I. Introduction to Natural Numbers

A. The counting numbers - 1, 2, 3, 4, ..., *N*

B. Positive integers and, sometimes, zero

C. No negative numbers

D. No fractions

1. Some fractions are composed of natural numbers

2. $\frac{N}{M}$ is a fraction, N and M are natural numbers

3. If N and M are natural numbers then the fraction $\frac{N}{M}$ is known as a ***rational number.***

E. Formal Definition

1. 1 is a natural number (sometimes the natural numbers start at 0)

2. If N is a natural number then $N+1$ is a natural number

3. Note: This is a ***recursive*** definition in which natural numbers are defined in terms of natural number.

F. Successor Operation

1. The successor operation applied to a natural number N generates the next natural number

2. $S(N) = N + 1$

3. The formal definition can now be stated as:
   a. 1 is a natural number (0 is a natural number)

   b. if N is a natural number then $S(N)$ is a natural number

G. ***The essence of the natural number concept is ... closure under the successor operation.*** Richard Dedekind(1888)

II. Defintion of the Natural Numbers: Peano's Postulates

A. The set S of undefined elements called natural numbers has the following properties:

1. S is nonempty.

2. Associated with each natural number $N \varepsilon S$ there is a unique natural number $N' \varepsilon S$ called the successor of N.

3. There is a natural number $N_0 \varepsilon S$ that is not the successor of any natural number.

4. Distinct natural numbers have distinct successors; that is, if N_2 is the successor of N_1, then there is no other N_i for which N_2 is the successor.

5. The only subset of S that contains N_0 and the successors of all its elements is S itself.

B. The axioms 1 - 5 are known as Peano's postulates.

C. Axiom 5 is the basis for the Principle of Mathematical Induction.

1. The natural number N that is not the successor of any other natural number is 1 (0?).

2. The set S of the natural numbers is, then, 1 and all of its successors.

III. Extended Definition of the Natural Number System

A. 1 and 0 are natural numbers.

B. Every natural number has a successor which is also a natural number

1. the successor is the 'next' natural number

2. if N is a natural number then $N+1$ is a natural number.

C. Every natural number except zero has a predecessor, the 'previous' natural number.

D. The definition of the natural numbers states that:

1. ***IF*** a subset S of the natural numbers has the properties:
   a. 0 and/or 1 are/is an element(s) of S

   b. if N + 1 is an element of S whenever N is an element of S

2. ***THEN*** S is equal to the set of natural numbers.

IV. Introduction to the Principle of Mathematical Induction

A. A simple statement embodying the essence of the principle:

If a flight of stairs is designed so that falling off any step inevitably leads to falling off the next, then falling off the top step is a sure way to end up at the bottom.

or

If a flight of stairs is designed so that falling off any step inevitably leads to falling off the next, then falling off the lowest step is a sure way to end up at the bottom.

B. A more sophisticated statement:

If a statement $P(N)$ is true for $N = 1$ and the truth of $P(N)$ implies the truth of the corresponding statement with N replaced by $N + 1$, i.e., $P(N + 1)$, then the statement is true for all positive integers N.

C. Stated as a Rule of Inference:

$$(P(1) \wedge (P(k) \rightarrow P(k+1))) \rightarrow \forall n (P(n)$$

D. The Principle of Mathematical Induction is, then:

1. ***Theorem:***

   Let $P(i)$ be propositions, one for each positive integer i such that:

   a. $P(1)$ is true.

   b. For each positive integer N, $P(N) \Rightarrow P(N+1)$

   Then $P(N)$ is true for each positive integer N.

2. ***Proof:*** Let $X = \{N | N \in S \text{ and } P(N) \text{ is true}\}$

From (a): Since $P(1)$ is true, 1 is an element of X, or $1 \in X$

From (b): $N + 1$ is an element of X whenever N is an element of X.

Therefore $P(N + 1)$ is true whenever $P(N)$ is true.
or
$N + 1 \in X$ whenever $N \in X$ and $P(N + 1)$ is true whenever $P(N)$ is true.

Therefore: Since the only subset of S that contains S and the successors of all its elements is S itself and $X \in S$, $X \equiv S$ by Postulate V.

