Chap 6 The Equation of Motion in Oceanography

6.1 The form of the equation of motion

Consider Newton's Second Law of Motion

$$\vec{F} = m\vec{a} \quad \text{or} \quad \vec{a} = \frac{d\vec{V}}{dt} = \frac{\vec{F}}{m}$$

where $\vec{F}$ =force, m=mass, $\vec{a}$ =acceleration, and $\vec{V}$ =velocity. Here if we consider four forces are important for ocean motion, the equation of motion can be written as

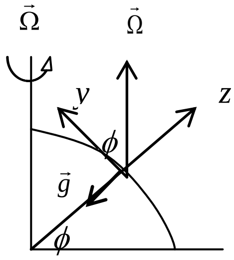
$$\frac{d\vec{V}}{dt} = \underset{\text{Pressure}}{-\alpha\nabla p} \underset{+\ \text{Coriolis}}{-\ 2\vec{\Omega}\times\vec{V}} \underset{+\ \text{Gravity}}{+\ g} \underset{+\ \text{Other forces}}{+\ \vec{F}} \tag{6.1}$$

where the magnitude Ω of $\vec{\Omega}$ is the ***rotation rate of Earth***, 2π radians per a sidereal day or $\Omega = 7.292\times10^{-5}$ radians/s.

Momentum Equation in Cartesian coordinates:

$$\begin{aligned}
(x) \quad & \frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\Omega\sin\phi\, v - 2\Omega\cos\phi\, w + F_x \\
(y) \quad & \frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - 2\Omega\sin\phi\, u + F_y \\
(z) \quad & \frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} + 2\Omega\cos\phi\, u - g + F_z
\end{aligned} \tag{6.2}$$

(6.2) appears under various names. Leonhard Euler (1707-1783) first wrote out the general form for fluid flow with external forces, so called the ***Euler equations***. Louis Marie Henri Navior (1785-1836) added the frictional terms, so called the ***Navior- Stokes equations***.



$$\vec{\Omega}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \Omega\cos\phi & \Omega\sin\phi \\ u & v & w \end{vmatrix}$$
$$= \vec{i}\left(w\Omega\cos\phi - v\Omega\sin\phi\right) + \vec{j}\left(u\Omega\sin\phi\right) - \vec{k}\left(u\Omega\cos\phi\right)$$

Note: The term $2\Omega\cos\phi\, u$ in the z-component equation of (6.2) is important for gravity measurements at sea. An east-west velocity produces a vertical acceleration.

6.2 Obtaining solutions to the equations, including boundary conditions

4 unknowns: u, v, w, and p, but only 3 equations. Add the equation of continuity (4.4) to obtain 4 equations and 4 unknowns.

Boundary conditions:

(1) There is no flow through the boundaries, so the component of flow normal (i.e. perpendicular) to the boundary must vanish.

(2) Next to solid boundaries the component of flow along boundary (i.e. tangential) must vanish too, i.e. there must be "no slip" at solid boundaries.

Solutions:

(1) There are no general solutions for non-linear equations with friction. "Non-linear" means that unknowns occur in combination in the equations, e.g. as $v(\partial u/\partial y)$.

(2) A very few solutions for non-linear equations without friction $\vec{F} = 0$ in above equations.

(3) Some approximate solutions have been developed.

6.3 The derivation of the terms in the equation of motion

6.31 The pressure term

Imagine a rectangular volume, in a fluid, of sides δx, δy and δz (Fig. 6.1). Then, the force in the x-direction on this volume due to the hydrostatic pressure will be $+p\delta y\delta z$ on the left and $-(p+\delta p)\delta y\delta z$ on the right. So, the net force in the x-direction is

$$F_x = p\delta y\delta z + [-(p+\delta p)]\delta y\delta z = -\delta p\delta y\delta z = -\frac{\partial p}{\partial x}\delta x\delta y\delta z = -\frac{\partial p}{\partial x}\delta V$$

The force per unit mass $= \dfrac{F_x}{\delta m} = -\dfrac{\partial p}{\partial x}\dfrac{\delta V}{\delta m} = -\alpha\dfrac{\partial p}{\partial x}$. Then considering all directions, the total pressure force/unit mass will be

$$-\alpha\left(\vec{i}\,\frac{\partial p}{\partial x}+\vec{j}\,\frac{\partial p}{\partial y}+\vec{k}\,\frac{\partial p}{\partial z}\right)=-\alpha\nabla p\,.$$

