# 1 Quaternions (Scalar-Vector)

| Step | Description | Expression |
|---|---|---|
| 1 | Composition | general: $q=\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}$ rotation: $q=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \hat{\boldsymbol{z}}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}$ identity:$q=\begin{pmatrix}1\\ \boldsymbol{0}\end{pmatrix}$ exponential: $q=\exp\left(\frac{\theta}{2}\hat{z}\right)$ |
| 2 | Magnitude | general: $\|q\|=\sqrt{q\otimes q^{\star}}=\sqrt{s^2+\|\boldsymbol{v}\|^2}$ rotation: $\|q\|=\sqrt{\cos^2\left(\frac{\theta}{2}\right)+\|\boldsymbol{z}\|^2\sin^2\left(\frac{\theta}{2}\right)}=1$ |
| 3 | Transformations | conjugate: $q^{\star}=\begin{pmatrix}s\\ -\boldsymbol{v}\end{pmatrix}$ inverse: $q^{-1}=\frac{q^{\star}}{\|q\|^2}=\frac{1}{s^2+\|\boldsymbol{v}\|^2}\begin{pmatrix}s\\ -\boldsymbol{v}\end{pmatrix}$ unit inverse: $q^{-1}=q^{\star}=\begin{pmatrix}s\\ -\boldsymbol{v}\end{pmatrix}$ |
| 4 | Inner Product (noted with $\cdot$) | $q_0\cdot q_1=\begin{pmatrix}s_0\\ \boldsymbol{v}_0\end{pmatrix}^{\intercal}\begin{pmatrix}s_1\\ \boldsymbol{v}_1\end{pmatrix}=s_0s_1+\boldsymbol{v}_0^{\intercal}\boldsymbol{v}_1$ |
| 5 | Cross Product (noted with $\times$) | $q_0\times q_1=\frac{1}{2}\left(q_0q_1-q_1^{\star}q_0^{\star}\right)$ <br> $\begin{bmatrix}0 & -\boldsymbol{0}^{\intercal}\\ \boldsymbol{0} & \boldsymbol{v}_0\times\end{bmatrix}\begin{pmatrix}s_1\\ \boldsymbol{v}_1\end{pmatrix}=\begin{pmatrix}0\\ \boldsymbol{v}_0\times\boldsymbol{v}_1\end{pmatrix}$ |
| 6 | Multiplication (noted with $\otimes$) | $q_0q_1=\begin{pmatrix}s_0\\ \boldsymbol{v}_0\end{pmatrix}\otimes\begin{pmatrix}s_1\\ \boldsymbol{v}_1\end{pmatrix}=\begin{bmatrix}s_0 & -\boldsymbol{v}_0^{\intercal}\\ \boldsymbol{v}_0 & s_0+\boldsymbol{v}_0\times\end{bmatrix}\begin{pmatrix}s_1\\ \boldsymbol{v}_1\end{pmatrix}=\begin{bmatrix}s_1 & -\boldsymbol{v}_1^{\intercal}\\ \boldsymbol{v}_1 & s_1-\boldsymbol{v}_1\times\end{bmatrix}\begin{pmatrix}s_0\\ \boldsymbol{v}_0\end{pmatrix}$ <br> $=\begin{pmatrix}s_0s_1-\boldsymbol{v}_0\cdot\boldsymbol{v}_1\\ s_0\boldsymbol{v}_1+s_1\boldsymbol{v}_0+\boldsymbol{v}_0\times\boldsymbol{v}_1\end{pmatrix}$ |
| 7 | Rotation by unit quaternion | $\begin{pmatrix}0\\ \boldsymbol{p}'\end{pmatrix}=\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}\otimes\begin{pmatrix}0\\ \boldsymbol{p}\end{pmatrix}\otimes\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}^{-1}=\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}\otimes\begin{pmatrix}0\\ \boldsymbol{p}\end{pmatrix}\otimes\begin{pmatrix}s\\ -\boldsymbol{v}\end{pmatrix}$ <br> $\boldsymbol{p}'=\boldsymbol{p}+2s\left(\boldsymbol{v}\times\boldsymbol{p}\right)+2\left(\boldsymbol{v}\times\left(\boldsymbol{v}\times\boldsymbol{p}\right)\right)$ <br> $\boldsymbol{p}=\boldsymbol{p}'-2s\left(\boldsymbol{v}\times\boldsymbol{p}'\right)+2\left(\boldsymbol{v}\times\left(\boldsymbol{v}\times\boldsymbol{p}'\right)\right)$ |
| 8 | Rotation Matrix | $\boldsymbol{R}=\boldsymbol{1}+2s[\boldsymbol{v}\times]+2[\boldsymbol{v}\times][\boldsymbol{v}\times]$ <br> $\boldsymbol{R}^{\intercal}=\boldsymbol{1}-2s[\boldsymbol{v}\times]+2[\boldsymbol{v}\times][\boldsymbol{v}\times]$ |
