Chap 7 The role of the non-linear terms and the magnitudes of terms in the equations of motion

7.1 The non-linear terms in the equation of motion

7.11 The friction terms in the equation of motion

The equation of motion per unit mass for the *x*-component is

$$\frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\Omega\sin\phi v - 2\Omega\cos\phi w + \mathit{friction} + \mathit{tidal\ forces} \tag{7.1}$$

Based on observational evidence, Newton hypothesized that the (tangential) frictional stress was related to the velocity shear as $\tau_x = \mu\partial u/\partial x$ where μ is a molecular viscosity coefficient. Then, it can be shown that the net friction force per unit mass in the *x*-direction on a small mass of fluid is given by

$$F_x = \nabla\cdot(\nu\nabla u) = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

where $\nu = \mu/\rho$ is the kinematic molecular viscosity, $\nu = \nu(s, t, p)$. A typical value for water is $0.8\sim1.8\times10^{-6}\ \mathrm{m^2s^{-1}}$.

Note: (1) The fluid is incompressible and (2) terms of the form $(\partial\nu/\partial x)(\partial u/\partial x)$ have been neglected because they are small compared with those retained in realistic oceanographic cases.

7.12 What is the source of the difficulty?

Non-linear terms (*x*-component)

$$\frac{du}{dt} = \underbrace{\frac{\partial u}{\partial t}}_{\text{local rate of change with time}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{advective rates of change due to motion}} \tag{7.2}$$

The advective terms are called “non-linear”.

Because of these non-linear terms a small perturbation (variation) may grow into a large fluctuation – these terms can cause instability and lie behind the presence of the turbulence which occurs whenever they are sufficiently large compared with the frictional terms which tend to remove velocity differences.

7.13 Scaling and the Reynolds Number

Reynolds number: $\mathrm{Re} = \dfrac{\text{non-linear term}}{\text{frictional term}} = \dfrac{U^2/L}{\nu U/L^2} = \dfrac{UL}{\nu}$

This dimensionless number, Re, is a very important quantity in determining the character of the flow.

(1) “dynamically similar” – flows have the same geometry and Re.

(2) small Re $\rightarrow$ laminar flow, large Re $\rightarrow$ turbulent flow

(3) Example: The Gulf Stream $U \sim 1$ ms^{-1} and $L \sim 100$ km and $\nu \sim 10^{-6}$ m^{2}s^{-1} so that Re $\sim 10^{11}$, and the flow will definitely be turbulent.

7.14 Reynolds stresses

When the motion is turbulent, so that it includes rapidly fluctuating components in addition to any mean flow, then the non-linear terms give rise to terms in the equations of motion which have the physical character of friction and they (similar terms in the heat- and salt-conservation equations) give rise to more rapid distribution of momentum, heat and salt than would occur with purely molecular processes. These are the so-called ***Reynolds stresses*** (forces/unit area) and ***fluxes*** (transports/unit area) which appear in the equations for the mean or average motion of a turbulent fluid.

7.2 Equations for the mean or average flow

Osborne Reynolds suggested that the variables *u*, *v*, *w* and *p* can be split into a mean and a ***fluctuating*** part, e.g. $u = \bar{u} + u'$ where the overbar denotes an average and $\bar{u}' = 0$ by definition.

(1) the average of $\partial u/\partial t$

$$\overline{\frac{\partial u}{\partial t}} = \frac{1}{T}\int_0^T \frac{\partial u}{\partial t}\,dt = \frac{[u(T) - u(0)]}{T} \quad \text{where } T \text{ is the average period.}$$

In practice we might wish to consider separation into a time-varying mean and fluctuations about it, if we were interested in variations due to tides or in seasonal changes. The sketch of Fig. 7.1 illustrates how we might do so.

