Math 487 Homework 6

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Section 7.2

Ch 7.2 Q3 Suppose again that $Z = X + Y$. Find f_Z if

(a)

$$f_X(x) = f_Y(x) = \begin{cases} \frac{x}{2} & \text{if } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Given $Z = X + Y$, $0 < X < 2$, and $0 < Y < 2$, we know that f_Z will take on the following,

$$\int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \quad \text{or} \quad \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx.$$

We will use the former. From the restrictions on X and Y, we can derive the following, $y < z < 2+y$ and $z-2 < y < z$. These will help us calculate our bounds of integration and ranges of the Z random variable. The Z ranges over which f_Z will have a value is $0 < z < 2$ and $2 < z < 4$. So, we will go ahead and calculate those,

$$f_Z(x) = \int_0^z \frac{z-2}{2} - \frac{y}{2}\, dy = \int_0^z \frac{zy}{4} - \frac{y^2}{4}\, dy = \left[\frac{zy^2}{8} - \frac{y^3}{12} \right]_0^z = \frac{z^3}{8} - \frac{z^3}{12} = \frac{z^3}{24},$$

$$
\begin{aligned}
f_Z(x) &= \int_{z-2}^{2} \frac{z-2}{2} - \frac{y}{2}\, dy \\
&= \int_{z-2}^{2} \frac{zy}{4} - \frac{y^2}{4}\, dy \\
&= \left[\frac{zy^2}{8} - \frac{y^3}{12}\right]_{z-2}^{2} \\
&= \frac{4}{8}z - \frac{8}{12} - \left[\frac{z(z-2)^2}{8} - \frac{(z-2)^3}{12}\right] \\
&= \frac{4}{8}z - \frac{8}{12} - \left[\frac{z^3-4z^2+4z}{8} - \frac{z^3-6z^2+12z-8}{12}\right] \\
&= \frac{4}{8}z - \frac{8}{12} - \frac{3z^3-12z^2+12z-2z^3+12z^2-24z+16}{24} \\
&= z - \frac{z^3}{24} - \frac{4}{3}.
\end{aligned}
$$

Thus,

$$
f_Z(x) = \begin{cases} \frac{x^3}{24} & \text{if } 0 \le x \le 2, \\ x - \frac{x^3}{24} - \frac{4}{3} & \text{if } 2 \le z \le 4. \end{cases}
$$

(b)

$$
f_X(x) = f_Y(x) = \begin{cases} \frac{1}{2}(x-3) & \text{if } 3 < x < 5, \\ 0 & \text{otherwise.} \end{cases}
$$

Given $Z = X + Y$, $3 < X, Y < 5$, we know that f_Z will take on the following,

$$
\int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \quad \text{or} \quad \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx.
$$

We will use the former. From the restrictions on X and Y, we can derive the following, $3 + y < z < 5 + y$ and $z - 5 < y < z - 3$. These will help us calculate our bounds of integration and ranges of the Z random variable. The Z ranges over which f_Z will have a value is $6 < z < 8$ and $6 < z < 8$. So, we will go ahead and calculate those,

$$
\begin{aligned}
f_Z(x) &= \int_3^{z-3} \frac{y^2 + z\,(3-y) - 9}{4}\ dy \\
&= \frac{1}{4}\left[\frac{y^3}{3} - \frac{zy^2}{2} + (3z-9)\,y\right]_3^{z-3} \\
&= -\frac{z^3 - 18z^2 + 81z - 108}{24} - \frac{9z-36}{8} \\
&= \frac{z^3 - 18z^2 + 108z - 216}{24}
\end{aligned}
$$

$$
\begin{aligned}
f_Z(x) &= \int_{z-5}^{5} \frac{y^2 + z\,(3-y) - 9}{4}\ dy \\
&= \frac{1}{4}\left[\frac{y^3}{3} - \frac{zy^2}{2} + (3z-9)\,y\right]_{z-5}^{5} \\
&= \frac{z^3 - 18z^2 + 69z - 20}{24} + \frac{15z-20}{24} \\
&= \frac{z^3 - 18z^2 + 84z - 40}{24}
\end{aligned}
$$

