The dynamics of a serial chain of bodies

Overview

For a serial chain with n bodies, there are vectors of length n containting the joint angle positions $\boldsymbol{q}$, their speeds $\dot{\boldsymbol{q}}$ accelerations $\ddot{\boldsymbol{q}}$ and torques $\boldsymbol{Q}$.

In this overview a reduced system of equations

$$\boldsymbol{Q} = \mathbf{A}\,\ddot{\boldsymbol{q}} + \boldsymbol{b} \tag{1}$$

is formulated for each system configuration defined by $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$.

Each body is connected to the previous body (or the ground) by a single DOF joint. More complex joints should be handled as a sequence of primitive joints with virtual bodies between them.

Position Kinematics

A lot of the quantities evaluated below are recursive in nature, where values depend on previous calculation results. Each *i*-th body pose is defind by the location vector $\overline{r}_i$ of the top of each joint, and the 3×3 rotation matrix R_i describing the body coordinates in the world basis vectors.

Jumping from one body to the next, there is a local vector $\overline{n}_{\mathrm{base}}$ that moves across the previous body to the base of the next joint. Then for a prismatic joint there is an additional translation along the joint axis $\hat{k}$ with displacement q_i, and for revolute joint there is rotation about the joint axis $\hat{k}$ with angle q_i.

$$\begin{array}{c|c} \text{prismatic} & \text{revolute} \\ \hline \overline{r}_i = \overline{r}_{i-1} + \mathrm{R}_{i-1}\left(\overline{n}_{\mathrm{base}} + \hat{k}\,q_i\right) & \overline{r}_i = \overline{r}_{i-1} + \mathrm{R}_{i-1}\overline{n}_{\mathrm{base}} \\ \mathrm{R}_i = \mathrm{R}_{i-1} & \mathrm{R}_i = \mathrm{R}_{i-1}\mathrm{rot}\left(\hat{k},\,q_i\right) \end{array} \tag{3}$$

Similarly to define the location of the center of mass, there is local vector $\overline{c}_{\text{body}}$ going from the joint to the CM

$$\overline{c}_i = \overline{r}_i + \mathrm{R}_i \overline{c}_{\text{body}} \tag{4}$$

Velocity Kinematics

Here we can move from cartesian vector formulation, to 6×1 spatial algebra vectors and matrices.
The joint axis $\boldsymbol{s}_i$ converts the joint speed $\dot{q}_i$ into the spatial velocity (called a twist) recursively for each joint

$$\boldsymbol{v}_i = \boldsymbol{v}_{i-1} + \boldsymbol{s}_i \dot{q}_i \tag{5}$$

Starting from the fixed ground with $\boldsymbol{v}_0 = 0$, each joint is defined as the a stack of two vectors, with the translational velocity first, and rotational velocity second.

$$\begin{array}{c|c} \text{prismatic} & \text{revolute} \\ \hline s_i = \begin{bmatrix} \hat{k} \\ \overline{0} \end{bmatrix} & s_i = \begin{bmatrix} \overline{r}_i \times \hat{k} \\ \hat{k} \end{bmatrix} \end{array} \tag{6}$$

Here $\times$ is the vector cross product. Mathmatically this can be represented as matrix-vector product with $a \times b = [a\times]b$ where $[a\times] = \begin{vmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{vmatrix}$. This notation is going to be used below.

For example, the first joint is a revolution about $\hat{k}$ located at $\overline{r}_1$ with rotational speed $\dot{\theta}$. The velocity twist is

$$\boldsymbol{v}_1 = \begin{bmatrix} \overline{r}_1 \times \hat{k} \\ \hat{k} \end{bmatrix} \dot{\theta} = \begin{bmatrix} \overline{r}_1 \times \overline{\omega} \\ \overline{\omega} \end{bmatrix}$$

The bottom vector $\overline{\omega}$ is just the rotational velocity of the body. The top vector $\overline{v} = \overline{r}_1 \times \overline{\omega}$ represents the velocity of the extended body of whatever particle happens to pass over the coordinate origin. If an actual particle isn't over the origin, the above is the velocity of the rotating frame at the origin. This is how the translating motion of the body is defined. The velocity of any other point not at the origin is calculated with the standard velocity transformation expression $\overline{v}_A = \overline{v} + \overline{\omega} \times \overline{r}_A$.

