_{100} < 55)$

$$\begin{aligned}
\mathbb{P}(S_{100} < 45) &= \mathbb{P}\left(\frac{45.5 - (100 \cdot .5)}{\sqrt{100 \cdot .5 \cdot .5)}} < \frac{S_{100}}{\sqrt{100}} < \frac{54.5 - (100 \cdot .5)}{\sqrt{100 \cdot .5 \cdot .5)}}\right) \\
&= \mathbb{P}\left(-.9 < \frac{S_{100}}{\sqrt{100}} < .9\right) \\
&= \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} < .9\right) - \mathbb{P}\left(-.9 < \frac{S_{100}}{\sqrt{100}}\right) \\
&= .815594 - .18406 \\
&= 0.6315
\end{aligned}$$

(c) $\mathbb{P}(S_{100} > 63)$

$$
\begin{aligned}
\mathbb{P}(S_{100} > 63) &= \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} > \frac{63.5 - (100 \cdot .5)}{\sqrt{100 \cdot .5 \cdot .5)}}\right) \\
&= 1 - \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} > 2.7\right) \\
&= 1 - .99653 \\
&= 0.00347
\end{aligned}
$$

(d) $\mathbb{P}(S_{100} < 57)$

$$
\begin{aligned}
\mathbb{P}(S_{100} < 57) &= \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} < \frac{56.5 - (100 \cdot .5)}{\sqrt{100 \cdot .5 \cdot .5)}}\right) \\
&= \mathbb{P}\left(\frac{S_{100}}{\sqrt{100}} < 1.3\right) \\
&= .9032
\end{aligned}
$$

Q2 Let S_{200} be the number of heads that turn up in 200 tosses of a fair coin. Estimate

(a) $\mathbb{P}(S_{200} = 100)$

$$
\begin{aligned}
\mathbb{P}(S_{200} = 100) &\sim \frac{\phi(X_{100})}{\sqrt{200 \cdot .5 \cdot .5}} = \frac{\phi(0)}{\sqrt{50}} = \frac{1}{\sqrt{50}}\left(\frac{1}{\sqrt{2\pi}}\right) \\
&= .056419
\end{aligned}
$$

(b) $\mathbb{P}(S_{200} = 90)$

$$
\begin{aligned}
\mathbb{P}(S_{200} = 90) &\sim \frac{\phi(X_{90})}{\sqrt{200 \cdot .5 \cdot .5}} = \frac{\phi(-\sqrt{2})}{\sqrt{50}} = \frac{1}{\sqrt{50}}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{2})^2}{2}}\right) \\
&= \frac{1}{\sqrt{50}}\left(\frac{1}{\sqrt{2\pi}} e^{-1}\right) \\
&= .020755
\end{aligned}
$$

(c) $\mathbb{P}(S_{200} = 80)$

$$
\begin{aligned}
\mathbb{P}(S_{200} = 80) &\sim \frac{\phi(X_{80})}{\sqrt{200 \cdot .5 \cdot .5}} = \frac{\phi(-\sqrt{8})}{\sqrt{50}} = \frac{1}{\sqrt{50}}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{8})^2}{2}}\right) \\
&= \frac{1}{\sqrt{50}}\left(\frac{1}{\sqrt{2\pi}} e^{-4}\right) \\
&= .001033
\end{aligned}
$$

Q3 A true-false examination has 48 questions. June has probability 3/4 of answering a question correctly. April just guesses on each question. A passing score is 30 or more correct answers. Compare the probability that June passes the exam with the probability that April passes it.

