Projective Articulated Dynamics

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Overview

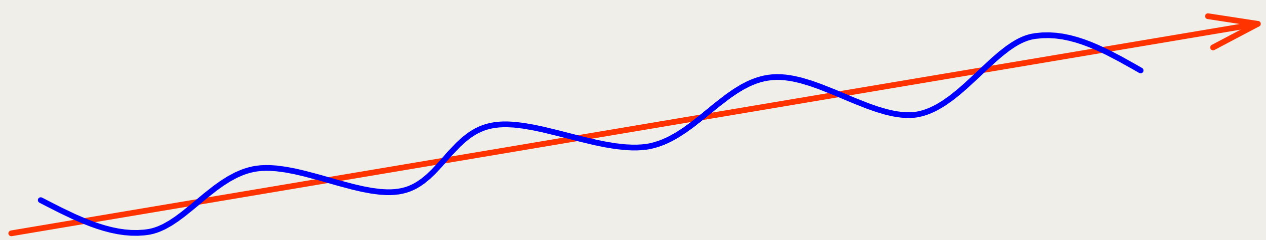
- Introduction
- Screw Theory Basics
- Kinematics and Equations of Motion
- Projective Geometry and Dynamics
- Subspace Decompositions
- Articulated Dynamics
- Stacked Form
- Conclusions and Future Work

Motivation

- Screws offer geometrical interpretations of dynamics and a compact notation
- Recursive methods offer simple formulations

What Is a Screw?

- Geometrically a screw is a line with a pitch
- Lines need 4 independent quantities
- Pitch is a scalar = 1 quantity
- 5 quantities needed to define a screw



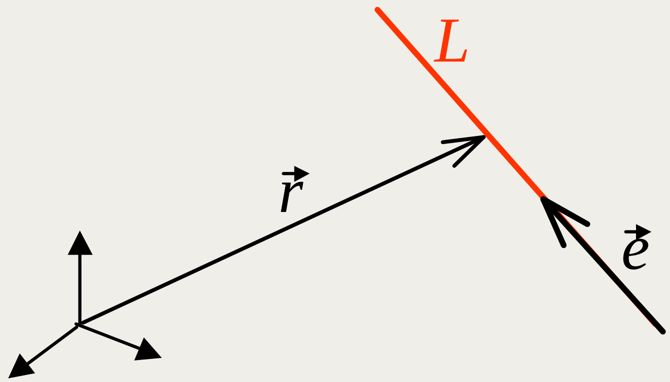
Screw Representations

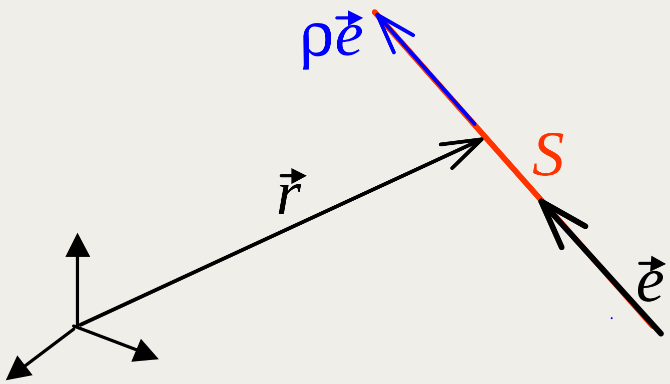
- Line - 6 homogeneous coordinates (4 dof)

$$L = \begin{bmatrix} \vec{r} \times \vec{e} \\ \vec{e} \end{bmatrix}$$

- Screw- 6 homogeneous coordinates (5 dof)

$$S = \begin{bmatrix} \vec{r} \times \vec{e} \\ \vec{e} \end{bmatrix} + \begin{bmatrix} \rho\vec{e} \\ \vec{0} \end{bmatrix}$$





Twists and Wrenches

- Twists - Spatial Velocity and Acceleration

$$v_P = \begin{bmatrix} \overrightarrow{v_P} \\ \vec{\omega} \end{bmatrix} \qquad a_P = \begin{bmatrix} \overrightarrow{a_P} \\ \vec{\alpha} \end{bmatrix}$$

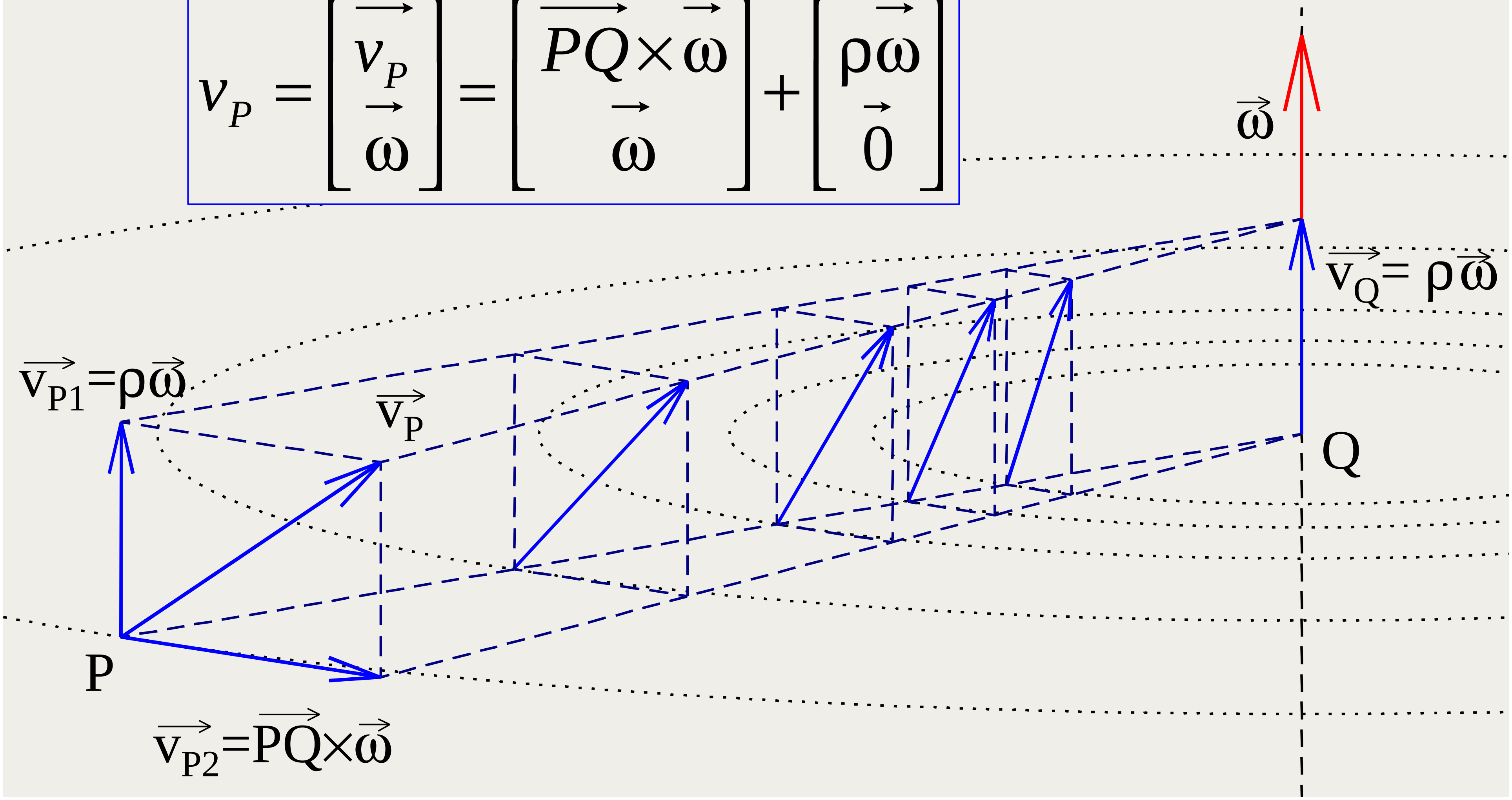
- Wrenches - Momentum and Loads

$$h_P = \begin{bmatrix} \vec{p} \\ \overrightarrow{h_P} \end{bmatrix} \qquad f_P = \begin{bmatrix} \vec{f} \\ \overrightarrow{\tau_P} \end{bmatrix}$$

Twist as a Screw

- Twist = screw with magnitude

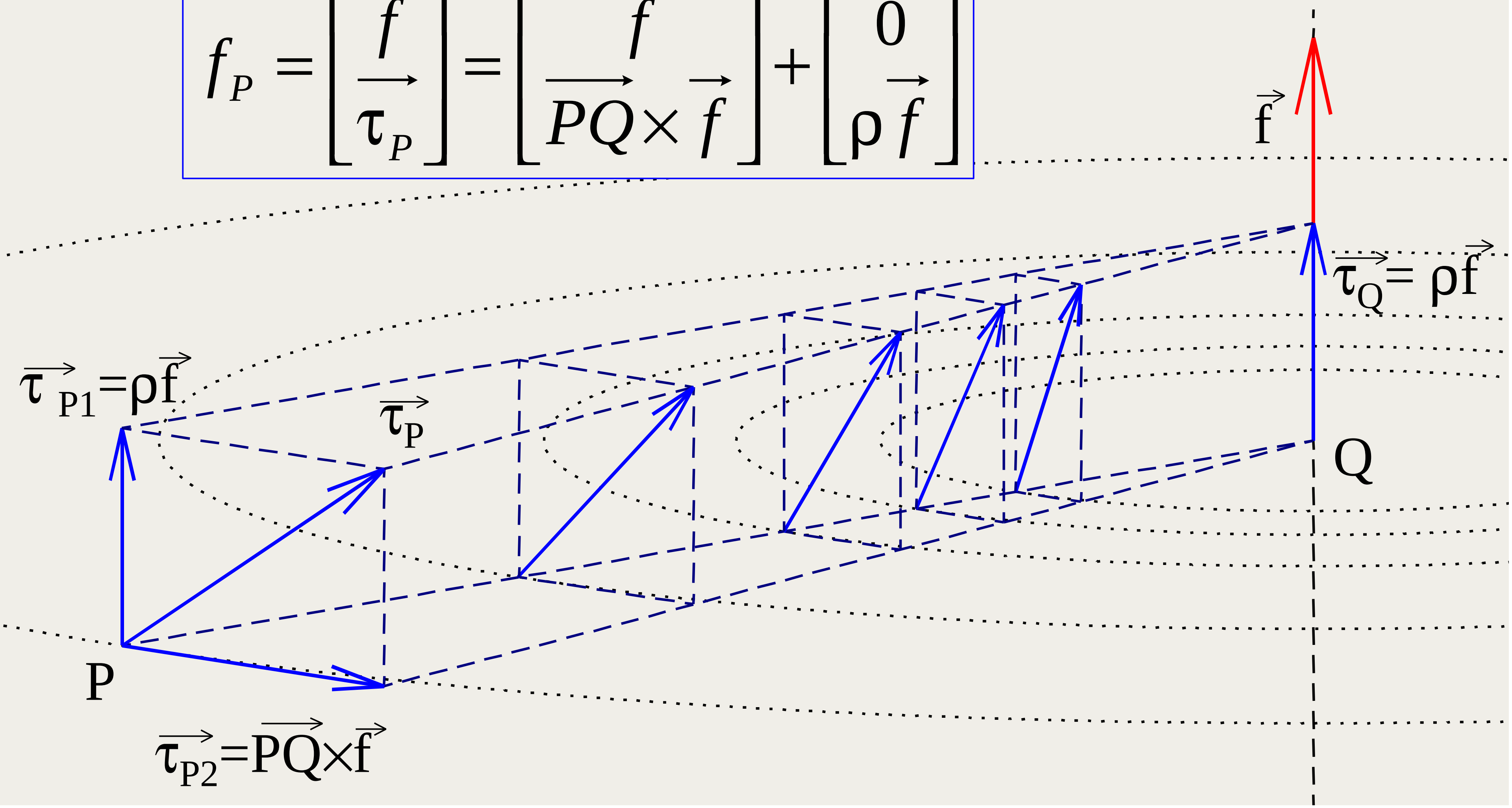
$$v_P = \begin{bmatrix} \overrightarrow{v_P} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} \overrightarrow{PQ} \times \vec{\omega} \\ \vec{\omega} \end{bmatrix} + \begin{bmatrix} \rho\vec{\omega} \\ \vec{0} \end{bmatrix}$$



