PHYS 121 – SPRING 2015

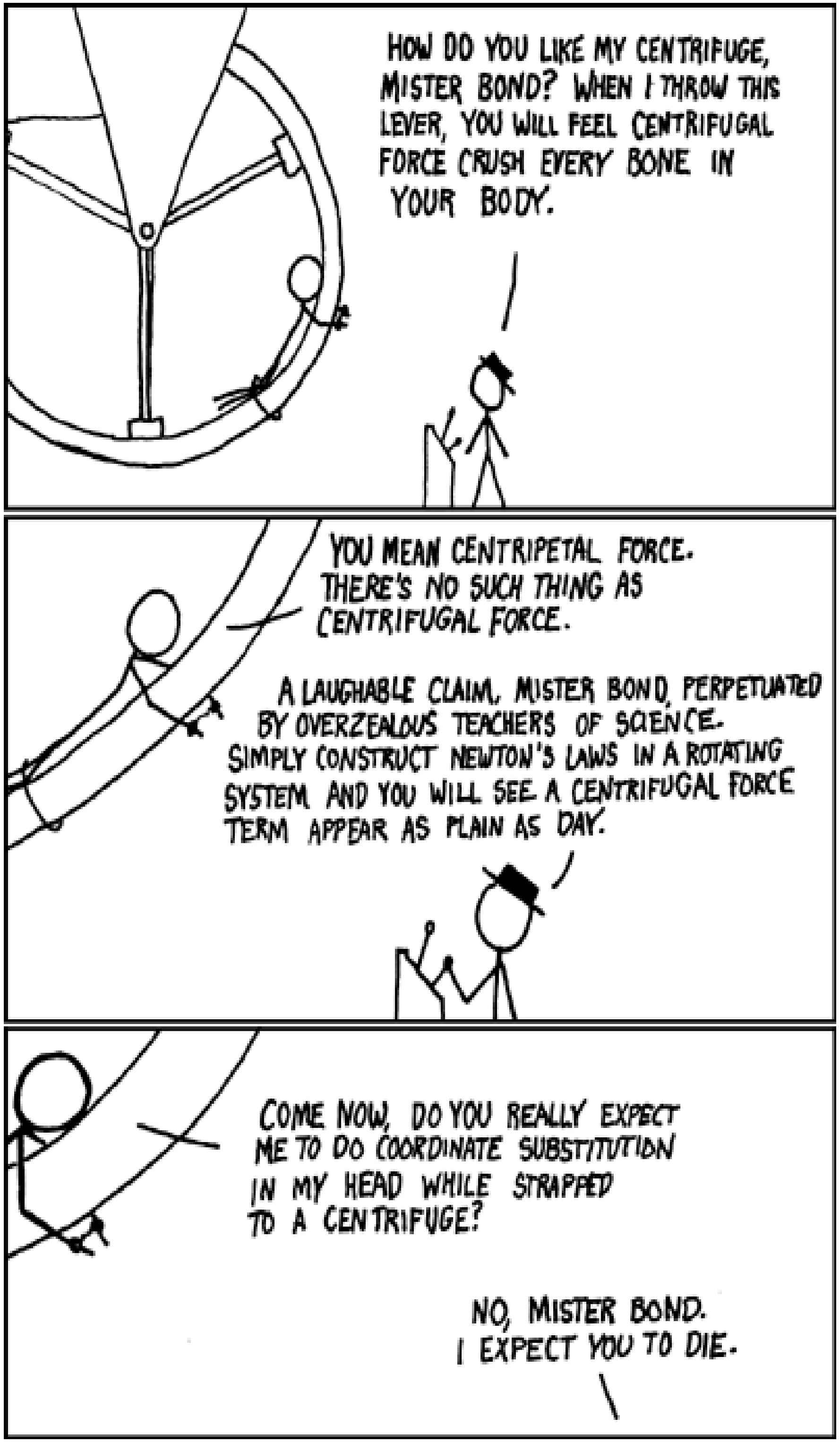
Chapter 12:

Rotation of a Rigid Body

version 04/03/2015

~112 slides

We finished this lecture on Friday, April 3.



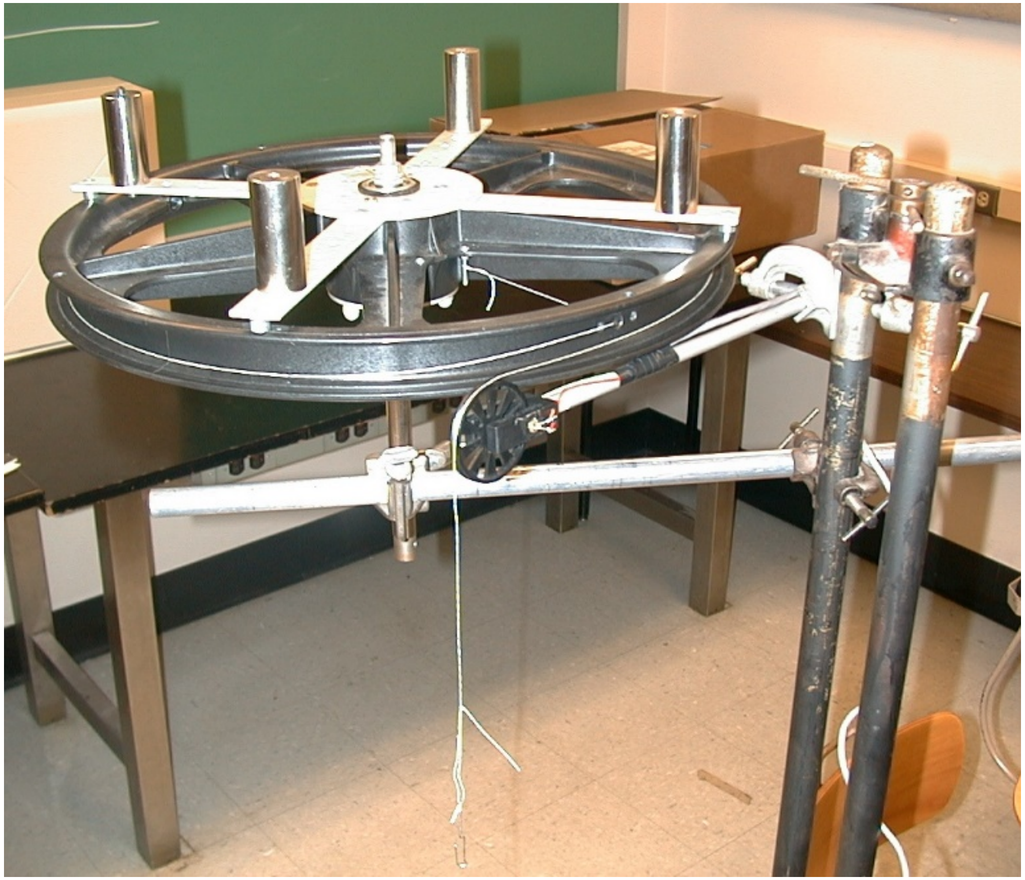
ANNOUNCEMENTS

- Clicker points were collected last Friday: 90 total!
- The grader for Group #1, A – Ch, is ill and was unable to return your homework this morning.
- We haven't set a deadline for requests to re-grade Exam #2, but we should!
- Course grades will be updated on Blackboard after requests for re-grading Exam #2 subside.

<u>Lab #4: Rotational Kinetic Energy</u>

March 18 - 26

You will use the principle of conservation of energy to determine the **<u>moment of inertia</u>** of a system of four identical masses symmetrically located on the circumference of a wheel which is rotating about its axis.



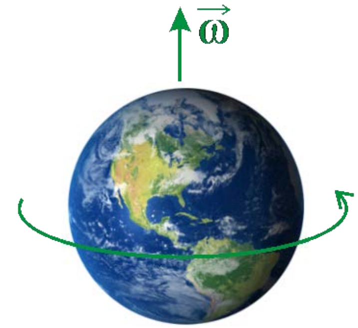
ROTATIONAL KINEMATICS

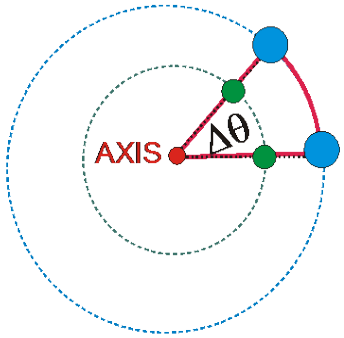
ROTATIONAL KINEMATICS is the rotational equivalent of *translational kinematics*.

- *"Kinematics" = study of motion*
- *"Dynamics" includes forces that cause motion.*
- *"Translational" motion* = change in position (x,y,z) or $\vec{r}$
  - Generally applied to a particle or CM of an object.
  - Classical (*point*) particles don't have spin/rotation.
  - Parameters of interest are $\vec{r}, \vec{v}, \vec{a}, t$

TERMINOLOGY

aka jargon





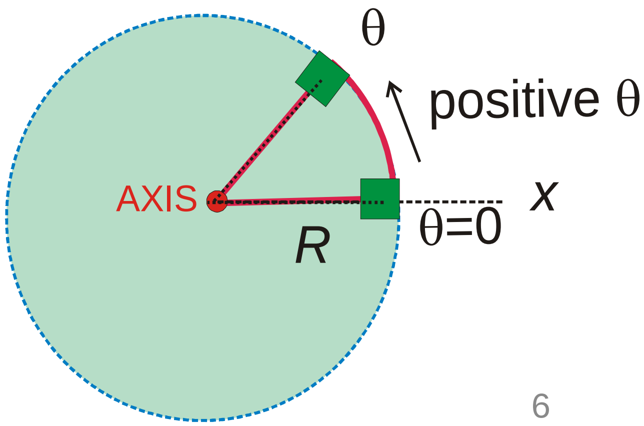
- ***RIGID BODY*** ≡ solid object whose parts are connected to each other & must move together ~ earth.
  - Normally invoked in the context of ***rotation*** about some ***axis***.
- ***AXIS of ROTATION*** ≡ line about which the parts of a rigid body rotate.
  - Every part of a rigid body rotates the same ***angular* distance $\Delta\theta$** in the same time interval Δt.
    - Every part moves around a circle centered on the axis but the radius of the circle can vary for different parts of a rigid body.
  - We'll focus first on systems with ***fixed axes*** = *the axis can move translationally in space but does not itself **rotate***).

ROTATION TERMINOLOGY

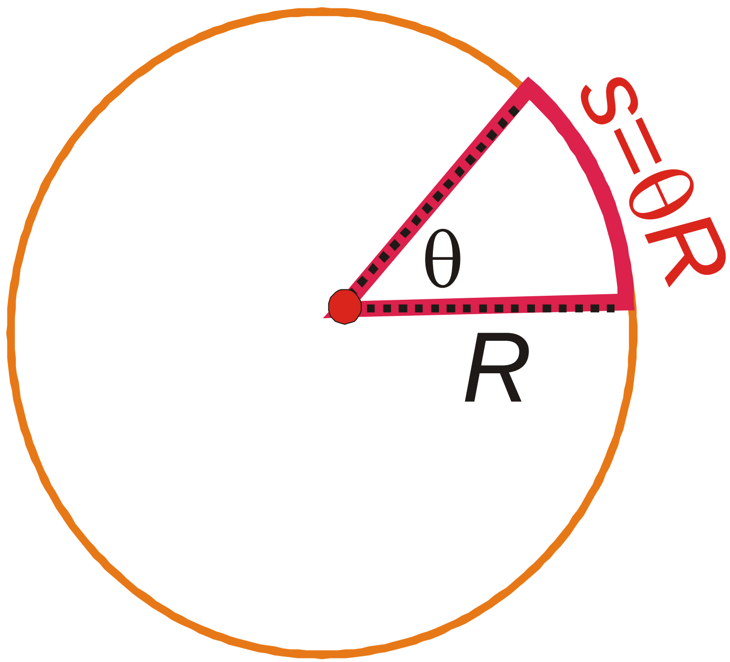
- *r or R* = radius of the circular path for a piece of a rigid body
- **θ = angular *position*** from the positive x-axis (*Ohanian uses* ϕ.)
  - Counterclockwise (*CCW*) is positive, clockwise (*CW*) is negative.
  - θ may be > 360° = 2π radians for > 1 rotation.
  - Angular *displacement*: $\Delta\theta = \theta_{final} - \theta_{initial}$
- **ω = angular *velocity*** (*Greek letter omega*)
  - **units of ω are radians/sec**
  - radians are dimensionless so the units are actually just sec^{-1}
  - positive = CCW motion, negative = CW motion
  - |ω| = ω angular *speed*; $\vec{\omega}$ = **ω** is a vector!

$$\omega_{average} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_{final} - \theta_{initial}}{\Delta t}$$

$$\omega_{\text{instantaneous}} = \omega = \frac{d\theta}{dt}$$



ROTATION TERMINOLOGY



s = arc length = θR with θ in radians

$$\theta = 2\pi \text{ radians in a circle} = 360°$$

Arc length of a complete circle = circumference:

$$C = \theta R = 2\pi R$$

FREQUENCY

f = conventional frequency = frequency

- dimensions are *cycles* or *revolutions per time*
- *time* is usually, but not necessarily, in *seconds*
- there are 2π radians/revolution
- $\omega = 2\pi f$ *since*

$$\begin{array}{ccccc} \omega & = & 2\pi & & f \\ \left(\dfrac{\#\text{ radians}}{\text{second}}\right) & = & \left(\dfrac{2\pi\text{ radians}}{\cancel{\text{revolution}}}\right) & & \left(\dfrac{\#\text{ }\cancel{\text{revolutions}}}{\text{second}}\right) \end{array}$$

PERIOD

PERIOD, ***T***, ≡ time for one complete revolution

time/revolution = (revolutions/time)$^{-1}$

$$T = 1/f = 2\pi/\omega$$

$$\omega = 2\pi f \;\rightarrow\; \omega = 2\pi/T$$

$v_{tan} = 2\pi R/T$ = circumference/period

$$v_{tan} = (2\pi/T)R = \omega R$$

"$v_{tan} = \omega R$" is worth remembering!

$v_{\text{tangential}}$ – *alternate description*

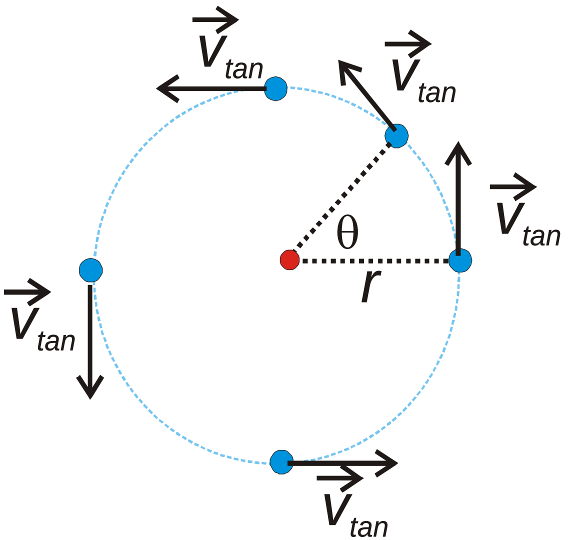
Take the time derivative of arc length $s = \theta R$ with fixed R.

$$\frac{ds}{dt} = \frac{d\theta}{dt} R \quad \rightarrow \quad v_{\text{tan}} = \omega R$$

$v = \omega R \equiv$ ***tangential velocity,*** $\boldsymbol{v_{tan}}$

$|\,\boldsymbol{v_{tan}}\,|$ = speed of the object at any instant.

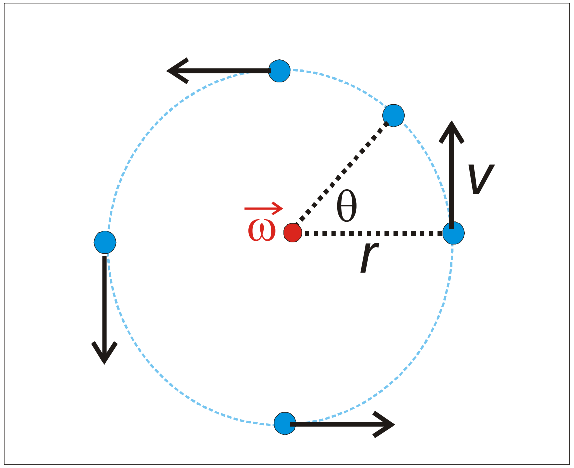
Direction of $\boldsymbol{v_{tan}}$ is always tangent to the circular path.

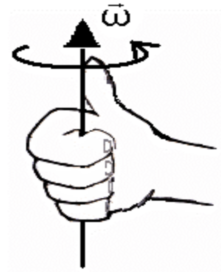


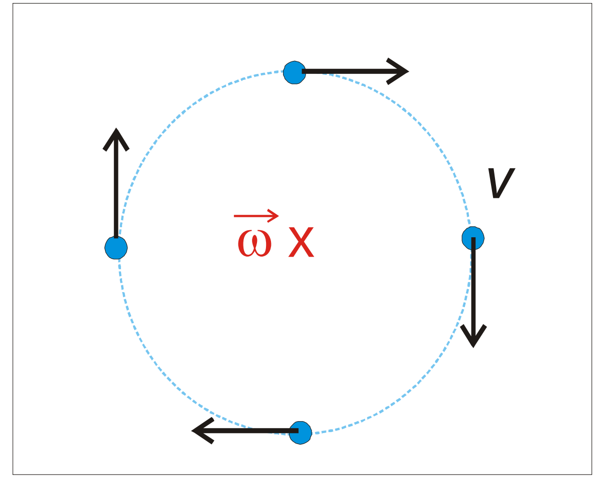
VECTOR $\vec{\omega}$

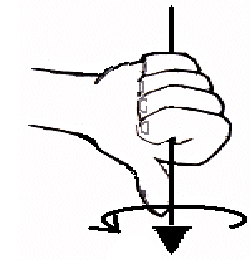
ω is a vector

Describing rotation as *clockwise* or *counter-clockwise* (*cw or ccw*) is like saying a velocity is positive or negative when you need to know the general direction of the motion first.



