MATH20111 Cheat Sheet

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definitions

sequence: a function $\mathbb{N} \to \mathbb{R}$
value set: $\{a_n | n \in \mathbb{N}\}$
subsequence: $(a_{n_k})_{n\in\mathbb{N}}$ with $n_1 < n_2 < n_3 < \cdots$
converges to r: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - r| < \epsilon$
divergent: not convergent

upper bound of S: $r \in \mathbb{R}$ with $S \leq r$
lower bound of S: $r \in \mathbb{R}$ with $r \leq S$
S is bounded from above: S has an upper bound
S is bounded from below: S has a lower bound
bounded: bounded from above and from below

supremum: least upper bound, sup(S)
infimum: greatest lower bound, inf(S) = -sup(-S)

increasing: $a_1 \leq a_2 \leq a_3 \leq \cdots$
decreasing: $a_1 \geq a_2 \geq a_3 \geq \cdots$
strictly increasing: $a_1 < a_2 < a_3 < \cdots$
strictly decreasing: $a_1 > a_2 > a_3 > \cdots$
monotone: increasing or decreasing
strictly monotone: strictly increasing or decreasing

null sequence: sequence that converges to 0

1.2.5 uniqueness of limits

if $(a_n)_{n\in\mathbb{N}} \to r$ and $(a_n)_{n\in\mathbb{N}} \to s$ then $r = s$

1.2.6 finite modification rule

if $(a_n)_{n\in\mathbb{N}} \to r$ and $(b_n)_{n\in\mathbb{N}} = (a_n)_{n\in\mathbb{N}}$ for all but finitely many n then $(b_n)_{n\in\mathbb{N}} \to r$

1.4.6 monotone convergence theorem

every monotone and bounded sequence has a limit

1.6.1 sandwich rule for sequences

if $(a_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \to r$ and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ then $(b_n)_{n\in\mathbb{N}} \to r$

1.6.2 compatibility of limits and order

(i) if $a_n \leq b_n$ FABFM n then $lim_{n\to\infty} a_n \leq lim_{n\to\infty} b_n$
(ii) if $lim_{n\to\infty} a_n < lim_{n\to\infty} b_n$ then FABFM n $a_n < b_n$

1.6.3 algebra of limits

sum rule: $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}(a_n) + \lim_{n\to\infty}(b_n)$
multiplication rule: $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}(a_n) \cdot \lim_{n\to\infty}(b_n)$
division rule: $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$
modulus rule: $\lim_{n\to\infty}|a_n| = |\lim_{n\to\infty} a_n|$
root rule: $\lim_{n\to\infty} \sqrt[p]{a_n} = \sqrt[p]{\lim_{n\to\infty} a_n}$, for $p \in \mathbb{N}$

propositions and friends

1.2.7: every subsequence of a convergent sequence $(a_n)_{n\in\mathbb{N}}$ converges to $\lim_{n\to\infty} a_n$
1.4.2: every convergent sequence is bounded
1.4.3: completeness axiom of $\mathbb{R}$: every nonempty subset S of $\mathbb{R}$ which has an upper bound, has a least upper bound.

properties of null sequences

Let $(a_n)_{n\in\mathbb{N}}$ be a null sequence.
(i) if $c \in \mathbb{R}$ then $(c \cdot a_n)_{n\in\mathbb{N}}$
(ii) if $(b_n)_{n\in\mathbb{N}}$ is null, then $(a_n + b_n)_{n\in\mathbb{N}}$ is null
(iii) if $|b_n| \leq |a_n|$ FABFM n, then $(b_n)_{n\in\mathbb{N}}$ is null
(iv) if $(b_n)_{n\in\mathbb{N}}$ is bounded, then $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is null
(v) if $a_n \geq 0$ for all $n, p \in \mathbb{R}$, then $(a_n^p)_{n\in\mathbb{N}}$ is null

$(a_n)_{n\in\mathbb{N}} \to r \iff (a_n - r)_{n\in\mathbb{N}}$ is null
$(a_n)_{n\in\mathbb{N}}$ is null $\iff (|a_n|)_{n\in\mathbb{N}}$ is null

standard list of null sequences

(i) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for every $p \in \mathbb{R}, p > 0$
(ii) $\lim_{n\to\infty} \frac{1}{c^n} = 0$ for every $c \in \mathbb{R}, |c| > 1$
(iii) $\lim_{n\to\infty} \frac{n^p}{c^n} = 0$ for every $p, c \in \mathbb{R}, |c| > 1$
(iv) $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for all $c \in \mathbb{R}$

definitions

n-th partial sum: $s_n = a_1 + \cdots + a_n$
the series $\sum_{n=1}^{\infty}$ **converges**: $(s_n)_{n\in\mathbb{N}}$ converges
$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$
absolutely convergent: $\sum_{n=1}^{\infty} |a_n|$ is convergent

