Kinematics Dynamics 1

## 1 Quaternions (Scalar-Vector)

Step	Description	Expression
1	Composition	general: $q = \begin{pmatrix} s \\ v \end{pmatrix}$ rotation: $q = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \hat{z}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$ identity: $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exponential: $q = \exp\left(\frac{\theta}{2}\hat{z}\right)$
2	Magnitude	general: $  q   = \sqrt{q \otimes q^*} = \sqrt{s^2 +   v  ^2}$ rotation: $  q   = \sqrt{\cos^2(\frac{\theta}{2}) +   z  ^2 \sin^2(\frac{\theta}{2})} = 1$
3	Transformations	conjugate: $q^* = \begin{pmatrix} s \\ -\boldsymbol{v} \end{pmatrix}$ inverse: $q^{-1} = \frac{q^*}{\ q\ ^2} = \frac{1}{s^2 + \ \boldsymbol{v}\ ^2} \begin{pmatrix} s \\ -\boldsymbol{v} \end{pmatrix}$ unit inverse: $q^{-1} = q^* = \begin{pmatrix} s \\ -\boldsymbol{v} \end{pmatrix}$
4	Inner Product (noted with $\cdot$ )	$q_0 \cdot q_1 = egin{pmatrix} s_0 \ v_0 \end{pmatrix}^{\intercal} egin{pmatrix} s_1 \ v_1 \end{pmatrix} = s_0 s_1 + oldsymbol{v}_0^{\intercal} oldsymbol{v}_1$
5	Cross Product (noted with ×)	$q_0 \times q_1 = \frac{1}{2} \left( q_0 q_1 - q_1^{\star} q_0^{\star} \right)$ $\begin{bmatrix} 0 & -0^{T} \\ 0 & \mathbf{v}_0 \times \end{bmatrix} \begin{pmatrix} s_1 \\ \mathbf{v}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_0 \times \mathbf{v}_1 \end{pmatrix}$ $q_0 q_1 = \begin{pmatrix} s_0 \\ \mathbf{v}_0 \end{pmatrix} \otimes \begin{pmatrix} s_1 \\ \mathbf{v}_1 \end{pmatrix} = \begin{bmatrix} s_0 & -\mathbf{v}_0^{T} \\ \mathbf{v}_0 & s_0 + \mathbf{v}_0 \times \end{bmatrix} \begin{pmatrix} s_1 \\ \mathbf{v}_1 \end{pmatrix} = \begin{bmatrix} s_1 & -\mathbf{v}_1^{T} \\ \mathbf{v}_1 & s_1 - \mathbf{v}_1 \times \end{bmatrix} \begin{pmatrix} s_0 \\ \mathbf{v}_0 \end{pmatrix}$
6	$\begin{array}{c} \text{Multiplication} \\ \text{(noted with } \otimes \text{)} \end{array}$	$=egin{pmatrix} s_0s_1-oldsymbol{v}_0\cdotoldsymbol{v}_1\ s_0oldsymbol{v}_1+s_1oldsymbol{v}_0+oldsymbol{v}_0 imesoldsymbol{v}_1 \end{pmatrix}$
7	Rotation by unit quaternion	$egin{aligned} egin{pmatrix} egin{pmatrix} egin{pmatrix} egin{pmatrix} s \ egin{pm$
8	Rotation Matrix	$egin{aligned} oldsymbol{R} = oldsymbol{1} + 2s[oldsymbol{v} imes] + 2[oldsymbol{v} imes][oldsymbol{v} imes] \\ oldsymbol{R}^\intercal = oldsymbol{1} - 2s[oldsymbol{v} imes] + 2[oldsymbol{v} imes][oldsymbol{v} imes] \end{aligned}$
		$s = rac{1}{2}\sqrt{rac{{{{{\left( {{olde R}_{32}} - {{olde R}_{23}}}  ight)}^2} + {{{\left( {{olde R}_{13}} - {{olde R}_{31}}}  ight)}^2} + {{{\left( {{olde R}_{21}} - {{olde R}_{12}}}  ight)}^2}}{3 - {{olde R}_{11}} - {{olde R}_{22}} - {{olde R}_{22}}}}$
9	Quaternion	7 - 1011 1022 1033
	$q = \begin{pmatrix} s \\ v \end{pmatrix}$ from	$v=rac{1}{4s}egin{pmatrix} R_{32}-R_{23}\ R_{13}-R_{31}\ R_{23}-R_{23} \end{pmatrix}$
	rotation matrix $R$	$(\mathbf{r}_{21} - \mathbf{r}_{12})$