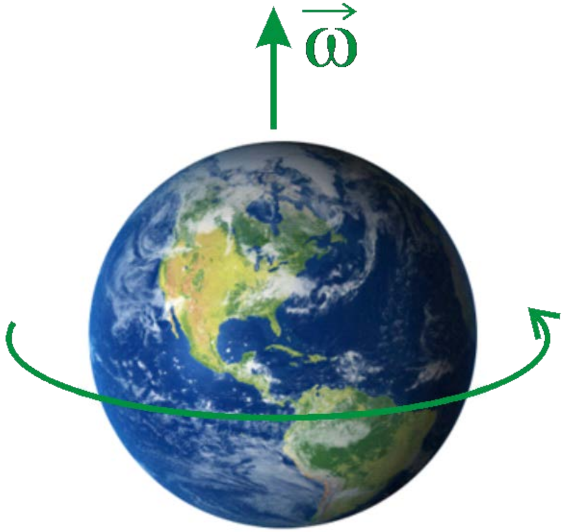






- The direction of the vector **ω** is defined as:
  - perpendicular to the plane of the circular path
  - pointing in a direction given by the **Right Hand Rule**, RHR

  RHR = the upright thumb of your **right** hand will point in the proper direction if you curl your fingers in the direction of the physical motion.
- In the figure above, **ω** points out of the page towards you.
- "towards you" is indicated with a dot • to represent the head of an arrow facing the reader.
- "away from you" is indicated with a cross × to represent the tail or feathers of an arrow facing away from the reader.



BE SURE TO USE YOUR RIGHT HAND

when applying the Right Hand Rule!

This is easy to forget when taking a test if you're right-handed.

We're going around a circle, due to the spin of the earth, while sitting in Strosacker Auditorium.

The earth spins on its axis with $\boldsymbol{v}_{tan}$ towards the east.

The direction of $\vec{\omega}$ points up out of the North Pole.

ANGULAR ACCELERATION

- If $v_{tangential}$ changes for circular motion

  $\Rightarrow$ angular speed ω must be changing too,

  since $v = \omega R$ and $\omega = v/R$

- A change in $\omega \equiv$ **angular acceleration**

  $\alpha =$ symbol for angular acceleration (*Greek letter alpha*)

$$\vec{\alpha}_{average} = \frac{\Delta\vec{\omega}}{\Delta t} = \frac{\vec{\omega}_{final} - \vec{\omega}_{initial}}{\Delta t}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

ANGULAR ACCELERATION

$$\alpha = \frac{d\omega}{dt} \qquad\qquad \alpha_{\ average} = \frac{\Delta\omega}{\Delta t}$$

ω as a function of time is given by

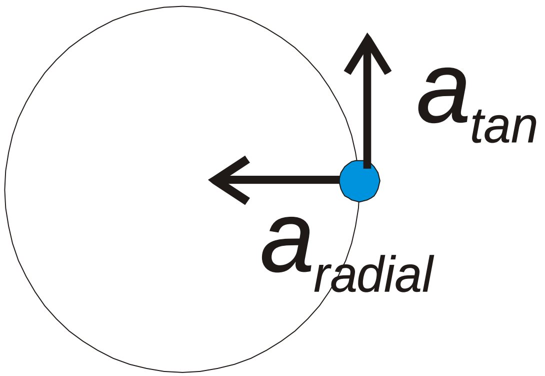
$$\omega = \int d\omega = \int \frac{d\omega}{dt} dt = \int \alpha(t)\, dt \underset{\text{if } \alpha \text{ is constant}}{=} \alpha t + \text{constant} = \omega_o + \alpha t$$

where ω_o is the constant of integration, corresponding to the angular velocity at $t = 0$.

TANGENTIAL ACCELERATION

An angular acceleration $\alpha = \dfrac{d\omega}{dt}$ on a circular path
must be accompanied by a
tangential acceleration a_{tan}
since $v_{tan} = \omega R$
and taking a time derivative yields

$$a_{tan} = \alpha R$$



Tangential acceleration for an object confined to a circular path (~ *rigid body*)

$\Rightarrow$ α is non-zero

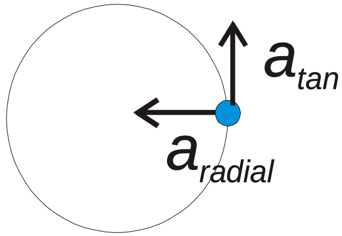
$\Rightarrow$ ω is changing

$\Rightarrow$ $a_{radial} = \omega^2 R$, the **centripetal acceleration**, will also have to change

a_{radial} changes the direction of the object's motion but not its speed.

$a_{tangential}$ changes the speed of the object but not its direction.

RADIAL + TANGENTIAL ACCELERATION



A tangential acceleration applied to an object that is **not** *constrained* to move in a circle will make that object leave its circular path.

This is how we travel between planets by rocket (*preview of coming attraction in PHYS 121*).

A tangential acceleration applied to an object that **is** *constrained* to move in a circle will change the object's angular velocity.

The constraint might be a string or light rod or a solid disc/turntable.

The centripetal force supplied by the constraint must change.

The magnitude of these forces is given by

$$F_{radial} = F_{centripetal} = ma_{centripetal} = mv^2_{tan}/R = m\omega^2 R$$

$$F_{tan} = ma_{tan} = m\alpha R$$

RADIAL + TANGENTIAL ACCELERATION *summary*

- An object traveling in a circular path of radius *R* at constant speed (*translational or angular*) given by

$$v_{tangential} \quad or \quad \omega = d\theta/dt$$

$$with \quad v_{tangential} = \omega R$$

experiences a centripetal or radial (*inward*) acceleration

$$a_{radial} = v^2_{tan}/R = \omega^2 R$$

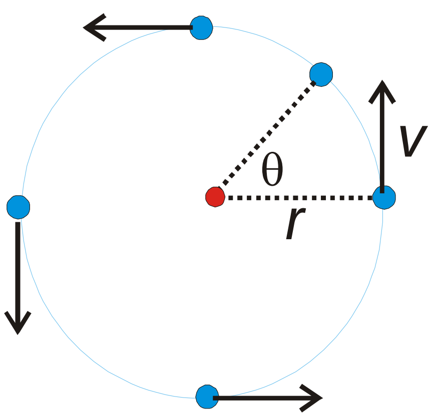
- A CHANGE in the angular velocity ω

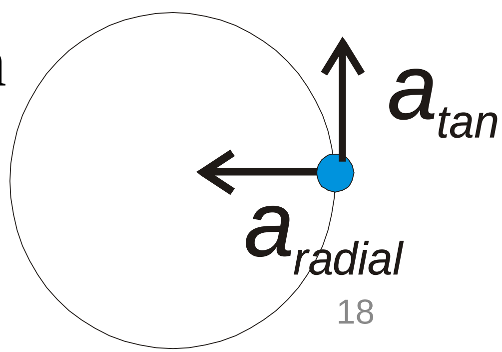
is an angular acceleration

$$\alpha = d\omega/dt$$

which requires a *tangential* acceleration

$$a_{tan} = \alpha R$$





ROTATIONAL KINEMATICS

Kinematics for circular motion with **constant** α

~ kinematics of translational motion with constant *a*.

There are four corresponding handy-dandy equations for each, derived from the same way from $\omega = \frac{d\theta}{dt}$ and $\alpha = \frac{d\omega}{dt}$

$$x \to \theta \quad v \to \omega \quad a \to \alpha$$

$v = v_o + at$	$\omega = \omega_o + \alpha t$
$x = x_0 + v_o t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_o t + \frac{1}{2}\alpha t^2$
$x = x_0 + \frac{1}{2}(v_o + v)t$	$\theta = \theta_0 + \frac{1}{2}(\omega_o + \omega)t$
$a(x - x_o) = \frac{1}{2}(v^2 - v_o^2)$	$\alpha(\theta - \theta_o) = \frac{1}{2}(\omega^2 - \omega_o^2)$

ROTATIONAL KINEMATICS

Rotational kinematics involves questions analagous to those asked for linear (*translational*) kinematics, like:

- Given α and ω_o, how much does θ change in a given time?

TRANSLATION: Given a & v_o, how far, x, does a particle move in time t ?

- Given α and some change in angle θ, how much has ω changed?
- Given α and an initial ω_o, how much time is required for a certain change in angle θ?

If you can answer the analogous questions for

x, v, a & t

you should be able to solve them with

θ, ω, α & t.

EXAMPLE

A ceiling fan is accelerating from rest (*at a constant rate*).

The angle of some point on the fan relative to a reference is described by

$$\boldsymbol{\theta(t) = A + Bt + Ct^2} \quad (A,\ B\ \&\ C \text{ are constants.})$$

Find the angle, the angular velocity and the angular acceleration at $t = 2$ s.

ANSWER

Think of this just like you would $x(t) = x_o + v_o t + \frac{1}{2} a t^2$

θ at $t = 2$ seconds is just $\theta(2) = A + B(2) + C(2)^2 = A + 2B + 4C$

You just plug t = 2 into $\theta(t)$.

$\omega = d\theta/dt = B + 2Ct = B + 2C2 = B + 4C$

You take the time derivative & plug in t = 2.

$\alpha = d\omega/dt = 2C$ independent of time (*as required for constant acceleration*)

You just take another derivative; plug in t = 2, if $\alpha \neq$ *constant .*

Note that you could find $A = \theta_o$ *and* $B = \omega_o$ *by plugging in t = 0 above.*

$$\boldsymbol{\theta(t) = \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2}$$

EXAMPLE

A compact disk changes speeds up as it plays so that the tangential velocity (& *rate of data transfer as the laser moves across the disc*) is roughly constant: v_{tan}= 1.3 m/s. See http://en.wikipedia.org/wiki/Compact_Disc.

The frequency of rotation in *rpm* (*revolutions per minute*) is:

Inside edge of disk at r_i = 25 mm: f_i = 500 rpm

Outside edge of disk at r_o = 58 mm: f_o = 200 rpm

A. If the CD player laser has to move from an outside track to an inside track in 3 seconds, what angular acceleration of the disk is required? (*Assume uniform angular acceleration or the four handy-dandy formulae don't apply!*)

B. How many complete revolutions will the CD make during this process?

EXAMPLE

v_{tan}= 1.3 m/s.

Inside edge of disk at r_i = 25 mm: f_i = 500 rpm

Outside edge of disk at r_o = 58 mm: f_o = 200 rpm

A. If the CD player laser has to move from an outside track to an inside track in 3 seconds, what angular acceleration of the disk is required?

$\alpha = \Delta\omega/\Delta t = 2\pi\ \Delta f/\Delta t =$

(2π radians/rev)(500 rpm – 200 rpm)(1 minute/60 seconds) / (3 sec)

= **10.5 s^{-2}**

B. How many complete revolutions will the CD make?

Using $\theta = \theta_0 + \tfrac{1}{2}(\omega_o + \omega)t$

$$\Delta\theta = \tfrac{1}{2}(\omega_o + \omega)t = \tfrac{1}{2}(2\pi f_i + 2\pi f_o)t = \pi(200\text{ rpm} + 500\text{ rpm})\left(\frac{1\text{ min}}{60\text{ sec}}\right)3\text{ sec} = 110$$

110 revolutions

- 5 options

If the earth's angular velocity is ω and its radius is *R*, what is the magnitude of the <u>*linear (tangential) velocity v*</u> of some object on the surface of the earth at latitude $\theta = \lambda$?

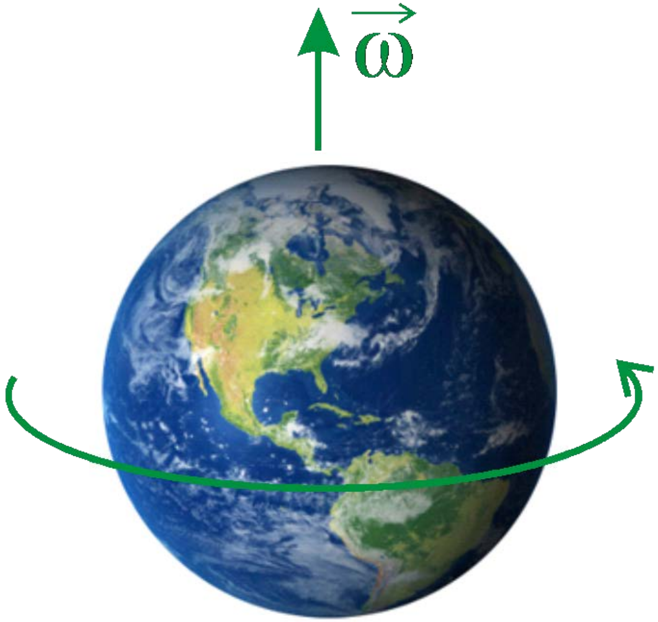
A. $\omega R \sin\lambda$

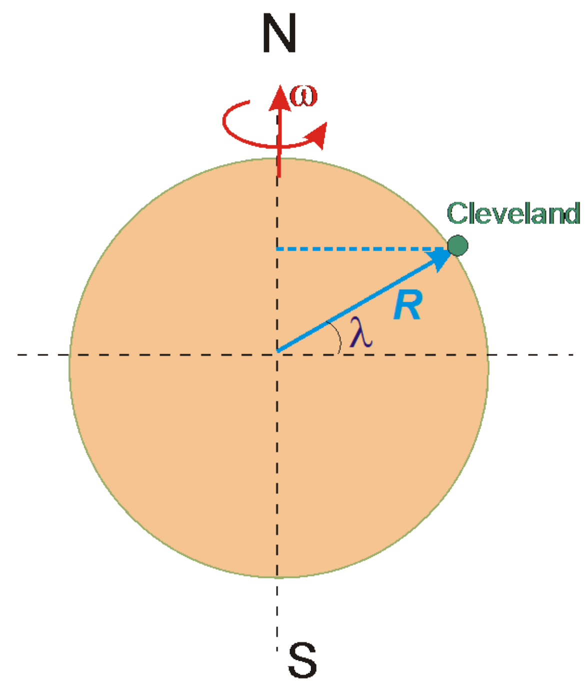
B. ωR

C. $\omega R \cos\lambda$

D. $\omega R \tan\lambda$

E. 0





- 5 options

If the earth's angular velocity is ω and its radius is R, what is the magnitude of the *linear* (*tangential*) *velocity v* of some object on the surface of the earth at latitude θ = λ?

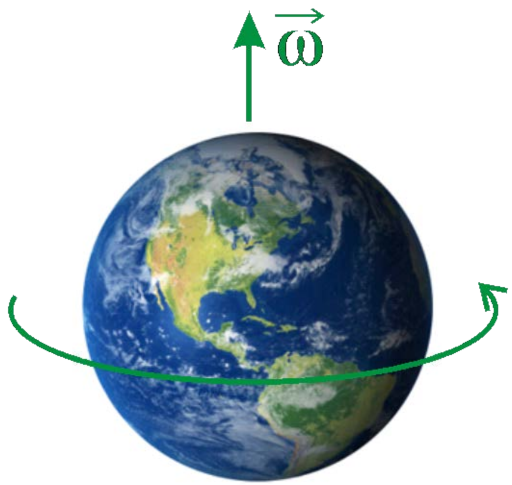
A. $\omega R\sin\lambda$

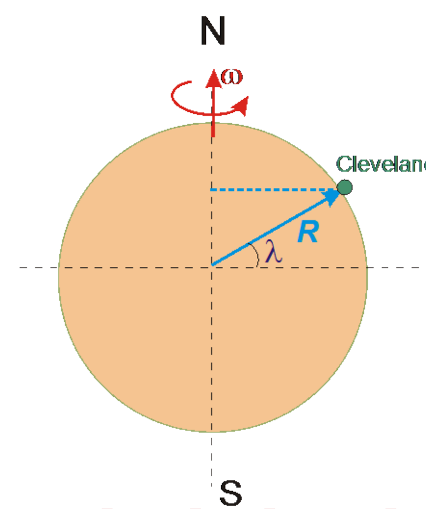
B. ωR

C. $\omega R\cos\lambda$

D. $\omega R\tan\lambda$

E. 0





$v = \omega r$ but in this case the radius of the circle which the object is following is $r = R\cos\lambda$.

