## I. Review:

A. Our previous work on logic gave us the form for proving theorems of the form: $P \to Q$

where both $P$ and $Q$ could be compound statements.

B. If the proof of this statement can be accomplished using only axioms and rules of inference then the theorem is true under all interpretations.

## II. Main Question of Logic - When does one sentence, statement, or claim follow logically from some others?

A. The study of logic is the study of the **proper form of an argument**

1. If one starts with valid premises and one follows the proper recipe one ends up with a conclusion that is called **logical**.
    a. Logic is a recipe for thinking correctly

    b. Logic, as in the ability to think correctly, is one, and not the only, attribute of intelligence, not the entirety of intelligence.

    c. Brilliant persons may reach a correct conclusion without the conscious application of logic or even reach a correct conclusion that the usual logical process does not uncover.

2. Proper form of an argument is **syntax**.

3. Truth or falsity of an argument is **semantics**.

B. The study of logic is **not** the study of what is truth.

1. A logical argument may not lead to a conclusion that is true.

2. An illogical argument may lead to a conclusion that is true.

3. The premises of a logical argument may be false, so the conclusion can be false.

4. Rules of a particular logical system may not apply to a given problem:
   a Euclidean versus non-Euclidean geometry

   b. Relativistic physics versus Galilean physics

   c. Quantum mechanics versus Classical mechanics.

5. ***IF*** the premises are true, then a proper argument will lead to a conclusion that is true.

C. Argument - definition
1. An argument is a series of statements.

2. One statement, the **conclusion**, is said to follow from all of the others.

3. The other statements are known as the **premises** of the argument.

4. Example 1:

| | | |
|---|---|---|
| Example 1: | statement 1 (premise): | All men are mortal. |
| | statement 2 (premise): | Socrates is a man. |
| | statement 3 (conclusion): | Socrates is mortal. |

   a. This example illustrates one proper form of an argument.

   b. The argument is **logically valid** in our frame of reference. All men, to the extent of our knowledge, are mortal.

   c. **If** the premises are true, the conclusion must be true.

   d. In any world in which the premises are true the conclusion must be true.

   e. Impossible for conclusion to be false **if** the premises are true.

f. In a different form this argument can be written:
Premise 1: $\forall X\,(Man(X) \;\rightarrow Mortal(X))$
Premise 2: $\underline{Man(Socrates)\qquad\qquad}$
Conclusion: $Mortal(Socrates)$

D. An argument is logically valid if and only if the conclusion must be true given the assumption that the premises are true.

1. The premises need not be true for the argument to be logically valid.

2. If premises are not true, the conclusion may not be true but this is not necessarily the case.

3. The argument will still be of the proper form, i.e., it is logically valid.

4. **If** the premises are true, **then** the conclusion is true.

5. If the premises are true then it is impossible for the conclusion of a logically valid argument to be false.

E. Example 2 - bad form:

Statement 1 (conclusion): Lucretious is a man.
Statement 2 (premise): Lucretious is mortal.
Statement 3 (premise): All men are mortal.

1. Order of statements is unimportant.

2. Must be able to identify the premises and the conclusion.

3. Consider the possibility that Lucretius is a a goldfish.

4. Goldfish are notoriously mortal - they die all the time.

5. The fact that a goldfish is mortal is unrelated to the fact that all men are mortal.

6. The statement that all men are mortal is true.

7. If Lucretius is a goldfish, he cannot be a man and the premise
All men are mortal.
says nothing regarding Lucretius.

8. Argument is not logically valid.

9. Even if premises are assumed to be true, the conclusion does not necessarily follow.

F. Example 3 - proper argument
Statement 1 (premise): *Cube(c)*
Statement 2 (premise): $c = b$
Statement 3 (conclusion): *Cube(b)*

1. $c = b$ means that the object $b$ is the same object as the object $c$.

2. If $b$ is the same object as $c$ then, if $c$ is a cube, $b$ must also be a cube.

3. **If** the premises are true, **then** the conclusion must be true.

G. Classical logic allows only two truth values, **true** and **false**.
1. **true** represents metaphysical truth without approximation

2. **false** represents metaphysical falsity

3. Referred to as **Law of Bivalence**
   a. All truth values must be either true or false

   b. No other truth values allowed.

H. Formal Definition of Logical Consequence:

> **Given predicates $F_1$, $F_2$, ... , $F_n$ and a predicate G,**
> **G is said to be a *logical consequence* of $F_1$, $F_2$, ... , $F_n$**
> **(or G *logically follows from* $F_1$, $F_2$, ... , $F_n$)**
> **if and only if for any world $W$ in which $F_1$and $F_2$ and ... and $F_n$ are all true, G is also true in $W$.**

1. In some cases the number of premises, i.e., the $F_i$, is very large.

2. We require that each of the $F_i$ be absolutely, without a doubt, true.

3. For degrees of truth we must study fuzzy logic.

## II. Proof Theory

A. Definition: A *proof* is a step-by-step demonstration that a given conclusion, say $G$, follows from some premises, i.e., $F_1$, $F_2$, $F_3$, ... , $F_n$.

B. Proofs come in two flavors, formal and informal.

C. Example: Informal proof:

Statement 1 (premise): Socrates is a man.
Statement 2 (premise): All men are mortal.
Statement 3 (premise): All mortals die sometime.
Statement 4 (premise) Everyone who will eventually die sometimes worries about it.
Statement 5 (conculsion):
Socrates sometimes worries about dying.

