math33011 - mathematical logic

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set theory

poset: a pair $(X, \leq)$ where X is a set and $\leq$ is a binary operation on X, such that:
(i) $\leq$ is reflective, i.e. $\forall x \in X : x \leq x$
(ii) $\leq$ is anti-symmetric, i.e. $\forall x, y \in X : x \leq y$ and $y \leq x \implies x = y$
(iii) $\leq$ is transitive, i.e. $\forall x, y, z \in X : x \leq, y \leq z \implies x \leq z$
partial order (on X): $\leq$ as defined above
comparable: $x, y \in X$ are comparable if either $x \leq y$ or $y \leq x$
strict partial order: $x < y$, i.e. $x \leq y$ and $x \neq y$
totality axiom: $\forall x, y \in X : x \leq$ or $y \leq x$
chain / totally ordered set: a poset that satisfies the totality axiom
trivial partial order: $x \leq x \iff x = x$
product of two posets: $(x, y) \leq (x', y') \iff x \leq_1 x'$ and $y \leq_2 y'$
lexicographic product of two posets: $(x, y) \leq_{lex} (x', y') \iff x <_1 x'$ or ($x = x'$ and $y \leq_2 y'$).
ordered sum of two posets: $X \cup Y$ with $\leq := \leq_1 \cup \leq_2 \cup (X \times Y)$

upper bound of S (in X): $x \in X$ such that $\forall s \in S : s \leq x$, i.e. $S \leq x$
lower bound of S (in X): $x \in X$ such that $\forall s \in S : x \leq s$, i.e. $x \leq S$
largest element of S: $x \in S$ such that $S \leq x$ (unique if it exists)
smallest element of S: $x \in S$ such that $x \leq S$ (unique if it exists)
supremum of S: the smallest upper bound of S (one at most)
infimum of S: the largest lower bound of S (one at most)
maximal element of S: $x \in S$ such that there is no $s \in S$ with $x < s$
minimal element of S: $x \in S$ such that there is no $s \in S$ with $s < x$

poset-homomorphism / monotone: $f : X \to Y$ such that $\forall x, x' \in X : x \leq_1 x' \implies f(x) \leq_2 f(x')$
poset embedding: $f : X \to Y$ such that $\forall x, x' \in X : x \leq_1 x' \iff f(x) \leq_2 f(x')$
isomorphism of posets: a poset-homomorphism which is bijective and an embedding

initial segment / down-set: $Y \subseteq X$ such that $\forall x, y \in X, x \leq y \in X \implies x \in Y$, denoted $Y \Subset X$
example down-set $X_{<a}$: $\{x \in X \mid x < a\}$ of X

well ordered set: a chain where every nonempty subset has a smallest element
proposition: if X, Y are well ordered, then so is the ordered sum and the lexicographic product of X and Y
lemma: a chain X is well ordered $\iff$ it does not possess infinite sequence $x_1 > x_2 > \dots$
observation: if X is well ordered then each $Y \Subset X, Y \neq X$ is of the form $Y = X_{<a}$ where $a = \min(X \backslash Y)$
observation: $X_{<a} = \emptyset$ if a is the smallest element of X
lemma of zorn: let $X = (X, \leq)$ be a nonempty, poset, such that each $W \subseteq X$ that is well ordered by $\leq$, has an upper bound in X. then X possesses at least one maximal element
well ordering principle: every set can be well ordered, i.e. for every set M there is a well order with universe M

notation: $X \sqsubset Y$ if there is a poset embedding $f : X \to Y$ such that $f(X) \Subset Y$ for well ordered sets X, Y.
in other words, $X \sqsubset Y \iff$ there is some $Z \Subset Y$ such that X and Z are isomorphic

theorem: if X, Y are well ordered sets and $X \sqsubset Y$, then the poset-embedding ($f : X \to Y$ such that $f(x) \Subset Y$) is unique
theorem: if X, Y are well ordered sets, then
(i) $X \sqsubset Y$ and $Y \sqsubset X \implies X \cong Y$
(ii) $X \sqsubset Y$ or $Y \sqsubset X$