 - 6 options

The *tangential acceleration* on the Earth's surface at the latitude λ of Cleveland is:

A. $\alpha R \sin\lambda$

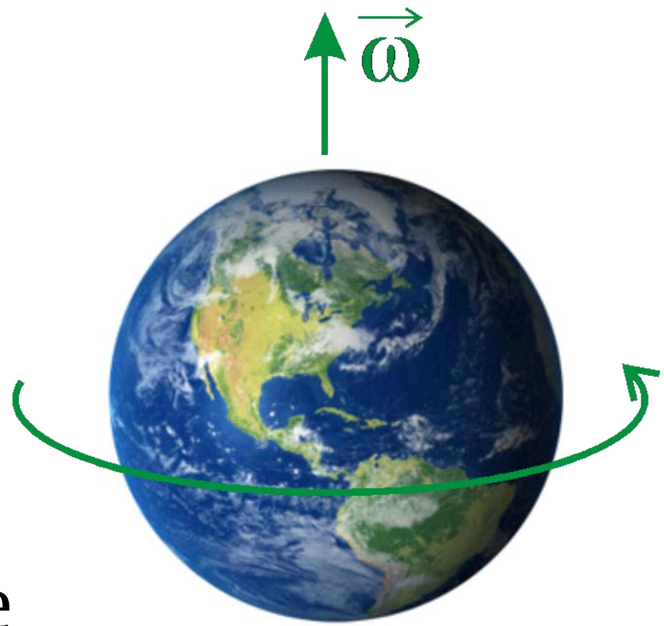
B. αR

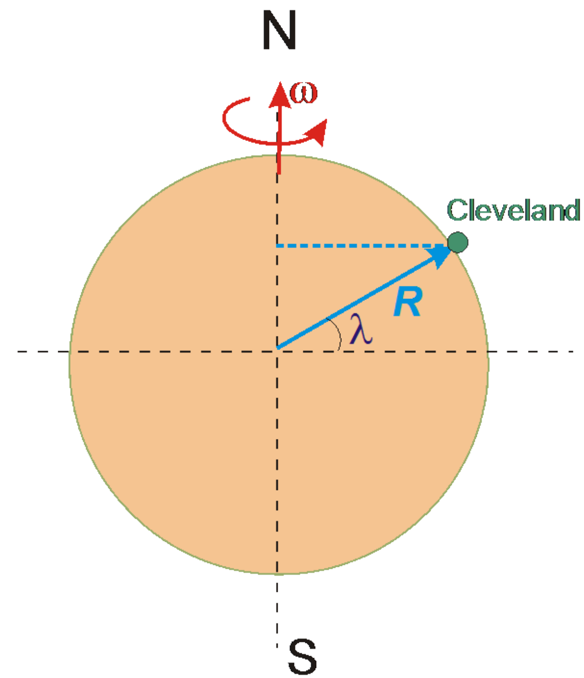
C. $\alpha R \cos\lambda$

D. $\alpha R \tan\lambda$

E. 0

F. something else





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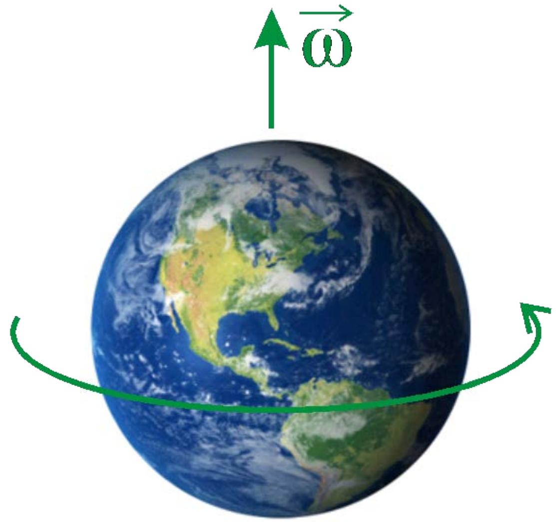
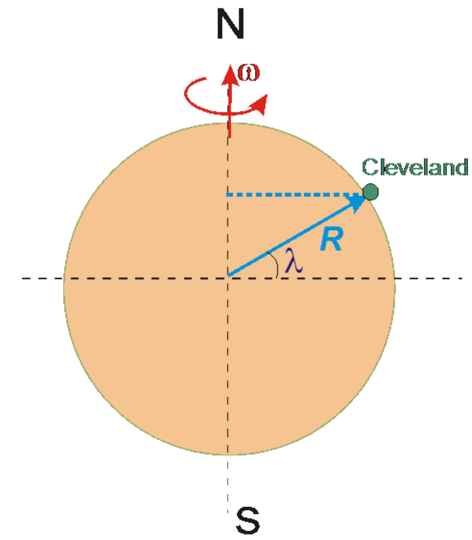




There's no tangential acceleration if an object is moving at constant angular velocity ω.

The earth is NOT speeding up or slowing down (*much*)!

YOU HOPE!

$a_{tan} = \alpha R = (d\omega/dt)R = 0.$

- 5 options

The magnitude of the *radial acceleration* experienced by an object on the Earth's surface at latitude λ is:

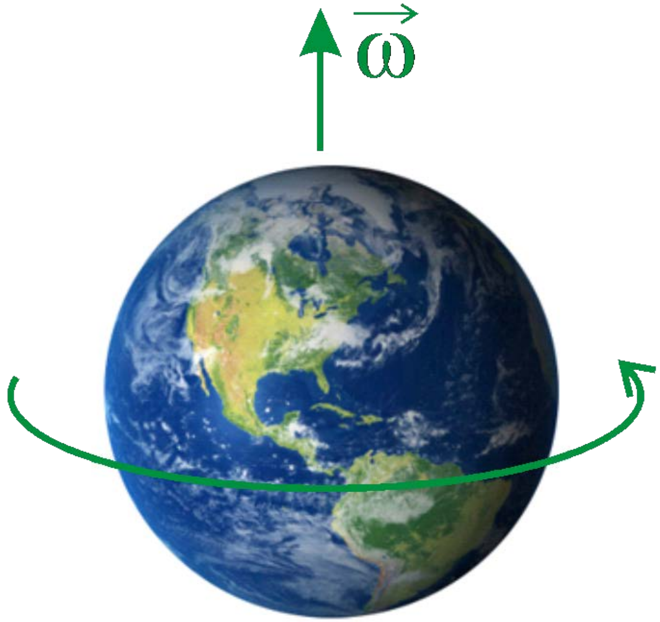
A. $\omega^2 R\sin\lambda$

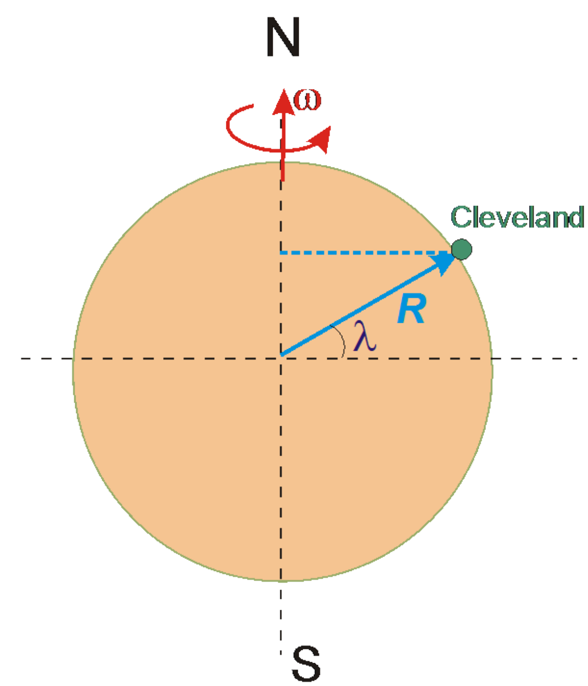
B. $\omega^2 R$

C. $\omega^2 R\cos\lambda$

D. $\omega^2 R\tan\lambda$

E. 0





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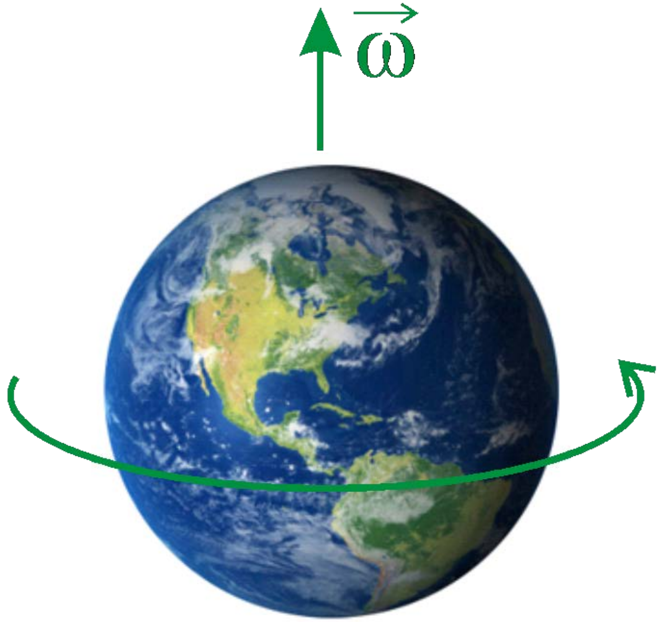
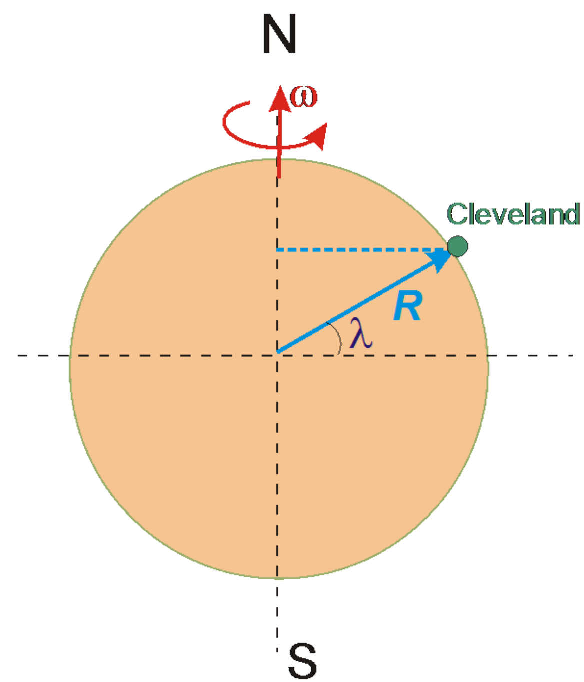




$a_{radial} = \omega^2 r = v^2/r = (\omega r)^2/r$ with $r = R\cos\lambda$.

- 4 options

A ladybug sits at the outer edge of a spinning merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation.

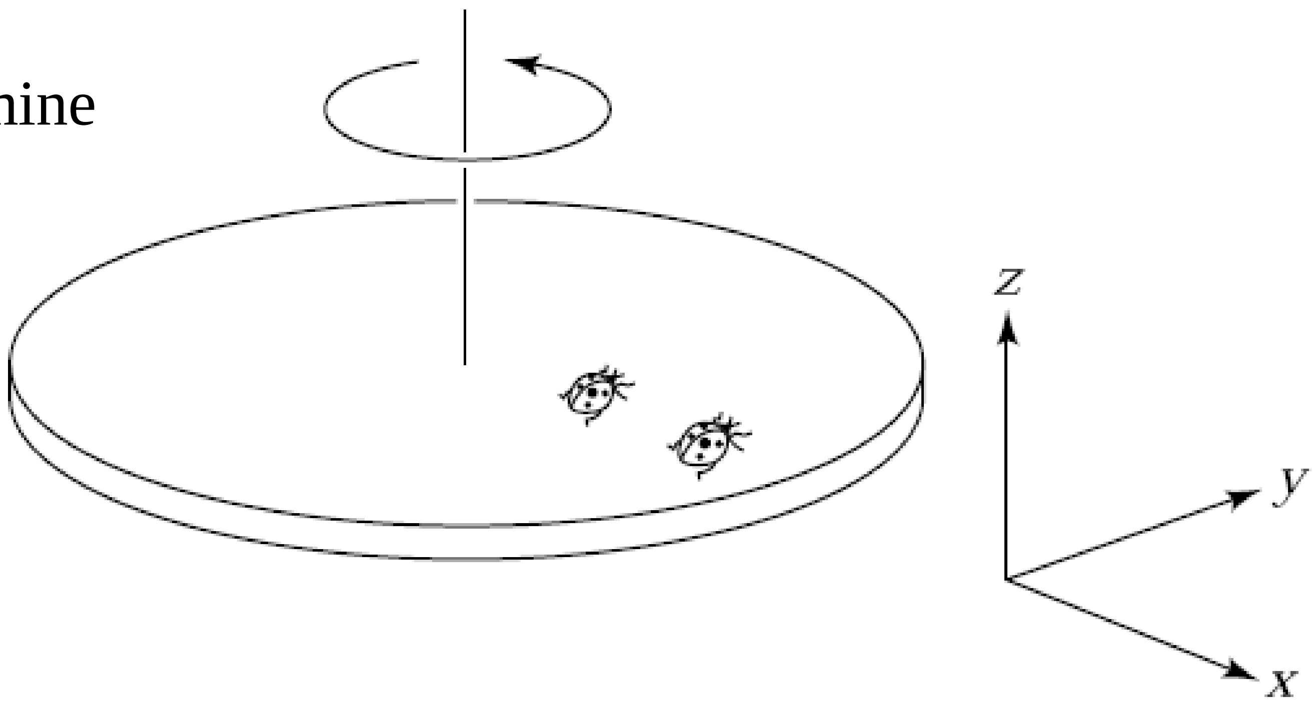
The gentleman bug's angular speed is:

A. half the ladybug's.

B. the same as the ladybug's.

C. twice the ladybug's.

D. impossible to determine



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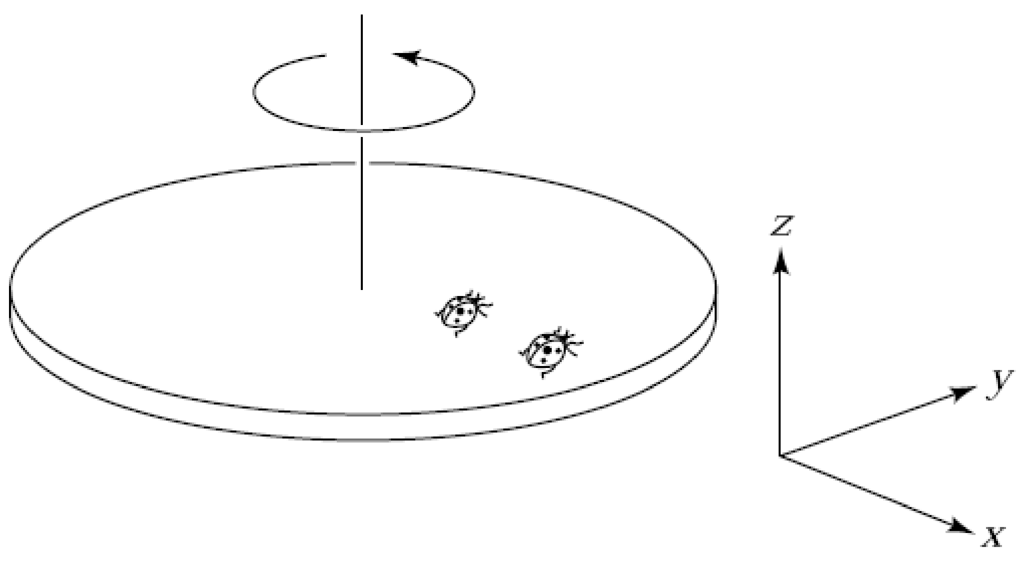
D. impossible to determine

This is a requirement of a rigid body like the turntable.



- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning but slowing down.

At the instant shown in the figure, the *radial acceleration* of the ladybug is

A. in the $+x$ direction.

B. in the $-x$ direction.

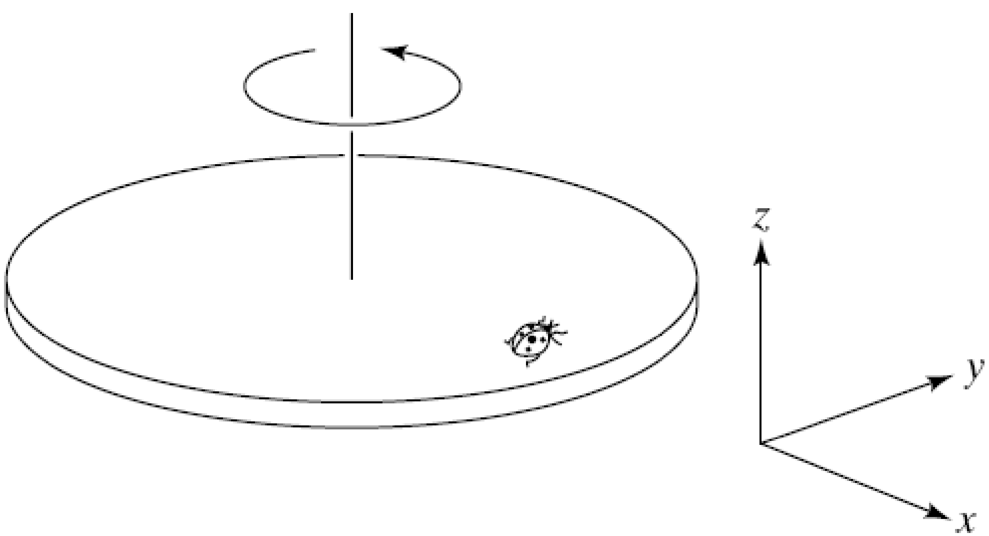
C. in the $+y$ direction.

D. in the $-y$ direction.

E. in the $+z$ direction.

F. in the $-z$ direction.

G. 0



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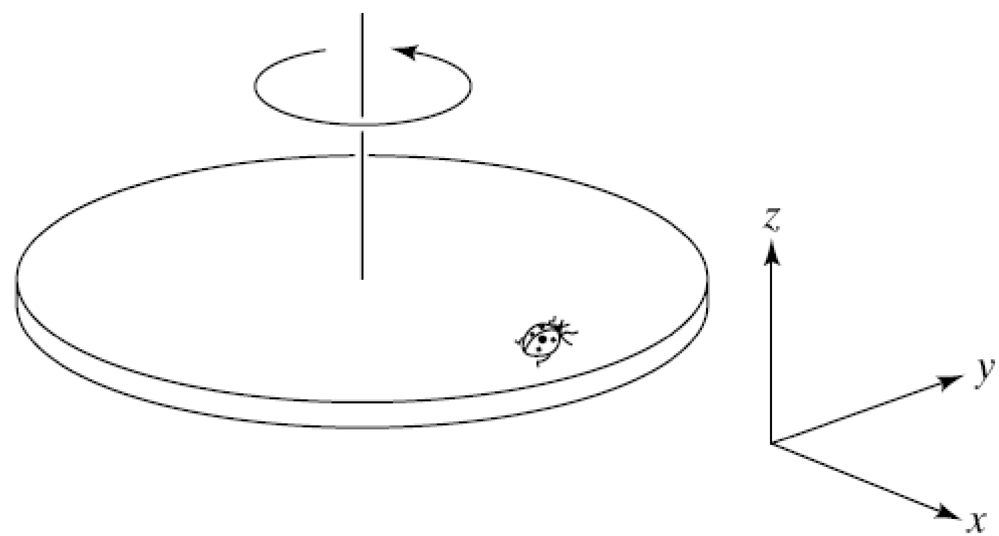
D. in the $-y$ direction.

