Math 325-001 - Fall 2016. Homework 4 - Solutions

Due: Friday, Oct 14, in class.

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1. (Properties of the absolute value). Problem #17 on page 36.

   Solution.

   (a) From the inequality $|a+b| \leq |a| + |b|$ we get

   $$|x| = |(x-y)+y| \leq |x-y| + |y|$$

   Subtract $|y|$ on each side of the inequality to get

   $$|x| - |y| \leq |x-y|$$

   as desired.

   (b) Cite the result of part (a) with y replaced by number $-y$. Then appeal to the fact that $|-y| = |y|$.

   (c) $|x-y| = |x+(-y)| \leq |x| + |-y| = |x| + |y|$

   (d) We already have from part (a) that

   $$|x| - |y| \leq |x-y|$$

   If we switch x and y then we end up with $|y| - |x| \leq |y-x|$ or, equivalently, $-(|x|-|y|) \leq |y-x|$. Note that on the right of this last inequality we have $|y-x| = |-(x-y)| = |x-y|$. Consequently,

   $$-(|x|-|y|) \leq |x-y|$$

   Summarizing, we know that one of $|x|-|y|$ and $-(|x|-|y|)$ equals $||x|-|y||$. So from the above two inequalities we conclude

   $$||x|-|y|| \leq |x-y|$$

   as claimed.

2. Recall the definitions of what it means for $S \subset \mathbb{R}$ to be *bounded above*, *bounded below* and just *bounded*. Prove that S is bounded if and only if there is a real number $M > 0$ such that

   $$\forall s \in S, \quad |s| \leq M.$$

   Solution.

   Suppose $\forall s \in S, |s| \leq M$. That is equivalent to saying that $\forall s \in S, -M \leq s \leq M$. The second inequality, namely, $\forall s \in S, s \leq M$, states that S is bounded above. And the first inequality $\forall s \in S, -M \leq s$, states that S is bounded below. Since the set S is bounded above and below then, by definition, it is bounded.

   Conversely, suppose S is bounded. We want to show that there is $M > 0$ such that $|s| \leq M$ for every $s \in S$. Since the set S is bounded, there are numbers m_1 and m_2 such that $\forall s \in S$, $m_1 \leq s \leq m_2$. **By the way, note that m_1 and m_2 may be negative!** For any $s \in S$ we therefore have

   $$-|m_1| \leq m_1 \leq s \leq m_2 \leq |m_2|$$

   So let's define

   $$M = \max\{|m_1|, |m_2|, 1\}.$$

   (The extra 1 is included in case $m_1 = m_2 = 0$. In this situation we have $S = \{0\}$ and in order to ensure that M is positive, we can just let $M = 1$).

   Then for all $s \in S$, we have

   $$-M \leq s \leq M$$

   or, equivalently, $\forall s \in S, |s| \leq M$.

3. We say that a function f is *invertible* if $f^{-1} = \{(b,a) : (a,b) \in f\}$ is also a function, in which case we call it the inverse function to f. Notice that

$$f^{-1}(b) = a \quad \Leftrightarrow \quad b = f(a),$$

assuming that f^{-1} is a function.

a) If f is invertible, what are the domain and range of f^{-1}?

Answer. Domain of f^{-1} is the range of f. And the range of f^{-1} is the domain of f.

b) Which of the following functions are invertible? For those that are invertible, give the inverse.

$$\ell = \{(x,y) \in \mathbb{R}^2 : y = 2x+1\} \qquad c = \{(x,y) \in \mathbb{R}^2 : y \geq 0, x^2+y^2 = 1\}$$
$$s = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \qquad \sqrt{} = \{(x,y) \in \mathbb{R}^2 : y \geq 0, y^2 = x\}$$
$$\sin = \{(x,y) \in \mathbb{R}^2 : y = \sin(x)\}$$

Answer. $\ell^{-1} = \{(x,y) \in \mathbb{R}^2 : y = \frac{1}{2}(x-1)\}$ and $(\sqrt{})^{-1} = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y = x^2\}$

Note that the last function is not the same as $y = x^2$ because here we also restrict the domain to non-negative numbers only.

4. Assume that g is a bounded function, that is, $|g(x)| < B$ for all $x \in \mathbb{R}$. Prove that $\lim_{x\to 0} x \cdot g(x) = 0$.

Solution. Since we are told g is defined at every x, then the function given by the formula $x \cdot g(x)$ is also defined for every x, in other words, on the open interval $(-\infty, \infty)$ which includes $p = 0$.

Next, let $\varepsilon > 0$ be given. Define $\delta_\varepsilon = \frac{\varepsilon}{B}$ (since B is strictly bigger than the absolute value of something then B itself is nonzero and positive).

Then if $0 < |x - 0| < \delta_\varepsilon$, we get

$$|xg(x) - 0| = |x||g(x)| \leq |x|B < \delta_\varepsilon B = \frac{\varepsilon}{B} B = \varepsilon$$

By the definition of the limit, the desired conclusion follows.