transitive set: a set X such that each of its elements is a subset of X, so $y \in x \in X \implies x \in X$
ordinal (number): a transitive set α such that the element relation is a strict well order on α, i.e. $x \leq y$ defined as $x = y$ or $x \in y$ is a well order on α
successor of α: $\alpha^* := \alpha \cup \{\alpha\}$

proposition: for ordinals α, β: $\alpha \sqsubset \beta \iff \exists$ poset-embedding $\alpha \to \beta \iff \alpha \subseteq \beta \iff \alpha \Subset \beta \iff \alpha = \beta$ or $\alpha \in \beta$
ordering on ordinals: for ordinals α, β, we write $\alpha \leq \beta$ instead of $\alpha \subseteq \beta$ and $\alpha < \beta$ for $\alpha \in \beta$
corollary: if α, β are ordinals, then $\alpha \Subset \beta$ or $\beta \Subset \alpha$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$, and $\alpha \leq \beta$ or $\beta \leq \alpha$
corollary: if I is an index set and α_i is an ordinal for all $i \in I$, then $\bigcup_{i \in I} \alpha_i$ is also an ordinal
corollary: every ordinal α is equal to the set of ordinals that are strictly less than α, $\alpha = \{\beta \mid \beta \text{ is an ordinal and } \beta < \alpha\}$
successor ordinal: α is called a successor ordinal if there is an ordinal β such that $\alpha = \beta^*$, else called a **limit ordinal**
theorem: if W is a well ordered set, then there is a unique ordinal α that is isomorphic to W (exactly one isomorphism)

corollary: every set is in bijection with some ordinal
ordinal minimisation principle: let P be a property of ordinals and assume there is an ordinal with property P, then there is a smallest ordinal with property P

cardinality / size of X: the smallest ordinal α that is in bijection with X. $card(X) = |X| = \alpha$
cardinal (number): an ordinal α whose cardinality is α. in particular, the size of any set is a cardinal number
proposition: for sets $X, Y, X \neq \emptyset$, $card(X) \leq card(Y) \iff \exists$ injective map $X \to Y \iff \exists$ surjective map $Y \to X$
theorem of bernstein: for sets X, Y, the following are equivalent:

(i) there are injective maps $X \to Y$ and $Y \to X$
(ii) there are surjective maps $X \to Y$ and $Y \to X$
(iii) there is a bijective map $X \to Y$
(iv) $card(X) = card(Y)$

size of a power set: for every set X, we have $card(X) < card(\mathcal{P}(X))$
corollary: if X is a set, then there is a cardinal $\kappa > card(X)$
pairing function: Pair: $\omega \times \omega \to \omega$, defined as $Pair(x, y) := \frac{1}{2}(x + y)(x + y + 1) + x$ is bijective
size of products: if $X, Y \neq \emptyset$ and at least one of them is infinite, then $card(X \times Y) = max\{card(X), card(Y)\}$
size of arbitrary unions: let I be an index set and for each $i \in I$, let X_i be a set. let κ be an infinite cardinal with $card(X_i) \leq \kappa$ for all i, then $card(\bigcup_{i\in I} X_i) \leq max\{card(I), \kappa\}$
corollary: if X, Y are sets and at least one of them is infinite, then $card(X \cup Y) = max\{card(X), card(Y)\}$

revision of predicate logic

for this section, let $\mathscr{L}$ be a language.

alphabet of $\mathscr{L}$ consists of: a set logical symbols $\{\neg, \to, \forall, \doteq,), (, , , v_0, v_1, v_2, \dots\}$ and three mutually disjoint sets $\mathscr{R}, \mathscr{F}, \mathscr{C}$ called the set of relation symbols, function symbols and constant symbols, respectively. Maps $\lambda : \mathscr{R} \to \mathbb{N}$ and $\mu : \mathscr{F} \to \mathbb{N}$, called the arity of relation symbols and arity of function symbols, respectively

letter / symbol: every logical element and every element from $\mathscr{R} \cup \mathscr{F} \cup \mathscr{C}$
variables: $Vbl = \{v_n \mid n \in \mathbb{N}_0\}$