Acceleration Kinematics

The time derivative of (6) gives us an expression for accelerations. This is spatial accelerations and not material accelerations because it does not track the acceleration of specific point, but the acceleration of of whatever particle goes over a fixed point (the coordinate origin).

$$\boldsymbol{a}_i = \boldsymbol{a}_{i-1} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i \tag{7}$$

Here $\times$ is *not* a vector cross product because these are 6×1 vectors. The spatial cross product represents the time derivative on a rotating frame and the definition depends on the right-hand-side of the expression. Specifically here we use the matrix-vector form the cross product to define

$$[\boldsymbol{v}\times] = \begin{bmatrix} \overline{\omega}\times & 0 \\ \overline{v}\times & \overline{\omega}\times \end{bmatrix} \tag{8}$$

where $\boldsymbol{v} = \begin{bmatrix} \overline{v} \\ \overline{\omega} \end{bmatrix}$ is a velocity twist, and $\overline{\omega}\times$, $\overline{v}\times$ are the matrix form of the vector cross product.

Force Balance

The spatial form of $\overline{F} = m\overline{a}$ is defined with 6×1 spatial vectors called wrenches, and a big 6×6 matrix representating the inertial and mass of a rigid body.

For example the weight of the body applied as an external force is described by the wrench

$$\boldsymbol{w}_i = \begin{bmatrix} m_i \overline{g} \\ \overline{c}_i \times m_i \overline{g} \end{bmatrix} \tag{9}$$

The top part represents the force applied, whereas the bottom part is the equipollent moment of the force about the coordinate origin.

The inertia matrix $\mathbf{I}_i$ is derived from the spatial momentum

$$\boldsymbol{h}_i = \mathbf{I}_i \boldsymbol{v}_i \tag{10}$$

and is defined as

$$\mathbf{I}_i = \begin{bmatrix} m_i\,1 & -m_i \overline{c}_i \times \\ m_i \overline{c}_i \times & \mathcal{I}_i - m_i \overline{c}_i \times \overline{c}_i \times \end{bmatrix} \tag{11}$$

The top left block is a diagonal 3×3 matrix with mass in the diagonals. The bottom right block is the parallel axis theorem in vector form of the body centered MMOI of $\mathcal{I}_i$ transformed onto the coordinate origin.

As a side note, the body centered MMOI matrix must be derived from the body aligned matrix $\mathcal{I}_{\rm body}$ and the rotation matrix R_i as follows

$$\mathcal{I}_i = \mathrm{R}_i\, \mathcal{I}_{\rm body}\, \mathrm{R}_i^\top \tag{12}$$

The force on each joint $\boldsymbol{f}_i$ is added to the reaction from the next body $\boldsymbol{f}_{i+1}$ and the external forces $\boldsymbol{w}_i$ producing the net force that moves the body. This is the spatial equations of motion that combines the translational laws of Newton with the rotational laws of Euler.

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} + \boldsymbol{w}_i = \mathbf{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times \mathbf{I}_i \boldsymbol{v}_i \tag{13}$$

Again $\boldsymbol{v}\times$ represents a spatial cross product which is a time derivative on a rotating frame, but in this case is acts on loads (wrenches) and has the form

$$[\boldsymbol{v}\times] = \begin{bmatrix} \overline{\omega}\times & \overline{v}\times \\ 0 & \overline{\omega}\times \end{bmatrix} \tag{14}$$

Joint Condition

The final component to the math model using spatial algebra is describing the force/torque provided by the joint. If the joint is revolute then q_i is an angle and Q_i is a torque. If the joint is prismatic then q_i is a distance and Q_i is a force.

