Chap 4 The Equation of Continuity of Volume

4.0 The Total Derivative (*d*/*dt*)

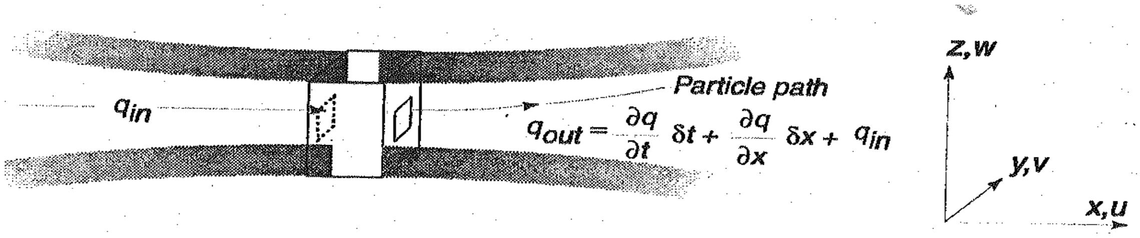

Fig. 4.0 Sketch of flow used for deriving the total derivative.

Consider the simple example of acceleration of flow in a small box of fluid (Fig. 4.0). The resulting equation is called the ***total derivative***. It relates the acceleration of a particle ***du/dt*** to derivatives of the velocity field at a fixed point in the fluid.

In fluid dynamics, a quantity $q = q(x, y, z, t)$. Initially if we suppose that there is motion only in the x-direction and variations only in the x-direction so that at time t a parcel is at point x with property $q(x)$ while at a slightly later time $(t+\delta t)$ it is at $(x+\delta x)$ with property $q(x+\delta x)$. Now using Taylor's series expansion we can write

$$q(x+\delta x) = q(x) + (\partial q/\partial x)\delta x + \mathrm{O}(\delta x)^2$$

These latter terms can be neglected (since in the limit as $\delta x \rightarrow 0$). Thus, the rate of change following the motion is

$$\frac{\text{property change}}{\text{time change}} = \frac{(\partial q/\partial x)\delta x}{\delta t} = \frac{\partial q}{\partial x}\frac{\delta x}{\delta t} = u\left(\frac{\partial q}{\partial x}\right).$$

In the more general case when there are also v and w components of velocity and variations in all three-component directions, and we include changes with time $(\partial q/\partial t)$ at the point itself in the fluid, so the total derivative is given by

$$\underbrace{\frac{dq}{dt}}_{\text{Total derivative}} = \underbrace{\frac{\partial q}{\partial t}}_{\text{Local term}} + \underbrace{\left(u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}\right)}_{\text{Advective terms}}$$

Therefore,

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}} \quad \text{or} \quad \boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\nabla(\ )}$$

4.1 The concept of continuity of volume

An oceanographer is studying a long, narrow coastal inlet which has a river at the inland end (Fig. 4.1). If we consider the horizontal flow only, there appears to be a lack of continuity of volume ($u_4 > u_3 > u_2 > u_1$). However, the inlet is observed not to be emptying and to produce a balance there must be upward flow (w) from the lower to the upper layer across CDD′C′ (Fig. 4.1b) so that we have

$$u_3 \times \text{area ABCD} + w \times \text{area CDD}'\text{C}' = u_4 \times \text{area A}'\text{B}'\text{C}'\text{D}'$$

expressing continuity of volume for the upper layer.

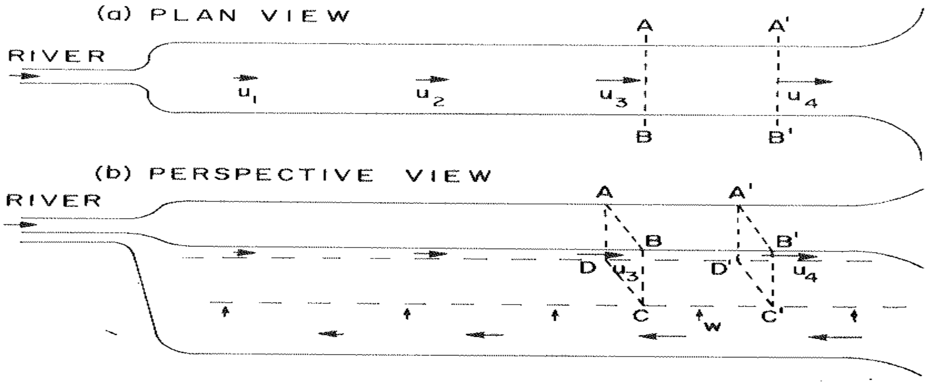

Fig. 4.1 Continuity of volume for an inlet: (a) plan view of upper layer, outflow (u) increasing to seaward; (b) perspective view – outflow in the upper layer, inflow in the lower layer and upward flow (w) from lower to upper layers.

