MATH 487: Probability Theory Exam II

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Jacob Shiohira

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Random Variables

In general, a random variable is a symbolic representation (variable) of the outcome of experiment that depends on chance.

Discrete Random Variables

A **discrete random variable** has a countable number of outcomes. The distribution of probabilities of a discrete random variable are measured by a *probability mass function*, $p_X(x)$, such that for any $x \in \text{Range}(X)$,

$$\mathbb{P}(X = x) = p_X(x)$$

where $p_X(x) >\geq 0$ and $\sum_x p_X(x) = 1$.

Continuous Random Variables

A **continuous random variable** has an infinitely uncountable number of outcomes. In other words, it can take on all values in a given interval. The distribution of probabilities of a continuous random variable are measured by a *probability distribution function*, $f_X(x)$ such that for any $a, b \in \text{Range}(X)$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

where $f_X(x) >\geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Unlike a **probability mass function**, the probability of $f_X(x)$ such that $X = x$ is not defined in a **probability distribution function**.

PMF versus PDF

A **probability mass function** measures the total mass with the dimension of probability. A **probability distribution function** is essentially the measurement of probability on some continuum where $\rho(\frac{\text{probability}}{unit})$ is a non-constant density function $f_X(x)$.

Cumulative Distribution Functions

A **cumulative distribution function**, or CDF, of a real-valued random variable X, or just distribution function of X, evaluated at x, is the probability that X will take a value less than or equal to x. In other words, $\text{CDF}(x) = F_X(x)$ is equivalent to

$$\mathbb{P}(X \leq x) = \sum_{x_i \leq x} p_X(x)dx \qquad \text{in the Discrete Case}$$

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)dx \qquad \text{in the Continuous Case}$$

General properties of the CDF are as follows,

1. $F(x) = \mathbb{P}(X \leq x)$,
2. $0 \leq F(x) \leq 1$,
3. $F(x)$ is a non-decreasing function. If $a < b$, then $F(a) < F(b)$,
4. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$,
5. $\mathbb{P}(a \leq X \leq b) = F(b) - F(a)$,
6. $F'(x) = f_X(x)$.

Distributions

Normal, Gaussian

Generally, the **normal** distribution is used $XXXX$. The probability distribution function is

$$N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \text{for } \sigma > 0.$$

while the cumulative distribution function is

$$\iota(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} du.$$

Thus, $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

Cauchy

Generally, the **Cauchy** distribution is used for resonance behavior and used to measure outliers in data sets. The probability distribution function is

$$C(\theta, \sigma) = \frac{1}{\pi\sigma}\left[\frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}\right].$$

while the cumulative distribution function is

$$F(X) = \frac{1}{\pi}\arctan\left(\frac{x - x_0}{\gamma}\right) + \frac{1}{2}$$

The $\mathrm{E}(X)$ and the $\mathrm{Var}(X)$ are not defined for the Cauchy distribution.

Exponential

Generally, the **Exponential** distribution is used to describe timing of events, such as the times between customer arrivals, times to equipment failure, etc. The probability distribution function is

$$E(\lambda) = \lambda e^{-\lambda x}$$

while the cumulative distribution function is

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Thus, $\mathrm{E}(X) = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

Gamma

Generally, the **Gamma** distribution is used to measure non-negative random variables such as rain fall amount, plant yields, time between earthquakes, etc. The probability distribution function is

$$\Gamma(\alpha, \beta) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

while the cumulative distribution function is

$$F(x) = \frac{\Gamma_x(\gamma)}{\Gamma(\gamma)} \qquad x \geq 0; \gamma > 0$$

Thus, $\mathrm{E}(X) = \frac{\alpha}{\lambda}$ and $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$. Note that $\Gamma(n) = (n-1)!$.

Bernoulli

Generally, the **Bernoulli** distribution is used $XXXX$. The probability distribution function is

$$f_X(x) = \begin{cases} \alpha & x = 1, \\ 1 - \alpha & x = 0. \end{cases}$$

while the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x \leq 0, \\ 1 - \alpha & 0 < x < 1, \\ 1 & x \geq 1. \end{cases}$$

Thus, $\mathrm{E}(X) = p$ and $\mathrm{Var}(X) = p(1-p)$.

Binomial

Generally, the **Binomial** distribution is used $XXXXX$. The probability distribution function is

$$b(n,p,m) = \binom{n}{m} p^n (1-p)^{n-m}$$

while the cumulative distribution function is

$$F(x;p,n) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$

Thus, $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = npq$.