| 9 | Quaternion $q=\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}$ from rotation matrix $\boldsymbol{R}$ | $s=\frac{1}{2}\sqrt{\frac{\left(\boldsymbol{R}_{32}-\boldsymbol{R}_{23}\right)^2+\left(\boldsymbol{R}_{13}-\boldsymbol{R}_{31}\right)^2+\left(\boldsymbol{R}_{21}-\boldsymbol{R}_{12}\right)^2}{3-\boldsymbol{R}_{11}-\boldsymbol{R}_{22}-\boldsymbol{R}_{33}}}$ <br> $\boldsymbol{v}=\frac{1}{4s}\begin{pmatrix}\boldsymbol{R}_{32}-\boldsymbol{R}_{23}\\ \boldsymbol{R}_{13}-\boldsymbol{R}_{31}\\ \boldsymbol{R}_{21}-\boldsymbol{R}_{12}\end{pmatrix}$ |
| 10 | Quaternion derivative (with $\boldsymbol{\omega}$ frame velocity) | $\dot{q}=\frac{1}{2}\omega\, q$ <br> $\begin{pmatrix}\dot{s}\\ \dot{\boldsymbol{v}}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\ \boldsymbol{\omega}\end{pmatrix}\otimes\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}=\frac{1}{2}\begin{vmatrix}0 & -\boldsymbol{\omega}^{\intercal}\\ \boldsymbol{\omega} & \boldsymbol{\omega}\times\end{vmatrix}\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\boldsymbol{\omega}^{\intercal}\boldsymbol{v}\\ s\boldsymbol{\omega}+\boldsymbol{\omega}\times\boldsymbol{v}\end{pmatrix}$ |
| 11 | Frame motion $\boldsymbol{\omega}$ from quaternion derivative | $\begin{pmatrix}0\\ \boldsymbol{\omega}\end{pmatrix}=2\dot{q}q^{-1}=2\begin{pmatrix}\dot{s}\\ \dot{\boldsymbol{v}}\end{pmatrix}\otimes\begin{pmatrix}s\\ -\boldsymbol{v}\end{pmatrix}=2\begin{bmatrix}s & \boldsymbol{v}^{\intercal}\\ -\boldsymbol{v} & s+\boldsymbol{v}\times\end{bmatrix}\begin{pmatrix}\dot{s}\\ \dot{\boldsymbol{v}}\end{pmatrix}=2\begin{pmatrix}s\dot{s}+\boldsymbol{v}^{\intercal}\dot{\boldsymbol{v}}\\ s\dot{\boldsymbol{v}}-\boldsymbol{v}\dot{s}+\boldsymbol{v}\times\dot{\boldsymbol{v}}\end{pmatrix}$ |
| 12 | Quaternion step (with $\boldsymbol{\omega}$ frame velocity and $h$ time step) | $q_{next}=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \hat{\boldsymbol{z}}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\otimes q$ $\qquad\hat{\boldsymbol{z}}=\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$ <br> $q_{next}=\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{z}}^{\intercal}\\ \sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{z}} & \cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)[\hat{\boldsymbol{z}}\times]\end{bmatrix}\begin{pmatrix}s\\ \boldsymbol{v}\end{pmatrix}$ $\qquad\theta=h\|\boldsymbol{\omega}\|$ |
| 13 | Quaternion Powers of $q=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \hat{\boldsymbol{z}}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}$ | $q^t=\|q\|^t\begin{pmatrix}\cos\left(t\frac{\theta}{2}\right)\\ \hat{\boldsymbol{z}}\sin\left(t\frac{\theta}{2}\right)\end{pmatrix}$ $\quad\exp(q)=\exp(s)\begin{pmatrix}\cos\left(\|\boldsymbol{v}\|\right)\\ \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\sin\left(\|\boldsymbol{v}\|\right)\end{pmatrix}=\exp\left(\cos\frac{\theta}{2}\right)\begin{pmatrix}\cos\left(\sin\frac{\theta}{2}\right)\\ \hat{z}\sin\left(\sin\frac{\theta}{2}\right)\end{pmatrix}$ |
| 14 | Interpolation with $\lambda=0\ldots1$ and integral with $\boldsymbol{\omega}$ | $q(\lambda)=q_1\left(q_1^{\star}q_2\right)^{\lambda}$ $\quad q(t)=\exp\left(\frac{t}{2}\begin{vmatrix}0 & -\boldsymbol{\omega}^{\intercal}\\ \boldsymbol{\omega} & \boldsymbol{\omega}\times\end{vmatrix}\right)q_0=\left(\cos\left(\omega\frac{t}{2}\right)+\frac{1}{\omega}\sin\left(\omega\frac{t}{2}\right)\begin{vmatrix}0 & -\boldsymbol{\omega}^{\intercal}\\ \boldsymbol{\omega} & \boldsymbol{\omega}\times\end{vmatrix}\right)q_0$ |

