Constrained Dynamics of a 3D Serial Chain of Rigid Bodies

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1 Chain Kinematics

Each body $i = 1 \dots n$ is defined by a mass m_i, center of mass relative disance c_i from the handle (pivot point) along the x-axis, and local mass moment of inertia tensor $\mathrm{I}_i^{\text{body}}$. Additionally each succesive joint is located a distance L_i along the local x-axis, defining the relative vector $L_i \hat{x}$.

1.1 Position Kinematics

Each joint is described by the local joint axis $\boldsymbol{z}_i$ and relative joint angle θ_i.

The orientation of the body is tracked by the 3×3 rotation matrix R_i and the location vector of the joint $\overline{r}_i$. Additionally each center of mass location vector is $\overline{r}_i^C$. These are found recursively going up the chain

$$\begin{aligned}
&\text{iterate } i{=}1 \text{ to } N \\
\mathrm{R}_i &= \mathrm{R}_{i-1} \mathrm{rot}(\boldsymbol{z}_i,\, \theta_i) && (1)\\
\overline{r}_i &= \overline{r}_{i-1} + \mathrm{R}_{i-1} L_i \hat{x} && (2)\\
\overline{r}_i^C &= \overline{r}_i + \mathrm{R}_i c_i \hat{x} && (3)\\
&\text{initial conditions} \\
\mathrm{R}_0 &= 1 && (4)\\
\overline{r}_1 &= 0 && (5)
\end{aligned}$$

1.2 Velocity Kinematics

Each joint is described by the joint axis twist $\boldsymbol{s}_i$ defined from the joint axis $\boldsymbol{z}_i$ and position $\overline{r}_i$. Then each body velocity twist $\boldsymbol{v}_i$ is defined recursively from the previous body and the relative twist $\boldsymbol{s}_i \dot{\theta}_i$

In addition each body is described by the spatial inertia matrix $\mathbf{I}_i$ defined from the mass m_i, the center of mass vector $\overline{r}_i^C$ and the mass moment of inertia tensor $\mathrm{I}_i^{\text{body}}$, but aligned about the inertial frame $\mathrm{I}_i = \mathrm{R}_i \mathrm{I}_i^{\text{body}} \mathrm{R}_i^{\intercal}$.

The following recursion can be combined with the position kinematics above

$$\begin{aligned}
&\text{iterate } i{=}1 \text{ to } N \\
\boldsymbol{s}_i &= \begin{bmatrix} \overline{r}_i \times \boldsymbol{z}_i \\ \boldsymbol{z}_i \end{bmatrix} && (6)\\
\boldsymbol{v}_i &= \boldsymbol{v}_{i-1} + \boldsymbol{s}_i \dot{\theta}_i && (7)\\
\mathrm{I}_i^C &= \mathrm{R}_i \mathrm{I}_i^{\text{body}} \mathrm{R}\ell_i^{\intercal} && (8)\\
\mathbf{I}_i &= \begin{bmatrix} m_i & -m_i \overline{r}_i^C \times \\ m_i \overline{r}_i^C \times & \mathrm{I}_i^C - m_i \overline{r}_i^C \times \overline{r}_i^C \times \end{bmatrix} && (9)\\
&\text{initial conditions} \\
\boldsymbol{v}_0 &= 0 && (10)
\end{aligned}$$

where $\overline{r}_i^C \times$ is the 3×3 skew-symmetric cross product matrix.

1.3 Impulse Propagation

Consider an impulse of magnitude J and direction $\boldsymbol{n}$ acting on the last N-th body. There is going to be a reaction impulse $\boldsymbol{\ell}_N$ on the N-th joint described by the following conservation of momentum

$$\boldsymbol{\ell}_N = \mathbf{I}_N \Delta \boldsymbol{v}_N + \boldsymbol{n}\, J \tag{11}$$

where $\Delta \boldsymbol{v}_N$ is the *change* in velocity twist on the last body. Note that with the convention above, the external impulse acts in a negative sense, as if the last body is impacting a fixed object, and thus experiences an equal and opposite reaction impulse along $\boldsymbol{n}$.

