I. Introduction to Functions

A. A function is often referred to as a ***mapping***.

B. Example 1: The assignment of Social Security Numbers to all citizens of the USA.

1. Let $C = \{c \mid x \text{ is a citizen of the USA}\}$

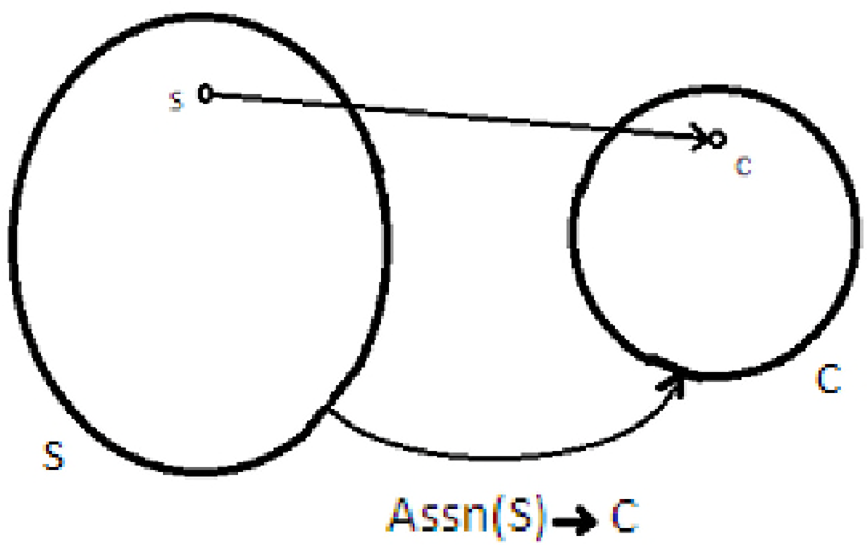
   $S = \{s \mid y \text{ is a Social Security Number}\}$

2. Then a function $Assn$ that assigns to each citizen of the USA ***exactly one*** social security number is a mapping from the set S to the set C.

3. Formally: The function $Assn : S \to C$
   maps a unique social number to each citizen

4. We can also write: $Assn(s) = c$
   where $s \in S \wedge c \in C$

5. Graphic Representation:



C. Example 2:

1. The Java method declaration

```
public static long flrTst(double X)
    {
    long   k  =  (long) X;
    return k;
    }
```

a. Describes a function which accepts a real value as a parameter and matches it to an integer.

b. Therefore the declaration describes a function matches an element of the set of real numbers to a single element of the set of integers.

D. Formal Definition of Functions:

If A and B are non-empty sets a function f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of the set A.

In this case we have a function from the set A to set B so we can write: $f : A \to B$

1. In the expression $f : A \to B$
   a. A is called the ***domain*** of f and B is called the ***codomain*** of f

   b. f is said to map A to B.

   c. B is said to be the range, or image, or codomain of f

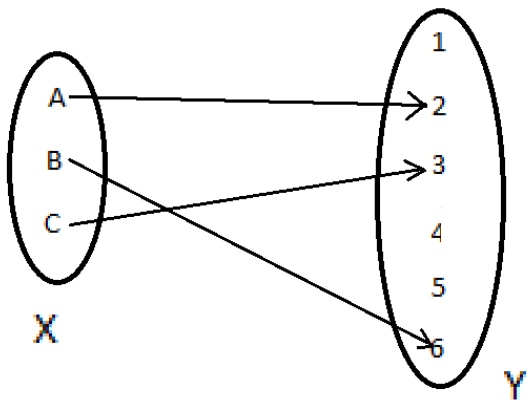
2. In the expression $f(a) = b$
   a. a is called the ***preimage*** of b under f

   b. b is called the ***image*** of a under f.