To prescribe this force of torque into the joint force $\boldsymbol{f}_i$ a projection is used using the joint axis $\boldsymbol{s}_i$

$$Q_i = \boldsymbol{s}_i^\top \boldsymbol{f}_i \tag{15}$$

The transpose $^\top$ is an alternate notation for the dot product $\cdot$ bewteen the two spatial vectors. This relationship is derived by equating the power balance across the joint with the power provided by the joint motor.

$$(\boldsymbol{v}_i - \boldsymbol{v}_{i-1}) \cdot \boldsymbol{f}_i = \dot{q}_i\, Q_i$$

Full System of Equations

Each link in the chain of n bodies, provides with 3 equations from (7), (13), (15) for a total of $13n$ components for 3D systems and $7n$ components for 2D systems,.

$$
\begin{array}{r|l}
\text{kinematics} & \boldsymbol{a}_1 = \boldsymbol{s}_1 \ddot{q}_1 + \boldsymbol{v}_1 \times \boldsymbol{s}_1 \dot{q}_1 \\
6n & \boldsymbol{a}_2 = \boldsymbol{a}_1 + \boldsymbol{s}_2 \ddot{q}_2 + \boldsymbol{v}_2 \times \boldsymbol{s}_2 \dot{q}_2 \\
 & \vdots \\
 & \boldsymbol{a}_n = \boldsymbol{a}_{n-1} + \boldsymbol{s}_n \ddot{q}_n + \boldsymbol{v}_n \times \boldsymbol{s}_n \dot{q}_n \\
\hline
\text{dynamics} & \boldsymbol{f}_1 - \boldsymbol{f}_2 = \mathbf{I}_1 \boldsymbol{a}_1 + \boldsymbol{v}_1 \times \mathbf{I}_1 \boldsymbol{v}_1 - \boldsymbol{w}_1 \\
6n & \boldsymbol{f}_2 - \boldsymbol{f}_3 = \mathbf{I}_2 \boldsymbol{a}_2 + \boldsymbol{v}_2 \times \mathbf{I}_2 \boldsymbol{v}_2 - \boldsymbol{w}_2 \\
 & \vdots \\
 & \boldsymbol{f}_n = \mathbf{I}_n \boldsymbol{a}_n + \boldsymbol{v}_n \times \mathbf{I}_n \boldsymbol{v}_n - \boldsymbol{w}_n \\
\hline
\text{joints} & Q_1 = \boldsymbol{s}_1^\top \boldsymbol{f}_1 \\
n & Q_2 = \boldsymbol{s}_2^\top \boldsymbol{f}_2 \\
 & \vdots \\
 & Q_n = \boldsymbol{s}_n^\top \boldsymbol{f}_n \\
\hline
\text{total} & \text{variables} : \ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix}, \boldsymbol{a} = \begin{Bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{Bmatrix}, \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_n \end{Bmatrix} \\
13n & 13n \checkmark
\end{array}
$$

The system *is* solvable, yet a bit cumbersome with having to deal with a largley sparse coefficients matrix. A subtle complexity in the above is the cross relationship between each force depending on the next body force, which in turn dends on its acceleration and the previous body acceleration. It turns out there is subtle coupling of the equations which makes a direct solution challenging.

Stacked Form

Each velocity twist depends on the previous body velocity twist and this relationship can be described with a big “parent” matrix P that has 1 on the first band under the diagonal

$$\mathrm{P} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

Then a big stack of velocity kinematics (5) is formulated as

$$\boldsymbol{v} = \mathrm{P}\boldsymbol{v} + \mathbf{s}\dot{q} \tag{15}$$

and expanded into

$$\begin{Bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{Bmatrix} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{Bmatrix} + \begin{bmatrix} \boldsymbol{s}_1 & & & \\ & \boldsymbol{s}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{s}_n \end{bmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix}$$

Notice that $\mathbf{s}$ is a block diagnonal matrix of $6n \times n$ dimensions.
The solution of (15) comes in terms of the tree matrix $\Delta = (1 - \mathrm{P})^{-1}$ which has 1 in the entire lower triangular region

$$\boldsymbol{v} = \Delta\, \mathbf{s}\dot{q} \tag{16}$$

with $\Delta = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & \ddots & \ddots & \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and the remaining quantities as defined above.

