Problem 1

1. $A = \{-1, 5.2, 1, 3.7\}$

   Upper bounds: 5.2, 6

   $\forall a \in A$, $a \leq 5.2$. Thus, 5.2 is an upper bound for A.

   A is a finite set with 4 elements. Thus, the maximum element must also be the *LUB(A)*. There cannot be another least upper bound because any number n less than the maximum element, 5.2 would mean that $\exists a \in A$, $a > n$, where that a is 5.2.

2. $B = [2, 3) = \{x \in \mathbb{R} : 2 \leq x < 3\}$

   Upper bounds: 3, 4

   $\forall b \in B$, $b < 3$ because B approaches 3 but does not actually contain 3. Thus, 3 is an upper bound for B.

   There is no *LUB(B)* < 3 because the elements of the set B approach 3 infinitely close. Thus, $\forall b \in B$, $b + \epsilon < 3$ meaming that there is always an element of B that is more than another element but still less than 3. Thus, 3 must be the *lub(B)*.

3. $C = (-\infty, 4.2]$

   Upper bounds: 4.2, 5

   $\forall c \in C$, $c < 4.2$. Thus, 4.2 is an upper bound for C.

   C is an infinite set with a defined maximum element. Thus, the maximum element must also be the *LUB(C)*. There cannot be another least upper bound because any number n less than the maximum element, 4.2 would mean that $\exists c \in C$, $c > n$, where that c is 4.2.

4. $D = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$

   Upper bounds: $\frac{1}{2}$, 1

   $\forall d \in D$, $d < \frac{1}{2}$. Thus, $\frac{1}{2}$ is an upper bound for D.

   We know $\frac{1}{2} \in D$ because when $n = 2$, $\frac{(-1)^2}{2} = \frac{1}{2}$. Let's assume there exists another least upper bound $e =$*LUB(D)* defined as $\frac{1}{2} - \epsilon$. By definition, $\frac{1}{2} - \epsilon$ must be greater than all elements in D, but we know $\frac{1}{2}$ is an element and greater than $\frac{1}{2} - \epsilon$. Thus, e contradicts our assumption and $\frac{1}{2}$ is the *LUB(D)*.

5. $E = \{-\frac{1}{n} : n \in \mathbb{N}\}$

   Upper bounds: 0, 1

   $\forall e \in E$, $e < 0$. Thus, 0 is an upper bound for E.

   Let's choose another upper bound f. Assume $f < 0$. Find a $k \in \mathbb{N}$ so that $-\frac{1}{k} < |f|$. Then $\frac{1}{k} > f$. Since $\frac{1}{k} \in E$, f cannot be a *LUB(E)*. Then, f is an uppber bound and $f \geq 0$. Thus, 0 is the *LUB(E)*.

6. $F = \{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\}$

   Upper bounds: *DNE*

   F is an infinite set with no upper bound. Since it is not bounded above, it cannot have a *LUB*.

7. $G = \{cos(n) : n \in \mathbb{N}\}$

   Upper bounds: 1, 2

$\forall g \in G$, $d < 1$. Thus, 1 is an upper bound for G.

The finite range of the cosine function is defined as $[-1, 1]$. The maximum element must also be the *LUB(G)* because $\forall g \in G$, such that $g < 1$, g is not even an upper bound of G and therefore cannot be a *LUB(G)*.

8. $H = \{100n - n^2 : n \in \mathbb{N}\}$

   Upper bounds: 2500, 2501

   $\forall h \in H$, $h < 2500$. The $\frac{d}{dn}100n - n^2$ shows us that there is a maxima at $n = 50$. When $n = 50$, the function takes the value 2500. Thus, 1 is an upper bound for H.

   We know 2500 is the maximum element in the defined range of the function $100n - n^2 : n \in \mathbb{N}$. The maximum element must also be the *LUB(H)* because $\forall h \in H$, such that $h < 2500$, h is not even an upper bound of H and therefore cannot be a *LUB(H)*.

Problem 2

If $S \subset \mathbb{R}$ has a *least upper bound*, then it is unique.

Proof

Let S be a subset of $\mathbb{R}$ and assume that $p = LUB(S)$ and therefore an upper bound for S as well. Let's also assume that there exists a number $q \in \mathbb{R}$ such that $q = LUB(S)$ and therefore an upper bound for S as well. We know that $p \leq q$ since p is a $LUB(S)$. Likewise, we know that $q \leq p$ since q is a $LUB(S)$. Because both $p \leq q$ and $q \leq p$, $b = c$ must be true. Therefore, the $LUB(S)$ must be unique.