2.1.2 geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1$$

2.1.4 (2.1.7 alternating) harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k} \text{ is divergent}$$

$$\text{(alternating): } \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} \text{ is convergent}$$

2.1.5 convergence of series with +ve terms

if $(a_n)_{n\in\mathbb{N}}$ is a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$ then

$$\sum_{n=0}^{\infty} a_n \text{ is convergent } \iff (s_n)_{n\in\mathbb{N}} \text{ is bounded}$$

2.2.1 algebra of series

(i) for every $c \in \mathbb{R}$, $\sum_{n=0}^{\infty}(c \cdot a_n) = c \cdot \sum_{n=0}^{\infty} a_n$
(ii) $\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$

2.2.5 comparison test

if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $|b_n| \leq |a_n|$ for all $n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} b_n$ is absolutely convergent

2.2.7 limit comparison test

if $a_n, b_n > 0$ for all $n \in \mathbb{N}$ and $(\frac{b_n}{a_n})_{n\in\mathbb{N}}$ is convergent with $lim_{n\to\infty} \frac{b_n}{a_n} \neq 0$ then

$$\sum_{n=1}^{\infty} a_n \text{ is convergent } \iff \sum_{n=1}^{\infty} b_n \text{ is convergent}$$

2.2.9 ratio test

if $(a_n)_{n\in\mathbb{N}}$ if a sequence with $a_n \neq 0$ for all $n \in \mathbb{N}$ such that $(|\frac{a_{n+1}}{a_n}|)_{n\in\mathbb{N}}$ is convergent.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is divergent}$$

propositions and friends

2.1.3: if $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n)_{n\in\mathbb{N}}$ is null
2.2.3: every absolutely convergent series is convergent
2.2.4: 2.2.1 also holds for absolutely convergent
2.2.6: if $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series and $(b_n)_{n\in\mathbb{N}}$ is bounded, then $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ is absolutely convergent

repertoire

$$\sum_{n=0}^{\infty} \frac{1}{n^k} \text{ is convergent for } k > 1$$

$$\sum_{n=0}^{\infty} \frac{c^n}{n!} \text{ is convergent for every c} \in \mathbb{R}$$

definitions

continuous at x_0: S $\in \mathbb{R}, f : S \to \mathbb{R}, x_0 \in S$, then $\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in S (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$
f tends to r **from above**: S $\in \mathbb{R}, f : S \to \mathbb{R}, a \in S$ with $(a, a+h) \in S$ for some $h > 0$. Let $r \in \mathbb{R}$ then $\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in S \, (a < x < a + \delta \Rightarrow |f(x) - r| < \epsilon)$
f tends to r **from below**: S $\in \mathbb{R}, f : S \to \mathbb{R}, a \in S$ with $(a - h, a) \in S$ for some $h > 0$. Let $r \in \mathbb{R}$ then $\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in S \, (a - \delta < x < a \Rightarrow |f(x) - r| < \epsilon)$

deleted neighbourhood of a: $(a - h, a) \cup (a, a + h)$

$\lim_{x \to a} f(x) := \lim_{x \nearrow a} f(x) = \lim_{x \searrow a} f(x)$ if equal

$\lim_{x \to \infty} f(x) = r$: $f : S \to \mathbb{R}, (h, +\infty) \in S$. Then $\forall \epsilon > 0 \, \exists d \in \mathbb{R} \, \forall x \in S \, (x > d \Rightarrow |f(x) - r| < \epsilon)$