Note that $\mathbb{P}(S_{48} \geq 30) = 1 - \mathbb{P}(S_{48} < 30) = 1 - \mathbb{P}(S_{48} \leq 29.5)$. We will use this fact to calculate the probability that June and April pass the exam.

$$\begin{aligned}
\mathbb{P}(S_{48} \leq 29.5) &= \mathbb{P}\left(\frac{S_{48}}{\sqrt{48}} \leq \frac{29.5 - (48 \cdot .75)}{\sqrt{48 \cdot .75 \cdot .25)}}\right) \\
&= 1 - \mathbb{P}\left(\frac{S_{48}}{\sqrt{48}} \leq -2.1667\right) \\
&= 1 - z(-2.1667) \\
&= 0.985
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(S_{48} \leq 29.5) &= \mathbb{P}\left(\frac{S_{48}}{\sqrt{48}} \leq \frac{29.5 - (48 \cdot .5)}{\sqrt{48 \cdot .5 \cdot .5)}}\right) \\
&= 1 - \mathbb{P}\left(\frac{S_{48}}{\sqrt{48}} \leq 1.5877\right) \\
&= 1 - z(1.5877) \\
&= 0.05705
\end{aligned}$$

So, it can be seen that June has a 98.5% chance of passing the exam while April only has a 5.7% chance of passing the exam.

Q5 A rookie is brought to a baseball club on the assumption that he will have a .300 batting average. (Batting average is the ratio of the number of hits to the number of times at bat.) In the first year, he comes to bat 300 times and his batting average is .267. Assume that his at bats can be considered Bernoulli trials with probability .3 for success. Could such a low average be considered just bad luck or should he be sent back to the minor leagues? Comment on the assumption of Bernoulli trials in this situation.

If his batting average was .267 out of 300 hits, we know he had approximately 80 hits. So, the probability we want to analyze is the probability that player is likely to have less than 80 hits out of 300 at bats with a success rate of .3.

$$\begin{aligned}
\mathbb{P}(S_{300} \leq 80) &= \mathbb{P}\left(\frac{S_{300}}{\sqrt{300}} \leq \frac{80 - (300 \cdot .3)}{\sqrt{300 \cdot .3 \cdot .7)}}\right) \\
&= \mathbb{P}\left(\frac{S_{300}}{\sqrt{300}} \leq -1.26\right) \\
&= .10383
\end{aligned}$$

Q10 Find the probability that among 10,000 random digits the digit 3 appears not more than 931 times.

$$\begin{aligned}
\mathbb{P}(S_{10,000} \leq 931) &= \mathbb{P}\left(\frac{S_{10,000}}{\sqrt{10,000}} \leq \frac{931 - (10,000 \cdot .1)}{\sqrt{10,000 \cdot .1 \cdot .9)}}\right) \\
&= \mathbb{P}\left(\frac{S_{10,000}}{\sqrt{10,000}} \leq -2.3\right) \\
&= .0107
\end{aligned}$$

Q12 A balanced coin is flipped 400 times. Determine the number x such that the probability that the number of heads is between $200 - x$ and $200 + x$ is approximately .80.

$$\begin{aligned}
\mathbb{P}(200 - x \leq S_{400} \leq 200 + x) &= \mathbb{P}\left(\frac{(200 - x) - (400 \cdot .5)}{\sqrt{400 \cdot .5 \cdot .5)}} \leq S_{400} \leq \frac{(200 + x) - (400 \cdot .5)}{\sqrt{400 \cdot .5 \cdot .5)}}\right) = .8 \\
&= \mathbb{P}\left(\frac{(200 - x) - 200}{10} \leq S_{400} \leq \frac{(200 + x) - 200}{10}\right) = .8 \\
&= \mathbb{P}\left(\frac{-x}{10} \leq S_{400} \leq \frac{x}{10}\right) = .8 \\
&= 2 \cdot \mathbb{P}\left(0 \leq S_{400} \leq \frac{x}{10}\right) = .8 \\
&= \mathbb{P}\left(S_{400} \leq \frac{x}{10}\right) - \mathbb{P}\left(S_{400} \leq 0\right) = .4 \\
&= \mathbb{P}\left(S_{400} \leq \frac{x}{10}\right) = .9 \\
&= \mathbb{P}\left(S_{400} \leq 1.28\right) = .9
\end{aligned}$$

So, we know that $\frac{x}{10} = 1.28$. Thus, $x \sim 13$.