E. in the $+z$ direction.

F. in the $-z$ direction.

G. 0



The radial component of the acceleration of a rotating object points towards the axis of rotation.

- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning but slowing down.

The *tangential acceleration* of the ladybug is

A. in the +*x* direction.

B. in the –*x* direction.

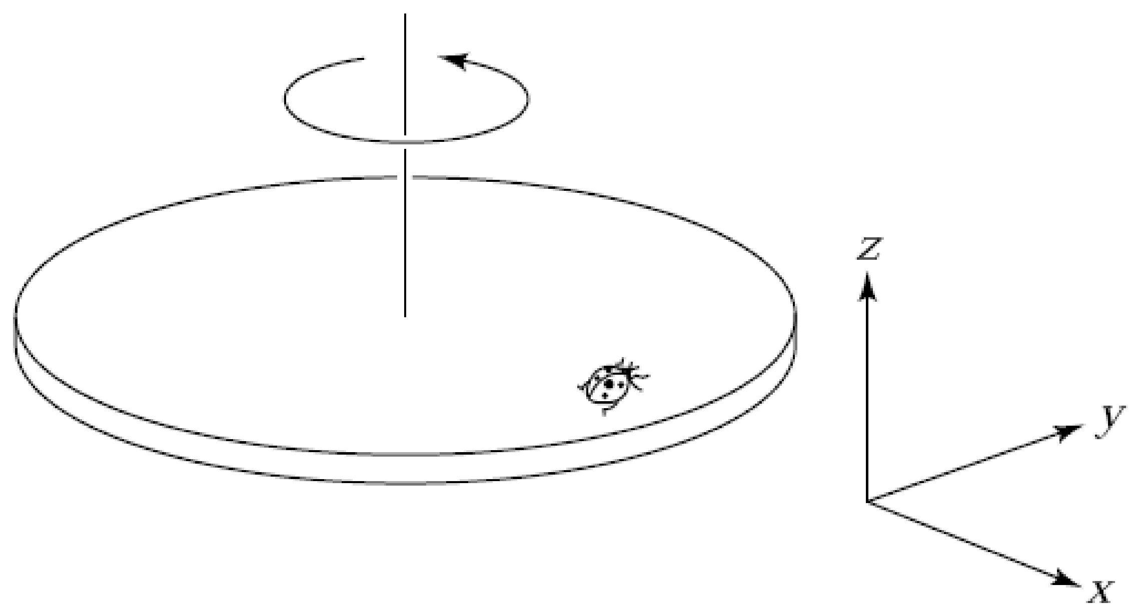
C. in the +*y* direction.

D. in the –*y* direction.

E. in the +*z* direction.

F. in the –*z* direction.

G. 0.



- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning but slowing down.

The ***tangential acceleration*** of the ladybug is

A. in the $+x$ direction.

B. in the $–x$ direction.

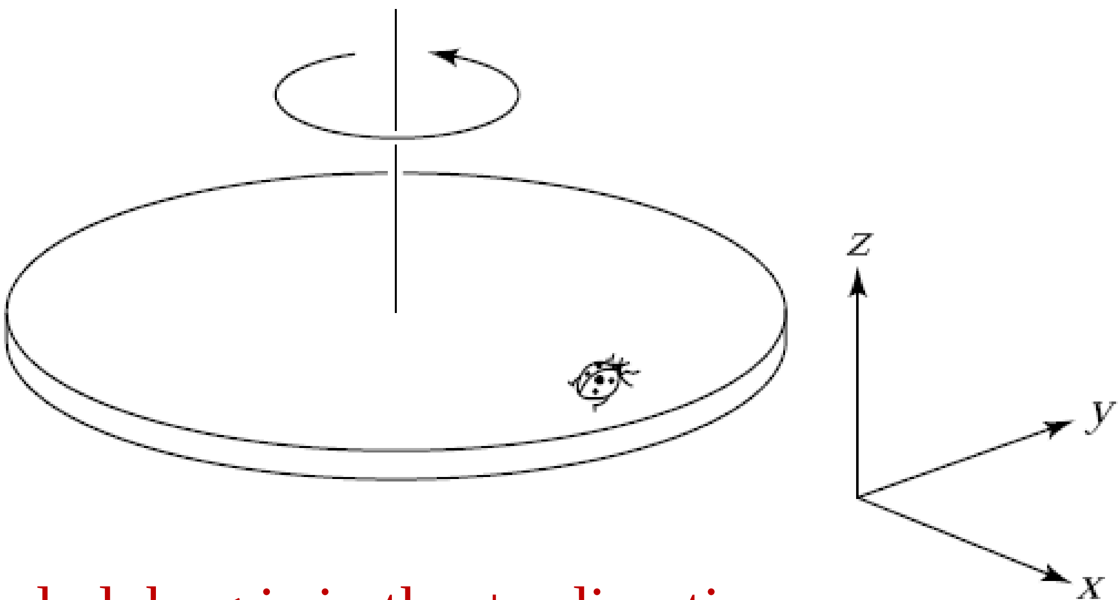
C. in the $+y$ direction.

D. in the $–y$ direction.

E. in the $+z$ direction.

F. in the $–z$ direction.

G. 0.



The tangential velocity of the ladybug is in the $+y$ direction.

However, because the merry-go-round is slowing down, the tangential acceleration of the ladybug is in the $–y$ direction.

- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning <u>but slowing down</u>.

The direction of her angular velocity vector is:

A. in the +*x* direction.

B. in the –*x* direction.

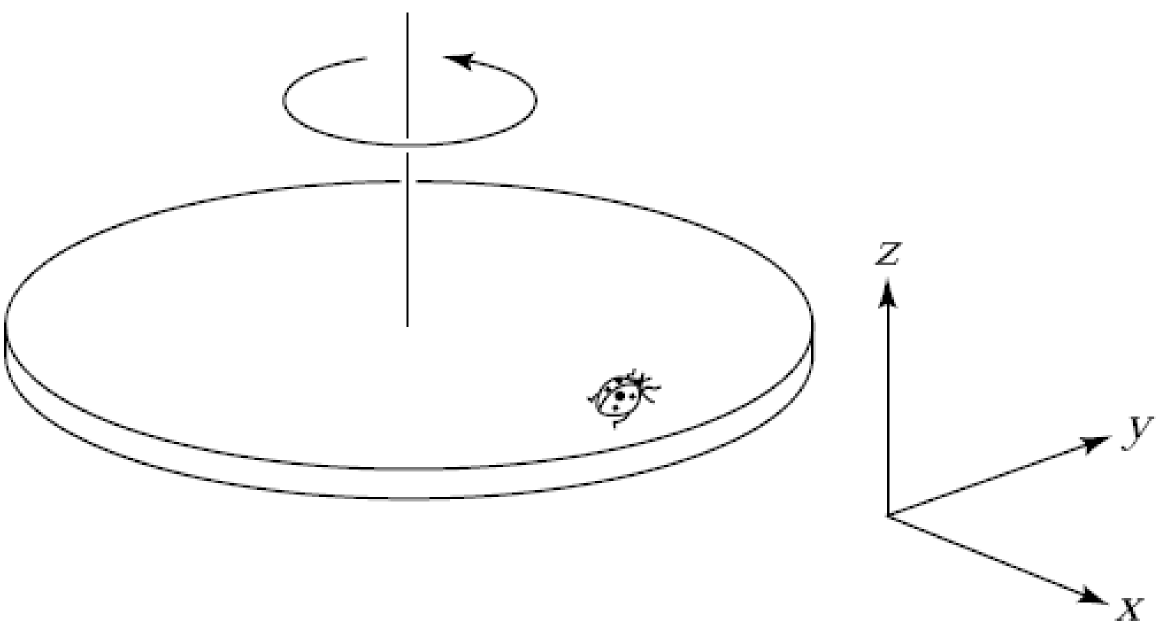
C. in the +*y* direction.

D. in the –*y* direction.

E. in the +*z* direction.

F. in the –*z* direction.

G. zero.



- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning but slowing down. The direction of her angular velocity vector is:

A. in the +*x* direction.

B. in the –*x* direction.

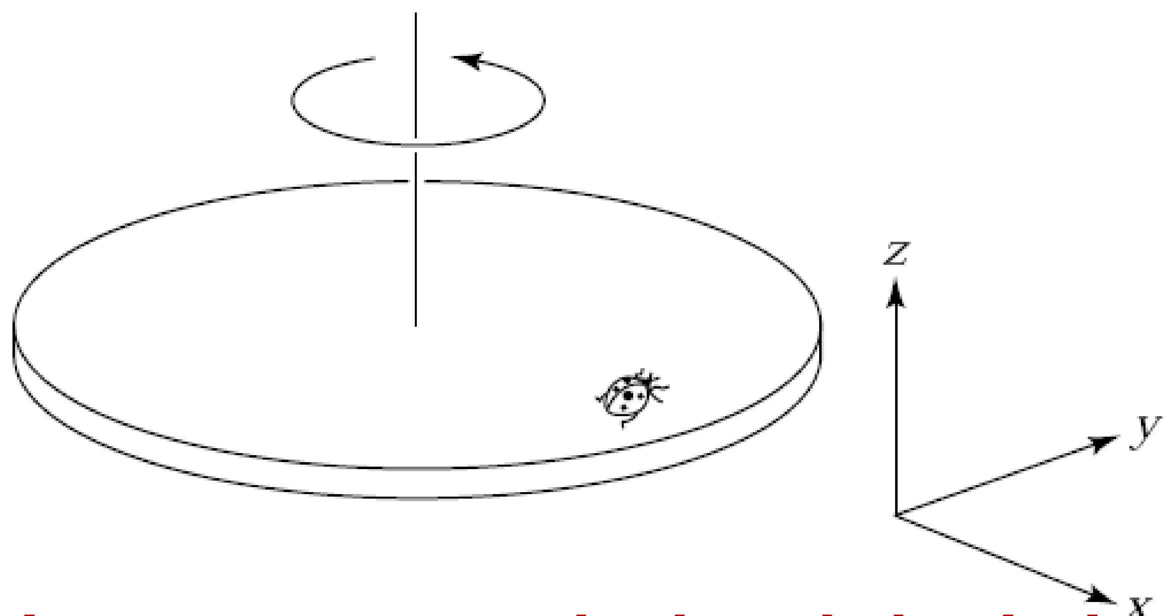
C. in the +*y* direction.

D. in the –*y* direction.

E. in the +*z* direction.

F. in the –*z* direction.

G. zero.



The direction of the angular velocity vector is given by the right-hand rule.

- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning but slowing down.

The direction of her angular acceleration vector is:

A. in the $+x$ direction.

B. in the $-x$ direction.

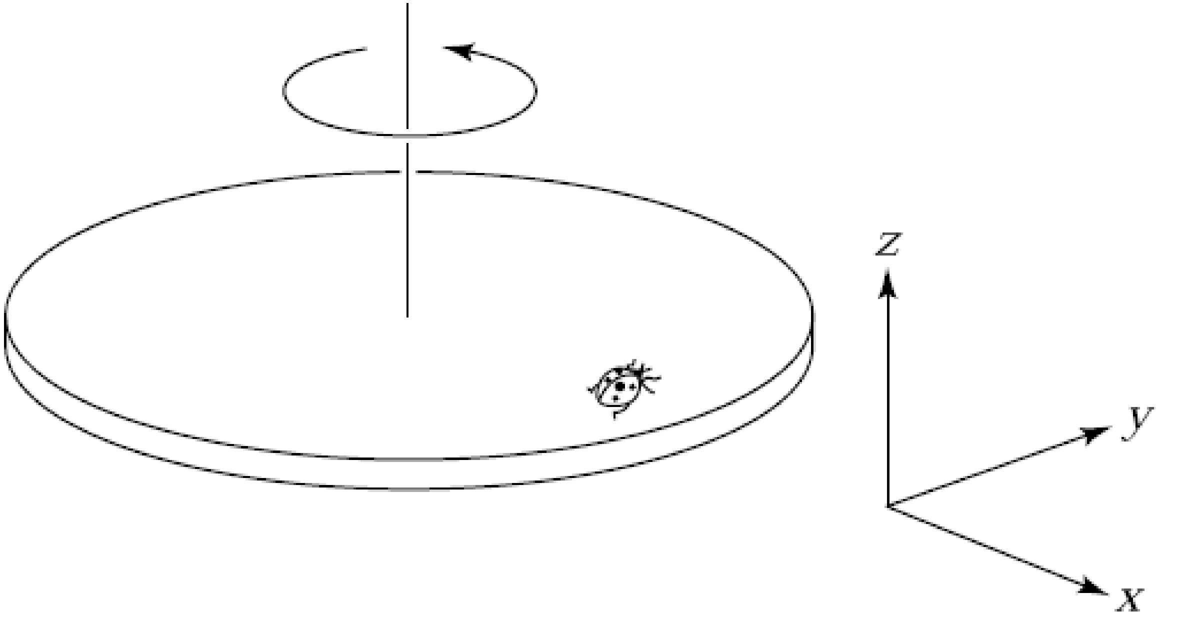
C. in the $+y$ direction.

D. in the $-y$ direction.

E. in the $+z$ direction.

F. in the $-z$ direction.

G. zero.



- 7 options

A ladybug sits at the outer edge of a merry-go-round that is spinning <u>but slowing down</u>.

The direction of her angular acceleration vector is:

A. in the +*x* direction.

B. in the –*x* direction.

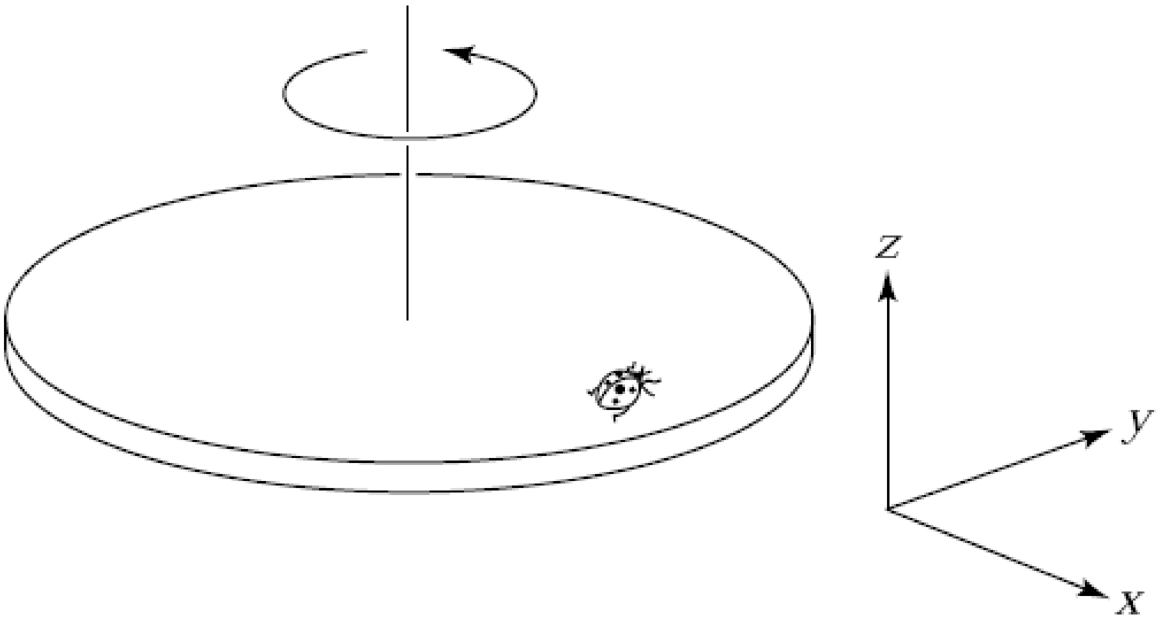
C. in the +*y* direction.

D. in the –*y* direction.

E. in the +*z* direction.

F. in the –*z* direction.

G. zero.



The direction of the angular velocity vector given by the right-hand rule points in the +*z* direction but this vector is getting smaller, so Δv points in the –*z* direction.