Hence, because (a) and (b) are true, $P(N)$ is true for all N. *QED*

G. Another Justification of The Principle of Mathematical Induction is:

1. ***Theorem:***
Let $P(i)$ be propositions, one for each positive integer i such that:
   a. $P(1)$ is true.
   b. For each positive integer N, $P(N) \Rightarrow P(N + 1)$

2. ***Proof By Contradiction:***
Assume:
   a. $P(1)$ is true.

   b. For each positive integer N, $P(N) \Rightarrow P(N + 1)$
   and
   c. there exists at least one positive integer M for which $P(M)$ is false

Therefore the subset of $S' \subset S$ of positive integers for which $P(N)$ is false is non-empty.

The well ordered property of the integers tells us that S' has a least element which we denote by M.

$M \neq 1$ since $P(1)$ is true by assumption.

Therefore: $M > 1$ so $M - 1$ is a positive integer.

$M - 1$ is not an element of S' so $P(M - 1)$ must be true.

Because of our conditional assumption (b),
$P(M - 1) \Rightarrow P((M - 1) + 1) = P(M)$

Therefore : Assuming $P(M)$ to be false leads to the conclusion that it is true.

Hence: The assumption that there exists M such that $P(M)$ is false is false. QED

H. We can state the principle of mathematical induction in pseudo-code as follows:

1. Consider the proposition $P(N)$ which is purported to be true for all positive integers, i.e., $\forall N \; P(N)$

2. if $P(1)$ is false then $P(N)$ is false since 1 is a positive integer

3. else begin:
   a. assume that $P(k)$ is true for an arbitrary positive integer k.

   b. if $P(k + 1)$ is false then $P(N)$ is false.

   c. else if $P(k + 1)$ is true then $P(N)$ is true for all positive integers.

V. Example 1:

A. **Theorem: Prove that if N is a positive integer:**

$$\mathbf{1+2+3+\ldots+(N-2)+(N-1)+N=\frac{N\times(N+1)}{2}}$$

or: $P(N) \equiv \sum_{k=1}^{N} k = \frac{N\times(N+1)}{2}$

B. Proof: Basis Step: For $N=1$, $\sum_{k=1}^{1} k = 1 = \frac{1\times(1+1)}{2}$

Inductive Step: If we assume that $P(N)$ is true, or that

$$\sum_{k=1}^{N} k = \frac{N\times(N+1)}{2}$$

we have: $\sum_{k=1}^{N+1} k = \sum_{k=1}^{N} k + (N+1)$

Substituting our inductive assumption $P(N)$: $\sum_{k=1}^{N} k = \frac{N\times(N+1)}{2}$

gives us:

$$\sum_{k=1}^{N+1} k = \sum_{k=1}^{N} k + (N+1) = \sum_{k=1}^{N} k + (N+1) = \frac{N\times(N+1)}{2} + (N+1)$$

So: $\sum_{k=1}^{N+1} k = \frac{N\times(N+1)}{2} + (N+1) = \frac{N\times(N+1)+2\times(N+1)}{2}$

Then: $\sum_{k=1}^{N+1} k = \frac{N^2+3\times N+2}{2} = \frac{(N+1)\times(N+2)}{2}$

so that: $\sum_{k=1}^{N+1} k = \frac{(N+1)\times[(N+1)+1]}{2} \equiv P(N+1)$

Therefore: $P(N) \Rightarrow P(N+1)$

We have shown, then, that: $P(1)$ is true

and

$P(N) \Rightarrow P(N+1)$

Therefore, according to the Principle of Mathematical Induction, $P(N)$ is true for all intergers N. QED

C. Explanation:

Since:

1. $P(1)$ was shown to be true

   and

2. The assumption that $P(N)$ was true was shown to lead to the conclusion that $P(N+1)$ was true

Then:

3. $P(N)$ has been proven to be true for all N according to the Principle of Mathematical Induction.

D. Crucial Point to Consider:

1. **We did NOT assume that**

$$\boldsymbol{P(N) \equiv \sum_{k=1}^{N} k = \frac{N \times (N+1)}{2}}$$

   was true for any particular values of $\boldsymbol{N}$.

2. We demonstrated that **IF** we hypothesized that $P(N)$ is true for some arbitrary and unspecifed value of N the conclusion that $P(N+1)$ must also be true followed.

E. A Bit of History

1. Anecdotal evidence records the nine year old Carl Friedrich Gauss solving this problem in a school classroom.
   a. The instructor had assigned to the class the problem of computing the sum of the first 100 integers.

   b. The teacher noted Gausss staring out a window and threatened to punish him for his misbehavior.

   c. Gauss protested, stating that he had already solved the problem, and displayed the correct answer (5050).

   d. In fact, according to this anecdote, he had solved the general problem of summing the first N integers.

