I. Sets

A. Formal Definition: No formal specification of the term ***set***.

B. The definition that is used is a definition of the operations that can be performed on a set - anything that is amenable to having these operations performed on it is a set.

C. Informal Definition: A set is a collection into a whole of definite, distinct objects of our intuitionperception, or our thought, which are called the elements of the set. Georg Cantor

D. Text Definition: A set is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

E. We will use, because your text does, capital letters to denote sets, and lower case letters to denote elements of a set.

F. Examples:

1. $E = \{x \mid x \text{ is an Englishman}\}$.
   a. The set is designated by E.

   b. An ***element*** of the set, i.e., an Englishman, is designated by x.

2. $X = \{x \mid x \text{ is an even integer}\}$
   a. X is the set of all even integers

   b. If x is an element of the set X, x is an even integer.

3. $Y = \{y \mid y \text{ is an odd integer where } y > 6 \text{ and } y < 122\}$
   a. Y contains the odd integers greater than 6 and less than 122.

   b. y is an odd integer, $6 < y < 122$

4. $P = \{p \mid p \text{ is a point on a line } l\}$
   a. P is the set of all of the points that make up the line l.

      Note: This set could have an infinite number of elements.

   b. l is an element of the set L of all lines on a plane.

5. $L = \{l \mid l \text{ is a line on a plane } X\}$
   a. L is the set of all lines that compose a plane X.

   b. P (defined above) is the set of all points that compose a line.

   c. Hence, L is the set of all sets P that form a line on the plane X.

6. $S = \{x \mid a \leq x \leq b\}$ S is the set of all elements x contained in the ***closed interval*** $a \leq x \leq b$

7. $S = \{x \mid a < x < b\}$ S is the set of all elements x contained in the ***open interval*** $a < x < b$

G. All of the above examples use what is know as the ***set builder*** notation to define a set.
   1. Chooose a property P such that all elements of the set X to te defined possess that property.

   2. Then the set definition is: $X \equiv \{x \mid P(x)\}$

H. Sometimes we don't want to, or can't, specify a general common property of all elements of the set that we wish to define, so we use the ***roster method*** to define the set by simply specifing the elements of the set directly, as follows:

$$Z \equiv \{1,\ 132,\ 56,\ A,\ 784,\ 0.0\}$$

I. ***Axiom of Specification:*** To every set A and to every logical predicate $P(x)$ there corresponds a set B whose elements are exactly those elements of A for which $P(x)$ is true.

or $B = \{x \mid x \in A \wedge P(x)\}$

1. The set A is referred to as the ***universe of discourse***.

2. The universe of discourse holds, for any particular situation, all of the objects pertinent to that situation.

3. The existance of the universe of discourse is usually taken for granted.

4. You must be very careful that the universe of discourse actually exists for your problem.

J. Finally:

1. A set is defined by defining those things that are elements of the set.

2. There are no duplicate set elements.
   a. A set element is represented once and only once in the set.

   b. If $a = b$ then :
      i. $\{a,\, b\} = \{a,\, a\} = \{a\}$

      ii. $\{a,\, b\} = \{b,\, b\} = \{b\}$

      iii. $\{a\} = \{b\}$

3. Set elements are not arranged in any defined order.

K. For completeness we also define the $Null$, or $Empty\,Set$ as that set that contains no elements. We normally designate the empty set by $\emptyset$ or, less often, by $\{\}$.

II. Often Used Sets:

A. $\boldsymbol{N} = \{0, 1, 2, 3, ...\}$ = the set of ***natural numbers***.
   Note: Some texts/authors do not include 0 in the set of natural numbers.