1. A logical consequence of statements 1 and 2 is:
Statement 2': (intermediate conclusion):
Socrates is mortal.

2. A logical consequence of statements 2' and 3 is:
Statement 3': (intermediate conclusion):
Socrates will die sometime.

3. A logical consequence of statements 3' and 4 is:
   Statement 5: (conclusion): Socrates sometimes worries about dying.
4. Complete argument:
   1. Socrates is a man.

   2. All men are mortal.
   2'. Socrates is mortal.

   3. All mortals die sometime.
   3'. Socrates will die sometime.

   4. Everyone who will eventually die sometimes worries about it.
   5. Socrates sometimes worries about dying.

5. Argument is logically valid - the conclusion must be true if the premises are true.

6. Argument is informal - given in English without using predefined arguments or rule.

D. Formal Proof:
   1. Uses a pre-determined set of rules
   2. Uses a highly stylized method of presentation.

E. Example of a rule: **Indiscernibility of Identicals (Ind Id):**

$$\begin{array}{l} \boldsymbol{P(n)} \\ \underline{\boldsymbol{n = m}} \\ \boldsymbol{P(m)} \end{array}$$

1. The premises, must be proven **before** we assert $P(m)$.

2. Using this rule we can generate a formal argument that if $a$ is a cube, and $a = b$, then $b$ is a cube.

$$\begin{array}{lll} 1. & Cube(a) & \\ \underline{2.} & \underline{a = b} & \\ 3. & Cube(b) & \text{1, 2, Ind Id} \end{array}$$

G. Other rules:

1. **Reflexivity of Identity (Refl = ):** $\boldsymbol{n = n}$

   This rule allows us to assert, at any stage of any proof, that something is equal to itself.

2. **Reiteration (Reit):** $\dfrac{\boldsymbol{P}}{\boldsymbol{P}}$

   This rule allows to reuse a previously asserted sentence or line at any following point in the proof.

H. Example Proof: Conclude *Likes(a,d)* from: *Likes(a,b), b = c,* and *c = d*.

$$
\begin{array}{lll}
1. & Likes(a,\, b) & \\
2. & b = c & \\
\underline{3.} & \underline{c = d} & \\
4. & b = d & 2, 3,\ Ind\,Id \\
5. & Likes(a,\, d) & 4,\ 1,\ Ind\,Id
\end{array}
$$

## III. Formal Rules of Proof Involving Conjunction

A. $\wedge$ Elimination - $\dfrac{\boldsymbol{P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n}}{\boldsymbol{P_i}}$

1. If one has proven a conjunction to be true, then **all** components of the conjunction must be true.

2. Note that a component may be an expression and not all components of that expression need be true.

   a. Assume: $A \wedge B \wedge (C \vee D)$

   b. By $\wedge$ Elimination we can conclude all of the following:

      i. $A$ is true

      ii. $B$ is true

      iii. $(C \vee D)$ is true

   c. We **cannot** conclude that $C$ and $D$ are both true, only that the disjunction $C \vee D$ is true.

B. Conjunction Introduction

1. If $A$ has been shown to be true and $B$ has been shown to be true, then we can conclude that $A \wedge B$ is true.

2. Once again, $A$ and/or $B$ can be expressions.

3. If either $A$ or $B$ is an expression it is sufficient to show that the expression is true, not the individual literals.

4. $\wedge$ Introduction :

$$\begin{array}{ll} 1. & P_1 \\ 2. & P_2 \\ 3. & P_3 \\ . & \\ . & \\ . & \\ n & P_n \\ \hline \multicolumn{2}{l}{P_1 \wedge P_2 \wedge P_3 \wedge ... \wedge P_n} \end{array}$$

## IV. Formal Rules of Proof Involving Disjunction

A. If: $A$ then: $A \vee B$

B. Therefore, if we have proven $A$ then we know that $A \vee B$ is true.

C. Disjunction Introduction Formal Rule of Proof

1. If $A$ has been shown to be true then we can conclude that $A \vee B$ is true.

2. If $B$ has been shown to be true then we can conclude that $A \vee B$ is true.

3. $\vee$ Introduction: $\dfrac{P_i}{P_1 \vee P_2 \vee P_3 \vee ... \vee P_n}$

D. If: $A \vee B$ then: either $A$ or $B$ or both

E. Therefore, if we have proven $A \vee B$ then we know that either $A$ is true, or $B$ is true, or both $A$ and $B$ are true.

F. Disjunction Elimination

1. If we wish to prove that a statement $S$ follows from the disjunction

$$P_1 \vee P_2 \vee P_3 \vee ... \vee P_n$$

then we must show that:

a. $S$ follows from $P_1$

**and**

b. $S$ follows from $P_2$

**and**

.

.

.