2. Gauss's Method:
   1. The sum of the first N integers can be written:

$$1 \ + \ \ 2 \ + \ 3 \ \ + \ .... \ + \ (N-2) + (N-1) + N = Sum$$

3. Since addition is commutative the order can be reversed and an equivalent expression for the sum is:

$$N \ + \ (N-1) \ + \ (N-2) \ + \ ... + \ \ 3 \ \ + \ \ 2 \ \ + \ \ 1 \ = \ Sum$$

4. If we add these two expression term by term we get:

$$1 \quad + \qquad 2 \qquad + \quad .... \ + (N-1) + N \qquad = Sum$$

$$\underline{N \ \ + (N-1) \ \ + \ \ .... \ + \qquad 2 \qquad + \ 1 \quad = Sum}$$

$$(N+1) + N + 1) + \ \ .... \ + (N+1) + (N+1) = 2 \times Sum$$

5. Since each addend contains N terms the sum can be written as:

$$N \times (N+1) = 2 \times Sum$$

or

$$Sum = \tfrac{N \times (N+1)}{2}$$

which is the result previously proven by mathematical induction.

VI. Example 2 Th: For all positive integers $\boldsymbol{N}$, $\boldsymbol{2N \geq N+1}$

Proof: $P(N) \equiv 2N \geq N+1$

Basis step: Let $N = 1$.

$2N = 2{*}1 = 2$ and $N + 1 = 2$

$P(1) \equiv 2 \geq 2$ which is true by definition

Inductive step: $P(k)$ is: $2k \geq k+1$

Assume that $P(k)$ is true for an arbitrary positive integer k.

$P(k+1) \equiv 2(k+1) \geq (k+1)+1$

$2(k+1) = 2k+2 \geq k+2 = (k+1)+1$

Therefore: $2(k+1) \geq (k+1)+1$

Therefore: $P(k+1)$ is true.

Therefore: $P(k) \Rightarrow P(k+1)$

Therefore: $2N \geq N+1$ for all positive integers $\square$

VII. Example 3: Th: For all positive integers N,

$$\boldsymbol{\sum_{i=1}^{N}(2i+1) = 1+3+5+\ \ldots + (2N-1) = N^2}$$

Note: The sum on the left side of the equation is simply the sum of all odd integers from 1 to $2N-1$

Proof: $P(N)$ is: $\sum_{i=1}^{N}(2i+1) = 1 + 3 + 5 + \ldots + (2N-1) = N^2$

Basis Step: Let $N = 1$. $N^2 = 1$

$P(1)$ is: $1 = 1$ which is true by definition

Inductive Step:

Assume: $P(N)$: $\sum_{i=1}^{N}(2i+1) = 1 + 3 + 5 + \ldots + (2N-1) = N^2$

$P(N+1)$:

$$\sum_{i=1}^{N+1}(2i+1) = 1 + 3 + 5 + \ldots + (2N-1) + 2(N+1) - 1$$

$$= \sum_{i=1}^{N}(2i-1) + 2(N+1) - 1$$

$$= N^2 + 2(N+1) - 1$$

$$= N^2 + 2N + 2 - 1$$

$$= N^2 + 2N + 1$$

$$= (N+1)^2$$

Therefore: $P(N+1)$ is true

Therefore: $P(N) \Rightarrow P(N+1)$

Therefore: $1 + 3 + 5 + \ldots + (2N-1) = N^2$ for all positive integers N $\square$

VIII. Example 4: Th: For all non-negative integers N,

$$\boldsymbol{\sum_{i=0}^{N} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^N = 2^{N+1} - 1}$$

Proof: $P(N)$ is: $2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^N = 2^{N+1} - 1$

Basis Step: $P(0)$ is: $2^0 = 2^1 - 1$ or: $1 = 1$

$P(1)$ is: $2^0 + 2^1 = 3 = 2^{1+1} - 1$ or: $3 = 3$

Inductive Step:

$P(N)$ is: $\sum_{i=0}^{N} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^N = 2^{N+1} - 1$

Assume: $P(N)$ is true.