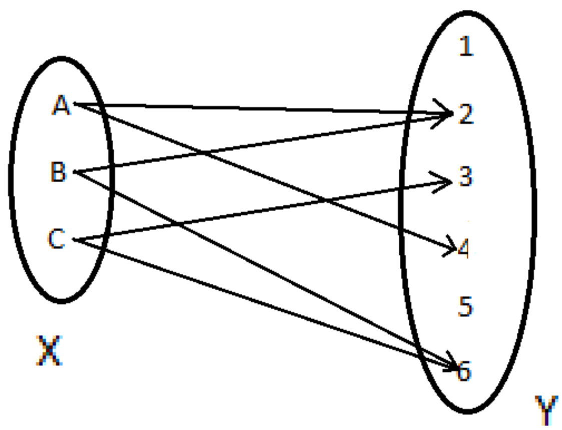
E. Restriction: ***A function is single-valued mapping***

1. In the case of $f : A \to B$ there is only a single $b \in B$ for which $F(a) = b$

2. The graphical representation of a function f shown below shows a single-valued mapping $f : X \to Y$.



3. The graphical representation of a mapping f shown below shows a multiple valued mapping and $f : X \to Y$ ***is not*** a function.



II. Ordered Pairs

A. Definition:

1. An ordered pair $(a,\, b)$ consists of two elements a and b stored in a specified order.

2. Since the two elements are stored in a specified order,

$$(a,\, b) \neq (b,\, a)$$

B. Representation

1. Collections are sets so a logical alternative would be to represent ordered pairs as sets.

2. Problem: Sets do not recognize order so $\{a,\, b\} = \{b,\, a\}$

3. Solution: Define $(a,\, b) \equiv \{\{a\},\, \{a,\, b\}\}$
   where a is the first coordinate and b the second.

4. Note: If $a = b$ then

$$(a,\, b) = (a,\, a) = \{\{a\},\, \{a,\, a\}\}$$

$$= \{\{a\},\, \{a\}\} = \{\{a\}\}$$

III. Theorem: If $(a,\, b)$ and $(x,\, y)$ are ordered pairs and $(a,\, b) = (x,\, y)$ then $a = x$ and $b = y$.

Proof: (by cases)

A. If $a = b$ then $(a,\, b) = (a,\, a) = \{\{a\},\, \{a,\, a\}\}$

$$= \{\{a\},\, \{a\}\} = \{\{a\}\}$$

1. Then $(a,\, b) = (x,\, y) = \{\{a\}\}$

   so that $(x,\, y) = \{\{x\},\, \{x,\, y\}\} = \{\{a\}\}$

2. $\{\{a\}\} = \{\{x\},\, \{x,\, y\}\}$ only if $\{\{x\},\, \{x,\, y\}\}$ is a singleton

3. This is true only if: $\{\{x\},\, \{x,\, y\}\} = \{\{x\},\, \{x,\, x\}\}$

$$= \{\{x\},\, \{x\}\} = \{\{x\}\}$$

4. Therefore: $\{\{a\}\} = \{\{x\}\}$ so $a = x$

B. If $a \neq b$ then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} = (x, y)$

1. Since $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$
   and the only singletons in each set are $\{a\}$ and $\{x\}$,
   respectively, we must have
   $\{a\} = \{x\}$ so $a = x$

2. Since $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$
   and the only duets in each set are $\{a, b\}$ and $\{x, y\}$,
   respectively, we must have $\{a, b\} = \{x, y\}$

3. Then, since $a = x$ we must have $b = y$

C. Therefore: If (a, b) and (x, y) are ordered pairs and
$(a, b) = (x, y)$ then $a = x$ and $b = y$.

IV. Ordered Pairs, Cartesian Products and Functions.

A. Definition of Cartesian Product

1. Given two sets, A and B, the Cartesian Product of A and B is denoted by $A \times B$

2. $A \times B = \{(a, b) \,|\, a \in A \wedge b \in B\}$ where (a, b) is an ***ordered pair***.

3. Note: $A \times B \neq B \times A$

B. Example:

1. If $X = \{3.1415,\ 8.9032,\ 1.8037\}$ and $Y = \{10,\ 45,\ 32\}$

   then $X \times Y = \{$ $(3.1415,\ 10)$, $(3.1415,\ 45)$, $(3.1415,\ 32\}$,
   $(8.9032,\ 10)$, $(8.9032,\ 45)$, $(8.9032,\ 32)$,
   $(1.8037,\ 10)$, $(1.8037,\ 45)$, $(1.8037,\ 32)\}$

2. Note: We have constructed a mapping of every element in X to every element in Y.

3. Therefore we have defined a function $f : X \to Y$

4. So: The Cartesian Product $X \times Y$ is a function $f : X \to Y$

V. Equal Functions

A. Definition: Two functions are ***equal*** if and only if:

1. they have the same domain.

