MATH20142 Cheat Sheet

1 Construction and Basic Properties of Complex Numbers

An expression $a + ib(a, b \in \mathbb{R})$ is called a **complex number**. We denote the set of complex numbers by $\mathbb{C}$. For $z = x + iy$, we use $x = \text{Re}z$ and $y = \text{Im}z$ and say that z is real if $\text{Im}z = 0$ and that z is imaginary if $\text{Re}z = 0$.

- $\text{Re}(z \pm w) = \text{Re}z \pm \text{Re}w$
- $\text{Im}(z \pm w) = \text{Im}z \pm \text{Re}w$
- $\overline{(z \pm w)} = \overline{z} \pm \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{(z/w)} = \overline{z}/\overline{w}$ if $w \neq 0$
- $z + \overline{z} = 2\text{Re}z$
- $z - \overline{z} = 2\text{Im}z$
- $|z| = 0 \iff z = 0$
- $|zw| = |z||w|$
- $|z/w| = |z|/|w|$ if $w \neq 0$
- $|z + w| \leq |z| + |w|$
- $|z - w| \geq ||z| - |w||$

2 Topology in $\mathbb{C}$

ε-neighbourhood of z_0: $N_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ (disc centred at z_0 containing points with distance $< \varepsilon$)

limit point: $z_0 \in \mathbb{C}$ is a limit point of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0, N_\varepsilon(z_0)$ contains a point in $S \setminus \{z_0\}$

interior point: let $S \subset C, z_0$ a limit point of S, then z_0 is an interior point of S if $\exists\, \varepsilon > 0, N_\varepsilon(z_0) \subset S$

boundary point: let $S \subset C, z_0$ a limit point of S, then z_0 is an boundary point of S if it is not a interior point

open: a set $S \subset \mathbb{C}$ is called open if it consists only of interior points

domain: let $S \subset \mathbb{C}, S \neq \emptyset$, then S is called a domain if S is open and every pair of points can be connected by a polygonal arc lying entirely in S

function: let $S \subset C, S \neq \emptyset$, a function $f : \subset \rightarrow \mathbb{C}$ is a rule which assigns to each $z \in S$, an image $f(z) \in \mathbb{C}$

$\lim_{z \to z_0} f(z)$: let $f : S \to \mathbb{C}$ be a function. if z_0 is a limit point of S then we say $\lim_{z \to z_0} f(z) = l$ if, $\forall \varepsilon > 0, \exists \delta > 0, s \in S$ and $0 < |z - z_0| < \delta \implies |f(z) - l| < \varepsilon$

continuity: $f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

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proposition a set $S \subset \mathbb{C}$ is closed $\iff$ its complement $\mathbb{C} \setminus S$ is open

proposition if $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = k$, then

1. $\lim_{z \to z_0}(f(z) \pm g(z)) = l \pm k$
2. $\lim_{z \to z_0}(f(z)g(z)) = lk$
3. $\lim_{z \to z_0}(f(z)/g(z)) = l/k$ (for $k \neq 0$)

proposition $\lim_{z \to z_0} f(z) = l = \alpha + i\beta\, (\alpha, \beta \in \mathbb{R}) \iff u(x, y) \to \alpha, v(x, y) \to \beta$, as $(x, y) \to (\text{Re}z, \text{Im}z)$

3 Differentiation and Cauchy-Riemann Equations

differentiable at a point: let $S \subset \mathbb{C}$ be a open set. we say that $f : S \to \mathbb{C}$ is differentiable at a point $z_0 \in S$ with derivative $f'(z_0)$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

differentiable function: if f is differentiable at every point of S, we say f is a differentiable function in S

partial derivatives: for $z = x + iy$, write $f(z) = u(x, y) + iv(x, y)$, where u, v are real-valued

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h, y) - u(x, y)}{h} \qquad v_x = \frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(x+h, y) - v(x, y)}{h}$$

$$u_y = \frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x, y+k) - u(x, y)}{k} \qquad v_y = \frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(x, y+k) - v(x, y)}{k}$$