4.2 The derivation of the equation of continuity of volume

We will now consider the conservation of mass in order to derive a general equation for applying continuity of volume. In Fig. 4.2 is represented a rectangular volume fixed in space with sides of lengths δx, δy and δz in a moving fluid.

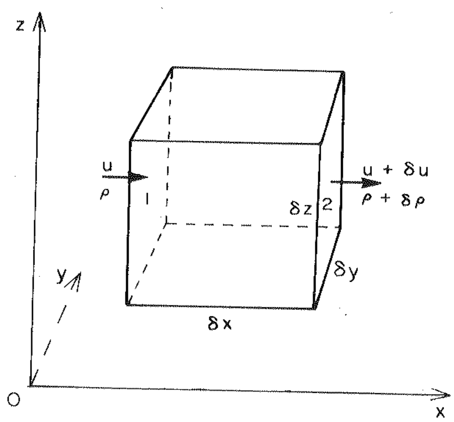

Fig. 4.2 Continuity of volume – components of flow in the x-direction.

The mass flow into the volume = $\rho\, u\delta y\delta z$ (mass/unit time)

The mass flow out of the volume = $\left(\rho + \dfrac{\partial \rho}{\partial x}\delta x\right)\left(u + \dfrac{\partial u}{\partial x}\delta x\right)\delta y\delta z$

$$= \left(\rho u + \rho\frac{\partial u}{\partial x}\delta x + u\frac{\partial \rho}{\partial x}\delta x + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\delta^2 x\right)\delta y\delta z$$

so that the net flow out of the volume in the x-direction is the difference

$$\left[u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\delta x\right]\delta x\delta y\delta z = \left[\frac{\partial(\rho\, u)}{\partial x} + \mathrm{O}(\delta x)\right]\delta x\delta y\delta z$$

By taking δx, δy, $\delta z \to 0$, in three dimensions the

$$\text{total flow out} = \left[\frac{\partial(\rho\, u)}{\partial x} + \frac{\partial(\rho\, v)}{\partial y} + \frac{\partial(\rho\, w)}{\partial z}\right]\delta x\delta y\delta z$$

The change of mass inside the volume is $(\partial\rho/\partial t)\delta x\delta y\delta z$. If mass is to be conserved, the sum of the effects must be zero, i.e.

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, u)}{\partial x} + \frac{\partial(\rho\, v)}{\partial y} + \frac{\partial(\rho\, w)}{\partial z} = 0} \tag{4.1}$$

This is the continuity equation for **compressible** flow.
The rate of change of density with the moving fluid is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \tag{4.2}$$

Combining (4.1) and (4.2) we have

$$\frac{1}{\rho}\frac{d\rho}{dt} + \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \tag{4.3}$$

For an incompressible fluid, then $(1/\rho)(\mathrm{d}\rho/\mathrm{d}t)=0$, and the equation of continuity becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.4}$$

Compressibility

Define compressibility = $\beta = -\dfrac{1}{V}\dfrac{dV}{dp} = -\dfrac{1}{V}\dfrac{dV/dt}{dp/dt}$

If $\beta = 0$, then $\dfrac{1}{V}\dfrac{dV}{dt} = 0$. Also, $\rho = M/V$

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{V}{M}\frac{d}{dt}\left(\frac{M}{V}\right)$$

But M = constant, so

$$\frac{1}{\rho}\frac{d\rho}{dt} = V\frac{d}{dt}\left(\frac{1}{V}\right) = -\frac{1}{V}\frac{dV}{dt}$$

$\therefore\ \beta = 0$ requires $\dfrac{1}{V}\dfrac{dV}{dt} = 0$ and $\dfrac{1}{\rho}\dfrac{d\rho}{dt} = 0.$

4.3 An application of the equation of continuity

Consider the simple case of calculating w from measurements of u and v

$$\frac{\partial w}{\partial z} = -\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$$

