Chap 8 Currents without friction: Geostrophic flow

8.1 Hydrostatic equilibrium

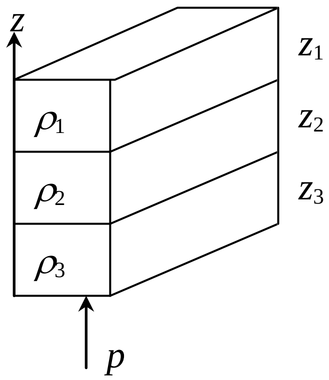
Assume that the flow is in steady state

$$\left.\begin{array}{l}(1)u=v=w=0,\text{stationary}\\[2ex](2)\dfrac{d\vec{V}}{dt}=0,\text{stationary}\\[2ex](3)\vec{F}=0\end{array}\right\}\quad\Rightarrow\quad\left\{\begin{array}{l}\alpha\dfrac{\partial p}{\partial x}=0\\\alpha\dfrac{\partial p}{\partial y}=0\\\alpha\dfrac{\partial p}{\partial z}=-g\end{array}\right. \tag{8.1}$$

The first two mean that the isobaric surfaces are horizontal, i.e. there is no pressure term to cause horizontal motion. The third can be written as

$$dp=-\rho\, g\, dz \tag{8.2}$$

which is the ***hydrostatic*** equation.



p = pressure, ρ = density

$$p=\rho_1 g z_1+\rho_2 g z_2+\rho_3 g z_3+\cdots$$

$$p=\sum_i \rho_i g z_i$$

or for a continuous density field

$$dp=\rho(z)gdz$$

$$\rho=g\int_{-z}^{0}\rho(z)dz$$

Usually, ρ is known only at a few measured points, so pressure is calculated from

$$p=\sum_i \rho_i g z_i$$

where ρ_i is the *in situ* value of density, and z_i is the thickness of the layer with density ρ_i.

8.2 Inertial motion

Assume that

$$\left.\begin{array}{l}(1)\partial p/\partial x=\partial p/\partial y=0\\(2)\vec{F}=0\\(3)w=0\end{array}\right\}\quad\Rightarrow\quad\left\{\begin{array}{l}\dfrac{du}{dt}=2\Omega\sin\phi\, v\\[2ex]\dfrac{dv}{dt}=-2\Omega\sin\phi\, u\end{array}\right.\tag{8.3}$$

$$\frac{d^2u}{dt^2}+f^2u=0 \quad\Rightarrow\quad u=A\sin ft+B\cos ft$$

According to the initial conditions, t = 0, u = 0

$$\therefore B=0\ ,\ u=A\sin ft \text{ and } v=A\cos ft$$

The equations have solutions

$$\begin{aligned}u&=V_H\sin(2\Omega\sin\phi\, t)\\v&=V_H\cos(2\Omega\sin\phi\, t)\end{aligned}\tag{8.4}$$

where $V_H^2=u^2+v^2$.

(1) These are the equations of motion for a body in the northern hemisphere traveling "clockwise" in a horizontal circle at constant linear speed V_H and angular speed $2\Omega\sin\phi$.

(2) If the radius of the circle is B, then $V_H^2/B=2\Omega\sin\phi V_H$, i.e. the centripetal acceleration V_H^2/B is provided by the Coriolis acceleration $2\Omega\sin\phi V_H$. Thus, $B=V_H/f$ (see Fig. 8.1).

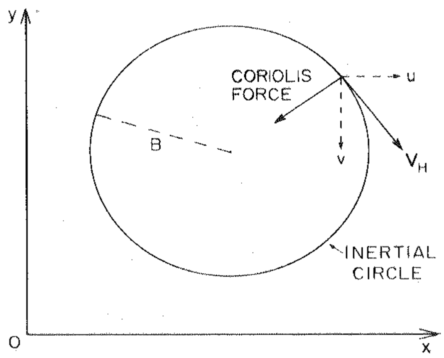


Fig. 8.1 Relationship of Coriolis force and velocity for inertial motion (northern hemisphere).

(3) Physically, such motion might be generated when a wind blows steadily in one direction for a time, causing the water to acquire a speed V_H, and then the wind stops and the motion continues without friction as a consequence of its "***inertia***", hence the term "***inertial motion***".

(4) The amplitudes vary depending on the strength of generating mechanisms and they decay due to friction when the generation stops.

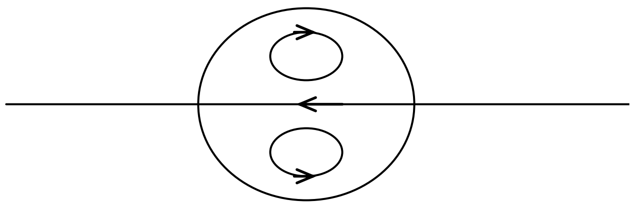
(5) The period of revolution is

$$T = \frac{2\pi}{f} = \frac{2\pi}{2\Omega \sin\phi} = \frac{1}{2}\frac{1\,\text{sidereal day}}{\sin\phi} = \frac{T_f}{2}$$

The quantity T_f = (1 sidereal day/sin ϕ) is called "one pendulum day" — it is the time required for the plane of vibration of a Foucault pendulum to rotate through 2π radians.

For example, the value of $0.5T_f$ is 11.97 hr at the pole, 16.93 hr at 45° latitude.

(6) The direction of rotation in the inertial circle is "clockwise" viewed from above in the northern hemisphere and "anticlockwise" in the southern hemisphere.



cum sole = anticyclonic (clockwise in the northern hemisphere)
contra solem = cyclonic (anticlockwise in the northern hemisphere)

(7) The term ***cyclonic*** comes from cyclones, a storm with low pressure as its center about which the winds are anticlockwise in the northern hemisphere. An ***anticyclonic*** system has high pressure at its center and winds circulate in the opposite direction (Fig. 8.2).

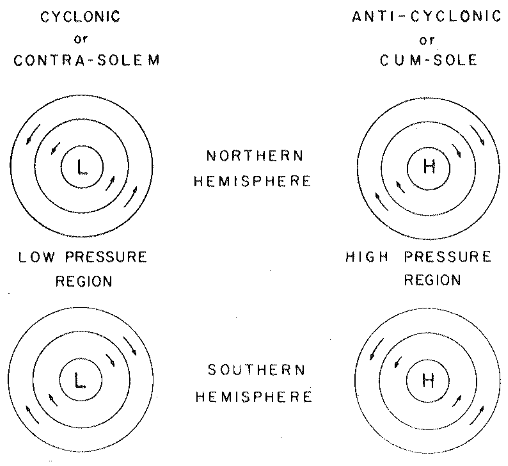


Fig. 8.2 Directions of rotation around low- and high-pressure regions in northern and southern hemispheres.