2. they have the same codomain.

3. they map each element of their common domain onto the same element in their common codomain.

B. Assuming $f_a : X \to Y$ and $f_b : A \to B$

then: $$(f_a = f_b) \leftrightarrow \Big((X = A) \wedge (Y = B) \\ \wedge \\ \big((f(x) = y) \wedge (f(a) = b) \big) \leftrightarrow \big((x = a) \wedge (y = b) \big) \Big)$$

VI. One-to-One Functions

A. Definition: A function f is said to be ***one-to-one***, or an ***injunction***, or an ***injective function***, if and only if:

a. $f(a) = f(b)$ implies that $a = b$ for all a and b in X, the domain of f.

b. $f(a) \neq f(b)$ implies that $a \neq b$ for any a or b in the domain of f.

B. Note that ***b.*** is the contrapositve expression of ***a.*** since the contrapositive of

$$(\forall a \in A)(\forall b \in B)\big((f(a) = f(b)) \to (a = b) \big)$$

is: $$\neg\big((\forall a \in X)(\forall b \in X)\,(a = b) \big) \to \neg\big(f(a) = f(b) \big)$$

or

$$\big((\exists a \in X) \vee (\exists b \in X)\,(a \neq b) \big) \to \big(f(a) \neq f(b) \big)$$

C. Examples

1. Set Definitions:
   a. $Y = \{y \mid y \text{ is a student in this Discrete Math Class}\}$

   b. $C = \{c \mid c \text{ is a mobile phone number}\}$

   c. $ID = \{id \mid id \text{ is a student ID number}\}$

   d. $G = \{g \mid g \text{ is a students grade for the class}\}$
      $= \{A, B, C, D, F, Inc\}$

   e. $H = \{h \mid h \text{ is a city/town in the US}\}$

2. Consider $(\forall y \in Y)(\exists c \in C)(f(y) = c)$
   where f assigns to each student in the class their mobile phone number.

   f is almost certainly one-to-one, failing in this classification only if the phone providers have severely malfunctioned.

3. Consider: $(\forall y \in Y)(\exists id \in ID)(i(y) = id)$
   where i assigns to each student in the class their student ID number.

   i is almost certainly one-to-one. If not, the registrar's office is suffering from some severe problem.

4. Consider: $(\forall y \in Y)(\exists g \in G)(gr(y) = g)$
   where gr assigns to each student their final grade.

   Since $|G| = 6$ and $|Y| = 80$ non-empty subsets of Y class will receive the same grade. Therefore gr is not 1-to-1.

5. Consider: $(\forall y \in Y)(\exists h \in H)(ht(y) = h)$
where ht assigns to each student their home town.

Since $|Y| = 80$ the probabilty that at least two students came to CWRU from the same town is quite high. Therefore it is most likely that ht is not 1-to-1.

D. To Prove a Function $f : A \rightarrow B$ is One-to-One

1. For $f : A \rightarrow B$ show that if $f(x) = f(y)$ for ***arbitrary*** $x,\ y \in A$ then $x = y$.

2. Example: $z = f(x) = \sqrt{x}$ where $A = N$, the set of natural numbers and $B = R$, the set of real numbers

   a. If $\sqrt{x} = \sqrt{y}$ then $(\sqrt{x})^2 = (\sqrt{y})^2$

   b. $(\sqrt{x})^2 = x = y = (\sqrt{y})^2$

   c. Therefore: $\Big(f(x) = f(y)\Big) \rightarrow x = y$

E. To Prove a Function $f : A \rightarrow B$ is Not One-to-One

1. For $f : A \rightarrow B$ find particular elements $x,\ y \in A$ such that $x \neq y$ and show that $f(x) = f(y)$

2. Example: $z = f(x) = x^2$ where $A = Z$, the set of integers and $B = R,$ the set of real numbers

   a. Consider $x = 2$ and $y = -2$

   b. Then $2 = x \neq y = -2$

   c. But $f(x) = x^2 = 4 = y^2 = f(y)$

   d. Therefore: $\neg\bigg(\Big(f(x) = f(y)\Big) \rightarrow x = y\bigg)$

VII. Strictly Increasing/Decreasing Functions

A. Definitions:

1. A function f whose domain and codomain is $\boldsymbol{R}$, the set of real numbers, is called ***increasing*** if $f(x) \leq f(y)$ whenever $x < y$ and $x,\ y\ \in \boldsymbol{R}$