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proposition if f is differentiable at z_0 then f is continuous at z_0

proposition if f is differentiable at $z = x + iy$ then u_x, u_y, v_x, v_y all exist and $u_x = v_y, v_x = -u_y$ (CRE)

theorem if $f(z) = u(x, y) + iv(x, y)$ is a complex function on an open set S and at $z_0 = x_0 + iy_0 \in S$, the partial derivatives u_x, v_x, u_y, v_y all exist, are continuous and satisfy the CRE then f is differentiable at z_0

theorem if f is differentiable in a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant in D

4 Power Series

convergence: we say a sequence $s_n \in \mathbb{C}$ converges to $s \in \mathbb{C}$ if, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon, \forall n \geq N$. the series $\sum_{k=0}^{\infty} z_k$ converges if the sequence of partial sums $s_n = \sum_{k=0}^{n} z_k$ converges and the limit of the sequence is called the sum of the series

divergent series: a series which does not converge is said to be divergent

absolute convergence: we say that $\sum_{k=0}^{\infty} z_k$ is absolutely convergent if the real series $\sum_{k=0}^{\infty} |z_k|$ is convergent

ratio test: consider $\sum_{k=0}^{\infty} z_k$ and suppose that $\lim_{n\to\infty} |z_{n+1}|/|z_n| = l$. if $l < 1$ then $\sum_{k=0}^{\infty} z_k$ is absolutely convergent and if $l > 1$ then $\sum_{k=0}^{\infty} z_k$ diverges

root test: consider $\sum_{k=0}^{\infty} z_k$ and suppose that $\lim_{n\to\infty} |z|^{1/n} = l$. if $l < 1$ then $\sum_{k=0}^{\infty} z_k$ is absolutely convergent and if $l > 1$ then $\sum_{k=0}^{\infty} z_k$ diverges

general principle of convergence: if a series $\sum_{n=1}^{\infty} s_n$ with $s_n \in \mathbb{C}$ converges, then $s_n \to 0$ as $n \to \infty$

power series about z_0: $\sum_{n=0}^{\infty} a_n z^n$

radius of convergence: $R = \sup\{r : \exists z \text{ such that } |z| = r \text{ and } \sum_{n=0}^{\infty} a_n z^n \text{ converges }\}$

disc of convergence: $\{z \in \mathbb{C} : |z| < R\}$, where R is the radius of convergence

computation of radius of convergence: $R = \lim_{n\to\infty} |a_{n-1}/a_n|$, provided the limit exists

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lemma if a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_1 \neq 0$, then it converges absolutely for all z with $|z| < |z_1|$

lemma if $\sum_{n=0}^{\infty} a_n z^n$ diverges for $z = z_2$, then is diverges for all z with $|z| > |z_2|$

theorem the radius of convergence R of $\sum_{n=0}^{\infty} a_n z^n$ is given by $1/R = \lim_{n\to\infty} |a_n|^{1/n}$

lemma if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ then $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ converges for $|z| < R$

theorem a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ may be differentiated term by term within its disc of convergence so that $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$

corollary all higher derivatives $f', f'', f''', \ldots, f^{(n)}, \ldots$ of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ exist for z within the disc of convergence and $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k} = \sum_{n=k}^{\infty} n!/(n-k)! \cdot a_n z^{n-k}$

corollary if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ has disc of convergence $|z-z_0| < R$ then $a_k = f^{(k)}(z_0)/k!$ and we can express f as a **Taylor series** $f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)/n! \cdot (z-z_0)^n$, valid for $|z-z_0| < R$

5 The Exponential Function and Its Friends

the exponential function: define $\exp z = \sum_{n=0}^{\infty} z^n/n!$, which converges absolutely for all $z \in \mathbb{C}$. we can check that $\exp(z_1+z_2) = \exp z_1 \exp z_2$ and by induction, $\exp nz = (\exp z)^n$, for all $n \in \mathbb{Z}^+$

the number e: define $e = \exp 1 = 2.7182818\ldots$ and we can also use the notation $\exp z = e^z$, for all $z \in \mathbb{C}$

trigonometric functions: define $\cos z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n)!$ and $\sin z = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)!$

hyperbolic functions: define $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$ so $\sin iz = i\sinh z$ and $\cos iz = \cosh z$

period: for a function $f : \mathbb{C} \to \mathbb{C}$, a nonzero number $k \in \mathbb{C}$ is called a period if $f(z+k) = f(z)$, for all $z \in \mathbb{C}$

logarithmic function: $\log z = u + iv = log|z| + i \arg z$ and $\mathrm{Log} z = \log z + i \arg z$ $(-\pi < \arg z \leq \pi)$

cut plane: the complex plane with the negative real axis, including zero, removed is called the cut plane and denoted $\mathbb{C}_\pi$