Now consider the second to last link which experiences and impulse $\boldsymbol{\ell}_N$ from the last link, and has its own reaction impulse, and so on down the chain.

$$\boldsymbol{\ell}_i = \mathbf{I}_i \Delta \boldsymbol{v}_i + \boldsymbol{\ell}_{i+1} \tag{12}$$

and each step change in motion twist $\Delta \boldsymbol{v}_i$ must be accomodated by a step change in joint speed $\Delta \dot{\theta}_i$. Similarly to the kinematics in (7), the response to the impulse is going to propagate throughtout the chain using the following relationship

$$\Delta\boldsymbol{v}_i = \Delta\boldsymbol{v}_{i-1} + \boldsymbol{s}_i \Delta\dot{\theta}_i \tag{13}$$

2 Impulse Response

2.1 Constrained Inverse Spatial Inertia

The constrained inverse spatial inertia matrix $\boldsymbol{\Lambda}_i^{-1}$ for the i-th link relates the reaction impulse coming from the next link to the step change in motion of the link

$$\Delta\boldsymbol{v}_i = -\boldsymbol{\Lambda}_i^{-1} \boldsymbol{\ell}_{i+1} \tag{14}$$

It is defined with the following recursion

$$\begin{aligned}
&\text{iterate } i{=}1 \text{ to } N \\
\boldsymbol{\Psi}_i &= \left(1 - \boldsymbol{s}_i \left(\boldsymbol{s}_i^\intercal \mathbf{I}_i \boldsymbol{s}_i\right)^{-1} \boldsymbol{s}_i^\intercal \mathbf{I}_i\right) \boldsymbol{\Lambda}_{i-1}^{-1} && (15)\\
\boldsymbol{\Lambda}_i^{-1} &= \left(1 + \boldsymbol{\Psi}_i \mathbf{I}_i\right)^{-1} \left(\boldsymbol{s}_i \left(\boldsymbol{s}_i^\intercal \mathbf{I}_i \boldsymbol{s}_i\right)^{-1} \boldsymbol{s}_i^\intercal + \boldsymbol{\Psi}_i\right) && (16)\\
&\text{initial conditions} \\
\boldsymbol{\Psi}_1 &= 0 && (17)\\
\boldsymbol{\Lambda}_1^{-1} &= \boldsymbol{s}_1 \left(\boldsymbol{s}_1^\intercal \mathbf{I}_1 \boldsymbol{s}_i\right)^{-1} \boldsymbol{s}_1^\intercal && (18)
\end{aligned}$$

the projection matrix $\boldsymbol{\Psi}_i$ is just used temporarily and can be discarded after each iteration.

2.2 Impulse Magnitude

Consider the impulse $\boldsymbol{n}J$ acting on the N-th link, the last in the chain, and the constrained inverse spatial inertia $\boldsymbol{\Lambda}_N^{-1}$ is known, the response of the last link to the impulse is given by

$$\Delta\boldsymbol{v}_N = -\boldsymbol{\Lambda}_N^{-1} \boldsymbol{n} J \tag{19}$$

which is used by the law of contact $v_{\text{bounce}} = -\epsilon\, v_{\text{imp}}$ to calculate the impulse magnitude J.

Consider the impact speed

$$v_{\text{imp}} = \boldsymbol{n}^\intercal \boldsymbol{v}_N \tag{20}$$

and the bounce speed

$$v_{\text{bounce}} = \boldsymbol{n}^\intercal \left(\boldsymbol{v}_N + \Delta\boldsymbol{v}_N\right) \tag{21}$$

then the law of contact is

$$\begin{aligned}
\boldsymbol{n}^\intercal \left(\boldsymbol{v}_N + \Delta\boldsymbol{v}_N\right) &= -\epsilon \left(\boldsymbol{n}^\intercal \boldsymbol{v}_N\right) \\
-\left(\boldsymbol{n}^\intercal \boldsymbol{\Lambda}_N^{-1} \boldsymbol{n}\right) J &= -\left(1+\epsilon\right) \left(\boldsymbol{n}^\intercal \boldsymbol{v}_N\right) \\
J &= \left(1+\epsilon\right) \left(\boldsymbol{n}^\intercal \boldsymbol{\Lambda}_N^{-1} \boldsymbol{n}\right)^{-1} \left(\boldsymbol{n}^\intercal \boldsymbol{v}_N\right) && (22)
\end{aligned}$$

2.3 Impulse Propagation

For the last link, equation (12) gives the reaction impulse with

$$\boldsymbol{\ell}_N = \mathbf{I}_N \Delta\boldsymbol{v}_N + \boldsymbol{n}\, J \tag{23}$$

For an arbitrary link, the impulse $\boldsymbol{n}\, J$ is projected down through the joints using a matrix $\boldsymbol{\Phi}_i$ such that equation (12) is solved with

$$\boldsymbol{\ell}_{i+1} = \boldsymbol{\Phi}_{i+1} \boldsymbol{n} J \tag{24}$$

By working the way down the chain, the following recursion calculates all of the projection matrices

$$\begin{aligned}
&\text{iterate } i{=}N \text{ down to } 1 \\
\boldsymbol{\Phi}_i &= \left(1 - \mathbf{I}_i \boldsymbol{\Lambda}_i^{-1}\right) \boldsymbol{\Phi}_{i+1} && (25)\\
&\text{initial conditions} \\
\boldsymbol{\Phi}_N &= \left(1 - \mathbf{I}_N \boldsymbol{\Lambda}_N^{-1}\right) && (26)
\end{aligned}$$

2.4 Joint Responses

Now that the reaction impulse at each joint is known from (24) and the link change in motrion is also known from (14) the response of each joint is a change in motion $\Delta\dot{\theta}_i$ which is calculated from the velocity kinematics up the chain.

