

MATH20142 Cheat Sheet

1 Construction and Basic Properties of Complex Numbers

An expression $a + ib$ ($a, b \in \mathbb{R}$) is called a **complex number**. We denote the set of complex numbers by \mathbb{C} . For $z = x + iy$, we use $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ and say that z is real if $\operatorname{Im} z = 0$ and that z is imaginary if $\operatorname{Re} z = 0$.

- $\operatorname{Re}(z \pm w) = \operatorname{Re} z \pm \operatorname{Re} w$
- $\operatorname{Im}(z \pm w) = \operatorname{Im} z \pm \operatorname{Im} w$
- $\overline{(z \pm w)} = \bar{z} \pm \bar{w}$
- $\overline{\bar{z}} = z$
- $\overline{(z/w)} = \bar{z}/\bar{w}$ if $w \neq 0$
- $z + \bar{z} = 2\operatorname{Re} z$
- $z - \bar{z} = 2i\operatorname{Im} z$
- $|z| = 0 \iff z = 0$
- $|zw| = |z||w|$
- $|z/w| = |z|/|w|$ if $w \neq 0$
- $|z + w| \leq |z| + |w|$
- $|z - w| \geq ||z| - |w||$
- $|\bar{z}| = |z|$
- $z\bar{z} = |z|^2$

2 Topology in \mathbb{C}

ε -neighbourhood of z_0 : $N_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ (disc centred at z_0 containing points with distance $< \varepsilon$)

limit point: $z_0 \in \mathbb{C}$ is a limit point of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0$, $N_\varepsilon(z_0)$ contains a point in $S \setminus \{z_0\}$

interior point: let $S \subset \mathbb{C}$, z_0 a limit point of S , then z_0 is an interior point of S if $\exists \varepsilon > 0$, $N_\varepsilon(z_0) \subset S$

boundary point: let $S \subset \mathbb{C}$, z_0 a limit point of S , then z_0 is a boundary point of S if it is not an interior point

open: a set $S \subset \mathbb{C}$ is called open if it consists only of interior points

domain: let $S \subset \mathbb{C}$, $S \neq \emptyset$, then S is called a domain if S is open and every pair of points can be connected by a polygonal arc lying entirely in S

function: let $S \subset \mathbb{C}$, $S \neq \emptyset$, a function $f : S \rightarrow \mathbb{C}$ is a rule which assigns to each $z \in S$, an image $f(z) \in \mathbb{C}$

$\lim_{z \rightarrow z_0} f(z)$: let $f : S \rightarrow \mathbb{C}$ be a function. if z_0 is a limit point of S then we say $\lim_{z \rightarrow z_0} f(z) = l$ if, $\forall \varepsilon > 0, \exists \delta > 0, z \in S$ and $0 < |z - z_0| < \delta \implies |f(z) - l| < \varepsilon$

continuity: $f(z)$ is continuous at z_0 if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

proposition a set $S \subset \mathbb{C}$ is closed \iff its complement $\mathbb{C} \setminus S$ is open

proposition if $\lim_{z \rightarrow z_0} f(z) = l$ and $\lim_{z \rightarrow z_0} g(z) = k$, then

1. $\lim_{z \rightarrow z_0} (f(z) \pm g(z)) = l \pm k$
2. $\lim_{z \rightarrow z_0} (f(z)g(z)) = lk$
3. $\lim_{z \rightarrow z_0} (f(z)/g(z)) = l/k$ (for $k \neq 0$)

proposition $\lim_{z \rightarrow z_0} f(z) = l = \alpha + i\beta$ ($\alpha, \beta \in \mathbb{R}$) $\iff u(x, y) \rightarrow \alpha, v(x, y) \rightarrow \beta$, as $(x, y) \rightarrow (\operatorname{Re} z_0, \operatorname{Im} z_0)$

3 Differentiation and Cauchy-Riemann Equations

differentiable at a point: let $S \subset \mathbb{C}$ be an open set. we say that $f : S \rightarrow \mathbb{C}$ is differentiable at a point $z_0 \in S$ with derivative $f'(z_0)$ if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

differentiable function: if f is differentiable at every point of S , we say f is a differentiable function in S

partial derivatives: for $z = x + iy$, write $f(z) = u(x, y) + iv(x, y)$, where u, v are real-valued

