I. Introduction to Relations

A. Definition: If A are B sets a ***binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$*** is a subset of $A \times B$.

B. Recall: $A \times B$, or the ***Cartesian product*** of A and B, is the set of **ordered** pairs $(a,\ b)$, where $a \in A$ and $b \in B$.

1. If $A = \{1,\ 2\}$ and $B = \{x,\ y,\ z\}$ then

   a. $A \times B = \{(1,\ x),\ (1,\ y),\ (1,\ z),\ (2,\ x), (2,\ y),\ (2,\ z)\}$

   b. $B \times A = \{(\ x,\ 1),\ (x,\ 2),\ (y,\ 1),\ (y,\ 2), (z,\ 1),\ (z,\ 2)\}$

2. Note that: $A \times B \neq B \times A$ unless $A = \emptyset$, $B = \emptyset$, or $A = B$

C. Second Definition: A binary relation R from A to B is a set R of ordered pairs $(a,\ b)$ in which $a \in A$ and $b \in B$.

1. Notation: $a\,R\,b$ denotes that $(a,\ b) \in R$

2. If $(a,\ b) \in R$ then a is said to be related to b by R.

D. Example 1: Let A be the set of students at CWRU and B the set of courses offered at CWRU. Then let R be the the relation that consists of all ordered pairs $(a,\ b)$ where a is a CWRU student and b a course in which a is enrolled.

1. If Sam and Mary are enrolled in EECS 302 then

$$(Sam,\ EECS\,302)$$

and

$$(Mary,\ EECS\,302)$$

are elements of R.

2. Note 1: If a student is not enrolled in any course then there are no pairs in $A \times B$ with that students name as the first element.

3. Note 2: If a particular course is not offered then there are no pairs in $A \times B$ with that course as the second element.

E. Example 2: Let Z be the set of all integers. Then let E be a relation from Z to Z such that :

1. $m, n \in Z$
2. $m\,E\,n \equiv m - n$ is even
3. $4\,E\,0$ because $4 - 0 = 4$ which is even
4. $(5\,,\,2) \notin E$ because $5 - 2 = 3$ which is not even

F. Example Problem: Prove that if n is any odd integer then $n\,E\,1$

1. If n is odd then $n = 2 \times k + 1$ where k is some integer.
2. Then: $n - 1 = (2 \times k + 1) - 1 = 2 \times k$
3. $2 \times k$ is even by definition.
4. Therefore: If n is any odd integer then $n\,E\,1$

G. Variation on the Example Problem: (% $\equiv$ modulus)
Prove that $n\,E\,m$ if and only if $m\,E\,n$

1. For any integer l, either $l\,\%\,2 = 0$ or $l\,\%\,2 = 1$
   If $l\,\%\,2 = 0$ then l is even and

   if $l\,\%\,2 = 1$ then l is odd
2. If $n\,E\,m$ then $n - m$ is even
   and $n - m = 2 \times k$
   a. Therefore $-(n - m) = -2 \times k$

   b. So: $m - n = 2 \times -k$

   c. Therefore: $m - n$ is even.

   d. Therefore: $m\,E\,n$

3. If $m\,E\,n$ then $m - n$ is even
   and $m - n = 2 \times j$
   a. Therefore $-(m - n) = -2 \times j$

   b. So: $n - m = 2 \times -j$

   c. Therefore: $n - m$ is even.

   d. Therefore: $n\,E\,m$

4. Therefore: $m\,E\,n \rightarrow n\,E\,m$ and $n\,E\,m \rightarrow m\,E\,n$
   so: $n\,E\,m \leftrightarrow m\,E\,n$

H. Example of a Relation on a Power Set.
1. Let $X = \{a, b, c\}$

2. Then:
   $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

3. Then if A and B are subsets of $P(X)$ we define the relation $\boldsymbol{S}$
   as $A\,\boldsymbol{S}\,B \equiv A$ has at least as many elements as B

4. Therefore:
   a. $\{a, b\}\ \boldsymbol{S}\ \{c\}$
   b. $\{a, b, c\}\ \boldsymbol{S}\ x$ where $x \in P(X)$
   c. $(\{c\},\ \{a, b, c\}) \notin \boldsymbol{S}$

I. Cardinality of Relations.
1. Recall: If A are B sets a ***binary relation from*** $\boldsymbol{A}$ ***to*** $\boldsymbol{B}$ is a subset of $A \times B$.
2. Therefore a relation R on a set A is a subset of $A \times A$

