Prove that (L) $(A \cap B) \cup C = A \cap (B \cup C)$ (R) if and only if $C \subset A$

PROOF

By definition, we know a set C is a subset of another set A iff all elements of C are contained in A. In other words, there cannot be a single element x that belongs to C but does not belong to A.

Suppose $x \in C$. We don't have to worry about the cases when x doesn't belong to C because our hypothesis assumes $C \subset A$.

(Case 1) if $x \in B$ and $x \in A$

(Case 2) if $x \in B$ and $x \notin A$

(Case 3) if $x \notin B$ and $x \in A$

(Case 4) if $x \notin B$ and $x \notin A$

(1) If an element x belongs to A, B, C then $L = R$ regardless of the combination of intersections and unions.

(2) We can evaluate at both L and R and then compare them. For L, an element x does not belong to A and B. However, it does belong to C, so x exists in L. For R, an element x belongs to B or C. However, it does not belong to A. Since it is not in both, x does not exist in R. $L \neq R$.

(3) Again, we can evaluate at both L and R and then compare them. For L, an element x does not belong to A and B. However, it does belong to C, so x exists in L. For R, an element x belongs to B or C. Additionally, it belongs to A as well. Since it is in both, x exists in R. $L = R$.

(4) Again, we can evaluate at both L and R and then compare them. For L, an element x does not belong to A and B. However, it does belong to C, so x exists in L. For R, an element x belongs to B or C. However, it does not belong to A. Since it is not in both, x does not exist in R. $L \neq R$.

CONSLUSION

From **Cases 1-4**, we saw that the only time when the hypothesis held true was **Case 1** and **Case 2**. In both instances, an element x belonged to both C and A. The value of the equation never depended on whether or not x belonged to B. So, by definition, $C \subset A$ because when x belongs to C, x also belongs to A.