math34001 - applied complex analysis

available at JTANG.DEV/RESOURCES

0 - revision of basics

complex exponential: $e^{iz} = \cos z + i \sin z$
triangle inequality: $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

order of functions
$f = O(g)$ as $z \to z_0 \iff f(z)/g(z)$ is bounded as $z \to z_0$
$f = o(g)$ as $z \to z_0 \iff f(z)/g(z) \to 0$ as $z \to z_0$
$f \sim g$ as $z \to z_0 \iff f(z)/g(z) \to 1$ as $z \to z_0$

bounds
$|z^\alpha| = |e^{\alpha \ln z}| = |e^{\alpha(\ln|z| + iargz)}| = |e^{\alpha \ln|z|}| = |z|^\alpha$
$|x^z| = |e^{z \ln x}| = |e^{(a+ib)\ln x}| = |e^{a \ln x}| = x^a = x^{Re(z)}$

useful identities
$\sin iz = i \sinh z$ and $\sinh iz = i \sin z$
$\cos iz = \cosh z$ and $\cosh iz = \cos z$

useful aside
suppose $a(z)$ and $b(z)$ have simple poles at $z = z_0$, and $c(z)$ has a simple zero at $z = z_0$, then $a(z)/b(z)$ and $a(z)c(z)$ are regular at $z = z_0$

1 - regular functions of a complex variable

complex differentiable at $z = a$: $\lim\limits_{z \to a} \left(\frac{f(z)-f(a)}{z-a}\right)$ exists

regular in $D \subseteq \mathbb{C}$: $\lim\limits_{z \to a} \left(\frac{f(z)-f(a)}{z-a}\right)$ exists $\forall a \in D$

entire: a function that is regular over the whole of $\mathbb{C}$

singularity: an isolated point where $f(z)$ fails to have a derivative

power series

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n \quad \text{for } |z - z_0| < R$$

laurent expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(z - z_0)^n} \quad \text{for } |z - z_0| > R$$

2 - the functions $\ln z$ and z^α

$$\ln z = \ln|z| + i(\arg(z) + 2m\pi) \quad (m \in \mathbb{Z}, \text{usually } m = 1)$$

$$z^\alpha = \exp(\alpha \ln z) = |z|^\alpha \exp(i\alpha(\arg(z) + 2m\pi)) \quad (\alpha \in \mathbb{R})$$

notation
$X + i0$ is the point just above the real axis at $x = X$
$X - i0$ is the point just below the real axis at $x = X$

principle branch: $-\pi \leq \arg(z) < \pi$
secondary branch: $0 \leq \arg(z) < 2\pi$

3 - contour integrals and cauchy's theorem

estimation lemma

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq (\text{length of } \gamma) \times \max_{\text{along } \gamma} \left| f(z) \right|$$

bounding a fraction

$$\max_\gamma \left| \frac{g(z)}{h(z)} \right| \leq \frac{\max_\gamma |g(z)|}{\min_\gamma |h(z)|} \quad \text{for } h(z) \neq 0 \text{ along } \gamma$$

cauchy's theorem: if γ is a simple closed curve in the complex z-plane and $f(z)$ is regular everywhere inside γ, then $\oint_\gamma f(z)\, \mathrm{d}z = 0$

cauchy's residue theorem: let C be a simple closed curve taken anti-clockwise and let $f(z)$ be regular inside and on C except for a finite number of poles $z_1, \ldots, z_m$ inside C, then

$$\oint_C f(z)\, \mathrm{d}z = 2\pi i \sum_{m=1}^{M} \mathrm{Res}\{f(z) : z = z_m\}$$

residue of a pole at z_0 of order n

$$\frac{1}{(n-1)!} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{n-1} \left\{ (z - z_0)^n f(z) \right\} \right\}_{z = z_0}$$

residue of simple pole at z_0

$$a_{-1} = \lim_{z \to z_0} \left\{ (z - z_0) f(z) \right\}$$

cauchy's integral formula: let C be a simple closed curve taken anti-clockwise and let $f(z)$ be regular inside and on C. then for any point x inside C, we have

$$f(x) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - x} \mathrm{d}z$$

4 - real definite integrals by contour integration

the strategy (almost all the time)
(1) evaualte the integral using cauchy's residue theorem
(2) split the integral into parts and argue some away

jordans lemma: if $s > 0$ and $f(z) \to 0$ as $z \to \infty$ then

$$\int_{C_R} f(z) e^{isz} \mathrm{d}z \to 0 \quad \text{as} \quad R \to \infty$$

common choices for contours
(1) D-contour: let $R \to \infty$
(2) keyhole contour: let $\varepsilon \to 0, R \to \infty$
(3) dumbbell contour: let $\varepsilon \to 0, \delta \to 0$