Thus,

$$
f_Z(x) = \begin{cases} \dfrac{z^3 - 18z^2 + 108z - 216}{24} & \text{if } 6 \le x \le 8, \\ \dfrac{z^3 - 18z^2 + 84z - 40}{24} & \text{if } 8 \le x \le 10. \end{cases}
$$

(c)

$$
f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
f_Y(x) = \begin{cases} \frac{x}{2} & \text{if } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}
$$

Given $Z = X + Y$, $0 < X < 2$, and $0 < Y < 2$, we know that f_Z will take on the following,

$$
\int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \quad \text{or} \quad \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx.
$$

We will use the former. From the restrictions on X and Y, we can derive the following, $y < z < 2+y$ and $z-2 < y < z$. These will help us calculate our bounds of integration and ranges of the Z random variable. The Z ranges over which f_Z will have a value is $0 < z < 2$ and $2 < z < 4$. So, we will go ahead and calculate those,

$$f_Z(x) = \int_0^z \frac{y}{4}\,dy = \left[\frac{y^2}{8}\right]_0^z = \frac{z^2}{8},$$

$$f_Z(x) = \int_{z-2}^2 \frac{y}{4}\,dy = \left[\frac{y^2}{8}\right]_{z-2}^2 = \frac{1}{2} - \frac{(z-2)^2}{8}.$$

Thus,

$$f_Z(x) = \begin{cases} \frac{z^2}{8} & \text{if } 0 \leq x \leq 2, \\ \frac{1}{2} - \frac{(z-2)^2}{8} & \text{if } 2 \leq z \leq 4. \end{cases}$$

Ch 7.2 Q9 Assume that the service time for a customer at a bank is exponentially distributed with mean service time 2 minutes. Let X be the total service time for 10 customers. Estimate the probability that $X > 22$ minutes.

We can view X as the sum of 10 independent, exponentially distributed random variables with mean service time of 2 minutes. Let X_i with $i = 0, \ldots, 10$ be these independent random variables. Then,

$$X = X_1 + X_2 + \cdots + X_{10}.$$

Since all the X_i are exponentially distributed with mean $\frac{1}{\lambda}$, then

$$X = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}.$$

Thus,

$$\mathbb{P}(X > 22) = 1 - \int_0^{22} \frac{.5e^{-.5x}(.5x)^9}{9!} dx = .341.$$

Ch 7.2 Q10 Let $X_1, X_2, \ldots, X_n$ be n independent random variables each of which has an exponential density with mean μ. Let M be the *minimum value* of the X_j . Show that the density for M is exponential with mean μ/n. Hint: Use cumulative distribution functions.

Since $X_1, X_2, \ldots, X_n$ are n independent random variables with exponential density and mean μ, the probability distribution of X_j for $j = 1, 2, \ldots, n$ is

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}.$$

It follows that the cumulative distribution function can be calculated from the integral of $f(x)$,

$$F(x) = \int \frac{1}{\mu} e^{-\frac{x}{\mu}} = e^{-\frac{x}{\mu}} + C.$$

We can plug in $F(0) = 0$ to find that $C = -1$ and $F(x) = 1 - e^{-\frac{x}{\mu}}$. Then, if M is the minimum value of X_j for $j = 1, 2, \ldots, n$, we can find $F_M(x)$,

$$P(\min(X_1, \ldots, X_n) > M) = \big[P(X_1 > x)\big]^n = \big[1 - F(x)\big]^n = \big[1 - (1 - e^{-x/\mu})\big]^n = 1 - e^{-\frac{n}{\mu}x}.$$

Then, we can find $f_M(X)$ by differentiating $F_M(x)$ to get

$$f_M(x) = \frac{n}{\mu} e^{-\frac{n}{\mu}x}.$$

This then satisfies that the minimum value of X_j for $j = 1, 2, \ldots, n$ has exponential distribution with mean n/μ.

Ch 7.2 Q11 A company buys 100 light bulbs, each of which has an exponential lifetime of 1000 hours. What is the expected time for the first of these bulbs to burn out? (See Exercise 10.)

