Angular Momentum and Torque on a Rigid Body

Angular momentum of a rigid body varies as the point of summation varies. Also, equipollent torque varies as the point of summation varies also. This posts examines the relationship between change in angular momentum and equipollent torque in the context of Newton's 2nd law. It is ~~expected~~ required that at the center of mass equipollent torque and change in angular momentum at simply equal to each other. But what about when measured at an arbitrary point **A** not at the center of mass **C**.

Note that all quantities are expressed on the same basis vectors, and only reference point being **C** or **A** or **O** the origin describing the properties of whichever particles happens to be passing under this reference point.

Particle Forces & Torques

The combined loading on the rigid body can be reduced down to an equipollent system of forces $\boldsymbol{F}$ and torques $\boldsymbol{\tau}_A$ at the reference point **A**. In dynamics it is important to consider the combined torque about the center of mass which is evaluated with the transformation law(s)

$$\begin{aligned} \boldsymbol{\tau}_C &= \boldsymbol{\tau}_A + (\boldsymbol{r}_A - \boldsymbol{r}_C) \times \boldsymbol{F} \\ \boldsymbol{\tau}_A &= \boldsymbol{\tau}_C + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{F} \end{aligned} \tag{1}$$

quantity	***description***
$\boldsymbol{F}$	combined of forces applied on body.
$\boldsymbol{\tau}_A$	combined torque applied on body about point **A**.
$\boldsymbol{\tau}_C$	combined torque applied on body about center of mass **C**.
$\boldsymbol{r}_A$	location of point **A** from the origin.
$\boldsymbol{r}_C$	location of point **C** from the origin.

Center of mass

It is important to define the center of mass not only as the weighted sum of the particle locations

$$\boldsymbol{r}_C = \frac{1}{m} \sum_i m_i \boldsymbol{r}_i \tag{3}$$

but also as the relative location of each particle being zero

$$\sum_i m_i (\boldsymbol{r}_i - \boldsymbol{r}_C) = \boldsymbol{0} \tag{4}$$

These two expressions are equivalent to each other. For simplicity you can define the relative position of each particle as $\boldsymbol{d}_{Ci}$, but at this stage I want to keep things as explicit as possible.

quantity	description	definition
m	combined masss	$\sum_i m_i$
$\boldsymbol{d}_{Ci}$	relative position of particle	$\boldsymbol{d}_{Ci} = \boldsymbol{r}_i - \boldsymbol{r}_C$

Momenta Definitions

Momenta are *defined* by summing up the following contributions of all the particles that move together on a rigid body. This summation happens an at arbitrary location **A** in space *at every instance* in time.

$$\begin{aligned} \boldsymbol{p} &= \sum_i m_i \boldsymbol{v}_i \\ \boldsymbol{L}_A &= \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_A) \times m_i \boldsymbol{v}_i \end{aligned} \tag{2}$$

where:

quantity	*description*
$\boldsymbol{p}$	~~linear~~ translational momentum vector.
$\boldsymbol{L}_A$	~~angular~~ rotational momentum vector of the body measured at **A**.

quantity	*description*
m_i	infinitesimal mass of each particle such that the total mass is $m = \sum_i m_i$.
$\boldsymbol{r}_i$	position vector of the particle from the origin.
$\boldsymbol{v}_i$	velocity vector of the particle.
$\boldsymbol{r}_A$	position vector of arbitrary point **A** from the origin.

Note that the point **A** might be fixed in space over time, or moving with constant velocity or riding on the body. At any time frame it might have non-zero velocity $\boldsymbol{v}_A$ with respect to the origin.

Kinematics

The motion of each particle can be decomposed into the motion of the center of mass and a rotation about the center of mass (Chasle's Theorem).

$$\boldsymbol{v}_i = \boldsymbol{v}_C + \boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_C) \tag{3}$$

Additionally the combined motion of the center of mass point **C** is expressed on the reference point **A** using the following transformation law(s)

$$\begin{aligned} \boldsymbol{v}_C &= \boldsymbol{v}_A + (\boldsymbol{r}_A - \boldsymbol{r}_C) \times \boldsymbol{\omega} \\ \boldsymbol{v}_A &= \boldsymbol{v}_C + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{\omega} \end{aligned} \tag{4}$$

Any resemblence of the above to (1) is not coincidence.

