MATH20122 Cheat Sheet

1 Definitions and Examples

metric space: a **metric space** (X, d) consists of a non-empty set X and a non-negative real valued **metric** (*distance function*) $d : X \times X \to \mathbb{R}^{\geq}$ which satisfies the following axioms:
(i) $d(x, y) = 0 \iff x = y$ for all $x, y \in X$
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the *triangle inequality*)

subspace: given any subset $W \subseteq X$, the restriction of d to W determines the subspace $(W, d := d|_W)$ of (X, d)

open ball: for any metric space (X, d), the open ball of radius $r > 0$ around any $x \in X$ is $B_r(x) := \{y : d(y, x) < r\}$

closed ball: for any metric space (X, d), the open ball of radius $r > 0$ around any $x \in X$ is $\bar{B}_r(x) := \{y : d(y, x) \leq r\}$

---

euclidean n-space: $(\mathbb{R}^n, d_2)$ consists of all real n-dimensional vectors $x = (x_1, \ldots, x_n)$, equipped with the **euclidean metric** $d_2(x, y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$ where the positive square root is understood

taxicab metric: d_1 is given on $\mathbb{R}^n$ is given by $d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$$\textbf{discrete metric} : d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

isometry: for any two metric spaces (X, d_X) and (Y, d_Y), a bijection $f : X \to Y$ is an isometry whenever $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$

standard metric: $d_{\mathbb{C}}$ on the complex numbers $\mathbb{C}$ is given by $d_{\mathbb{C}}(z, z') = |z - z'|$

---

graph: $\Gamma := (V, E)$ consists of a set V of **vertices** and a set E of **edges**

complete graph: K_n is a complete graph if it contains n vertices and one edge between every pair

path: a **path** in Γ from u to w is a finite sequence of edges $\pi(u, w) = (uv_1, v_1v_2, \ldots, v_{n-2}v_{v-1}, v_{n-1}w)$ with length n

path connected: a graph is path connected whenever there is a path joining *any* pair of vertices

edge metric: e on the vertex set V of a path connected graph is defined by $e(u, w) = min_{\pi(u,w)} l(\pi(u, w))$

alphabet: a finite set A of **letters** and a finite sequence of letters is a **word** in A. the vertex set W of the associated **word graph** $\Gamma(A)$ consists of all possible words in A. word w_1 and w_2 are joined by an edge iff they differ by one of (i) inserting or deleting a letter (ii) swapping two adjacent letters (iii) replacing one letter with another

word metric: d_w on W is the edge metric on the associated word graph

---

binary sequences: $X = \{0, 1\}^\infty$ is the set of all infinite binary sequences $x = x_0x_1 \ldots$ where $x_n = 0$ or 1 for all $n \geq 0$

$$d_{min}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{if } n = min\{m : x_m \neq y_m\} \end{cases}$$

$$d^*(x, y) = \sum_{j=0}^{\infty} \frac{|x_j - y_j|}{2^j}$$

---

bounded: a real valued function f on a closed interval $[a, b] \subset \mathbb{R}$ is bounded whenever $\exists K, |f(x)| \leq K, \forall x \in [a, b]$

let X denote the set of all bounded $f : [a, b] \to \mathbb{R}$ then

$$d_{sup}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| \text{ with } (X, d_{sup}) \text{ denoted by } \mathcal{B}[a, b]$$

let Y denote the set of all continuous $f : [a, b] \to \mathbb{R}$ then

$$d_1(f, g) = \int_a^b |f(t) - g(t)|dt \text{ with } (Y, d_1) \text{ denoted by } \mathcal{L}_1[0, 1]$$

---

let X denote the set of all closed intervals $[a, b]$ in the euclidean line

interval metric: d_H on X is given by $d_H([a, b], [r, s]) = max\{|r - a|, |s - b|\}$

$d_\infty((x_1, x_2), (y_1, y_2)) = max\{|x_1 - y_1|, |x_2 - y_2|\}$

---

let X be the set of infinite sequences $(a_i : i \geq 0)$ of reals, such that $\sum_i a_i$ is absolutely convergent

$$d_1((a_i), (b_i)) = \sum_{i \geq 0} |a_i - b_i|$$

---

cartesian product: of two metric spaces (X, d) and (X', d') is the set $X \times X'$ with one of the metrics