$\lim_{x \searrow a f(x)} = \infty$: $f : S \to \mathbb{R}, (a, a + h) \in S$ for some $a, h \in \mathbb{R}$. Then $\forall A > 0 \, \exists \delta \in \mathbb{R} \, \forall x \in S \, (a < x < a + \delta \Rightarrow f(x) > A)$

characterisation of continuity via sequences

let S $\in \mathbb{R}$, $x_0 \in S$, the following are equivalent:
(i) f is continuous at x_0
(ii) for every sequence $(a_n)_{n \in \mathbb{N}}$ in S, if $(a_n)_{n \in \mathbb{N}} \to x_0$ then $f((a_n))_{n \in \mathbb{N}} \to f(x_0)$
(iii) for every monotone sequence $(a_n)_{n \in \mathbb{N}}$ in S, if $(a_n)_{n \in \mathbb{N}} \to x_0$ then $f((a_n))_{n \in \mathbb{N}} \to f(x_0)$

algebra of limits for continuity at a point

let S $\in \mathbb{R}$, $x_0 \in S$, $f, g : S \to \mathbb{R}$ which are continuous at x_0 then
(i) $f + g$ is continuous at x_0
(ii) $f \cdot g$ is continuous at x_0
(iii) if $g(x) \neq 0, \forall x \in S$, then $\frac{f}{g}$ is continuous at x_0

composite rule for continuous functions

let S, T $\subseteq \mathbb{R}$, $f : S \to \mathbb{R}, g : T \to \mathbb{R}$ with $f(S) \subseteq T$. if f is continuous at x_0 and g is continuous at $f(x_0)$ then $g \circ f : S \to \mathbb{R}$ is continuous at x_0

intermediate value theorem

if $f : [a, b] \to \mathbb{R}$ is continuous, then every number between $f(a)$ and $f(b)$ is attained by f. $\forall r$ between $f(a)$ and $f(b)$, $\exists c \in [a, b]$ with $f(c) = r$

the boundedness theorem

if $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and f attains a global maximum and global minimum

algebra of limits of functions

let $S \subseteq \mathbb{R}, f, g : S \to \mathbb{R}$. let $a \in \mathbb{R}$ such that $(a, a+h) \in S$ for some $h > 0$. if $\lim_{x \searrow a} f(x)$ and $\lim_{x \searrow a} g(x)$ exists, then:
(i) $\lim_{x \searrow a}(f + g)(x) = \lim_{x \searrow a} f(x) + \lim_{x \searrow a} g(x)$
(ii) $\lim_{x \searrow a}(f \cdot g)(x) = (\lim_{x \searrow a} f(x)) \cdot (\lim_{x \searrow a} g(x))$
(iii) if , $\forall x \in S, g(x) \neq 0$ and $\lim_{x \searrow a} g(x) \neq 0$, then $\lim_{x \searrow a}(\frac{f}{g})(x) = \frac{\lim_{x \searrow a} f(x)}{\lim_{x \searrow a} g(x)}$

sandwich rule for limits of functions

let S $\in \mathbb{R}, f, g, h : S \to \mathbb{R}$. let $a \in \mathbb{R}$ such that $(a, a + b)$ for some $b > 0$. if $f(x) \leq h(x) \leq g(x)$ in $(a, a + b)$ and $lim_{x \searrow a} f(x) = \lim_{x \searrow a} g(x) = r$ then $\lim_{x \searrow a} h(x) = r$

propositions and friends

3.2.3: let $S \in \mathbb{R}$, $f, g : S \to R$ be continuous then $f + g : S \to R, f \cdot g : S \to R$ and $\frac{f}{g} : S \to R$ are continuous
3.3.3: let $f : [a, b] \to \mathbb{R}$ be a continuous and injective function. let $f^{-1} : f([a, b]) \to [a, b]$ be the compositional inverse of f then:
(i) f is strictly monotone
(ii) f^{-1} is strictly monotone (precisely, f^-1 is strictly increasing if f is strictly increasing and vice versa)
(iii) f^{-1} is continuous
3.4.2: all general statements regarding limits from above also hold for translated version from below
3.4.4: let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. let $a \in \mathbb{R}$ be such that $(a, a + h) \in S$ for some $h > 0$. let $r \in \mathbb{R}$. define $\hat{f}$ by $\hat{f} = f(x)$ if $a < x$ and $\hat{f} = r$ is $x = a$. then

$$\lim_{x \searrow a} f(x) = r \iff \hat{f} \text{ is continuous at a}$$

3.4.5: let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. let $a \in \mathbb{R}$ be such that $(a, a + h) \in S$ for some $h > 0$. let $r \in \mathbb{R}$. then $\lim_{x \to a} f(x) = r \iff$ for every (monotone) sequence $(a_n)_{n \in \mathbb{N}}$ in $(a, a+h)$ that converges to a, $(f(a_n))_{n \in \mathbb{N}}$ converges to r
3.4.6: let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. let $a \in \mathbb{R}$ be such that $(a - h, a + h) \in S$ for some $h > 0$. then f is continuous at $a \iff \lim_{x \to a} f(x) = f(a)$
3.4.11: $\lim_{x \to \infty} f(x) = \lim_{t \searrow 0} f(\frac{1}{t})$
3.4.13: $\lim_{x \searrow a} f(x) = \infty \iff$ there is some $h > 0$ with $f(x) > 0$ for all $x \in (a, a + h)$ and $\lim_{x \searrow a} \frac{1}{f(x)} = 0$