3. If $|A| = n$ then $|A \times A| = n^2$

4. The number of subsets of A is 2^n and the number of subsets of $A \times A$ is 2^{n^2}

5. Therefore: $|R| \leq 2^{n^2}$

II. Relations versus Functions

A. Recall:

1. $f : A \to B$ assigns exactly one element of the codomain B to each element of the domain A.
   a. If $a \in A$ then $f(a) = b \in B$

   b. Therefore: A function is a subset of $A \times B$

2. Hence: A function is a relation.

B. If A are B sets a ***binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$*** is a subset of $A \times B$.

1. If $A = (a,\ b,\ c)$ and $B = (1,\ 2,\ 3\}$
   then a relation R could be $R = \{(a,\ 1),\ (a,\ 2),\ (a,\ 3)\}$

2. In the language of functions, element a of the domain is being assigned to three elements of the codomain

3. Hence, R cannot be a function.

C. So: All functions are relations but not all relations are functions.

III. Properties of Relations: Reflexivity

A. Definition:

1. A relation R on a set A is called ***reflexive*** if and only if
   $(a,\ a) \in R$ for all $a \in A$

   or

2. A relation R on a set A is called ***reflexive*** if and only if $a\ R\ a$ for all $a \in A$

   or

3. A relation R on a set A is called ***reflexive*** if and only each element of A is related to itself.

   or

4. A relation R on a set A is called ***reflexive*** if and only if
   $\forall a \in A((a,\ a) \in R)$

B. Example 1:

If $A = \{a,\, b,\, c\}$ and

$R = \{(a,\, a),\, (a,\, b),\, (a,\, c),\, (b,\, b),\, (b,\, c),\, (c,\, c)\}$

then R is reflexive because each element is related to itself.

C. Example 2:

If $A =$ is the set of all integers then $R \equiv a$ divides y is reflexive because each integer divides itself, or $\frac{x}{x} = 1$.

D. Example Problem: How many reflexive relations are there on a set S where $|S| = n$

1. R is a subset of $S \times S$

2. Therefore the relation is determined by choosing ordered pairs from the n^2 ordered pairs that are the elements of $S \times S$

3. If R is reflexive, each of the pairs $(x,\, x)$ must be elements of R.

4. Each of the other $n \times (n - 1)$ ordered pairs may or may not be elements R.

5. The number of ways to choose an ordered pair with x as the first element from S and the second element $y \neq x$ is the number of ways to pick an ordered pair from $S \times (S - x)$ or $2^{n(n-1)}$

IV. Properties of Relations: Symmetry

A. Definition:

1. A relation R on a set A is called ***symmetric*** if and only if for all $x,\, y\, \in A$ if $x\, R\, \, y$ then $y\, R\, X$

   or

2. A relation R on a set A is called ***symmetric*** if and only if for all $x,\, y\, \in A$ if $(x,\, y)\, \in\, R\, \, y$ then $(y,\, x)\, \in\, R$

   or

3. A relation R on a set A is called ***symmetric*** if and only if any one element is related to any other element then the second element is related to the first.

   or

4. A relation R on a set A is called ***symmetric*** if and only if $\forall x \forall y ((x,\, y) \in R \rightarrow (y,\, x) \in R)$

B. Examples:

1. If $x,\ y \in Z$, the set of all integers, and $R \equiv x + y \leq 6$ then R is symmetric because if $x + y \leq 6$ then $y + x \leq 6$

2. If $A = \{1,\ 2,\ 3,\ 4\}$ and $R = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1)\}$ then R is symmetric because whenever $(x,\ y) \in R$ we have that $(y,\ x) \in R$

V. Properties of Relations: AntiSymmetry

A. Definition:

1. A relation R on a set A is called ***antisymmetric*** if and only if for all $x,\ y \in A$ if $x\,R\,y$ and $y\,R\,x$ then $x = y$

or

2. A relation R on a set A is called ***antisymmetric*** if and only if for all $x,\ y \in A$ if $(x,\ y) \in R$ and $(y,\ x) \in R$ then $x = y$

or

3. A relation R on a set A is called ***antisymmetric*** if and only if the only way to have x related to y and y related to x is if $x = y$

or

4. A relation R on a set A is called ***antiymmetric*** if and only if $\forall x \forall y(((x,\ y) \in R \wedge (y,\ x) \in R) \rightarrow x = y)$

B. Example: If $A = Z$, the set of all integers then

1. $R_1 = \{(a,\ b) \,|\, a \leq b\}$ is antisymmetric because

$$a \leq b \wedge a \leq b \rightarrow a = b$$

2. $R_2 = \{(a,\ b) \,|\, a = b + 1\}$ is antisymmetric because

$$a = b + 1 \rightarrow a \neq b$$

VI. Properties of Relations: Transitivity

A. Definition:

1. A relation R on a set A is called ***transitive*** if and only if for all $x,\ y,\ z\ \in A$ if $x\,R\ y$ and $y\,R\,z$ then $x\,R\,z$

or

2. A relation R on a set A is called ***transitive*** if and only if for all $x,\ y\ \in A$ if $(x,\ y)\ \in\ R$ and $(y,\ z)\ \in\ R$ then $(x,\ z)\ \in\ R$

or

3. A relation R on a set A is called ***transitive*** if one element x is related to any other element y and the second element y is related to a third element z then the first element x must be related to the third element z.

