Name:______________________________ **CWRU ID:**________________

This is an open book, open notes examination. No electronic devices allowed. Any communication with anyone except the instructor while taking this exam is forbidden and will result in a failing grade. **Put your name and CWRU ID on every page.**

1. Prove or disprove $(\forall x\, P(x)) \vee (\forall x\, Q(x)) \equiv \forall x\, (P(x) \vee Q(x))$

Answer: Consider the predicates:
$P(x) \equiv x$ is even and $Q(x) \equiv x$ is odd for $x \in Z$.

$\forall x\, P(x)$ states that all $x \in Z$ are even, which is ***false***.

$\forall x\, Q(x)$ states that all $x \in Z$ are odd, which is ***false***.

Therefore: $(\forall x\, P(x)) \vee (\forall x\, Q(x))$ is ***false***.

$\forall x\, (P(x) \vee Q(x))$ states that, for all $x \in Z$, x is either even or odd, which is ***true***.

Therefore $(\forall x\, P(x)) \vee (\forall x\, Q(x))$ is not logically equivalent to $\forall x\, (P(x) \vee Q(x))$

As a more general case, consider $P(x) = \neg Q(x)$

Then: $\forall x\, (P(x) \vee Q(x)) \equiv \forall x\, (\neg Q(x) \vee Q(x))$
which is a tautology.

$(\forall x\, P(x)) \vee (\forall x\, Q(x)) \equiv (\forall x\, \neg Q(x)) \vee (\forall x\, Q(x))$
which is true only in a domain in which $Q(x)$ is true (or false) for all elements.

Name:____________________________ **CWRU ID:**________________

2. Prove that if $A \times B = \emptyset$ then either $A = \emptyset$ or $B = \emptyset$

Answer: Proof by cases:

Case 1: Neither A nor B are empty.

Assume $A = \{a\}$ and $B = \{b\}$

Then: $A \times B = \{(a,\, b)\} \neq \emptyset$

Case 2: A is empty and B is not empty.

Assume $A = \emptyset$ and $B = \{b\}$

Then: $A \times B = \emptyset$

Case 3: A is not empty and B is empty.

Assume $A = \{a\}$ and $B = \emptyset$

Then: $A \times B = \emptyset$

Therefore if $A \times B = \emptyset$ then either $A = \emptyset$ or $B = \emptyset$

Name:______________________________ **CWRU ID:**________________

3. Prove the following theorem:

An integer n is odd if and only if n^2+2n is odd.

Answer: left-to-right: n is odd $\rightarrow n^2+2n$ is odd.

If n is odd then $n=2k+1$ where k is some integer.

Then n^2+2n $=(2k+1)^2+2(2k+1)$

$=4k^2+8k+2+1$

$=2\times(2k^2+4k+1)+1$

$=2m+1$ where m is some integer.

Therefore: If n is odd then n^2+2n is odd.

right-to-left: n^2+2n is odd $\rightarrow n$ is odd

If n^2+2n is odd then $n^2+2n=2k-1$

Then: $n^2+2n+1=(n+1)^2=2k$

Therefore: $(n+1)^2$ is even and n must be odd.

Therefore: An integer n is odd if and only if n^2+2n is odd.

Name:________________________________ **CWRU ID:**_________________

4. A ***perfect square*** is an integer whose square root is also an integer. Therefore 4 is a perfect square since $\sqrt{4} = 2$, 9 is a perfect square since $\sqrt{9} = 3$, and 16 is a perfect square since $\sqrt{16} = 4$

Prove or disprove the following theorem:

Th: if $k > 1$ then $2^k - 1$ is not a perfect square.

Answer: Proof by contradiction:
Assume that $2^k - 1$ is a perfect square for $k > 1$.

If $2^k - 1$ is a perfect square then $2^k - 1 = n^2$

Then, since 2^k must be even, $n^2 = n \times n$ must be odd.

Therefore: n is odd and $n = 2 \times j + 1$ where j is some integer.

Therefore: $2^k - 1 = (2 \times j + 1)^2 = 4 \times j^2 + 4 \times j + 1$

and: $2^k = 4 \times j^2 + 4 \times j + 2 = 2 \times (2 \times j^2 + 4 \times j + 1)$

Since $2 \times j^2 + 4 \times j + 1$ is the sum of two even numbers and 1,

then $2 \times j^2 + 4 \times j + 1$ must be odd.

Therefore: $2 \times (2 \times j^2 + 4 \times j + 1)$ is divisible by 2 but not by 4

Therefore: 2^k is divisible by 2 but not by 4.

This is a contradiction for $k \geq 2$, hence for $k > 1$.

Therefore: if $k > 1$ then $2^k - 1$ is not a perfect square.