Translational Momentum

Translational momentum is the easiest to evaluate from (2) and (3). Center of mass expression (4) is used for simplification below

$$\begin{aligned} \boldsymbol{p} &= \sum_i m_i (\boldsymbol{v}_C + \boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_C)) \\ &= (\sum_i m_i) \boldsymbol{v}_C + \boldsymbol{\omega} \times \cancel{\sum_i m_i (\boldsymbol{r}_i - \boldsymbol{r}_C)} \end{aligned}$$

$$\boldsymbol{p} = m\,\boldsymbol{v}_C \tag{5}$$

The is a well known result, undesputed as long as $\boldsymbol{p}$ and $\boldsymbol{v}_C$ are measued form *the same coordinate frame*. Again, $\boldsymbol{v}_C$ is the velocity vector of the center of mass, as in

$$\boldsymbol{v}_C = \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_C$$

Rotational Momentum about Center of Mass

Rotational momentum is also evaluated from (2) and (3) but we start with summing about the center of mass first

$$\begin{aligned}\boldsymbol{L}_C &= \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_C) \times m_i v_i \\ &= \sum_i m_i(\boldsymbol{r}_i - \boldsymbol{r}_C) \times (\boldsymbol{v}_C + \boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_C)) \\ &= \cancel{\sum_i m_i(\boldsymbol{r}_i - \boldsymbol{r}_C)} \times \boldsymbol{v}_C + \sum_i m_i \boldsymbol{d}_{Ci} \times (\boldsymbol{\omega} \times \boldsymbol{d}_{Ci}) \\ &= \sum_i \left[-m_i \boldsymbol{d}_{Ci} \times (\boldsymbol{d}_{Ci} \times \boldsymbol{\omega}) \right]\end{aligned}$$

$$\boldsymbol{L}_C = \mathbf{I}_C \boldsymbol{\omega} \tag{6}$$

where $\mathbf{I}_C$ is the 3×3 mass moment of inertia tensor (inertia dyatic) derived from factoring $\boldsymbol{\omega}$ from the rotational inertia expression above.

Practically this is done by evaluating the following sum. For simplicity consider each relative position vector having components $\boldsymbol{d}_{Ci} = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ and evaluate

$$\mathbf{I}_C = \sum_i m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix} \tag{7}$$

Another common alterative to the above is to declare the 3×3 cross product matrix as

$$[\boldsymbol{d}_{Ci} \times] \equiv \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

and evaluating

$$\mathbf{I}_C = \sum_i \left(-m_i [\boldsymbol{d}_{Ci} \times][\boldsymbol{d}_{Ci} \times] \right) \tag{8}$$

Rotational Momentum about Arbitrary Point

Rotational momentum is also evaluated from (2) and (3) but we sum about the arbitrary point **A**.

$$\begin{aligned} \boldsymbol{L}_A &= \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_A) \times m_i v_i \\ &= \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_C) \times m_i v_i + \sum_i (\boldsymbol{r}_C - \boldsymbol{r}_A) \times m_i v_i \\ &= \boldsymbol{L}_C + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \sum_i m_i v_i \end{aligned}$$

Which leads the the following transformation law(s)

$$\begin{aligned} \boldsymbol{L}_A &= \boldsymbol{L}_C + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{p} \\ \boldsymbol{L}_C &= \boldsymbol{L}_A + (\boldsymbol{r}_A - \boldsymbol{r}_C) \times \boldsymbol{p} \end{aligned} \tag{9}$$

Again, any resemblance to (1) and (4) is not a coincidence. This is because these are Plücker coordinates of different lines in space. The force line is called the *line of action*. The motion line is called the *rotation axis*. And the momentum line is called *axis of percussion*.

Spatial Momentum

In spatial form the above is

$$\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{L}_A \end{bmatrix} = \begin{bmatrix} m & -m\boldsymbol{c}\times \\ m\boldsymbol{c}\times & \mathbf{I}_A \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_A \\ \boldsymbol{\omega} \end{bmatrix} \tag{10}$$

where $\boldsymbol{c} = \boldsymbol{r}_C - \boldsymbol{r}_A$ is the location of the CM from the arbitrary point and $\mathbf{I}_A = \mathbf{I}_C - m\,[\boldsymbol{c}\times][\boldsymbol{c}\times]$ is the MMOI matrix at the arbitrary point

The above is to be interpreted as spatial momentum (also known as the momentum wrench) equals a 6×6 mass matrix times the spatial motion (also known as the velocity twist). The mass matrix is symmetric and contains cross terms due to the movement of the center of mass.