5. Using the "ε-δ" definition of the limit prove that

a) $\lim_{x\to 2}(4x+1) = 9$

Solution. Note that the function inside the limit is defined for every x in $\mathbb{R}$. Let $\varepsilon > 0$ be given and choose $\delta = \frac{\varepsilon}{4}$. Then if $0 < |x-2| < \delta$, we get

$$|4x+1-9| = |4x-8| = 4|x-2| < 4 \cdot \delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

By the definition of the limit the conclusion follows.

b) $\lim_{x\to 5}\sqrt{x+4} = 3$

Solution. The function inside the limit is defined on the open interval $(-4, \infty)$ which includes the target point $p = 5$. Let $\varepsilon > 0$ be given. Define $\delta = \min\{9, \varepsilon\}$. Suppose now $0 < |x-5| < \delta$. First of all, this inequality implies $|x-5| < 9$, which is, in turn, equivalent to $-4 < x < 14$. Consequently

$$\sqrt{x+4}+3 > 3 \quad \Leftrightarrow \quad \frac{1}{\sqrt{x+4}+3} < \frac{1}{3}$$

Secondly,

$$|\sqrt{x+4}-3| = \left|\frac{(\sqrt{x+4}-3)\cdot(\sqrt{x+4}+3)}{\sqrt{x+4}+3}\right| = \left|\frac{x+4-9}{\sqrt{x+4}+3}\right| = \frac{|x-5|}{\sqrt{x+4}+3} \leq \frac{|x-5|}{3}$$

where in the last step we used the previous estimate on $\frac{1}{\sqrt{x+4}+3}$. Thus

$$|\sqrt{x+4}-3| \leq \frac{|x-5|}{3} < \frac{\delta}{3} \leq \frac{\varepsilon}{3} < \varepsilon.$$

By the definition of the limit, the conclusion follows.

c) $\lim_{x\to 3} \dfrac{1}{8-4x} = -\dfrac{1}{4}$

Solution.

The function in the limit is defined on $\mathbb{R} \setminus \{2\}$, which includes the open interval $(2, \infty)$ that contains $p = 3$. So it remains to check the ε-δ part. Let $\varepsilon > 0$ be given, and define the corresponding δ by

$$\delta = \min\left\{\frac{1}{4}, 3\varepsilon\right\}$$

Likewise can choose, for example, $\min\{\frac{1}{2}, \varepsilon\}$.

Suppose now $0 < |x-3| < \delta$. Then, first, we have $|x-3| < \frac{1}{4}$ whence $\frac{11}{4} < x < \frac{13}{4}$. In particular, take note of the ensuing inequality

$$-5 < 8 - 4x < -3$$

which implies $3 < |8-4x| < 5$ and, consequently,

$$\frac{1}{|8-4x|} < \frac{1}{3}.$$

With this inequality in mind, proceed to estimate "$|f(x) - L|$", namely:

$$\left|\frac{1}{8-4x} - \left(-\frac{1}{4}\right)\right| = \left|\frac{4+8-4x}{4(8-4x)}\right| = \left|\frac{3-x}{8-4x}\right| = \frac{|x-3|}{|8-4x|} < \frac{\delta}{3} \leq \frac{3\varepsilon}{3} = \varepsilon$$

This completes the proof.

6. Use the "ε-δ" definition of the limit to prove that:

a) $\lim_{x\to 2}(x^2 - x) = 2$

Solution.

The function inside the limit is defined on $(-\infty, \infty)$. Next, let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{4}\}$. Suppose $0 < |x-2| < \delta$. As one consequence, we have $|x-2| < 1$, whence $1 < x < 3$ and $2 < x+1 < 4$. In particular, $|x+1| < 4$. With this in mind, estimate

$$|x^2 - x - 2| = |(x-2)(x+1)| = |x-2| \cdot |x+1| < |x-2| \cdot 4 < \delta \cdot 4 \leq \frac{\varepsilon}{4} \cdot 4 = \varepsilon$$

Appeal to the definition of the limit to complete the proof.

b) $\lim_{x\to 0} x^2(\sin(x) + \cos(x)) = 0$

Solution.

The function inside the limit is defined on $(-\infty, \infty)$ so the first part of the definition of the limit holds. Pick any $\varepsilon > 0$ and choose

$$\delta = \frac{\sqrt{\varepsilon}}{10}$$

Suppose $0 < |x-0| < \delta$ and estimate:

$$|x^2(\sin(x) + \cos(x))| = |x|^2|\sin(x) + \cos(x)| \leq |x|^2(|\sin(x)| + |\cos(x)|) \leq |x|^2 \cdot 2 < \delta^2 \cdot 2 = \frac{\varepsilon}{100} \cdot 2 = \frac{\varepsilon}{50} < \varepsilon$$

Remark. *Optionally, it can be shown that*

$$-\sqrt{2} < \sin(x) + \cos(x) < \sqrt{2}$$

by either noting that the critical points of this function on the real line occur only at $\frac{\pi}{4} + m\pi$, $m \in \mathbb{Z}$, *or via the identity*

$$\sin(x) + \cos(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

If we appeal to this observation, then, it follows

$$|\sin(x) + \cos(x)| < \sqrt{2}$$

So, for a given $\varepsilon > 0$, *a larger value for* δ *could be specified as* $\sqrt{\varepsilon/\sqrt{2}} = \sqrt{\varepsilon}/\sqrt[4]{2}$.