Completeness of the tree test for validity of arguments in SL

Completeness is the property of "getting all the correct positives". If an argument IS valid, the tree test will SAY it is valid. Let us mark that off as a theorem.

COMPLETENESS THEOREM FOR VALIDITY IN SL: If an argument is valid in SL, then the tree for that argument will say it is. I.e. that tree will close.

The first step toward a proof is to turn this around. I mean, notice it is equivalent to saying: If the tree for an SL argument does not close, then the argument is not valid.

Since every tree for SL does finish eventually, we only need to show every finished open branch gives a CE. In fact we will show something about trees in general, not only trees for arguments. Make sure you understand why this is what we need:

LEMMA: Going from the bottom to the top of a finished open branch on a tree in SL; assign T to each atomic sentence you meet, and each time you meet a negated atomic sentence you assign F to that atomic sentence. (And just to make a decision, assign F to every atomic that did not occur either as such or as negated on the branch.) You get a row of the truth table that makes every sentence on that branch True.

First, this T/F assignment is well defined—that is, you will never be asked to assign both True and False to the same atomic sentence. How do you know that?

We show that as you take this assignment up the branch there cannot be any *first* sentence that it makes False, so there cannot be any sentence at all that it makes False. We do this by looking at each kind of sentence:

1-2. An atomic or negated atomic sentence cannot be made False at all. (Why?)

3a. A conjunction ($\mathscr{P}$ & $\mathscr{Q}$) cannot be the *first* sentence made False, since below it we have already met both $\mathscr{P}$ and $\mathscr{Q}$. If the assignment makes both of those True, then by the truth table for & it also makes ($\mathscr{P}$ & $\mathscr{Q}$) True.

3b. A negated conjunction ~($\mathscr{P}$ & $\mathscr{Q}$) cannot be the *first* sentence made False, since below it we have already met either ~$\mathscr{P}$ or ~$\mathscr{Q}$, (could we met both?). If the assignment makes one of those True, then by the truth table for ~ and & it also makes ~ ($\mathscr{P}$ & $\mathscr{Q}$) True.

I leave it to you to finish the cases ($\mathscr{P} \vee \mathscr{Q}$), ~($\mathscr{P} \vee \mathscr{Q}$), ($\mathscr{P} \supset \mathscr{Q}$), ~($\mathscr{P} \supset \mathscr{Q}$), ($\mathscr{P} \equiv \mathscr{Q}$), ~($\mathscr{P} \equiv \mathscr{Q}$). Do one or two of these for yourself, in full, as exercise.

In short, this T/F assignment makes the bottom sentence of the branch True, and makes each new sentence we come to True. So it makes them all True.