8.3 Geopotential

The quantity $dw = Mgdz$ is the amount of work done (= potential energy gained) in raising a mass M through a vertical distance dz against the force of gravity. We then define a quantity called ***geopotential*** (Φ) is given by

$$M d\Phi = dw = Mgdz \text{ (joules)}$$

or
$$\begin{aligned} d\Phi &= gdz \text{ (joules kg}^{-1} = \text{m}^2\text{s}^{-2}\text{) (potential energy change / unit mass)} \\ &= -\alpha\, dp \end{aligned}$$

Integrating from z_1 to z_2 we have

$$\int_1^2 d\Phi = \int_1^2 gdz = -\int_1^2 \alpha dp$$

Now writing $\alpha = \alpha_{35,0,p} + \delta$ we get

$$\begin{aligned} \Phi_2 - \Phi_1 = g(z_2 - z_1) &= -\int_1^2 \alpha_{35,0,p}\, dp - \int_1^2 \delta\, dp \\ &= -\Delta\Phi_{\text{std}} \qquad\;\; - \Delta\Phi \end{aligned} \tag{8.5}$$

The quantity $(\Phi_2 - \Phi_1)$ is called the ***geopotential distance*** between the levels z_2 and z_1 where the pressure will be p_2 and p_1.

(1) $\Delta\Phi_{\text{std}}$ is called "***standard*** geopotential distance" (a function of p only).

(2) $\Delta\Phi$ is called "geopotential ***anomaly***" (a function of S, T and p).

(3) $\Delta\Phi_{std} \sim 1000 \times \Delta\Phi$ (The unit is energy per unit mass J kg^{-1} or m^{2}s^{-2}).

(4) For g = 9.8 ms^{-2}, δz=1 m, then $d\Phi$ = 9.8 J Kg^{-1}. Thus, the "dynamic meter" 1 dyn m = 10 J Kg^{-1}. The geopotential distance (D_2–D_1) is then numerically almost equal to (z_2–z_1) in meters.

(5)

		SI units	Mixed units
at a geometrical depth in the sea		= + 100 m	+ 100 m
then	z_2	= – 100 m	– 100 m
the pressure will be about	p	= +1005kPa	+100.5 dbar

and the geopotential distance relative to the surface Φ_2–Φ_1 = –980 J kg^{-1}, D_2–D_1 =–98 dyn m.

8.31Geopotential surfaces and isobaric surfaces

(1) a geopotential surface — a surface to which the force of gravity is everywhere perpendicular. That is, the value of the geopotential must be the same everywhere on the surface, or called "level surface".

(2) an ***isobaric*** surface — is one which the pressure is everywhere the same.

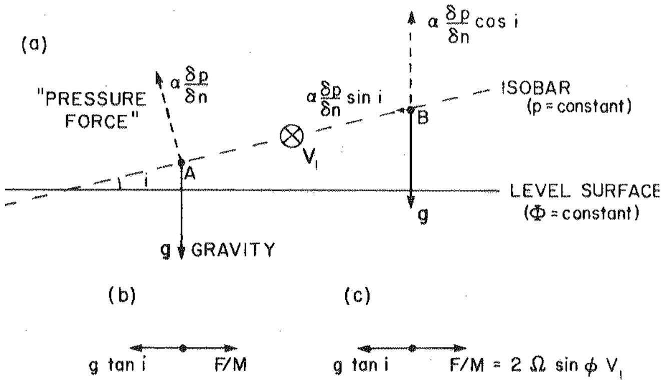


Fig. 8.3 Pressure terms in relation to isobaric and level surfaces for the northern hemisphere, $\vec{V}_1$ is into the paper.

There are two forces act on a water particle; pressure force and gravity. From Fig. 8.3 we can find that

(1) a vertical component $\alpha(\partial p/\partial n)\cos i$ which balances g.

(2) a horizontal component $\alpha(\partial p/\partial n)\sin i$ which is unbalanced and would cause accelerated motion to the left, i.e. the situation is not stable.

(3) the component to the left is

$$\alpha \frac{\partial p}{\partial n}\sin i = \left(\alpha \frac{\partial p}{\partial n}\cos i\right)\frac{\sin i}{\cos i} = g \tan i$$

To stop the acceleration to the left it is necessary to apply the right a force/unit mass equal to g tan i.

(4) So, one way would be to generate a Coriolis force by having the water move "into the paper" at speed V_1 so that $2\Omega\sin\phi\, V_1 = g \tan i$.

8.4 The geostrophic equation

The Coriolis force is sometimes called the "geostrophic" (=earth turned) force and the equation

$$2\Omega \sin\phi V_1 = g \tan i \qquad (8.6)$$

is one version of the ***geostrophic equation***, which expresses a balance between the pressure force and the Coriolis force.

(1) In principle, this geostrophic equation should permit us to determine the speed V_1 by measuring the slope i of the isobaric surface. In practice we

cannot do this because we cannot determine p directly with the necessary accuracy.

(2) In fact, if there are currents in the surface waters the sea surface will not be level because the geostrophic equation applies there, and motion gives rise to a Coriolis force which requires the water surface to be sloping so that the horizontal component of the pressure gradient can act to balance the Coriolis force.

(3) The slope are small, e.g. $2\Omega \sin \phi \cong 10^{-4}$ at 45^{o} latitude and for $V_1 = 1$ ms^{-1}, tan $i \cong 10^{-5}$, i.e. the surface rises by 1 m in 100 km, a distance typical of the width of a strong current such as the Gulf Stream.

(4) A technique for determining the absolute slope of the sea surface, which is receiving much attention, is to use radar or laser altimetry from satellites. Cheney and March (1981) demonstrated the estimation from Seasat satellite radar altimeter observations of the change of sea surface elevation across the Gulf Stream of 140 ± 35 cm or a slope of $(1.2 \pm 0.3) \times 10^{-5}$ for three months in 1978.

8.41 Why worry about the geostrophic equation?

–Direct measurement of ocean currents in sufficient quantity to be useful is technically difficult and expensive.

–The geostrophic method for calculating the current requires information on the distribution of density in the ocean. It is easier to obtain this information than it is to measure currents directly.

8.42 The geostrophic method for calculating relative velocities

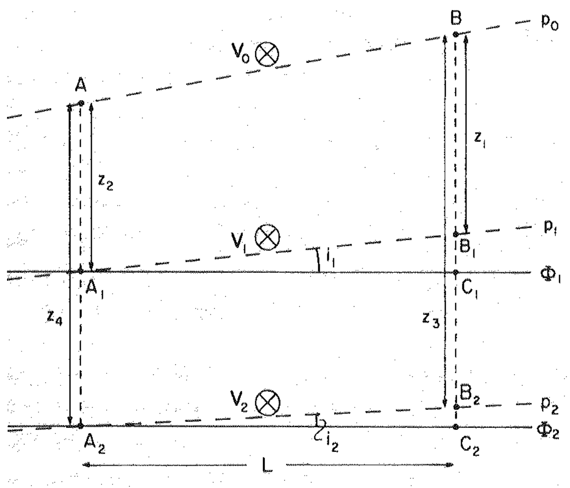


Fig. 8.4 For the derivation of the geostrophic equation.