$P(N+1)$ is:

$$\sum_{i=0}^{N+1} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^N + 2^{N+1}$$

$$= \sum_{i=0}^{N} 2^i + 2^{N+1} = 2^{N+1} - 1$$

Using our inductive assumption we have:

$$\sum_{i=0}^{N+1} 2^i = \sum_{i=0}^{N} 2^i + 2^{N+1} = 2^{N+1} - 1 + 2^{N+1}$$

$$= 2\left(2^{N+1}\right) - 1 = 2^{(N+1)+1} - 1$$

Therefore: $P(N+1)$ is true

and: $P(N) \Rightarrow P(N+1)$

Therefore: $\sum_{i=0}^{N} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^N = 2^{N+1} - 1$

$\square$

IX. Example 5:

$$\boldsymbol{\sum_{i=1}^{N} i^2 = 1^2 + 2^2 + 3^2 + ... + N^2 = \frac{N(N+1)(2N+1)}{6}}$$

for all N.

So: $P(N) \equiv \sum_{i=1}^{N} i^2 = 1^2 + 2^2 + 3^2 + ... + N^2 = \frac{N(N+1)(2N+1)}{6}$

Proof:

1. **Basis Step:** $P(1) \equiv 1^2 = \frac{1(1+1)(2\times 1+1)}{6} = \frac{1\times 2\times 3}{6} = 1$

2. **Inductive Step:**

If we assume :

$P(N) \equiv 1^2 + 2^2 + 3^2 + ... + N^2 = \frac{N(N+1)(2N+1)}{6}$

we have: $P(N+1) \equiv 1^2 + 2^2 + 3^2 + ... + N^2 + (N+1)^2$

$$= \frac{N(N+1)(2N+1)}{6} + (N+1)^2$$

$$= \frac{N(N+1)(2N+1) + 6\times(N+1)^2}{6}$$

$$= \frac{(N^2+N)(2N+1) + 6N^2 + 12N + 6}{6}$$

$$= \frac{2N^3 + 2N^2 + N^2 + N + 6N^2 + 12N + 6}{6}$$

$$= \frac{2N^3 + 9N^2 + 13N + 6}{6} = \frac{(N+1)(2N^2 + 7N + 6)}{6}$$

$$= \frac{(N+1)(N+2)(2N+3)}{6} = \frac{(N+1)\left[(N+1)+1\right]\left[2(N+1)+1\right]}{6}$$

We have shown, then, that: a. $P(1)\ \ is\, true$

b. $P(N) \Rightarrow P(N+1)$

Therefore :

$P(N) \equiv 1^2 + 2^2 + 3^2 + ... + N^2 = \frac{N(N+1)(2N+1)}{6}$

is true for all integers N. $\square$

X. Example in Problem Definition:

A. In the previous examples the expression to be proven was given, e.g., as:

$$P(N) \equiv 1^2 + 2^2 + 3^2 + ... + N^2 = \frac{N(N+1)(2N+1)}{6}$$

or

$$P(N) \equiv \sum_{k=1}^{N} k = \frac{N \times (N+1)}{2}$$

B. In many cases the problem is stated as:

Find a formula for: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ ... \ + \frac{1}{2^N}$
and prove it to be correct.

C. In these cases, in addition to proving $P(N)$, one must devise the expression for $P(N)$.

D. To find an expression for the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ ... \ + \frac{1}{2^N}$ consider the following trials:

1. $$\sum_{i=1}^{N=2} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4} = \frac{2^2-1}{2^2} = \frac{2^N-1}{2^N}$$

2. $$\sum_{i=1}^{N=3} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$$
$$= \frac{2^3-1}{2^3} = \frac{2^N-1}{2^N}$$

In general, then, it would seem that:

$$P(N) \equiv \sum_{i=1}^{N} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ ... \ + \frac{1}{2^N} = \frac{2^N-1}{2^N}$$

Proof:

Basis step: $N = 1$ $\qquad \frac{1}{2^N} = \frac{1}{2} = \frac{2^1-1}{2^1}$

Inductive Step:

For $P(N+1)$ we have:

$$P(N+1) \equiv \sum_{i=1}^{N+1} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^N} + \frac{1}{2^{N+1}}$$

If we assume the truth of $P(N)$ we have

$$\sum_{i=1}^{N+1} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^N} + \frac{1}{2^{N+1}} = \sum_{i=1}^{N} \frac{1}{2^i} + \frac{1}{2^{N+1}}$$

$$= \frac{2^N-1}{2^N} + \frac{1}{2^{N+1}} = \frac{2\times(2^N-1)+1}{2^{N+1}}$$

$$= \frac{2^{N+1}-2+1}{2^{N+1}} = \frac{2^{N+1}-1}{2^{N+1}}$$

So: $P(N) \Rightarrow P(N+1)$

Therefore: $P(1) \wedge P(N) \Rightarrow P(N+1)$ $\square$

XI. The Use of The Principle of Mathematical Induction With a Subset of the Natural Numbers

A. Consider the problem stated below:

Use the Principle of Mathematical Induction to prove that a set with N elements has $\frac{N(N-1)(N-2)}{6}$ subsets containing exactly three elements whenever N is an integer greater than or equal to three.