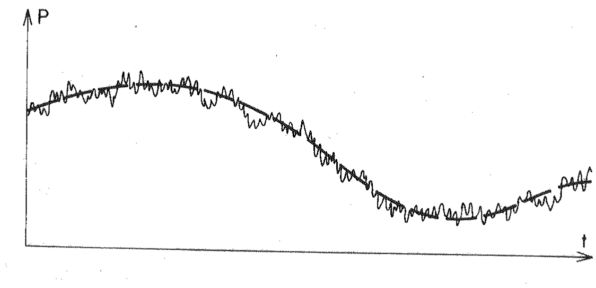

Fig. 7.1 Time variation of some property P at a point. The solid curve is the total quantity $P(t)$ as a function of time t, the dashed curve is a possible mean $\overline{P}$ while the difference between the two at any instant would be taken as the fluctuation part P'.

(2) the pressure term

$$\overline{(\overline{\alpha}+\alpha')\frac{\partial(\overline{p}+p')}{\partial x}} = \overline{\alpha}\frac{\partial \overline{p}}{\partial x} + \underset{\Downarrow \atop 0}{\overline{\overline{\alpha}\frac{\partial p'}{\partial x}}} + \underset{\Downarrow \atop 0}{\overline{\alpha'\frac{\partial \overline{p}}{\partial x}}} + \overline{\alpha'\frac{\partial p'}{\partial x}}$$

$\because$ $\alpha' << \overline{\alpha}$ in the ocean, so $\overline{\alpha'(\partial p'/\partial x)}$ is negligible compared with $\overline{\alpha}(\partial\overline{p}/\partial x)$.

(3) the Coriolis term

$$\overline{2\Omega\sin\phi(\overline{v}+v')} = 2\Omega\sin\phi(\overline{v}+\overline{v}') = 2\Omega\sin\phi\,\overline{v}$$

(4) the frictional term

$$\nu\frac{\partial^2(\overline{u}+\overline{u}')}{\partial x^2} = \nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}'}{\partial x^2}\right) = \nu\frac{\partial^2\overline{u}}{\partial x^2}$$

(5) the advective (non-linear) acceleration terms

$$\overline{(\overline{u}+u')\frac{\partial(\overline{u}+u')}{\partial x}} + \overline{(\overline{v}+v')\frac{\partial(\overline{u}+u')}{\partial y}} + \overline{(\overline{w}+w')\frac{\partial(\overline{u}+u')}{\partial z}} =$$

$$\left(\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}\right) + \underbrace{\left(\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right)}_{\text{additional terms}}$$

These new terms must represent effect of the velocity fluctuations or “turbulence” on the mean motion. Note that they arise from the non-linear terms in the Navier-Stokes equation.

7.21 Reynolds stresses and viscosity

(1) the equation of continuity for an incompressible fluid

$$\overline{\frac{\partial(\overline{u}+u')}{\partial x}}+\overline{\frac{\partial(\overline{v}+v')}{\partial y}}+\overline{\frac{\partial(\overline{w}+w')}{\partial z}}=0$$

Thus,

$$\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}+\frac{\partial\overline{w}}{\partial z}=0 \qquad \text{and} \qquad \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

(2) The additional terms can be expressed as

$$\overline{u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+w'\frac{\partial u'}{\partial z}+u'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)}=\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}$$

(3) The Reynolds equation for the x-component of velocity is

$$\begin{aligned}\frac{du}{dt}&=-\overline{\alpha}\frac{\partial\overline{p}}{\partial x}+2\Omega\sin\phi\,\overline{v}-2\Omega\cos\phi\,\overline{w}+\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)\\&\quad-\frac{\partial\overline{u'u'}}{\partial x}-\frac{\partial\overline{u'v'}}{\partial y}-\frac{\partial\overline{u'w'}}{\partial z}\end{aligned} \tag{7.3}$$

Stresses such as $-\rho\overline{u'u'},-\rho\overline{u'v'},-\rho\overline{u'w'}$ are termed as Reynolds stresses, which are stresses due to the turbulence.