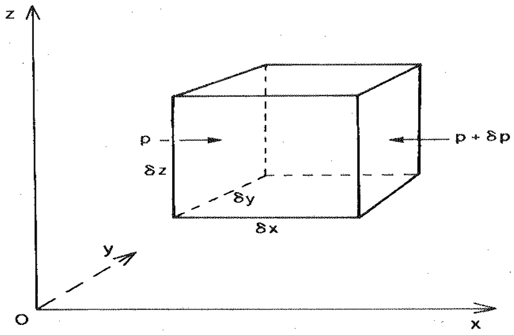
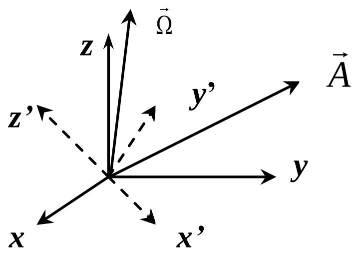


Fig. 6.1 For the derivation of the pressure term in the equation of motion.

6.32 Transforming from axes fixed in space to axes fixed in the rotating earth

–The Coriolis term arises because we normally make observations relative to axes fixed to the earth which is itself rotation about its axis.

–An inertial coordinate system, which the axes are fixed in space, is one whose origin is not accelerating. For practical, this is a system relative to the distant star.

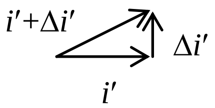


$$\vec{A}=\vec{i}A_x+\vec{j}A_y+\vec{k}A_z \quad \text{: in an inertial system}$$

$$=\vec{i}'A'_x+\vec{j}'A'_y+\vec{k}'A'_z \quad \text{: in a system rotating with an angular velocity } \vec{\Omega}$$

$$\frac{d_f\vec{A}}{dt}=\left(\frac{d\vec{A}}{dt}\right)_f=\vec{i}\,\frac{dA_x}{dt}+\vec{j}\,\frac{dA_y}{dt}+\vec{k}\,\frac{dA_z}{dt}$$

$$=\vec{i}'\frac{dA'_x}{dt}+\vec{j}'\frac{dA'_y}{dt}+\vec{k}'\frac{dA'_z}{dt}+A'_x\frac{d\vec{i}'}{dt}+A'_y\frac{d\vec{j}'}{dt}+A'_z\frac{d\vec{k}'}{dt}$$



$$\boxed{\frac{d\vec{i}'}{dt}=\vec{\Omega}\times\vec{i}',\qquad \frac{d\vec{j}'}{dt}=\vec{\Omega}\times\vec{j}',\qquad \frac{d\vec{k}'}{dt}=\vec{\Omega}\times\vec{k}'}$$

So, $\dfrac{d_f\vec{A}}{dt}=\dfrac{d_e\vec{A}}{dt}+\vec{\Omega}\times\vec{A}.\quad \Rightarrow\ \vec{V}_f=\dfrac{d_f\vec{R}}{dt}=\dfrac{d_e\vec{R}}{dt}+\vec{\Omega}\times\vec{R}=\vec{V}_e+\vec{\Omega}\times\vec{R}\,.$

$$\vec{a}_f = \frac{d_f \vec{V}_f}{dt} = \left(\frac{d\vec{V}_f}{dt}\right)_e + \vec{\Omega} \times \vec{V}_f = \left(\frac{d\vec{V}_e}{dt}\right)_e + \frac{d_e}{dt}\left(\vec{\Omega} \times \vec{R}\right) + \vec{\Omega} \times \left(\vec{V}_e + \vec{\Omega} \times \vec{R}\right)$$

$$= \left(\frac{d\vec{V}}{dt}\right)_e + \vec{\Omega} \times \left(\frac{d\vec{R}}{dt}\right)_e + \vec{\Omega} \times \vec{V}_e + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\right)$$

Therefore, a transformation from ideal axes fixed in space to practical rotating axes (e.g. see Neumann and Pierson, 1966; Lacombe, 1965; Batchelor, 1967) gives in vector form

$$\vec{a}_f = \left(\frac{d\vec{V}}{dt}\right)_e + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\right) \tag{6.3}$$

where the subscript f means relative to fixed axes and the subscript e means relative to axes fixed to the earth.