B. $\boldsymbol{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ = the set of ***integers***.

C. $\boldsymbol{Z}^{+} = \{1, 2, 3, ...\}$ = the set of ***positive integers***.

D. $\boldsymbol{Q} = \{\frac{p}{q} \mid p \in Z,\ q \in Z,\ q \neq 0\}$ = the set of ***rational numbers.***

E. $\boldsymbol{\Re}$ = the set of ***real numbers***.

F. $\boldsymbol{\Re}^{+}$ = the set of ***positive real numbers***.

G. $\boldsymbol{C}$ = the set of ***complex numbers***.

H. U = the universal set.

I. $\emptyset$ = the empty set.

III. Properties of Sets:

A. Equality: Two sets are equal if and only if they each contain the same elements.

   1. $A = B$ if $(\forall x)(x \in A \leftrightarrow x \in B)$

   2. If $A = \{1, 2, 3, 4\}$ and $B = \{4, 2, 1, 3\}$
      then $A = B$

   3. If $A = (1, 2, 3, 4, 5\}$ and $B = \{4, 2, 1, 3\}$
      then $A \neq B$

   4. Note: $\emptyset \neq \{\emptyset\}$
      a. $\emptyset$ denotes the ***empty set***, the set with no elements.

      b. $\{\emptyset\}$ denotes a set that contains the empty set as its only element.

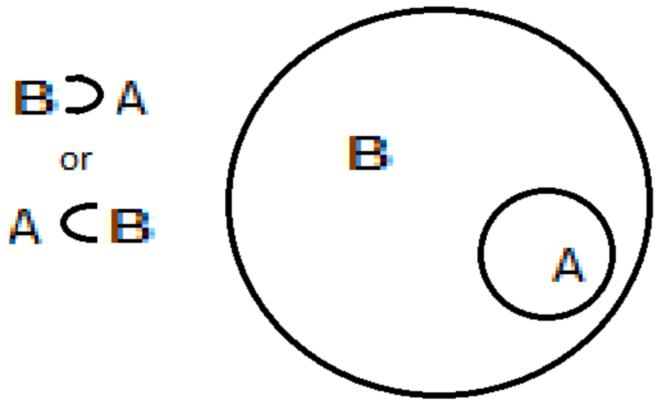
   5. ***Axiom of Extension:*** ***Two sets are equal if and only if they contain the same elements.***

B. Subsets

1. A set A is called a ***subset*** of a set B if every element of A is also and element of the set B. B is then called a ***super set*** of A.

2. A is a ***subset*** of a set B $\equiv$ $A \subset B$
   $\equiv$ $(\forall x)(x \in A \rightarrow x \in B)$

3. Venn Diagram:



4. $A = B$ $\equiv$ $(\forall x)(x \in A \leftrightarrow x \in B)$

   In this case, A is an ***improper subset*** of B.

5. If B contains one or more elements b such that $b \notin A$ then:
   a. $A \neq B$

   b. A is a ***proper*** subet of B.

   c. Therefore every set is an improper subset of itself.

6. Note: The empty set $\emptyset$ is a subset of every other set.

7. Note: Inclusion and belonging are very different concepts:
   1. $A \subset A$ is always true - Every set is a subset of itself.

   2. $A \in A$ has not been found to be true in any case.

C. Size

1. The size of a set S is defined as the number of elements contained in S.
   a. If the set S has N elements then the size of S is N.

   b. Normally written as: $N = |S|$

   c. N is often referred to as the ***cardinality*** of S.

2. Identical elements are counted only once.
   Example: $a \in S$, $b \in S$, $a = b$

   a and b are counted as one when computing cardinality.

3. If N is finite then S is a ***finite set***.
   Example: $S = \{a, b, c, d\}$ then $|S| = 4$
   S is a finite set.

4. If N is infinite then S is an ***infinite set***.
   Example: $S = \{x \mid x \text{ is a real number}\}$ and $|S| = \infty$
   S is an infinite set.

IV. The Power Set

A. Definition: The ***power set*** of the set S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S.

B. Examples:

1. If $S = \{a, b, c\}$ then
   $\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

2. If $S = \emptyset$ then $\mathcal{P}(S) = \{\emptyset\}$

3. $\mathcal{P}(\mathcal{P}(S)) = \{\emptyset, \{\emptyset\}\}$

C. When we cover Mathematical Induction we will prove that:
   If $|S| = N$ then $|\mathcal{P}(S)| = 2^N$

V. Ordered Pairs:

A. Preliminary Definitions:

1. Ordered Pair: An ***ordered pair*** $(x,\ y)$ has x as its first element and y as its second.
$(x,\ y) \neq (y,\ x)$

2. An ***un-ordered*** pair is a set, denoted by $\{x,\ y\} = \{y,\ x\}$

3. Ordered n-tuple: An ***ordered sequence*** $(a_1,\ a_2,\ a_3, \ldots, a_n)$ in which a_1 is the first element, a_2 is the second element, ..., and a_n is the nth element.