In Fig. 8.4, A and B represent the positions where oceanographic stations have been taken so that the distribution of ρ or α is known along each vertical AA_1A_2 and BB_1B_2. The line AB represents the sea surface which is assumed not to be level. The two isobaric surfaces p_1 and p_2. Lines Φ_1 and Φ_2 represent two level surfaces. The geostrophic equations for the two levels are

$$2\Omega \sin\phi V_1 = g \tan i_1$$
$$2\Omega \sin\phi V_2 = g \tan i_2$$

Subtracting

$$\begin{aligned} 2\Omega \sin\phi(V_1 - V_2) &= g\,(\tan i_1 - \tan i_2) \\ &= g\left(\frac{B_1C_1}{A_1C_1} - \frac{B_2C_2}{A_2C_2}\right) \\ &= \frac{g}{L}(B_1B_2 - C_1C_2) \\ &= \frac{g}{L}(B_1B_2 - A_1A_2) \\ &= \frac{g}{L}\left[(z_1 - z_3) - (z_2 - z_4)\right] \end{aligned} \tag{8.7}$$

Now from the hydrostatic equation $g\ dz = -\alpha\ dp$

$$\begin{aligned} \therefore \int_{B_1}^{B_2} g dz = g(z_3 - z_1) &= -\int_{p_1}^{p_2} \alpha_B dp \\ &= -\left[\int_{p_1}^{p_2} \alpha_{35,0,p} dp + \int_{p_1}^{p_2} \delta_B dp\right] \end{aligned} \tag{8.8}$$

Similarly $\qquad g(z_4 - z_2) = -\left[\int_{p_1}^{p_2} \alpha_{35,0,p} dp + \int_{p_1}^{p_2} \delta_A dp\right]$

Therefore,

$$\frac{g}{L}\left[(z_1 - z_3) - (z_2 - z_4)\right] = \frac{1}{L}\left[\int_{p_1}^{p_2} \delta_B dp - \int_{p_1}^{p_2} \delta_A dp\right]$$

Then, we can obtain

$$\begin{aligned} (V_1 - V_2) &= \frac{1}{L 2\Omega \sin\phi}\left[\int_{p_1}^{p_2} \delta_B dp - \int_{p_1}^{p_2} \delta_A dp\right] \\ &= \frac{1}{L 2\Omega \sin\phi}\left[\Delta\Phi_B - \Delta\Phi_A\right] \end{aligned} \tag{8.9A}$$

(In texts using mixed units this is written

$$(V_1 - V_2) = \frac{10}{L 2\Omega \sin\phi}\left[\Delta D_B - \Delta D_A\right] \quad \text{where} \quad \Delta D = \int_{p_1}^{p_2} \delta\, dp. \tag{8.9B}$$

(1) The result of the calculation with the geostrophic equations is a value for

(V_1–V_2), the difference between the current at pressure p_1 from that at pressure p_2 *averaged* between the stations A and B.

(2) Its direction is perpendicular to the line AB, i.e. it is the component in the direction perpendicular to AB of the actual current difference in the sea. Thus, it would be directed "into the paper" in the northern hemisphere in Fig. 8.4.

(3) The current will be along the slope in such a direction that the surface is higher on the right. In other words, the water on the right is lighter (less dense) than on the left.

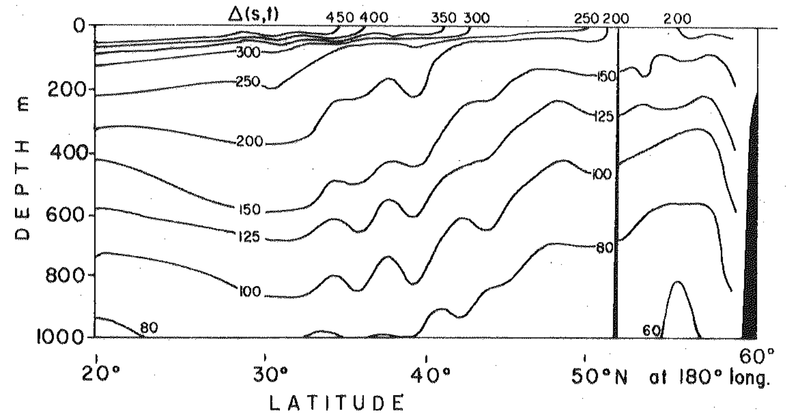


Fig. 8.5 Vertical section of specific volume (as $\Delta_{s,\,t}$) for a real ocean region to show the irregular character of isopleths, (Norpac Atlas, 1955).

As the isopycnals in a section of the real ocean are generally wavy rather than straight and vary irregularly in angle relative to each other over finite depth ranges (e.g. Fig 8.5).

In Fig. 8.5, if one assumes that the thermosteric anomaly ($\Delta_{s\,,t}$) curves become level in deep water (well below 1000 m):

(1) 20°N~30°N, the general flow below 300 m would be "into the paper", i.e. to the west (the North Equatorial Current),

(2) 30°N~48°N, the general flow would be to the east (the North Pacific Drift or Subarctic Current),

(3) 48°N~52°N, (the Aleutian Island chain) the general flow would be to the west (the Aleutian Stream which forms the northern side of the cyclonic Aleutian Gyre of the northeast Pacific),

(4) 52°N~the continental slope at about 59°N, the general flow is the cyclonic circulation of the deep basin of the Bering Sea.

(5) 33°N~45°N, the wave-like $\Delta_{s,\,t}$ curves might be due to eddies within the general flow.

8.43 An example of the calculation of a geostrophic velocity profile

–Table 8.1(a,b) show the depths and observed temperatures and salinities, A and B, in the region of the North Atlantic Drift.

–Note that $\Delta p = -\Delta z \times 10^4$Pa = $10^4\times$depth difference in meters has been used in calculating $\Delta\Phi$.

–In Table 8.2. the values calculated for $\Delta\Phi$ for station B and A are listed and then the difference between them ($\Delta\Phi_B - \Delta\Phi_A$).

–The relative speed is calculated for each depth (relative to zero speed assumed at 1000 m) from (8.9A).

–The values of the speed relative to that at 1000 m are plotted against depth in Fig. 8.6(a). The velocity component is directed to the east because relative to 1000 m depth the isopycnals slope down from left to right (Fig. 8.6(b)).

–$(\Delta\Phi_B - \Delta\Phi_A)/g$ gives an estimate of the height difference (***dynamic topography***) of isobaric surfaces at two stations, e.g. at the sea surface the difference is 0.13 m, that is the depth of the water column from the sea surface to the pressure level of 104 kPa (about 1000 m depth) differ by 0.13 m.