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} = \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h} & v_x &= \frac{\partial v}{\partial x} = \lim_{h \rightarrow 0} \frac{v(x+h, y) - v(x, y)}{h} \\ u_y &= \frac{\partial u}{\partial y} = \lim_{k \rightarrow 0} \frac{u(x, y+k) - u(x, y)}{k} & v_y &= \frac{\partial v}{\partial y} = \lim_{k \rightarrow 0} \frac{v(x, y+k) - v(x, y)}{k} \end{aligned}$$

proposition if f is differentiable at z_0 then f is continuous at z_0

proposition if f is differentiable at $z = x + iy$ then u_x, u_y, v_x, v_y all exist and $u_x = v_y, v_x = -u_y$ (CRE)

theorem if $f(z) = u(x, y) + iv(x, y)$ is a complex function on an open set S and at $z_0 = x_0 + iy_0 \in S$, the partial derivatives u_x, v_x, u_y, v_y all exist, are continuous and satisfy the CRE then f is differentiable at z_0

theorem if f is differentiable in a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant in D

4 Power Series

convergence: we say a sequence $s_n \in \mathbb{C}$ converges to $s \in \mathbb{C}$ if, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon, \forall n \geq N$. the series $\sum_{k=0}^{\infty} z_k$ converges if the sequence of partial sums $s_n = \sum_{k=0}^n z_k$ converges and the limit of the sequence is called the sum of the series

divergent series: a series which does not converge is said to be divergent

absolute convergence: we say that $\sum_{k=0}^{\infty} z_k$ is absolutely convergent if the real series $\sum_{k=0}^{\infty} |z_k|$ is convergent

ratio test: consider $\sum_{k=0}^{\infty} z_k$ and suppose that $\lim_{n \rightarrow \infty} |z_{n+1}|/|z_n| = l$. if $l < 1$ then $\sum_{k=0}^{\infty} z_k$ is absolutely convergent and if $l > 1$ then $\sum_{k=0}^{\infty} z_k$ diverges

root test: consider $\sum_{k=0}^{\infty} z_k$ and suppose that $\lim_{n \rightarrow \infty} |z|^{1/n} = l$. if $l < 1$ then $\sum_{k=0}^{\infty} z_k$ is absolutely convergent and if $l > 1$ then $\sum_{k=0}^{\infty} z_k$ diverges

general principle of convergence: if a series $\sum_{n=1}^{\infty} s_n$ with $s_n \in \mathbb{C}$ converges, then $s_n \rightarrow 0$ as $n \rightarrow \infty$

power series about z_0 : $\sum_{n=0}^{\infty} a_n z^n$

radius of convergence: $R = \sup\{r : \exists z \text{ such that } |z| = r \text{ and } \sum_{n=0}^{\infty} a_n z^n \text{ converges}\}$

disc of convergence: $\{z \in \mathbb{C} : |z| < R\}$, where R is the radius of convergence

computation of radius of convergence: $R = \lim_{n \rightarrow \infty} |a_{n-1}/a_n|$, provided the limit exists

lemma if a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_1 \neq 0$, then it converges absolutely for all z with $|z| < |z_1|$

lemma if $\sum_{n=0}^{\infty} a_n z^n$ diverges for $z = z_2$, then it diverges for all z with $|z| > |z_2|$

theorem the radius of convergence R of $\sum_{n=0}^{\infty} a_n z^n$ is given by $1/R = \lim_{n \rightarrow \infty} |a_n|^{1/n}$

lemma if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ then $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ converges for $|z| < R$

theorem a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ may be differentiated term by term within its disc of convergence so that $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$

corollary all higher derivatives $f', f'', f''', \dots, f^{(n)}, \dots$ of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ exist for z within the disc of convergence and $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n z^{n-k} = \sum_{n=k}^{\infty} n!/(n-k)! \cdot a_n z^{n-k}$

corollary if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has disc of convergence $|z - z_0| < R$ then $a_k = f^{(k)}(z_0)/k!$ and we can express f as a **Taylor series** $f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)/n! \cdot (z - z_0)^n$, valid for $|z - z_0| < R$