Dynamics

We establish Newton's second law about the center of mass **C** relating forces/torques to rate of change of momenta

$$\begin{aligned} \boldsymbol{F} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} \\ \boldsymbol{\tau}_C &= \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_C \end{aligned} \tag{11}$$

Using the definitions above from particle summation the expression of equations of motion are straigtforward when using rhe center of mass as a reference point.

$$\begin{aligned} \boldsymbol{F} &= m\boldsymbol{a}_C \\ \boldsymbol{\tau}_C &= \mathbf{I}_C\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{L}_C \end{aligned} \tag{12}$$

where

quantity	description	definition
$\boldsymbol{a}_C$	translational acceleration of the center of mass	$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_C$
$\boldsymbol{\alpha}$	rotational acceleration of the body	$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}$

In spatial form the above is

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{\tau}_C \end{bmatrix} = \begin{bmatrix} m & \\ & \boldsymbol{I}_C \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_C \\ \boldsymbol{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{\omega} \times \boldsymbol{L}_A \end{bmatrix} \tag{13}$$

But what about when expressed at a different point?

Arbitrary Point

The question is can the equations of motion above can be derived from Newton's 2nd law *at an arbitrary location*?

Is the following valid?

$$\boldsymbol{\tau}_A \overset{?}{=} \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_A \tag{14}$$

We can use the transformation laws (1) and (9) described above to see if (12) can lead to (11) which we know is correct.

$$\begin{aligned}
\boldsymbol{\tau}_C + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{F} &= \tfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}_C + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{p}) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_C + \tfrac{\mathrm{d}}{\mathrm{d}t}\left((\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{p}\right) \\
\cancel{\boldsymbol{\tau}_C} + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{F} &= \cancel{\boldsymbol{\tau}_C} + \tfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{p} + (\boldsymbol{r}_C - \boldsymbol{r}_A) \times \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} \\
\cancel{(\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{F}} &= (\boldsymbol{v}_C - \boldsymbol{v}_A) \times \boldsymbol{p} + \cancel{(\boldsymbol{r}_C - \boldsymbol{r}_A) \times \boldsymbol{F}} \\
\boldsymbol{0} &= \boldsymbol{p} \times (\boldsymbol{v}_A - \boldsymbol{v}_C) \\
\boldsymbol{0} &= m\,\boldsymbol{v}_C \times \boldsymbol{v}_A
\end{aligned}$$

The *necessary* condition(s) for (12) to be correct are as follows

- Reference point **A** is fixed in space.
- Reference point **A** is co-moving with the center of mass.
- Center of mass **C** is fixed in space.

The general rule connecting equipollent torque to change in rotational momentum is thus

$$\boldsymbol{\tau}_A = \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_A + \boldsymbol{v}_A \times \boldsymbol{p} \tag{15}$$

where

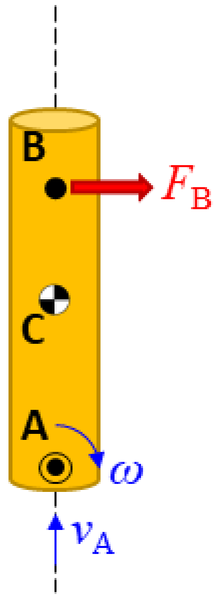
quantity	description
$\boldsymbol{\tau}_A$	equipollent torque acting on the body (instanteneously) summed on the reference point **A**.
$\boldsymbol{L}_A$	rotational momentum of rigid body summed on the reference point **A**.
$\boldsymbol{p}$	translational momentum of the rigid body.
$\boldsymbol{v}_A$	instanteneous velocity of reference point **A**.

In spatial form the above is

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{\tau}_A \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{L}_A \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{v}_A \times \boldsymbol{p} \end{bmatrix} \tag{16}$$

Example

Consider a rod sliding vertically along a rail with speed v_A and pivoting with rotational speed ω about a point **A** away from the center of mass **C**. A force of magnitude F_B is applied horizontally at another point **B**.



The variable c is the distance between the pivot and the center of mass, and ℓ the distance between the pivot and the force application point.

Can (12) produce the correct equation of motion?

First we develop the equations of motion about the center of mass

1. **Kinematics** the center of mass **C** moves under two degrees of freedom, the sliding velocity v_A and the rotation ω
   - $\boldsymbol{v}_C = v_A\,\boldsymbol{\hat{\jmath}} + \omega\boldsymbol{\hat{k}} \times c\boldsymbol{\hat{\jmath}} = \begin{pmatrix} -c\,\omega \\ v_A \\ 0 \end{pmatrix}$
   - $\boldsymbol{a}_C = \dot{v}_A\boldsymbol{\hat{\jmath}} + \dot{\omega}\boldsymbol{\hat{k}} \times c\boldsymbol{\hat{\jmath}} + \omega\boldsymbol{\hat{k}} \times \boldsymbol{v}_C = \begin{pmatrix} -c\,\dot{\omega} - \omega\,v_A \\ \dot{v}_A - c\,\omega^2 \\ 0 \end{pmatrix}$
2. **Momenta** directly calculated from the motion of the center of mass