Table 8.1(a) Oceanographic data, etc., and calculation of geopotential anomalies ($\Delta\Phi$) for station A (Lafond, 1951)

Station A 41°55′N, 50°09′W				Units of 10^{-8} m^{3}kg^{-1}					Units of m^{3}kg^{-1}Pa = m^{2}s^{-2}	
Depth (m)	T°C	S	σ_t	$\Delta_{s,t}$	$\delta_{s,p}$	$\delta_{t,p}$	δ	$\bar{\delta}$	$\bar{\delta}\times\Delta p$	$\sum(\bar{\delta}\times\Delta p) = \Delta\Phi_A$
0	5.99	33.71	26.56	148	0	0	148			6.638
								146	0.365	
25	6.00	33.78	26.61	144	0	0	144			6.273
								135	0.338	
50	10.30	34.86	26.81	125	0	1	126			5.935
								126	0.315	
75	10.30	34.88	26.83	123	0	2	125			5.620
								122	0.305	
100	10.10	34.92	26.89	117	0	2	119			5.315
								112	0.560	
150	10.25	35.17	27.06	101	0	3	104			4.755
								99	0.455	
200	8.85	35.03	27.19	89	0	4	93			4.300
								83	0.830	
300	6.85	34.93	27.41	68	0	5	73			3.470
								65	0.650	
400	5.55	34.93	27.58	52	0	5	57			2.820
								52	1.040	
600	4.55	34.95	27.71	39	0	7	46			1.780
								45	0.900	
800	4.25	34.95	27.74	37	0	8	45			0.880
								44	0.880	
1000	3.90	34.95	27.78	33	0	10	43			0

Table 8.1(b) Oceanographic data, etc., and calculation of geopotential anomalies ($\Delta\Phi$) for station B (Lafond, 1951)

Station B 41°28′N, 50°09′W				Units of 10^{-8} m^3kg^{-1}					Units of $m^3kg^{-1}Pa = m^2s^{-2}$	
Depth (m)	T°C	S	σ_t	$\Delta_{s,t}$	$\delta_{s,p}$	$\delta_{t,p}$	δ	$\bar{\delta}$	$\bar{\delta} \times \Delta p$	$\sum(\bar{\delta} \times \Delta p) = \Delta\Phi_A$
0	13.04	35.62	26.88	118	0	0	118			7.894
								119	0.298	
25	13.09	35.63	26.88	118	0	1	119			7.596
								119	0.298	
50	13.07	35.63	26.88	118	0	1	119			7.298
								119	0.298	
75	13.05	35.64	26.89	117	0	2	119			7.000
								120	0.300	
100	13.05	35.62	26.88	118	0	3	121			6.700
								122	0.610	
150	13.00	35.61	26.88	118	0	4	122			6.090
								122	0.610	
200	12.65	35.54	26.90	116	0	5	121			5.480
								117	1.170	
300	11.30	35.36	27.02	105	0	7	112			4.310
								98	0.980	
400	8.30	35.09	27.32	76	0	7	83			3.330
								70	1.400	
600	5.20	34.93	27.61	49	0	8	57			1.930
								52	1.030	
800	4.20	34.92	27.73	38	0	8	46			0.900
								45	0.900	
1000	4.20	34.97	27.77	34	0	10	44			0

Table 8.2 Geopotential anomalies from Table 8.1(a, b) and calculated mean relative velocities between stations A and B at various depths.

Depth (m)	$\Delta\Phi_B$ (m^2s^{-2})	$\Delta\Phi_A$ (m^2s^{-2})	($\Delta\Phi_B - \Delta\Phi_A$) ($m^2s^{-2}$)	V_{rel} (ms^{-1})	
0	7.894	6.638	1.256	0.26	Stn. A: 41°55′N, 50°09′W
25	7.596	6.273	1.323	0.27	Stn. B: 41°28′N, 50°09′W
50	7.298	5.935	1.363	0.28	Diff. = 27′, 0′
75	7.000	5.620	1.380	0.29	i.e. stations are 27 n ml apart,
100	6.700	5.315	1.385	0.29	= 50 km = 5×10^4 m.
150	6.090	4.755	1.335	0.28	
200	5.480	4.300	1.180	0.24	sin 41°28′ = 0.662
300	4.310	3.470	0.840	0.17	sin 41°55′ = 0.668
400	3.330	2.820	0.510	0.11	mean sin ϕ = 0.665
600	1.930	1.780	0.150	0.03	
800	0.900	0.880	0.020	0.005	$2\Omega \sin\phi = 9.70 \times 10^{-5}$ s^{-1}
1000	0	0	0	0	

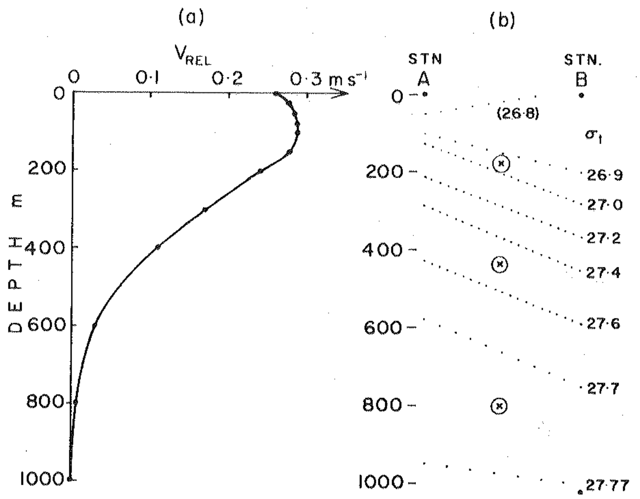


Fig. 8.6 (a) Profile of velocity relative to that at 1000 m depth as calculated from data of Tables 8.1 and 8.2. (b) Mean slopes of isopycnal surfaces between stations A and B for the same data.

8.44 An alternative derivation of the geostrophic equation

Assume that

(1) no acceleration, i.e. $\dfrac{du}{dt} = \dfrac{dv}{dt} = \dfrac{dw}{dt} = 0$

(2) w = small, so that $2\Omega \cos \phi w$ may be neglected

(3) $\boldsymbol{F} = 0$

Thus, $$\left.\begin{aligned} 0 &= 2\Omega \sin \phi\, v - \alpha \frac{\partial p}{\partial x} \\ 0 &= -2\Omega \sin \phi\, u - \alpha \frac{\partial p}{\partial y} \end{aligned}\right\}$$ component geostrophic equations. (8.10)

That is, for purely horizontal motion

Coriolis force = – Pressure force.

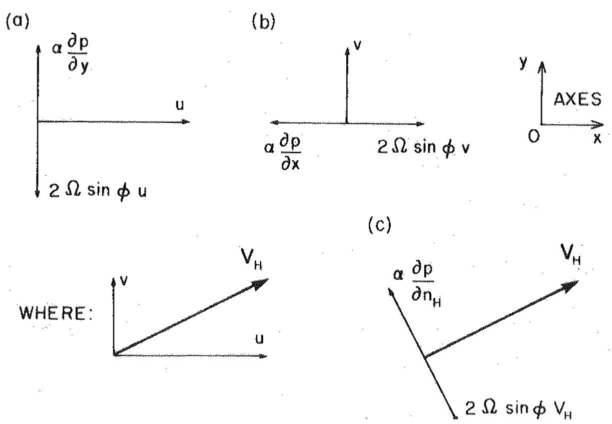


Fig. 8.7 (a, b) Directional relationships of velocity components (u, v) to pressure and Coriolis force terms (northern hemisphere), (c) directional relationship of total horizontal velocity ($\vec{V}_H$) to pressure and Coriolis force terms (northern hemisphere).