**and**

n. $S$ follows from $P_n$

2. This is commonly known as **Proof by Cases**

3. $\vee$ Elimination:

$$
\begin{array}{llll}
1. & P_1 \vee P_2 \vee P_3 \vee ... \vee P_n & & \\
 & & & \\
2. & 2.a & P_1 & \\
 & 2.b & ... & Case\,1 \\
 & . & & \\
 & \underline{2.i} & \underline{...} & \\
 & & S & \\
 & & & \\
3. & 3.a & P_2 & \\
 & 3.b & ... & Case\,2 \\
 & . & & \\
 & \underline{3.j} & \underline{...} & \\
 & & S & \\
 & & & \\
4. & 4.a & P_3 & \\
 & 4.b & ... & Case\,3 \\
 & . & & \\
 & \underline{4.k} & \underline{...} & \\
 & & S & \\
 & & & \\
 & . & & \\
 & . & & \\
 & . & & \\
 & & & \\
n. & n.a & P_3 & \\
 & n.b & ... & Case\,n \\
 & . & & \\
 & \underline{n.m} & \underline{...} & \\
 & & S & \\
\hline
 & S & &
\end{array}
$$

Note that a proof must be done for **each** element of the disjunction.

**VI. The Proper Use of SubProofs**

A. The previous rule used used subproofs to show that $S$ followed from each of the $P_i$.

B. When a subproof is used in the main proof **only the final result can be cited.**

1. In a subproof by contradiction the assumed premises results in a contradiction
   a. intermediate results are based on what is finally proved to be a contradiction

   b. intermediate results are suspect and cannot be assumed to be proven.

2. In a proof by cases one is assuming that the premise of each case is true for the proof of that case.
   a. One does not know that each premise is true.

   b. But, to prove the theorem one must show that the desired result holds for each case.

   c. What can be proven from the premise for one case may not be amenable to proof from the premise of another case.

   d. The intermediate result from one subproof may not be proveable from the premises of another
      - Cannot be used.

C. None of the intermediate results of the subproof are valid outside the subproof.

## VII. Subject Specific Proofs

A. Normally, however, we want a proof that is valid for a particular set of circumstances at hand and we do not need a proof valid in all interpretations.

B. In many cases we know a priori that the theorem is NOT valid in all interpretations.

C. We don't care, we need a theorem that is valid in the interpretations that corresponds to our circumstances. These proofs are normally labeled ***subject specific***.

D. Proofs of this sort are generally pursued by using proof techniques somewhat less formal than the rules of logic and the proof sequences that we have just studied. Three of these methods are:
   1. direct proof
   2. contraposition
   3. contradiction

E. All of these methods can involve either ***induction*** or ***deduction***.
   1. Deduction
      a. Reasoning from facts or axioms to a conclusion.

      b. Also known as ***data-directed reasoning*** or ***forward chaining***.

   2. Induction
      a. Reasoning by justification, or tracing a conclusion back to the justifying axioms.

      b. Also known as ***goal-oriented reasoning*** or ***backward chaining***.

   3. **Note:** The theory of mathematical induction (which we will study later) is a deductive, rather than an inductive, proof technique.

## VIII. Direct Proof

A. Assume the hypothesis and deduce the conclusion (deduction).

B. Example 1: Theorem: The product of two even integers is even.

1. Formal Statement:
   a. Let: $E(x) \equiv x$ is even
   b. $(E(x) \equiv x$ is even$) \to x = 2 \times k$ where $k$ is some integer.
   c. Theorem: $E(x) \wedge E(y) \to E(x \times y)$

2. Proof:
   a. If $x$ is even then it can be represented by $x = 2 \times N$ where $N$ is an integer.

      Formally: $E(x) \to x = 2 \times N$ where $N$ is some integer.

   b. If $y$ is even then it can be represented by $y = 2 \times M$ where $M$ is an integer.

   c. Then: $x \times y = (2 \times N)(2 \times M) = 4 \times N \times M$

      $= 4 \times K$ where $K = N \times M$

      $= 2 \times (2 \times K) = 2 \times L$

   d. Therefore $x \times y = 2 \times L$ is even by definition.

C. Example 2: Theorem: The sum of two odd integers is even.

1. Formal Statement:
   a. Let: $F(x) \equiv x$ is odd
   b. $F(x) \equiv x$ is odd $\to x = (2 \times k) + 1$ where $k$ is some integer.
   c. Theorem: $F(x) \wedge F(y) \to F(x + y)$

2. Proof:
   a. If $x$ is odd then it can be represented by

      $x = (2 \times N) + 1$ where $N$ is an integer.

   b. If $y$ is odd then it can be represented by

      $y = (2 \times M) + 1$ where $M$ is an integer.

   c. Then: $x + y = \Big[(2 \times N) + 1\Big] + \Big[(2 \times M) + 1\Big]$

      $= (2 \times N) + (2 \times M) + 2$

      $= 2 \times (N + M + 1)$

      $= 2 \times L$ where $L = N + M + 1$ and $L$ is an integer.

   d. Therefore $x + y = 2 \times L$ is even by definition.

**IX. Example 3: A Case in Which One Must Determine a Result Before Using a Direct Proof to Prove the Result Correct.**

A. ***Theorem:*** ***Every odd integer $X$ can be represesented as $A^2 - B^2$ where $A$ and $B$ are integers.***

B. Problem: We need to know something about the integers $A$ and $B$ in order to prove that $X = A^2 - B^2$ when $X$ is odd.