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eulers theorem: $e^{iz} = \cos z + i \sin z$

corollary

- $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
- $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
- $\cos^2 z + \sin^2 z = 1$
- $\sin(z+w) = \sin z \cos w + \cos z \sin w$
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$

lemma: the functions $arg(z)$ and $Log(z)$ are continuous on the cut plane

theorem: let $z \neq 0$ be a complex and let n be a positive integer, then

$$z^{\frac{1}{n}} = \{|z|^{\frac{1}{n}} e^{i(\frac{Argz + 2k\pi}{n})} \mid k = 0, 1, \ldots, n-1\}$$

6 Integration

path: a path is a function $\gamma : [a,b] \to \mathbb{C}$, where $[a,b]$ is a real interval

closed path: γ is a closed path if $\gamma(a) = \gamma(b)$ (it starts and ends at the same point)

smooth path: a path γ is smooth if $\gamma : [a,b] \to \mathbb{C}$ is differentiable and γ' is continuous (one-sided derivatives at a and b)

length of a path: $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$

contour: a contour is a collection of smooth paths $\gamma_1, \ldots, \gamma_n$ where the end point of γ_r coincides with the start point of γ_{r+1} for $r = 1, \ldots, n-1$. if the end point of γ_n coincides with the start point of γ_1, then γ is a closed contour.

length of contour: $\gamma = \gamma_1 + \cdots + \gamma_n$ is $L(\gamma) = L(\gamma_1) + \cdots + L(\gamma_n)$

opposite path: if $\gamma : [a,b] \to \mathbb{C}$ is path then $-\gamma : [b,a] \to \mathbb{C}$ defined by $-\gamma(t) = \gamma(a+b-t)$ is called the opposite path

the integral of f along γ: $\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b U(t)\, dt + i\int_a^b V(t)\, dt$ where $U, V : [a,b] \to \mathbb{R}$

winding number of γ around z_0: $w(\gamma, z_0)$, the number of times γ winds around z_0, with anticlockwise as $+ve$

simply connected: a domain D is simply connected if $w(\gamma, z) = 0$ for every closed contour γ in D and $z \notin D$

analytic: a function $f : D \to \mathbb{C}$ is called analytic if it can be expanded into a Taylor series around any point in D

bounded: we say that a function $f : D \to \mathbb{C}$ is bounded if there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$

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properties of contour integration

- $\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$
- $\int_\gamma (f_1 + f_2) = \int_\gamma f_1 + \int_\gamma f_2$
- $\int_\gamma cf = c \int_\gamma f$
- $\int_{-\gamma} f = -\int_\gamma f$

fundamental theorem of contour integration: if $f : D \to \mathbb{C}$ is continuous, $F : D \to \mathbb{C}$ satisfies $F' = f$ and γ is a contour in D from z_0 to z_1, then $\int_\gamma f = F(z_1) - F(z_0)$

cauchys theorem: let f be differentiable in a domain D and γ a closed contour in D which does not wind around any point outside D, then $\int_\gamma f = 0$

generalised cauchys theorem: suppose that $\gamma_1, \ldots, \gamma_n$ are closed contour in a domain such that $w(\gamma_1, z) + \cdots + w(\gamma_n, z) = 0, \forall z \notin D$. if f is differentiable in D then $\int_{\gamma_1} f + \cdots + \int_{\gamma_n} f = 0$

corollary: let f be differentiable in a simply connected domain D and let γ a closed contour in D, then $\int_\gamma f = 0$

cauchys integral formula for a circle: let f be differentiable in the disc $\{z \in \mathbb{C} : |z - z_0| < R\}$. for $0 < r < R$, let C_r be the path $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ then for $|w - z_0| < r$, $f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z-w} \, dz$