Note: the total expression of Reynolds stresses

$$\begin{bmatrix}-\rho\overline{u'u'} & -\rho\overline{u'v'} & -\rho\overline{u'w'}\\ -\rho\overline{v'u'} & -\rho\overline{v'v'} & -\rho\overline{v'w'}\\ -\rho\overline{w'u'} & -\rho\overline{w'v'} & -\rho\overline{w'w'}\end{bmatrix}$$

We might suppose that these stresses are related to the mean velocity gradients by some sort of "viscosity" (an eddy or turbulent viscosity)

$$-\overline{u'u'}=A_x\frac{\partial\overline{u}}{\partial x};\ -\overline{u'v'}=A_y\frac{\partial\overline{u}}{\partial y};\ -\overline{u'w'}=A_z\frac{\partial\overline{u}}{\partial z} \tag{7.4}$$

where A's are called "eddy" viscosity. Unlike coefficients of molecular viscosity, the eddy-viscosity coefficients are not constant for a particular fluid and temperature, salinity and pressure but vary with the particular motion involved. Values are up to 10^{11} times those for kinematic molecular viscosity.

If we assume that A's are constant, the equations of motion for the x- and y-components are

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=$$
$$-\alpha\frac{\partial p}{\partial x}+fv-2\Omega\cos\phi\, w+A_x\frac{\partial^2 u}{\partial x^2}+A_y\frac{\partial^2 u}{\partial y^2}+A_z\frac{\partial^2 u}{\partial z^2} \qquad (7.6x)$$
$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=$$
$$-\alpha\frac{\partial p}{\partial y}-fu \qquad +A_x\frac{\partial^2 v}{\partial x^2}+A_y\frac{\partial^2 v}{\partial y^2}+A_z\frac{\partial^2 v}{\partial z^2} \qquad (7.6y)$$

where u, v, w, α and p are average quantities the overbar having been omitted for simplicity.

7.3 Scaling the equations of motion; Rossby number, Ekman number

Let us examine the various terms to make rough estimates of their sizes.
horizontal length scale $L \sim 10^6$ m
horizontal speeds $U \sim 0.1$ ms^{-1}
vertical scale length $H \sim 10^3$ m
vertical speeds W ~ $UH/L \sim 10^{-4}$ ms^{-1} (from continuity Eq. $\partial w/\partial z \cong \partial u/\partial x$)
Values estimated for A_x and A_y vary from 10 to 10^5 m^{2}s^{-1}, taking 10^5

$$\frac{U^2}{L}\approx A_x\frac{U}{L^2}\approx A_y\frac{U}{L^2}\approx A_z\frac{U}{H^2} \quad \text{or} \quad A_x\cong UL \quad \text{and} \quad A_z\cong\frac{H^2}{L^2}A_x$$

So, A_z estimates range from 10^{-5} to 10^{-1} m^{2}s^{-1}, taking 10^{-1}

(i) The z-component equation of motion is

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=$$
$$-\alpha\frac{\partial p}{\partial z}+2\Omega\cos\phi\, u-g+A_x\frac{\partial^2 w}{\partial x^2}+A_y\frac{\partial^2 w}{\partial y^2}+A_z\frac{\partial^2 w}{\partial z^2} \qquad (7.7)$$

scaling

$$\frac{W}{T}+\frac{UW}{L}+\frac{VW}{L}+\frac{W^2}{H}=\alpha\frac{10^7}{H}+2\Omega\cos\phi U-g+10^5\frac{W}{L^2}+10^5\frac{W}{L^2}+10^{-1}\frac{W}{H^2}$$

i.e.

$$10^{-10}+10^{-11}+10^{-11}+10^{-11}=10+10^{-5}-10+10^{-11}+10^{-11}+10^{-11}$$

So, we can ignore all terms except that the pressure term and g will be left with the ***hydrostatic equation***, i.e.

$$\alpha \frac{\partial p}{\partial z} = -g \quad \text{or} \quad dp = -\rho\, g\, dz \tag{7.8}$$

