MATH20142 Cheat Sheet

1 Construction and Basic Properties of Complex Numbers

An expression $a + ib(a, b \in \mathbb{R})$ is called a **complex number**. We denote the set of complex numbers by $\mathbb{C}$. For $z = x + iy$, we use $x = \text{Re}z$ and $y = \text{Im}z$ and say that z is real if $\text{Im}z = 0$ and that z is imaginary if $\text{Re}z = 0$.

- $\text{Re}(z \pm w) = \text{Re}z \pm \text{Re}w$
- $\text{Im}(z \pm w) = \text{Im}z \pm \text{Re}w$
- $\overline{(z \pm w)} = \overline{z} \pm \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{(z/w)} = \overline{z}/\overline{w}$ if $w \neq 0$
- $z + \overline{z} = 2\text{Re}z$
- $z - \overline{z} = 2\text{Im}z$
- $|z| = 0 \iff z = 0$
- $|zw| = |z||w|$
- $|z/w| = |z|/|w|$ if $w \neq 0$
- $|z + w| \leq |z| + |w|$
- $|z - w| \geq ||z| - |w||$

2 Topology in $\mathbb{C}$

ε-neighbourhood of z_0: $N_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ (disc centred at z_0 containing points with distance $< \varepsilon$)

limit point: $z_0 \in \mathbb{C}$ is a limit point of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0, N_\varepsilon(z_0)$ contains a point in $S \setminus \{z_0\}$

interior point: let $S \subset C, z_0$ a limit point of S, then z_0 is an interior point of S if $\exists\, \varepsilon > 0, N_\varepsilon(z_0) \subset S$

boundary point: let $S \subset C, z_0$ a limit point of S, then z_0 is an boundary point of S if it is not a interior point

open: a set $S \subset \mathbb{C}$ is called open if it consists only of interior points

domain: let $S \subset \mathbb{C}, S \neq \emptyset$, then S is called a domain if S is open and every pair of points can be connected by a polygonal arc lying entirely in S

function: let $S \subset C, S \neq \emptyset$, a function $f : \subset \to \mathbb{C}$ is a rule which assigns to each $z \in S$, an image $f(z) \in \mathbb{C}$

$\lim_{z \to z_0} f(z)$: let $f : S \to \mathbb{C}$ be a function. if z_0 is a limit point of S then we say $\lim_{z \to z_0} f(z) = l$ if, $\forall \varepsilon > 0, \exists \delta > 0, s \in S$ and $0 < |z - z_0| < \delta \implies |f(z) - l| < \varepsilon$

continuity: $f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

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proposition a set $S \subset \mathbb{C}$ is closed $\iff$ its complement $\mathbb{C} \setminus S$ is open

proposition if $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = k$, then

1. $\lim_{z \to z_0}(f(z) \pm g(z)) = l \pm k$
2. $\lim_{z \to z_0}(f(z)g(z)) = lk$
3. $\lim_{z \to z_0}(f(z)/g(z)) = l/k$ (for $k \neq 0$)

proposition $\lim_{z \to z_0} f(z) = l = \alpha + i\beta\, (\alpha, \beta \in \mathbb{R}) \iff u(x, y) \to \alpha, v(x, y) \to \beta$, as $(x, y) \to (\text{Re}z, \text{Im}z)$

3 Differentiation and Cauchy-Riemann Equations

differentiable at a point: let $S \subset \mathbb{C}$ be a open set. we say that $f : S \to \mathbb{C}$ is differentiable at a point $z_0 \in S$ with derivative $f'(z_0)$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

differentiable function: if f is differentiable at every point of S, we say f is a differentiable function in S

partial derivatives: for $z = x + iy$, write $f(z) = u(x, y) + iv(x, y)$, where u, v are real-valued

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h, y) - u(x, y)}{h} \qquad v_x = \frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(x+h, y) - v(x, y)}{h}$$
$$u_y = \frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x, y+k) - u(x, y)}{k} \qquad v_y = \frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(x, y+k) - v(x, y)}{k}$$

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proposition if f is differentiable at z_0 then f is continuous at z_0

proposition if f is differentiable at $z = x + iy$ then u_x, u_y, v_x, v_y all exist and $u_x = v_y, v_x = -u_y$ (CRE)

theorem if $f(z) = u(x, y) + iv(x, y)$ is a complex function on an open set S and at $z_0 = x_0 + iy_0 \in S$, the partial derivatives u_x, v_x, u_y, v_y all exist, are continuous and satisfy the CRE then f is differentiable at z_0

theorem if f is differentiable in a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant in D