- $\boldsymbol{p} = m\boldsymbol{v}_C = \begin{pmatrix} -m\,c\,\omega \\ m\,v_A \\ 0 \end{pmatrix}$
- $\boldsymbol{L}_C = \mathbf{I}_C \boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ I_C\,\omega \end{pmatrix}$

3. **Equipollent forces and moments** at the center of mass **C** are a combination of the appled force F_B and the pin reaction force A_x.
   - $\boldsymbol{F} = \begin{pmatrix} F_B + A_x \\ 0 \\ 0 \end{pmatrix}$
   - $\boldsymbol{\tau}_C = \begin{pmatrix} 0 \\ 0 \\ cA_x - (\ell - c)F_B \end{pmatrix}$
4. **Equations of motion**
   - $\begin{pmatrix} F_B + A_x \\ 0 \\ 0 \end{pmatrix} = m \begin{pmatrix} -c\,\dot{\omega} - \omega\,v_A \\ \dot{v}_A - c\,\omega^2 \\ 0 \end{pmatrix}$
   - $\begin{pmatrix} 0 \\ 0 \\ cA_x - (\ell - c)F_B \end{pmatrix} = I_C \begin{pmatrix} 0 \\ 0 \\ \dot{\omega} \end{pmatrix}$
5. **Solution**
   - $\dot{\omega} = -\dfrac{\ell F_B + m\,c\,\omega\,v_A}{I_C + mc^2}$
   - $\dot{v}_A = c\,\omega^2$
   - $A_x = -\left(1 - \dfrac{m\,c\,\ell}{I_C + mc^2}\right) F_B - \dfrac{\omega\,v_A}{\frac{1}{m} + \frac{c^2}{I_C}}$

Now we apply equation (12) directly after the rotational momentum about **A** is calculated from (9).

- $\boldsymbol{L}_A = \begin{pmatrix} 0 \\ 0 \\ (I_C + mc^2)\omega \end{pmatrix}$
- $\boldsymbol{\tau}_A = \begin{pmatrix} 0 \\ 0 \\ -\ell F_B \end{pmatrix}$

- $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_A = \begin{pmatrix} 0 \\ 0 \\ (I_C + mc^2)\dot{\omega} \end{pmatrix}$
- $\boldsymbol{v}_A \times \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ m\,c\,\omega\,v_A \end{pmatrix}$

All together equation (12) is

$$-\ell F_B = (I_C + mc^2)\dot{\omega} + mc\omega v_A$$

which is solved for $\dot{\omega}$ to produce **the exact same same solution as above**

$$\dot{\omega} = -\frac{\ell F_B + m\,c\,\omega\,v_A}{I_C + mc^2} \;\; \checkmark$$

Rotational momemtnum rate

We need to expand on the rate of momentum and find the spatial form of dynamics.

Translational momentum

Use the transformation of velocity from the center of mass to the arbitrary point to relate the change in momentum with the motion of **A**.

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}_C\right) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}_A + \boldsymbol{\omega}\times m\,\boldsymbol{c}\right) \\ &= m\boldsymbol{a}_A + \boldsymbol{\alpha}\times m\,\boldsymbol{c} + \boldsymbol{\omega}\times m\left(\boldsymbol{v}_C - \boldsymbol{v}_A\right) \end{aligned} \tag{17}$$

Note that $\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{c} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_C - \boldsymbol{r}_A\right) = \boldsymbol{v}_C - \boldsymbol{v}_A$

Rotational momentum

Similarly differentiate (10) to get

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_A &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{c}\times m\boldsymbol{v}_A + \mathbf{I}_A\boldsymbol{\omega}\right) \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{\omega}\times\mathbf{I}_A\boldsymbol{\omega} + \left(\boldsymbol{v}_C - \boldsymbol{v}_A\right)\times m\boldsymbol{v}_A + \boldsymbol{c}\times m\boldsymbol{a}_A \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c}\times m\boldsymbol{a}_A + \boldsymbol{p}\times\boldsymbol{v}_A + \boldsymbol{\omega}\times\mathbf{I}_A\boldsymbol{\omega} \end{aligned} \tag{18}$$

Spatial Acceleration

The above expressions are not very intuitive or elegant. But the situation improves if instead of using $\boldsymbol{a}_A$ the material acceleration of of **A**, we use the *spatial acceleration* of the body resolved at **A**. This is defined as

$$\boldsymbol{a}_A^\star = \boldsymbol{a}_A - \boldsymbol{\omega} \times \boldsymbol{v}_A \tag{19}$$

The above is inspired by fluid dynamics, and it uses the material acceleration $\boldsymbol{a}_A$ *minus* the certifugal parts $\boldsymbol{\omega} \times \boldsymbol{v}_A$. What remains is sometimes called the Eulerian acceleration. Spatial accelerations transform similarly to velocities $\boldsymbol{a}_C^\star = \boldsymbol{a}_A^\star + \boldsymbol{\alpha} \times \boldsymbol{c}$.