Since we are looking for the first bulb to burn out, we can refer to the result of the last problem. Let X_1 denote the first bulb to burn out. Then, $\mathrm{E}(X_1) = \mu/n = 1000/100 = 10$ hours.

Ch 7.2 Q13 Particles are subject to collisions that cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction is uniformly distributed between 0 and 1. Following a single particle through several splittings we obtain a fraction of the original particle $Z_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$ where each X_j is uniformly distributed between 0 and 1. Show that the density for the random variable Z_n is

$$f_n(z) = \frac{1}{(n-1)!}(-\log z)^{n-1}.$$

Hint: Show that $Y_k = -\log X_k$ is exponentially distributed. Use this to find the density function for $S_n = Y_1 + Y_2 + \cdots + Y_n$, and from this the cumulative distribution and density of $Z_n = e^{S_n}$.

Since $Y_k = -\log X_k$, we see that

$$P(Y_k \le y_k) = P(-\log X_k \le y_k) = P(X_k \ge e^{-y_k}) = 1 - P(X_k \le e^{-y_k}) = 1 - e^{-y_k}.$$

Thus, Y_k is exponentially distributed. Then, it follows that $Y_k \sim \exp(1)$ for all $k = 1, 2, \ldots, n$. So, S_n is distributed according to the gamma function,

$$f_{S_n}(x) = \frac{e^{-x}x^{n-1}}{(n-1)!}.$$

Then, per the hint, $Z_n = e^{S_n}$ can be used to yield the cumulative distribution of f_A. So, we will start with

$$F_{Z_n}(z) = P(Z_n \leq z) = P(e^{-S_n} \leq z) = P(S_n \geq -\ln z) = 1 - P(Z_n \leq -\ln z) = 1 - F_s(-\ln z).$$

By the properties of probability density functions, we can differentiate $F_{Z_n}(z)$ with respect to z to get,

$$f_{Z_n}(z) = \frac{d}{dz}1 - F_s(-\ln z) = \frac{(-\ln z)^{n-1}}{(n-1)!}.$$

Ch 7.2 Q14 Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density

$$f_Z(z) = \frac{1}{2}\lambda e^{-\lambda|z|}.$$

We can think of $Z = X_1 - X_2$ as $Z = X_1 + (-X_2)$. Then, it is easily seen that X_1 and X_2 have probability density functions

$$f_{X_1}(x) = \begin{cases} 0 & \text{for } x < 0, \\ \lambda e^{-\lambda x} & \text{for } x > 0. \end{cases} \quad f_{X_2}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases}$$

The convolution of $Z = X_1 + (-X_2)$ is therefore

$$f_Z(z) = (f_1 \cdot f_2)(z) = \int_{-\infty}^{\infty} f_1(z-x)f_2(x)\ dx.$$

As a result of the ranges of X_1 and X_2, the analysis on Z can be broken down into the two cases: $z > 0$ and $z < 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{z} f_1(z-x)f_2(x)\ dx \\ &= \lambda^2 e^{-\lambda z}\int_{-\infty}^{z} e^{2\lambda x}\ dx \\ &= \lambda^2 e^{-\lambda z}\frac{e^{2\lambda x}}{2\lambda}\ dx \\ &= \frac{\lambda}{2}e^{-\lambda z} \end{aligned}$$

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{0} f_1(z-x)f_2(x)\ dx \\
&= \lambda^2 e^{-\lambda z} \int_{-\infty}^{0} e^{2\lambda x}\ dx \\
&= \lambda^2 e^{-\lambda z} \frac{1}{2\lambda} \\
&= \frac{\lambda}{2} e^{-\lambda z}
\end{aligned}
$$

Since $f_Z(z)$ has the same definition when $z < 0$ and $z > 0$, we can

$$f_Z(z) = \frac{\lambda}{2} e^{-\lambda |z|}.$$

Ch 7.2 Q20 Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables, all having a common density function f_X with support $[a, b]$ (see Exercise 19). Let $S_n = X_1 + X_2 + \cdots + X_n$, with density function f_{S_n}. Show that the support of f_{S_n} is the interval $[na, nb]$. Hint: Write $f_{S_n} = f_{S_{n-1}} \cdot f_X$. Now use Exercise 19 to establish the desired result by induction.