See Fig. 8.7 (a,b). The *x*- and *y*-equations can be combined into a single one

$$2\Omega \sin\phi V_H = \alpha \frac{\partial p}{\partial n_H} \tag{8.11}$$

where V_H = magnitude of the vector sum of u and v = $(u^2+v^2)^{1/2}$,

$\partial p/\partial n_H$ = the horizontal pressure term perpendicular to the direction of $\vec{V}_H$ (see Fig. 8.7(c)).

One way to remember the relative directions of the pressure force and the velocity is to think of the sequence:

(1) the pressure gradient is initiated somehow,

(2) the fluid starts to move down the gradient,

(3) the fluid then experiences the Coriolis force to the right (in the northern hemisphere) and therefore swings to the right,

(4) the fluid eventually moves along the isobars, i.e. ***along*** the slope, not down it, with the pressure force ***down*** the slope balanced by the Coriolis force ***up*** the slope.

8.45 The "*thermal wind*" equations

These are another variation of the geostrophic equations originally derived to show temperature differences in the horizontal could lead to vertical variations in the geostrophic wind velocity, hence the term ***thermal wind equations***.

$$\frac{\partial}{\partial z}(\rho f v) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right)$$

$$\frac{\partial}{\partial z}(\rho f u) = \frac{\partial}{\partial z}\left(-\frac{\partial p}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right)$$

Then, using the hydrostatic equation, $\partial p/\partial z = -\rho g$, gives

$$\begin{aligned}\frac{\partial}{\partial z}(\rho f v) &= -g\frac{\partial \rho}{\partial x}\\ \frac{\partial}{\partial z}(\rho f u) &= \quad g\frac{\partial \rho}{\partial y}\end{aligned} \tag{8.14}$$

(1) these equations show that from the density field we can only determine the vertical variation of velocity, i.e. the velocity shear $\partial u/\partial z$ and $\partial v/\partial z$.

(2) taking the Boussinesq approximation or assuming that the effect of density

variation is small compared with the effect of vertical gradients of u and v, $\partial(\rho f v)/\partial z \approx \rho f(\partial u/\partial z)$.

(3) In the upper 1000 m or so, as a first approximation $\partial\sigma_t/\partial x$ and $\partial\sigma_t/\partial y$ can probably be used for $\partial\rho/\partial x$ and $\partial\rho/\partial y$, respectively.

(4) In deep water, the thermal wind equations are not likely to give useful results if temperature gradients are the dominant contribution to density gradients.

(5) The rule "light water on the right" relative to the water below when looking in the flow direction comes from the thermal wind equations (8.14).

(6) Suppose ρ decreases to the east, then $\partial\rho/\partial x<0$ and $\partial(\rho f v)/\partial z>0$, that is v increases as we go upward in the water column. If ρ decreases to the south $\partial\rho/\partial y>0$ and $\partial(\rho f u)/\partial z>0$.

8.5 Deriving absolute velocities

The geostrophic calculation gives the relative velocity component (V_1-V_2) between two depths, i.e. the velocity shear dV/dz. Therefore, if we know the value of either V_1 or V_2, we will know the absolute value of the other. There are several possibilities:

(1) assume that there is a ***level*** or ***depth of no motion (reference level)*** e.g. that $V_2=0$ in deep water, and then calculate V_1 for various levels above this (the classical method);

(2) when there are stations available across the full width of a strait or ocean, calculate the velocities and then apply the equation of continuity to see if the resulting flow is reasonable, i.e. complies with all facts already known about the flow and also satisfies conservation of heat and salt;

(3) use a "level of ***known*** motion", e.g. if surface currents are known or if the currents have been measured at some depth(s) by current meters or neutrally buoyant floats (preferably while the density measurements for the geostrophic calculations were being made). In the future, it is possible that satellite measurements of the surface slope may enable the surface currents to be calculated, at least in regions of strong currents.

Since surface velocities are important and can be inferred quickly from the slope of the sea surface (which is assumed to be isobaric) it is common to plot the geopotential (or "dynamic") topography of the sea surface relative to some deeper surface, if a sufficient grid of station data is available.

The relative current directions will be inversely proportional to the spacing of the lines (i.e. close spacing = steep slope = large speed, e.g. Fig. 8.8).

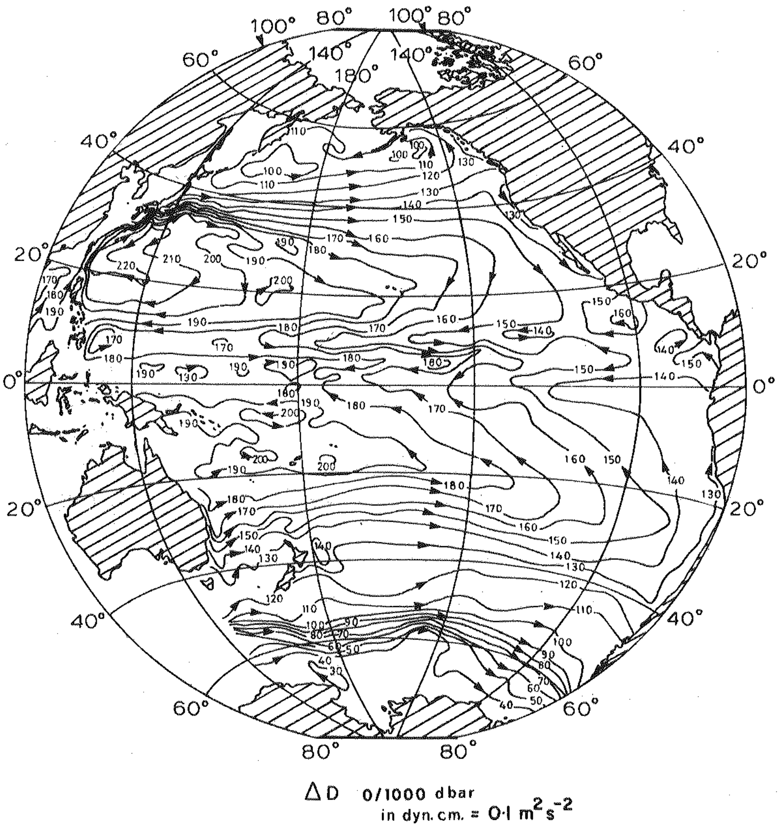


Fig. 8.8 Annual mean dynamic topography of the Pacific Ocean sea surface relative to 1000 dbar (10,000 kPa) in dynamic centimeters (=0.1 m^2s^{-2}), i.e. ΔD = 0/1000 dbar; 36,356 observations. Arrows indicate flow directions deduced from topography. (Wyrtki, 1974).

8.6 Relations between isobaric and level surfaces

In the Pacific, the uniformity of properties in the deep water suggests that assuming a level of no motion at 1000 m or so is reasonable, with very slow motion below this. In the Atlantic, there is evidence of a level of no motion at 1000-2000 m (between the upper waters and the North Atlantic Deep Water) with significant currents above and below this depth. A selection of relations between isobaric and constant geopotential or level surfaces is shown in Fig. 8.9.

(1) Fig. 8.9(a) is typical of the west Pacific;

(2) Fig. 8.9(b) is typical of the west Atlantic (Gulf Stream region) with characteristics described above;

(3) Fig. 8.9(c) would indicate little motion at the surface but increasing speed into the deep water, which is unlikely in the real ocean;

(4) Fig. 8.9(d) shows a situation where all the isobaric surfaces are parallel and equally inclined to level surfaces – the so-called "slope current" situation. In this case, the application of the geostrophic calculation would yield zero relative velocity at all depths which would be correct although the absolute velocity would not be zero. This situation is unlikely in the ocean because density variations due to temperature and salinity variations are likely to lead to changes in the isobaric slopes with depth.

(5) In Fig. 8.9(a), (b) and (c) are examples of "baroclinic" situations while (d) is a "barotropic" one.

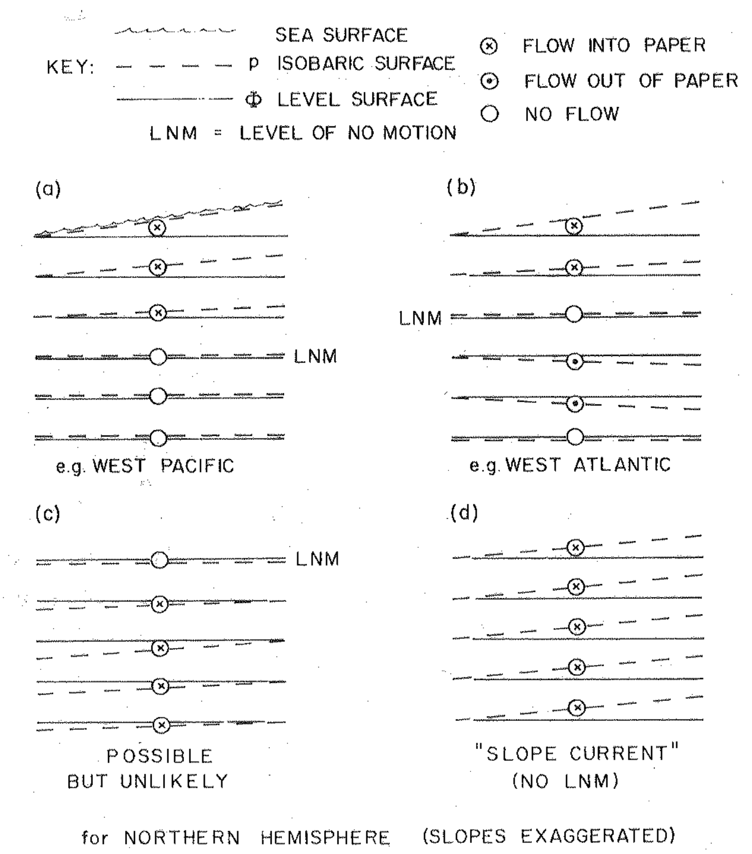


Fig. 8.9 Relationship between isobaric and level surfaces and flow directions for northern hemisphere.

8.7 Relations between isobaric and isopycnal surfaces and currents

–An ***isobaric*** surface in a fluid is one on which the hydrostatic pressure is constant.

–An ***isopycnal*** surface (sometimes called ***isosteric***) is one on which the density of the fluid is constant.

(1) When the density of a fluid is a function of pressure only (i.e. $\rho=\rho(p)$), as in fresh water of uniform potential temperature, the isobaric and isopycnal surfaces are parallel to each other—this is called a ***barotropic*** field of mass.

(2) If the density is a function of other parameters as well and actually varies horizontally with them, the isobaric and isopycnal surfaces may be inclined to each other—the ***baroclinic*** field. Such as in a freshwater lake $\rho=\rho(t,p)$; in the sea $\rho=\rho(s,t,p)$.

(3) In the ocean, the barotropic case is most common in deep water; the baroclinic case is most common in the upper 1000 m where most of the faster currents occur.

(4) In the barotropic case the isopycnals will be parallel to the isobars (see Fig. 8.10(a, b)).

(5) In the baroclinic case, there is no simple relation between the isobars and isopycnals (see Fig. 8.10(c, d)). From the geostrophic equation the slopes of isobars are proportional to the velocity.

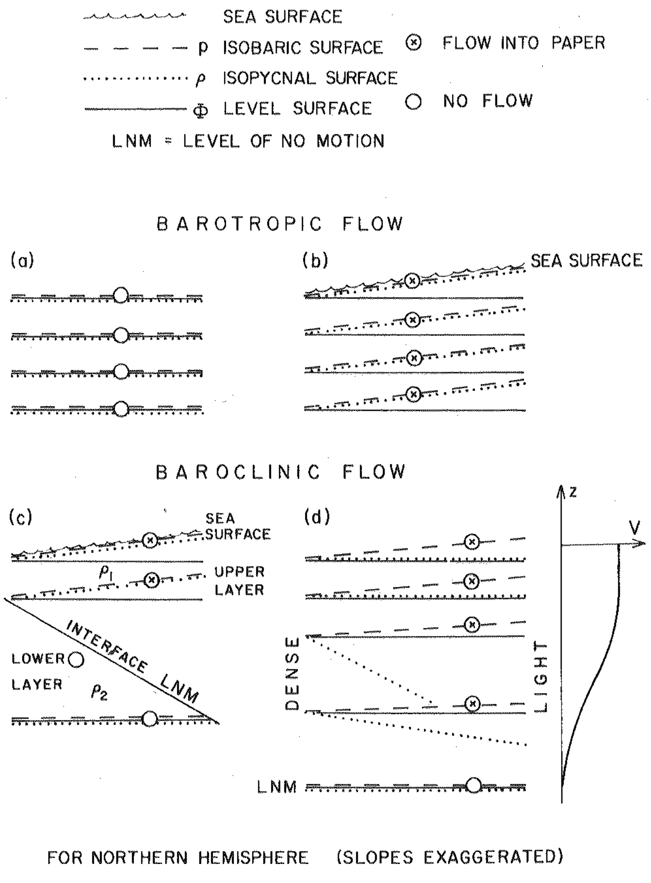


Fig. 8.10 Schematic examples of: (a, b) barotropic and (c, d) baroclinic fields of mass and pressure with related flow directions for the northern hemisphere. Slopes are exaggerated: p and ρ by about 10^5 in (b, c), interface by 10^3 in (c), p by 10^5 in (d), ρ by 10^3 in (d). With this exaggeration, maximum speeds implied are about 0.1 ms^{-1}.

Fig. 8.11 shows a reasonably realistic and more complicated case.

In low- and mid-latitudes, temperature is the main factor determining density and a section of temperature may be used as a reasonable approximation for a density section. The rule “light water on right when looking in the flow direction” becomes “warm water on the right” with isotherms sloping down to the right.

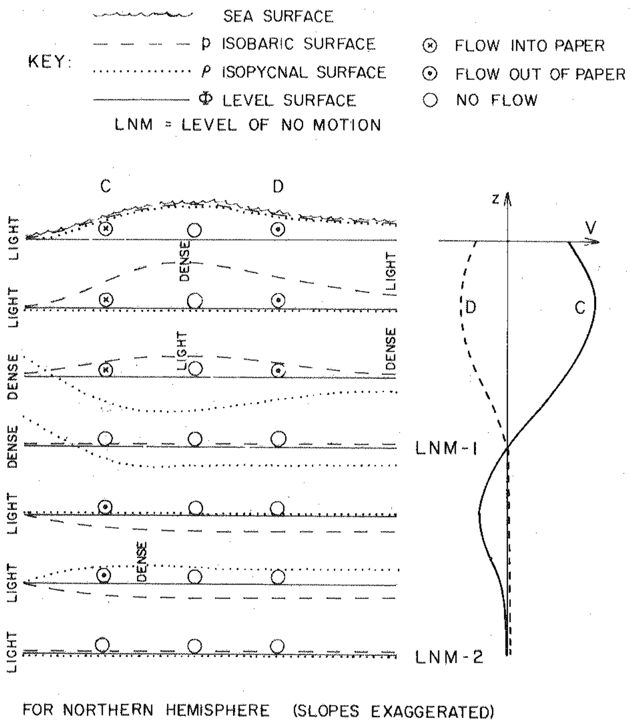


Fig. 8.11 A more realistic (and complicated) schematic example of relations between baroclinic fields of mass and pressure with related flow directions, and velocity profiles for stations C and D for the northern hemisphere. Slopes are exaggerated: p by about 10^5 and ρ by about 10^3.

Often in oceanography,

(a) Barotropic flow is thought of as the flow due to a uniform tilt of the pressure surfaces like that of the deep water where the density essentially depends only on pressure, and the velocity is uniform with depth.

(b) Baroclinic flow is the part due to additional tilts of the pressure surfaces caused by density variations.

(c) Both are geostrophic flows in the sense that there is a balance between the pressure force and Coriolis force. For example, Fig. 8.12.

Note: The baroclinic part may be obtained from a geostrophic calculation but the barotropic part will not appear in this calculation, it must be obtained by some other means.

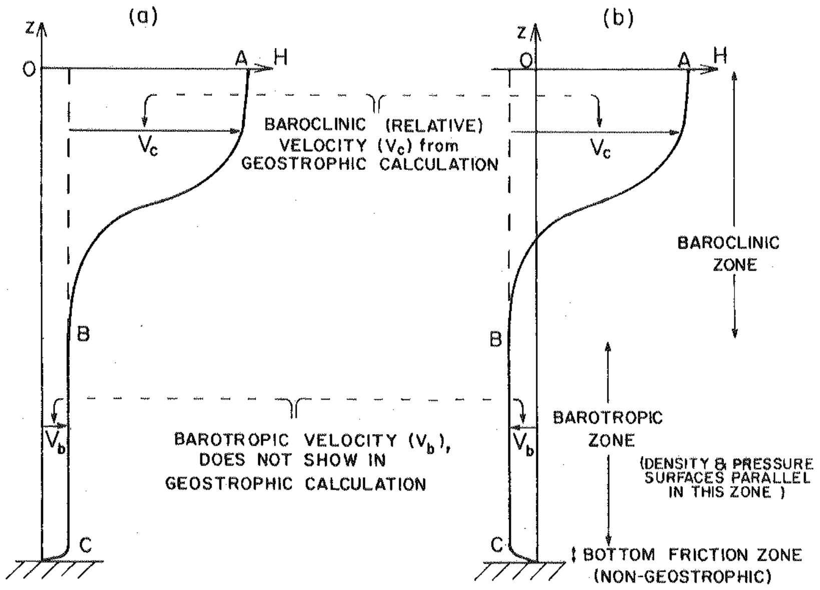


Fig. 8.12 Horizontal speed as a combination of barotropic (V_b) and baroclinic (V_c) parts of the geostropic velocity; V_b and V_c in (a) the same direction, (b) opposite directions. (Note that in reality V_b and V_c need not be colinear.)

8.8 Comments on the geostrophic equation

The procedure of calculating the geostrophic currents from the oceanographic data at two stations, e.g. A and B (Fig. 8.13(a)), yields only the component V_1 of current perpendicular to the line AB. From B and C, we can get the component V_2. These may then be added vectorially to obtain the total current ($\vec{V}_H$).

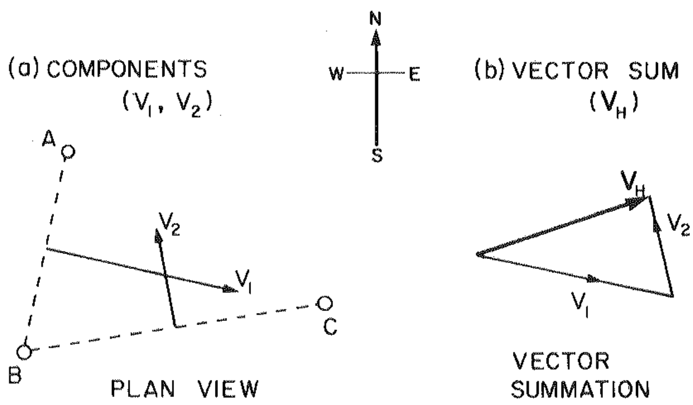


Fig. 8.13 Stations AB yield velocity component V_1, stations BC yield component V_2; the total horizontal velocity $\vec{V}_H$ is the vector sum of these.

The geostrophic method for calculating currents suffers from several disadvantages:

(1) It yields only relative currents and the selection of an appropriate level of no motion always presents a problem.

(2) One is faced with a problem when the selected level of no motion reaches the ocean bottom as the stations get close to shore.

(3) It only yields mean values between stations which are usually many tens of kilometers apart.

(4) Friction has been ignored in deducing the geostrophic equation. It may actually be significant near the bottom or where there is current shear, and therefore the equation does not apply in such situations.

(5) The equation breaks down near the equator where the Coriolis force becomes so small that the friction forces may be important.

(6) The calculated geostrophic current will include any long-period transient currents. It is possible to separate the transients from the "steady" ocean currents if the geostrophic current is calculated from only two stations.

Despite all these disadvantages, it must be admitted that application of the geostrophic equation has provided us with much of our present knowledge of the velocities of ocean currents.

8.9 The beta-spiral

Schott and Stommel (1978) proposed a method for obtaining the absolute velocity from measurements of the density field alone.

–The method is applicable in regions where the geostrophic equations are a good approximation.

–Assume that the density does not change as the water flows along, i.e. $d\rho/dt = 0$; with $\partial\rho/\partial t = 0$.

$$u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}=0 \qquad (8.15)$$

Use the thermal wind equations (8.14) with the Boussinesq approximation

$$\left(\rho_0 f\frac{\partial v}{\partial z}=-g\frac{\partial\rho}{\partial x};\ \rho_0 f\frac{\partial u}{\partial z}=g\frac{\partial\rho}{\partial y}\quad \text{where}\,\rho_0\text{ is an average density}\right)$$

Then (8.15) becomes

$$-u\frac{\partial v}{\partial z}+v\frac{\partial u}{\partial z}=-\left(\frac{g}{f\rho_0}\right)w\frac{\partial \rho}{\partial z} \tag{8.16}$$

If we write the velocity in polar form components as $u = V\cos\theta$, $v = V\sin\theta$ then (8.16) becomes

$$\frac{\partial \theta}{\partial z}=w\frac{\partial \rho}{\partial z}\left(\frac{g}{\rho_0 f V^2}\right) \tag{8.17}$$

(1) Now $\partial\rho/\partial z<0$ and $f>0$ in the northern hemisphere so, if $w>0$ (upward), $\partial\theta/\partial z<0$ and the current rotates to the right (θ becomes smaller) as we go upward in the water column (or to the left as we go downward).