Similarly the acceleration kinematics are described by the equation

$$\boldsymbol{a} = \mathrm{P}\boldsymbol{a} + \mathbf{s}\,\ddot{q} + \boldsymbol{\kappa}$$

where $\boldsymbol{\kappa} = \begin{Bmatrix} \boldsymbol{v}_1 \times \boldsymbol{s}_1 \dot{q}_1 \\ \boldsymbol{v}_2 \times \boldsymbol{s}_2 \dot{q}_2 \\ \vdots \\ \boldsymbol{v}_n \times \boldsymbol{s}_n \dot{q}_n \end{Bmatrix}$ is the stacked vector for bias accelerations.

The solution is

$$\boldsymbol{a} = \Delta \left(\mathbf{s}\ddot{q} + \boldsymbol{\kappa} \right) \tag{17}$$

The force balance equations are also produced in stacked form by recognizing that the next body is selected with $\mathrm{P}^\top$. The system of force equations in stacked form is

$$\boldsymbol{f} - \mathrm{P}^\top \boldsymbol{f} = \mathbf{I}\boldsymbol{a} + \boldsymbol{p}$$

where $\boldsymbol{p} = \begin{Bmatrix} \boldsymbol{v}_1 \times \mathbf{I}_1 \boldsymbol{v}_1 - \boldsymbol{w}_1 \\ \boldsymbol{v}_2 \times \mathbf{I}_2 \boldsymbol{v}_2 - \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{v}_n \times \mathbf{I}_n \boldsymbol{v}_n - \boldsymbol{w}_n \end{Bmatrix}$ is the stacked vector of bias and external forces. The solution is

$$\boldsymbol{f} = \Delta^\top (\mathbf{I}\boldsymbol{a} + \boldsymbol{p}) \tag{18}$$

And finally the joint conditions are the easiest to group in a stacked form

$$\boldsymbol{Q} = \mathbf{s}^\top \boldsymbol{f} \tag{19}$$

Composite Inertia Formulation

Equations (17), (18) and (19) can be combined into a single equation that produces the reduced linear system using the composite inertia method. Why it is called so will becomes obvious shortly.

Plug (17) into (18) and then into (19) to produce

$$\boldsymbol{Q} = \left(\mathbf{s}^\top \mathbf{I}^C \mathbf{s}\right) \ddot{\boldsymbol{q}} + \mathbf{s}^\top \left(\mathbf{I}^C \boldsymbol{\kappa} + \boldsymbol{p}^C\right) \tag{20}$$

where $\mathbf{I}^C = \Delta^\top \mathbf{I} \Delta$ and $\boldsymbol{p}^C = \Delta^\top \boldsymbol{p}$.

The term $\mathbf{I}^C$ is called the composite intertia matrix because if you expand it the terms contain the sum of interias from a body and above.

For example with a 3 body problem:

$$\mathbf{I}^C = \begin{bmatrix} \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 & \mathbf{I}_2 + \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_2 + \mathbf{I}_3 & \mathbf{I}_2 + \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}$$

and

$$\boldsymbol{p}^C = \begin{bmatrix} \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 \\ \boldsymbol{p}_2 + \boldsymbol{p}_3 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Equation (20) is finally in the form $\boldsymbol{Q} = \mathrm{A}\ddot{\boldsymbol{q}} + \boldsymbol{b}$ as promised in the begining of this paper,. allowing for “easy” solution for of the joint accelerations $\ddot{\boldsymbol{q}}$ for use in an ODE scheme, $\ddot{\boldsymbol{q}} = \mathbf{f}\left(\boldsymbol{q},\, \dot{\boldsymbol{q}},\, \boldsymbol{Q}\right)$.