This is useful for calculating vertical velocities over large area using measurements of horizontal velocity. $w \ll u, v$ and it is difficult to measure directly. Hence it is usually estimated from measurements of other variables which are easier to measure, such as E in Fig. 4.3.

$$\left.\frac{\partial u}{\partial x}\right|_{\mathrm{A}} = \frac{[-0.25-(-0.25)]\ \mathrm{m/s}}{5\times10^{5}\ \mathrm{m}} = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{\mathrm{B}} = \frac{(0.25-0.30)\ \mathrm{m/s}}{5\times10^{5}\ \mathrm{m}} = -10\times10^{-8}\ \mathrm{s}^{-1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{\mathrm{E}} = \frac{1}{2}\left[\left.\frac{\partial u}{\partial x}\right|_{\mathrm{A}} + \left.\frac{\partial u}{\partial x}\right|_{\mathrm{B}}\right] = -5\times10^{-8}\ \mathrm{s}^{-1}$$

$$\left.\frac{\partial v}{\partial y}\right|_{\mathrm{D}} = \frac{-0.01-0.05}{5.6\times10^{5}} = -11\times10^{-8}\ \mathrm{s}^{-1}$$

$$\left.\frac{\partial v}{\partial y}\right|_{\mathrm{C}} = \frac{0-0.03}{5.6\times10^{5}} = -5.3\times10^{-8}\ \mathrm{s}^{-1}$$

$$\left.\frac{\partial v}{\partial y}\right|_{\mathrm{E}} = \frac{-11-5.3}{2}\times10^{-8}\ \mathrm{s}^{-1}$$

$$\frac{\partial w}{\partial z} = -\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right] = -(-5-8.3)\times10^{-8}\ \mathrm{s}^{-1} = 13.3\times10^{-8}\ \mathrm{s}^{-1}$$

Since $w = 0$ at the surface, the vertical velocity, w_h, at depth h below the surface (i.e. at $z = -h$) is given by

$$w_{-h} = \int_0^{-h} dw = \int_0^{-h}\frac{\partial w}{\partial z}dz = -\int_0^{-h}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]dz$$

If u, v are independent of depth to a depth h = 50 m, then

$$w_{-50} = \int_0^{-50\mathrm{m}}(13.3\times10^{-8})dz = -6.7\times10^{-6}\ \mathrm{m\,s}^{-1} = 0.58\ \mathrm{m/day\ downward}$$

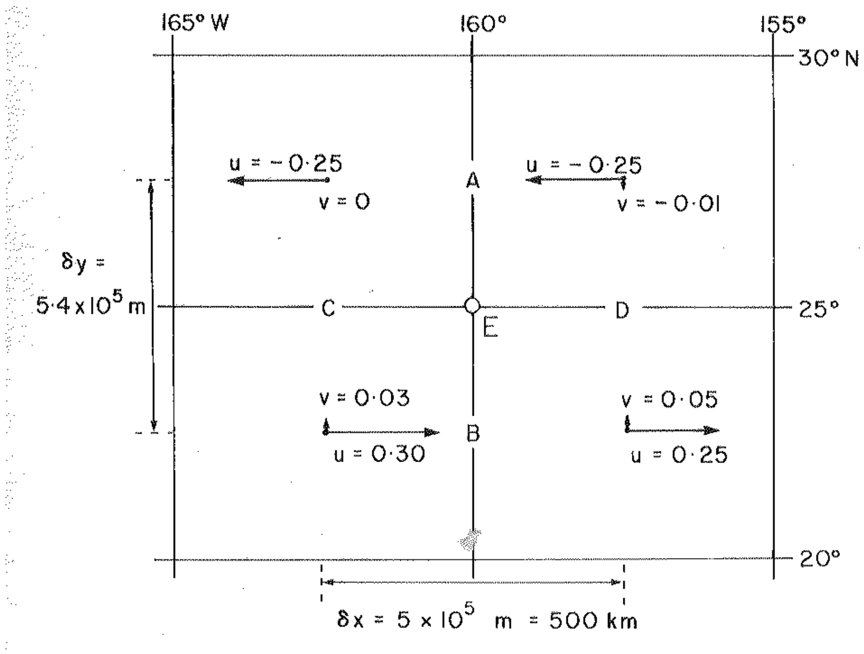

Fig. 4.3 Example of horizontal flows for calculation of vertical flow from continuity of volume (speeds in ms^{-1}).