B. The set of numbers involved in this proof, i.e.: $S' = \{3, 4, 5, \ldots\}$

does not constitute the entire set of natural numbers: $S = \{1, 2, 3, \ldots\}$

C. We note that: $S' \subset S$ or S' is a subset of S

D. We note further that all of Peano's Postulates hold for S', i.e.:

I. S' is nonempty.

II. Associated with each natural number $N \varepsilon S$' there is a unique natural number N' called the successor of N.

III. There is a natural number N_0 that is not the successor of any natural number in S'.

IV. Distinct natural numbers have distinct successors; that is, if N_2 is the successor of N_1, then there is no other N_i for which N_2 is the successor.

V. The only subset of S' that contains S' and the successors of all its elements is S' itself.

E. Consequently, the Principle of Mathematical Induction applies to $S' = \{3,\, 4,\, 5,\, ...\}$ just as it did to $S = \{1,\, 2,\, 3,\, ...\}$

F. Consequently:

1. Basis Step(s):

a. $S = \{a,\, b,\, c\}$ $|S| = 3$

Subsets: $S_0 = \emptyset$ $S_1 = \{a,\, b,\, c\}$

Not counting the empty set we have one subset for $|S| = 3$.

$$\frac{n(N-1)(N-2)}{6} = \frac{3\times(3-1)\times(3-2)}{6} = \frac{6}{6} = 1$$

b. $S = \{a,\, b,\, c,\, d\}$ $|S| = 4$

Subsets: $S_1 = \{a,\, b,\, c\}$ $S_2 = \{a,\, b,\, d\}$
$S_3 = \{a,\, c,\, d\}$ $S_4 = \{b,\, c,\, d\}$

$$\frac{N(N-1)(N-2)}{6} = \frac{4\times(4-1)\times(4-2)}{6} = \frac{24}{6} = 4$$

2. Inductive Step:

We assume $P(N)$ or that the number of ways three subsets of exactly three elements can be chosen from a set of N elements is $\frac{N(N-1)(N-2)}{6}$.

We now create a set S' by adding one additional element x to the original set S, so:

$$S' = S \cup \{x\} \qquad \text{and} \qquad |S'| = N + 1.$$

We can now pick the first element of a subset in $N + 1$ different ways.

Since we have N elements left we can choose the second element in N different ways, leaving $N - 1$ different elements from which to choose our third value.

There are, then, $(N + 1) \times N \times (N - 1)$ different sequences in which we can pick three elements from the $N + 1$ elements of S'.

The elements chosen may be ordered in $3! = 6$ different ways, as shown below for the three elements $a,\ b,$ and c:

1. $\{a,\ b,\ c\}$
2. $\{a,\ c,\ b\}$
3. $\{b,\ a,\ c\}$
4. $\{b,\ c,\ a\}$
5. $\{c,\ a,\ b\}$
6. $\{c,\ b,\ a\}$

(A general proof for N elements is given as Example XIV on page 196)

Order is irrelevant to subset construction since a set is defined only by the elements that it contains regardless of order. There are, then, six ways to choose three distinct elements from the set S to create a subset of three elements, so we must divide the number of choices by six to get for the number of possible subsets of three elements that can be chosen from $N+1$ elements.

Then: $\frac{(N+1)\times N\times(N-1)}{6} = \frac{(N+1)\times\left[(N+1)-1\right]\times\left[(N+1)-2\right]}{6}$
which is $P(N+1)$

3. Therefore we have shown that $P(3)$ is true and
$P(N) \Rightarrow P(N+1)$ $\square$

XII. Example 7: Th: For all integers $\boldsymbol{N}$ such that $\boldsymbol{N \geq 3}$, $\boldsymbol{2N+1 \leq N^2}$

Proof: $P(N)$ is: $2N+1 \leq N^2$

Basis Step: In this case we are only concerned with the integers that are greater than or equal to three. Our basis step, then, is a proof of $P(3)$.

$2{*}3+1=7$ $3^2=9$
P(3) is: $7 \leq 9$ which is true by definition.

