MATH20142 Cheat Sheet

1 Construction and Basic Properties of Complex Numbers

An expression $a + ib(a, b \in \mathbb{R})$ is called a **complex number**. We denote the set of complex numbers by $\mathbb{C}$. For $z = x + iy$, we use $x = \text{Re}z$ and $y = \text{Im}z$ and say that z is real if $\text{Im}z = 0$ and that z is imaginary if $\text{Re}z = 0$.

- $\text{Re}(z \pm w) = \text{Re}z \pm \text{Re}w$
- $\text{Im}(z \pm w) = \text{Im}z \pm \text{Re}w$
- $\overline{(z \pm w)} = \overline{z} \pm \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{(z/w)} = \overline{z}/\overline{w}$ if $w \neq 0$
- $z + \overline{z} = 2\text{Re}z$
- $z - \overline{z} = 2\text{Im}z$
- $|z| = 0 \iff z = 0$
- $|zw| = |z||w|$
- $|z/w| = |z|/|w|$ if $w \neq 0$
- $|z + w| \leq |z| + |w|$
- $|z - w| \geq ||z| - |w||$

2 Topology in $\mathbb{C}$

ε-neighbourhood of z_0: $N_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ (disc centred at z_0 containing points with distance $< \varepsilon$)

limit point: $z_0 \in \mathbb{C}$ is a limit point of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0, N_\varepsilon(z_0)$ contains a point in $S \setminus \{z_0\}$

interior point: let $S \subset C, z_0$ a limit point of S, then z_0 is an interior point of S if $\exists\, \varepsilon > 0, N_\varepsilon(z_0) \subset S$

boundary point: let $S \subset C, z_0$ a limit point of S, then z_0 is an boundary point of S if it is not a interior point

open: a set $S \subset \mathbb{C}$ is called open if it consists only of interior points

domain: let $S \subset \mathbb{C}, S \neq \emptyset$, then S is called a domain if S is open and every pair of points can be connected by a polygonal arc lying entirely in S

function: let $S \subset C, S \neq \emptyset$, a function $f : \subset \rightarrow \mathbb{C}$ is a rule which assigns to each $z \in S$, an image $f(z) \in \mathbb{C}$

$\lim_{z \to z_0} f(z)$: let $f : S \to \mathbb{C}$ be a function. if z_0 is a limit point of S then we say $\lim_{z \to z_0} f(z) = l$ if, $\forall \varepsilon > 0, \exists \delta > 0, s \in S$ and $0 < |z - z_0| < \delta \implies |f(z) - l| < \varepsilon$

continuity: $f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

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proposition a set $S \subset \mathbb{C}$ is closed $\iff$ its complement $\mathbb{C} \setminus S$ is open

proposition if $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = k$, then

1. $\lim_{z \to z_0}(f(z) \pm g(z)) = l \pm k$
2. $\lim_{z \to z_0}(f(z)g(z)) = lk$
3. $\lim_{z \to z_0}(f(z)/g(z)) = l/k$ (for $k \neq 0$)

proposition $\lim_{z \to z_0} f(z) = l = \alpha + i\beta\, (\alpha, \beta \in \mathbb{R}) \iff u(x, y) \to \alpha, v(x, y) \to \beta$, as $(x, y) \to (\text{Re}z, \text{Im}z)$

3 Differentiation and Cauchy-Riemann Equations

differentiable at a point: let $S \subset \mathbb{C}$ be a open set. we say that $f : S \to \mathbb{C}$ is differentiable at a point $z_0 \in S$ with derivative $f'(z_0)$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

differentiable function: if f is differentiable at every point of S, we say f is a differentiable function in S

partial derivatives: for $z = x + iy$, write $f(z) = u(x,y) + iv(x,y)$, where u, v are real-valued

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} \qquad v_x = \frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h}$$
$$u_y = \frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{k} \qquad v_y = \frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(x,y+k) - v(x,y)}{k}$$

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proposition if f is differentiable at z_0 then f is continuous at z_0

proposition if f is differentiable at $z = x + iy$ then u_x, u_y, v_x, v_y all exist and $u_x = v_y, v_x = -u_y$ (CRE)

theorem if $f(z) = u(x,y) + iv(x,y)$ is a complex function on an open set S and at $z_0 = x_0 + iy_0 \in S$, the partial derivatives u_x, v_x, u_y, v_y all exist, are continuous and satisfy the CRE then f is differentiable at z_0

theorem if f is differentiable in a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant in D