B. Addition and multiplication are unordered operations (commutative)

1. $x + y = y + x$ and $x \times y = y \times x$

2. These operations can use the concept of an ***unordered pair*** of set elements.

C. Subtraction and division require ***ordered pairs*** since:

$$x \div y \neq y \div x$$

and

$$x - y \neq y - x$$

D. Example: $(2 \times x - y,\ x + y) = (7,\ -1)$

1. This is an equality of ordered pairs.

2. Therefore, we must have: $2 \times x - y = 7$ and $x + y = -1$

3. With an unordered pair you could not get unique solutions.

V. Cartesian Product : An Operation Performed on Sets:

A. Definition: If A and B are sets then the ***cartesian product*** (or cross product) of A and B, denoted $A \times B$, is the set defined by

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

where (x, y) denotes an ***ordered pair.***

B. Example: $A = \{1, 2, 3\}$ and $B = \{a, b\}$

a. $A \times B = \{1, 2, 3\} \times \{a, b\}$
$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

b. $B \times A = \{a, b\} \times \{1, 2, 3\}$
$= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

c. Note: $A \times B \neq B \times A$

C. Definition: If $A_1, A_2, A_3, \ldots$, and A_N are sets then the ***cartesian product*** of $A_1, A_2, A_3, \ldots$, and A_N is the set of ordered n-tuples defined by

$$A_1 \times A_2 \times A_3 \times \ldots \times A_N$$
$$= \{(a_1, a_2, a_3, \ldots, a_N) \mid a_i \in A_i, i = 1, 2, \ldots N\}$$

VI. Union: An Operation Performed on Sets

A. Definition: The ***union*** of two subsets A and B of some universe, designated by $A \cup B$, is the set of all elements x where $x \in A$ and all elements y where $y \in B$.

B. The union of two sets is, therefore, the set of all the elements of both of the sets.

C. Examples:

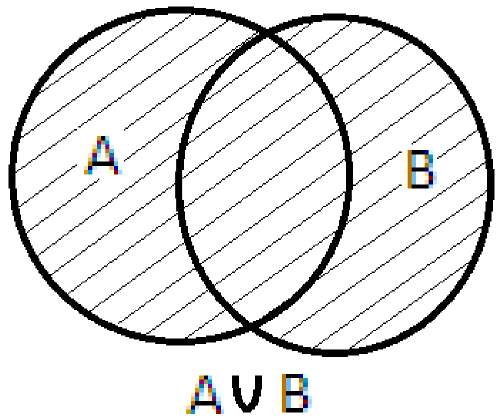
1. If: $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$

then: $A \cup B = \{a, b, c, d, 1, 2, 3\}$

2. If: $A = \{x \mid x$ is an Engineering student at Case$\}$
   and
   $B = \{y \mid y$ is a Mathematics student at Case$\}$

   then: $A \cup B = \{z \mid z$ is either an Engineering or a Mathematics student at Case$\}$

D. Venn Diagram for $A \cup B$:



VII. Intersection: An Operation Performed on Sets

A. Definition: The ***intersection*** of two subsets A and B of some universe designated by $A \cap B$, is the set of all elements x where $x \in A$ and $x \in B$.

B. The intersection of two sets is, therefore, the set of all the elements of either set which is also an element of the other set.

C. Examples:

1. If: $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$
   then: $A \cap B = \emptyset$

   Since A and B have no elements in common the intersection of A and B is the empty set.

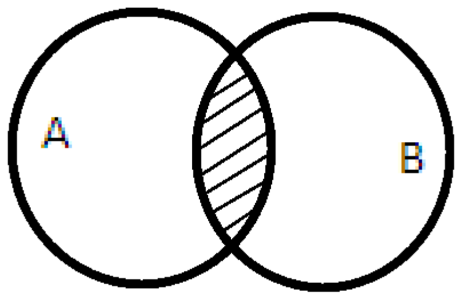
   In this case A and B are said to be ***disjoint***.