C. Examine Test Cases to Solve This Problem:
   1. Generate test cases or create a Java program that performs a brute-force search for integers $A$ and $B$ such that, if $X$ is odd then $X = A^2 - B^2$

2. Java Source Code:

```
package odddifsqs;

public class OddDifSqs
  {
  public static void main(String[] args)
    {
    int A = 0; int B = 0; int C = 0; int X = 0;

    for (X = 3; X < 110; X = X + 2)
      {
      for (A = X; A >= 1; A--)
        {
        for (B = 1; B <= X; B++)
          {
          C = A*A - B*B;
          if (C == X) break;
          }
        if (C == X) break;
        }
      System.out.println(X + " = " + A + "*" + A + " - " + B + "*" + B);
      }
    }
  }
```

C. Program Output:

```
run:
3 = 2*2 - 1*1
5 = 3*3 - 2*2
7 = 4*4 - 3*3
9 = 5*5 - 4*4
11 = 6*6 - 5*5
13 = 7*7 - 6*6
15 = 8*8 - 7*7
17 = 9*9 - 8*8
19 = 10*10 - 9*9
21 = 11*11 - 10*10
23 = 12*12 - 11*11
```

25 = 13*13 - 12*12
27 = 14*14 - 13*13
29 = 15*15 - 14*14
31 = 16*16 - 15*15
33 = 17*17 - 16*16
35 = 18*18 - 17*17
37 = 19*19 - 18*18
39 = 20*20 - 19*19
41 = 21*21 - 20*20
43 = 22*22 - 21*21
45 = 23*23 - 22*22
47 = 24*24 - 23*23
49 = 25*25 - 24*24
51 = 26*26 - 25*25
53 = 27*27 - 26*26
55 = 28*28 - 27*27
57 = 29*29 - 28*28
59 = 30*30 - 29*29
61 = 31*31 - 30*30
63 = 32*32 - 31*31
65 = 33*33 - 32*32
67 = 34*34 - 33*33
69 = 35*35 - 34*34
71 = 36*36 - 35*35
73 = 37*37 - 36*36
75 = 38*38 - 37*37
77 = 39*39 - 38*38
79 = 40*40 - 39*39
81 = 41*41 - 40*40
83 = 42*42 - 41*41
85 = 43*43 - 42*42
87 = 44*44 - 43*43
89 = 45*45 - 44*44
91 = 46*46 - 45*45
93 = 47*47 - 46*46$N$
95 = 48*48 - 47*47
97 = 49*49 - 48*48
99 = 50*50 - 49*49

101 = 51*51 - 50*50
103 = 52*52 - 51*51
105 = 53*53 - 52*52
107 = 54*54 - 53*53
109 = 55*55 - 54*54
BUILD SUCCESSFUL (total time: 1 second)

D. **Note:** The computer output above does not constitute a proof. It does, however, provide data from which one can reason regarding possible values for $X = A^2 - B^2$

E. Analysis:

1. In all cases shown above we have that, for odd $X$,
$X = (N+1)^2 - N^2$ for some $N$.

2. We recall that the definition of an odd integer $X$ is that:
$X = 2 \times N + 1 = (N + 1) + N$ for some $N$.

3. Therefore we will investigage the possibility that, if
$X = (N + 1) + N$ then $X = (N+1)^2 - N^2$

F. Direct Proof:

1. If $X$ is odd then, by definition $X = (2 \times N) + 1$
where $N$ is some integer.

2. $X = (2 \times N) + 1 = N + N + 1 = N + (N+1)$

3. Then: $(N+1)^2 - N^2 \quad = N^2 + 2 \times N + 1 - N^2$
$= 2 \times N + 1$
$= (2 \times N) + 1$

4. Therefore: $X = 2 \times N + 1 \quad = (N+1)^2 - N^2 = A^2 - B^2$

where: $A = N + 1$ and $B = N$ □

## X. Contraposition

A. Recall: $P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg\neg Q \vee \neg P$

$$\equiv \neg Q \to \neg P$$

1. If a proof of $P \to Q$ is not apparent, try a proof of $\neg Q \to \neg P$

2. The proof of $\neg Q \to \neg P$ is equivalent to a proof of $P \to Q$

B. Example 4: ***If $X + Y \geq 2$ then $X \geq 1$ or $Y \geq 1$***

1. Formal Statement:

   a. $E(X, Y) \equiv X + Y \geq 2$

   b. $F(X) \equiv X \geq 1$

   c. Theorem: $E(X, Y) \to F(X) \vee F(Y)$

2. Contrapositive statement: $\neg(F(X) \vee F(Y)) \to \neg E(X, Y)$

   a. $\neg(F(X) \vee F(Y)) \equiv \neg F(X) \wedge \neg F(Y)$

   b. $\neg F(X) \equiv \neg(X \geq 1) \equiv X < 1$

   c. $\neg E(X, Y) \equiv \neg(X + Y \geq 2) \equiv X + Y < 2$

   d. $\neg F(X) \wedge \neg F(Y) \equiv$
   $(X < 1) \wedge (Y < 1) \to \neg E(X, Y) \equiv X + Y < 2$

3. Proof: $X + Y < 2$ whenever $X < 1$ and $Y < 1$
   from the definition of integer addition.

   Therefore: If $X + Y \geq 2$ then $X \geq 1$ or $Y \geq 1$ $\square$

B. Example 5: **If $M$ and $N$ are integers and $M \times N$ is even, then $M$ is even or $N$ is even.**

1. Definitions:
   a. If an integer $M$ is even then $M = 2 \times K$ where $K$ is some integer.

   b. If an integer $M$ is odd then $M = 2 \times K + 1$ where $K$ is some integer.

   c. Therefore:
      a. If $M \times N$ is even then $M \times N = 2 \times K$ where $K$ is some integer.

      b. If $M \times N$ is odd then $M \times N = 2 \times K + 1$ where $K$ is some integer.