10	Quaternion derivative (with $\omega$ frame velocity)	$\dot{q} = \frac{1}{2}\omega q$ $\begin{pmatrix} \dot{s} \\ \dot{v} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ \omega \end{pmatrix} \otimes \begin{pmatrix} s \\ v \end{pmatrix} = \frac{1}{2}\begin{vmatrix} 0 & -\omega^{T} \\ \omega & \omega \times \end{vmatrix} \begin{pmatrix} s \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\omega^{T} v \\ s\omega + \omega \times v \end{pmatrix}$
11	Frame motion $\omega$ from quaternion derivative	$\begin{pmatrix} 0 \\ \omega \end{pmatrix} = 2\dot{q}q^{-1} = 2\begin{pmatrix} \dot{s} \\ \dot{\boldsymbol{v}} \end{pmatrix} \otimes \begin{pmatrix} s \\ -\boldsymbol{v} \end{pmatrix} = 2\begin{bmatrix} s & \boldsymbol{v}^{T} \\ -\boldsymbol{v} & s + \boldsymbol{v} \times \end{bmatrix} \begin{pmatrix} \dot{s} \\ \dot{\boldsymbol{v}} \end{pmatrix} = 2\begin{pmatrix} s\dot{s} + \boldsymbol{v}^{T}\dot{\boldsymbol{v}} \\ s\dot{\boldsymbol{v}} - \boldsymbol{v}\dot{s} + \boldsymbol{v} \times \dot{\boldsymbol{v}} \end{pmatrix}$
12	Quaternion step (with $\omega$ frame velocity and $h$ time step)	$q_{next} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \hat{z}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \otimes q \qquad \qquad \hat{z} = \frac{\omega}{\ \omega\ }$ $q_{next} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\hat{z}^{T} \\ \sin\left(\frac{\theta}{2}\right)\hat{z} & \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)[\hat{z}\times] \end{bmatrix} \begin{pmatrix} s \\ v \end{pmatrix}  \theta = h\ \omega\ $
13	Quaternion Powers of $q = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \hat{z}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$	$q^{t} =  q ^{t} \begin{pmatrix} \cos\left(t\frac{\theta}{2}\right) \\ \hat{z}\sin\left(t\frac{\theta}{2}\right) \end{pmatrix}  \exp(q) = \exp(s) \begin{pmatrix} \cos\left(\ \boldsymbol{v}\ \right) \\ \frac{\boldsymbol{v}}{\ \boldsymbol{v}\ }\sin\left(\ \boldsymbol{v}\ \right) \end{pmatrix} = \exp\left(\cos\frac{\theta}{2}\right) \begin{pmatrix} \cos\left(\sin\frac{\theta}{2}\right) \\ \hat{z}\sin\left(\sin\frac{\theta}{2}\right) \end{pmatrix}$
14	Interpolation with $\lambda = 0 \dots 1$ and integral with $\boldsymbol{\omega}$	$q(\lambda) = q_1 \left( q_1^{\star} q_2 \right)^{\lambda}  q(t) = \exp\left( \frac{t}{2} \begin{vmatrix} 0 & -\boldsymbol{\omega}^{T} \\ \boldsymbol{\omega} & \boldsymbol{\omega} \times \end{vmatrix} \right) q_0 = \left( \cos\left( \boldsymbol{\omega} \frac{t}{2} \right) + \frac{1}{\omega} \sin\left( \boldsymbol{\omega} \frac{t}{2} \right) \begin{vmatrix} 0 & -\boldsymbol{\omega}^{T} \\ \boldsymbol{\omega} & \boldsymbol{\omega} \times \end{vmatrix} \right) q_0$

Kinematics Dynamics 2

## 2 Quaternions (Vector-Scalar)

Step	Description	Expression
1	Composition	general: $q = \begin{pmatrix} v \\ s \end{pmatrix}$ rotation: $q = \begin{pmatrix} \hat{z} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$ identity: $q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ exponential: $q = \exp\left(\frac{\theta}{2}\hat{z}\right)$