2. If: $A = \{a, b, c, d\}$ and $B = \{d, f, g\}$
   then: $A \cap B = \{d\}$

3. If: $A = \{x \mid x$ is a Case student majoring in Engineering$\}$
and
$B = \{y \mid y$ is a Case student majoring in Mathematics$\}$

then: $A \cap B = \{z \mid z$ is majoring in both Engineering and Mathematics student at Case$\}$

D. Venn Diagram for $A \cap B$:



A ∩ B

IX. Cardinality of $A \cup B$

A. Recall the definition of ***set***:

Given a set A an object x can either be an element of A or not, i.e., $x \in A$ is either true or false.

1. Therefore, all we know about x is whether $x \in A$ or $x \notin A$

2. We have no knowledge of the number of occurrences of $x \in A$.

3. When computing $|A|$, if $x,\ y \in A$ and $x = y$, x and y are counted once and only once.

B. Therefore, when computing $|A \cup B|$ for the case of $A \cap B \neq \emptyset$ we adjust $|A \cup B|$ to account for the elements $x \in A \cap B$ as in:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

XI. Difference: An Operation Performed on Sets

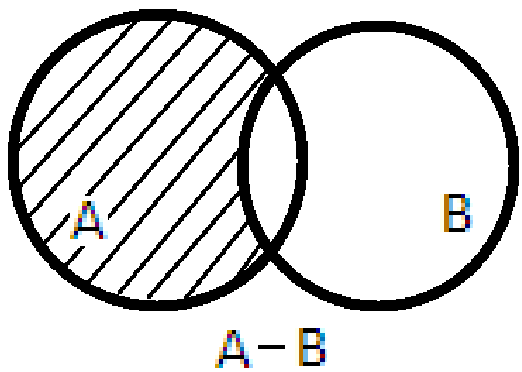
A. Definition: The ***difference*** of two subsets A and B of some universe, designated by $A - B$, is the set of all elements x where $x \in A$ and $x \notin B$.

B. Then: $A - B = \{x \mid x \in A \land x \notin B\}$

C. Example: If: $A = \{a, b, c, d\}$ and $B = \{d, f, g\}$
then: $A - B = \{a, b, c\}$

E. $A - B$ is also referred to as the ***complement of B with respect to A***.

F. Venn Diagram for $A - B$:



XII. Universal Set (Again) and Complements of a Set

A. The universal set U contains ***everything***.

B. Generally assumed to exist.

C. The complement of a set X, written $\overline{X}$, are all elements y such that $y \in U$ and $y \notin X$.

D. Therefore: $\overline{X} = \{y \mid y \in U \wedge y \notin X\} = U - X$

XIII. Illustrative Example:

A. Consider: $A - B$ and $A \cap \overline{B}$

B. $A - B = \{a \mid a \in A \wedge a \notin B\}$
1. $x \in A - B \equiv x \in A \wedge x \notin B$
2. $x \notin B \equiv x \in \overline{B}$
3. Therefore: $x \in A - B \equiv x \in A \wedge x \in \overline{B} \equiv x \in A \cap \overline{B}$

C. $A \cap \overline{B} = \{a \mid a \in A \wedge a \notin B\}$
1. $x \in A \cap \overline{B} \equiv x \in A \wedge x \notin B \equiv x \in A - B$

2. Therefore: $x \in A \cap \overline{B} \equiv x \in A \wedge x \notin B \equiv x \in A - B$

D. Therefore: $A - B = A \cap \overline{B}$

XIV. Commonly Used Set Identities

Identity	Name
$A \cap U = A$	Identity Laws
$A \cup \emptyset = A$	
$A \cup U = U$	Domination Laws
$A \cap \emptyset = \emptyset$	
$A \cup A = A$	Idempotent Laws
$A \cap A = A$	
$\overline{\overline{(A)}} = A$	Complementation Law
$A \cup B = B \cup A$	Commutative Laws
$A \cap B = B \cap A$	
$(A \cup B) \cup C = A \cup (B \cup C)$	Associative Laws
$(A \cap B) \cap C = A \cap (B \cap C)$	
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's Laws
$\overline{A \cap B} = \overline{A} \cap \overline{B}$	
$A \cup (A \cap B) = A$	Absorption Laws
$A \cap (A \cup B) = A$	
$A \cup \overline{A} = U$	Complement Laws
$A \cap \overline{A} = \emptyset$	