KINETIC ENERGY IN CIRCULAR MOTION

For translational/linear motion

$$K = \tfrac{1}{2}mv^2$$

Rotating objects also have kinetic energy

(*energy due to motion*)

$$v \rightarrow \omega \qquad m \rightarrow ?$$

$m \rightarrow I$ = Moment of Inertia

$$\boldsymbol{K = \tfrac{1}{2} I\omega^2}$$

PROOF: $K = \frac{1}{2} I\omega^2$

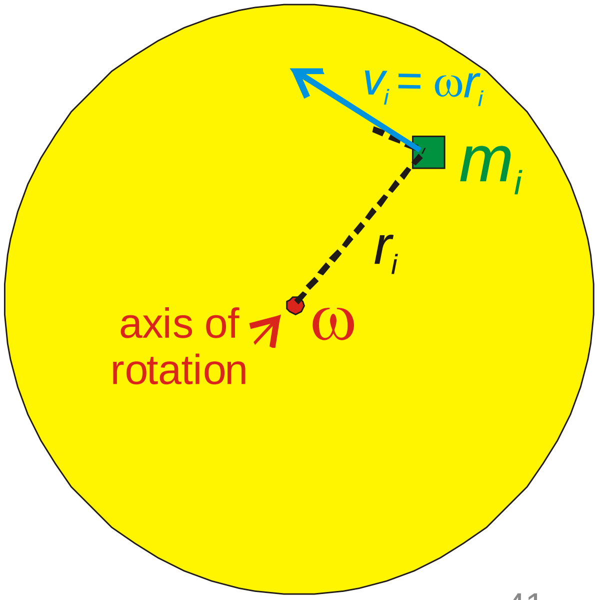
$K = \Sigma K_i$ for all the small elements of mass m_i moving at $v_{tan} = v_i = \omega r_i$ around the axis of rotation.

$$K = \Sigma \tfrac{1}{2} m_i v^2_i = \Sigma \tfrac{1}{2} m_i (\omega r_i)^2 = \tfrac{1}{2}\Sigma(m_i r_i^2)\omega^2$$

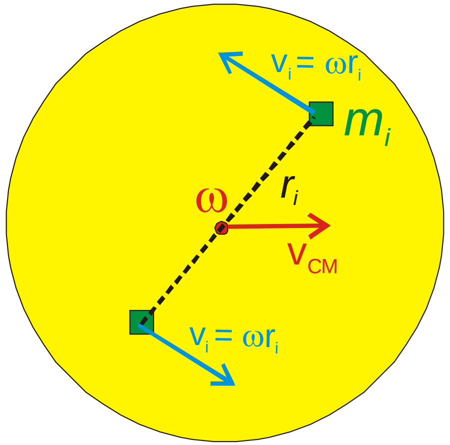
$$\boldsymbol{K = \tfrac{1}{2} I\omega^2}$$

with

$$\boldsymbol{I \equiv \Sigma m_i r_i^2}$$



$K_{total} = K_{rotation} + K_{translation}$
assuming the axis goes through the CM



Total kinetic energy = sum of the translational K of the CM plus K due to rotation about the CM.

$\mathbf{v}_{i\text{-}tan} \equiv$ tangential v of a small piece m_i rotating about the CM at ω

$\mathbf{v}_{CM}$ = velocity of the CM as seen from the lab

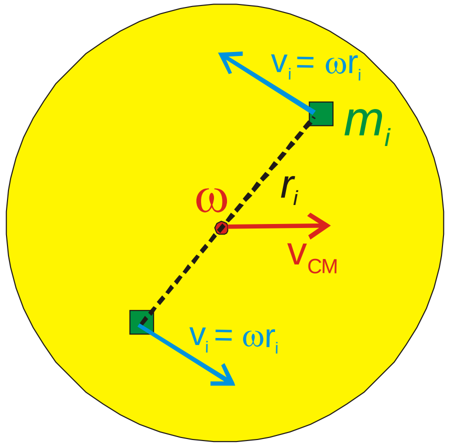
VELOCITY ADDITION: $\mathbf{v}_{\mathbf{total}}$ for piece $i \equiv \mathbf{v}_{\mathbf{i}} = \mathbf{v}_{\text{i-lab}}$

$$\mathbf{v}_{\text{i-lab}} = \mathbf{v}_{\text{i-CM}} + \mathbf{v}_{\text{CM-lab}} = \mathbf{v}_{\text{i-tan}} + \mathbf{v}_{\text{CM}}$$

$$K = \Sigma\tfrac{1}{2}m_i v^2{}_i = \Sigma\tfrac{1}{2}m_i(\mathbf{v}_{\text{i-tan}} + \mathbf{v}_{\text{CM}})^2$$

$$K = \Sigma\tfrac{1}{2}m_i(\mathbf{v}^2{}_{\text{i-tan}} + 2\,\mathbf{v}_{\text{i-tan}} \cdot \mathbf{v}_{\text{CM}} + \mathbf{v}^2{}_{\text{CM}})$$

$$K_{total} = K_{rotation} + K_{translation}$$



$$K = \Sigma \tfrac{1}{2} m_i \left(\mathbf{v}^2_{\text{i-tan}} + \cancel{2\,\mathbf{v}_{\text{i-tan}} \cdot \mathbf{v}_{\text{CM}}} + \mathbf{v}^2_{\text{CM}}\right)$$

but the middle term sums to zero because the $\mathbf{v}_{\text{i-tan}} \cdot \mathbf{v}_{\text{CM}}$ terms have opposite signs on opposite sides of the axis or rotation,

so

$$K = \Sigma \tfrac{1}{2} m_i \left(\mathbf{v}^2_{\text{i-tan}} + \mathbf{v}^2_{\text{CM}}\right) =$$

$$\Sigma \tfrac{1}{2} m_i \left[(\omega r_i)^2 + \mathbf{v}_{\text{CM}}^2\right]$$

$$= \tfrac{1}{2} I\omega^2 + \tfrac{1}{2} M v^2_{CM}$$

$$K_{total} = K_{rotation} + K_{translation}$$

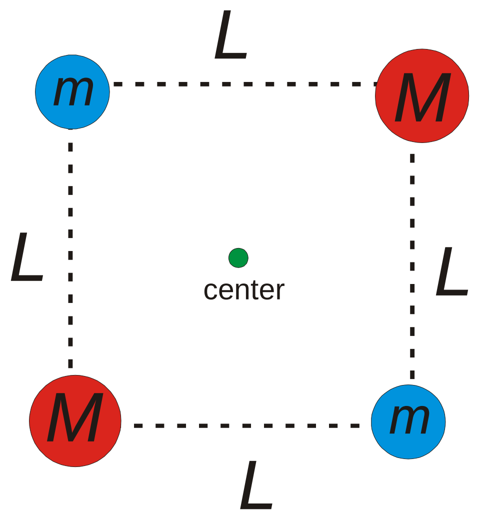
MOMENT OF INERTIA for a SYSTEM of PARTICLES

$$I \equiv \Sigma m_i R_i^2$$

A pair of point masses ***M*** are placed at diagonally opposite corners on a massless square of side *L*.

Another pair of point masses ***m*** are placed at the other 2 corners.

What is the moment of inertia for a rotational axis through the center of the square?



<u>TRICK QUESTION!</u>

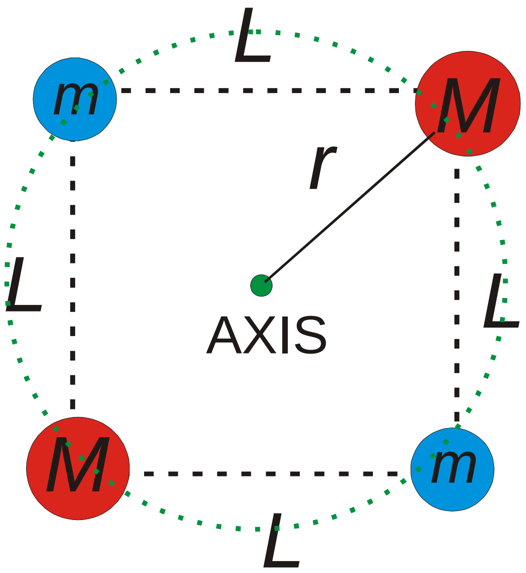
The correct answer is "*I don't know!*"

You can't calculate I without determining the precise axis of rotation.

The distance to the center of the square is NOT the issue, it's the distance to the **<u>axis</u>** of rotation.

If the axis is <u>perpendicular</u> to this page,

Then the distance from each mass to the axis is



$$r = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2} = \frac{L}{\sqrt{2}}$$

$$I = \sum m_i r_i^2 = M\frac{L^2}{2} + M\frac{L^2}{2} + m\frac{L^2}{2} = m\frac{L^2}{2} = (M+m)L^2$$

If the axis of rotation cuts through the square along a diagonal,

r is the radius of the circular path each mass follows as the system rotates about the axis.

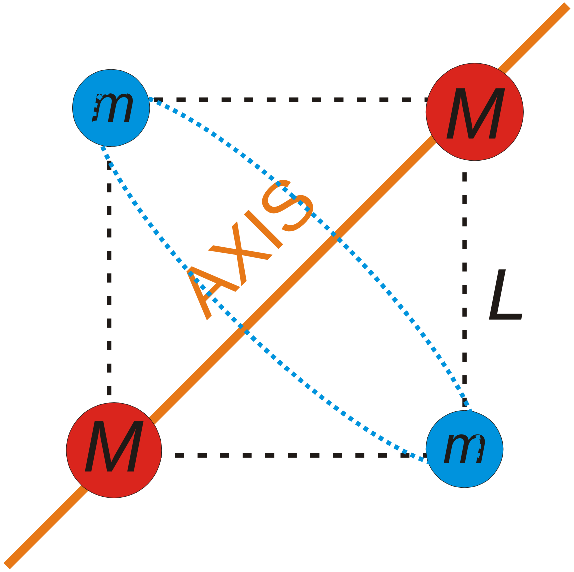
$\Rightarrow r = 0$ for the larger masses M.

$$r = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2} = \frac{L}{\sqrt{2}} \text{ for the other masses}$$

$$I = \sum m_i r_i^2 = m\frac{L^2}{2} + m\frac{L^2}{2} = mL^2$$

OR, if you use the other diagonal

$$I = \sum m_i r_i^2 = M\frac{L^2}{2} + M\frac{L^2}{2} = ML^2$$

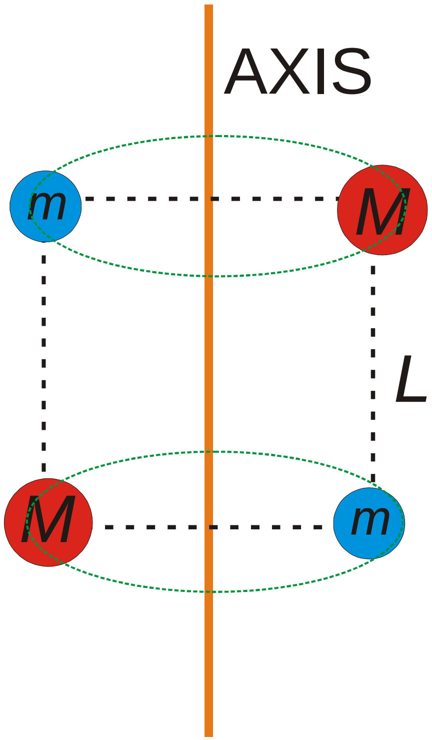


If the axis of rotation bisects the square vertically or horizontally

$r = ½L$ instead of $\frac{L}{\sqrt{2}}$

$$I = \sum m_i r_i^2 = M\left(\frac{L}{2}\right)^2 + M\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2$$

$$I = \frac{1}{2}(M + m)L^2$$



MORAL

Rotational dynamics is MUCH more complicated than linear/translational dynamics!

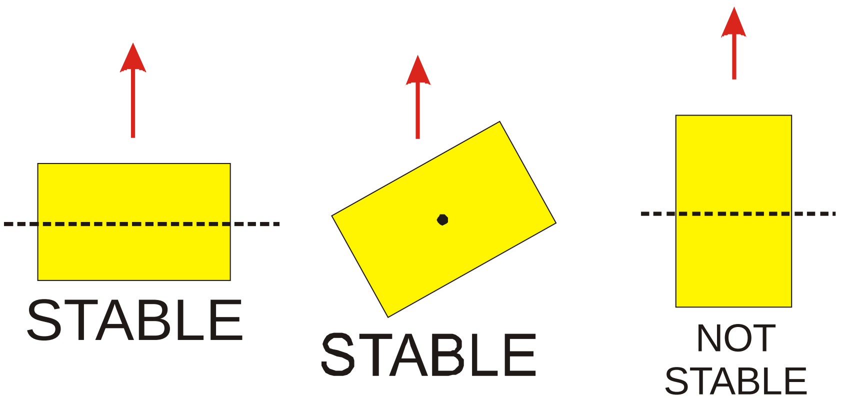
&

you're only seeing the easy things!

Brain-teaser & advertisement for PHYS 310

Try to throw a book into the air and make it spin stably about each of the three axes through the middle of the book, perpendicular to each surface.

Two options will work but one will not!



CULTURE

The moment of inertia is actual a 2nd rank tensor,

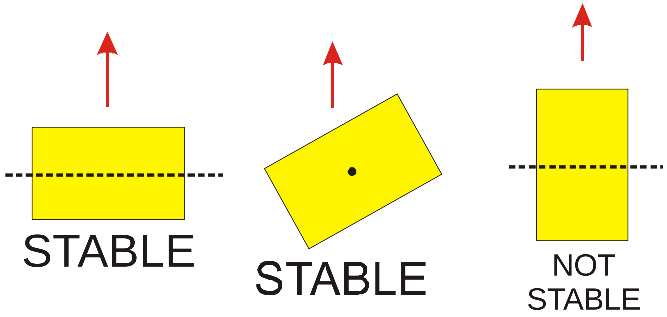
~ 3 x 3 matrix.

Angular Momentum **L** & **ω** are vectors

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

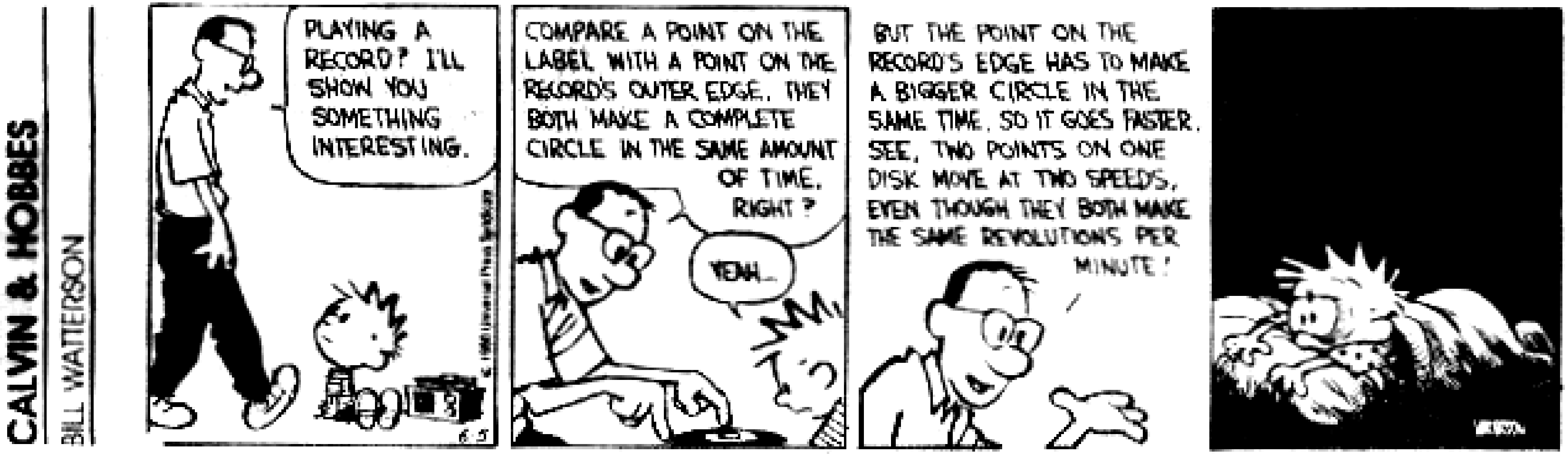
An angular velocity around the *x* axis can be coupled to motion around the other axes.

But symmetry can make I simpler.



We made it to slide #49 on Monday, March 30.

PHYS 121 – SPRING 2015



Chapter 12: Rotation of a Rigid Body

version 04/01/2015, ~112 slides

We made it to slide #51 on Monday, March 30.

Get your clickers ready.

ANNOUNCEMENTS

- Homework Group #1, A – Ch, that was missing Monday is available now.
- Blackboard grades have been updated, including revised grades as of Monday, March 30.
  - Blackboard will not be updated again until after the 3rd exam on Friday, April 17.
  - Formal course grades are kept separately, in an Excel file.