Inductive step: $P(k)$ is: $2k+1 \leq k^2$

Assume that $P(k)$ is true for arbitrary $k \geq 3$

$P(k+1)$ is: $2(k+1)+1 \leq (k+1)^2$

$2(k+1)+1 = 2k+3 \leq k^2+3$ from $P(k)$
$< k^2+2k+1 = (k+1)^2$

Therefore: $2(k+1)+1 \leq (k+1)^2$
Therefore: $P(k+1)$ is true.
Therefore: $P(k) \Rightarrow P(k+1)$

Therefore: $2N+1 \leq N^2$ for all $N \geq 3$ $\square$

XI. Example 8

A. Problem: Suppose that m and n are positive integers such that:

1. $m > n$

2. $S_m = \{1, 2, 3, ..., m\}$

3. $S_n = \{1, 2, 3, ..., n\}$

4. $f : S_m \to S_n$.

Prove that f cannot one-to-one.

B. Theorem: $P(n)$: $f : S_m \to S_n$ cannot be one-to-one for $m > n$

C. Basis Step: Let $n = 1$.

1. Therefore $S_n = \{1\}$ and $S_m = \{1, 2, 3, ..., m\}$

2. Therefore, for $x \in S_m$ and $y \in S_m$, $x \neq y$, we must have
   $f(x) = f(y)$

3. Therefore $f : S_m \to S_n$ is not one-to-one by definition.

D. Inductive Assumption:
$f : S_m \to S_n$ cannot be a one-to-one function for $m > n$

E. Let $f : S_m \to S_n$ be a function from $S_m = \{1, 2, 3, ..., m\}$ to
$S_n = \{1, 2, 3, ..., n, n+1\}$
where $m > n + 1$

1. Case 1: If $x = n + 1$ and $f(x) \notin S_n$ then:
   a. f is not a function from $S_m \to S_n$.

   b. Therefore $f : S_m \to S_n$ is not a one-to-one function from $S_m \to S_n$

2. Case 2: $f : S_m \to S_n$ is a function from $S_m \to S_n$
   a. We have, for some $x \in S_m$ and $y \in S_m$, $x \neq y$,
   $f(x) = f(y)$

   b. Therefore $f : S_m \to S_n$ is not one-to-one.

3. Case 3: $f(x) = n+1$ for exactly one element $x \in S_m$
    a. Let $S_m^{'} = S_m - \{x\}$

    b. Then $|S_m^{'}| = n$ and we can consider $f' : S_m^{'} \to S_n$

    c. But, but by the inductive hypothesis $f' : S_m \to S_n$ cannot be a one-to-one function for $m > n$

    d. Therefore $f : S_m \to S_n$ is not one-to-one.

E. $f : S_m \to S_n$ cannot be one-to-one for $m > n$

XII. Applications of Mathematical Induction

A. For our work we will use, primarily, the principle of strong induction.

B. Note that the theory of mathematical induction works ONLY for the natural numbers, i.e., the counting numbers and 0.
   1. It does not apply to the reals.
   2. Many Computer Science problems involve only the natural numbers
      a. The number of bytes of memory of a computer is an integer
      b. The number of times that a given loop is executed is an integer
      c. The pattern of bits held in any word of RAM/data memory can be interpreted as an integer.

C. Example:
   1. Consider the code:

      $$\boldsymbol{int \quad Sum = 0, i, N;}$$
      $$\boldsymbol{for\,(i = 1;\, i < \,= N;\, i+\,+\,)\, Sum = Sum + i;}$$

      which performs N additions.

   2. The same result could be, from Theorem I, obtained by the code:

      $$\boldsymbol{int \quad Sum = 0, i, N;}$$
      $$\boldsymbol{Sum = (N/2){*}(N+1);}$$

      which performs one addition, one division, and one multiplication.

XIII Th: There are N! Permutations of a List S of N Distinct Elements

I. $P(N) \equiv N!$ permutations can be constructed from a list S of N elements.

II. Basis Step(s):

A. $N = 1$ $S_1 = [a]$ Number of permutations $= 1 = N!$

$N = 2$ $S_1 = [a,\, b]$ Number of permutations $= 2 = N!$
$S_2 = [b,\, a]$

$N = 3$ $S_1 = [a,\, b,\, c]$ Number of permutations = 6
$S_2 = [a,\, c,\, b]$ $6 = N!$
$S_3 = [b,\, a,\, c]$
$S_4 = [b,\, c,\, a]$
$S_5 = [c,\, a,\, b]$
$S_6 = [c,\, b,\, a]$

B. Inductive Step:

1. Assume that a list S with N distinct elements has $N!$ permutations.

2. We insert a new element x, which is distinguishable from any of the existing elements of S, in one of the $N!$ permutations of of S.