2. Contrapositive Statement of Theorem:
   $\neg Even(M) \wedge \neg Even(N) \rightarrow \neg Even(M \times N))$

3. Proof by Contraposition:
   a. $\neg Even(M) \equiv M = 2 \times K + 1$

   b. $\neg Even(N) \equiv N = 2 \times J + 1$

   c. Therefore:
   $$\begin{aligned} M \times N &= (2 \times K + 1) \times (2 \times J + 1) \\ &= (2 \times K \times 2 \times J) + (2 \times K) + (2 \times J) + 1 \\ &= 2 \times (2 \times K \times J + K + J) + 1 \\ &= 2 \times I + 1 \quad \text{where} \\ & \qquad I = 2 \times K \times J + K + J \end{aligned}$$

   f. Therefore: $M \times N = 2 \times I + 1$ where $I$ is an integer.
   g. Therefore: $M \times N$ is not even.

   h. Therefore: $\neg Even(M) \wedge \neg Even(N) \rightarrow \neg Even(M \times N)$

   i. Therefore: $Even(M \times N) \rightarrow Even(M) \vee Even(N)$ $\square$

## XI. Contradiction

A. A proof by contradiction involves assuming that your conclusion is false and trying to derive a result that contradicts known axioms or facts.

B. Example 6: **If $M$ and $N$ are integers and $M \times N$ is even, then $M$ is even or $N$ is even.**

C. Proof By Contradiction:

1. Assume that :
   a. $Even(M) \vee Even(N)$ is false.
   b. $Even(M \times N)$ is true.

2. Then: $\neg(Even(M) \vee Even(N)) \equiv \neg Even(M) \wedge \neg Even(N)$ is true.

3. $\neg Even(M) \wedge \neg Even(N)$ states that:
   a. $M$ is not even, hence must be odd
   and
   b. $N$ is not even, hence must be odd

4. Therefore, both $M$ and $N$ must be odd.
5. If $M$ is odd, then $M = 2 \times J + 1$ for some $J$.
6. If $N$ is odd, then $N = 2 \times K + 1$ for some $K$.

7. Then:
$$\begin{aligned} M \times N &= (2 \times J + 1) \times (2 \times K + 1) \\ &= (2 \times J \times 2 \times K) + 2 \times J + 2 \times K + 1 \\ &= 2 \times ((2 \times J \times K) + J + K) + 1 \\ &= 2 \times (2 \times J \times K + J + K) + 1 \\ &= 2 \times I + 1 \quad \text{where } I = 2 \times J \times K + J + K \end{aligned}$$

8. Therefore, $M \times N$ is odd.

9. So: Assuming that $\neg(Even(M) \vee Even(N))$ leads to the conclusion that $M \times N$ is odd.

12. This contradicts our assumption $Even(M \times N)$

13. Therefore: $Even(M \times N) \rightarrow Even(M) \vee Even(N)$ □

D. Classic Example: **Example 7:** $\boldsymbol{\sqrt{2}}$ **is not a rational number.**

E. Definition: A rational number $X$ is the quotient of two integers with no factors in common.

or

$X = \frac{L}{M}$ where $L$ and $M$ are integers with no factors in common.

F. Proof by contradiction:

1. Assume that $\sqrt{2}$ is a rational number.

2. Therefore: $\sqrt{2} = \frac{L}{M}$
   where $L$ and $M$ are integers with no factors in common.

3. Squaring both sides of the equation gives us:
   $(\sqrt{2}\,)^2 = 2 = \left(\frac{L}{M}\right)^2 = \frac{L^2}{M^2}$

4. Therefore: $2 \times M^2 = L^2$

5. Therefore: $L^2 = 2 \times M^2$ is divisible by 2.

6. Therefore: $L^2 = L \times L$ is divisible by 2 from Example 2.

7. Therefore: $L$ must be divisible by 2 so $L = 2 \times K$
   where $K$ is some integer.

8. Then: $L^2 = (2 \times K)^2 = 4 \times K^2 = 2 \times M^2$

9. So: $4 \times K^2 = 2 \times M^2$
   or: $2 \times K^2 = M^2$

10. Therefore: Both $L$ and $M$ are divisible by 2, contradicting our original assumption.

11. Therefore: $\sqrt{2}$ is not a rational number. ☐

## XII. Proof By Cases:

A. Example 8: $Even(M \times N) \rightarrow Even(M) \vee Even(N)$

B. Logical Statement:

1. Predicates:
   a. $Even(K) \equiv$ $K$ is even
   b. $Integer(K) \equiv$ $K$ is an integer.

2. The predicate $Even(K)$ requires that $K$ be an integer so we will dispense with the predicate $Integer(K)$ in the statement of the theorem and assume that our domain of discourse is the integers.

3. This theorem is, unlike many others, amenable to a proof by cases since there are only a small number of cases to be processed, i.e.:
   a. $M$ and $N$ are both odd.
   b. $M$ is even and $N$ is odd.
   c. $M$ is odd and $N$ is even.
   d. $M$ and $N$ are both even.

