Problem 1

Several useful inequalities involving absolute values can be derived from the triangle inequality by a judicious choice for a and b.

For any of these examples, we can redefine x and y to be a and b, which will help us reduce to the following variants of the triangle inequalities. For each of the following inequalities, I use the following format $|a| + |b| \leq |a + b|$.

(a) $|x| - |y| \leq |x - y|$

Here we will let $a = (x - y)$ and $b = y$. Then, we see that

$$|x - y + y| \leq |x - y| + |y| \implies |x| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$$

(b) $|x| - |y| \leq |x + y|$

Here we will let $a = (x + y)$ and $b = -y$. Then, we see that

$$\big|x + y + (-y)\big| \leq |x + y| + |-y| \implies |x| \leq |x + y| + |y| \implies |x| - |y| \leq |x + y|$$

(c) $|x| + |y| \geq |x - y|$

Here we will let $a = x$ and $b = -y$. Then, we see that

$$\big|x + (-y)\big| \leq |x| + |-y| \implies |x - y| \leq |x| + |y|$$

(d) $\big||x| - |y|\big| \leq |x - y|$

To prove this, we will set up 2 different equations.

$$(1)\ |y - x + x| \leq |y - x| + |x| \qquad (2)\ |x - y + y| \leq |x - y| + |y|$$

(1) We can move the $|x|$ from the left side to the right side and take negative one out of the left hand side, which gives us $-(|x| - |y|) \leq |y - x|$.
(2) We can move the $|y|$ from the left side to the right side, which gives us $|x| - |y| \leq |x - y|$.

From the properties of absolute values, we know $|x - y| = |y - x|$ and if $x \geq a$ and $x \geq -a$ then $x \geq |a|$. From the combination of these two facts, we now have $\big||x| - |y|\big| \leq |x - y|$.

Problem 2

Recall the definition of what it means for $S \subset \mathbb{R}$ to be *bounded above, bounded below* and just *bounded.* Prove that S is bounded if and ony if there is a real number $M > 0$ such that

$$\forall s \in S, |s| \leq M$$

Proof

Let's assume $S \subset \mathbb{R}$, then $|s| \leq M$ means that $-M < s < M$, $\forall s \in S$. S is bounded above if there exists some $X \in \mathbb{R}$ such that $\forall s \in S$, $s < X$. Likewise, S is bounded below if there exists some $x \in \mathbb{R}$ such that $\forall s \in S$, $x < s$. Thus, we see that M is in fact an upper bound for S because $s < M$, $\forall s \in S$ and that $-M$ is in fact a lower bound for S because $-M < s$, $\forall s \in S$. Therefore, since S is bounded above and below, S is, by definition, bounded.

Problem 3

We say that a function f is *invertible* if $f^{-1} = \{(b,a) : (a,b) \in f\}$ is also a function, in which case we call it the inverse function to f. Notice that

$$f^{-1}(b) = (a) \leftrightarrow b = f(a),$$

assuming that f^{-1} is a function.

(a) If f is invertible, what are the domain and range of f^{-1}?

(b) Which of the following functions are invertible? For those that are invertible, give the inverse.

$l = \{(x,y) \in \mathbb{R}^2 : y = 2x+1\}$	Inverse: $y = \frac{x-1}{2}$, Domain: $\mathbb{R}$, Range: $\mathbb{R}$
$c = \{(x,y) \in \mathbb{R}^2 : y \geq 0, x^2+y^2 = 1\}$	Not invertible
$s = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$	Not invertible
$\sqrt{=}(x,y) \in \mathbb{R}^2 : y \geq 0, y^2 = x\}$	Inverse: $y = x^2$, Domain: $\mathbb{R}$, Range: $[0,\infty)$
$sin = \{(x,y) \in \mathbb{R}^2 : y = sin(x)\}$	Not invertible

Problem 4

Assume that g is bounded function, that is, $\big|g(x)\big| < B$ for all $x \in \mathbb{R}$. Prove that $\lim\limits_{x\to 0} x \cdot g(x) = 0$.

$$c = 0,\ L = 0,\ f(x) = x \cdot g(x)$$

$\big|f(x) - L\big| < \epsilon \implies \big|x \cdot g(x) - 0\big| < \epsilon \implies \big|x \cdot g(x)\big| < \epsilon.$

At its max, $g(x)$ can be no greater than B. Thus, we know B is an upper bound for $f(x)$. We can rewrite $f(x)$ as $|x \cdot B|$. Thus, $|x \cdot B| < \epsilon \implies -\epsilon < x \cdot B < \epsilon$. By dividing by B on both sides, we have $\frac{-\epsilon}{B} < x < \frac{\epsilon}{B}$.