REMINDER

KINETIC ENERGY IN CIRCULAR MOTION

Rotating objects have kinetic energy:

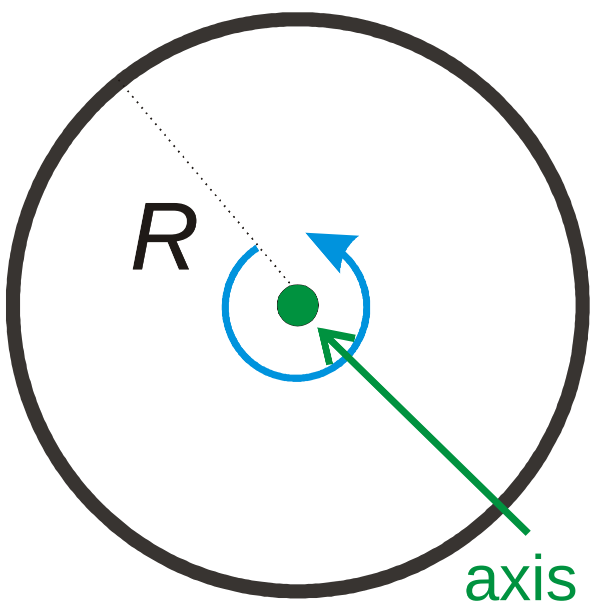
$$K = \tfrac{1}{2} I\omega^2$$

For a collection of point masses:

$$I \equiv \Sigma m_i r_i^2$$

BEYOND THE POINT MASS

Calculate I for a circular hoop, perhaps a bicycle rim of mass M and radius R, which can rotate about an axis through its center, perpendicular to the plane of the rim? (*Ignore spokes et al.*)

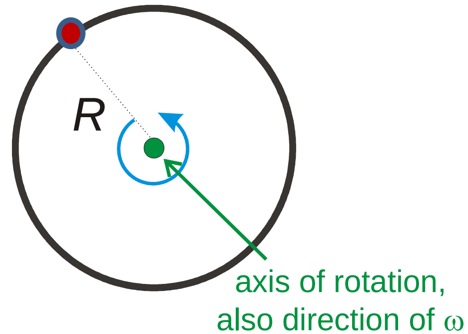


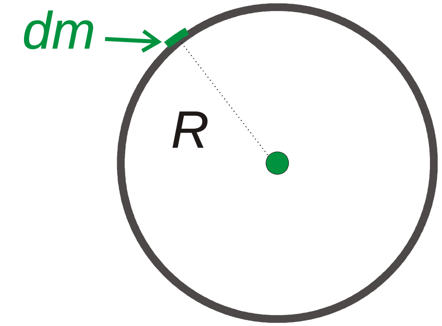
axis of rotation,
also direction of ω

http://media.performancebike.com/images/performance/products/1500/50-0106-angle.jpg

$I \equiv \Sigma m_i r_i^2$

I_{HOOP}





- $I = mr^2$ for a point mass m at R from the axis.
- For N point masses with $M = Nm$ all at R,

  $I = (Nm)R^2 = MR^2$ since *moments of inertia add.*
- Break the hoop into many tiny pieces, N is very large, so that each piece is essentially a point mass.
- Each piece has a mass $\Delta m = dm = m_i = M/N$ and each piece is a distance R from the axis, so

$$I = \sum_i m_i r_i^2 \to \sum_{i=1}^{i=N} \left(\frac{M}{N}\right) R^2 = \left(\frac{M}{N}\right) R^2 \sum_{i=1}^{i=N} 1 = \left(\frac{M}{N}\right) R^2 N = MR^2$$

$$I=\sum_i m_i r_i^2 \rightarrow \sum_{i=1}^{i=N}\left(\frac{M}{N}\right)R^2=\left(\frac{M}{N}\right)R^2\sum_{i=1}^{i=N}1=\left(\frac{M}{N}\right)R^2 N=MR^2$$

We can take this even further,

going to the limit of $N \rightarrow \infty$

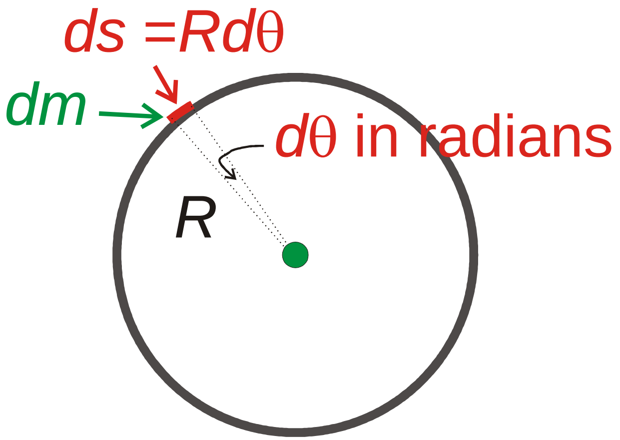
and setting up an integral in place of a sum.

In this case the m_i terms $\rightarrow dm$

$$I=\sum_i m_i r_i^2 \rightarrow \int_{\substack{over\\ alldm}} R^2 dm = R^2 \int_{\substack{over\\ alldm}} dm = MR^2$$

A hoop of radius R has the same moment of inertia as an equivalent point mass a distance R from the axis of rotation.

CULTURAL INTERLUDE



If you'd like to write this integral more elegantly, you'd use cylindrical/polar coordinates.

(*Have you seen this yet in math; it's supposed to be covered in MATH 122?*)

$$dm = (\text{mass per length})\,ds = \left(\frac{M}{2\pi R}\right)ds = \left(\frac{M}{2\pi R}\right)R\,d\theta = \frac{M}{2\pi}d\theta$$

$$I = \sum_i m_i r_i^2 \rightarrow \int_{\theta=0}^{2\pi} R^2 dm = R^2 \int_{\theta=0}^{2\pi} \frac{M}{2\pi} d\theta = \frac{MR^2}{2\pi} \int_{\theta=0}^{2\pi} d\theta = MR^2$$

[END OF CULTURAL INTERLUDE]

$2D \rightarrow 3D$: $I_{hoop} = MR^2 \rightarrow I_{cylinder}$

We've solved for I for a *hoop*

but the same result applies for a hollow *cylinder*

since, again,

all the mass is concentrated a distance R

from the axis of rotation.

$$I_{hollow\ cylinder} = MR^2$$

No matter what the shape is for some system, if ALL of its mass is the same distance R from an axis, the moment of inertia about that axis is

$$I = MR^2.$$

(*for the purposes of PHYS 121*)

$I \equiv \Sigma m_i r_i^2$

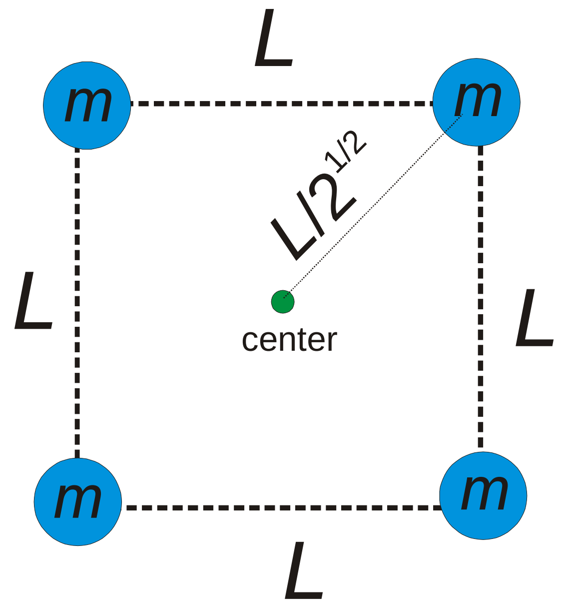
7 options in each of the following 4 systems

Four identical particles of mass M are arranged at the corners of a square of side L.

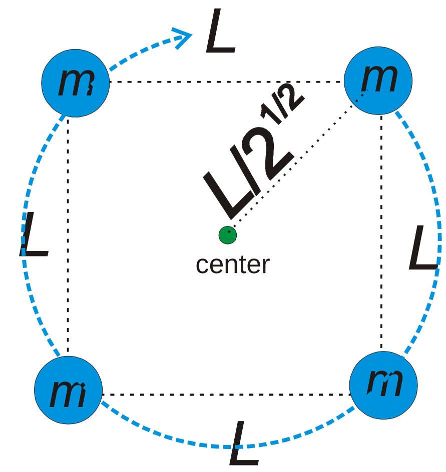
Calculate the moment of inertia of for this system about the axis identified in each of the following slides, choosing from the following possibilities.

A. $4ML^2$ B. $4\text{x}2^{1/2}\,ML^2$ C. $2^{1/2}\,ML^2$ D. $2ML^2$

E. $(2 + 2\text{x}2^{1/2})\,ML^2$ F. $(2\text{x}2^{1/2})\,ML^2$ G. None of the above



- 7 options



$I \equiv \Sigma m_i r_i^2$

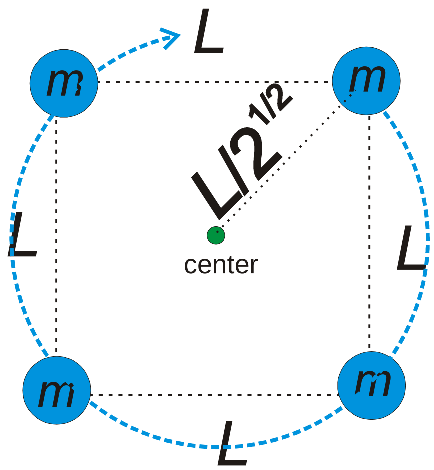
The axis of rotation is perpendicular to the plane of this page and goes through the middle of the square.

Your options are:

A. $4ML^2$ B. $4 \mathrm{x} 2^{1/2} ML^2$ C. $2^{1/2} ML^2$ D. $2ML^2$

E. $(2 + 2 \mathrm{x} 2^{1/2}) ML^2$ F. $(2 \mathrm{x} 2^{1/2}) ML^2$ G. None of the above

- 7 options



$I \equiv \Sigma m_i r_i^2$

The axis of rotation is perpendicular to the plane of this page and goes through the middle of the square.

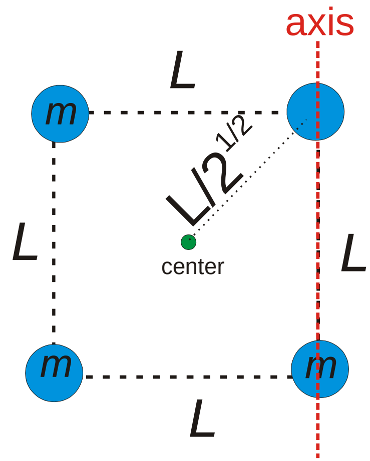
Your options are:

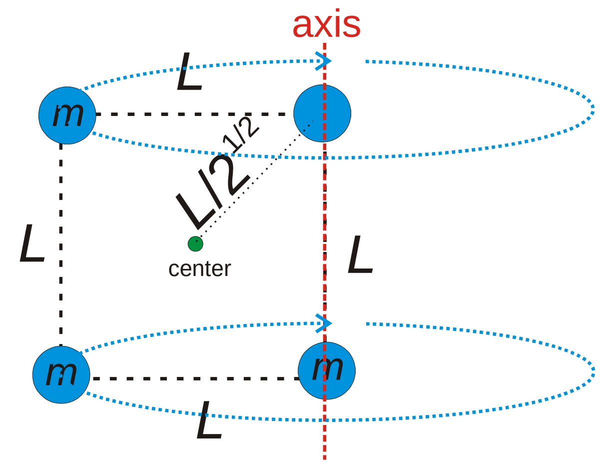
A. $4ML^2$ B. $4\text{x}2^{1/2}\,ML^2$ C. $2^{1/2}\,ML^2$ D. $2ML^2$

E. $(2 + 2\text{x}2^{1/2})\,ML^2$ F. $(2\text{x}2^{1/2})\,ML^2$ G. None of the above

$$I = \sum_i M_i R_1^2 = 4\left[M \left(\frac{L}{\sqrt{2}} \right)^2 \right] = 2ML^2$$

- 7 options





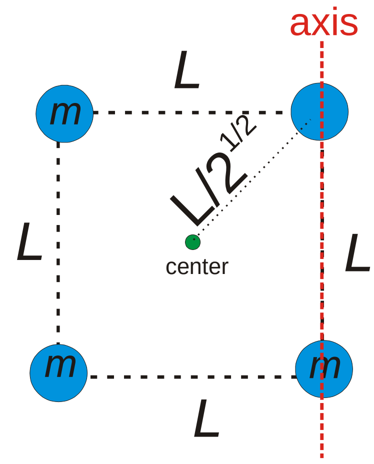
The axis of rotation lies in the plane of the page <u>along one side of the square</u>.

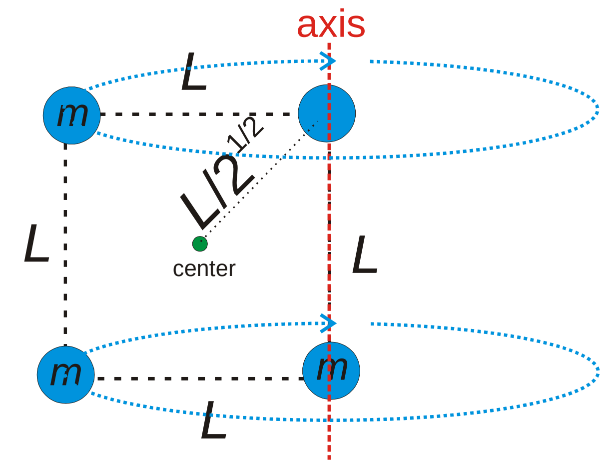
Your options are:

A. $4ML^2$ B. $4\text{x}2^{1/2}\, ML^2$ C. $2^{1/2}\, ML^2$ D. $2ML^2$

E. $(2 + 2\text{x}2^{1/2})\, ML^2$ F. $(2\text{x}2^{1/2})\, ML^2$ G. None of the above

- 7 options





The axis of rotation lies in the plane of the page along one side of the square.

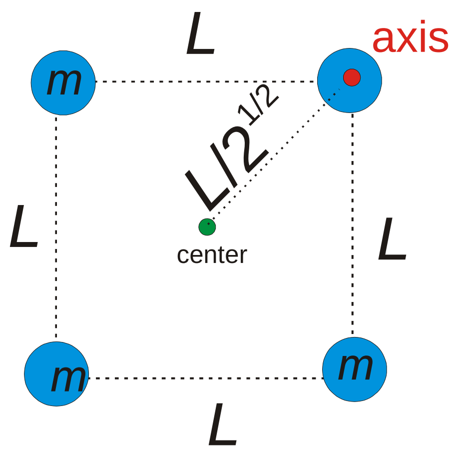
Your options are:

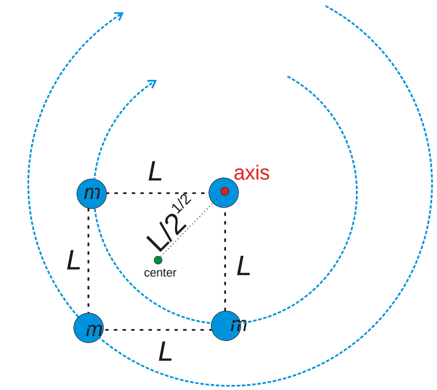
A. $4ML^2$ B. $4\text{x}2^{1/2}\,ML^2$ C. $2^{1/2}\,ML^2$ **D. $2ML^2$**

E. $(2 + 2\text{x}2^{1/2})\,ML^2$ F. $(2\text{x}2^{1/2})\,ML^2$ G. None of the above

$$I = ML^2 + ML^2 = 2ML^2$$

- 7 options





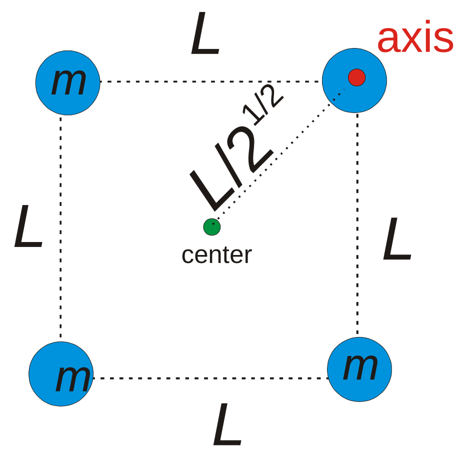
The axis of rotation is <u>perpendicular to the plane of this page</u> and goes <u>through one of the masses</u>.

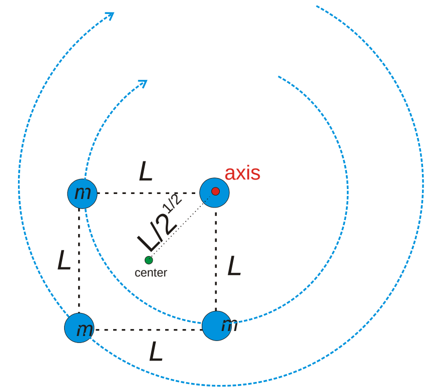
Your options are:

A. $4ML^2$ B. $4\text{x}2^{1/2}\,ML^2$ C. $2^{1/2}\,ML^2$ D. $2ML^2$

E. $(2 + 2\text{x}2^{1/2})\,ML^2$ F. $(2\text{x}2^{1/2})\,ML^2$ G. None of the above

- 7 options





The axis of rotation is <u>perpendicular to the plane of this page</u> and goes <u>through one of the masses</u>.

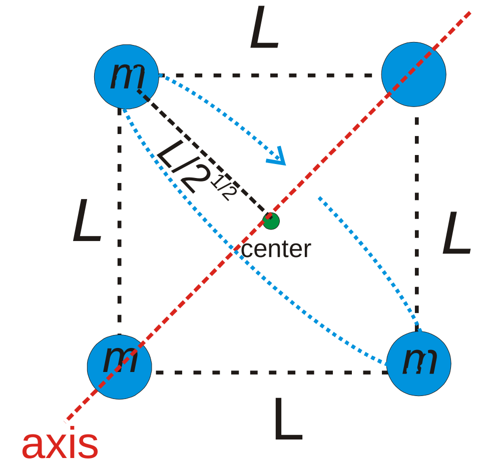
Your options are:

A. $4ML^2$ B. $4\text{x}2^{1/2}\,ML^2$ C. $2^{1/2}\,ML^2$ D. $2ML^2$

E. $(2 + 2\text{x}2^{1/2})\,ML^2$ F. $(2\text{x}2^{1/2})\,ML^2$ G. None of the above

$$I = M\left(2\frac{L}{\sqrt{2}}\right)^2 + ML^2 + ML^2 = 4ML^2$$

- 7 options



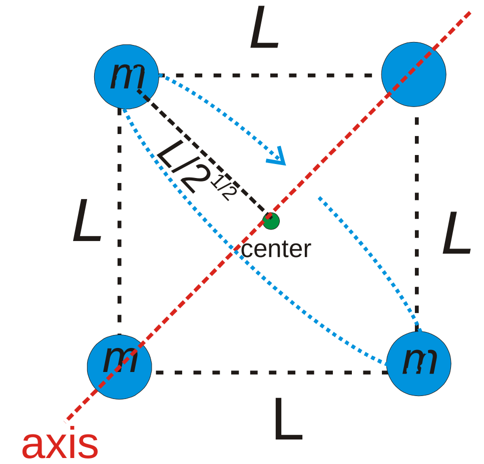
The axis of rotation lies in the plane of the page <u>along a diagonal of the square</u>.

Your options are:

A. $4ML^2$ B. $4 \mathrm{x} 2^{1/2} ML^2$ C. $2^{1/2} ML^2$ D. $2ML^2$

E. $(2 + 2\mathrm{x}2^{1/2}) ML^2$ F. $(2\mathrm{x}2^{1/2}) ML^2$ G. None of the above

- 7 options



The axis of rotation lies in the plane of the page along a diagonal of the square.

