$$r_k = \frac{\sum_{t=1}^{N-k} (x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1}^{N} (x_t - \bar{x})^2}$$

$\omega_p = \frac{2\pi \times 0}{N} = 0$ $\quad \omega_p = \frac{2\pi \times \cancel{\frac{N}{2}}}{\cancel{N}} = \pi \Rightarrow \because 0 < \omega < \pi \Rightarrow p \neq \frac{N}{2}$

$\omega_p = 2\pi p/N \qquad p = 1, \ldots, N/2$

$$\Sigma \cos \omega_p t = \Sigma \sin \omega_p t = 0 \tag{7.2}$$

$$\Sigma \cos \omega_p t \cos \omega_q t = \begin{cases} 0 & p \neq q \\ N & p = q = N/2 \\ N/2 & p = q \neq N/2 \end{cases} \tag{7.3}$$

$$r_k = \frac{\sum_{t=1}^{N-k} (x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1}^{N} (x_t - \bar{x})^2} \tag{2.6}$$

 $0 < \omega < \pi$

W4:HW

2.3 If $x_t = a \cos t\omega$ where a is a constant and ω is a constant in $(0, \pi)$, show that $r_k \to \cos k\omega$ as $N \to \infty$.
(Hint: You will need to use the trigonometrical results listed in Section 7.2. Using equation (7.2) it can be shown that $\bar{x} \to 0$ as $N \to \infty$, so that $r_k \to \Sigma \cos \omega t \cos \omega(t+k)/\Sigma \cos^2 \omega t$. Now use the result that $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$ together with the result that $\Sigma \cos^2 \omega t = N/2$ for a suitably chosen N.)

$$r_k = \frac{\sum_{t=1}^{N-k} \left(a \cos t\omega - \underbrace{\frac{\Sigma x_t}{N}}_{=0}\right)\left(a \cos (t+k)\omega - \underbrace{\frac{\Sigma x_t}{N}}_{=0}\right)}{\sum_{t=1}^{N} \left(a \cos t\omega - \underbrace{\frac{\Sigma x_t}{N}}_{=0}\right)^2}$$

$$= \frac{\Sigma \left[a \cos t\omega \times a \cos (t+k)\omega\right]}{\Sigma a^2 \cos^2 t\omega}$$

$$= \frac{\frac{1}{2} \times \cancel{a^2}\, \Sigma \left[\cos(2t+k)\omega + \cos k\omega\right]}{\cancel{a^2}\, \Sigma \cos^2 t\omega}$$

$$= \sum_{t=1}^{N} \frac{\cos(2t+k)\omega + \cos k\omega}{2 \cos^2 t\omega}$$

$$= \frac{\sum \cos(2t+k)w}{2\sum \cos^2 tw} + \frac{N \cdot \cos kw}{2\sum \cos^2 tw}$$

$$= \frac{\sum \cos(2t+k)w}{2 \times \sum\left(\frac{\cos 2tw + 1}{2}\right)} + \frac{N \cdot \cos kw}{2 \times \sum\left(\frac{\cos 2tw + 1}{2}\right)}$$

$$\doteq \frac{0}{\cancel{2} \times \left(\frac{0 + N}{\cancel{2}}\right)} + \frac{\cancel{N} \cdot \cos kw}{\cancel{2} \times \left(\frac{0 + \cancel{N}}{\cancel{2}}\right)}$$

$$= \cos kw$$

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