or

4. A relation R on a set A is called ***transitive*** if and only if $\forall x \forall y(((x,\ y) \in R \wedge (y,\ z) \in R) \rightarrow (x,\ z) \in R)$

B. Examples:

1. $R = \{(x,\ y)\,|\,x \leq y\}$ is transitive since

$$x \leq y \wedge y \leq z \rightarrow x \leq z$$

2. Suppose: $R \equiv x\,\%\ y = 0$ or y divides x with no remainder.
   a. $x\ \%\ y = 0$ means that $x = i \times y$
   b. $y\ \%\ z = 0$ means that $y = k \times z$
   c. Therefore: $x = i \times y = i \times k \times z = j \times z$
   d. Therefore: $x\ \%\ z = 0$
   e. So: $(x\ \%\ y = 0) \wedge (y\ \%\ z = 0) \rightarrow x\ \%\ z = 0$

VII. Equivalence Relations

A. Definition: If A is a set and R a relation on A, $R$is an equivalence relation if and only if R is reflexive, symmetric, and transistive.

B. Example: Let S be the set of all digital circuits with a fixed number n of inputs.

Let the relation E be defined for all circuits $C_1,\ C_2 \in S$ as $C_1\ E\ C_2 \equiv C_1$ has the same input/output table as C_2

1. E is reflexive, or $C\ E\ C$ for any digital circuit C.

   This is true because any circuit C has the same input/output table as itself.

2. E is transitive, or $C_1\ E\ C_2 \wedge C_2\ E\ C_3 \rightarrow C_1\ E\ C_3$
   a. $C_1\ E\ C_2$ means that C_1 has the same input/output table as C_2

      and

   b. $C_2\ E\ C_3$ means that C_2 has the same input/output table as C_3.
   c. Hence, we must have that C_1 has the same input/output table as C_3.

   d. Therefore $C_1\ E\ C_3$

   e. Hence: $C_1\ E\ C_2 \wedge C_2\ E\ C_3 \rightarrow C_1\ E\ C_3$

   f. Therefore E is transitive.

2. E is symmetric, or $C_1\,E\,C_2 \rightarrow C_2\,E\,C_1$
   a. Suppose $C_1\,E\,C_2$

   b. Therefore C_1 has the same input/output table as C_2.

   c. Since C_1 has the same input/output table as C_2 the two input/output tables must be the same.

   d. Therefore C_2 has the same input/output table as C_1.

   e. Therefore E is symmetric.

3. Since E is reflexive, transitive, and symmetric E is an equivalence relation.

VIII. Combining Relations or Composite Relations

A. Definition 1: If R is a relation from the set A to a set B and S is a relation from the set B to a set C then the ***composite of R and S*** is arelation consisting of ordered pairs $(a,\,c)$ where $a \in A$ and $c \in C$ and for which there exixts an element $b \in B$ such that $(a,\,b) \in R$.

   The composite of R and S is denoted by $S \circ R$

B. Methodology: Computing the composite of two relations R and S requires that one finds elements that are the second element of ordered pairs in the first relation and first element of ordered pairs in the second relation

C. Example:

1. Let:
   a. $A = \{1,\ 2,\ 3\}$ and $B = \{a,\ b,\ c,\ d\}$ and $C = \{0,\ 1,\ 2\}$
   b. Let $R = \{(1,\ a),\ (1,\ d),\ (2,\ c),\ (3,\ a),\ (3,\ d)\}$
   c. Let $S = \{(a,\ 0),\ (b,\ 0),\ (c,\ 1),\ (c,\ 2),\ (d,\ 1)\}$

2. $S \circ R$ is computed by finding all ordered pairs in S where the second element of the ordered pair in R agrees with the first element of the ordered pair in S.

3. The$n$: $S \circ R = \{(1,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 0),\ (3,\ 1)\}$

C. Definition 2: Since relations from A to B are subsets of $A \times B$ two relations R and S can be combines in any way that two sets can be combined.

1. Example 1:
   a. Let: $A = \{1,\ 2,\ 3\}$ and $B = \{1,\ 2,\ 3,\ 4\}$
   b. Let: $R_1 = \{(1,\ 1),\ (2,\ 2),\ (3,\ 3)\}$ and

      $R_2 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4)\}$
   c. Then $R_1 \cup R_2 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 2),\ (3,\ 3)\}$

2. Example 2:
   a. Let: $A =$ the set of all students at CWRU and $B =$ the set of all courses at CWRU
   b. Let: $R_1 =$ all ordered pairs $(a,\ b)$ where $a \in A$ is a student who has taken course $b \in B$

      and

      $R_2 =$ all ordered pairs $(a,\ b)$ where a is a student who requires course b to graduate.
   c. Then $R_1 \cap R_2 =$ is the set of all ordered pairs $(a,\ b)$ where a is a student who has taken curse b and needs this course to graduate.