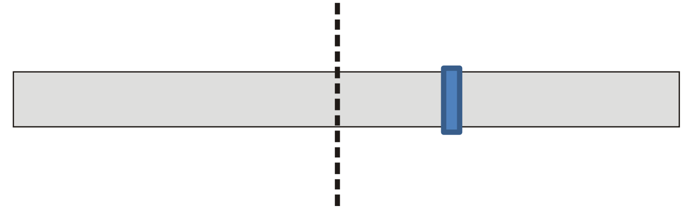
Your options are:

A. $4ML^2$ B. $4\text{x}2^{1/2}\,ML^2$ C. $2^{1/2}\,ML^2$ D. $2ML^2$

E. $(2+2\text{x}2^{1/2})\,ML^2$ F. $(2\text{x}2^{1/2})\,ML^2$ **G. None of the above**

$$I = M\left(\frac{L}{\sqrt{2}}\right)^2 + M\left(\frac{L}{\sqrt{2}}\right)^2 = ML^2$$

I of a ROD



What is the moment of inertia of a uniform rod about an axis perpendicular to the rod through its center?

Break the rod into point masses m_i or dm.

Add the contribution to I from each dm using

$$I = \sum_i m_i r_i^2 \rightarrow \int_{\substack{over \\ all dm}} r^2 dm$$

This should be reminiscent of the calculation for the CM

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M} \rightarrow \frac{\int_{\substack{over \\ all dm}} \vec{r} dm}{M}$$

The calculation for *I* is similar **BUT**

I is a scalar while $\mathbf{r}_{CM}$ is a vector.

$\mathbf{r}_{CM}$ is the *vector displacement* from a point in space

while r_i is the *scalar distance* from an axis of rotation.

I of a ROD

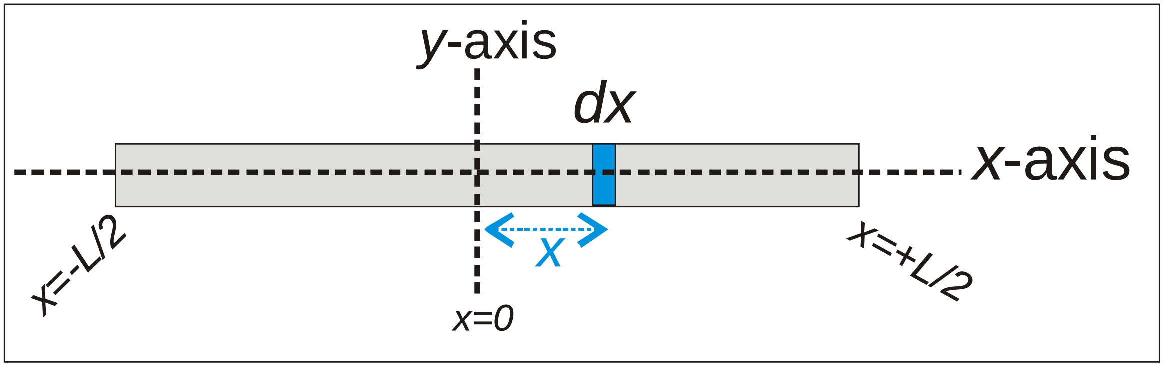
$$I = \sum_i m_i r_i^2 \rightarrow \int_{\substack{over\\ alldm}} r^2 dm$$

The rod has length L and mass M and is centered on an x-axis.

We want the moment of inertia about the y axis through the center of the rod. (*We'll shift to other axes soon*.)

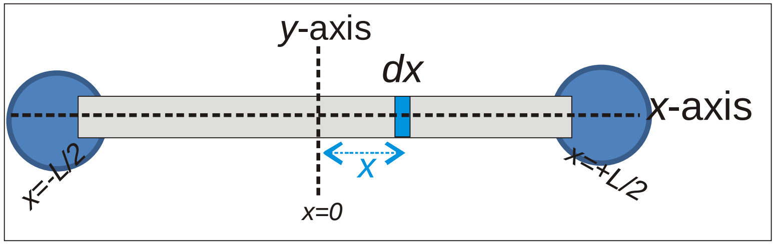
Break the rod into infinitesimal slices of width dx located a distance x from the axis of rotation.

Define a *linear mass density* $\lambda = M/L$ so $dm = \lambda dx$



$$I = \sum_i m_i r_i^2 \rightarrow \int_{\substack{over\\ alldm}} r^2 dm = \int_{x=-L/2}^{x=+L/z} x^2 \lambda dx = \lambda \left. \frac{x^3}{3} \right|_{-L/2}^{+L/2} = \left(\frac{M}{L}\right)\left(\frac{L^3}{24} + \frac{L^3}{24}\right) = \frac{1}{12} ML^2$$

I of a ROD

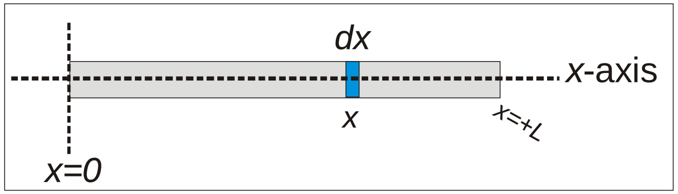


Note that $I_{ROD} = ML^2/12 << I = (2)(M/2)(L/2)^2 = \frac{1}{4} ML^2$ for two point masses $M/2$ placed at the ends of a massless rod $L/2$ from the axis.

We can easily modify our calculation to find *I* about an axis through one end of the rod. What changes from our previous calculation?

Only the limits of integration change.

$$I = \int_{x=-L/2}^{x=+L/z} x^2 \lambda dx = \lambda \left.\frac{x^3}{3}\right|_{-L/2}^{+L/2} = \frac{1}{12} ML^2$$



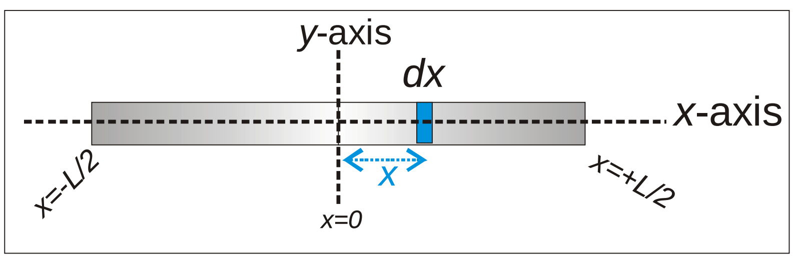
$$I = \sum_i m_i r_i^2 \rightarrow \int_{\substack{over \\ alldm}} r^2 dm = \int_{x=0}^{x=L} x^2 \lambda dx = \lambda \left.\frac{x^3}{3}\right|_0^L = \left(\frac{M}{L}\right)\left(\frac{L^3}{3}\right) = \frac{1}{3} ML^2$$

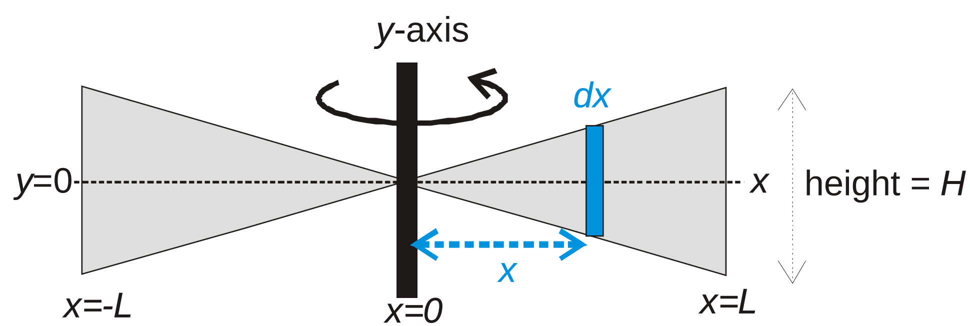
I of NON-UNIFORM OBJECTS

We can easily modify our calculation to find *I* about an axis through non-uniform rods & related objects. What changes?

Only $\lambda(x)$: the linear mass density varies as a function of x.

If $\lambda(x) = \alpha x$ where α is a constant:

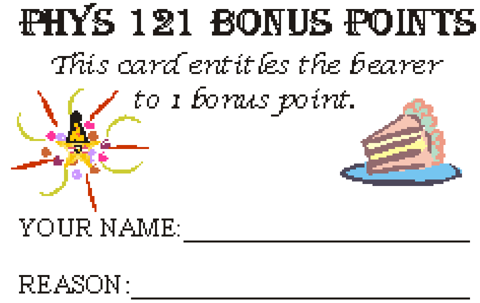




$$I = \int_{x=-L/2}^{x=+L/2} x^2 \lambda dx = \int_{x=-L/2}^{x=+L/2} x^2 (\alpha x) dx = \alpha \left. \frac{x^4}{4} \right|_{-L/2}^{+L/2} = \frac{1}{32} \alpha L^4$$

DEMO

bonus points are available



I need two volunteers, 1 ‘dainty’ and 1 ‘muscular’ person to twirl a plastic tube.

The tubes are identical & have the same overall length & mass.

Your challenge is to flip your rod back & forth 10 times as quickly as you can.

One of the tubes is more difficult to twirl.

What could explain this?

The distribution of weight within the tubes is different.

The one that is easier to spin has more mass closer to its center so that it has a smaller moment of inertia.

r is different in $I = \Sigma m r^2$

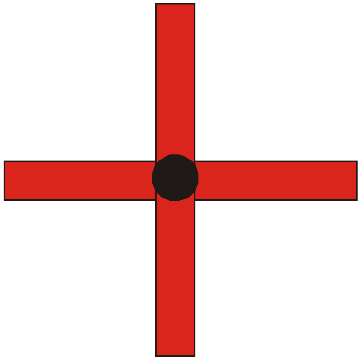
http://www.youtube.com/watch?v=EQzv4fI1ycg&feature=em-share_video_user

I of ROD-LIKE OBJECTS

Moments of inertia about a common axis add.

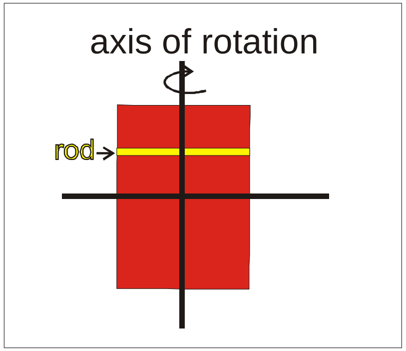
If you have N pieces of an object each contributing I_i to some total I_{total} about some common axis

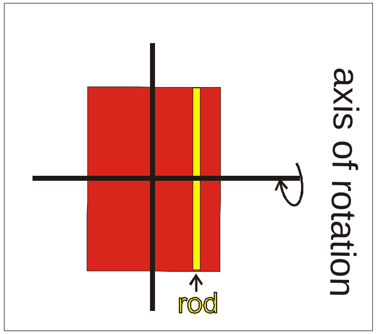
$$I_{total} = \sum_{i=0}^{i=N} I_i = \sum_i m_i r_i^2$$

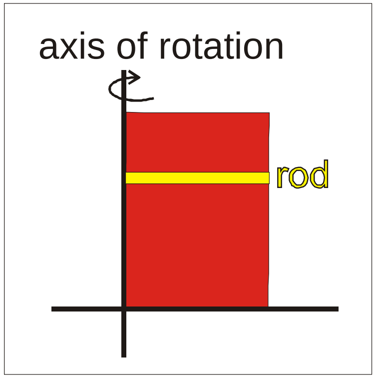


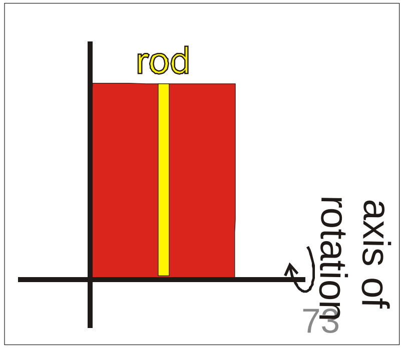
$\Rightarrow$ We can use our results for a rod to find I for:

- combinations of rods, like a propeller
- thin sheets constructed from combinations of rods laid out in parallel that share a common axis
  - $I = ML^2/12$ through center (L = *length of side perpendicular to the axis*)
  - $I = ML^2/3$ about an edge

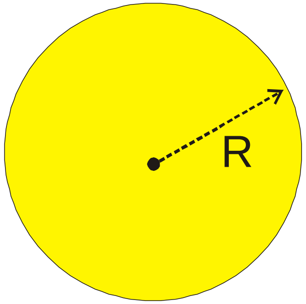








I for DISCS

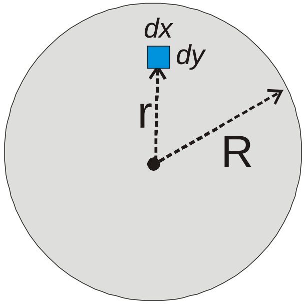


- Recall that for a point mass $I = mr^2$
- For a collection of point masses $I = \sum_{i=1}^{i=N} m_i r_i^2$
- If all the point masses are the same distance R from the origin $I = \sum_{i=1}^{i=N} m_i r_i^2 = \sum_{i=1}^{i=N} m_i R^2 = R^2 \sum_{i=1}^{i=N} m_i = MR^2$
- If the point masses form a uniform hoop of mass M and radius R $I_{hoop\ \text{w. axis through CM} \perp \text{plane of hoop}} = MR^2$
- If the elements of masses form a uniform **disc** of mass M and radius R

??? What then ???

What is I for an axis through the CM perpendicular to the plane of a uniform disc?

MOMENT of INERTIA of a UNIFORM DISC



- The brute force method requires that you construct a uniform disc from something similar to point masses.
- You start by defining infinitesimal mass elements *dm*.
  - Each *dm* has an area *da* = *dxdy*
  - You convert area to mass using the *surface mass density*

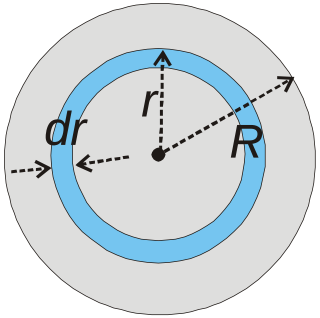
$$\sigma = \text{mass/area} = M/\pi R^2$$

  - The contribution to *I* from each *dm* is $dI = r^2 dm = r^2 \sigma dxdy$
  - You 'add' *dI* for each *dm* with a 2D integral.

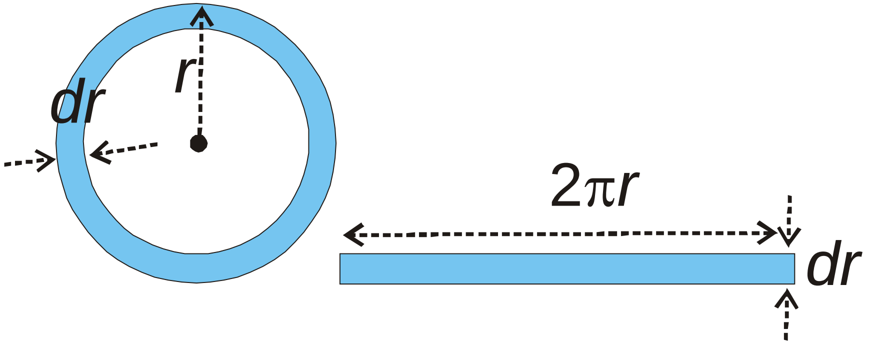
$$I = \sum_{i=1}^{i=N} m_i r_i^2 \xrightarrow[\text{about z-axis}]{} \iint_{disc} \left(x^2 + y^2\right) dm = \iint_{disc} \left(x^2 + y^2\right) \sigma dxdy$$

- The hard part is writing out the limits of integration. $x = -\sqrt{R^2 - y^2}$ to $x = +\sqrt{R^2 - y^2}$, $y = -R$ to $+R$
- But PHYS 121 students aren't expected to know 2D integrals yet!
  - Besides, there's an easier & better approach.

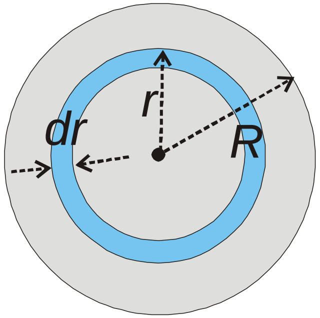
MOMENT of INERTIA of a UNIFORM DISC



- Use our knowledge that $I_{hoop}=MR^2$ to reduce this to a 1-D problem.
- Construct a disc by adding concentric hoops of radius r and infinitesimal thickness dr, starting with $r = 0$ and ending when $r = R$.
- Each individual hoop has a surface area $(2\pi r)dr$ & mass $dm = \sigma\,(2\pi r)dr$, with σ = overall mass/overall area = $M/(\pi R^2) \equiv$ *surface mass density.*
  - The formula *area = circumference* x *thickness* works because our hoops are infinitesimally thin.
  - Imagine unwrapping an infinitesimally thin hoop. What is its surface area?
  - **Do NOT use $\pi(r+dr)^2 - \pi r^2$** *although* **$[(\pi r^2 + 2\pi r dr + \pi dr^2) - \pi r^2] \sim 2\pi r dr$**



MOMENT of INERTIA of a UNIFORM DISC



- Each individual hoop has a surface area $(2\pi r)dr$ and a mass $dm = \sigma\,(2\pi r)dr$.
- Each individual hoop contributes $dI=(dm)r^2$ to the overall moment of inertia.

$$I = \sum_{i=1}^{i=N} m_i r_i^2 \underset{\text{about z-axis}}{\longrightarrow} \int_{r=0}^{r=R} r^2\,dm$$

$$= \int_{r=0}^{r=R} r^2 \left[\sigma\left(2\pi r\right)dr\right]$$

$$= 2\pi\left(\frac{M}{\pi R^2}\right)\int_{r=0}^{r=R} r^3\,dr$$

$$= 2\pi\left(\frac{M}{\pi R^2}\right)\frac{R^4}{4}$$

$$= \frac{1}{2}MR^2$$

I for DISCS

$$I_{disc} = \tfrac{1}{2} MR^2$$

- Notice that the thickness of the disc doesn't matter

  $\Rightarrow$ it's okay to use surface density σ rather than volume density ρ.