$\vec{R}$ = vector distance from the center of the earth

$\vec{\Omega}$ =7.3 × 10^{-5} rad/s = angular velocity of the earth (per sidereal day)

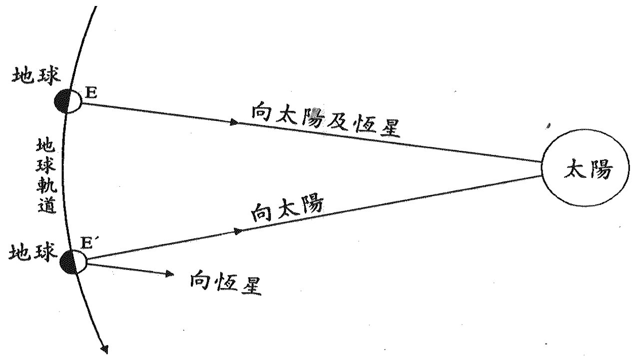
$-2\vec{\Omega} \times \vec{V}$ = Coriolis force

$-\vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\right)$ = Centrifugal acceleration (included in gravity)

1 sidereal day = rotation relative to the fixed stars

= 23 hr 56 min 4 s = 86164 s

Since the earth revolves about the sun it must turn a little further to point back to the sun and complete one solar day – hence the solar day is a little longer than the sidereal day.



A sketch showing the difference between a sidereal day and a solar day.

The equation of motion relative to fixed axes is

$$\left(\frac{d\vec{V}}{dt}\right)_f = -\alpha \nabla p + \vec{g}_f + \vec{F}\,. \tag{6.4}$$

When transformed to earth axes using (6.3) we get

$$\left(\frac{d\vec{V}}{dt}\right)_e = -\alpha\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g}_f - \vec{\Omega}\times\left(\vec{\Omega}\times\vec{R}\right) + \vec{F}\,. \tag{6.5}$$

6.33 Gravitation and gravity

– ***Gravitation*** is the attractive force between masses, recognized first by Newton.

– $F_g = G(M_1M_2)/r^2$ where M_1 and M_2 are the two masses, r is the distance between their centers, and G is the gravitational constant.

– The gravitational force provides the gravitational acceleration $\vec{g}_f$ in (6.4).

– The difference,$[\vec{g}_f - \vec{\Omega}\times(\vec{\Omega}\times\vec{R})]$, is referred to as the ***acceleration due to gravity,*** i.e. it is the familiar acceleration $\vec{g}$ of a body falling freely near the earth (Fig. 6.2).

– $\vec{g}$ is a maximum at the poles, where the needed centripetal acceleration vanishes and $\vec{g}_f$ is also a maximum because the polar radius is slightly less than the equatorial radius.

– $\vec{g}$ is a minimum at the equator, where the needed centripetal acceleration is a maximum and $\vec{g}_f$ is a minimum.

– The variation of $\vec{g}$ from pole to equator is only about 0.5%.

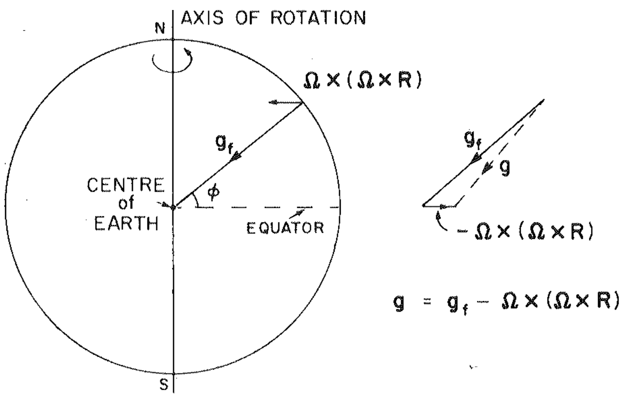


Fig. 6.2 Showing how the gravitational acceleration $(\vec{g}_f)$ is reduced to the acceleration due to gravity $(\vec{g})$ in providing the centripetal acceleration $(\vec{\Omega}\times(\vec{\Omega}\times\vec{R}))$ required. Note that the size of the centripetal acceleration is exaggerated. Its magnitude is $(\Omega^2 R\cos\phi)$ and it acts perpendicular to the axis of rotation.

6.34 The Coriolis terms

$-2\vec{\Omega}\times\vec{V}$ term in (6.5) is called the ***Coriolis acceleration*** term (named after G. Coriolis, 1835)

–Only two Coriolis terms are left and these are in the *x*– and *y*–component equations, which only depend only on the horizontal components of velocity.

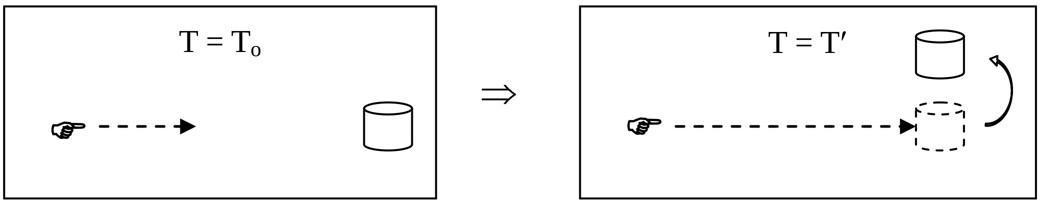
$-\vec{C}_H = 2\Omega\sin\phi\vec{V}_H\times\vec{k}$, $\vec{V}_H = \vec{i}\,u + \vec{j}\,v =$ the horizontal component of the total velocity.