theorem: if f is a differentiable in a domain D, then all the higher derivatives of f exist in D and, for any disc $\{z \in \mathbb{C} : |z - z_0| < R\}$, f has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$

the estimation lemma: let D be a domain. if $f : D \to \mathbb{C}$ is continuous, γ is a contour in D, $|f(z)| \leq M$ for all z on γ, then $|\int_\gamma f| \leq M \cdot L(\gamma)$

cauchys estimate: suppose that f is differentiable in $\{z \in \mathbb{C} : |z - z_0| < R\}$. if $0 < r < R$ and $|f(z)| \leq M$ for $|z - z_0| = r$, then for all $n \geq 0, |f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$

liouvilles theorem: if f if differentiable and bounded in the whole complex plane then f is constant

corollary: suppose $f : \mathbb{C} \to \mathbb{C}$ is differentiable in of $\mathbb{C}$ and there exists $C > 0$ such that $|f(z)| \leq C|z|$, $\forall z \in \mathbb{C}$, then $f(z) = az$ for some $a \in \mathbb{C}$

fundamental theorem of algebra: let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be a polynomial with $n \geq 1$ and $a_1, \ldots, a_n \in \mathbb{C}$, then there exists $w \in \mathbb{C}$ with $P(w) = 0$

corollary: each polynomial of degree n with complex coefficients has exactly n complex roots, taken with their multiplicity

7 Laurent Series

isolated singularities: if f is differentiable in a punctured disc $0 < |z - z_0| < R$ then we say that z_0 is a isolated singularity of f. such f has a Laurent expansion: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$, for $|z - z_0| < R$

1. $b_n = 0$ for all $n \geq 1$. if we define $f(z_0) = a_0$, we obtain a function which is differentiable in the whole disc $|z - z_0| < R$, with Taylor series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. in this case, we say that z_0 is a **removable singularity**
2. only finitely many b_n are non-zero, then we can write $f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n$ where $b_m \neq= 0$. in this case, we say that f has a **pole of order m** at z_0. a pole of order one is called a **simple pole**

3. infinitely many b_n are non-zero. then we say that z_0 is an **isolated essential singularity**

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laurents theorem: if f is differentiable in the annulus $\{z \in \mathbb{C} : R_1 \leq |z - z_0| \leq R_2\}$ where $0 \leq R_1 \leq R_2 \leq \infty$ then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_n)^n + \sum_{n=1}^{\infty} b_n(z - z_n)^{-n}$, where $\sum_{n=0}^{\infty} a_n(z - z_n)^n$ converges for $|z - z_0| < R_2$ and $\sum_{n=1}^{\infty} b_n(z - z_n)^{-n}$ converges for $|z - z_0| > R_1$. in particular, both series converge in $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$

furthermore, if $C_R(t) = z_0 + re^{it}$ with $R_1 < r < R_2$, $0 \leq t \leq 2\pi$ then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_{C_r} f(z)(z - z_0)^{n-1} dz$$

8 Residues and Evaluation of Integrals

the residue of f at z_0: $res(f, z_0) = b_1$ (i.e. the $(z - z_0)^{-1}$ coefficient in Laurent expansion

simple loop: a closed path γ is a simple loop if for every point z not on γ, either $w(\gamma, z) = 0$ or $w(\gamma, z) = 1$. if $w(\gamma, z) = 1$, we say z is inside γ

infinite real integral: we say that $\int_{-\infty}^{\infty}$ exists if $\lim_{A,B \to +\infty} f(x)\, dx$ converges, where the limits can be taken in either order

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cauchys residue theorem: let D be a domain containing a simple loop γ and the points inside γ. if f is differentiable in D except for finitely many isolated singularities at $z_1, \ldots, z_n$ inside γ then $\int_\gamma f(z)\, dz = 2\pi i \sum_{r=1}^{n} res(f, z_r)$

lemma: if z_0 is a simple of f then $res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$

lemma: if $f(z) = p(z)/q(z)$ where $p(z_0) \neq 0, q(z_0) = 0$ and $q'(z_0) \neq 0$, then $res(f, z_0) = p(z_0)/q'(z_0)$

lemma: if f has a pole of order m at z_0, then $res(f, z_0) = \lim_{z \to z_0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \big((z - z_0)^m f(z) \big) \right)$