Wrench as a Screw

- Wrench = screw with magnitude

$$\boxed{f_P = \begin{bmatrix} \vec{f} \\ \vec{\tau}_P \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \overrightarrow{PQ} \times \vec{f} \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \rho \vec{f} \end{bmatrix}}$$



Spatial Transformations

- Twists and wrenches transform linearly for both translations and rotations

$$
{}_P X_Q = \begin{bmatrix} E & -E\left(\overrightarrow{QP}\times\right) \\ 0 & E \end{bmatrix}
$$

translation vector between Q and P

3×3 rotation matrix between Q and P

Twists

$$v_P = {}_P X_Q v_Q$$

Wrenches

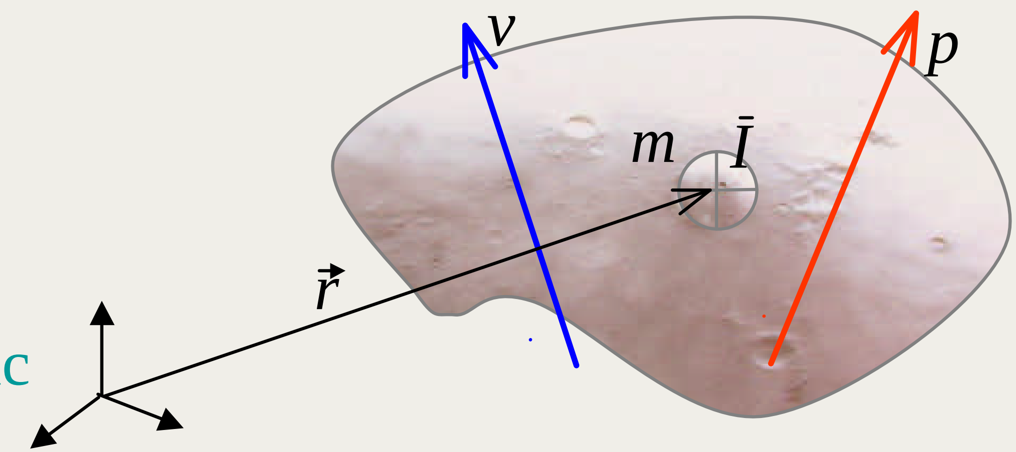
$$f_P = \left({}_P X_Q\right)^{-T} f_Q$$

Spatial Inertia

- Spatial inertia maps velocity twist into momentum wrench

$$I = \begin{bmatrix} m1 & -m\vec{r}\times \\ m\vec{r}\times & \bar{I} - m\vec{r}\times\vec{r}\times \end{bmatrix}$$

NOTE:
Use 3×3 skew symmetric cross operator for $r\times$



Power and Energy

- Both power and kinetic energy:
  - have compact notation
  - are invariant to coordinate representation

Rigid Body Power

$$P = v^T f$$

Kinetic Energy

$$K = \frac{1}{2} v^T I v$$

Planar Screws

Spatial (6 components)

- Twists and Wrenches

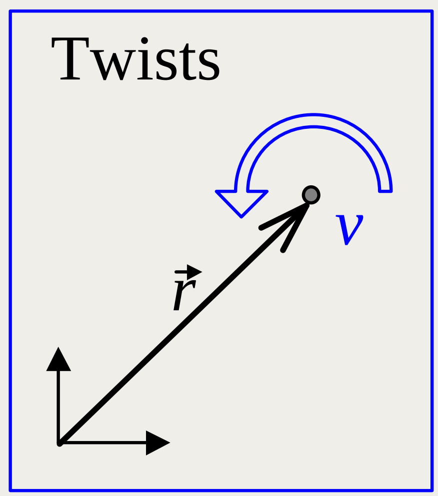
$$v_{3D} = \begin{bmatrix} v_x \\ v_y \\ 0 \\ 0 \\ 0 \\ w_z \end{bmatrix} \qquad f_{3D} = \begin{bmatrix} f_x \\ f_y \\ 0 \\ 0 \\ 0 \\ t_z \end{bmatrix}$$

Planar (3 components)

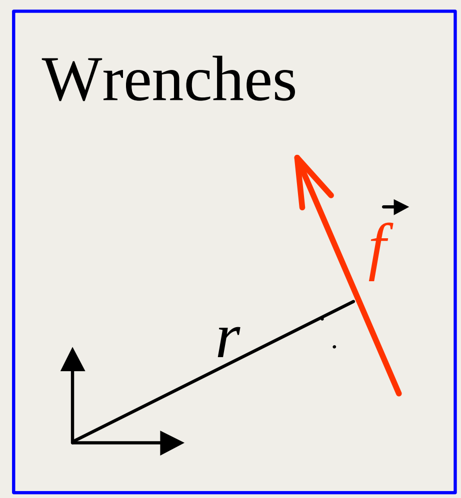
- Twists and Wrenches

$$v_{2D} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} \qquad f_{2D} = \begin{bmatrix} f_x \\ f_y \\ \tau \end{bmatrix}$$

Point



Line



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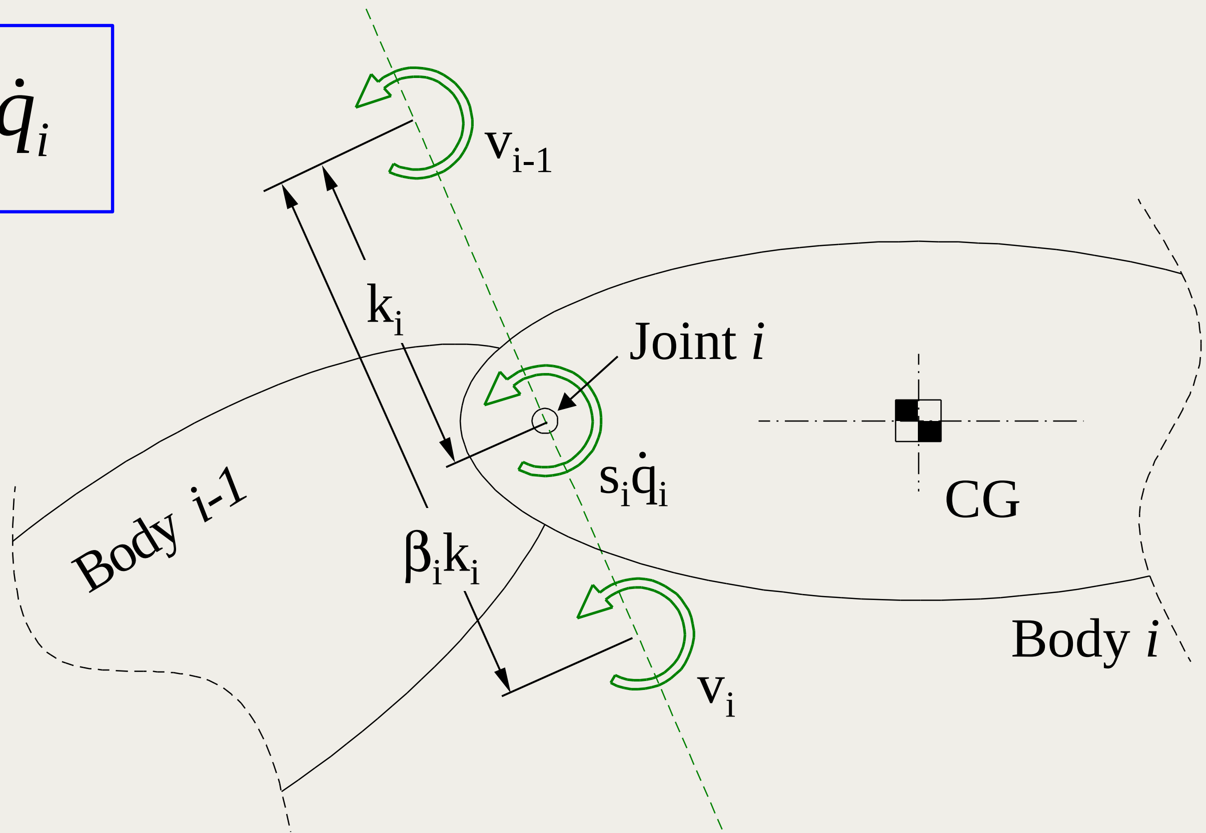
Velocity Kinematics

- Spatial Equation

$$\boxed{v_i = v_{i-1} + s_i \dot{q}_i}$$

- Planar Interpretation



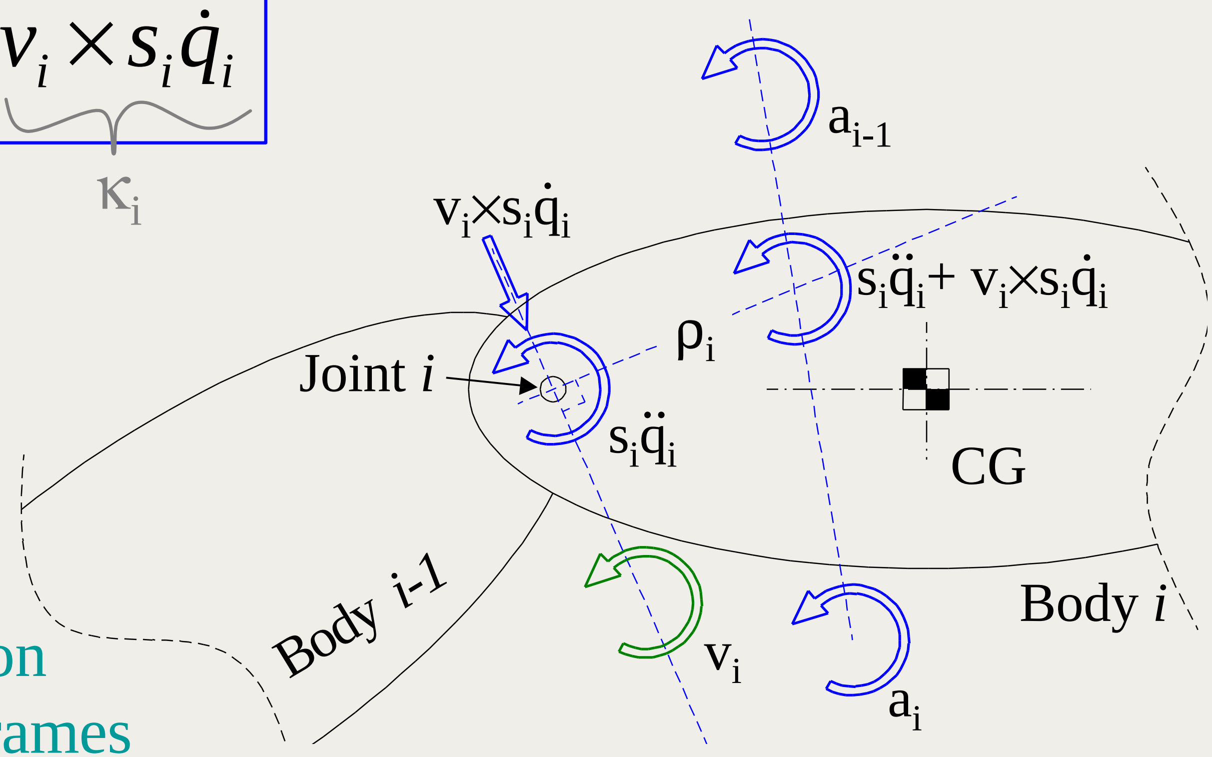
Cylindoid

Acceleration Kinematics

- Spatial Equation

$$\boxed{a_i = a_{i-1} + s_i \ddot{q}_i + \underbrace{v_i \times s_i \dot{q}_i}_{\kappa_i}}$$