Suppose that $n = 1$. Then, $f_{S_n} = f_{S_1} = f_{X_1}$ has support in the interval $[na, nb] = [a, b]$ because f_X is defined to have support $[a, b]$.

Next, suppose that for some $k \geq n$, f_{S_k} has support on $[ka, kb]$. Then, since we know $f_{S_n} = f_{S_{n-1}} \cdot f_X$, we can say that $f_{S_{k+1}} = f_{S_k} \cdot f_X$. $f_{S_{k+1}}$ then has support on $\big[ka + a, kb + b\big] = \big[(k+1)a, (k+1)b\big]$.

Thus, by induction, we have shown that the support of f_{S_n} is the interval $[na, nb]$ for all n.

Ch 7.2 Q21 Let $X_1, X_2, ..., X_n$ be a sequence of independent random variables, all having a common density function f_X. Let $A = S_n/n$ be their average. Find f_A if

(a) $f_X(x) = (1/\sqrt{2\pi})e^{-\frac{x^2}{2}}$ (normal density).

Since we know that $A = S_n/n$, $\mathrm{E}(A) = \mathrm{E}(S_n/n) = \frac{1}{n}\mathrm{E}(S_n)$. Then, since $X_1, X_2, ..., X_n$ is a sequence of independent random variables, the expected value of the sums is the sum of the expected values of each X_i for $i = 1, 2, \ldots$. So, $\mathrm{E}(S_n) = n \cdot \mathrm{E}(X_i) = n\mu$. By plugging in $\mathrm{E}(X_i)$, we can see that $\mathrm{E}(A) = \mu_{X_i} = 0$.

A similar approach can be taken with the variance such that $\mathrm{V}(A) = \mathrm{V}(S_n/n) = \frac{1}{n^2}\mathrm{V}(S_n) = \frac{1}{n^2}n = \frac{1}{n}$.

Then, we use the fact that the sum of independent normally distributed random variables is normally distributed to plug in our values of μ and λ for A,

$$f_A(x) = \frac{1}{\sqrt{\frac{n}{2\pi}}} e^{\frac{-nx^2}{2}}.$$

(b) $f_X(x) = e^{-x}$ (exponential density). *Hint*: Write $f_A(x)$ in terms of $f_{S_n}(x)$.

We know that the sum of exponentially distributed random variables is gamma distributed. So, we need to find parameters α and β that satisfy $f_A(x)$ based on the distribution of $f_X(x) = e^{-x}$. This is clearly exponentially distributed with parameter $\lambda = 1$. The resulting expected value is $\mathrm{E}(x) = \frac{1}{\lambda}$ and the resulting variance is $\mathrm{E}(x) = \frac{1}{\lambda^2}$.

The resulting distributions of the sums of random variables, $S_n = \frac{e^{-x}x^{n-1}}{(n-1)!}$. The parameters of this distribution are thus $\alpha = n$ and $\beta = 1$. Since we know that $A = S_n/n$, $\mathrm{E}(A) = \mathrm{E}(S_n/n) = \frac{1}{n}\mathrm{E}(S_n) = \frac{1}{n}\frac{\alpha}{\beta} = \frac{1}{n}\frac{n}{1} = 1$.

A similar approach can be taken with the variance such that $\mathrm{V}(A) = \mathrm{V}(S_n/n) = \frac{1}{n^2}\mathrm{V}(S_n) = \frac{1}{n^2}n = \frac{1}{n} = \frac{\alpha}{\beta^2}$. Since $\mathrm{E}(A) = \frac{\alpha}{\beta} = 1$, it is seen that that $\alpha = \beta$. By plugging this into our variance equation, we can solve for both α and β such that $\alpha = \beta = n$.

Then, we use the fact that the sum of independent exponentially distributed random variables is gamma distributed to plug in our values of α and β for A,

$$f_A(x) = \frac{n^n x^{n-1} e^{-xn}}{(n-1)!}.$$