Geometric

Generally, the **Geometric** distribution is used $XXXXX$. The probability distribution function is

$$g(p,k) = q^{k-1} p$$

while the cumulative distribution function is

$$F(x) = 1-(1-p)^k$$

Thus, $\mathrm{E}(X) = \frac{1}{p}$ and $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

Poisson

Generally, the **Poisson** distribution is used XXX. The probability distribution function is

$$P(k,\lambda) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

The $\mathrm{E}(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$.

Joint, Marginal, Conditional Functions

Discrete Random Variables

Consider the two probability mass functions, $f_X(x) = \mathbb{P}(X = x)$ and $f_Y(y) = \mathbb{P}(Y = y)$. The **joint probability mass function** for X, Y is

$$\mathbb{P}_{XY}(x,y) = \mathbb{P}(X = x \cap Y = y).$$

Note that

$$\sum_x \sum_y \mathbb{P}_{XY}(x,y) = \sum_{(x_i,y_j)\in R_{XY}} \mathbb{P}_{XY}(x_i,y_j) = 1,$$

where $R_{XY} = \{(x_i,y_j)|x_i \in R_X, y_j \in R_Y\} = R_X \times R_Y$ where $R_X = x_1, x_2, \ldots$ and $R_Y = y_1, y_2, \ldots$.

The **marginal probability mass function** for any $x \in R_X$ is

$$\mathbb{P}_X(x) = \mathbb{P}(X = x) = \sum_{y_j \in R_Y} \mathbb{P}(X = x, Y = y_j) = \sum_{y_j \in R_Y} \mathbb{P}_{XY}(x, y_j).$$

Likewise, for any $y \in R_Y$,

$$\mathbb{P}_Y(y) = \mathbb{P}(Y = y) = \sum_{x_i \in R_X} \mathbb{P}(X = x_i, Y = y) = \sum_{x_i \in R_X} \mathbb{P}_{XY}(x_i, y).$$

The **conditional probability mass function** of Y given $X = x$ is

$$f_{Y|x} \frac{f_{XY}(x,y)}{f_X(x)} \qquad \text{for } f_X(x) > 0.$$

In other words, $f_{Y|x}$ is equivalent to the joint probability of X and Y divided by the marginal probability for X. A few common properties of the **conditional probability mass function** are as follows,

1. $f_{Y|x}(y) \geq 0$,
2. $\sum_y f_{Y|x} = 1$,
3. $f_{Y|x}(y) = \mathbb{P}(Y = y|X = x)$.

Continuous Random Variables

Law of Total Probability

$$\mathbb{P}(x) = f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x,y) f_Y(y) dy.$$

Joint Cumulative Distribution Function

For two Random Variables X, Y, the joint Cumulative Distribution Function is

$$F_{XY}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}((X \leq x) \cap (Y \leq y)).$$

Marginal Cumulative Distribution Function

For any $x \in X$,

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} F_{XY}(x, y).$$

For any $y \in Y$,

$$F_Y(y) = F_{XY}(\infty, y) = \lim_{x \to \infty} F_{XY}(x, y).$$

Independent Random Variables

Random variables X, Y are independent *if and only if*

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

for any subsets $A, B \in \mathbb{R}^2$.

Expected Value of Random Variables

The expected value of a random variable is also its mean. So, $\mathrm{E}(X) = \mu$. Common properties of the expected value of random variables are as follows,

1. $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$,
2. $\mathrm{E}(cX) = c\mathrm{E}(X)$,
3. If X, Y are independent, $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

Discrete Random Variables

In the case of a discrete random variable,

$$\mathrm{E}(X) = \sum_{k \in \Omega} k m(x)$$

for distribution function $m(x)$.

Martingales If X is a random variable, Ω is a sample space, and $F_1, F_2, \ldots, F_r$ are events such that $F_i \cap F_j = \emptyset$ for $i \neq j$ and $\Omega = \bigcup_j F_j$, then

$$\mathrm{E}(X) = \sum_j \mathrm{E}(X|F_j)\mathbb{P}(F_j).$$

Continuous Random Variables

Variance of Random Variables

Variance is how much the outcome of an experiment varies from the expected value, or mean, $\mathrm{E}(X)$. Common properties of the variance of random variables are as follows,

1. $\mathrm{Var}(X+b) = \mathrm{Var}(X)$,
2. $\mathrm{Var}(cX) = c^2\mathrm{E}(X)$,
3. If X, Y are independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Discrete Random Variables

Generally, for a distribution function $m(x)$,

$$\mathrm{Var}(X) = \mathrm{E}((X-\mu)^2) = \sum_x (x-\mu)^2 m(x) = \mathrm{E}(X^2) - \Big[\mathrm{E}(X)^2\Big].$$

Continuous Random Variables

Convolutions