XV. Example: Proof of $(A \cap B) \cap C = A \cap (B \cap C)$:

1. Identity: $(A \cap B) \cap C = A \cap (B \cap C)$

2. $(A \cap B) \cap C = \{x \mid (x \in A \cap B) \wedge (x \in C)\}$
   Definition of intersection

3. $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$
   Definition of intersection

4. Hence: $(A \cap B) \cap C = \{x \mid (x \in A) \wedge (x \cap B) \wedge (x \in C)\}$
   from statements 2. and 3.

5. Therefore: $(A \cap B) \cap C = \{x \mid (x \in A) \wedge (x \in B \cap C)\}$
   Definition of intersection and meaning of set builder notation.

6. Therefore: $(A \cap B) \cap C = A \cap (B \cap C)$
   by meaning of set builder notation.

XVI. Membership Tables

1. Definition: A table displaying sets and combinations of sets and all combinations of membership possibilities (i.e., either $True/1$ or $False/0$)

2. Example 1: $A \cap B$

Row	A	B	$A \cap B$
1	1	1	1
2	1	0	0
3	0	1	0
4	0	0	0

   a. Row 1: An element x is a member of A, a member of B, **and** a member of $A \cap B$
   b. Row 2: An element x is a member of A, is not a member of B, **and** is not a member of $A \cap B$
   c. Row 3: An element x is not a member of A, is a member of B, **and** is not a member of $A \cap B$
   d. Row 4: An element x is not a member of A, is not a member of B, **and** is not a member of $A \cap B$

3. Example 2: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Row	A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$
1	1	1	1	0	0	0	0
2	1	0	0	1	0	1	1
3	0	1	0	1	1	0	1
4	0	0	0	1	1	1	1

a. Row 1: An element x is a member of A, a member of B, a member of $A \cap B$, not a member of $\overline{A \cap B}$, not a member of $\overline{A}$, not a member of $\overline{B}$, and, hence, not a member of $\overline{A} \cup \overline{B}$

b. Row 2: An element x is a member of A, is not a member of B, and is not a member of $A \cap B$, is a member of $\overline{A \cap B}$, is not a member of $\overline{A}$, is a member of $\overline{B}$, and is a member of $\overline{A} \cup \overline{B}$

c. Row 3: An element x is not a member of A, is a member of B, **and** is not a member of $A \cap B$, is a member of $\overline{A \cap B}$, is a member of $\overline{A}$, is not a member of $\overline{B}$, and is a member of $\overline{A} \cup \overline{B}$

d. Row 4: An element x is not a member of A, is not a member of B, **and** is not a member of $A \cap B$, is a member of $\overline{A \cap B}$, is a member of $\overline{A}$, is a member of $\overline{B}$, and is a member of $\overline{A} \cup \overline{B}$

e. Since the membership values in the colum for $\overline{A \cap B}$ are identical to those for $\overline{A} \cup \overline{B}$ we have proven the identity $A \cap B = \overline{A} \cup \overline{B}$

XVII. Generalized Unions and Intersections

A. The union of a collection of sets is denoted by:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

1. $\bigcup_{i=1}^{n} A_i$ denotes the set that contains those elements that are members of at least one of the sets A_i.

2. If $A_1 = \{1,\ 2,\ 3\}$ and $A_2 = \{2,\ 3,\ 4\}$

   and $A_3 = \{5,\ 6,\ 7,\ 8\}$ then

   $\bigcup_{i=1}^{3} A_i = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$

B. The intersection of a collection of sets is denoted by:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

1. $\bigcap_{i=1}^{n} A_i$ denotes the set that contains those elements that are members of all of the A_i.