   Alternatively, f is increasing if

   $$\forall x \forall y \Big((x < y) \rightarrow (f(x) \leq f(y)) \Big)$$

2. A function f whose domain and codomain is $\boldsymbol{R}$, the set of real numbers, is called ***strictly increasing*** if $f(x) < f(y)$ whenever $x < y$ and $x,\ y\ \in \boldsymbol{R}$

   Alternatively, f is strictly increasing if

   $$\forall x \forall y \Big((x < y) \rightarrow (f(x) < f(y)) \Big)$$

3. A function f whose domain and codomain is $\boldsymbol{R}$, the set of real numbers, is called ***decreasing*** if $f(x) \geq f(y)$ whenever $x > y$ and $x,\ y\ \in \boldsymbol{R}$

   Alternatively, f is decreasing if

   $$\forall x \forall y \Big((x > y) \rightarrow (f(x) \geq f(y)) \Big)$$

4. A function f whose domain and codomain is $\boldsymbol{R}$, the set of real numbers, is called ***strictly decreasing*** if $f(x) > f(y)$ whenever $x > y$ and $x,\ y\ \in \boldsymbol{R}$

   Alternatively, f is strictly increasing if

   $$\forall x \forall y \Big((x > y) \rightarrow (f(x) > f(y)) \Big)$$

B. Example Problem 1:
Prove that a strictly increasing function f from R to itself must be one-to-one.

1. Premise: Definition of strictly increasing function, or:
$\forall x \forall y \Big((x < y) \rightarrow (f(x) < f(y)) \Big)$
2. $x < y$ requires that $x \neq y$
3. Similarly, $f(x) < f(y)$ requires that $f(x) \neq f(y)$
4. Therefore $\forall x \forall y \Big((x < y) \rightarrow (f(x) < f(y)) \Big)$
is a special case for: $\forall x \forall y \Big((x \neq y) \rightarrow (f(x) \neq f(y)) \Big)$
5. Therefore the definition of strictly increasing is a special case of a contrapositive statement of the definition of a 1-to-1 function.
6. Therefore a strictly increasing function f from R to itself must be one-to-one.

C. Example Problem 2:
Give an example of an increasing function from $\boldsymbol{R}$ to $\boldsymbol{R}$ that is not one-to-one.

1. A very simple example is $f(x) = c$ where $c \in \boldsymbol{R}$ is some constant.

Then, if $a > b$ we must have $f(a) = c = f(b)$
so f cannot be 1-to-1 by definition.

2. A slightly more complicated example is defined as follows:
a. $f(x) = x$ for $x < -1$
b. $f(x) = 0$ for $-1 \leq x \leq 1$
c. $f(x) = x - 4$ $x > 1$

Then $f(2) = -2 = f(-2)$

so f cannot be 1-to-1 by definition.

VIII. Onto Functions

A. Definition: A function $f : A \to B$ is called ***onto***, or a ***surjection***, or a ***surjective function***, if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$

B. Alternative Definition:

$f : A \to B$ is ***onto*** $\leftrightarrow \forall a \in A\, \exists b \in B \Big(f(a) = b \Big)$

C. Example 1: (From V. One-to-One Functions)

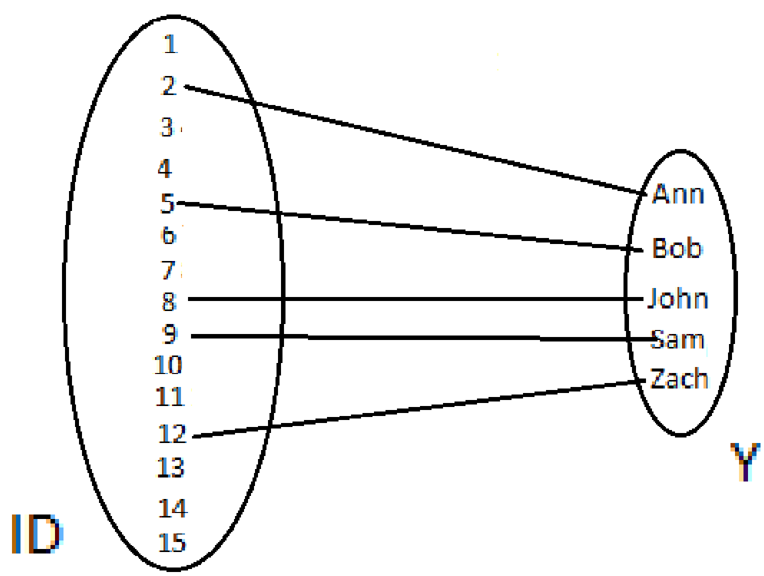
1. $Y = \{y \mid y$ is a student in this Discrete Math Class$\}$

