MATH20122 Cheat Sheet

1 Definitions and Examples

metric space: a **metric space** (X, d) consists of a non-empty set X and a non-negative real valued **metric** (*distance function*) $d : X \times X \to \mathbb{R}^{\geq}$ which satisfies the following axioms:
(i) $d(x, y) = 0 \iff x = y$ for all $x, y \in X$
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the *triangle inequality*)

subspace: given any subset $W \subseteq X$, the restriction of d to W determines the subspace $(W, d := d|_W)$ of (X, d)

open ball: for any metric space (X, d), the open ball of radius $r > 0$ around any $x \in X$ is $B_r(x) := \{y : d(y, x) < r\}$

closed ball: for any metric space (X, d), the open ball of radius $r > 0$ around any $x \in X$ is $\bar{B}_r(x) := \{y : d(y, x) \leq r\}$

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euclidean n-space: $(\mathbb{R}^n, d_2)$ consists of all real n-dimensional vectors $x = (x_1, \dots, x_n)$, equipped with the **euclidean metric** $d_2(x, y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$ where the positive square root is understood

taxicab metric: d_1 is given on $\mathbb{R}^n$ is given by $d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$$\textbf{discrete metric} : d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

isometry: for any two metric spaces (X, d_X) and (Y, d_Y), a bijection $f : X \to Y$ is an isometry whenever $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$

standard metric: $d_{\mathbb{C}}$ on the complex numbers $\mathbb{C}$ is given by $d_{\mathbb{C}}(z, z') = |z - z'|$

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graph: $\Gamma := (V, E)$ consists of a set V of **vertices** and a set E of **edges**

path: a **path** in Γ from u to w is a finite sequence of edges $\pi(u, w) = (uv_1, v_1v_2, \dots, v_{n-2}v_{v-1}, v_{n-1}w)$ with length n

path connected: a graph is path connected whenever there is a path joining *any* pair of vertices

edge metric: e on the vertex set V of a path connected graph is defined by $e(u, w) = min_{\pi(u,w)} l(\pi(u, w))$

alphabet: a finite set A of **letters** and a finite sequence of letters is a **word** in A. the vertex set W of the associated **word graph** $\Gamma(A)$ consists of all possible words in A. word w_1 and w_2 are joined by an edge iff they differ by one of (i) inserting or deleting a letter (ii) swapping two adjacent letters (iii) replacing one letter with another

word metric: d_w on W is the edge metric on the associated word graph

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binary sequences: $X = \{0, 1\}^\infty$ is the set of all infinite binary sequences $x = x_0x_1 \dots$ where $x_n = 0$ or 1 for all $n \geq 0$

$$d_{min}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{if } n = min\{m : x_m \neq y_m\} \end{cases}$$

$$d^*(x, y) = \sum_{j=0}^{\infty} \frac{|x_j - y_j|}{2^j}$$

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bounded: a real valued function f on a closed interval $[a, b] \subset \mathbb{R}$ is bounded whenever $\exists K, |f(x)| \leq K, \forall x \in [a, b]$

let X denote the set of all bounded $f : [a, b] \to \mathbb{R}$ then

$$d_{sup}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| \text{ with } (X, d_{sup}) \text{ denoted by } \mathcal{B}[a, b]$$

let Y denote the set of all continuous $f : [a, b] \to \mathbb{R}$ then

$$d_1(f, g) = \int_a^b |f(t) - g(t)|dt \text{ with } (Y, d_1) \text{ denoted by } \mathcal{L}_1[0, 1]$$

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let X denote the set of all closed intervals $[a, b]$ in the euclidean line

interval metric: d_H on X is given by $d_H([a, b], [r, s]) = max\{|r - a|, |s - b|\}$

$d_\infty((x_1, x_2), (y_1, y_2)) = max\{|x_1 - y_1|, |x_2 - y_2|\}$

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let X be the set of infinite sequences $(a_i : i \geq 0)$ of reals, such that $\sum_i a_i$ is absolutely convergent

$$d_1((a_i), (b_i)) = \sum_{i \geq 0} |a_i - b_i|$$

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cartesian product: of two metric spaces (X, d) and (X', d') is the set $X \times X'$ with one of the metrics

1. $d_a((x, x'), (y, y')) = d(x, y) + d'(x', y')$
2. $d_b((x, x'), (y, y')) = (d(x, y)^2 + d'(x', y')^2)^{1/2}$
3. $d_c((x, x'), (y, y')) = max\{d(x, y), d'(x', y')\}$

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lipschitz equivalent: two metrics d and e on a given set X are lipschitz equivalent whenever there exists positive constants $h, k \in \mathbb{R}$ such that $he(x, y) \leq d(x, y) \leq ke(x, y)$ for every $x, y \in X$

theorem: the metrics d_a, d_b, d_c on $X \times X'$ are lipschitz equivalent

2 Open and Closed Sets

in this section, let (X, d) be a any metric space and $U \subseteq X$

interior point: $u \in U$ such that $\exists \varepsilon > 0, B_\varepsilon(u) \subseteq U$, the **interior** of U is the subset $U^\circ \subseteq U$ of all interior points and if $U^\circ = U$, then U is **open** in X

proposition: every open ball $B_r(x)$ is open in X

theorem: given any two subsets $U, V \subseteq X$, the following hold:

- $U \subseteq V \implies U^\circ \subseteq V^\circ$
- $(U^\circ)^\circ = U^\circ$
- U° is open in X
- U° is the largest subset of U, open in X

theorem: the sets X and $\emptyset$ are open in X, so are an arbitrary union $U = \bigcup_{i \in I} U_i$ of open sets U_i, and a finite intersection $U' = U'_1 \cap \cdots \cap U'_m$ of open sets U'_j

closure point: $x \in X$ is a closure point of $U \subseteq X$ if $B_\varepsilon(x) \cap U$ is non-empty for every $\varepsilon > 0$, the **closure** of U is the superset $\overline{U} \supseteq U$ of all closure points and if $\overline{U} = U$, then U is **closed** in X

proposition: a set V is closed in X iff its complement $U := X \backslash V$ is open

corollary: every closed ball $\overline{B}_r(x)$ is closed in X

partially open ball: in X is a set $P_r(x) := B_r(x) \cup P$, where P is a proper subset of $\{p : d(x, p) = r\}$

theorem: given any two subsets $U, V \subseteq X$, the following hold:

- $U \subseteq V \implies \overline{U} \subseteq \overline{V}$
- $\overline{\overline{V}} = \overline{V}$
- $\overline{V}$ is closed in X
- $\overline{V}$ is the smallest set containing V, closed in X

theorem: the sets X and $\emptyset$ are closed in X, so are an arbitrary intersection $V = \bigcap_{i\in I} V_i$ of closed sets V_i and a finite union $V' = V_1' \cup \cdots \cup V_m'$ of closed sets V_j'

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sequence: a sequence in X is a function $s : \mathbb{N} \to X$ and it is standard practise to write $s(n)$ as x_n and display the sequence as $(x_n : n \geq 1)$

converges: a sequence (x_n) converges to the point $x \in X$ whenever $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $n \geq N \implies d(x, x_n) < \varepsilon$ and in this situation, we say that x is the **limit** of (x_n)

theorem: in any metric space (X, d), the limit of a convergent sequence is unique

theorem: suppose that $Y \subseteq X$ and $y \in X$, then y lies in $\overline{Y}$ iff there exists a sequence (y_n) in Y such that $y_n \to y$ as $n \to \infty$

cauchy sequence: in any metric space (X, d), a cauchy sequence (x_n) satisfies $\forall \varepsilon < 0, \exists N \in \mathbb{N}$, such that $m, n \geq N \implies d(x_n, x_m) < \varepsilon$

dense: a subset Y is dense in (X, d) whenever $\overline{Y} = X$

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bounded: a subset A of the metric space (X, d) is bounded whenever there exists $x_0 \in X$ and $M \in \mathbb{R}$ such that $d(x, x_0) \leq M$ for every $x \in A$. a function $f : S \to X$ is bounded whenever its image $f(S) \subset X$ is a bounded, for any set

diameter: the diameter, $diam(A)$ of a bounded non-empty set $A \subseteq X$ is the real number $sup\{d(x, y) : x, y \in A\}$

eucildean (n - 1)-sphere of radius r: $\{x : |x| = r\}$

boundary point: $x \in X$ of A is one for which every open ball $B_\varepsilon(x)$ meets both A and $X \setminus A$, the **boundary** δA of A is the set of all such boundary points

theorem: any subset A of (X, d) satisfies:

- $A \setminus \delta A = \overline{A} \setminus \delta A = A^\circ$
- $\delta A = \delta(X \setminus A)$
- δA is closed in X

3 Uniform Convergence

in this section, the functions under consideration are of the form $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ denotes a generic **domain** in the Euclidean line (usually D is a interval)

pointwise convergence: a sequence $(f_n : n \geq 1)$ of functions converges pointwise to f on D whenever the sequence of real numbers $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$ for every $x \in D$

uniform convergence: a sequence $(f_n : n \geq 1)$ of functions converges uniformly to f on D whenever $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ such that $n \geq N(\varepsilon) \implies |f_n(x) - f(x)| < \varepsilon$ for every $x \in D$

proposition: let f and $f_n : D \to \mathbb{R}$ be functions on the domain D, then $f_n \to f$ uniformly on D iff $sup_{x\in D}|f_n(x) - f(x)|$ exists for sufficiently large n, and tends to 0 and $n \to \infty$

theorem: if $f_n : [a, b] \to \mathbb{R}$ is continuous for every $n \in N$, and $f_n \to f$ uniformly on $[a, b]$, then f is continuous on $[a, b]$

corollary: suppose that $f_n : [a, b] \to \mathbb{R}$ is continuous for every $n \in N$ and that the pointwise limit of the sequence (f_n) is discontinuous on $[a, b]$, then the convergence cannot be uniform

theorem: if $f_n : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ for every $n \in N$ and (f_n) converges uniformly on $[a, b]$ then

$$\lim_{n\to\infty} \int_a^b f_n(t)\ dt = \int_a^b \lim_{n\to\infty} \int_a^b f_n(t)\ dt$$