1. $d_a((x, x'), (y, y')) = d(x, y) + d'(x', y')$
2. $d_b((x, x'), (y, y')) = (d(x, y)^2 + d'(x', y')^2)^{1/2}$
3. $d_c((x, x'), (y, y')) = max\{d(x, y), d'(x', y')\}$

---

lipschitz equivalent: two metrics d and e on a given set X are lipschitz equivalent whenever there exists positive constants $h, k \in \mathbb{R}$ such that $he(x, y) \leq d(x, y) \leq ke(x, y)$ for every $x, y \in X$

theorem: the metrics d_a, d_b, d_c on $X \times X'$ are lipschitz equivalent

2 Open and Closed Sets

in this section, let (X, d) be a any metric space and $U \subseteq X$

interior point: $u \in U$ such that $\exists \varepsilon > 0, B_\varepsilon(u) \subseteq U$, the **interior** of U is the subset $U^\circ \subseteq U$ of all interior points and if $U^\circ = U$, then U is **open** in X

proposition: every open ball $B_r(x)$ is open in X

theorem: given any two subsets $U, V \subseteq X$, the following hold:

- $U \subseteq V \implies U^\circ \subseteq V^\circ$
- $(U^\circ)^\circ = U^\circ$
- U° is open in X
- U° is the largest subset of U, open in X

theorem: the sets X and $\emptyset$ are open in X, so are an arbitrary union $U = \bigcup_{i \in I} U_i$ of open sets U_i, and a finite intersection $U' = U'_1 \cap \cdots \cap U'_m$ of open sets U'_j

closure point: $x \in X$ is a closure point of $U \subseteq X$ if $B_\varepsilon(x) \cap U$ is non-empty for every $\varepsilon > 0$, the **closure** of U is the superset $\overline{U} \supseteq U$ of all closure points and if $\overline{U} = U$, then U is **closed** in X

proposition: a set V is closed in X iff its complement $U := X \backslash V$ is open

corollary: every closed ball $\overline{B}_r(x)$ is closed in X

partially open ball: in X is a set $P_r(x) := B_r(x) \cup P$, where P is a proper subset of $\{p : d(x, p) = r\}$

theorem: given any two subsets $U, V \subseteq X$, the following hold:

- $U \subseteq V \implies \overline{U} \subseteq \overline{V}$
- $\overline{\overline{V}} = \overline{V}$
- $\overline{V}$ is closed in X
- $\overline{V}$ is the smallest set containing V, closed in X

theorem: the sets X and $\emptyset$ are closed in X, so are an arbitrary intersection $V = \bigcap_{i \in I} V_i$ of closed sets V_i and a finite union $V' = V_1' \cup \cdots \cup V_m'$ of closed sets V_j'

---

sequence: a sequence in X is a function $s : \mathbb{N} \to X$ and it is standard practise to write $s(n)$ as x_n and display the sequence as $(x_n : n \geq 1)$

converges: a sequence (x_n) converges to the point $x \in X$ whenever $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $n \geq N \implies d(x, x_n) < \varepsilon$ and in this situation, we say that x is the **limit** of (x_n)

theorem: in any metric space (X, d), the limit of a convergent sequence is unique

theorem: suppose that $Y \subseteq X$ and $y \in X$, then y lies in $\overline{Y}$ iff there exists a sequence (y_n) in Y such that $y_n \to y$ as $n \to \infty$

cauchy sequence: in any metric space (X, d), a cauchy sequence (x_n) satisfies $\forall \varepsilon < 0, \exists N \in \mathbb{N}$, such that $m, n \geq N \implies d(x_n, x_m) < \varepsilon$

dense: a subset Y is dense in (X, d) whenever $\overline{Y} = X$

---

bounded: a subset A of the metric space (X, d) is bounded whenever there exists $x_0 \in X$ and $M \in \mathbb{R}$ such that $d(x, x_0) \leq M$ for every $x \in A$. a function $f : S \to X$ is bounded whenever its image $f(S) \subset X$ is a bounded, for any set

diameter: the diameter, $diam(A)$ of a bounded non-empty set $A \subseteq X$ is the real number $sup\{d(x, y) : x, y \in A\}$

eucildean (n - 1)-sphere of radius r: $\{x : |x| = r\}$

boundary point: $x \in X$ of A is one for which every open ball $B_\varepsilon(x)$ meets both A and $X \setminus A$, the **boundary** δA of A is the set of all such boundary points

theorem: any subset A of (X, d) satisfies:

- $A \setminus \delta A = \overline{A} \setminus \delta A = A^\circ$
- $\delta A = \delta(X \setminus A)$
- δA is closed in X

3 Uniform Convergence

in this section, the functions under consideration are of the form $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ denotes a generic **domain** in the Euclidean line (usually D is a interval)

pointwise convergence: a sequence $(f_n : n \geq 1)$ of functions converges pointwise to f on D whenever the sequence of real numbers $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$ for every $x \in D$

uniform convergence: a sequence $(f_n : n \geq 1)$ of functions converges uniformly to f on D whenever $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ such that $n \geq N(\varepsilon) \implies |f_n(x) - f(x)| < \varepsilon$ for every $x \in D$

proposition: let f and $f_n : D \to \mathbb{R}$ be functions on the domain D, then $f_n \to f$ uniformly on D iff $sup_{x \in D}|f_n(x) - f(x)|$ exists for sufficiently large n, and tends to 0 and $n \to \infty$

theorem: if $f_n : [a, b] \to \mathbb{R}$ is continuous for every $n \in N$, and $f_n \to f$ uniformly on $[a, b]$, then f is continuous on $[a, b]$

corollary: suppose that $f_n : [a, b] \to \mathbb{R}$ is continuous for every $n \in N$ and that the pointwise limit of the sequence (f_n) is discontinuous on $[a, b]$, then the convergence cannot be uniform

theorem: if $f_n : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ for every $n \in N$ and (f_n) converges uniformly on $[a, b]$ then

$$\lim_{n \to \infty} \int_a^b f_n(t)\ dt = \int_a^b \lim_{n \to \infty} \int_a^b f_n(t)\ dt$$

4 Continuous Functions

continuous at x_0 in X: given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is continuous at x_0 in X whenever $\forall \varepsilon > 0, \exists \delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$

continuous: if f is continuous at every $x_0 \in X$, then f is continuous. a continuous function is often called a **map**

theorem: the function $f : X \to Y$ is continuous at w in X iff (w_n) converges to w in $X \implies f(w_n)$ converges to $f(w)$ in Y

corollary: the function $f : X \to Y$ is continuous iff (w_n) converges to w in $X \implies f(w_n)$ converges to $f(w)$ in Y for every $w \in X$

inverse image: for any function $f : X \to Y$ and any subset $U \subseteq Y$, the inverse image $f^{-1}(U)$ is the subset $\{x : f(x) \in U\}$. this notation does *not* imply that f is invertible, or a bijection - although it sometimes may be!

theorem: given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is continuous iff U open in $Y \implies f^{-1}(U)$ open in X

theorem: given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is continuous iff V closed in $Y \implies f^{-1}(V)$ closed in X

theorem: if $f : X \to Y$ and $g : Y \to Z$ are continuous functions defined on the metric spaces $(X, d_X), (Y, d_Y)$, and (Z, d_Z), then the composition $g \cdot f : X \to Z$ is also continuous

lipschitz equivalence: for any two metric spaces (X, d_X) and (Y, d_Y), a bijection $f : X \to Y$ is a lipschitz equivalence whenever there exists positive constants $h, k \in \mathbb{R}$ such that $hd_Y(f(w), f(x)) \leq d_X(w, x) \leq kd_Y(f(w), f(x))$

homeomorphism: a bijection $f : X \to Y$ is a homeomorphism whenever f and f^{-1} are both continuous

proposition: the identity map 1_X is an isometry iff $d = e$, it is a lipschitz equivalence iff d and e are lipschitz equivalent (in the sense of section 1 definition)

topologically equivalent: two metrics d and e on a set X are topologically equivalent whenever the identity function 1_X is a homeomorphism

proposition: two metrics d and e on X are topologically equivalent iff they give rise to precisely the same open sets

path connected: a metric space X is path connected if every two points $x_0, x_1 \in X$ admit a continuous function $\sigma : [0, 1] \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$, then σ is a path from x_0 to x_1 in X

proposition: if $f : X \to Y$ is a homeomorphism, then X is path connected iff Y is path connected

corollary: as subspaces of the euclidean plane, the interval $[0, 1]$ and the unit circle S_1 are not homoemorphic