## 2 Quaternions (Vector-Scalar)

| Step | Description | Expression |
|---|---|---|
| 1 | Composition | general: $q = \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix}$ rotation: $q = \begin{pmatrix} \hat{\boldsymbol{z}} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$ identity: $q = \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$ exponential: $q = \exp\left(\frac{\theta}{2}\hat{z}\right)$ |
| 2 | Magnitude | general: $\|q\| = \sqrt{q \otimes q^\star} = \sqrt{s^2 + \|\boldsymbol{v}\|^2}$ rotation: $\|q\| = \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \|\boldsymbol{z}\|^2 \sin^2\left(\frac{\theta}{2}\right)} = 1$ |
| 3 | Transformations | conjugate: $q^\star = \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$ inverse: $q^{-1} = \frac{q^\star}{\|q\|^2} = \frac{1}{s^2+\|\boldsymbol{v}\|^2} \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$ unit inverse: $q^{-1} = q^\star = \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$ |
| 4 | Inner Product (noted with $\cdot$) | $q_0 \cdot q_1 = \begin{pmatrix} \boldsymbol{v}_0 \\ s_0 \end{pmatrix}^\intercal \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = s_0 s_1 + \boldsymbol{v}_0^\intercal \boldsymbol{v}_1$ |
| 5 | Cross Product (noted with $\times$) | $q_0 \times q_1 = \frac{1}{2}\left(q_0 q_1 - q_1^\star q_0^\star\right)$<br>$\begin{bmatrix} \boldsymbol{v}_0\times & \boldsymbol{0} \\ -\boldsymbol{0}^\intercal & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_0 \times \boldsymbol{v}_1 \\ 0 \end{pmatrix}$ |
| 6 | Multiplication (noted with $\otimes$) | $q_0 q_1 = \begin{pmatrix} \boldsymbol{v}_0 \\ s_0 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = \begin{bmatrix} s_0 + \boldsymbol{v}_0\times & \boldsymbol{v}_0 \\ -\boldsymbol{v}_0^\intercal & s_0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = \begin{bmatrix} s_1 - \boldsymbol{v}_1\times & \boldsymbol{v}_1 \\ -\boldsymbol{v}_1^\intercal & s_1 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_0 \\ s_0 \end{pmatrix}$<br>$= \begin{pmatrix} s_0 \boldsymbol{v}_1 + s_1 \boldsymbol{v}_0 + \boldsymbol{v}_0 \times \boldsymbol{v}_1 \\ s_0 s_1 - \boldsymbol{v}_0 \cdot \boldsymbol{v}_1 \end{pmatrix}$ |
| 7 | Rotation by unit quaternion | $\begin{pmatrix} \boldsymbol{p}' \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$<br>$\boldsymbol{p}' = \boldsymbol{p} + 2s\left(\boldsymbol{v} \times \boldsymbol{p}\right) + 2\left(\boldsymbol{v} \times \left(\boldsymbol{v} \times \boldsymbol{p}\right)\right)$ |
| 8 | Rotation Matrix | $\boldsymbol{R} = \boldsymbol{1} + 2s[\boldsymbol{v}\times] + 2[\boldsymbol{v}\times][\boldsymbol{v}\times] \quad \boldsymbol{R}^\intercal = \boldsymbol{1} - 2s[\boldsymbol{v}\times] + 2[\boldsymbol{v}\times][\boldsymbol{v}\times]$ |
| 9 | Quaternion $q = \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix}$ from rotation matrix $\boldsymbol{R}$ | $s = \frac{1}{2}\sqrt{\frac{\left(\boldsymbol{R}_{32} - \boldsymbol{R}_{23}\right)^2 + \left(\boldsymbol{R}_{13} - \boldsymbol{R}_{31}\right)^2 + \left(\boldsymbol{R}_{21} - \boldsymbol{R}_{12}\right)^2}{3 - \boldsymbol{R}_{11} - \boldsymbol{R}_{22} - \boldsymbol{R}_{33}}}$<br>$\boldsymbol{v} = \frac{1}{4s}\begin{pmatrix} \boldsymbol{R}_{32} - \boldsymbol{R}_{23} \\ \boldsymbol{R}_{13} - \boldsymbol{R}_{31} \\ \boldsymbol{R}_{21} - \boldsymbol{R}_{12} \end{pmatrix}$ |
