Math 487 Homework 3

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September 22, 2017

Ch5. Q1: Let A and B be disjoint events, with $0 < \mathbb{P}(A) + \mathbb{P}(B) < 1$. Express in terms of $x = \mathbb{P}(A)$ and $y = \mathbb{P}(B)$ the probabilities of the events:

(a) A but B does not occur;

$$x(1-y)$$

(b) either A does not occur or B does not occur;

$$(1-x)+(1-y)-((1-x)\cdot(1-y))$$

(c) either A occurs or B does not occur;

$$x+(1-y)-(x\cdot(1-y))$$

(d) neither the event A nor the event B occurs;

$$(1-x)\cdot(1-y)$$

(e) either A but not B occurs, or B but not A occurs;

$$x+y-2xy$$

(f) A but not B occurs; B but not A fails to occur.

$$(x\cdot(1-y)\cdot((1-y)+x-(1-y)\cdot x))$$

Ch5. Q5: In a certain city, 53% of the adults are female and 15% are unemployed males.

(a) What is the probability that an adult chosen at random in this city is an employed male?

$$\begin{aligned}\mathbb{P}(\text{unemployed males}) &= \big(1-\mathbb{P}(\text{female})\big)\cdot\big(1-\mathbb{P}(\text{unemployed males})\big)\\ &= 47\%\cdot 85\%\\ &\sim 40\%\end{aligned}$$

(b) If the overall unemployment rate is 22%, what is the probability that an adult is an employed female?

$$\begin{aligned}
\mathbb{P}(\text{employed female}) &= \mathbb{P}(\text{female adult}) \cdot \Big(1 - \big(\mathbb{P}(\text{total unemployment}) - \mathbb{P}(\text{unemployed males})\big)\Big) \\
&= 53\% \cdot 93\% \\
&\sim 49\%
\end{aligned}$$

(c) What is the probability that an adult is employed or female (or both)?

$$\begin{aligned}
\mathbb{P}(\text{employed or female}) &= \mathbb{P}(\text{female}) + \mathbb{P}(\text{employed adult}) - \mathbb{P}(\text{both}) \\
&= 53\% + 78\% - (53\% \cdot 78\%) \\
&\sim 90\%
\end{aligned}$$

Ch5. Q8: In five tosses of a fair coin, what is the probability that the first head does not appear until the third toss or there are at least three straight heads (or both)?

Since we are dealing with a fair coin,

$$\mathbb{P}(\text{H}) = \mathbb{P}(\text{T}) = \frac{1}{2}.$$

Then, the probability of all possible combinations is $\frac{1}{2^5} = \frac{1}{32}$. The probability that the first head does not appear until the third toss or there are at least three straight heads is then,

$$\begin{aligned}
\mathbb{P}(3^{\text{rd}}) =& \big(\mathbb{P}(\text{TTHTT}) + \mathbb{P}(\text{TTHTH}) + \mathbb{P}(\text{TTHHT}) + \mathbb{P}(\text{TTHHH})\big) + \\
& \big(\mathbb{P}(\text{HHHTT}) + \mathbb{P}(\text{HHHTH}) + \mathbb{P}(\text{HHHHT}) + \mathbb{P}(\text{HHHHH}) + \\
& \mathbb{P}(\text{THHHT}) + \mathbb{P}(\text{THHHH}) + \mathbb{P}(\text{TTHHH}) + \mathbb{P}(\text{HTHHH})\big) - \\
& \mathbb{P}(\text{TTHHH}) \\
=& \frac{4}{32} + \frac{8}{32} - \frac{1}{32} \\
=& \frac{11}{32}
\end{aligned}$$

Ch5. Q9: The symmetric difference between two sets A and B is defined by the equation $A \triangle B = A \cap \overline{B} + B \cap \overline{A}$. Prove that, for events A and B, we have $\mathbb{P}(A \triangle B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$. (Do not rely on Venn Diagrams).

We can reinterpret the definition of the symmetric difference as probability,

$$
\begin{aligned}
\mathbb{P}(A \cap \overline{B} \cup B \cap \overline{A}) &= \mathbb{P}(A \cap \overline{B}) + \mathbb{P}(B \cap \overline{A}) \\
&= \big(\mathbb{P}(A) - \mathbb{P}(A \cap B))\big) + \big(\mathbb{P}(B) - \mathbb{A} \cap \mathbb{B})\big) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{A} \cap \mathbb{B} - \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)
\end{aligned}
$$

Ch6. Q3: How many ordered samples of size five, without replacement, are there in a set containing 30 elements? Solve this problem in two ways: directly and using Stirling's formula.