Problem 3

Suppose that $\lambda = lub(A)$. Let $B = \{ka | \in A\}$, where $k > 0$.

a) Show that $k\lambda$ is an upper bound for the set B.

If λ is *lub(A)*, then $\forall a \in A$, $a \leq \lambda$. If $k > 0$ and $a \leq \lambda$, we know $ka \leq k\lambda$. Thus $k\lambda$ is an upper bound of B.

b) Show that $k\lambda$ is the least upper bound for B.

Let's assume there is another element $\gamma \in B$ such that $\gamma < \lambda$. Then,

$$\begin{aligned} ka &\leq \gamma < k\lambda \\ a &\leq \gamma/k < \lambda \\ \gamma/k &< \lambda \end{aligned}$$

We end up with γ/k as greater than $a \in A$ and smaller than λ. However, we defined λ as *lub(A)*. No such element γ can exist, and we end up with a contradiction. Thus, $k\lambda =$ *lub(B)*.

c) What can happen if $k < 0$?

If $k < 0$, then $a \leq \lambda$ means that $ka \geq k\lambda$. In that case, the proof above does **not** hold.

Problem 4

Prove the following facts about real numbers, stating explicitly which field axioms/theorems you are using at each stage.

a) $\forall a, x \in \mathbb{R}$ if $a \neq 0$ and $ax = a$ then $x = 1$.

By A10, we know we can get x alone on the left hand side of the equation ax because the existence of a multiplicative inverse defined by $a^{-1} = 1/a$. So, we have

$$\begin{aligned} a^{-1} \cdot ax &= a^{-1} \cdot a \\ \tfrac{1}{a} \cdot \tfrac{a}{1} x &= \tfrac{1}{a} \cdot \tfrac{a}{1} \\ \tfrac{a}{a} \cdot x &= \tfrac{a}{a} \\ 1 \cdot x &= 1 \\ x &= 1 \end{aligned}$$

We know see that after multiplying both sides by the multiplicative inverse of a becomes $\frac{a}{a}$, which reduces to 1. We are left with $x = 1$, which is what we were trying to find.

b) $\forall a, b \in \mathbb{R}$ if $ab = 0$ then $a = 0$ or $b = 0$.

Again, by A10, we know we can utilize the existence of a multiplicative inverse to separate the variables a and b defined as $a^{-1} = \frac{1}{a}$ and $b^{-1} = \frac{1}{b}$, respectively.

$$\begin{array}{rrrl} & a^{-1}ab = 0 & b^{-1}ab = 0 & \\ & \frac{1}{a} \cdot ab = 0 & b^{-1}ba = 0 & \text{By A7} \\ & \frac{1}{a} \cdot \frac{a}{1} b = 0 & \frac{1}{b}\frac{b}{1} a = 0 & \\ & \frac{a}{a} \cdot b = 0 & \frac{b}{b} \cdot a = 0 & \\ & 1 \cdot b = 0 & 1 \cdot a = 0 & \\ \text{By A9} & b = 0 & a = 0 & \text{By A9} \end{array}$$

We know see that regardless of whether we multiply $ab = 0$ by the multiplicative inverse of a then b, the resulting variable is equal to 0.

c) $\forall x, y \in \mathbb{R}$ if $x^2 = y^2$ then $x = y$ or $x = -y$.

Let's assume that $x \neq y$ and $x \neq -y$. Then, by definition of the square of a number a, we get $a \cdot a = a^2$.

In the case of $x = y$, $$\begin{aligned} x \cdot x &\neq y \cdot y \\ x^2 &\neq y^2 \end{aligned}$$

In the case of $x = -y$, $$\begin{aligned} x \cdot x &\neq (-y)(-y) \\ x^2 &\neq (-1)(y)(-1)(y) \\ x^2 &\neq (-1)(-1)(y)(y) \\ x^2 &\neq (y)(y) \\ x^2 &\neq y \cdot y \\ x^2 &\neq y^2 \end{aligned}$$

According to our assumption, we see that in either case, we get $x^2 \neq y^2$. However, that contradicts the original hypothesis. Thus, $x = y$ or $x = -y$.