finite: the alphabet of $\mathscr{L}$ is finite if $\mathscr{R}, \mathscr{F}$ and $\mathscr{C}$ are finite. otherwise, infinite
countable: the alphabet of $\mathscr{L}$ is countable if $\mathscr{R}, \mathscr{F}$ and $\mathscr{C}$ is countable or finite. otherwise, uncountable.
cardinality of the alphabet of $\mathscr{L}$: the cardinality of $\mathscr{R} \cup \mathscr{F} \cup \mathscr{C}$
similarity type of $\mathscr{L}$: $(\lambda : \mathscr{R} \to \mathbb{N}, \mu : \mathscr{F} \to \mathbb{N}, \mathscr{C})$

given the similarity type $(\lambda : \mathscr{R} \to \mathbb{N}, \mu : \mathscr{F} \to \mathbb{N}, \mathscr{C})$, we define $tm_k(\mathscr{L})$ by induction on $k \in \mathbb{N}_0$ as follows:

$$tm_0(\mathscr{L}) = Vbl \cup \mathscr{C} \quad \text{and} \quad tm_{k+1}(\mathscr{L}) = tm_k(\mathscr{L}) \cup \left\{ F(t_1, t_2, \dots, t_n) \mid n \in \mathbb{N}, F \in \mathscr{F}, \mu(F) = n, t_1, \dots, t_n \in tm_k(\mathscr{L}) \right\}$$

terms: $tm(\mathscr{L}) = \cup_{k\in\mathbb{N}_0} tm_k(\mathscr{L})$
complexity of a term $c(t)$ is the least $k \in \mathbb{N}_0$ such that $t \in tm_k(\mathscr{L})$

unique readability theorem for terms: if t is an $\mathscr{L}$-term, then either t is a variable or t is a constant symbol or there are uniquely determined $n \in \mathbb{N}, F \in \mathscr{F}$ of arity n and $t_1, \dots, t_n \in tm(\mathscr{L}$ such that $t = F(t_1, \dots, t_n)$

corollary: for $n \in \mathbb{N}$, all terms $t_1, \dots, t_n$ and each $F \in \mathscr{F}, \mu(F) = n$, we have $c(F(t_1, \dots, t_n)) = 1 + max\{c(t_1), \dots, c(t_n)\}$

atomic formula: a string of the alphabet of $\mathscr{L}$ of the form $t_1 \doteq t_2$ where t_1, t_2 are $\mathscr{L}$-terms or $R(t_1, \dots, t_n)$ where $R \in \mathscr{R}, \lambda(R) = n$ and $t_1, \dots, t_n$ are $\mathscr{L}$-terms. the set of atomic $\mathscr{L}$-formulas is denoted at-$Fml(\mathscr{L})$

we define Fml_k by induction on $k \in \mathbb{N}_0$ as follows:

$$Fml_(\mathscr{L}) = \text{at-}Fml(\mathscr{L}) \quad \text{and} \quad Fml_{k+1}(\mathscr{L}) = Fml_k(\mathscr{L}) \cup \{(\neg\varphi), (\varphi \to \psi), (\forall x \varphi) \mid \varphi, \psi \in Fml_k(\mathscr{L}), x \in Vbl\}$$

formulas: $Fml(\mathscr{L}) = \cup_{k\in\mathbb{N}_0} Fml_k(\mathscr{L})$
quantifier free: a formula φ is quantifier free if the letter $\forall$ does not occur in it

unique readability theorem for formulas: let $\mathscr{L} = (\lambda : \mathscr{R} \to \mathbb{N}, \mu : \mathscr{F} \to \mathbb{N}, \mathscr{C})$ be a language and let φ be an $\mathscr{L}$-formula. then exactly one of the following holds true:

(i) φ is atomic and there are unique determined $t_1, t_2 \in tm(\mathscr{L})$ such that φ is $t_1 \doteq t_2$

(ii) φ is atomic and there is a unique $n \in \mathbb{N}$, $R \in \mathscr{R}$ and uniquely determined $\mathscr{L}$-terms $t_1, \ldots, t_n$ such that φ is $R(t_1, \ldots, t_n)$
(iii) φ is equal to a string of the form $(\neg\psi)$ for a uniquely determined $\psi \in Fml(\mathscr{L})$
(iv) φ is equal to a string of the form $(\varphi_1 \to \varphi_2)$ for uniquely determined $\varphi_1, \varphi_2 \in Fml(\mathscr{L})$
(v) φ is a string of the form $(\forall x\psi)$ for uniquely determined $\psi \in Fml(\mathscr{L})$ and $x \in Vbl$

language $\mathscr{L}$: the triple consisting of the alphabet of $\mathscr{L}$, the set of $\mathscr{L}$ and the set of $\mathscr{L}$-formulas.
finite / infinite / countable / uncountable: $\mathscr{L}$ has this property if its alphabet has this property
cardinality: $card(\mathscr{L})$, is the cardinality of the alphabet of $\mathscr{L}$

model theory

let $\mathscr{L}$ be a language and $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures

map between $\mathscr{M}$ and $\mathscr{N}$: a map $f : |\mathscr{M}| \to |\mathscr{N}|$, but we write $f : \mathscr{M} \to \mathscr{N}$ instead
preserved by a map: a formula $\varphi(x_1, \ldots, x_n) \in \mathrm{Fml}(\mathscr{L})$ is preserved by a map $f : \mathscr{M} \to \mathscr{N}$ if for all $a_1, \ldots, a_n$

$$\mathscr{M} \models \varphi(a_1, \ldots, a_n) \implies \mathscr{N} \models \varphi(f(a_1), \ldots, f(a_n))$$

f **respects** φ: φ is preserved by f

homomorphism: a map $f : \mathscr{M} \to \mathscr{N}$ between $\mathscr{L}$-structures which respects all atomic formulas
lemma: let $f : \mathscr{M} \to \mathscr{N}$ be a map between $\mathscr{L}$-structures. the following are equivalent:
(i) f is an $\mathscr{L}$-homomorphism
(ii) f satisfies each of the following conditions:
  (a) for all $R \in \mathscr{R}$ of arity n and all $a_1, \ldots, a_n \in |\mathscr{M}|$ we have $(a_1, \ldots, a_n) \in R^{\mathscr{M}} \implies (f(a_1), \ldots, f(a_n)) \in R^{\mathscr{N}}$
  (b) for all $F \in \mathscr{F}$ of arity n and all $a_1, \ldots, a_n \in |\mathscr{M}|$ we have $f(F^{\mathscr{M}}(a_1, \ldots, a_n)) \implies F^{\mathscr{N}}(f(a_1), \ldots, f(a_n))$
  (c) for all $c \in \mathscr{C}$ we have $f(c^{\mathscr{M}}) = c^{\mathscr{N}}$
(iii) f respects each of the following formulas:
  (a) all formulas of the form $R(v_1, \ldots, v_n)$ where $R \in \mathscr{R}$ is a relation symbol of $\mathscr{L}$ or arity n
  (b) all formulas of the form $v_0 \doteq F(v_1, \ldots, v_n)$ where $F \in \mathscr{F}$ is a function symbol of $\mathscr{L}$ of arity n
  (c) all formulas of the form $v_0 \doteq c$, where $c \in \mathscr{C}$ is a constant symbol of $\mathscr{L}$

embedding: a map $f : \mathscr{M} \to \mathscr{N}$ between $\mathscr{L}$-structures which respects all quantifier free formulas
$\mathscr{M}$ is a substructure of $\mathscr{N}$: if $|\mathscr{M}| \subseteq |\mathscr{N}|$ and the inclusion map $|\mathscr{M}| \to |\mathscr{N}|$ is an embedding, then $\mathscr{M}$ is called a substructure of $\mathscr{M}$. in addition, if $|\mathscr{M}| \neq |\mathscr{N}|$, then $\mathscr{M}$ is called a **proper substructure** of $\mathscr{N}$
lemma: let $f : \mathscr{M} \to \mathscr{N}$ be a map between $\mathscr{L}$-structures. the following are equivalent:
(i) f is an embedding
(ii) f is an injective $\mathscr{L}$-homomorphism such that for all $a_1, \ldots, a_n \in |M|$ we have

$$(a_1, \ldots, a_n) \in R^{\mathscr{M}} \iff (f(a_1), \ldots, f(a_n)) \in R^{\mathscr{N}}$$

(iii) for all $\varphi(x_1, \ldots, x_n) \in \text{at-Fml}(\mathscr{L})$ and all $a_1, \ldots, a_n \in |\mathscr{M}|$ we have

$$\mathscr{M} \models \varphi(a_1, \ldots, a_n) \iff \mathscr{N} \models \varphi(f(a_1), \ldots, f(a_n))$$

corollary: let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. if $c^{\mathscr{M}}$ and for each n-ary function symbol F of $\mathscr{L}$, the function $F^{\mathscr{M}}$ maps A^n to A, then A is the universe of a unique substructure $\mathscr{A}$ of $\mathscr{M}$, which is called the **substructure of $\mathscr{M}$ induced on** A, which interprets the non-logical symbols as follows:
(i) $R^{\mathscr{A}} = R^{\mathscr{M}} \cap A^n$ for all $R \in \mathscr{R}$ of arity n
(ii) $F^{\mathscr{A}}(a_1, \ldots, a_n) = F^{\mathscr{M}}(a_1, \ldots, a_n)$ for all $F \in \mathscr{F}$ or arity n
(iii) $c^{\mathscr{A}} = c^{\mathscr{M}}$ for all $c \in \mathscr{C}$

corollary: let $\mathscr{M}$ be an $\mathscr{L}$-structure. then any nonempty intersection of universes of substructures of $\mathscr{M}$ is again the universe of a substructure of $\mathscr{M}$. consequently, if $A \subseteq |\mathscr{M}|$ is nonempty, then there is a smallest (for inclusion) universe U of a substructure of $\mathscr{M}$ containing A, namely the intersection of all the universes of substructures of $\mathscr{M}$ containing A, and the substructure with universe U is called the **substructure of $\mathscr{M}$ generated by** A

elementary embedding: a map $f : \mathscr{M} \to \mathscr{N}$ between $\mathscr{L}$-structures which respects all formulas
$\mathscr{M}$ is a elementary substructure of $\mathscr{N}$: if $|\mathscr{M}| \subseteq |\mathscr{N}|$ and the inclusion map $|\mathscr{M}| \to |\mathscr{N}|$ is an elementary embedding, then $\mathscr{M}$ is called an elementary substructure of $\mathscr{N}$, denoted $\mathscr{M} \prec \mathscr{N}$, and $\mathscr{N}$ is called an **elementary extension** of $\mathscr{M}$

isomorphism: a map $f : \mathscr{M} \to \mathscr{N}$ between $\mathscr{L}$-structures which is a bijective embedding.
isomorphic: two $\mathscr{L}$-structures $\mathscr{M}$ and $\mathscr{N}$ are isomorphism, denoted $\mathscr{M} \cong \mathscr{N}$, if there is an isomorphism $\mathscr{M} \to \mathscr{N}$
lemma: every $\mathscr{L}$-isomorphism is an elementary embedding

elementary equivalent: two $\mathscr{L}$-structures $\mathscr{M}$ and $\mathscr{N}$ that satisfy the same $\mathscr{L}$-sentences, denoted $\mathscr{M} \equiv \mathscr{N}$
lemma: if $f : \mathscr{M} \to \mathscr{N}$ is an elementary embedding then $\mathscr{M} \equiv \mathscr{N}$. if particular, isomorphic structures are elementary equivalent.
proposition: if $\mathscr{M}$ is finite and $\mathscr{N} \equiv \mathscr{M}$, then $\mathscr{M} \cong \mathscr{N}$

tarski-vaught test: let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. the following are equivalent:
(i) A is the universe of an elementary substructure of $\mathscr{M}$
(ii) for every $\mathscr{L}$-formula $\varphi(x, \overline{y})$ and all $\overline{a} \in A^{\overline{y}}$, if $\mathscr{M} \models (\exists x \varphi)(\overline{a})$, then there is some $b \in A$ with $\varphi(b, \overline{a})$

lemma: for any language $\mathscr{L}$, the cardinality of $Fml(\mathscr{L})$ is $max\{\aleph_0, \text{card}(\mathscr{L})\}$

skolem-löwenheim downwards: let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. then there is an elementary substructure $\mathscr{N}$ of $\mathscr{M}$ with $A \subseteq |\mathscr{N}|$ such that $\text{card}(\mathscr{N}) \leq max\{\aleph_0, \text{card}(A), \text{card}(\mathscr{L})\}$

$\mathscr{L}$-theory: a set of $\mathscr{L}$-sentences
model $\mathscr{M}$ of an $\mathscr{L}$-theory: an $\mathscr{L}$-structure $\mathscr{M}$ with $\mathscr{M} \models \varphi$ for all $\varphi \in T$, denoted $\mathscr{M} \models T$
consistent / satisfiable: a theory is consistent or satisfiable if it has a model
complete: a theory is complete if all its models are elementary equivalent
theory of $\mathscr{M}$: defined as $Th(\mathscr{M}) = \{\varphi \in Sen(\mathscr{L}) \mid \mathscr{M} \models \varphi\}$ is always complete

compactness theorem: if T is a set of $\mathscr{L}$-sentences such that any finite subset of T has a model, then T itself has a model

lemma: let $\mathscr{M}$ be an infinite $\mathscr{L}$-structure and let κ be any cardinal. then there is an elementary extension $\mathscr{N}$ of $\mathscr{M}$ with $\text{card}(|\mathscr{N}|) \geq \kappa$

skolem-löwenheim upwards: let $\mathscr{M}$ be an infinite $\mathscr{L}$-structure and let κ be an cardinal $\geq \text{card}(\mathscr{M}), \text{card}(\mathscr{L})$. then there is an elementary extension $\mathscr{N} \succ \mathscr{M}$ of cardinality κ

definable in $\mathscr{M}$: a subset S of $|\mathscr{M}|^n$ is called definable in $\mathscr{M}$ if there is some $\mathscr{L}$-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and a k-tuple $\overline{a} \in |\mathscr{M}|^k$ such that

$$S = \varphi(\mathscr{M}^n, \overline{a}) := \{(m_1, \ldots, m_n) \in \mathscr{M}^n \mid \mathscr{M} \models \varphi(m_1, \ldots, m_n, a_1, \ldots, a_k)\}$$

we say that S is **defined by** $\varphi(\overline{x}, \overline{a})$ in $\mathscr{M}$ and the elements $a_1, \ldots, a_k$ are called **parameters**

proposition: let $\mathscr{M}$ be an $\mathscr{M}$-structure with universe $M = |\mathscr{M}|$, then
(i) if S, T are definable subsets of M^n, then also $S \cap T, S \cup T$ and $S \backslash T$ are definable. if p is the projection $M^n \to M^k$ and S is a definable subset of M^n, then $p(S)$ is a definable subset of M^k
(ii) if $f : \mathscr{M} \to \mathscr{N}$ is an isomorphism between $\mathscr{L}$-structures and $S \subseteq M^n$ is defined by $\varphi(\overline{x}, \overline{a})$, then $f(S)$ is defined by $\varphi(\overline{x}, f(\overline{a}))$, here we also consider f as a map $M^n \to |\mathscr{N}|^n$ obtained from f by applying f coordinate wise; thus $f(S) \subseteq |\mathscr{N}|^n$ and $f(\overline{a}) \in |\mathscr{N}|^n$

definable in $\mathscr{M}$: let $\mathscr{M}$ be an $\mathscr{L}$-structure with universe M and let $S \subseteq M^n$. a function $f : S \to M^k$ is called definable in M if its graph is a subset of $M^n \times M^k$ that is definable in $\mathscr{M}$

proposition: let $\mathscr{M}$ be an $\mathscr{L}$-structure with universe M and let $S \subseteq M^n$. let $f : S \to M^k$ be a function
(i) f is definable if and only if each component of f is a definable map $S \to M$
(ii) if f is definable, then S and the image of f are definable
(iii) the composition of definable maps (when well-defined) is definable

proposition: any two countable and dense total orders without endpoints are isomorphic

categorial in an infinite cardinal κ: an $\mathscr{M}$-theory T that has an infinite model is called categorial in an infinite cardinal κ, or simply **κ-categorial**, if all models of T of cardinality κ are isomorphic

theorem: if T has no finite models and T is categorial in some infinite cardinal $\geq \text{card}(\mathscr{M})$, then T is complete