3. x can be inserted:
   a. As the first element, i.e., before any existing element of S, creating one new list.
   b. After the last element of S creating one more new list.
   c. Between any two existing elements of S, creating $N - 1$ new lists.

4. For each of the existing permutations of S the addition of a new element has created
$$1 + 1 + (N - 1) = N + 1 \qquad \text{new permutations.}$$

5. The total number of permutations of a list S with $N + 1$ elements is, then: $(N + 1) \times N! = (N + 1)!$

C. So: $P(1)$ is true and $P(N) \Rightarrow P(N + 1)$
Therefore $P(N)$ is true for all N.

XIV: Th: DeMorgan's Law: $\boldsymbol{\overline{\bigcup_{j=1}^{n} A_j} = \bigcap_{j=1}^{n} \overline{A_j}}$

A. Basis step: $n = 2$ $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ DeMorgan's Law for two sets.

Recall: $x \in A_1 \cup A_2 \rightarrow x \in A_1 \vee x \in A_2$

Therefore: $x \in \overline{A_1 \cup A_2} \rightarrow \neg\left(x \in A_1 \vee x \in A_2\right)$

$$\equiv x \notin A_1 \wedge x \notin A_2$$

$$\equiv x \in \overline{A_1} \wedge x \in \overline{A_2}$$

$$\equiv x \in \overline{A_1} \cap \overline{A_2}$$

B. Inductive step:

Assume: $\overline{\bigcup_{j=1}^{n} A_j} = \bigcap_{j=1}^{n} \overline{A_j}$

Then: $\overline{\bigcup_{j=1}^{n+1} A_j} = \overline{\bigcup_{j=1}^{n} A_j \cup A_{n+1}}$ by the definition of union

So: $\overline{\bigcup_{j=1}^{n+1} A_j} = \overline{\bigcup_{j=1}^{n} A_j} \cap \overline{A_{n+1}}$ by DeMorgan's law for two sets.

Here, one set is $\bigcup_{j=1}^{n} A_j$ and the other is A_{n+1}

Then: $\overline{\bigcup_{j=1}^{n+1} A_j} = \bigcap_{j=1}^{n} \overline{A_j} \cap \overline{A_{n+1}}$ according to our inductive assumption.

and: $\overline{\bigcup_{j=1}^{n+1} A_j} = \bigcap_{j=1}^{n+1} \overline{A_j}$ by the associative law for disjunction

□

XV: Detecting a Counterfeit Coin

A. Problem Description

1. You are given a collection of coins with the proviso that one coin is a counterfeit.

2. Your scale is a simple balance scale that detects only that the object in one pan is lighter, heavier, or the same weight as the object in the other pan.

3. The counterfeit coin is lighter than any of the others.

4. Prove by induction that N weighings are sufficient to detect one counterfeit coin among 3^N coins.

B. Basis Step:

1. For three coins, one weighing is sufficient.
   a. If the two coins, one in each pan of the balance, are of equal weight, the remaining coin is the counterfeit. Otherwise, the lighter coin is the counterfeit.

   b. 3^N coins $\equiv 3^1$ coins $\equiv 1$ weighing

2. For four coins two weighings are required.

   If two coins are placed in each pan and one pan is lighter one additional weighing is sufficent to detect the counterfeit in the lighter pan.

3. For nine coins, two weighings can be necessary.
   a. If two coins on the balance are of equal weight, one of the remaining two coins is the counterfeit and one more weighing will detect the lighter coin which is the counterfeit.

   b. Otherwise, one weighing will detect the lighter coin which is the counterfeit.

   c. 3^N coins $\equiv 3^2$ coins $\equiv 9$ coins $\equiv 2$ weighings

4. For ten coins, three weighings may be necessary.

   If three coins on the balance are of equal weight, four coins remain, requiring two additional weighings. Otherwise, one additional weighing is required to determine which of the lighter three is counterfeit.

C. Inductive Assumption: N weighings are sufficient to detect a single counterfeit coin among 3^N coins.

D. Inductive Step:

1. Consider 3^{N+1} coins.

2. $3^{N+1} = 3 \times 3^N$

3. Therefore we can divide our set of 3^{N+1} coins into three sets of 3^N coins.

4. One weighing is required to determine which of the three sets contains the counterfeit coin.
   a. If, on the first weighing with 3^N coins on each pan of the scales, one side is lighter than the other, the lighter side contains the counterfeit coin.

   b. If both sides balance, the remaining set of 3^N coins contains the counterfeit coin.