The material accelration can always be recovered from the spatial acceleration by reversing (19) into

$$\boldsymbol{a}_A = \boldsymbol{a}_A^\star + \boldsymbol{\omega} \times \boldsymbol{v}_A \tag{20}$$

Translational Momentum Rate

Use (20) into (17) to get

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = m\boldsymbol{a}_A^\star + \boldsymbol{\alpha} \times m\,\boldsymbol{c} + \boldsymbol{\omega} \times \boldsymbol{p} \tag{21}$$

Rotational Momentum Rate

and use (20) into (18) to get (with use of the vector triple products)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_A &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c} \times m\boldsymbol{a}_A + \boldsymbol{p} \times \boldsymbol{v}_A + \boldsymbol{\omega} \times \mathbf{I}_A\boldsymbol{\omega} \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c} \times m\left(\boldsymbol{a}_A^\star + \boldsymbol{\omega} \times \boldsymbol{v}_A\right) - m\boldsymbol{v}_A \times (\boldsymbol{\omega} \times \boldsymbol{c}) + \boldsymbol{\omega} \times \mathbf{I}_A\boldsymbol{\omega} \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c} \times m\boldsymbol{a}_A^\star - m\boldsymbol{v}_A \times (\boldsymbol{\omega} \times \boldsymbol{c}) - \boldsymbol{c} \times m\left(\boldsymbol{v}_A \times \boldsymbol{\omega}\right) + \boldsymbol{\omega} \times \mathbf{I}_A\boldsymbol{\omega} \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c} \times m\boldsymbol{a}_A^\star + \boldsymbol{\omega} \times \left(\boldsymbol{c} \times m\boldsymbol{v}_A\right) + \boldsymbol{\omega} \times \mathbf{I}_A\boldsymbol{\omega} \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c} \times m\boldsymbol{a}_A^\star + \boldsymbol{\omega} \times \left(\mathbf{I}_A\boldsymbol{\omega} + \boldsymbol{c} \times m\boldsymbol{v}_A\right) \\ &= \mathbf{I}_A\boldsymbol{\alpha} + \boldsymbol{c} \times m\boldsymbol{a}_A^\star + \boldsymbol{\omega} \times \boldsymbol{L}_A \end{aligned} \tag{22}$$

Spatial Momentum Rate

Combine (21) and (22) to the following expression for the rate of momentum in spatial form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{p}\\ \boldsymbol{L}_A\end{bmatrix} = \begin{bmatrix} m & -m\boldsymbol{c}\times \\ m\boldsymbol{c}\times & \mathbf{I}_A\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_A^{\star}\\ \boldsymbol{\alpha}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\omega}\times\boldsymbol{p}\\ \boldsymbol{\omega}\times\boldsymbol{L}_A\end{bmatrix} \tag{23}$$

The first term is contains the spatial mass matrix just as momentum (10) does and the second term is the change in momentum due to the rotating frame.

Spatial Dynamics

Combine (16) and (23) to get the spatial form of the equations of motion

$$\begin{bmatrix}\boldsymbol{F}\\ \boldsymbol{\tau}_A\end{bmatrix} = \begin{bmatrix} m & -m\boldsymbol{c}\times \\ m\boldsymbol{c}\times & \mathbf{I}_A\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_A^{\star}\\ \boldsymbol{\alpha}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\omega}\times & 0\\ \boldsymbol{v}_A\times & \boldsymbol{\omega}\times\end{bmatrix}\begin{bmatrix}\boldsymbol{p}\\ \boldsymbol{L}_A\end{bmatrix} \tag{24}$$

The above is valid at *any* arbitrary point **A**, being inertial, co-moving or completely independent of the motion of the center of mass. All of the 6×1 quantities above transform with standard twist and wrench rules allowing the above equation to be applied to kinematically meaningful points, such as on joints where motions might be if not prescribed, but constrained geometrically.

The first part is obviously the effect of body acceleration (as in rate of change of velocity twist) throught the spatial mass matrix. The second part is the effect of the rotating momentum wrench has on the forces and torques required for such motion.