4. Case 1: Both $M$ and $N$ are odd.
   a. Therefore: $M = 2 \times J + 1$ and $N = 2 \times K + 1$ where $K$ and $J$ are integers

   b. Then:
   $$\begin{aligned} M \times N &= (2 \times J + 1) \times (2 \times K + 1) \\ &= (2 \times J \times 2 \times K) \\ &\qquad + (2 \times K) + (2 \times J) + 1 \\ &= 4 \times J \times K + 2 \times K \\ &\qquad + 2 \times J + 1 \\ &= 2 \times (2 \times J \times K \\ &\qquad + K + J) + 1 \\ &= 2 \times I + 1 \quad \text{where} \\ &\qquad I = 2 \times J \times K + K + J \end{aligned}$$

   c. Since $I$ is an integer and $M \times N = 2 \times I + 1$ then $M \times N$ is odd by definition.

   d. Therefore, if both $M$ and $N$ are odd then $M \times N$ is odd.

3. Case 2: $M$ is even and $N$ is odd.
   a. Therefore: $M = 2 \times J$ and $N = 2 \times K + 1$
   where $K$ and $J$ are integers

   b. Then: $M \times N$ $= (2 \times J) \times (2 \times K + 1)$

   $= (2 \times J \times 2 \times K) + (2 \times J)$

   $= 2 \times (2 \times J \times K + J)$

   $= 2 \times I$ where
   $I = 2 \times J \times K + J$

   c. Since $I$ is an integer and $M \times N =$ $2 \times I$ then
   $M \times N$ is even by definition.

   4. Therefore, if $M$ is even and $N$ is odd then $M \times N$ is even.

4. Case 3: $M$ is odd and $N$ is even.
   a. Therefore: $M = 2 \times J + 1$ and $N = 2 \times K$
   where $K$ and $J$ are integers

   b. Then: $M \times N$ $= (2 \times J + 1) \times (2 \times K)$

   $= (2 \times J \times 2 \times K) + (2 \times K)$

   $= 2 \times (2 \times J \times K + K)$

   $= 2 \times I$
   where $I = 2 \times J \times K + K$

   c. Since $I$ is an integer and $M \times N =$ $2 \times I$
   then $M \times N$ is even by definition.

   4. Therefore, if $M$ is odd and $N$ is even then $M \times N$ is even.

5. Case 4: $M$ is even and $N$ is even.
   a. Therefore: $M = 2 \times J$ and $N = 2 \times K$ where $K$ and $J$ are integers

   b. Then: $$\begin{aligned} M \times N &= (2 \times J) \times (2 \times K) \\ &= (2 \times J \times 2 \times K) \\ &= 2 \times 2 \times J \times K \\ &= 2 \times I \end{aligned}$$ where $I = 2 \times J \times K$

   c. Since $I$ is an integer and $M \times N = 2 \times I$ then $M \times N$ is even by definition.

   4. Therefore, if $M$ is even and $N$ is even then $M \times N$ is even.

6. Therefore, in every case for which $M$ and $N$ are integers and $M \times N$ is even, we have shown that $M$ is even or $N$ is even or both $M$ and $N$ are even.

   The only case in our exhaustive list of cases for which neither $M$ nor $N$ is even requires that $M \times N$ be odd. $\square$

## XIII. Example Proofs - Proof by Cases

A. Example 9: If $P^2$ is even (divisible by 2) then $P^2$ is divisible by 4.

**Proof:**

1. Case 1. Assume that $P$ is odd.
   Therefore: $P = 2 \times k + 1$ where $k$ is some integer.

   So: $$P^2 = (2k + 1) \times (2k + 1) = 4k^2 + 4k + 1 = 2 \times (2k^2 + 2k) + 1$$
   Therefore: $P^2$ is odd.

   So, if $P$ is odd we must have that $P^2$ is odd.

   Therefore, if $P$ is odd our premises are not satisfied. It is sufficient, then, to consider only those cases for which $P$ is even.

2. Case 2. Assume that $P$ is even.
Therefore: $P = 2 \times m$
where $m$ is some integer

So: $P^2 = 2m \times 2m = 4m^2$

Therefore: $P^2$ is divisible by 4. □

B. Example 10: **There exist irrational numbers $b$ and $c$ such that $b^c$ is rational.**

**Proof:** Consider: $b^c = \sqrt{2}^{\sqrt{2}}$

Both $b = \sqrt{2}$ and $c = \sqrt{2}$ are irrational.

Either: $b^c = \sqrt{2}^{\sqrt{2}}$ is rational or it is not.

Case 1: If $b^c = \sqrt{2}^{\sqrt{2}}$ is rational then:

a. $b = \sqrt{2}$ and $c = \sqrt{2}$

b. Both $b = \sqrt{2}$ and $c = \sqrt{2}$ are irrational (from Example 7)

c. We have an instance $b^c$ which is rational when both $b$ and $c$ are irrational

Case 2: If $b^c = \sqrt{2}^{\sqrt{2}}$ is irrational then we consider

$$(b^c)^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$$

So, when: $b^c = \sqrt{2}^{\sqrt{2}}$, which is irrational by assumption

and: $c = \sqrt{2}$, which is irrational, from Example 7,

we have that: $(b^c)^c = 2$ which is rational.

Therefore: There exist irrational numbers $b^c$ and $c$ such that $(b^c)^c$ is rational. □

## XIV. Prolog Problem Solving

A. Problem Description

1. Consider the double digit decimal multiplication shown below in which letters are used to represent unique decimal digits.

```
          H   E

      ×   M   E
      -----------
          B   E

  +   Y   E
  ---------------
      E   X   E
```

2. Problem: What assignment of single digits to letters will make the arithmetical operation valid?

3. Premises:
   a. Each unique letter represents a single unique digit, not a multi-digit integer value.
   b. Each distinct letter represents a single unique digit, no duplicates.

B. Problem History: Presented in the Mensa Problem section of an airline magazine, the flight was boring and long, and a laptop computer with Arity Prolog installed was available.

C. Note:

1. The problem is carefully structured, as befits a Mensa problem, so that all of the information needed to solve the problem is presented or implied in the problem statement.
2. Enough information is presented, as befits a Mensa problem, to allow the application of logic, rather than simple exhaustive search, to the problem solving process.
3. A single or only a few solutions possible.
4. A solution requires the instantiation of only six digits.

D. Implied Prerequisite Information

1. The definition of decimal addition and multiplication, including the process of carrying a digit from one column of the sum to the next.

$$
\begin{array}{rcccc}
 & & & 8 & 6 \\
 & & \times & 3 & 1 \\
\hline
 & & & 8 & 6 \\
+ & 2 & 5 & 8 & \\
\hline
 & \mathbf{2} & \mathbf{6} & \mathbf{6} & \mathbf{6}
\end{array}
$$

2. The definition of the set $D$ of decimal digits,i.e.,

   $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

E. First Step in Problem Solution:

1. Define the decimal digits, or define the domain from which values for all of the variables can be chosen.

   Since there are only ten digits, the definition can be a simple exhaustive definition:

   ***digit(0).***
   ***digit(1).***
   ***digit(2).***
   ***digit(3).***
   ***digit(4).***
   ***digit(5).***
   ***digit(6).***
   ***digit(7).***
   ***digit(8).***
   ***digit(9).***

2. For each digit ***i*** the predicate ***digit(i)*** returns ***true***.

3. In other words, the decimal digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$ **and no others,** have the property of ***digit***.

4. A query: ***digit(7).*** would return ***true***.

5. A query: ***digit(X).*** would return ***X*** $=$ ***0*** ->

6. If user enters ***;*** for '*find another solution*' the response would be

   ***X = 1*** ->

   a. Repetively entering ***;*** would instantiate ***X*** to each decimal digit in turn.

   digit(X).
   X = 0 ->; X = 1 ->;
   X = 2 ->; X = 3 ->;
   X = 4 ->; X = 5 ->;
   X = 6 ->; X = 7 ->;
   X = 8 ->; X = 9
   yes
   ?-

   b. The ***yes*** after X = 9 indicates that the list of legal values for which ***digit(X)*** is ***true*** has been exhausted.

F. Second Step in Problem Solution: Provide a definition of decimal digit addition

1. Format:

$$\begin{array}{ccc} & X & Y \\ +U & V & \\ \hline A & B & C \end{array}$$

   a. $C = Y$
   b. $X + V = carrydigit\ B$
   c. $A = U + carrydigit$

2. Modified definition to account for the displacement to the left that occurs as a result of multiplication.

   a. Example:

$$\begin{array}{cccc} & & 9 & 3 \\ + & 5 & 8 & \\ \hline & 6 & 7 & 3 \end{array}$$

   b. Since $9 + 8 = 17$ we have: $B = 7$
   $carrydigit = 1$

   c. Then: $U + carrydigit = 5 + 1 = 6 = A$

3. Each letter $\boldsymbol{U}$, $\boldsymbol{B}$, $\boldsymbol{X}$, $\boldsymbol{Y}$ must represent a unique decimal digit.

4. The right-most digit of the sum must be the right-most digit of the top addend, so:
   $\boldsymbol{C = Y}$ and $\boldsymbol{C = 3}$

5. The operation $\boldsymbol{X + Y}$ can generate a two-digit sum so the operation of ***carry*** must be implemented.
   a. $9 + 8 = 17$ so $B = 7$ with a carry of $1$

   b. Hence: $X + Y$ $= carrydigit$ $B$

6. Logic: ***add(X, Y, U, V, A, B, C) :-***

```
C is Y,
M is X + V,
Carry is M // 10,
N is Carry*10,
B is M - N,
A is U + Carry,
digit(A),
digit(B),
digit(C),
A \= B,
A == C.
```

   a. The numerical value of the right-most digit is instantiated
      $\boldsymbol{C\, is\, Y}$ so, in this case, $\boldsymbol{C = 3}$

   b. The numerical value of the sum of the next digits is computed:
      $\boldsymbol{M\, is\, X + V}$ so $\boldsymbol{9 + 8 = 17}$

   c. The value of the carry digit, if any, is given by the quotient of the integer division indicated by:
      $\boldsymbol{Carry\, is\, M\, //\, 10}$ or
      $Carry\, is\, 17\, //\, 10 = 1$

d. The first digit of the sum is been calcluated as:
***N is Carry*10,***
***B is M - N,***
***A is U + Carry,***
i. ***N is Carry*10*** $N = 1 \times 10 = 10$
ii. ***B is M - N*** $B = 17 - 10 = 7$
iii. ***A is U + Carry*** $A = 5 + 1 = 6$

e. The requirement that the values of $A$, $B$, and $C$ be single digits is enforced by the statements:
***digit(A), digit(B), digit(C)***

f. The requirement that the value of $A$ is different from that of $B$ and that the value of $C$ is that of $A$ is enforced by the statements:
***A \= B*** and ***A == C.***

G. Third Step in Problem Solution: Provide a definition of decimal digit multiplication

1. Format:

$$\begin{array}{cc} X & Y \\ \times & \text{A} \\ \hline M & N \end{array}$$

2. Each letter $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{X}$, $\boldsymbol{Y}$ must represent a unique decimal digit.