2. If $A_1 = \{1,\ 2,\ 3\}$ and $A_2 = \{2,\ 3,\ 4\}$

   and $A_3 = \{4,\ 5,\ 6,\ 7,\ 8\}$ then $\bigcup_{i=1}^{3} A_i = \{\ 4\}$

XIX. Computer Representation of Sets

A. Singly Dimensioned Array

1. Store the elements of each set of interest as a separate set A as a list (singly-dimensioned array) of dimension $|A| = N_A$.

2. Create separate arrays for the sets $B,\ C,\ D,\ C \cup D,\ C \cap D$, etc..

3. If the sets being used are relatively small subsets of the universal set, or the domain of discourse, this method is probably the most memory efficient.

B. Text Book Solution

1. Store universal set U as a list (singly-dimensioned array) of dimension $|U| = N$.

2. Represent each subsets as a bit-string.
   a. String element is $'1'B$ ($\boldsymbol{True}$) if corresponding element in U-array is an element of the set.

   b. String element is $'0'B$ ($\boldsymbol{False}$) if corresponding element in U-array is not an element of the set.

C. Text Example:

1. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

2. $A = \{1, 3, 5, 7, 9\} = \{x \,|\, x$ is an odd integer, $1 \leq x \leq 10\}$

3. $B = \{2, 4, 6, 8, 10\} = \{x \,|\, x$ is an even integer, $1 \leq x \leq 10\}$

4. Bit-String Representation

U	1	2	3	4	5	6	7	8	9	10
Bits for U	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$
Bits for A	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$
Bits for B	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$

D. Bit-String Representations of Union and Intersection

1. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

2. $C = \{1, 4, 7\}$ and $D = \{3, 7, 8, 9\}$

3. $C \cup D = \{1, 3, 4, 7, 8, 9\}$ and $C \cap D = \{7\}$

U	1	2	3	4	5	6	7	8	9	10
Bits for U	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$	$'1'B$
Bits for C	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$
Bits for D	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$	$'0'B$	$'1'B$
Bits for $C \cup D$	$'1'B$	$'0'B$	$'1'B$	$'1'B$	$'0'B$	$'0'B$	$'1'B$	$'1'B$	$'1'B$	$'0'B$
Bits for $C \cap D$	$'0'B$	$'0'B$	$'0'B$	$'0'B$	$'0'B$	$'0'B$	$'1'B$	$'0'B$	$'0'B$	$'0'B$

XX. A Warning Regarding the Set Builder Technique and the Axiom of Specification

A. Using the set builder technique a set S is defined as:

$$S = \{X \mid P(X)\}$$

where $P(X)$ is some logical condition.

B. The Axiom of Specification is: To every set A and to every logical condition $P(x)$ there corresponds a set S whose elements are exactly those elements x of A for which $P(x)$ holds.

C. Define $P(X)$ to be: $P(X) \equiv (X \in A) \wedge (X \notin X)$

so: $S = \{X \mid P(X)\} \equiv \{X \mid (X \in A) \wedge (X \notin X)\}$

D. The set S is composed of all sets X in the set A that are not elements of themselves.

Note: S is a set of sets, such as the ***power set*** of the set S, or $\mathcal{P}(S)$, the set of all subsets of S.

E. Using the defintion of S, i.e.: $S = \{X \mid (X \in A) \wedge (X \notin X)\}$
we note that:

1. If $S \in S$ then the assumption that $S \in A$ leads to the conclusion that $S \notin S$.

2. If $S \notin S$ then the assumption that $S \in A$ leads to the conclusion that $S \in S$.

F. In both cases, the assumption that $S \in A$ leads to a contradiction so the assumption must be false.

G. Since we have placed no restriction on the set A, A could be the universe of discourse U and we have proven that there cannot be a universe of discourse!!!!

H. The moral of this exercise is that it is not sufficient to arbitrarily compose a condition $P(X)$ and define a set $S = \{X \mid P(X)\}$. It is also necessary to have a set to which $P(X)$ applies.

Ignoring this caution can lead to some very misleading conclusions.

I. A situation described in English that corresponds to $S(X)$ is that of the village barber, described below:

> In a certain isolated village all men of the village who do not shave themselves are shaved by the barber, who is a man of the village.
>
> Question: Who shaves the barber?