–The direction of $\vec{C}_H$ must be perpendicular to both $\vec{k}$ and to $\vec{V}_H$, i.e. it is horizontal and directed at right angles to and to the right of $\vec{V}_H$ in the northern hemi- sphere, to the left in the southern hemisphere.

$-\,f = 2\Omega\sin\phi =$ Coriolis parameter.

6.35 The Coriolis terms – an intuitive derivation

To obtain a physical picture of the apparent effect of the rotation of the earth on objects moving near to it, consider the following hypothetical situation.



A long-range gun mounted at the north pole is aimed along a meridian directly at a target fixed on earth and some distance to the south. In plan view, a projectile fired from the gun will travel in a plane fixed relative to the "fixed star" but the target will be carried to the east by the rotating earth during the flight of the projectile. From the point of view of the gunner, also rotating with the earth, the projectile will appear to curve to the west, i.e. to the right, of its aimed direction. Since the change of direction represents acceleration, the earthbound observer interprets the apparent curved path of the projectile as due to some force acting across its direction of motion. This is so-called "Coriolis force."

–Any moving body will have a Coriolis force acting on it but other forces, e.g. road friction on a car's tires, may overcome it and prevent the body from following a curved path in coordinates fixed to the earth.

–Indeed the Coriolis force is too small to be directly experienced in everyday life.

–For the ocean and atmosphere the horizontal forces are small and the Coriolis force is important.

–Fig. 6.3 shows the factor 2 in the Coriolis terms. If *AB* is the initial direction to the target *B* and the flight path of the projectile in space *AB* = *Vt* and the target will be displaced $BB' = Vt(\Omega\sin\phi)t$. The apparent acceleration will then be the second differential of the apparent displacement, i.e.

$$\frac{d^2}{dt^2}(\Omega\sin\phi)Vt^2 = 2\Omega\sin\phi V.$$

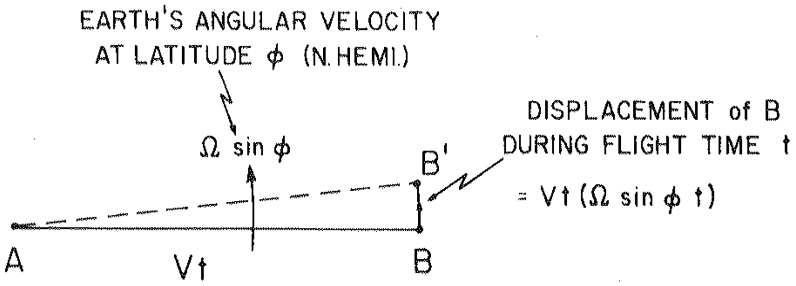


Fig. 6.3 For explanation of the origin of the factor 2 in the Coriolis terms (e.g. $2\Omega\sin\phi\, v$).

6.36 Other accelerations

The final term $\vec{F}$ reminds us that there are other forces to be taken into account, such as the gravitational attraction of the moon and sun, friction between wind and water, friction at solid boundaries, friction within the water mass, etc.

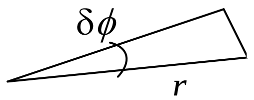
6.4 Coordinate systems

–***rectangular*** or ***Cartesian*** coordinates. If the horizontal area being considered is not too large then we can work on a plane tangent to the sphere and use a rectangular system with negligible errors.

–Spherical coordinates for the global ocean.

–For phenomena of relatively small scale, e.g. 100 km or so, the tangent plane is called the ***f–plane*** because for such small north–south distances the Coriolis parameter may be taken to be constant.

–For relatively larger areas, with ϕ varying over a few tens of degrees, between mid-latitudes and the equator, the tangent plane approximation is called the ***β(beta)–plane***. The variation of f with latitude is taken as $\boldsymbol{f = (f_0+\beta y+\ldots)}$, where $\boldsymbol{f_0}$ is the value of $\boldsymbol{f}$ at the mid-latitude of the region.



$\delta y = r\delta\phi$

$$\beta = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} = \frac{1}{r}\frac{\partial f}{\partial \phi} = \frac{2\Omega\cos\phi}{r}$$