5 The Exponential Function and Its Friends

the exponential function: define $\exp z = \sum_{n=0}^{\infty} z^n/n!$, which converges absolutely for all $z \in \mathbb{C}$. we can check that $\exp(z_1 + z_2) = \exp z_1 \exp z_2$ and by induction, $\exp nz = (\exp z)^n$, for all $n \in \mathbb{Z}^+$

the number e : define $e = \exp 1 = 2.7182818\dots$ and we can also use the notation $\exp z = e^z$, for all $z \in \mathbb{C}$

trigonometric functions: define $\cos z = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)!$ and $\sin z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)!$

hyperbolic functions: define $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$ so $\sin iz = i \sinh z$ and $\cos iz = \cosh z$

period: for a function $f : \mathbb{C} \rightarrow \mathbb{C}$, a nonzero number $k \in \mathbb{C}$ is called a period if $f(z + k) = f(z)$, for all $z \in \mathbb{C}$

logarithmic function: $\log z = u + iv = \log|z| + i \arg z$ and $\text{Log} z = \log z + i \arg z$ ($-\pi < \arg z \leq \pi$)

cut plane: the complex plane with the negative real axis, including zero, removed is called the cut plane and denoted \mathbb{C}_π

eulers theorem: $e^{iz} = \cos z + i \sin z$

corollary

- $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
- $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
- $\cos^2 z + \sin^2 z = 1$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$
- $\sin iz = i \sinh z$
- $\cos iz = \cosh z$
- $\sinh iz = i \sin z$
- $\cosh iz = \cos z$

lemma: the functions $\arg(z)$ and $\text{Log}(z)$ are continuous on the cut plane

theorem: in the cut plane, $\frac{d}{dz} \text{Log} z = \frac{1}{z}$

theorem: let $z \neq 0$ be a complex and let n be a positive integer, then

$$z^{\frac{1}{n}} = \{|z|^{\frac{1}{n}} e^{i(\frac{\arg z + 2k\pi}{n})} \mid k = 0, 1, \dots, n-1\}$$

6 Integration

path: a path is a function $\gamma : [a, b] \rightarrow \mathbb{C}$, where $[a, b]$ is a real interval

closed path: γ is a closed path if $\gamma(a) = \gamma(b)$ (it starts and ends at the same point)

smooth path: a path γ is smooth if $\gamma : [a, b] \rightarrow \mathbb{C}$ is differentiable and γ' is continuous (one-sided derivatives at a and b)

length of a path: $L(\gamma) = \int_a^b |\gamma'(t)| dt$

contour: a contour is a collection of smooth paths $\gamma_1, \dots, \gamma_n$ where the end point of γ_r coincides with the start point of γ_{r+1} for $r = 1, \dots, n-1$. if the end point of γ_n coincides with the start point of γ_1 , then γ is a closed contour.

length of contour: $\gamma = \gamma_1 + \dots + \gamma_n$ is $L(\gamma) = L(\gamma_1) + \dots + L(\gamma_n)$

opposite path: if $\gamma : [a, b] \rightarrow \mathbb{C}$ is path then $-\gamma : [b, a] \rightarrow \mathbb{C}$ defined by $-\gamma(t) = \gamma(a + b - t)$ is called the opposite path

the integral of f along γ : $\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt = \int_a^b U(t) dt + i \int_a^b V(t) dt$ where $U, V : [a, b] \rightarrow \mathbb{R}$

winding number of γ around z_0 : $w(\gamma, z_0)$, the number of times γ winds around z_0 , with anticlockwise as $+ve$

simply connected: a domain D is simply connected if $w(\gamma, z) = 0$ for every closed contour γ in D and $z \notin D$

analytic: a function $f : D \rightarrow \mathbb{C}$ is called analytic if it can be expanded into a Taylor series around any point in D

bounded: we say that a function $f : D \rightarrow \mathbb{C}$ is bounded if there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$

properties of contour integration

- $\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$
- $\int_{\gamma} cf = c \int_{\gamma} f$
- $\int_{\gamma} (f_1 + f_2) = \int_{\gamma} f_1 + \int_{\gamma} f_2$
- $\int_{-\gamma} f = - \int_{\gamma} f$

fundamental theorem of contour integration: if $f : D \rightarrow \mathbb{C}$ is continuous, $F : D \rightarrow \mathbb{C}$ satisfies $F' = f$ and γ is a contour in D from z_0 to z_1 , then $\int_{\gamma} f = F(z_1) - F(z_0)$

cauchys theorem: let f be differentiable in a domain D and γ a closed contour in D which does not wind around any point outside D , then $\int_{\gamma} f = 0$

generalised cauchys theorem: suppose that $\gamma_1, \dots, \gamma_n$ are closed contour in a domain such that $w(\gamma_1, z) + \dots + w(\gamma_n, z) = 0, \forall z \notin D$. if f is differentiable in D then $\int_{\gamma_1} f + \dots + \int_{\gamma_n} f = 0$