5. According to our inductive assumption we can isolate the counterfeit coin from a set of 3^N coins in N weighings.

6. Since it requires at most one additional weighing to isolate the counterfeit coin from a set of 3^{N+1} coins the isolation will require at most $N + 1$ weighings.

E. Test: Suppose that we have a set of $3^{N+1}+1$ coins.

1. We can place two sets of 3^N coins on each pan.
   a. If the two sets are of equal weight there will be more than 3^N coins remaining to be tested.

   b. If the two sets are of unequal weight we can test the lighter set of 3^N coins in N weighings.

2. For the case of equal weights we can divide the set of 3^N+1 into two sets of 3^{N-1} coins and one set of $3^{N-1}+1$ coins and repeat step E.1.

3. Proceeding in this way for a total of N repetitions leaves us with four coins.
   a. One more weighing is not guaranteed to isolate the counterfeit coin, two may be required.

   b. Therefore $(N+1)+1$ weighings may be required.

4. Therefore a set of $3^{N+1}+1$ coins may require more than $N+1$ weighings to determine the counterfeit coin.

F. Therefore: N weighings are sufficient to detect a single counterfeit coin among 3^N coins.

XVI: Detecting a Counterfeit Coin II

A. Problem Description

1. You are given a collection of coins with the proviso that one coin is a counterfeit.

2. Your scale is a simple balance scale that detects only that the object in one pan is lighter, heavier, or the same weight as the object in the other pan.

3. The counterfeit coin is lighter than any of the others.

4. Prove by induction that N weighings may not be sufficient to detect a single counterfeit coin among 3^N+k coins, where k is some integer such that $3^N < 3^N+k < 3^{N+1}$.

B. Basis Step:

1. For three coins, 3^1, one weighing is sufficient.
   a. If the two coins, one in each pan of the balance, are of equal weight, the remaining coin is the counterfeit. Otherwise, the lighter coin is the counterfeit.

   b. 3^N coins $\equiv 3^1$ coins $\equiv 1$ weighing

2. For four coins two weighings are required.
   a. If two coins are placed in each pan and one pan is lighter one additional weighing is sufficent to detect the counterfeit in the lighter pan.

   b. $3^N + 1$ coins $\equiv 3^1 + 1$ coins $\equiv 2$ weighings

   c. Therefore: Number of weighings $> N$ when the number of coins is greater than 3^N.

3. For nine coins, two weighings can be necessary.
   a. If the six coins on the balance (three on each pan) are of equal weight, one of the remaining three coins is the counterfeit and one more weighing will detect the lighter coin which is the counterfeit.

   b. Otherwise, one additional weighing will detect the lighter coin which is the counterfeit.

   c. 3^N coins $\equiv 3^2$ coins $\equiv 9$ coins $\equiv 2$ weighings

4. For ten coins, three weighings may be necessary.
   a. If three coins on the balance are of equal weight, four coins remain, requiring two additional weighings. Otherwise, one additional weighing is required to determine which of the lighter three is counterfeit.

   b. $3^N + 1$ coins $\equiv 3^2 + 1$ coins $\equiv 10$ coins $\equiv 3$ weighings
   Therefore the number of weighings is greater than N when the number of coins is $3^N + 1$.

C. Inductive Assumption: N weighings may not be sufficient to detect a single counterfeit coin among a maximum of $3^N + 1$ coins.

D. Inductive Step:

1. Consider $3^{N+1} + k$ coins.

2. $3^{N+1} + k = 3 \times 3^N + k$

3. Therefore we can divide our set of $3^{N+1} + k$ coins into two sets of 3^N coins and one set of $3^N + k$ coins.

4. The worst case is that in which the two sets of 3^N coins are of equal weight, requiring that the counterfeit coin be contained in the third set of $3^N + k$ coins. One weighing determines this state.

5. Our inductive assumption states that N weighings may not be sufficient to isolate the counterfeit coin from $3^N + k$ coins.

6. We used one weighing to isolate the set of $3^N + k$ coins that contained the countefeit.

7. Our inductive assumption states that N weighings may not be sufficient to isolate the counterfeit coin from $3^N + k$ coins.

8. Therefore we have demonstrated that N weighings may not be sufficient to isolate the counterfeit coin from $3^{N+1} + k$ coins.

E. Therefore: N weighings may not be sufficient to detect a single counterfeit coin among a maximum of $3^N + k$ coins.