- Planar Interpretation

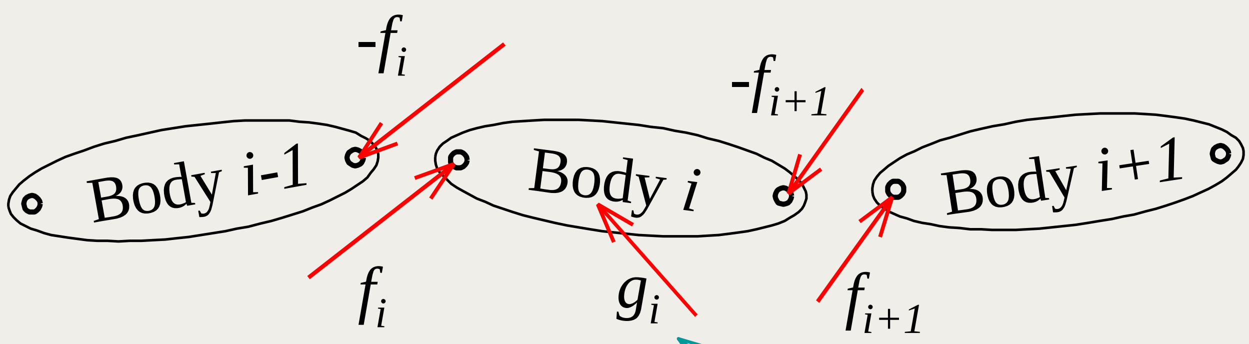


NOTE:
Spatial $v\times$ is derived from differentiation on moving coordinate frames

Free Body Diagram

(Linear Chain of Rigid Bodies)

- Three loads acting on each body



Joint internal forces

Applied forces

$$\boxed{f_i^{net} = f_i - f_{i+1} + g_i}$$

Equal and opposite internal forces

Newton-Euler Equations of Motion

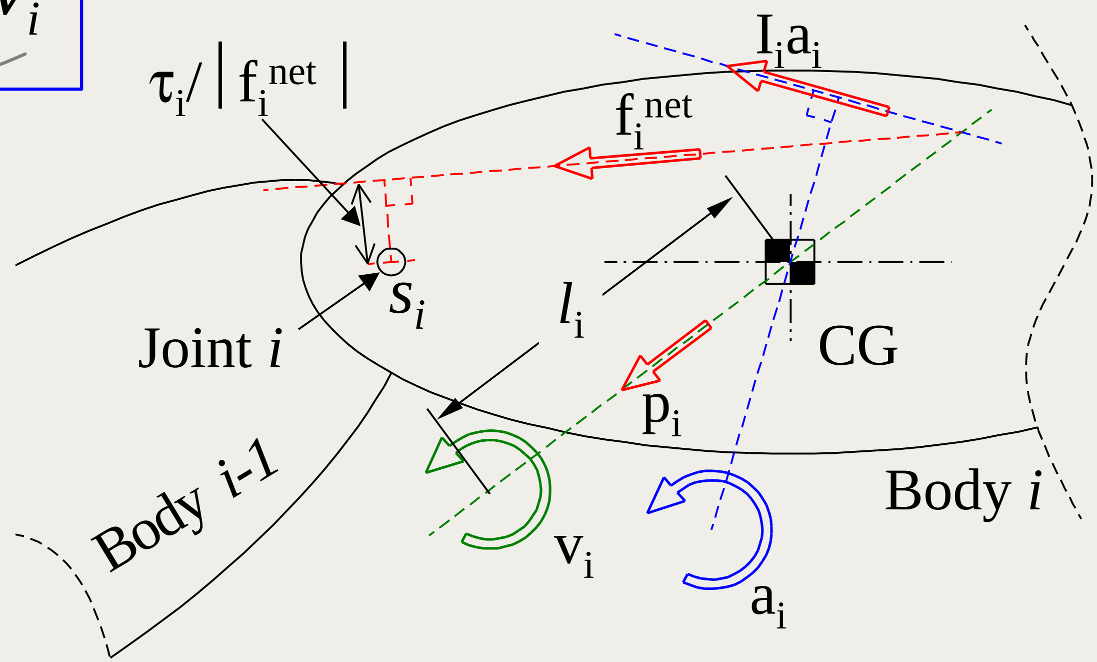
- Spatial Equation

$$\boxed{f_i^{\,net} = I_i a_i \underbrace{-(v_i \times)^T I_i v_i}_{p_i}}$$

- Joint Torque

$$\boxed{\tau_i = s_i^{\,T} f_i}$$

- Planar Interpretation



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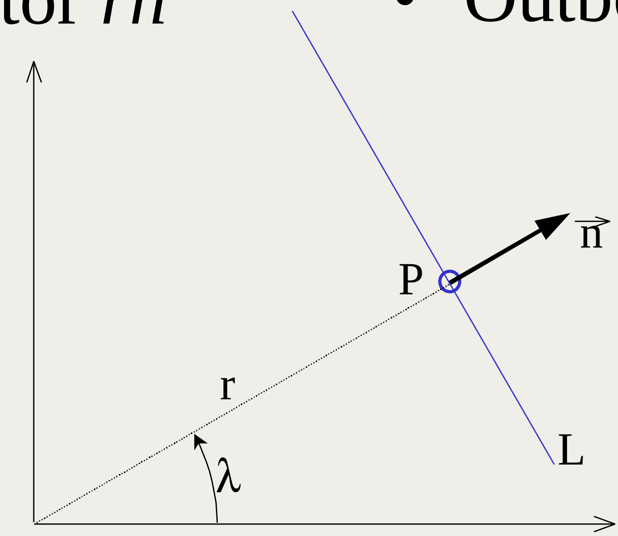
Planar Homogeneous Coordinates

Points (easy)

- Unit 1
- Position vector $r\vec{n}$

$$P = \begin{bmatrix} 1 \\ r\vec{n} \end{bmatrix}$$



Lines (difficult)

- Minus of distance r
- Outboard normal $\vec{n}$

$$L = \begin{bmatrix} -r \\ \vec{n} \end{bmatrix}$$

$P^T L=0$ Û *Point on Line*

Alternative Coordinates

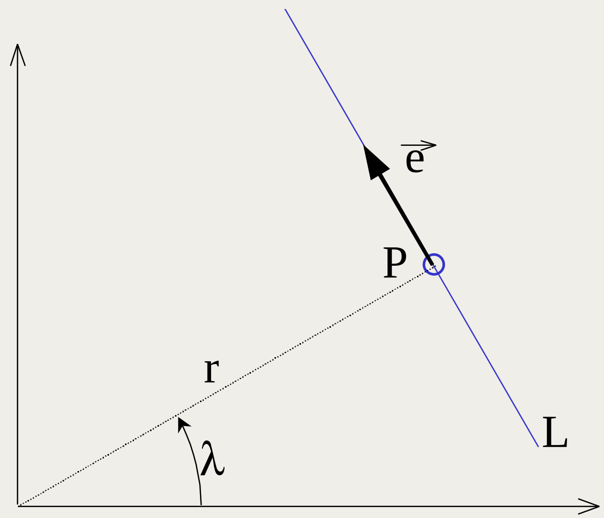
Points (difficult)

- Vector $-r\vec{e}$
- Unit 1

$$P = \begin{bmatrix} -r\vec{e} \\ 1 \end{bmatrix}$$

Lines (easy)