5 Compactness

covering: a covering of A is a collection of sets $\mathcal{U} = \{U_i : i \in I\}$ for which $A \subseteq \bigcup_{i \in I} U_i$. a subcovering of $\mathcal{U}$ is a subcollection $\{U_i : i \in J\}$ which also covers A, for some $J \subseteq I$. if every U_i is open, the $\mathcal{U}$ is an opening covering of A

compact: a subset $A \subseteq X$ is compact if every open covering of A contains a finite subcovering

proposition: if A is finite, then it is compact

theorem: if A is compact, then it is bounded

proposition: any finite closed interval $[a, b]$ of the euclidean line is compact

theorem: if $f : X \to Y$ is continuous, and $A \subseteq X$ is compact, then the image $f(A) \subseteq Y$ is also compact

theorem: if $A_1, \dots, A_r$ are compact subsets of X, so is $A := A_1 \cup \dots \cup A_r$

---

sequentially compact: a subspace $S \in X$ is sequentially compact if any infinite sequence $(x_n : n \geq 0)$ in S has a subsequence $(x_{n_r} : r \geq 1)$ that converges to a point in S

lemma: let (x_n) be an infinite sequence in X, and let $x \in X$. if, for any $\varepsilon > 0$, the ball $B_\varepsilon(x)$ contains x_n for infinitely many values of n, then (x_n) contains a subsequence that converges to x in X

lemma: suppose that the infinite sequence (x_n) contains no convergent subsequences. then for every $x \in X$ there exists an $\varepsilon(x) > 0$ such that $B_{\varepsilon(x)}(x)$ contains x_n for at most finitely many n

proposition: if a subspace $A \subseteq X$ is compact, then it is closed

theorem: if a subspace $A \subseteq X$ is compact, then it is sequentially compact

---

proposition: any closed subspace $V \subseteq X$ of a compact metric space (X, d) is itself compact

theorem: if a subspace $A \subseteq X$ is compact, then it is closed and bounded

heine-borel theorem: if a subspace $A \subseteq \mathbb{R}^n$ is closed and bounded, then it is compact

corollary: a subspace $A \subset \mathbb{R}^n$ is compact $\iff$ it is closed and bounded

proposition: given any compact metric space (X, d), every continuous function $f : X \to \mathbb{R}$ attains its bounds

---

cantor set: $K \subset \mathbb{R}$ is defined as $K := K_0 \cap K_1 \cap \dots \cap K_n \cap \dots$, where $K_n = [0, 1/3^n] \cup \dots \cup [(3^n - 1)/3^n, 1]$, so K_n is the union of 2^n closed intervals, each of length $1/3^n$

theorem: the cantor set K consists of all real numbers which have ternary expansions containing only $0s$ and $2s$

corollary: the cantor set K is uncountable

proposition: the cantor set K is closed

corollary: the cantor set K is compact

proposition: the cantor set has boundary $\delta K = K$ in $\mathbb{R}$

6 Completeness

complete: a metric space (X, d) is complete if every cauchy sequence tends to a limit in X

proposition: euclidean n-space $\mathbb{R}^n$ is complete, for every $n \geq 1$

lemma: suppose that a cauchy sequence $(x_n : n \geq 1)$ in X has a convergent subsequence $(x_{n_r} : r \geq 1)$ and that $\lim_{r \to \infty} x_{n_r} = x$, then (x_n) is also convergent and $\lim_{n \to \infty} = x$

proposition: a closed subspace Y of a complete metric space X is itself complete

proposition: any compact metric space X is complete

proposition: any complete subspace $Y \subseteq X$ is closed in X

contraction: given any metric space (X, d), a self-map $f : X \to X$ is a contraction whenever there exists a constant $0 < K < 1$ such that $d(f(x), f(y)) \leq K d(x, y), \forall x, y \in X$

lemma: any contraction $f : X \to X$ is continuous

fixed point: a fixed point of a self-map $f : X \to X$ is a point $x \in X$ for which $f(x) = x$

contraction mapping theorem: let X be complete, and $f : X \to X$ a contraction, then f has a unique fixed point