2. $ID = \{id \mid id$ is a student ID number$\}$

3. $f : ID \to Y$ attaches a student ID number to a student in Y.

4. Therefore: $f : ID \to Y$ is ***onto*** since, for every student $y \in Y$ there exists a student ID number $id \in ID$ such that $f(y) = id$

5. Graphical Representation



D. Example 2: (From V. One-to-One Functions)

1. $Y = \{y \mid y$ is a student in this Discrete Math Class$\}$
2. $G = \{g \mid g$ is a student's grade for the class$\}$
   $= \{A,\ B,\ C,\ D,\ F,\ Inc\}$
3. $f : Y \to G$ assigns to a student in Y one of the grades in G.
4. $f : Y \to G$ may not be ***onto*** since it is possible that no student in this class will be awarded a D (or an A).
5. Therefore the statement: $\forall g \in G\, \exists y \in Y\Big(f(y) = g\Big)$
   may not be true and, hence, the statement
   $f : Y \to G$ is ***onto*** $\leftrightarrow \forall g \in G\, \exists y \in Y\Big(f(y) = g\Big)$
   requires that $f : Y \to G$ is not onto.

E. To Prove a Function $f : A \to B$ is surjective, or Onto

1. For $f : A \to B$ consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = f(y)$.

2. Example: $y = f(x) = x^2$ where $A = R$, the set of real numbers, and $B = Z^+$, the set of positive integers.
   a. For any $y \in B x = \sqrt{y}$ exists.
   b. Then $x = \sqrt{y} = \sqrt{x^2}$
   c. Therefore: An arbitrary selection of $y \in B$ leads to a value for $x \in A$ such that $f(x) = y$

F. To Prove a Function $f : A \to B$ is surjective, or Onto

1. For $f : A \to B$ find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$

2. Example: $y = f(x) = x^2$ where $A = Z$, the set of real numbers, and $B = Z$.

   a. There is no integer x such that $x^2 = -1$

   c. Therefore: There exists a $y \in B$ such that there is no x such that $f(x) \neq y$

IX. One-to-One and Onto Functions

A. Definition: A function $f : X \to Y$ is a ***one-to-one correspondence***, or a ***bijection***, or is a ***bijective function***, if it both one-to-one and onto

1. Every element x in the domain X must be assigned by f to a unique element y in the codomain Y.
2. Every element y in the codomain Y is an image under f of a unique element x of the domain X.

B. Example: $A = \{2, 4, 6, 8\}$ and $B = \{4, 16, 36, 64\}$

1. Let $f : A \to B$ be $f(a) = a^2$

2. Therefore: $\forall a \in A, \exists b \in B$ such that $f(a) = b$

and

$\forall b \in B, \exists a \in A$ such that $f(a) = b$

X. The Identity Function

A. Definition: The ***identity function*** $\equiv \iota_A : A \to A$

where $\iota_A(a) = a$

B. $\iota_A : A \to A$ maps each element of A onto itself.

C. Theorem: $\iota_A : A \to A$ is one-to-one

Proof:

$(\forall a, b) \in A \; \iota_A(a) = a$ and $\iota_A(b) = b$

$(\forall a, b) \in A \left(\iota_A(a) = \iota_A(b)\right) \to (a = b)$

$(\forall a, b) \in A \; (a \neq b) \to \left(\iota_A(a) \neq \iota_A(b)\right)$

Therefore: $\iota_A : A \to A$ is one-to-one.