(2) With $f<0$ in the southern hemisphere, the rotation is to the left as we go upward for $w<0$.

(3) If $w = 0$, there is no rotation of the flow direction as the depth changes.

In (8.10) we use the Boussinesq approximation, i.e. treat α as a constant. Then

$$(a)\quad \frac{\partial(f\,u)}{\partial x}=f\frac{\partial u}{\partial x}=-\alpha\frac{\partial^2 p}{\partial x\partial y};\quad (b)\quad \frac{\partial(f\,v)}{\partial y}=f\frac{\partial v}{\partial y}+v\frac{\partial f}{\partial y}=\alpha\frac{\partial^2 p}{\partial y\partial x}$$

(a)+(b) $\Rightarrow$ $$f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+v\frac{\partial f}{\partial y}=0$$

Thus,

$$\beta v = f\frac{\partial w}{\partial z} \tag{8.18}$$

(1) If $v \neq 0$, i.e. the flow is not just east–west, $\partial w/\partial z \neq 0$, i.e. w changes with z and cannot be zero everywhere.

(2) f changes with latitude, so that $\beta \neq 0$, makes rotation of the geostrophic flow with depth changes likely; hence the term ***β–spiral.***

(3) (8.18) is the vorticity-conservation equation for geostrophic flow.

How do we use this approach to calculate absolute velocities?

(1) Let $z = -h_0 + h(x, y)$ be an isopycnal surface; $h(x, y)$ is the height above $z = -h_0$.

(2) Suppose that we go along this surface in the x–direction, then $dz=(\partial h/\partial x)dx$ and $d\rho = [(\partial\rho/\partial x)dx+(\partial\rho/\partial z)dz] = 0$.

(3) Thus $\partial h/\partial x = -[(\partial\rho/\partial x)/(\partial\rho/\partial z)]$ is the slope of the surface in the x–direction.

Likewise $\partial h/\partial y = -[(\partial \rho/\partial y)/(\partial \rho/\partial z)]$.

(4) Using these values for $\partial h/\partial x$ and $\partial h/\partial y$ in (8.15) gives

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = w \tag{8.19}$$

Taking $\partial/\partial z$ of this equation gives

$$\frac{\partial}{\partial z}\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = \frac{\partial w}{\partial z} = \frac{\beta\, v}{f}$$

and rewriting (8.4) in terms of $\partial h/\partial x$, $\partial h/\partial y$ as

$$\frac{\partial v}{\partial z} = \frac{g}{\rho_0 f}\left(\frac{\partial h}{\partial x}\frac{\partial \rho}{\partial z}\right); \quad \frac{\partial u}{\partial z} = -\frac{g}{\rho_0 f}\left(\frac{\partial h}{\partial y}\frac{\partial \rho}{\partial z}\right)$$

we see that

$$\frac{\partial u}{\partial z}\frac{\partial h}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial h}{\partial y} = 0 \quad \text{and} \quad u\frac{\partial^2 h}{\partial x \partial z} + v\left(\frac{\partial^2 h}{\partial y \partial z} - \frac{\beta}{f}\right) = 0 \tag{8.20}$$

–Now $\partial h/\partial x$ and $\partial h/\partial y$ can be determined as functions of z from the observations of ρ if there are enough oceanographic stations in an area and $\partial/\partial z$ of $\partial h/\partial x$ and $\partial h/\partial y$ can be calculated.

–Suppose that we have calculated u' and v', the relative geostrophic velocities based on some reference level where velocity components are u_0 and v_0, then $u = u_0 + u'$ and $v = v_0 + v'$ gives

$$u_0\frac{\partial^2 h}{\partial x \partial z} + v_0\left(\frac{\partial^2 h}{\partial y \partial z} - \frac{\beta}{f}\right) + u'\frac{\partial^2 h}{\partial x \partial z} + v'\left(\frac{\partial^2 h}{\partial y \partial z} - \frac{\beta}{f}\right) = 0 \tag{8.21}$$

–If we use N levels to determine u', v', $(\partial^2 h/\partial x \partial z)$ and $(\partial^2 h/\partial y \partial z)$ we get N equations for u_0, v_0 in the form of (8.21).

–Use least-squares techniques to find u_0 and v_0 such that the sum of the squares of the left-hand sides of the (8.21) will be minimum.

–Schott and Stommel (1978) tested the method with historical data at a number of locations. The u_0, v_0 values obtained at a particular location often depended on the range of depths used for calculating $(\partial^2 h/\partial x \partial z)$ and $(\partial^2 h/\partial y \partial z)$ in (8.21).

–Wunsch (1978) used an inverse procedure to calculate the absolute velocity from the density field.

8.10 Justification for using the geostrophic approach to obtain the speeds of strong currents

Consider the Gulf Stream or Kuroshio as an example.
x-axis across the stream and *y*-axis along the stream.
The width of the stream $\boldsymbol{L_x}$ = 100 km = 10^5 m
The length of the stream $\boldsymbol{L_y}$ = 1000 km = 10^6 m
Along-stream V = 1 ms^{-1}, cross-stream U = 0.1 ms^{-1} and $H = 10^3$ m
$A_x = A_y = 10^5$ m^{2}s^{-1}, A_z = 0.1 m^{2}s^{-1}

The *x*-equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\alpha\frac{\partial p}{\partial x}+fv-2\Omega\cos\phi\, w+A_x\frac{\partial^2 u}{\partial x^2}+A_y\frac{\partial^2 u}{\partial y^2}+A_z\frac{\partial^2 u}{\partial z^2}$$

Scaling

$$10^{-3} + 10^{-3} + 10^{-3} = \quad ? \quad + 1 \quad - \quad 10^{-3} + \quad 10^{-2} + \quad 10^{-4} \quad + 10^{-4}$$

The *x*- or cross-stream equation therefore remains geostrophic.

The *y*-equation

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\alpha\frac{\partial p}{\partial y}-fu+A_x\frac{\partial^2 v}{\partial x^2}+A_y\frac{\partial^2 v}{\partial y^2}+A_z\frac{\partial^2 v}{\partial z^2}$$

Scaling

$$10^{-6} + 10^{-6} + 10^{-6} = -\,?\, - 10^{-5} + 10^{-5} + \quad 10^{-7} + 10^{-7}$$

The non-linear terms now cannot be neglect and the largest friction term is of about the same order as the Coriolis term *fu*. Thus, the geostrophic approximation is not good for the *y*-equation.