- Distance r
- Direction vector $\vec{e}$



$$L = \begin{bmatrix} \vec{e} \\ r \end{bmatrix}$$

Planar Twists and Wrenches

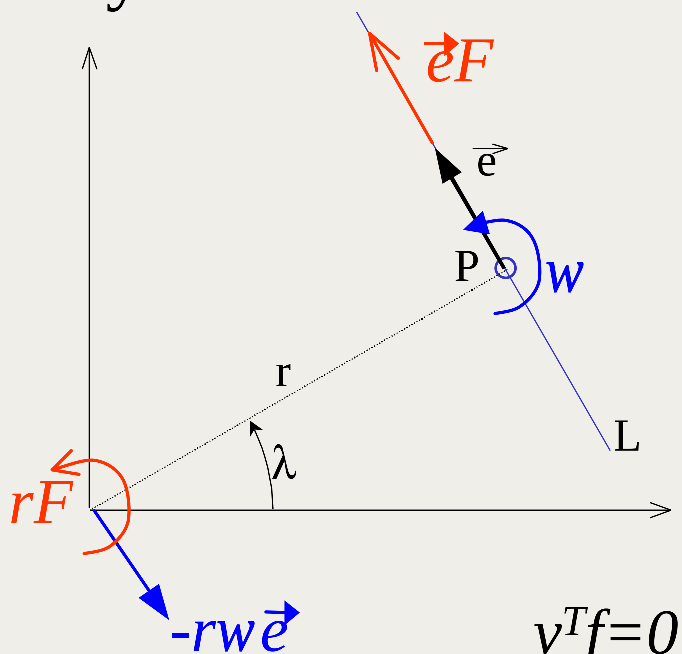
Twists

- Velocity vector $-r\vec{e}\omega$
- Angular velocity ω

Wrenches

- Moment rF
- Force vector $\vec{e}F$

$$v = \begin{bmatrix} -r\vec{e}\omega \\ \omega \end{bmatrix}$$

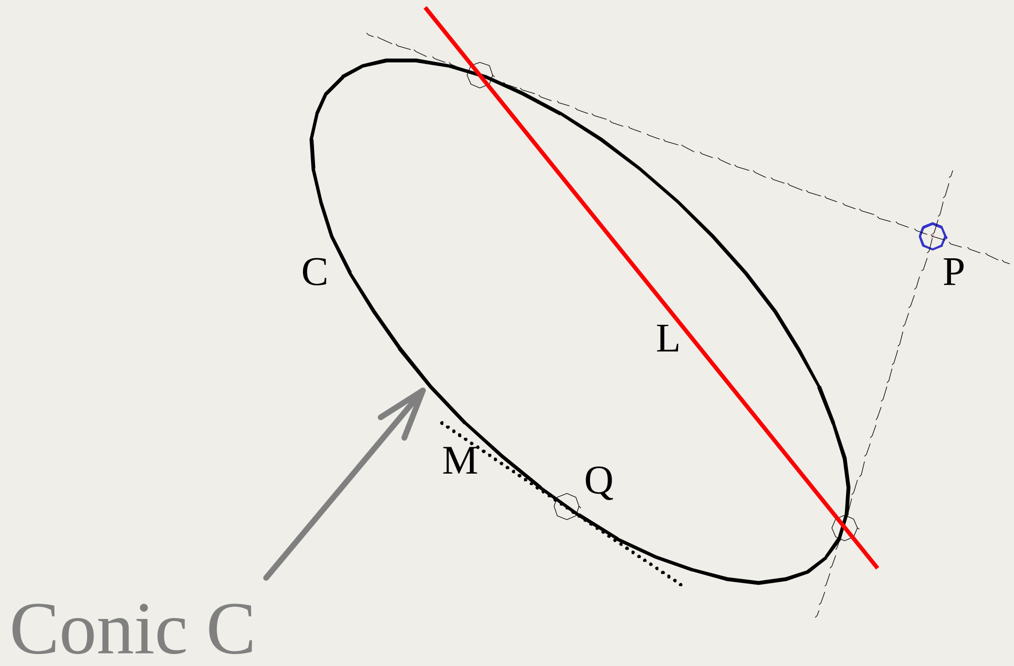


$$f = \begin{bmatrix} \vec{e}F \\ rF \end{bmatrix}$$

$v^T f = 0$ Û *powerless force*

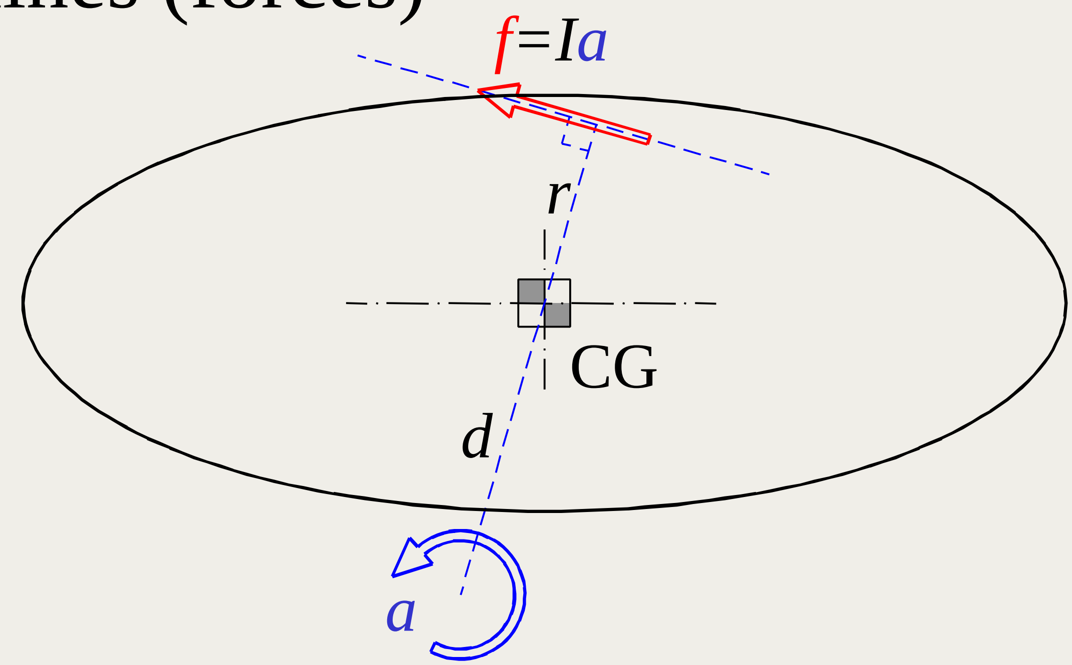
Planar Mapping

$$L = CP$$



Inertia Mappings

- Planar inertia maps points (accelerations) into lines (forces)

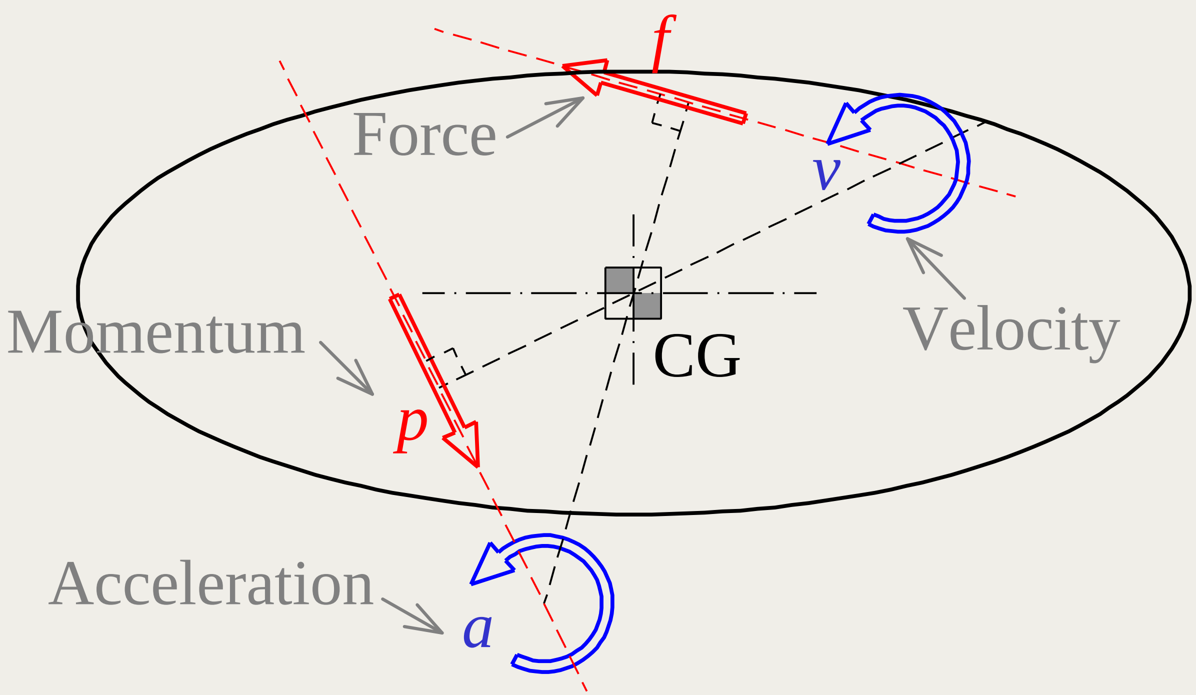


- Inertia mapping of a point is an axis of percussion for that point (sweet spot!)

Power Relationships

- Zero power ($v^{T}f=0$) $\Leftrightarrow$ Point(v) on Line(f)
- Zero power ($a^{T}p=0$) $\Leftrightarrow$ Point(a) on Line(p)

- Map with $p=Iv$ and $a=I^{-1}f$



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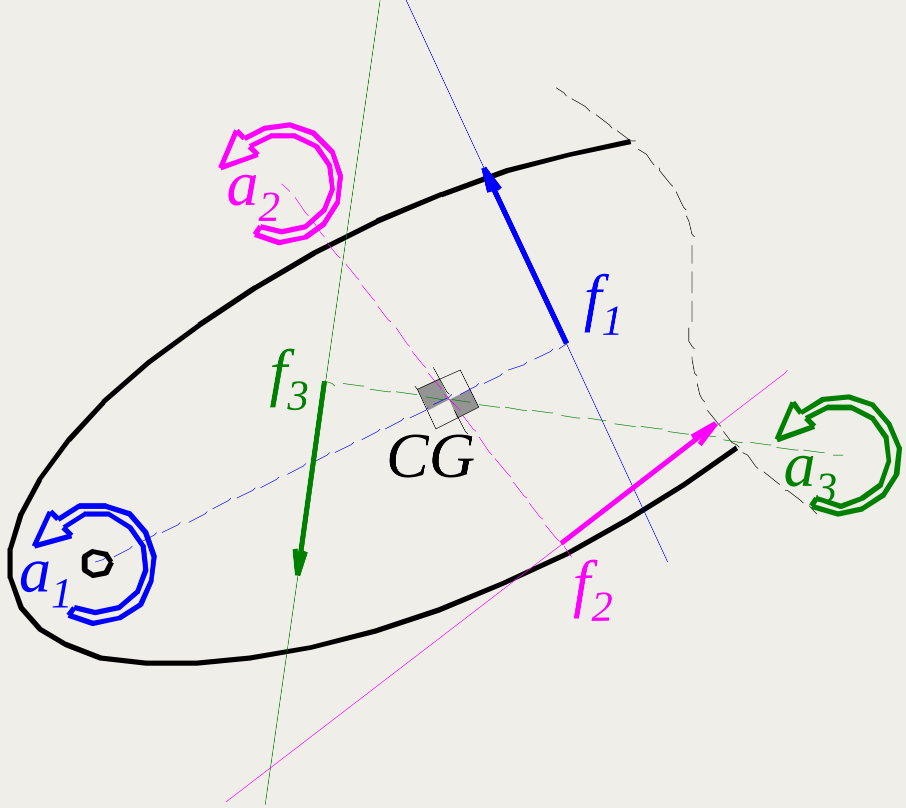
Arbitrary Decomposition

- Twists
  - 3 accelerations
    - a_1
    - a_2
    - a_3

- Wrenches
  - 3 forces
    - $f_1 = Ia_1$
    - $f_2 = Ia_2$
    - $f_3 = Ia_3$



NOTE: Spatial case needs 6 base twists + 6 base wrenches

Basic Decomposition

- Twists

U_1 , U_2 - reactive accelerations

(resulting from reaction forces)

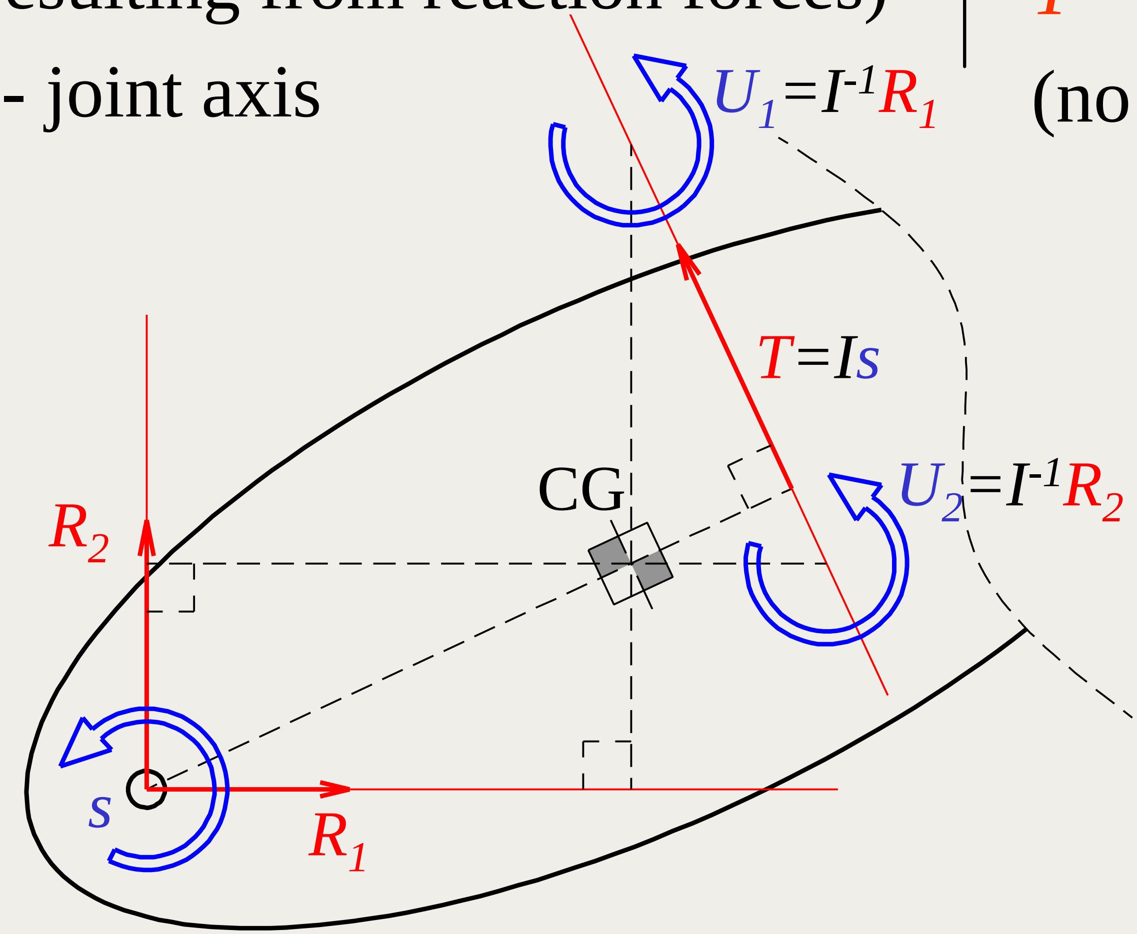
s - joint axis

- Wrenches

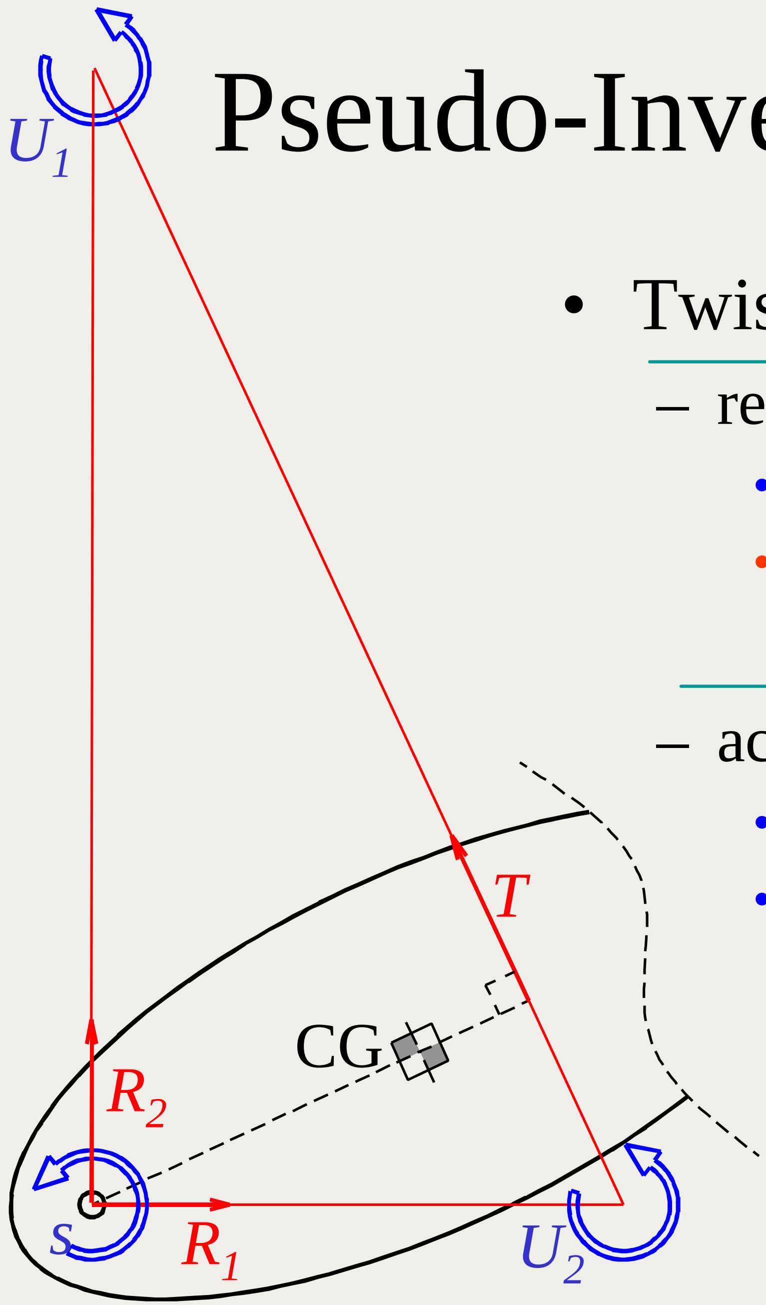
R_1 , R_2 - reaction forces

T - axis of percussion

(no reaction forces)



Pseudo-Inverse Decomposition



- Twists
  - reactive
    - $U = I^{-1}R(R^T I^{-1} R)^{-1}$
    - $R^T U = 1$
  - active
    - s
    - $s^T T = 1$
- Wrenches
  - reactive
    - R
    - $R^T s = 0$
  - active
    - $T = Is(s^T Is)$
    - $U^T T = 0$

Subspaces

- Planar subspaces
  - Active acceleration

    $s=[\, s_1 \,]$
  - Reactive Acceleration

    $U=[\, U_1 \; U_2 \,]$
  - Active Force

    $T=[\, T_1 \,]$
  - Reaction Forces

    $R = [\, R_1 \; R_2 \,]$

- Spatial subspaces
  - Active acceleration (k-dof)

    $s=[\, s_1$ ¼ $s_k \,]$
  - Reactive Acceleration

    $U=[\, U_1$ ¼ $U_{6-k} \,]$
  - Active Force

    $T=[\, T_1$ ¼ $T_k \,]$
  - Reaction Forces

    $R = [\, R_1$ ¼ $R_{6-k} \,]$

Component Projections

- Twists (accelerations)

Active Accelerations

Reactive Accelerations

- $a = s\psi + U\gamma$
  - $\psi = T^T a$
  - $\gamma = R^T a$

- Wrenches (forces)

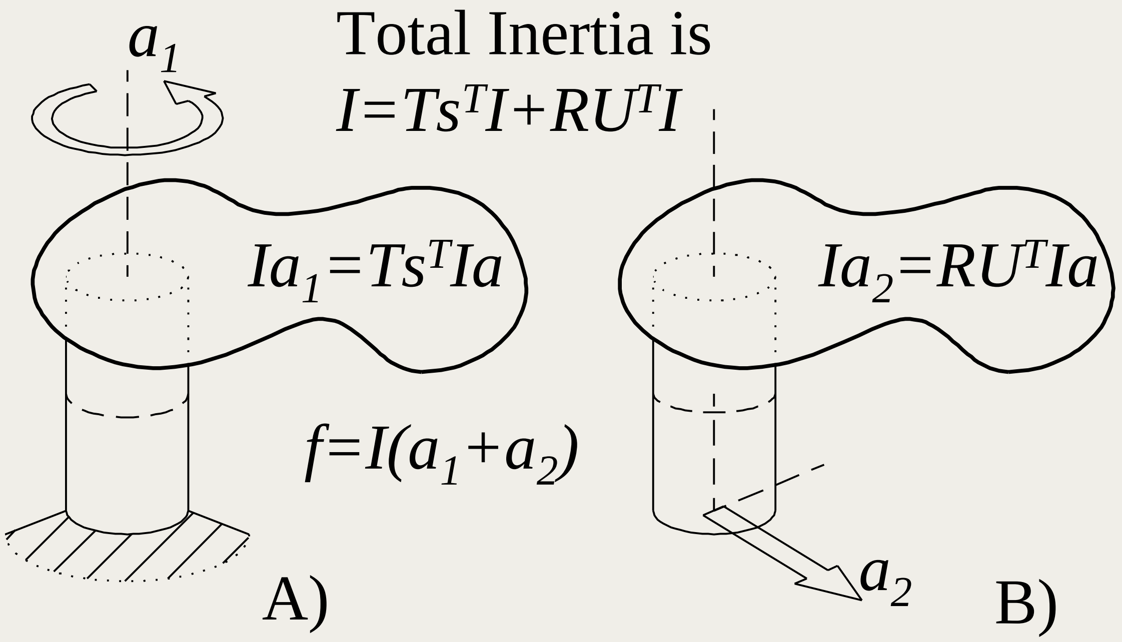
Active Forces

Reaction Forces

- $f = T\mathrm{Q} + R\mu$
  - $\mathrm{Q} = s^T f$
  - $\mu = U^T f$

Inertia Decomposition

- A) a_1 along the active space
- B) a_2 along the reactive space

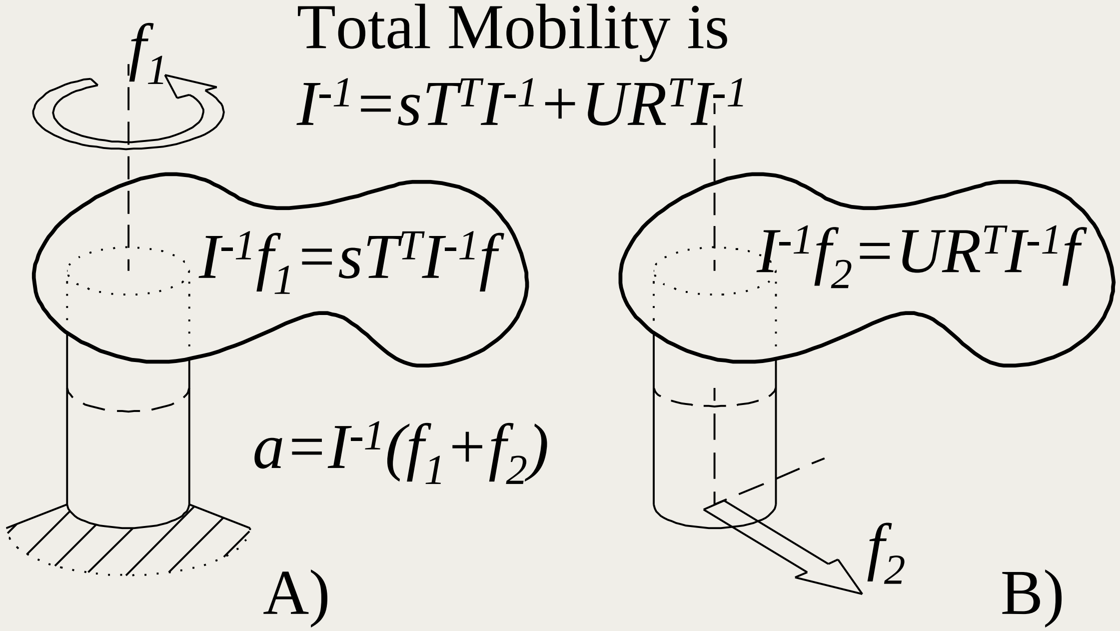


Mobility Decomposition

- A) f_1 along the active space
- B) f_2 along the reactive space

Total Mobility is

$$I^{-1}=sT^TI^{-1}+UR^TI^{-1}$$



Projected Accelerations

- Spatial equation

$$\boxed{a_i = a_{i-1} + s_i \ddot{q}_i + \kappa_i}$$

- Projected equation

$$\boxed{a_i = \overbrace{s_i T_i^T I_i^{-1}\left(T_i Q_i - p_i - f_{i+1}\right)}^{\text{active acceleration}} + \overbrace{U_i R_i^T \left(a_{i-1} + \kappa_i\right)}^{\text{reactive acceleration}}}$$

1 equation, 2 unknowns

Projected Forces

- Spatial equation

$$\boxed{f_i = f_{i+1} + I_i a_i + p_i}$$

- Projected equation

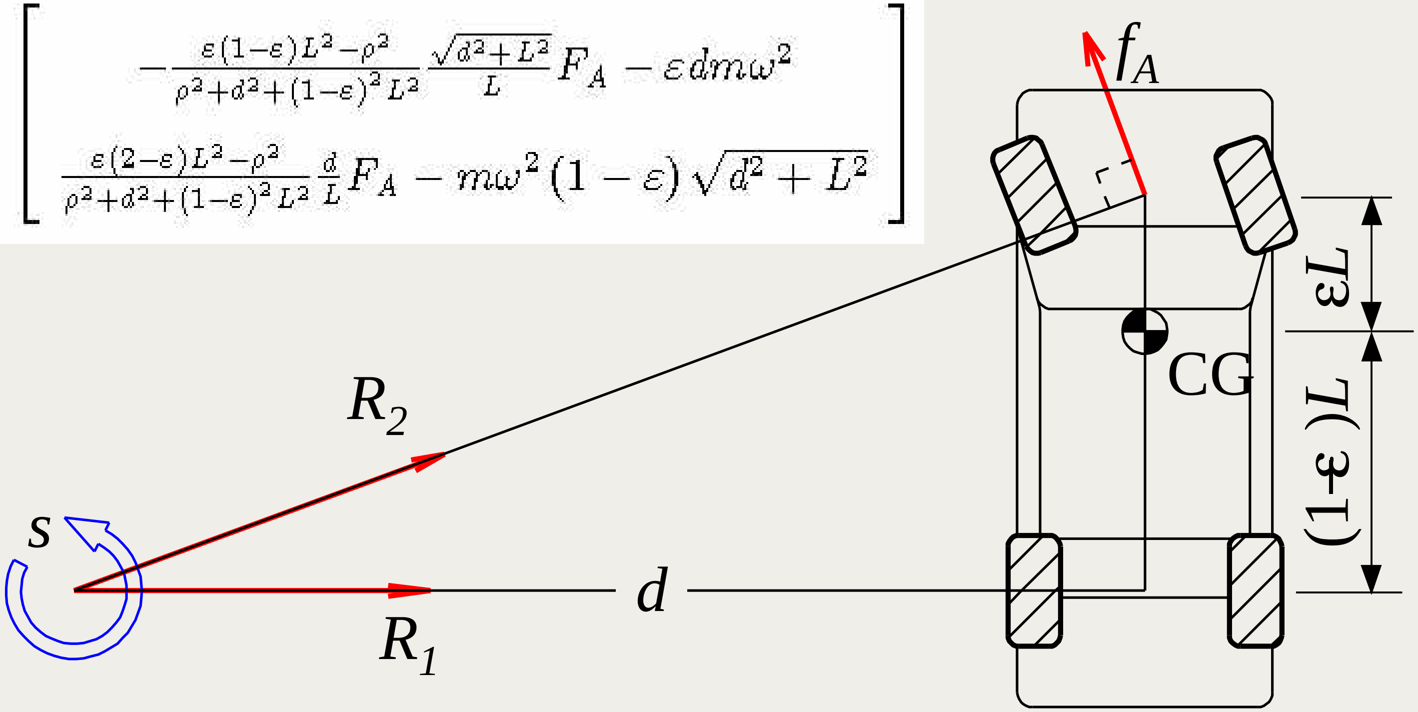
$$\boxed{f_i = \overbrace{T_i Q_i}^{\text{active forces}} + \overbrace{R_i U_i^T \left(I_i \left(a_{i-1} + \kappa_i \right) + f_{i+1} + p_i \right)}^{\text{reaction forces}}}$$