Proof

Let $\delta_\epsilon = \frac{\epsilon}{B}$. Then $|x - 0| < \delta_\epsilon \implies \delta_\epsilon < x - 0 < \delta_\epsilon$.

$$\frac{-\epsilon}{B} < x < \frac{\epsilon}{B}$$

$$-\epsilon < x \cdot B < \epsilon$$

$$|x \cdot B| < \epsilon$$

Since we said $g(x) < B$, $\forall x \in \mathbb{R}$, we know the following inequality to be true: $x \cdot g(x) < |x \cdot B| < \epsilon$. This completes the proof.

Problem 5

Using the "$\epsilon - \delta$" definition to prove that

(a) $\lim\limits_{x\to 2}(4x+1) = 9$

$$\big|f(x) - L\big| < \epsilon \implies \big|(4x+1) - 9\big| < \epsilon \implies |4x - 8| < \epsilon \implies \big|4(x-2)\big| \implies |x-2| < \frac{\epsilon}{4}$$

Proof

Let $\delta_\epsilon \implies \frac{\epsilon}{4}$. Then, $0 < |x-2| < \frac{\epsilon}{4} \implies -\frac{\epsilon}{4} < x-2 < \frac{\epsilon}{4}$.

$$\frac{\epsilon}{4} < x-2 < \frac{\epsilon}{4}$$
$$-\epsilon < 4(x-2) < \epsilon$$
$$-\epsilon < 4x-8 < \epsilon$$
$$-\epsilon < (4x+1)-9 < \epsilon$$
$$\big|(4x+1)-9\big| < \epsilon$$

This completes the proof because we showed $\big|f(x)-L\big| < \epsilon$ for a specific δ.

(b) $\lim\limits_{x\to 5} \sqrt{x+4} = 3$

$$\big|f(x)-L\big| < \epsilon \implies \left|\sqrt{x+4}-3\right| < \epsilon \text{ implies}$$
$$-\epsilon < \sqrt{x+4}-3 < \epsilon$$
$$-\epsilon+3 < \sqrt{x+4} < \epsilon+3$$
$$(-\epsilon+3)^2 < x+4 < (\epsilon+3)^2$$
$$(-\epsilon+3)^2-9 < x-5 < (\epsilon+3)^2-9$$

You can see that we subtracted 9 to both sides at the end. That is because our $|x-p|$ value is $|x-5|$.

Proof

Let $\delta_\epsilon = (\epsilon+3)^2-9$. Then, $0 < |x-5| < (\epsilon+3)^2+9$ implies

$$(-\epsilon+3)^2-9 < x-5 < (\epsilon+3)^2-9$$
$$(-\epsilon+3)^2 < x+4 < (\epsilon+3)^2$$
$$-\epsilon+3 < \sqrt{x+4} < \epsilon+3$$
$$-\epsilon < \sqrt{x+4}-3 < \epsilon$$

This completes the proof because we showed $\big|f(x)-L\big| < \epsilon$ for a specific δ.

(c) $\lim\limits_{x\to 3} \dfrac{1}{8-4x} = \dfrac{1}{4}$

$$\big|f(x)-L\big| < \epsilon \implies \left|\tfrac{1}{8-4x}-\tfrac{1}{4}\right| < \epsilon \text{ implies}$$
$$-\epsilon < \tfrac{1}{8-4x}-\tfrac{1}{4} < \epsilon$$
$$-\epsilon+\tfrac{1}{4} < \tfrac{1}{-4(x-2)} < \epsilon+\tfrac{1}{4}$$
$$4\epsilon-1 > \tfrac{1}{x-2} > -4\epsilon-1$$
$$\tfrac{1}{4\epsilon-1} > x-2 > \tfrac{1}{-4\epsilon-1}$$
$$\tfrac{1}{4\epsilon-1}-1 > x-3 > \tfrac{1}{-4\epsilon-1}-1$$

You can see that we subtracted 1 to on sides at the end. That is because our $|x-p|$ value is $|x-3|$.

Proof

Let $\delta_\epsilon = \frac{1}{-4\epsilon-1} - 1$. Then, $0 < |x-3| < \frac{1}{-4\epsilon-1} - 1$ implies

$$\begin{gathered}
\tfrac{1}{4\epsilon-1} - 1 > x - 3 > \tfrac{1}{-4\epsilon-1} - 1 \\
\tfrac{1}{4\epsilon-1} > x - 2 > \tfrac{1}{-4\epsilon-1} \\
4\epsilon - 1 > \tfrac{1}{x-2} > -4\epsilon - 1 \\
-\epsilon + \tfrac{1}{4} < \tfrac{1}{-4(x-2)} < \epsilon + \tfrac{1}{4} \\
-\epsilon < \tfrac{1}{-4(x-2)} - \tfrac{1}{4} < \epsilon \\
-\epsilon < \tfrac{1}{8-4x} - \tfrac{1}{4} < \epsilon
\end{gathered}$$

This completes the proof because we showed $\left|f(x) - L\right| < \epsilon$ for a specific δ.