definition

differentiable at x_0: S open, $f : S \to \mathbb{R}$, $x_0 \in S$ then $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists
differentiable: differentiable at each x_0 in S
derivative of f at x_0: $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

local maximum: S $\in \mathbb{R}$, $f : S \to \mathbb{R}$, $x_0 \in S$ then $\exists h > 0 \, \forall x \in S \, (x_0 - h < x < x_0 + h \Rightarrow f(x_0) \geq f(x))$
local minimum: S $\in \mathbb{R}$, $f : S \to \mathbb{R}$, $x_0 \in S$ then $\exists h > 0 \, \forall x \in S \, (x_0 - h < x < x_0 + h \Rightarrow f(x_0) \leq f(x))$
local extremum: local minimum or local maximum

chain (composite) rule

let f:$S \to \mathbb{R}$ be differentiable at x_0, $f(S) \subseteq T$ and let $g : T \to \mathbb{R}$ be differentiable at $f(x_0)$. then also $g \circ f : S \to \mathbb{R}$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

algebra of differentiable functions

let $f, g : S \to \mathbb{R}$ be differentiable at x_0, then:
(i) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
(ii) $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$
(iii) if $g(x) \neq 0$ for all $x \in S$ then

$$(\frac{f}{g})'(x_0) = \frac{g(x_0)f'(x_0) - g'(x_0)f(x_0)}{g^2(x_0)}$$

some theorem

let $f : S \to \mathbb{R}$ be a continuous function and differentiable at $x_0 \in S$. if f is injective with $f'(x_0) \neq 0$, then f^{-1} defined on $T = f(s)$ is differentiable at $y_0 := f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

some other theorem

if $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S$ and x_0 is a local extremum of f, then $f'(x_0) = 0$

rolle's theorem

let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable in (a, b). if $f(a) = f(b)$, then there is some $\xi \in (a, b)$ with $f'(\xi) = 0$.

and another theorem

let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable in (a, b). then there is some $\xi \in (a, b)$ with $f(\xi) \cdot (g(b)g(a)) = g(\xi) \cdot (f(b)f(a))$

mean value theorem

let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable in (a, b), then there is some $\xi \in (a, b)$ with $f'(\xi) = \frac{f(b)-f(a)}{b-a}$

cauchy mean value theorem

let $f, g : [a, b] \to \mathbb{R}$ be a continuous functions which are differentiable in (a, b). then there is some $\xi \in (a, b)$ with $\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

l'hopital's rule

let $f, g : [a, b) \to \mathbb{R}$ be continuous functions which are differentiable in (a, b). suppose
(i) $g'(x) \neq 0$ for all $x \in (a, b)$
(ii) $f(a) = g(a) = 0$
(iii) $\lim_{x \searrow a} \frac{f'(x)}{g'(x)}$ exists
then

$$\lim_{x \searrow a} \frac{f(x)}{g(x)} = \lim_{x \searrow a} \frac{f'(x)}{g'(x)}$$

propositions and friends

4.1.2: if $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S$, then f is continuous at x_0
4.1.3: let $f : S \to \mathbb{R}$ be differentiable at x_0, then
(i) if $f'(x_0) > 0$, then these is some $h > 0$ such that for all $x_1, x_2 \in (x_0 - h, x_0 + h)$, we have

$$x_1 < x_0 < x_2 \Rightarrow f(x_1) < f(x_0) < f(x_2)$$

(ii) if $f'(x_0) < 0$, then these is some $h > 0$ such that for all $x_1, x_2 \in (x_0 - h, x_0 + h)$, we have

$$x_1 < x_0 < x_2 \Rightarrow f(x_1) > f(x_0) > f(x_2)$$

4.2.2: let $S, T \subseteq \mathbb{R}$ be open intervals and let $f : S \to T, g : T \to \mathbb{R}$ be functions. if f and g are differentiable, then also $g \circ f : S \to \mathbb{R}$ is differentiable and

$$(g \circ f)' = (g' \circ f) \cdot f'$$