1 equation, 2 unknowns

Planar Example

- Car cornering and accelerating

$$\mu = \begin{bmatrix} -\frac{\varepsilon(1-\varepsilon)L^2-\rho^2}{\rho^2+d^2+(1-\varepsilon)^2L^2}\frac{\sqrt{d^2+L^2}}{L}F_A - \varepsilon d m\omega^2 \\ \frac{\varepsilon(2-\varepsilon)L^2-\rho^2}{\rho^2+d^2+(1-\varepsilon)^2L^2}\frac{d}{L}F_A - m\omega^2(1-\varepsilon)\sqrt{d^2+L^2} \end{bmatrix}$$



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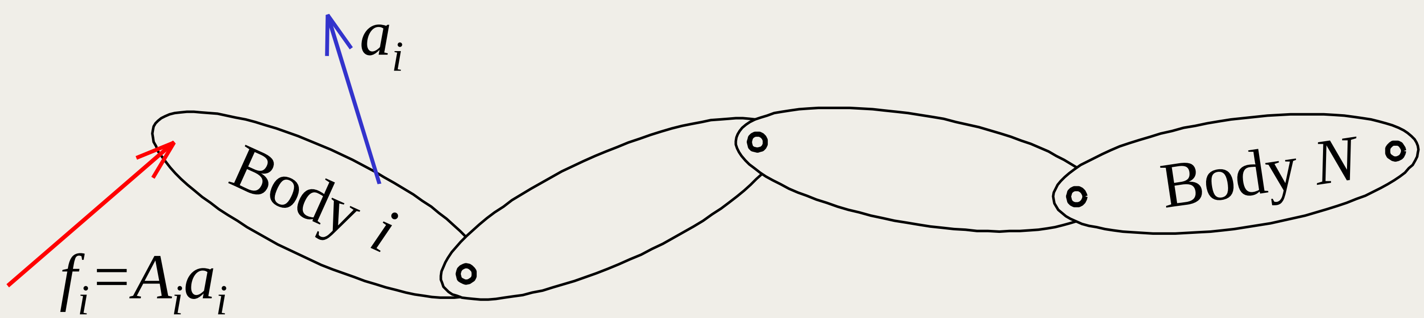
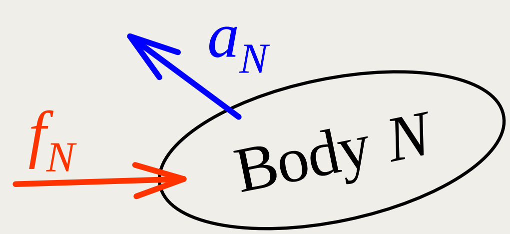
Articulated Inertia Concept

- Free floating chain from i to N
- Effect of body acceleration a_i on force f_i
- A_i replaces I_i in equations of motion

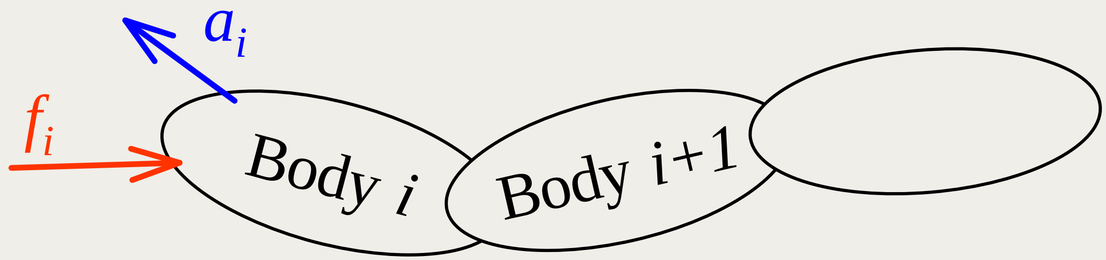


Articulated Inertia Recursion

- Tip of chain ($A_N = I_N$)

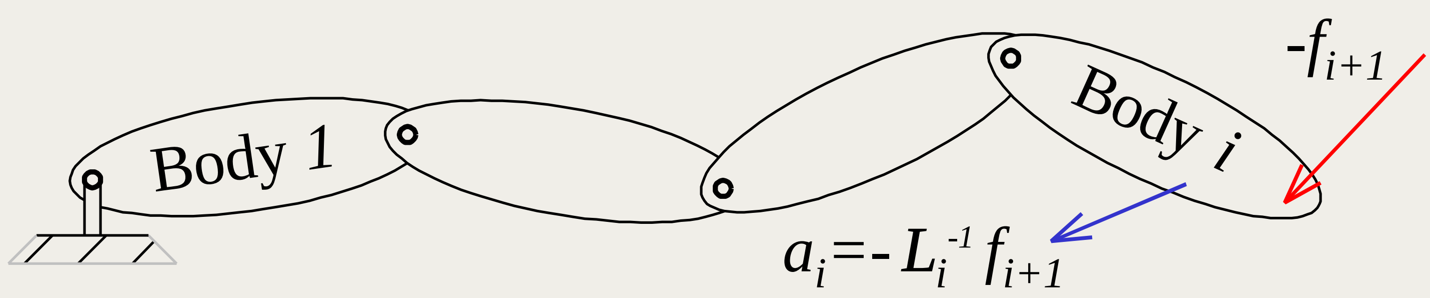


- Recursively down the chain (A_{i+1}® A_i)



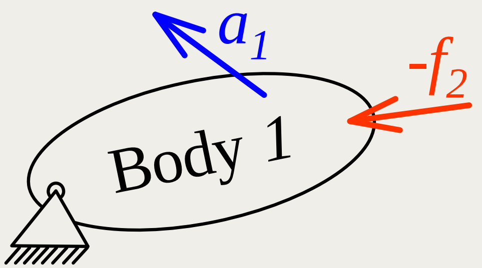
Articulated Mobility Concept

- Constrained chain from base to i
- Effect of body acceleration a_i on force f_{i+1}
- L_i^{-1} replaces I_i^{-1} in equations of motion

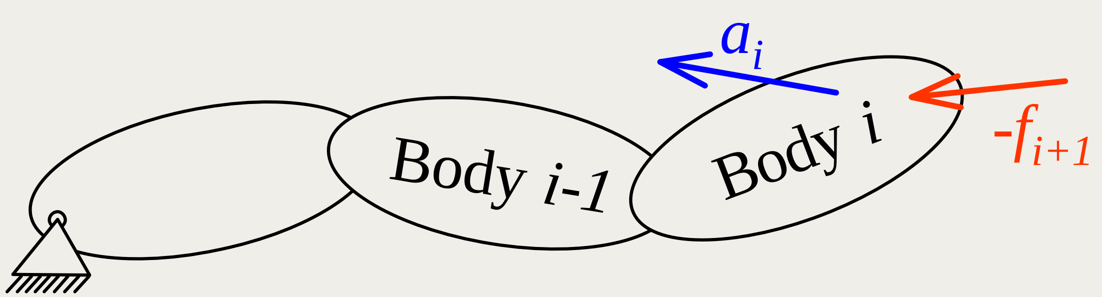


Articulated Mobility Recursion

- First body ($L_1^{-1}=s_1 T_1^T I_1^{-1}$)



- Recursively up the chain (L_{i-1}^{-1} ® L_i^{-1})



Articulated Equations of Motion

Inertia

- Spatial equation

$$\boxed{f_i = f_{i+1} + I_i a_i + p_i}$$

- Articulated equation

$$\boxed{f_i = A_i a_i + d_i}$$

Mobility

- Spatial equation

$$\boxed{a_i = I_i^{-1}\left(f_i - f_{i+1} - p_i\right)}$$

- Articulated equation

$$\boxed{a_i = b_i - \Lambda_i^{-1} f_{i+1}}$$

singular ?

Recursive Accelerations

- From base to tip ($a_0=0$)

$$\boxed{a_i = \overbrace{s_i T_i^T A_i^{-1}\left(T_i Q_i - d_i\right)}^{\text{active acceleration}} + \overbrace{U_i R_i^T\left(a_{i-1} + \kappa_i\right)}^{\text{reactive acceleration}}}$$

recursive

Recursive Forces

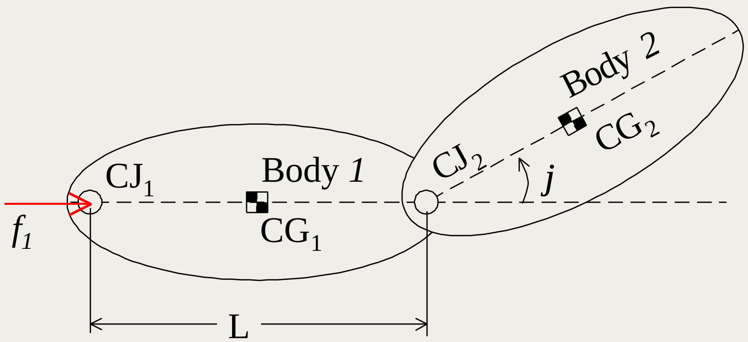
- From base to tip ($a_0 = 0$)

