Group Problem 5.5

Recall that a function $f \in \mathbb{R}^2$ is called a polynomial if there is a non-negative integer n (called "the degree" of f) and real numbers $c0, c1, ..., cn$, such that for every $x \in \mathbb{R}$.

$$f(x) = \sum_{i=0}^{n} c_i x^i$$

Use induction and the theorem about combining the limits to prove that every polynomial is continuous on $\mathbb{R}$. (Note: the results about combining limits apply to at most two functions at a time. Also don't forget that you must check the case when $n = 0$, specifically, when f is a constant function).

Proof

First, we'll prove the base case of $n = 1$. When $n = 1$,

$$\sum_{i=0}^{1} c_i x^i = c_0 + c_1 x.$$

Since the limit of the resulting function is constructed based on the two limit definitions of

$$\lim_{x \to a} c = c \quad \lim_{x \to a} x = a,$$

we can combine them to show that $c_0 + c_1 x$ is continuous if the limit exists for $\lim_{x \to a} c_0 + c_1 x$. Thus, we know the polynomial of $n = 1$ is also continuous.

Next, for the induction step, we assume a polynomial P_n is continuous as defined $f(x) = c_0 + c_1 x + ... + c_n x^n$. Let $g(x) = c_{n+1} x^{n+1}$. We know $c_1 x$ and $c_n x^n$ are continuous from our initial assumption about P_n, so we can see that $c_{n+1}(x^n \cdot x)$ is also continuous for any $c \in \mathbb{R}$ as a result of

$$\lim_{x \to a} (f \cdot g)(x)$$
$$\lim_{x \to a} (cf)(x), \ \forall c \in \mathbb{R}.$$

Since $g(x)$ is also continuous, we know by $\lim_{x \to a} (f + g)(x)$ that P_{n+1} is also continuous. This completes the proof.