4 Power Series

convergence: we say a sequence $s_n \in \mathbb{C}$ converges to $s \in \mathbb{C}$ if, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon, \forall n \geq N$. the series $\sum_{k=0}^{\infty} z_k$ converges if the sequence of partial sums $s_n = \sum_{k=0}^{n} z_k$ converges and the limit of the sequence is called the sum of the series

divergent series: a series which does not converge is said to be divergent

absolute convergence: we say that $\sum_{k=0}^{\infty} z_k$ is absolutely convergent if the real series $\sum_{k=0}^{\infty} |z_k|$ is convergent

ratio test: consider $\sum_{k=0}^{\infty} z_k$ and suppose that $\lim_{n\to\infty} |z_{n+1}|/|z_n| = l$. if $l < 1$ then $\sum_{k=0}^{\infty} z_k$ is absolutely convergent and if $l > 1$ then $\sum_{k=0}^{\infty} z_k$ diverges

root test: consider $\sum_{k=0}^{\infty} z_k$ and suppose that $\lim_{n\to\infty} |z|^{1/n} = l$. if $l < 1$ then $\sum_{k=0}^{\infty} z_k$ is absolutely convergent and if $l > 1$ then $\sum_{k=0}^{\infty} z_k$ diverges

general principle of convergence: if a series $\sum_{n=1}^{\infty} s_n$ with $s_n \in \mathbb{C}$ converges, then $s_n \to 0$ as $n \to \infty$

power series about z_0: $\sum_{n=0}^{\infty} a_n z^n$

radius of convergence: $R = \sup\{r : \exists z \text{ such that } |z| = r \text{ and } \sum_{n=0}^{\infty} a_n z^n \text{ converges }\}$

disc of convergence: $\{z \in \mathbb{C} : |z| < R\}$, where R is the radius of convergence

computation of radius of convergence: $R = \lim_{n\to\infty} |a_{n-1}/a_n|$, provided the limit exists

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lemma if a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_1 \neq 0$, then it converges absolutely for all z with $|z| < |z_1|$

lemma if $\sum_{n=0}^{\infty} a_n z^n$ diverges for $z = z_2$, then is diverges for all z with $|z| > |z_2|$

theorem the radius of convergence R of $\sum_{n=0}^{\infty} a_n z^n$ is given by $1/R = \lim_{n\to\infty} |a_n|^{1/n}$

lemma if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ then $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ converges for $|z| < R$

theorem a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ may be differentiated term by term within its disc of convergence so that $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$

corollary all higher derivatives $f', f'', f''', \ldots, f^{(n)}, \ldots$ of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ exist for z within the disc of convergence and $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k} = \sum_{n=k}^{\infty} n!/(n-k)! \cdot a_n z^{n-k}$

corollary if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has disc of convergence $|z - z_0| < R$ then $a_k = f^{(k)}(z_0)/k!$ and we can express f as a **Taylor series** $f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)/n! \cdot (z - z_0)^n$, valid for $|z - z_0| < R$

5 The Exponential Function and Its Friends

the exponential function: define $\exp z = \sum_{n=0}^{\infty} z^n/n!$, which converges absolutely for all $z \in \mathbb{C}$. we can check that $\exp(z_1 + z_2) = \exp z_1 \exp z_2$ and by induction, $\exp nz = (\exp z)^n$, for all $n \in \mathbb{Z}^+$

the number e: define $e = \exp 1 = 2.7182818\ldots$ and we can also use the notation $\exp z = e^z$, for all $z \in \mathbb{C}$

trigonometric functions: define $\cos z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n)!$ and $\sin z = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)!$

hyperbolic functions: define $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$ so $\sin iz = i \sinh z$ and $\cos iz = \cosh z$

period: for a function $f : \mathbb{C} \to \mathbb{C}$, a nonzero number $k \in \mathbb{C}$ is called a period if $f(z + k) = f(z)$, for all $z \in \mathbb{C}$

logarithmic function: $\log z = u + iv = log|z| + i \arg z$ and $\mathrm{Log} z = \log z + i \arg z$ $(-\pi < \arg z \leq \pi)$

cut plane: the complex plane with the negative real axis, including zero, removed is called the cut plane and denoted $\mathbb{C}_\pi$

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eulers theorem: $e^{iz} = \cos z + i \sin z$

corollary

- $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
- $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
- $\cos^2 z + \sin^2 z = 1$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$

lemma: the functions $arg(z)$ and $Log(z)$ are continuous on the cut plane

theorem: let $z \neq 0$ be a complex and let n be a positive integer, then

$$z^{\frac{1}{n}} = \{|z|^{\frac{1}{n}} e^{i(\frac{Argz + 2k\pi}{n})} \mid k = 0, 1, \ldots, n-1\}$$