MATH20122 Cheat Sheet

1 Definitions and Examples

metric space: a **metric space** (X, d) consists of a non-empty set X and a non-negative real valued **metric** (*distance function*) $d : X \times X \to \mathbb{R}^{\geq}$ which satisfies the following axioms:
(i) $d(x, y) = 0 \iff x = y$ for all $x, y \in X$
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the *triangle inequality*)

subspace: given any subset $W \subseteq X$, the restriction of d to W determines the subspace $(W, d := d|_W)$ of (X, d)

open ball: for any metric space (X, d), the open ball of radius $r > 0$ around any $x \in X$ is $B_r(x) := \{y : d(y, x) < r\}$

closed ball: for any metric space (X, d), the open ball of radius $r > 0$ around any $x \in X$ is $\bar{B}_r(x) := \{y : d(y, x) \leq r\}$

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euclidean n-space: $(\mathbb{R}^n, d_2)$ consists of all real n-dimensional vectors $x = (x_1, \dots, x_n)$, equipped with the **euclidean metric** $d_2(x, y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$ where the positive square root is understood

taxicab metric: d_1 is given on $\mathbb{R}^n$ is given by $d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$$\textbf{discrete metric} : d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

isometry: for any two metric spaces (X, d_X) and (Y, d_Y), a bijection $f : X \to Y$ is an isometry whenever $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$

standard metric: $d_{\mathbb{C}}$ on the complex numbers $\mathbb{C}$ is given by $d_{\mathbb{C}}(z, z') = |z - z'|$

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graph: $\Gamma := (V, E)$ consists of a set V of **vertices** and a set E of **edges**

path: a **path** in Γ from u to w is a finite sequence of edges $\pi(u, w) = (uv_1, v_1v_2, \dots, v_{n-2}v_{v-1}, v_{n-1}w)$ with length n

path connected: a graph is path connected whenever there is a path joining *any* pair of vertices

edge metric: e on the vertex set V of a path connected graph is defined by $e(u, w) = min_{\pi(u,w)} l(\pi(u, w))$

alphabet: a finite set A of **letters** and a finite sequence of letters is a **word** in A. the vertex set W of the associated **word graph** $\Gamma(A)$ consists of all possible words in A. word w_1 and w_2 are joined by an edge iff they differ by one of (i) inserting or deleting a letter (ii) swapping two adjacent letters (iii) replacing one letter with another

word metric: d_w on W is the edge metric on the associated word graph

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binary sequences: $X = \{0, 1\}^\infty$ is the set of all infinite binary sequences $x = x_0x_1 \dots$ where $x_n = 0$ or 1 for all $n \geq 0$

$$d_{min}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{if } n = min\{m : x_m \neq y_m\} \end{cases}$$

$$d^*(x, y) = \sum_{j=0}^{\infty} \frac{|x_j - y_j|}{2^j}$$

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bounded: a real valued function f on a closed interval $[a, b] \subset \mathbb{R}$ is bounded whenever $\exists K, |f(x)| \leq K, \forall x \in [a, b]$

let X denote the set of all bounded $f : [a, b] \to \mathbb{R}$ then

$$d_{sup}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| \text{ with } (X, d_{sup}) \text{ denoted by } \mathcal{B}[a, b]$$

let Y denote the set of all continuous $f : [a, b] \to \mathbb{R}$ then

$$d_1(f, g) = \int_a^b |f(t) - g(t)|dt \text{ with } (Y, d_1) \text{ denoted by } \mathcal{L}_1[0, 1]$$

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let X denote the set of all closed intervals $[a, b]$ in the euclidean line

interval metric: d_H on X is given by $d_H([a, b], [r, s]) = max\{|r - a|, |s - b|\}$

$d_\infty((x_1, x_2), (y_1, y_2)) = max\{|x_1 - y_1|, |x_2 - y_2|\}$

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let X be the set of infinite sequences $(a_i : i \geq 0)$ of reals, such that $\sum_i a_i$ is absolutely convergent

$$d_1((a_i), (b_i)) = \sum_{i \geq 0} |a_i - b_i|$$

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cartesian product: of two metric spaces (X, d) and (X', d') is the set $X \times X'$ with one of the metrics

1. $d_a((x, x'), (y, y')) = d(x, y) + d'(x', y')$
2. $d_b((x, x'), (y, y')) = (d(x, y)^2 + d'(x', y')^2)^{1/2}$
3. $d_c((x, x'), (y, y')) = max\{d(x, y), d'(x', y')\}$

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lipschitz equivalent: two metrics d and e on a given set X are lipschitz equivalent whenever there exists positive constants $h, k \in \mathbb{R}$ such that $he(x, y) \leq d(x, y) \leq ke(x, y)$ for every $x, y \in X$

theorem: the metrics d_a, d_b, d_c on $X \times X'$ are lipschitz equivalent