| 10 | Quaternion derivative (with $\boldsymbol{\omega}$ frame velocity) | $\dot{q} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\omega} \\ 0 \end{pmatrix} q$<br>$\begin{pmatrix} \dot{\boldsymbol{v}} \\ \dot{s} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\omega} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} = \frac{1}{2}\begin{vmatrix} \boldsymbol{\omega}\times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\intercal & 0 \end{vmatrix} \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} = \frac{1}{2}\begin{pmatrix} s\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{v} \\ -\boldsymbol{\omega}^\intercal \boldsymbol{v} \end{pmatrix}$ |
| 11 | Frame motion $\boldsymbol{\omega}$ from quaternion derivative | $\begin{pmatrix} \boldsymbol{\omega} \\ 0 \end{pmatrix} = 2\dot{q}q^{-1} = 2\begin{pmatrix} \dot{\boldsymbol{v}} \\ \dot{s} \end{pmatrix} \otimes \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix} = 2\begin{bmatrix} s + \boldsymbol{v}\times & -\boldsymbol{v} \\ \boldsymbol{v}^\intercal & s \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{v}} \\ \dot{s} \end{pmatrix} = 2\begin{pmatrix} s\dot{\boldsymbol{v}} - \boldsymbol{v}\dot{s} + \boldsymbol{v} \times \dot{\boldsymbol{v}} \\ s\dot{s} + \boldsymbol{v}^\intercal \dot{\boldsymbol{v}} \end{pmatrix}$ |
| 12 | Quaternion step (with $\boldsymbol{\omega}$ frame velocity and $h$ time step) | $q_{next} = \begin{pmatrix} \hat{\boldsymbol{z}} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \otimes q \qquad \hat{\boldsymbol{z}} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$<br>$q_{next} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)[\hat{\boldsymbol{z}}\times] & \sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{z}} \\ -\sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{z}}^\intercal & \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} \qquad \theta = h\|\boldsymbol{\omega}\|$ |
| 13 | Quaternion Powers of $q = \begin{pmatrix} \hat{\boldsymbol{z}} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$ | $q^t = \|q\|^t \begin{pmatrix} \hat{\boldsymbol{z}} \sin\left(t\frac{\theta}{2}\right) \\ \cos\left(t\frac{\theta}{2}\right) \end{pmatrix} \quad \exp(q) = \exp(s)\begin{pmatrix} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\sin\left(\|\boldsymbol{v}\|\right) \\ \cos\left(\|\boldsymbol{v}\|\right) \end{pmatrix} = \exp\left(\cos\frac{\theta}{2}\right)\begin{pmatrix} \hat{z}\sin\left(\sin\frac{\theta}{2}\right) \\ \cos\left(\sin\frac{\theta}{2}\right) \end{pmatrix}$ |
| 14 | Interpolation with $\lambda = 0 \ldots 1$ and integral with $\boldsymbol{\omega}$ | $q(\lambda) = q_1\left(q_1^\star q_2\right)^\lambda \quad q(t) = \exp\left(\frac{t}{2}\begin{vmatrix} \boldsymbol{\omega}\times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\intercal & 0 \end{vmatrix}\right) q_0 = \left(\cos\left(\omega\frac{t}{2}\right) + \frac{1}{\omega}\sin\left(\omega\frac{t}{2}\right)\begin{vmatrix} \boldsymbol{\omega}\times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\intercal & 0 \end{vmatrix}\right) q_0$ |