2	Magnitude	general: $  q   = \sqrt{q \otimes q^*} = \sqrt{s^2 +   v  ^2}$ rotation: $  q   = \sqrt{\cos^2\left(\frac{\theta}{2}\right) +   z  ^2 \sin^2\left(\frac{\theta}{2}\right)} = 1$
3	Transformations	conjugate: $q^* = \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$ inverse: $q^{-1} = \frac{q^*}{\ q\ ^2} = \frac{1}{s^2 + \ \boldsymbol{v}\ ^2} \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$ unit inverse: $q^{-1} = q^* = \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix}$
4	Inner Product (noted with $\cdot$ )	$q_0 \cdot q_1 = inom{v_0}{s_0}^{\intercal} inom{v_1}{s_1} = s_0 s_1 + oldsymbol{v}_0^{\intercal} oldsymbol{v}_1$
5	Cross Product (noted with ×)	$q_0 \times q_1 = \frac{1}{2} \left( q_0 q_1 - q_1^{\star} q_0^{\star} \right)$ $\begin{bmatrix} \boldsymbol{v}_0 \times & 0 \\ -0^{T} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_0 \times \boldsymbol{v}_1 \\ 0 \end{pmatrix}$ $q_0 q_1 = \begin{pmatrix} \boldsymbol{v}_0 \\ s_0 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = \begin{bmatrix} s_0 + \boldsymbol{v}_0 \times & \boldsymbol{v}_0 \\ -\boldsymbol{v}_0^{T} & s_0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ s_1 \end{pmatrix} = \begin{bmatrix} s_1 - \boldsymbol{v}_1 \times & \boldsymbol{v}_1 \\ -\boldsymbol{v}_1^{T} & s_1 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_0 \\ s_0 \end{pmatrix}$
6	$\begin{array}{c} \text{Multiplication} \\ \text{(noted with } \otimes \text{)} \end{array}$	$=egin{pmatrix} s_0oldsymbol{v}_1+s_1oldsymbol{v}_0+oldsymbol{v}_0 imesoldsymbol{v}_1\ s_0s_1-oldsymbol{v}_0\cdotoldsymbol{v}_1 \end{pmatrix}$
7	Rotation by unit	$egin{pmatrix} egin{pmatrix} oldsymbol{p}' \\ 0 \end{pmatrix} = egin{pmatrix} oldsymbol{v} \\ s \end{pmatrix} \otimes egin{pmatrix} oldsymbol{p} \\ 0 \end{pmatrix} \otimes egin{pmatrix} oldsymbol{v} \\ s \end{pmatrix} = oldsymbol{p} + 2s \left( oldsymbol{v}  imes oldsymbol{p} \right) + 2 \left( oldsymbol{v}  imes \left( oldsymbol{v}  imes oldsymbol{p} \right) \right)$
8	Rotation Matrix	$R = 1 + 2s[\underline{v}\times] + 2[\underline{v}\times][\underline{v}\times] R^{T} = 1 - 2s[\underline{v}\times] + 2[\underline{v}\times][\underline{v}\times]$
9	Quaternion	$s = \frac{1}{2} \sqrt{\frac{(\boldsymbol{R}_{32} - \boldsymbol{R}_{23})^2 + (\boldsymbol{R}_{13} - \boldsymbol{R}_{31})^2 + (\boldsymbol{R}_{21} - \boldsymbol{R}_{12})^2}{3 - \boldsymbol{R}_{11} - \boldsymbol{R}_{22} - \boldsymbol{R}_{33}}}$
	$q = \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} \text{ from}$ rotation matrix $\boldsymbol{R}$	$oldsymbol{v} = rac{1}{4s}egin{pmatrix} oldsymbol{R}_{32} - oldsymbol{R}_{23} \ oldsymbol{R}_{13} - oldsymbol{R}_{31} \ oldsymbol{R}_{21} - oldsymbol{R}_{12} \end{pmatrix}$
		$\dot{q} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\omega} \\ 0 \end{pmatrix} q$