The following recursion establishes the joint reposnses

$$\begin{aligned}
&\text{iterate } i{=}1 \text{ to } N \\
\Delta\dot{\theta}_i &= -\left(\boldsymbol{s}_i^\intercal \mathbf{I}_i \boldsymbol{s}_i\right)^{-1} \boldsymbol{s}_i^\intercal \left(\boldsymbol{\ell}_{i+1} + \mathbf{I}_i \Delta\boldsymbol{v}_{i-1}\right) && (27)\\
\Delta\boldsymbol{v}_i &= \Delta\boldsymbol{v}_{i-1} + \boldsymbol{s}_i \Delta\dot{\theta}_i && (28)\\
&\text{initial conditions} \\
\Delta\boldsymbol{v}_0 &= 0 && (29)
\end{aligned}$$

3 Example Calculation

Consider a 2D chain with $N = 2$ links of length L, all horizontal, each with mass m and mass moment of inertia $\mathrm{I}_i^C = \frac{1}{12} m L^2$ at each center of mass, which is located at $c = \frac{1}{2} L$. Since the links are horizontal $\theta_1 = 0$ and $\theta_2 = 0$ at this instant.

Each joint has arbitrary velocity $\dot{\theta}_1$ and $\dot{\theta}_2$ and the end of the last link hits a immovable wall. Caclulate the impulse magnitude J and the joint resposnses $\Delta\dot{\theta}_1$ and $\Delta\dot{\theta}_2$.

The impact direction wrench is defined as

$$\boldsymbol{n} = \begin{bmatrix} 0 \\ 1 \\ 2L \end{bmatrix} \tag{30}$$

3.1 Velocity Kinematics

From (6) and (13) the kinematics are

$$\boldsymbol{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ -L \\ 1 \end{bmatrix} \tag{31}$$

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -L\dot{\theta}_2 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \tag{32}$$

Additionally the spatial inertia matrices are defined from (9)

$$\mathbf{I}_1 = \begin{bmatrix} m & & \\ & m & \frac{1}{2} mL \\ & \frac{1}{2} mL & \frac{1}{3} mL^2 \end{bmatrix} \quad \mathbf{I}_2 = \begin{bmatrix} m & & \\ & m & \frac{3}{2} mL \\ & \frac{3}{2} mL & \frac{7}{3} mL \end{bmatrix} \tag{33}$$

The impact speed from (20) is

$$v_{\mathrm{imp}} = L \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \tag{34}$$

3.2 Constrained Inverse Spatial Inertia

The constrained inverse inertia matrix from (16) is

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{3}{mL^2} \end{bmatrix} \quad \boldsymbol{\Lambda}_2 = \begin{bmatrix} 0 & & \\ & \frac{96}{7m} & -\frac{66}{7mL} \\ & -\frac{66}{7mL} & \frac{48}{7mL^2} \end{bmatrix} \tag{35}$$

3.3 Impulse Response

The impulse magnitude is calculated from (22) as

$$J = (\epsilon + 1) \, \tfrac{7m}{24} L \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \tag{36}$$

3.4 Impulse Propagation and Joint Response

The impulse propagation matrix from (25) is

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} 1 & & \\ & -\frac{5}{7} & \frac{3}{7L} \\ & 0 & 0 \end{bmatrix} \quad \boldsymbol{\Phi}_2 = \begin{bmatrix} 1 & & \\ & \frac{10}{7} & -\frac{6}{7L} \\ & \frac{10L}{7} & -\frac{6}{7} \end{bmatrix} \tag{37}$$

and so the response at each joint from (27) and (28) is

$$\Delta\dot{\theta}_1 = (\epsilon + 1) \left(\tfrac{1}{4}\right) \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \quad \Delta\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ (\epsilon + 1) \, \frac{1}{4} \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \end{bmatrix} \tag{38}$$

$$\Delta\dot{\theta}_2 = (\epsilon + 1) \left(-\tfrac{3}{2}\right) \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \quad \Delta\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ (\epsilon + 1) \left(\frac{3L}{2}\right) \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \\ (\epsilon + 1) \left(-\frac{5}{4}\right) \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \end{bmatrix} \tag{39}$$

As a result the bounce speed from (21) is

$$v_{\mathrm{bounce}} = -\epsilon \, L \left(2\dot{\theta}_1 + \dot{\theta}_2 \right) \tag{40}$$

which **is** equal to $v_{\mathrm{bounce}} = -\epsilon \, v_{\mathrm{tmp}}$ and hence obeys the law of contact.