D. Theorem: $\iota_A : A \to A$ is onto

Proof:

$(\forall a, b) \in A \; \iota_A(a) = a$ and $\iota_A(b) = b$

$\left((\forall b) \in A\right)\left((\exists a) \in A\right)\left(\iota_A(a) = b\right)$

since: $(a = b) \to \iota_A(a) = b$

Therefore: $\iota_A : A \to A$ is onto.

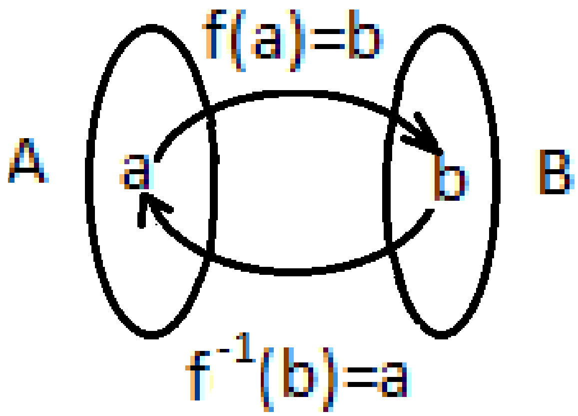
XI. Inverse Functions

A. The ***inverse*** of the one-to-one and onto function $f : A \to B$ is $f^{-1} : B \to A$.

$f^{-1} : B \to A$ assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

Therefore: If: $f(a) = b$ then $f^{-1}(b) = a$

B. Graphic Illustration:



C. Conditions:

1. $f : A \to B$ must be one-to-one for $f^{-1} : B \to A$ to exist.
   a. If $f : A \to B$ is not one-to-one then $\exists(a_1,\, a_2) \in A$ such that $f(a_1) = b$ and $f(a_2) = b$
   b. Therefore: $f^{-1}(b) = a_1$ and $f^{-1}(b) = a_2$
   c. The definition of a function $f : A \to B$ states that $f(a) = b$ specifies a unique element $b \in B$.
   d. Therefore $f^{-1}(b) = a_1$ and $f^{-1}(b) = a_2$ is impossible and f^{-1} cannot be a function if f is not one-to-one.

2. $f : A \to B$ must be onto for $f^{-1} : B \to A$ to exist.
   a. If $f : A \to B$ is not onto then $\exists b \in B$ such that there is no $a \in A$ such that $f(a) = b$
   b. Therefore: $f^{-1}(b)$ has no value.
   c. Therefore $f : A \to B$ must be onto for $f^{-1} : B \to A$ to exist.

D. Example 1:

1. Theorem: If $f : R \to R^+$ where $f(x) = |x|$ then f is not invertible.

2. Proof:

   a. If $a, -a \in R$ then $f(a) = a$ and $f(-a) = a$

   b. Therefore, if $f^{-1} : R \to R^+$ exists we must have: $f^{-1}(a) = a$ and $f^{-1}(a) = -a$

   c. This contradicts the definition of a function.

   d. Therefore $f^{-1}(x)$ does not exist for $f(x) = |x|$

   e. Therefore $f : R \to R^+$ where $f(x) = |x|$ is not invertible.

E. Example 2:

1. Theorem: If $f : R^+ \to R^+$ where $f(x) = |x|$ then f is invertible.

2. Proof:

   a. The domain of f is R^+ so $(\forall a \in R^+) f(a) = |a| = a$

   b. The codomain of f is R^+ so $(\forall |a| \in R^+) f^{-1}(|a| = a) \equiv (\forall a \in R^+) f^{-1}(a) = a)$

   c. $(\forall |a| \in R^+) f^{-1}(a) = a$ is the unique element of its domain R^+ assigned by f^{-1} to a unique element of its codomain R^+

   d. Therefore $f^{-1}(x)$ exists for $f(x) = |x|$ and $f : R^+ \to R^+$

3. Note: A much shorter proof would simply state that $f : R^+ \to R^+$ where $f(x) = |x|$

   is the identity function and, therefore, invertible.

XII. Compositions of Functions

A. Definition: If $g : A \to B$ and $f : B \to C$ then the ***composition*** of f and g, denoted by $f \circ g$, is defined by:

$$(f \circ g)(a) = f(g(a))$$

B. Alternative statement: $(f \circ g)(a)$ is the function that:

1. Maps to $b \in B$ the value specified by $g(a) = b$

2. Maps to $c \in C$ the value specified by $f(b) = c$

C. Graphical Illustration: