MATH32001 Midterm Cheat Sheet

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alg structures definitions

group: $(G, *), G \neq \emptyset$ such that:
(i) $\forall a, b \in G, a * b \in G$
(ii) $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
(iii) $\exists e \in G, \forall a \in G, a * e = e * a = a$
(iv) $\forall a \in G, \exists a' \in G, a * a' = e = a' * a$

abelian: $(G, *)$ where $*$ is commutative binary operation

subgroup criterion: if G is a group with $H \subseteq G$, then $H \leq G$ if $H \neq \emptyset$ and $\forall a, b \in H$, we have $ab^{-1} \in H$

right coset: suppose G is a group with $H \leq G$, for $a \in G$, $Ha = \{ha \mid h \in H\} \subseteq G$

index of H in G: number of right cosets of H in G

lagranges theorem: suppose G is a finite group with $H \leq G$, then $|G| = [G : H]|H|$

centraliser: let $g \in G$. $C(g) = \{x \in G \mid xg = gx\} \leq G$

centre of G: $Z(G) = \{x \in G \; xg = gx, \forall g \in G\} \leq G$

homomorphism: $\varphi : G \to H$ such that $\forall a, b \in G \varphi(ab) = \varphi(a)\varphi(b)$

isomorphism: a bijective homomorphism

conjugate: if G is a group, $x \in G$. $y \in G$ is a conjugate of x if $y = g^{-1}xg$ for some $g \in G$, denoted $y \sim x$

conjugacy class: $x^G = \{g^{-1}xg \mid g \in G\}$

class theorem: suppose G is a finite group with $g \in G$. then $|g^G| = [G : C(g)] = \frac{|G|}{|C(g)|}$

alg structures theorems and friends

(i) if $H \leq G, K \leq G$ and $K \subseteq K$, then $k \leq H$
(ii) if $K \leq G$ and $K \subseteq H$, then $[G : K] = [G : H][H : K]$
(iii) $(S_n, *)$ is a group and $|S_n| = n!$
(iv) any permutation in S_n can be written as product of pairwise disjoint cycles
(v) suppose G is a group with $H \leq G$. let $x, y \in G$ then either $Hx = Hy$ or $Hx \cap Hy = \emptyset$

examples of groups

direct product: suppose $(H, *)$ and $(K, \odot)$ are groups. define $(h, k)(h', k') = (h * h', k \odot k') \in H \times K$

lemma: every permutation in S_n can be written as a product of transpositions

odd / even permutations: we say a permutation is even (odd) if it can be expressed as an even (odd) number of transpositions

c(σ): the number of cycles when we express σ as product of disjoint cycles

lemma: let $\sigma, \tau \in S_n$, τ transposition, then $c(\sigma\tau) = c(\sigma) \pm 1$

s(σ) $= (-1)^{n-c(\sigma)} (= \pm 1)$ for $\sigma \in S_n$

lemma: let $n \in \mathbb{N}, n \geq 2$. if $\sigma \in S_n$ can be written as a product of r transpositions, then $s(\sigma) = (-1)^r$

corollary: if σ can be written as a product of r_1 and r_2 transpositions, then r_1 and r_2 have the same parity

A_n: the set of all even permutations in S_n, i.e. $A = \{\sigma \in S_n \mid \sigma \text{ is an even permutation}\} \leq S_n$

remark: $|A_n| = \frac{1}{2}n!$

subgroups

definitions: suppose G is a group, $g \in G, A, B \subseteq G$.
(i) $A^g = \{g^{-1}ag \mid a \in A\}$
(ii) $AB = \{ab \mid a \in A, b \in B\} \subseteq G$
(iii) $A^- = \{a^{-1} \mid a \in A\}$

definitions: suppose G is a group and $\emptyset \neq S \subseteq G$.
(i) $C_G(S) = \{g \in G \mid xg = gx, \forall x \in S\}$
(ii) $N_G(S) = \{g \in G \mid S^g = S\}$
(iii) $\langle S \rangle = \{x_1 x_2 \ldots x_n \mid x_i \in S \cup S^-, m \in \mathbb{N}\}$

remarks
(i) $C_G(S) \subseteq N_G(S)$
(ii) if $S \leq G$, then $S \subseteq N_G(S)$
(iii) $S \cup S^- \subseteq \langle R \rangle$ and if $R \subseteq S$, then $\langle R \rangle \leq \langle S \rangle$
(iv) if $S = \{x\}$, then $C_G(S) = C_G(x)$ and $\langle S \rangle = \langle x \rangle$
(v) $C_G(S) \leq N_G(S) \leq G$
(vi) if $R \subseteq S$, then $\langle R \rangle \leq \langle S \rangle$
(vii) if $S \langle G$, then $S \subseteq N_G(S)$

lemma: suppose G is a group, $\emptyset \neq S \subseteq G$, then $C_G(S), N_G(S), \langle S \rangle$ are all subgroups of G

lemma: since $Z(G) = C_G(G)$, we have $Z(G) \leq G$

lemma: suppose G is a group and $H \leq G, K \leq G$. then $HK \leq G \iff HK = KH$