Name:______________________________ **CWRU ID:**________________

5. Prove that if f and $f \circ g$ are one-to-one functions then g is one-to-one.

Answer: Contrapositive proof

A contrapositive statement of
If f and $f \circ g$ are one-to-one functions then g is one-to-one.

is: If g is not one-to-one then f and $f \circ g$ are not one-to-one

If: $g : A \to B$ and $f : B \to C$ then $f \circ g : A \to C$

If $g : A \to B$ is not one-to-one then, by definition, there are distinct elements $a_1 \in A$ and $a_2 \in A$ such that $g(a_1) = g(a_2)$.

Under these conditions $f(g(a_1)) = f(g(a_2))$ and $f \circ g : A \to C$ is not one-to-one.

This is true regardless of whether $f : B \to C$ is one-to-one or not.

Therefore: If f and $f \circ g$ are one-to-one functions then g is one-to-one.

Name:________________________________ **CWRU ID:**_________________

6. Prove or disprove the statement:

   Every positive integer can be written as the sum of the squares of three integers.

   Example: $9 = 2^2 + 2^2 + 1^2$

Answer: A counter example is 7, i.e., 7 cannot be written as the sume of three squares.

Proof by cases:

1. The inclusion of a single instance of the square of any single integer $n \geq 3$ will result in a square that is greater than 7.

2. The inclusion of 2 twice results in $2^2 + 2^2 + i^2 = 8 + i^2$ which is greater than 7.

3. We are left with:
   1. $0^2 + 0^2 + 0^2 = 0 \neq 7$
   2. $1^2 + 0^2 + 0^2 = 1 \neq 7$
   3. $1^2 + 1^2 + 0^2 = 2 \neq 7$
   3. $1^2 + 1^2 + 0^2 = 2 \neq 7$
   4. $1^2 + 1^2 + 1^2 = 3 \neq 7$
   5. $2^2 + 1^2 + 0^2 = 5 \neq 7$
   6. $2^2 + 1^2 + 1^2 = 6 \neq 7$

Therefore 7 cannot be written as the sum of three integers squared.

Name:________________________________ **CWRU ID:**_________________

7. A person deposits \$$A$ in an account that yields R% interest, with the interest compounded annually at the end of each year.

a. Set up a recurrence relation for the total amount in account at the end of n years. Be sure to specify the initial term.

Answer: The amount after $n-1$ years is multiplied by $1+\frac{R}{100.0}$ to give the amount after n years, since R% of the value must be added to account for the interest. Thus we have
$a_n = (1+\frac{R}{100.0}) \times a_{n-1}$

The initial condition is $a_0 = A$

b. Find an explicit formula for the amount in the account at the end of n years.

Answer: Since we multiply by $(1+\frac{R}{100.0})$ for each year, the solution is
$a_n = A \times (1+\frac{R}{100.0})^n$

Name:______________________________ **CWRU ID:**________________

8. Prove that if x^3 is irrational then x is irrational.

Answer:

a. Use a proof by contraposition:

The contrapositive equivalent of : If x^3 is irrational then x is irrational.

is: If x is rational then x^3 is rational.

If x is rational then $x = \frac{m}{n}$ where m and n are integers.

Then $x^3 = (\frac{m}{n})^3 = \frac{m^3}{n^3}$ where both m^3 and n^3 are integers because both m and n are integers.

Therefore $x^3 = \frac{m^3}{n^3}$ is the quotient of integers and is, therefore, rational.

Therefore if x is rational then x^3 is rational.

Therefore if x^3 is irrational then x is irrational. $\square$

Name:________________________________ **CWRU ID:**_________________

9. Prove that, if x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is not an integer and $\lceil x \rceil - \lfloor x \rfloor = 0$ if x is an integer.

Answer: If x is not an integer, then $\lceil x \rceil$ is the integer just larger than x and $\lfloor x \rfloor$ is the integer just smaller than x.

Therefore $\lceil x \rceil - \lfloor x \rfloor = 1$.

If x is an integer, then $\lceil x \rceil = x$ and $\lfloor x \rfloor = x$ so $\lceil x \rceil - \lfloor x \rfloor = 1$.

Name:________________________________ **CWRU ID:**________________

10. If $f : A \rightarrow B$ where A and B are finite sets with $|A| = |B|$ prove that f is one-to-one if and only if f is onto.

Answer:
Proof of: f is one-to-one $\rightarrow f$ is onto

If f is one-to-one then every $a \in A$ is assigned to a unique $b \in B$.

Then every $a \in A$ is assigned to a different $b \in B$

Since $|A| = |B|$ every unique element $b \in B$ is an image of a unique $a \in A$.

Therefore f is onto.

Proof by contradiction of: f is onto $\rightarrow f$ is one-to-one

Since f is onto every $b \in B$ is an image of some $a \in A$.

If f is not one-to-one then at least two elements of A must have the same $b \in B$ as their image.

Then $|A| \geq |B| + 1$ and $|A| \neq |B|$

Since $|A| = |B|$ we must have that f is one-to-one.