Problem 1

Assume that $f(x) \leq g(x) \leq h(x)$ for all x in some open interval containing a, except perhaps at a itself. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x)$ exists and equals L, also. Justify each step in the following proof of the squeeze theorem.

a) Given $\epsilon > 0$ show that there exists a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $\left|f(x) - L\right| < \epsilon$.

b) Show that if $0 < |x - a| < \delta_1$, then $L - \epsilon < f(x)$.

c) Show that there exists a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $h(x) < L + \epsilon$

d) Let $\delta = min \delta_1, \delta_2$. Show: If $0 < |x - a| < \delta$, then $L - \epsilon < g(x) < L + \epsilon$

e) Complete the proof by showing that if $0 < |x - a| < \delta$, then $\left|g(x) - L\right| < \epsilon$.

Problem 2

a) a) Suppose that $f(xx) \leq 0$ for all x (except perhaps at $x = a$). Show: if $\lim_{x \to a} f(x) = L$, then $L \leq 0$. Hint: Assume instead that $L > 0$. Let $\epsilon = \frac{L}{2}$ and derive a contradiction

   b) State and prove the analogue for $f(x) \geq 0$.

b) Assume that $g(x) \leq h(x)$ for all x (except perhaps at a). If $\lim_{x \to a} g(x) = M$ and $\lim_{x \to a} h(x) = N$, prove that $M \leq N$. (Hint: Let $f(x) = g(x) - h(x)$ and then use problem 2.2.9).

Problem 3

Prove (from the $\epsilon - \delta$ definition) that $\lim_{x \to 3} \sqrt{3-x}$ does not exist, but $\lim_{x \to 3^-} \sqrt{3-x} = 0$.

Problem 4

Consider the set of numbers $a_n : n \in \mathbb{N}$ where each a_n is determined via this recursive definition:

$$a_1 = 2 \text{ and } a_n = 2 - \tfrac{1}{a_{n-1}} \text{ for } n \geq 2.$$

For example,

$$a_1 = 2,\ a_2 = 2 - \tfrac{1}{a_1} = 2 - \tfrac{1}{2} = \tfrac{3}{2},\ a_3 = 2 - \tfrac{1}{\frac{3}{2}} = \tfrac{4}{3}, \text{ etc.}$$

Use induction to prove that for all $n \in \mathbb{N}$ we have $a_n = \frac{n+1}{n}$.