3. Logic: ***mult(X, Y, A, M, N)*** **:-**
***P is A * Y,***
***Carry is P // 10,***
***Q is Carry * 10,***
***N is P - Q,***
***digit(N),***
***V is A * X,***
***M is V + Carry,***
***digit(M),***
***M \= N.***

4. The operation $\boldsymbol{A \times Y}$ can generate a two digit product
a. $5 \times 7 = 35$
b. Hence: $M = A \times X + carrydigit$

5. The values of $A,\ Y,\ M,$ and $N$ must be instantiated to values such that $M$ an $N$ are decimal digits.
   a. $X\ and\ Y$ were instantiated as decimal digits by the invoking predicate.
   b. Example: $18 \times 4 = 72$

6. The numerical value of the first product is instantiated as:
   ***P is A * Y***
   so: $P = A \times Y$ or $32 = 8 \times 4$

7. The value of the carry digit, if any, is given by: ***Carry is P // 10***
   so: $32 \,//\, 10 = 3$

8. The numerical value of the first digit of the product is instantiated as:

   | | |
   |---|---|
   | ***Q is Carry * 10*** | $Q = 3 \times 10$ |
   | ***N is P − Q,*** | $N = 32 - 30 = 2$ |
   | ***digit(N),*** | 2 is a digit |

9. The numerical value of the second digit of the product is instantiated as:

   | | |
   |---|---|
   | ***V is A * X,*** | $V = 1 \times 4 = 4$ |
   | ***M is V + Carry,*** | $M = 4 + 3 = 7$ |
   | ***digit(M),*** | 7 is a digit |

10. Product $= 72$ and $M = 7$ and $N = 2$

H. Complete Prolog Program Code

```
/*   Ths program solves the problem of associating with each character given
     below the appropriate unique digit so that the computation indicated
     below is correct:
                    H    E

               ×    M    E
               ---------------
                    B    E
          +    Y    E
          --------------------
               E    X    E
```

```
First, we need a predicate that defines the term 'digit', in this case
        by complete enumeration.                                */
digit(0).
digit(1).
digit(2).
digit(3).
digit(4).
digit(5).
digit(6).
digit(7).
digit(8).
digit(9).

/*      The high level query predicate is:                      */
digitchar    :-    digit(E), digit(H), H \= E,
                   mult(H, E, E, B, E),      /* H E multiplied by E = B E    */

                   digit(B), B \= H, B \= E, /* B is unique                  */
                   digit(M), M \= E, M \= H, M \= B,
                   digit(Y), Y \= E, Y \= H, Y \= B, Y \= M,

                   mult(H, E, M, Y, E),      /* H E X M = Y E */

                   add(B, E, Y, E, E, X, E), /* B E +  Y E  0 = E X E

                   write('       '), write(H), write('   '), write(E), nl,
                   write('    X   '), write(M), write('   '), write(E), nl,
                   write('----------------------'), nl,

                   write('       '), write(B), write('   '), write(E), nl,
                   write('+    '), write(Y), write('   '), write(E), nl,
                   write('----------------------'), nl,

                   write('    '), write(E), write('   '), write(X), write('   '),
                   write(E), nl.
```

```
/*      The mult predicate instantiate values that satisfy the case where:

                    X      Y
          ×                V
          ------------------
                    M      N             */

mult(X, Y, A, M, N)       :-      P is A * Y,
                                  Carry is P // 10,
                                  Q is Carry * 10,
                                  N is P - Q,
                                  digit(N),
                                  V is A * X,
                                  M is V + Carry,
                                  digit(M),
                                  M \= N.

/*      The add predicate instantiate values that satisfy the case where:
                    X      Y
          +         U      V
          ------------------
          A         B      C             */

add(X, Y, U, V, A, B, C) :-       C is Y,
                                  M is X + V,
                                  Carry is M // 10,
                                  N is Carry*10,
                                  B is M - N,
                                  A is U + Carry,
                                  digit(A),
                                  digit(B),
                                  digit(C),
                                  A \= B,
                                  A == C.
```

I. Solution: ?- digitchar.

```
      1  5
  ×   3  5
-----------
      7  5
+  4  5
-----------
   5  2  5
```

**XV. Discrepancy between PROLOG (Programming in Logic) and Classical Logic**

A. The double negation, $\neg(\neg A)$, is not supported by the formal definition of PROLOG

B. Consider a predicate: $member(X, Set)$.
where: $X$ is a possible set element and
$Set$ denotes some set.

C. If we were to type the query: $member(X, [a, b, c, d])$.
PROLOG would, in turn, instantiate $X$ to $a$, $b$, $c$ and $d$.

D. $not(member(X, [a, b, c, d]))$. must, then, fail.
1. Upon failure, existing instantiations are erased, or forgotten
2. Therefore, any instantiation of $X$ no longer exists.

E. Then: $not(not(member(X, [a, b, c, d])))$.
must also fail because $X$ is no longer instantiated.

F. So, in PROLOG, $\neg\neg A \equiv A$ does not hold.