5 - analytic continuation

definition (a): suppose
(1) $f(z)$ is regular in a domain $D \subseteq \mathbb{C}$
(2) $g(z)$ is regular in a domain $E \subseteq \mathbb{C}$
(3) $f(z) = g(z)$ is regular in $D \cap E \subseteq \mathbb{C}$
then $g(z)$ **is the analytic continuation of** $f(z)$

theorem: suppose that $f(z)$ is regular in a domain D and $f(a) = 0$ for some internal point $a \in D$, then either a is an isolated zero of $f(z)$ or $f(z) \equiv 0$ in D.

definition (b): suppose
(1) $f(z)$ is regular in a domain $D \subseteq \mathbb{C}$
(2) $g(z)$ is regular in a domain $E \subseteq \mathbb{C}$
(3) $f(z) = g(z)$ on the line $L \in D \cap E \subseteq \mathbb{C}$
then $g(z)$ is the analytic continuation of $f(z)$ into E, and $f(z)$ is the analytic continuation of $g(z)$ into D

regularity of a function defined by an integral
the function $f(z) = \int_a^b F(z,t)\,\mathrm{d}t$ will be regular if the integral exists and $F(z,t)$ is *suitably well defined*

schwarz's reflection principle (weak form): suppose
(1) $f(z)$ is regular in a domain D which is symmetrical about the real axis
(2) $f(z)$ is real on some section of the real axis lying in D
then $f(z) = \overline{f(\overline{z})}$ (also written as $\overline{f(z)} = f(\overline{z})$)

definition (c) - contact continuation: suppose
(1) regions D_1 and D_2 are in contact along the line γ
(2) functions $f_1(z)$ and $f_2(z)$ are regular in the regions D_1 and D_2 respectively, and are continuous in the regions $D_1 \cup \gamma$ and $D_1 \cup \gamma$ respectively
(3) $f_1(z) = f_2(z)$ on γ
then f_1 and f_2 are analytic continuations of each other

schwarz's reflection principle (strong form)
(1) $f(z)$ behaves as $|z| \to \infty$ in $Im(z) > 0$
(2) $f(z)$ is well defined on the real axis
then we can use $f(z) = \overline{f(\overline{z})}$ to analytically continue $f(z)$ into the lower half plane $Im(z) < 0$

6 - the gamma function $\Gamma(z)$

the **gamma function** $\Gamma(z)$ is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t$$

and is regular for $Re(z) > 0$

for $n \in \mathbb{Z}^+$, $\Gamma(n+1) = n!$

we can analytically continue $\Gamma(z)$ into the whole plane except for simple poles at $z = 0, -1, -2, \ldots$ and the residue of the simple pole $z = -n$ is $\frac{(-1)^n}{n!}$

recurrence relation: $\Gamma(z+1) = z\Gamma(z)$
reflection formula: $\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi$

7 - integral transforms

fourier cosine

$$F(k) = \int_0^\infty f(x)\cos(kx)\,\mathrm{d}x$$
$$f(x) = \frac{2}{\pi}\int_0^\infty F(k)\cos(kx)\,\mathrm{d}k$$

fourier sine

$$F(k) = \int_0^\infty f(x)\sin(kx)\,\mathrm{d}x$$
$$f(x) = \frac{2}{\pi}\int_0^\infty F(k)\sin(kx)\,\mathrm{d}k$$

complex fourier

$$F(k) = \int_{-\infty}^\infty f(x)e^{ikx}\,\mathrm{d}x$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(k)e^{-ikx}\,\mathrm{d}k$$

laplace

$$F(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t$$
$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(t)e^{pt}\mathrm{d}p \quad (c > a, Re(p) < a)$$

in order for $F(k)$ to exist, the condition that $f(x)$ is **absolutely integrable** is sufficient, that is, $\int_{-\infty}^\infty |f(x)|\mathrm{d}x < \infty$

heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

complex fourier summary

$$\mathcal{F}\{f(x)\} = F(k)$$
$$\mathcal{F}\{f'(x)\} = -ikF(k)$$
$$\mathcal{F}\{f''(x)\} = -k^2F(k)$$
$$\mathcal{F}\{ixf(x)\} = \frac{\mathrm{d}}{\mathrm{d}k}F(k)$$
$$\mathcal{F}\{-x^2f(x)\} = \frac{\mathrm{d}^2}{\mathrm{d}k^2}F(k)$$

laplace summary

$$\mathcal{L}\{f(x)\} = F(p)$$
$$\mathcal{L}\{f'(x)\} = -f(0) + p\hat{F}(p)$$
$$\mathcal{L}\{f''(x)\} = -f'(0) + pf(0) + p^2\hat{F}(p)$$
$$\mathcal{L}\{-xf(x)\} = \frac{\mathrm{d}}{\mathrm{d}p}\hat{F}(p)$$
$$\mathcal{L}\{x^2f(x)\} = \frac{\mathrm{d}^2}{\mathrm{d}p^2}\hat{F}(p)$$