10	Quaternion derivative (with $\omega$ frame velocity)	$ \begin{pmatrix} \dot{\boldsymbol{v}} \\ \dot{s} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\omega} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} = \frac{1}{2} \begin{vmatrix} \boldsymbol{\omega} \times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{vmatrix} \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix} = \frac{1}{2} \begin{pmatrix} s\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{v} \\ -\boldsymbol{\omega}^T \boldsymbol{v} \end{pmatrix} $
11	Frame motion $\omega$ from quaternion derivative	$\begin{pmatrix} \boldsymbol{\omega} \\ 0 \end{pmatrix} = 2\dot{q}q^{-1} = 2\begin{pmatrix} \dot{\boldsymbol{v}} \\ \dot{s} \end{pmatrix} \otimes \begin{pmatrix} -\boldsymbol{v} \\ s \end{pmatrix} = 2\begin{bmatrix} s + \boldsymbol{v} \times & -\boldsymbol{v} \\ \boldsymbol{v}^{T} & s \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{v}} \\ \dot{s} \end{pmatrix} = 2\begin{pmatrix} s\dot{\boldsymbol{v}} - \boldsymbol{v}\dot{s} + \boldsymbol{v} \times \dot{\boldsymbol{v}} \\ s\dot{s} + \boldsymbol{v}^{T}\dot{\boldsymbol{v}} \end{pmatrix}$
		$q_{next} = egin{pmatrix} \hat{z} \sin\left(rac{ heta}{2} ight) \ \cos\left(rac{ heta}{2} ight) \end{pmatrix} \otimes q \qquad \qquad \hat{z} = rac{oldsymbol{\omega}}{\ oldsymbol{\omega}\ }$
12	Quaternion step (with $\omega$ frame velocity and $h$ time step)	$q_{next} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \left[\hat{\boldsymbol{z}}\times\right] & \sin\left(\frac{\theta}{2}\right) \hat{\boldsymbol{z}} \\ -\sin\left(\frac{\theta}{2}\right) \hat{\boldsymbol{z}}^{T} & \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{pmatrix} \boldsymbol{v} \\ s \end{pmatrix}  \theta = h\ \boldsymbol{\omega}\ $
13	Quaternion Powers of $q = \begin{pmatrix} \hat{z} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$	$q^{t} =  q ^{t} \begin{pmatrix} \hat{z} \sin\left(t\frac{\theta}{2}\right) \\ \cos\left(t\frac{\theta}{2}\right) \end{pmatrix}  \exp(q) = \exp(s) \begin{pmatrix} \frac{\boldsymbol{v}}{\ \boldsymbol{v}\ } \sin\left(\ \boldsymbol{v}\ \right) \\ \cos\left(\ \boldsymbol{v}\ \right) \end{pmatrix} = \exp\left(\cos\frac{\theta}{2}\right) \begin{pmatrix} \hat{z} \sin\left(\sin\frac{\theta}{2}\right) \\ \cos\left(\sin\frac{\theta}{2}\right) \end{pmatrix}$
14	Interpolation with $\lambda = 0 \dots 1$ and integral with $\omega$	$q(\lambda) = q_1 \left( q_1^{\star} q_2 \right)^{\lambda}  q(t) = \exp \left( \frac{t}{2} \begin{vmatrix} \boldsymbol{\omega} \times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{T} & 0 \end{vmatrix} \right) q_0 = \left( \cos \left( \boldsymbol{\omega} \frac{t}{2} \right) + \frac{1}{\omega} \sin \left( \boldsymbol{\omega} \frac{t}{2} \right) \begin{vmatrix} \boldsymbol{\omega} \times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{T} & 0 \end{vmatrix} \right) q_0$