(ii) x–component equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cdots = -\alpha \frac{\partial p}{\partial x} + f\, v - 2\Omega \cos\phi\, w + A_x \frac{\partial^2 u}{\partial x^2} + \cdots$$

scaling

$$\frac{U}{T} + \frac{U^2}{L} + \cdots = -\alpha \frac{\partial p}{\partial x} + fU - 10^{-4} W + 10^5 \frac{U}{L^2} + \cdots$$

or
$$10^{-2} + 10^{-3} + \cdots = \;?\; + 1 - 10^{-3} + 10^{-3} + \cdots$$

Therefore, we have

$$\begin{aligned} 0 &= -\alpha \frac{\partial p}{\partial x} + f\, v \\ 0 &= -\alpha \frac{\partial p}{\partial y} - f\, u \qquad \text{for the interior of the ocean a few} \\ 0 &= -\alpha \frac{\partial p}{\partial z} - g \qquad \text{degrees or more away from the equator.} \end{aligned} \tag{7.9}$$

In the scaling of the large-scale flow we find that both non-linear and friction effects were very small. In other regions they may be more important. The Coriolis term turns out to be important for almost all large-scale flow phenomena.

Rossby number $R_o = \dfrac{\text{Non-linear term}}{\text{Coriolis term}} = \dfrac{U^2}{L}\dfrac{1}{f_0 U} = \dfrac{U}{f_0 L}$

Ekman number $E_x = \dfrac{\text{Friction term}}{\text{Coriolis term}} = A_x \dfrac{U}{L^2}\dfrac{1}{f_0 U} = \dfrac{A_x}{f_0 L^2}$

$$\text{or} \quad E_y = \frac{A_y}{f_0 L^2} \quad \text{or} \quad E_z = \frac{A_z}{f_0 H^2}$$

7.4 Dynamic stability

(1) The Reynolds number determines the ***dynamic stability***. If Re>10^6, then turbulent flow is likely. If U=0.01ms^{-1}, ν≈10^{-6}m^{2}s^{-1}, L=100 m. It would seem

that turbulent flow is likely to occur everywhere.

(2) The effect of density variations on dynamic stability — when density variations occur in a fluid they may enhance or diminish the mechanical effects. The ***static stability*** gives a measure of the effect. If it is negative (unstable) the vertical component of velocity fluctuations is enhanced. If it is positive (stable) the vertical component is diminished. If the turbulence persists it will tend to mix the fluid, that is, make the density more uniform in the vertical.

$$\boxed{\text{mean flow}} \xrightarrow{\text{input}} \boxed{\text{turbulence}} \xrightarrow{\text{output}} \boxed{\text{potential heat}}$$

When turbulence occurs, light fluid is mixed down and heavy fluid up, raising the center of gravity and increasing the gravitational potential energy. The turbulent flow loses some energy to heat (internal energy) through molecular viscosity effects.

(3) A measure of the relative importance of mechanical and density effects is the dimensionless ***Richardson Number***

$$Ri = \frac{N^2}{(\partial u / \partial z)^2} \quad \text{where} \quad N^2 = gE \quad \textbf{\textit{Brunt-Väisälä frequency}}\ (\boldsymbol{N})$$

If $Ri < 0$, density variations enhance the turbulence; if $Ri > 0$ they tend to reduce it. Miles (1961) showed that a stratified shear flow is stable if $Ri > 1/4$ everywhere in the flow.

(4) ***Boussinesq approximation*** — Boussinesq said that, if the density variations are fairly small, to a first approximation we can neglect their effect on the ***mass*** of the fluid but we must retain their effect on the ***weight***. That is, we must include the buoyancy effects but can neglect the variations in the horizontal accelerations for a given force due to the mass variations with density.

Use average density in the *x*- and *y*-components momentum equations.

Use *in situ* density in the hydrostatic equation.