corollary: let f be differentiable in a simply connected domain D and let γ a closed contour in D , then $\int_{\gamma} f = 0$

cauchys integral formula for a circle: let f be differentiable in the disc $\{z \in \mathbb{C} : |z - z_0| < R\}$. for $0 < r < R$, let C_r be the path $C_r(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$ then for $|w - z_0| < r, f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} dz$

theorem: if f is a differentiable in a domain D , then all the higher derivatives of f exist in D and, for any disc $\{z \in \mathbb{C} : |z - z_0| < R\}$, f has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$

the estimation lemma: let D be a domain. if $f : D \rightarrow \mathbb{C}$ is continuous, γ is a contour in D , $|f(z)| \leq M$ for all z on γ , then $|\int_{\gamma} f| \leq M \cdot L(\gamma)$

cauchys estimate: suppose that f is differentiable in $\{z \in \mathbb{C} : |z - z_0| < R\}$. if $0 < r < R$ and $|f(z)| \leq M$ for $|z - z_0| = r$, then for all $n \geq 0, |f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$

liouvilles theorem: if f is differentiable and bounded in the whole complex plane then f is constant

corollary: suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is differentiable in of \mathbb{C} and there exists $C > 0$ such that $|f(z)| \leq C|z|, \forall z \in \mathbb{C}$, then $f(z) = az$ for some $a \in \mathbb{C}$

fundamental theorem of algebra: let $P(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$ be a polynomial with $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{C}$, then there exists $w \in \mathbb{C}$ with $P(w) = 0$

corollary: each polynomial of degree n with complex coefficients has exactly n complex roots, taken with their multiplicity

7 Laurent Series

isolated singularities: if f is differentiable in a punctured disc $0 < |z - z_0| < R$ then we say that z_0 is a isolated singularity of f . such f has a Laurent expansion: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$, for $|z - z_0| < R$

1. $b_n = 0$ for all $n \geq 1$. if we define $f(z_0) = a_0$, we obtain a function which is differentiable in the whole disc $|z - z_0| < R$, with Taylor series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. in this case, we say that z_0 is a **removable singularity**

2. only finitely many b_n are non-zero, then we can write $f(z) = \frac{b_m}{(z-z_0)^m} + \dots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$ where $b_m \neq 0$. in this case, we say that f has a **pole of order m** at z_0 . a pole of order one is called a **simple pole**
3. infinitely many b_n are non-zero. then we say that z_0 is an **isolated essential singularity**

laurents theorem: if f is differentiable in the annulus $\{z \in \mathbb{C} : R_1 \leq |z - z_0| \leq R_2\}$ where $0 \leq R_1 \leq R_2 \leq \infty$ then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$, where $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for $|z - z_0| < R_2$ and $\sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$ converges for $|z - z_0| > R_1$. in particular, both series converge in $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$

furthermore, if $C_R(t) = z_0 + re^{it}$ with $R_1 < r < R_2$, $0 \leq t \leq 2\pi$ then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_{C_r} f(z)(z - z_0)^{n-1} dz$$

8 Residues and Evaluation of Integrals

the residue of f at z_0 : $\text{res}(f, z_0) = b_1$ (i.e. the $(z - z_0)^{-1}$ coefficient in Laurent expansion)

simple loop: a closed path γ is a simple loop if for every point z not on γ , either $w(\gamma, z) = 0$ or $w(\gamma, z) = 1$. if $w(\gamma, z) = 1$, we say z is inside γ

infinite real integral: we say that $\int_{-\infty}^{\infty} f(x) dx$ exists if $\lim_{A, B \rightarrow +\infty} \int_A^B f(x) dx$ converges, where the limits can be taken in either order

cauchys residue theorem: let D be a domain containing a simple loop γ and the points inside γ . if f is differentiable in D except for finitely many isolated singularities at z_1, \dots, z_n inside γ then $\int_{\gamma} f(z) dz = 2\pi i \sum_{r=1}^n \text{res}(f, z_r)$

lemma: if z_0 is a simple pole of f then $\text{res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z)$

lemma: if $f(z) = p(z)/q(z)$ where $p(z_0) \neq 0, q(z_0) = 0$ and $q'(z_0) \neq 0$, then $\text{res}(f, z_0) = p(z_0)/q'(z_0)$

lemma: if f has a pole of order m at z_0 , then $\text{res}(f, z_0) = \lim_{z \rightarrow z_0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right)$