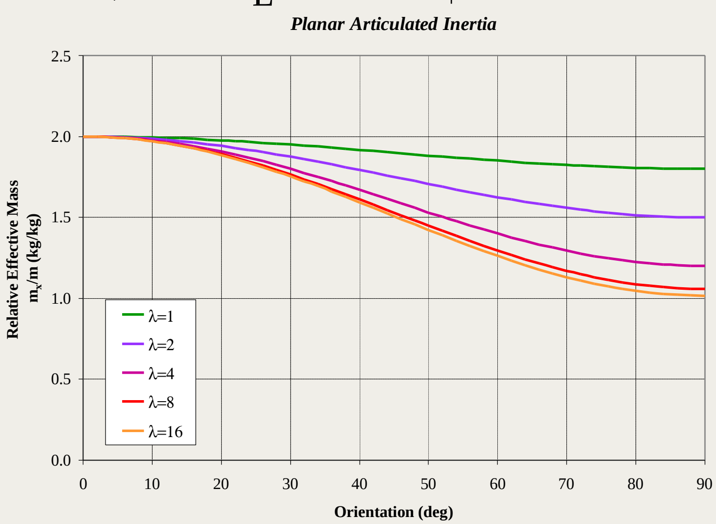
$$\boxed{f_i = \overbrace{T_i Q_i}^{\text{active forces}} + \overbrace{R_i U_i^{T}\left(A_i\left(a_{i-1} + \kappa_i\right) + d_i\right)}^{\text{reaction forces}}}$$

recursive

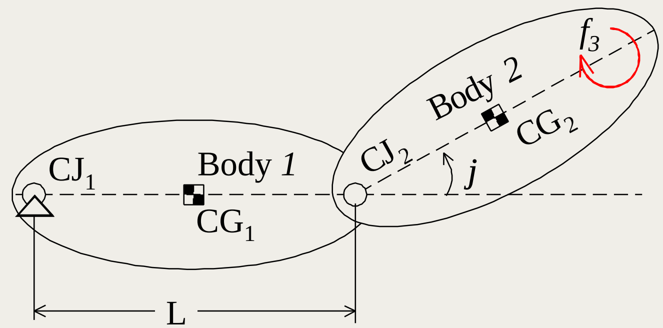
Planar Articulated Example

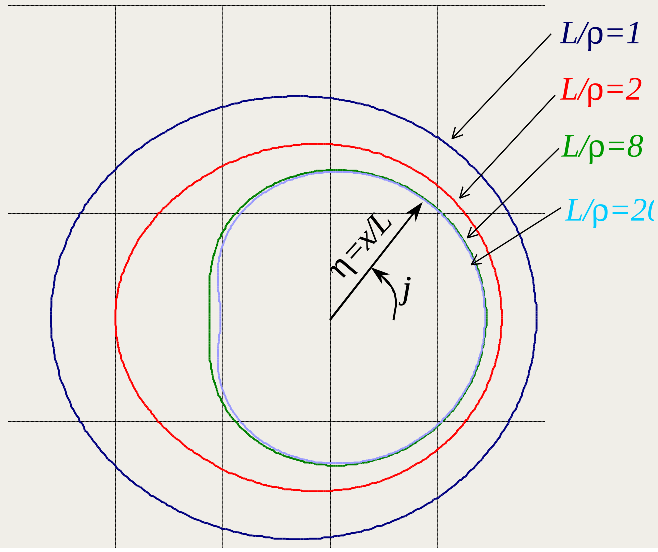
- Symbolic Inertia





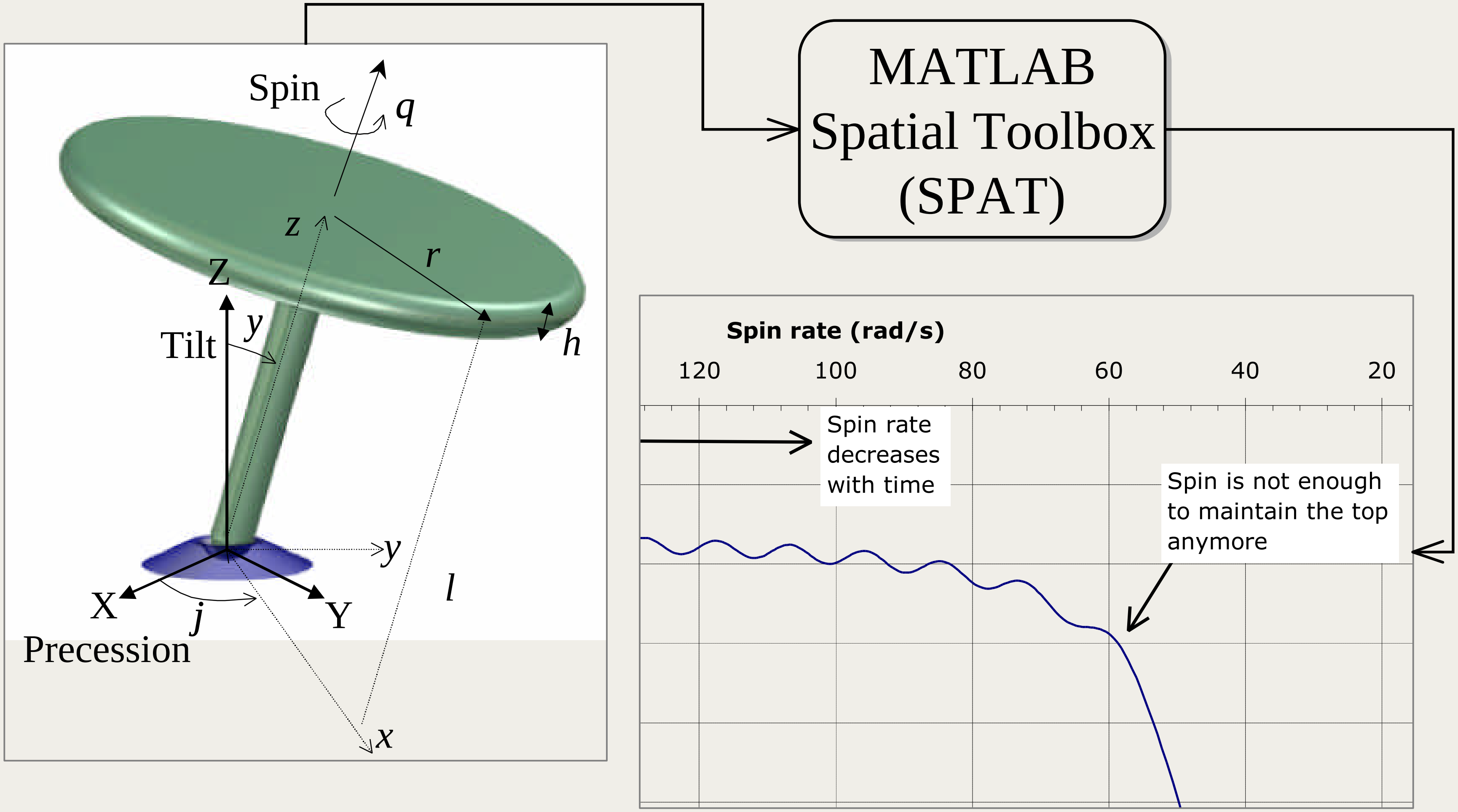
- Symbolic Mobility





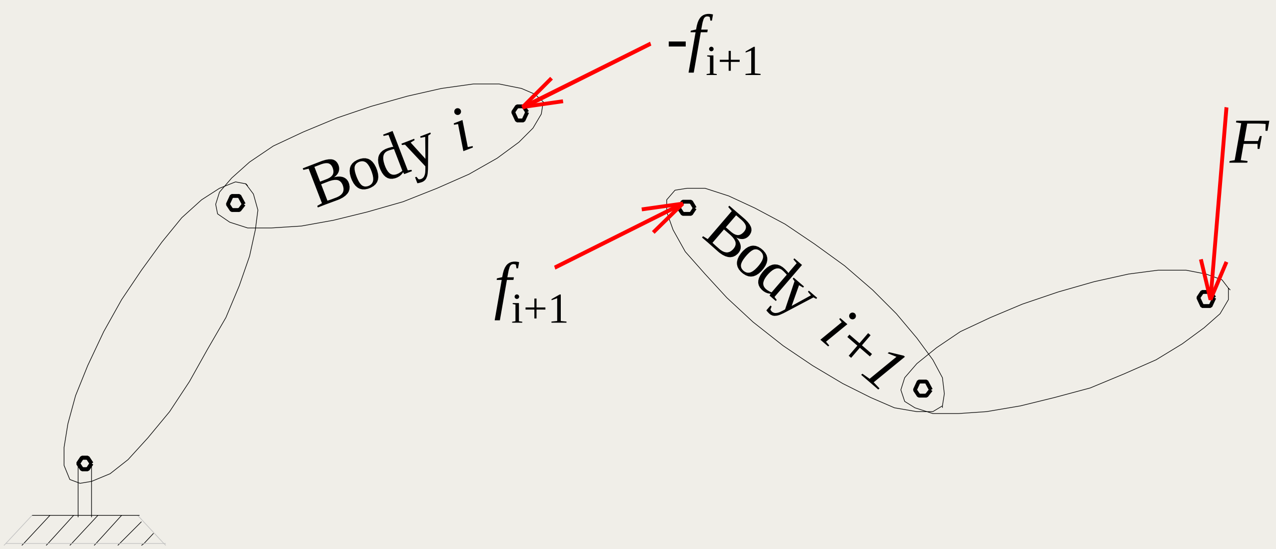
Spatial Example

- Spinning top with viscous friction



Chain Splitting

- Combine articulated inertia and articulated mobility for direct solution



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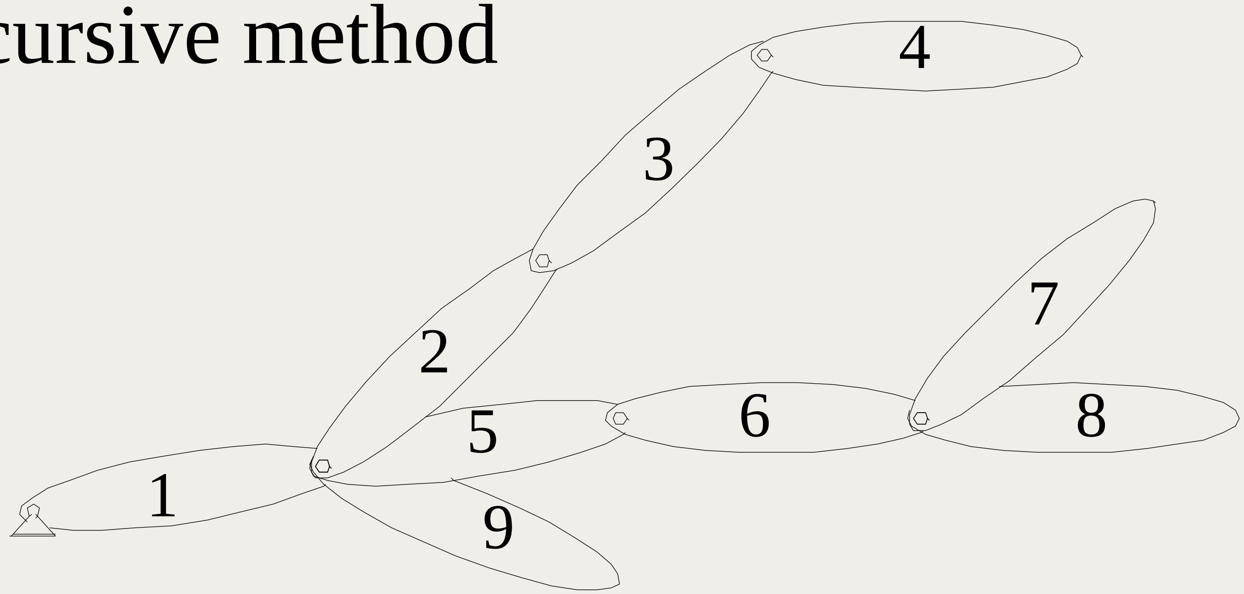
Open Tree Structures

- Extension of recursive methods to systems with multiple open subchains
- Topology of bodies (adjacency matrix)
- Non-recursive method

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$$



Contacts Between Bodies

- Topology of contacts (contact matrix *G*)

