Math 325-001 - Fall 2016. Homework 2 (individual problems)

Solutions

$\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{R}$ denotes the set of real numbers.

1. Prove that $\sqrt{15}$ is irrational.

   Solution. Proceed as we did in class by contradiction. Assume $\sqrt{15}$ is rational. Then we can find two integers M, N, with $N \neq 0$, that are relatively prime (have no common factors), such that

   $$\sqrt{15} = \frac{M}{N}\,.$$

   That implies $N^2 15 = M^2$. Thus, for example, prime 3 (which is a factor of 15), divides M^2, and therefore divides M. **Careful! We of course have that 15 divides M^2 as well, but we can't immediately conclude from that that 15 divides M because 15 is not prime. That's why we are going to do it one prime at a time, first 3, then 5.**

   For the same reason prime 5 divides M. Since 3 and 5 are primes in the prime factorization of M, then $3 \cdot 5 = 15$ must divide M. Consequently, we can write $M = 15K$ which leads to the equation

   $$N^2 15 = 15^2 K^2$$

   Cancel 15 on each side

   $$N^2 = 15K^2$$

   Now we get that 15, and in particular, both 3 and 5 divide N^2. Just as with M^2 and M, here we have that because 3 and 5 are prime they each divide N.

   We conclude that M and N do have common factors (3 and 5 in particular), which contradicts the way we chose them. Thus $\sqrt{15}$ cannot be rational.

2. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

   Solution.

   Method 1. Let $a = \sqrt{2} + \sqrt{3}$ and $b = \sqrt{2} - \sqrt{3}$. First, note that if you multiply a and b and simplify the result, you get -1. In particular $a = -1/b$ and $b = -1/a$. Consequently, if one of a, b is rational, then so is the other number (since negative reciprocals of non-zero rational numbers are rational).

   In short, either a, b are both rational or they are both irrational. Assume they are both rational, then $(a + b)/2 = \sqrt{2}$ would also be rational, whereas we know $\sqrt{2}$ is not. That's a contradiction. Thus both a, b, and in particular a, must be irrational.

   Method 2. Alternatively, you could look at $a^2 = 2 + 2\sqrt{6} + 3$. Then prove that $\sqrt{6}$ is irrational using the same method as in problem 1. It would follow that a^2 must be irrational too.

   Then note that if a were rational, $a = p/q$ for integers p, q with $q \neq 0$, then so would be its square p^2/q^2, thus contradicting the fact that a^2 is not rational. Hence a itself must be irrational too.

3. Prove that for any $n \in \mathbb{N}$, the number $3^n - 3$ is divisible by 6.

   Solution.

   Let $a_n = 3^n - 3$. For every $n \in \mathbb{N}$, we can factor $a_n = 3 \cdot (3^{n-1} - 1)$ where $n - 1$ is an integer bigger or equal to 0 (that is why 3^{n-1} is still an integer). Thus 3 divides a_n.

   Next, since neither 3^n nor 3 have prime number 2 in their prime factorizations, then they are both odd. So a_n is the difference of two odd numbers. A difference of two odd numbers must be even (**why?**). Then a_n is even, whence 2 divides a_n. In conclusion, a_n has primes 2 and 3 in its prime factorization. Consequently $6 = 2 \cdot 3$ divides a_n.

4. Negate the following statements. Also say which, the original or the negation, is true (provide a brief explanation).

a) For all $x \in \mathbb{R}$ if $x^2 + 3x + 2 = 0$ then $x = -1$.

Answer. The negation reads: *There exists $x \in \mathbb{R}$ such that $x^2 + 3x + 2 = 0$ and $x \neq -1$.* This statement is true since number $x = -2$ has this property. So the original statement is false.

b) For all $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ such that y divides x.

Answer.

The negation is: *There exists $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$ the number y does not divide x.*

This statement is false, because every natural number is divisible by itself and by 1 (also a natural number). So the original is true.

c) There exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have: $m^2 - 6m + 8 = 0$ implies $m < n$.

Answer.

Negation: *For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m^2 - 6m + 9 = 0$ and $m \geq n$.*

This statement is false (so the original is true) because the equation $m^2 - 6m + 9 = 0$ has only one solution $m = 3$ (repeated root). Hence, if we choose any natural number $n > 3$, then $n > m$.