$$\lim_{x \to 2} (x^2 - x) = 2$$

We know that $c = 2$, $L = 2$, $f(x) = x^2 - x$.

$$\left|f(x) - L\right| = \left|x^2 - x - 2\right| < \epsilon \implies \left||x-2|\,|x+1|\right| < \epsilon$$

We cannot define δ_ϵ in terms of x, but $|x-2| = \frac{\epsilon}{|x+1|}$ leaves δ_ϵ in terms of x. We need to find a fixed M such that $\frac{\epsilon}{M} < \frac{\epsilon}{|x+1|}$. We only care about x *around* 2, so we can restrict x to the open interval $(1, 3)$.

$$\begin{array}{cc} 1 < x < 3 & 1 < x < 3 \\ -1 < x - 2 < 1 & 2 < x + 1 < 4 \\ |x-2| < 1 & \frac{1}{2} < \frac{1}{|x+1|} < \frac{1}{4} \end{array}$$

Proof

Let $\epsilon > 0$ be given. Let $\delta_\epsilon = min\{1, \frac{\epsilon}{4}\}$. Then $0 < |x-2| < \delta_\epsilon$ implies

$$\begin{array}{l} (1)\ |x-2| < 1 \implies 2 < |x+1| < 4 \\ (2)\ \left|f(x) - L\right| = \left|x^2 - x - 2\right| = |x-2|\,|x+1| < \frac{\epsilon}{4} \cdot 4 = \epsilon \end{array}$$

This completes the proof.

$$\lim_{x \to 0} x^2(sin(x) + cos(x)) = 0$$

We know that $c = 0$, $L = 0$, $f(x) = x^2(sin(x) + cos(x))$.

From the combination of the transcendental functions $\left|sin(x) + cos(x)\right|$, we know the maximum value that the functions can reach is $\sqrt{2}$. Therefore, we can modify $f(x)$ to be $\sqrt{2}x^2$ since we know that the $\left|sin(x) + cos(x)\right|$ part of the function is bounded above. Thus, we have

$$\begin{aligned} \left|\sqrt{2}x^2 - 0\right| &< \epsilon \\ \left|\sqrt{2}x^2\right| &< \epsilon \\ \sqrt{2}x^2 &< \epsilon \\ x &< \frac{\sqrt{\epsilon}}{\sqrt[4]{2}} \end{aligned}$$

Proof

Let $\epsilon > 0$ and $\delta_\epsilon = \frac{\sqrt{\epsilon}}{\sqrt[4]{2}}$. Then, $0 < x < \frac{\sqrt{\epsilon}}{\sqrt[4]{2}}$ implies

$$\begin{gathered} -\frac{\sqrt{\epsilon}}{\sqrt[4]{2}} < x < \frac{\sqrt{\epsilon}}{\sqrt[4]{2}} \\ x^2 < \frac{\epsilon}{\sqrt{2}} \\ \sqrt{2}x^2 < \epsilon \end{gathered}$$

Thus, $\left|\sqrt{2}x^2\right| < \epsilon$. This satisfies the proof because we stated that $sin(x) + cos(x)$ is bounded above by $\sqrt{2}$. By definition of an upper bound, there can never exist a value that exceeds $\left|sin(x) + cos(x)x^2\right|$. Then, $\left|sin(x) + cos(x)x^2\right| < \left|x^2\sqrt{2}\right| < \epsilon$.