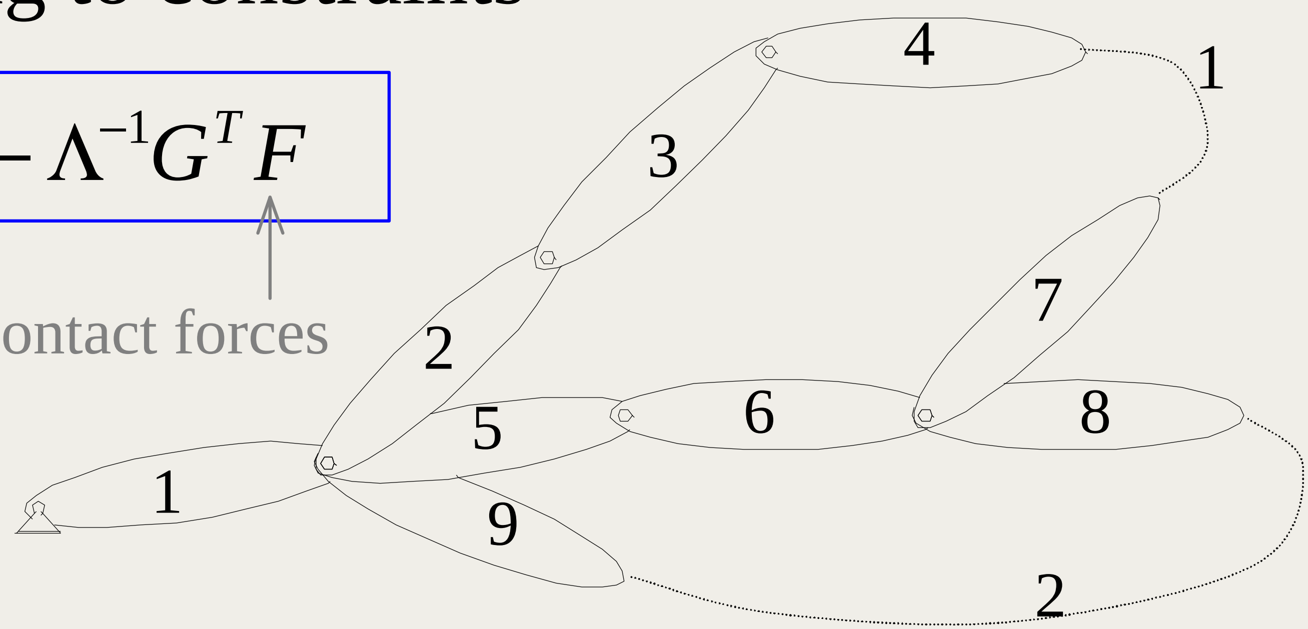
- Contact forces adjust accelerations according to constraints

$$\boxed{a = u - \Lambda^{-1} G^T F}$$

Closed loop

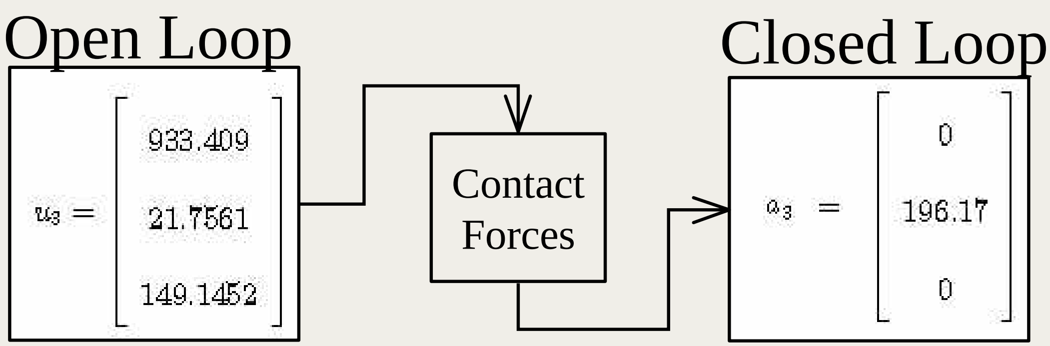
Open loop

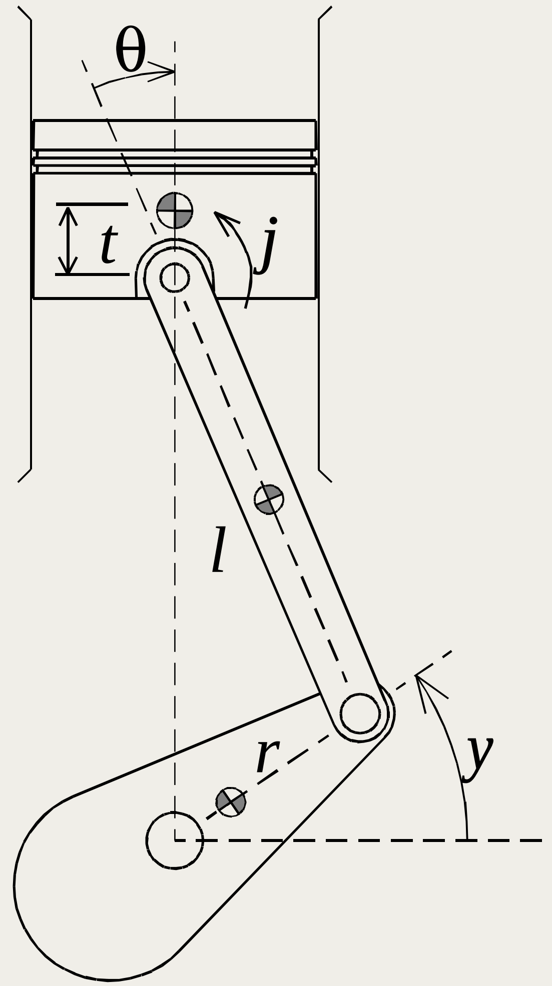
Contact forces



Planar Constrained Example

- Piston-crank mechanism
  - Ignore piston contact
  - Calculate contact forces
  - Constrained acceleration





Overview

- Introduction
- Screw Theory Basics
- Kinematics and Equations of Motion
- Projective Geometry and Dynamics
- Subspace Decompositions
- Articulated Dynamics
- Stacked Form
- Conclusions and Future Work

Conclusions

- Projective geometry and screw theory offer interpretations
- Duality between forces and accelerations
- Articulated analysis simplifies equations
- Implementation with MATLAB toolbox
- Stacked form model constrained systems

Future Work

- Special cases using projections
- Visualization of articulated quantities
- Eigenstructure of articulated inertia
- Convergence in recursions

Acknowledgments

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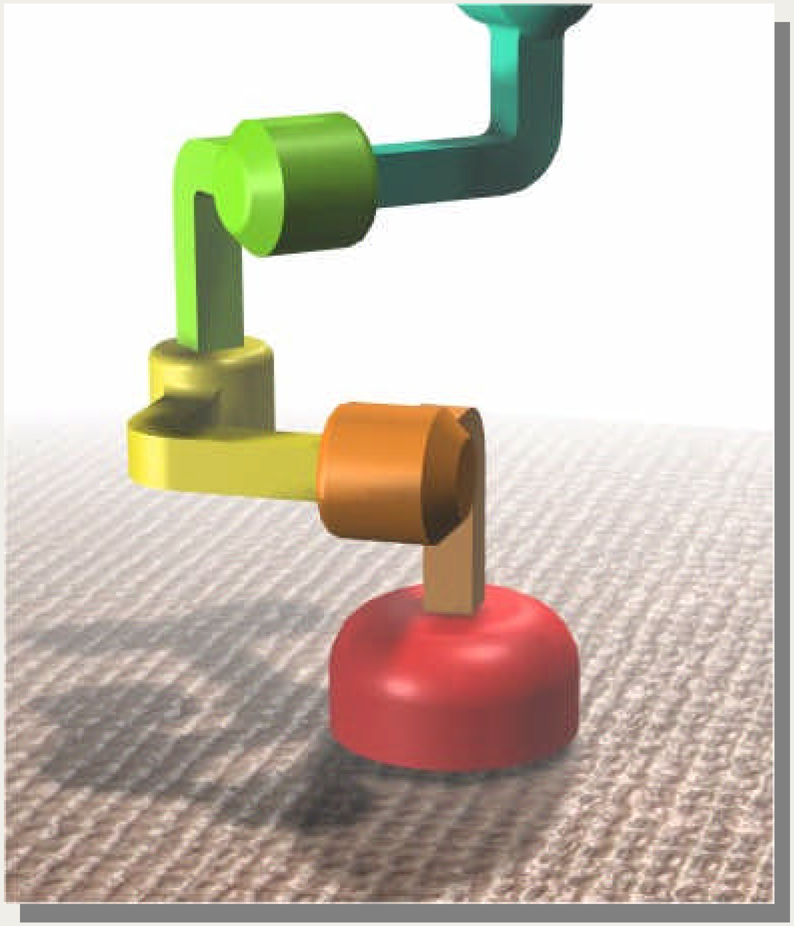
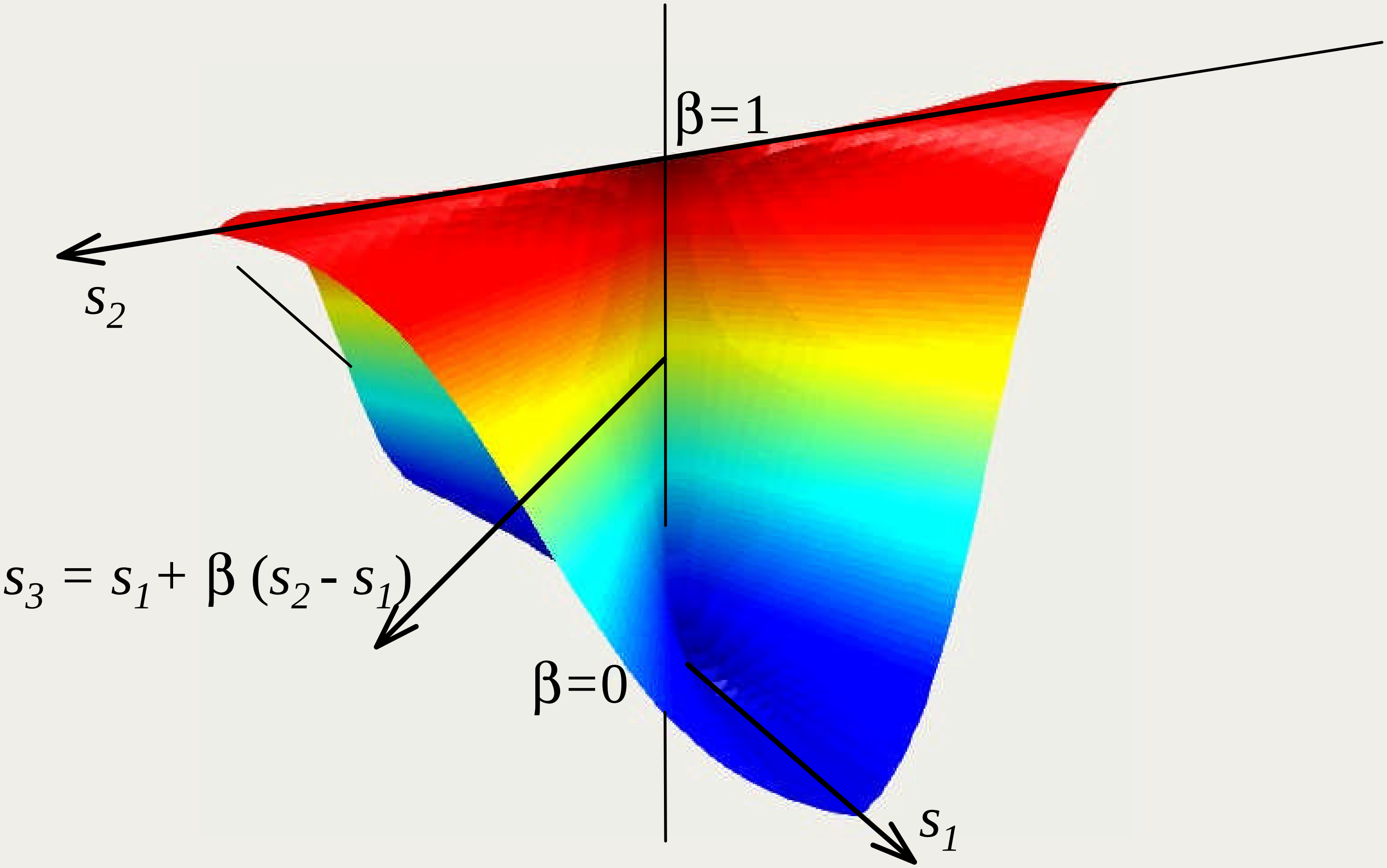
- Committee, Dr. Papastavridis and Dr. Ferri
- Family and friends

Questions ?



Screw Cylindroid



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