To model the sampling a subset of a sample space without replacement, we can use $(n)_m$, which models permutations of n items taken k at a time.

$$(n)_m = n \cdot (n-1) \cdot \ldots \cdot (n-m+1) = \frac{n!}{(n-k)!} = \frac{30!}{25!} = 17,100,720.$$

Using Stirling's Approximation, we see that

$$
\begin{aligned}
(n)_m = \frac{n!}{(n-k)!} &\sim \frac{(2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}}{(2\pi)^{\frac{1}{2}} (n-k)^{(n-k)+\frac{1}{2}} e^{-(n-k)}} \\
&\sim \frac{30^{30.5} e^{-30}}{25^{25.5} e^{-25}} \\
&\sim 17,110,222
\end{aligned}
$$

Ch6. Q5:

(a) Suppose that an ordered sample, with replacement, of size three is selected from a set of five elements. What is the probability that the sample obtained is an ordered sample without replacement (that is, that all the terms of the 3-tuple are different)?

Since 3 elements are selected from a set of 5 elements with replacement, we know that the total number of combinations can be found with $n^k = 5^3 = 125$. Then, the number of ordered samples without replacement can be found with $(n)_k$,

$$(n)_k = \frac{n!}{(n-k)!} = \frac{5!}{2!} = 60.$$

So, we find the probability that the sample obtained is an ordered sample without replacement by taking $\frac{60}{125} \sim 48\%$.

(b) What is the probability that an ordered sample, with replacement, of size m selected from a set of n elements ($n \geq m > 1$) is an ordered sample without replacement?

The number ordered samples of size m selected from a set of n elements with replacement is calculated with n^m. Then, the number of ordered samples without replacement is found with $\frac{n!}{(n-m)!}$. So, by combining those, we find that the probability of an ordered sample with replacement is an ordered sample without replacement with

$$\frac{n!}{n^m(n-m)!}$$

Ch6. Q6: How many words can be created from the word SAMPLE? A created word does not have to be an actual English word, but it may contain at most as many instances of a letter as there are in the original word (for example, 'ama' is not acceptable, whereas 'pma' is).

$$\sum_{i=1}^{6} \frac{6!}{(6-i)!} = \frac{6!}{5!} + \frac{6!}{4!} + \frac{6!}{3!} + \frac{6!}{2!} + \frac{6!}{1!} + \frac{6!}{0!} = 1,956.$$

Ch6. Q8: The standard car license plate in a certain state has seven characters. The first character is one of the digits $1, 2, 3$, or 4; the next three characters are letters (repetitions allowed); and thee final three characters are digits $(0, 1, ..., 9)$; repetitions are allowed.

(a) How many license plates are possible?

$$4 \cdot 26^3 \cdot 9^3 = 70,304,000 \text{ combinations.}$$

(b) How many have no repeated characters?

$$4 \cdot (26 \cdot 25 \cdot 24) \cdot (9 \cdot 8 \cdot 7) = 31,449,600 \text{ combinations}$$

(c) How many have a vowel (A, E, I, O, or U) as the first letter or the sequence 222 as the final three digits?

There likelihood that the first letter is a vowel is $\frac{5}{26}$, and the likelihood of 222 being the final three digits is $\frac{1}{10^3}$. So, we can now find the probability of either of those events happening,

$$\frac{5}{26} + \frac{1}{1000} - (\frac{5}{26} \cdot \frac{1}{1000}) \sim 19.3\%$$

(d) Assuming that each license plate is equally likely, what is the probability that a license plate chosen at random ends with your favorite digit?

From part (a), we know that there are $70,304,000$ possible license number. We can find the number of combinations of license plates that might end with our favorite digit with

$$4 \cdot 26^3 \cdot 10^2 \cdot 1 = 7,030,400 \text{ combinations}$$

Here we can find the probability of the of our event happening by dividing the cardinality of our event by the cardinality of all license plate possibilities,

$$\frac{\mathbb{P}(\text{license plate ending with favorite number})}{\mathbb{P}(\text{all license plate combinations})} = \frac{7,030,400}{70,304,000} = 10\%$$