- This formula also applies to a uniform cylinder (*not just a disc*) of mass M and radius R.

  The length L of the cylinder doesn't matter!

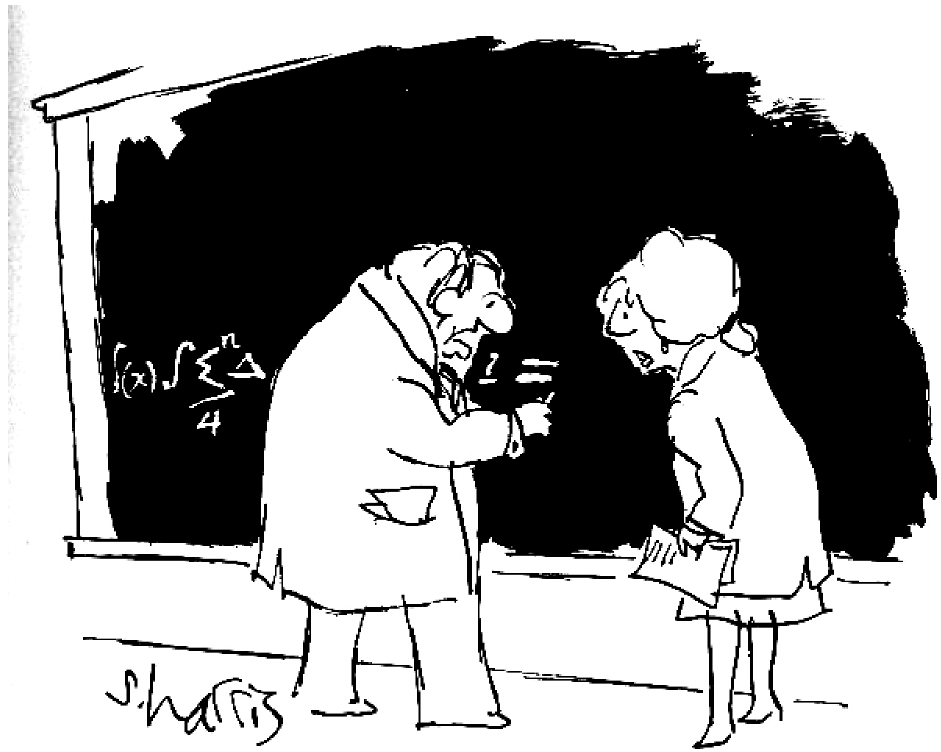
- Use $I_{disc} = \tfrac{1}{2} MR^2$ for pulleys & wheels ~ uniform discs

but

- Use $I = MR^2$ for pulleys & wheels ~ hoops with their mass concentrated on their perimeters.

We made it to slide #79 on Wednesday, April 1.

PHYS 121 – SPRING 2015



"This is the part I always hate."

Chapter 12: Rotation of a Rigid Body

version 04/03/2015, ~112 slides

We made it to slide #79 on Wednesday, April 1.

Get your clickers ready.

ANNOUNCEMENTS

- Welcome high school visitors!
- Sunday’s SI session is delayed until 7 PM because of Easter.

See Ohanian Table 12. 3 or
http://en.wikipedia.org/wiki/List_of_moments_of_inertia
for more moments of inertia – but ignore the “inertia tensors” list!

TABLE 12.3 SOME MOMENTS OF INERTIA

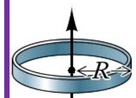
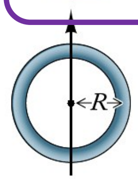
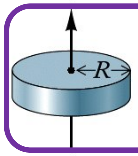
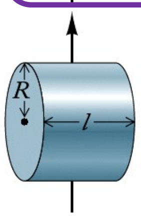
BODY		MOMENT OF INERTIA
	Thin hoop about symmetry axis	MR^2
	Thin hoop about diameter	$\frac{1}{2}MR^2$
	Disk or cylinder about symmetery axis	$\frac{1}{2}MR^2$
	Cylinder about diameter through center	$\frac{1}{4}MR^2 + \frac{1}{12}Ml^2$

Table 12-3 part 1 Physics for Engineers and Scientists 3/e

TABLE 12.3 SOME MOMENTS OF INERTIA

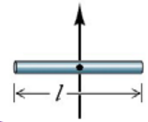
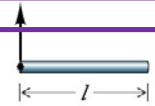
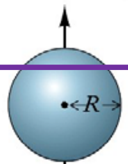
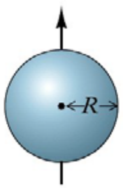
BODY		MOMENT OF INERTIA
	Thin rod about perpendicular axis through center	$\frac{1}{12}Ml^2$
	Thin rod about perpendicular axis through end	$\frac{1}{3}Ml^2$
	Sphere about diameter	$\frac{2}{5}MR^2$
	Thin spherical shell about diameter	$\frac{2}{3}MR^2$

Table 12-3 part 2 Physics for Engineers and Scientists 3/e

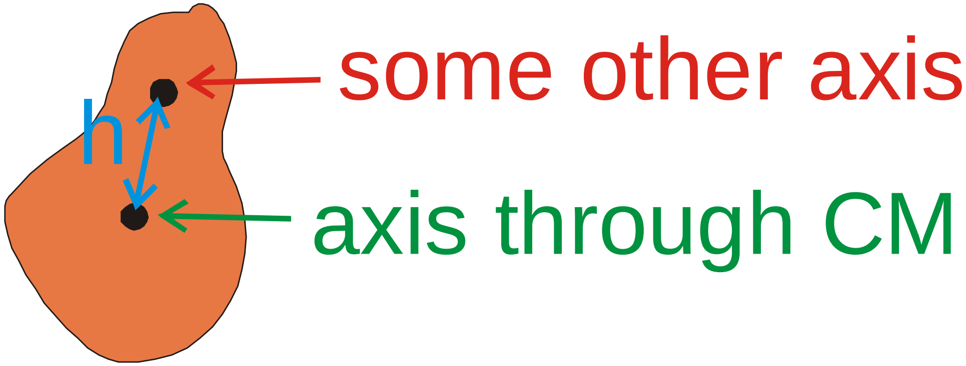
PARALLEL AXIS THEOREM

Consider an axis of rotation through the CM that is parallel to some other axis.

If the distance between these two axes $\equiv h$ and the total mass of the object is M, then

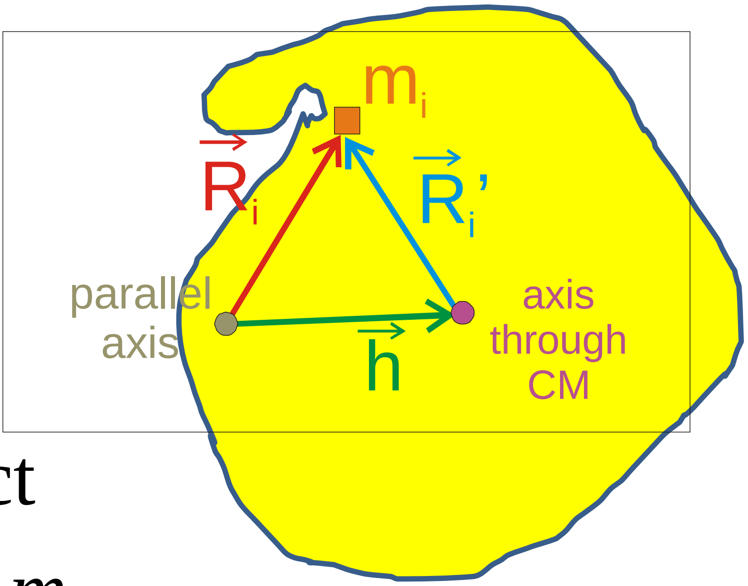
$$I_{any\ parallel\ axis} = I_{CM} + Mh^2$$

PROOF ON NEXT SLIDE



PARALLEL AXIS THEOREM

proof (*not in Ohanian*)



m_i = small element of mass in an object

$\mathbf{R}_i$ = vector from the parallel axis p to m_i

$\mathbf{R}_i'$ = vector from the CM to m_i

$\mathbf{h}$ = vector from the parallel axis to the CM

$$I_p = \sum_i m_i R_i^2 = \sum m_i \vec{R}_i \bullet \vec{R}_i = \sum m_i \left(\vec{R}_i' + \vec{h}\right) \bullet \left(\vec{R}_i' + \vec{h}\right)$$

$$= \sum m_i R_i'^2 + 2\vec{h} \bullet \sum m_i \vec{R}_i' + h^2 \sum m_i$$

But $\sum m_i \vec{R}_i' = 0$ defines the CM

$$I_p = \sum m_i R_i'^2 + Mh^2$$

$$\boxed{I_p = I_{CM} + Mh^2}$$

USING THE PARALLEL AXIS THEOREM

Remember our calculations for the moment inertia of a rod.

$I_{center} = (1/12)ML^2$ $\qquad$ $I_{end} = (1/3)ML^2$

We could have found I_{end} from I_{center} or *vice versa*

using the Parallel Axis Theorem

$$I_p = I_{CM} + mh^2$$

which in this case would give

$$I_{end} = (1/12)ML^2 + M(L/2)^2$$

$$= (1/12 + 3/12)ML^2$$

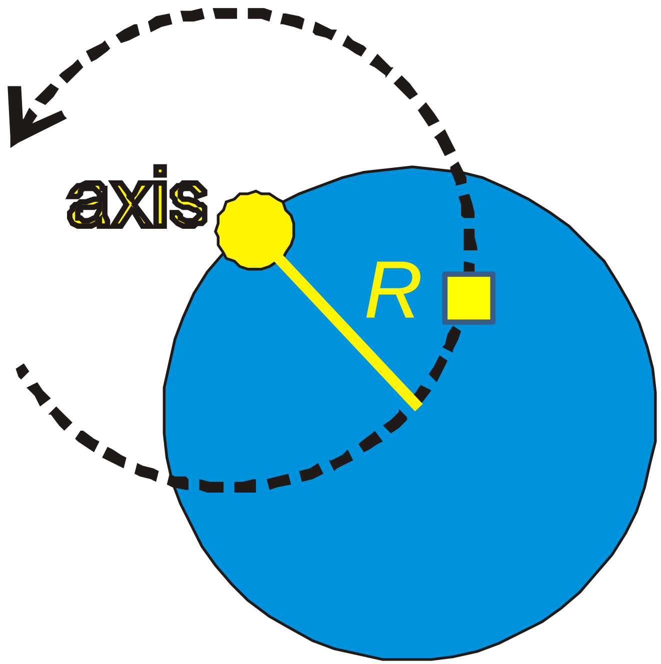
$$= (1/3)ML^2$$

as with the brute force method.

USING THE PARALLEL AXIS THEOREM
for a "PHYSICAL PENDULUM"

Calculate *I* for an axis near the edge of a solid disc.

$$I = \sum_i m_i r_i^2 \rightarrow \int_{\substack{over\\ alldm}} r^2 dm$$ is nasty!



The parallel axis theorem is easy!

$$I_p = I_{CM} + mh^2$$

$$I_{edge} = ½\, MR^2 + MR^2$$

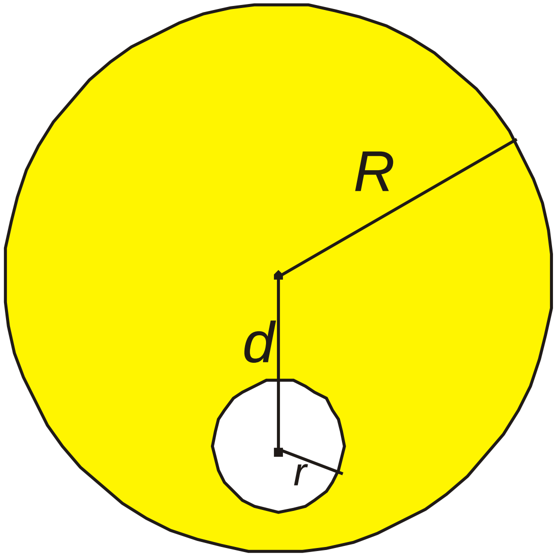
$$I_{edge} = (3/2)MR^2$$

HOLE-Y DISC *Ohanian 12.56*

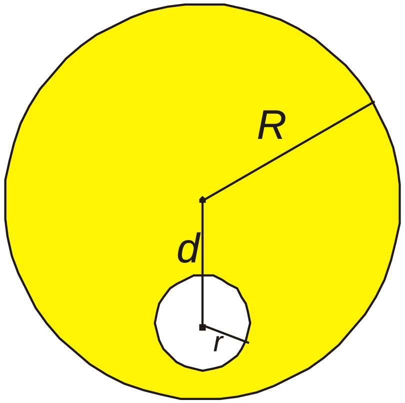
Calculate the moment of inertia about an axis through the center of a disc that has an off-center hole centered a distance d from the center of the disc.

The mass of the disc, before the hole is removed, is M.

The radius of the disc is R and the radius of the hole is r.



HOLE-Y DISC



Use $I_{total} = I_{disc\text{-}without\ hole} - I_{hole\text{-}CM}$

- $I_{disc\text{-}without\ the\ hole} = ½MR^2$
- Use the Parallel Axis Theorem to find I_{hole} about axis at CM

$$I_{hole\text{-}CM} = ½m_{hole}r^2 + m_{hole}d^2$$

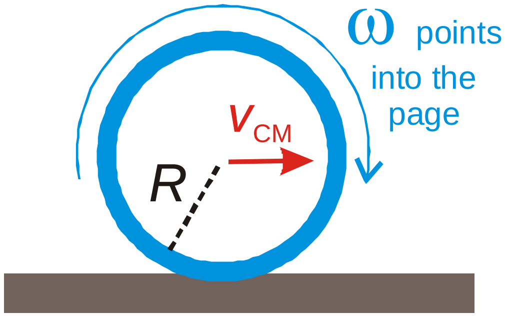
- We need m_{hole}. Use Mass/Area $\sigma = M/\pi R^2$

$$m_{hole} = \sigma(\pi r^2) = (M/\pi R^2)(\pi r^2) = Mr^2/R^2$$

$$I_{hole\text{-}CM} = Mr^2/R^2\ (½r^2 + d^2)$$

- $I_{total} = ½MR^2 - Mr^2/R^2\ (½r^2 + d^2)$
- Plug in the numerical values if necessary.

ROLLING WHEELS



Consider a wheel of radius R

rolling along a horizontal surface

without slipping

with translational velocity $\boldsymbol{v}_{\mathbf{CM}}$.

The CM of the wheel is on its axle.

The angular velocity about the CM, $\boldsymbol{\omega}_{\mathbf{CM}}$,

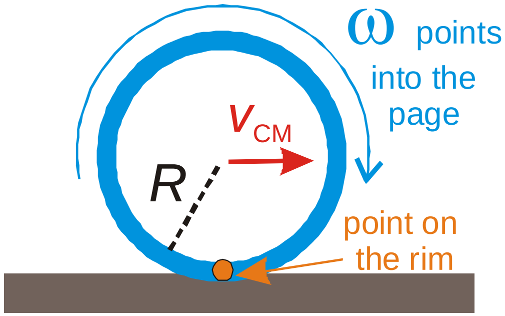
is "*obviously*" related to v_{CM} by

$$v_{CM} = \omega_{CM} R$$

This might not be obvious!

ROLLING WHEELS

$v_{CM} = \omega_{CM} R$ *justification*

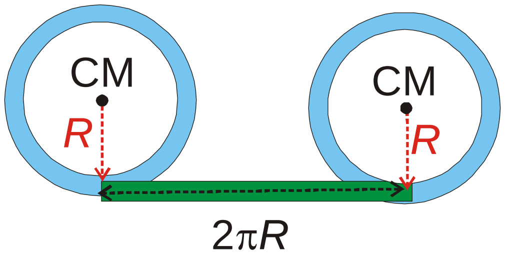


Picture yourself sitting on the axle of the wheel, at the CM.

- Rotation of a point on the rim by an angle θ (*say* 2π) corresponds to an arc length $s = \theta R$ (*say* $2\pi R$)
- *Imagine the rim 'unwrapping ' as you roll down the road.*
- The wheel's CM travels a distance $s = \theta R$ ($2\pi R$) when the wheel has rotated by θ (2π) about the CM.

$$s_{CM} = \theta_{CM} R$$

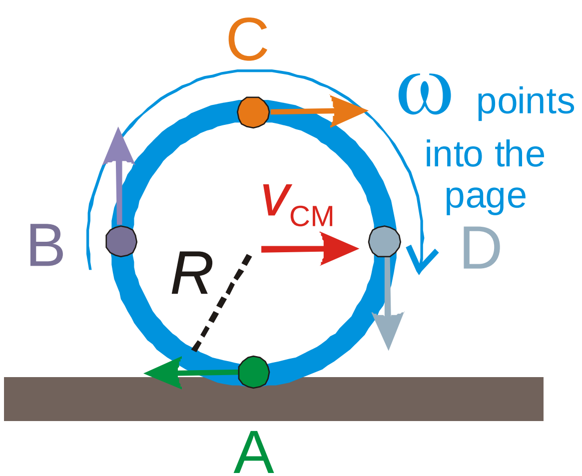
- Take a time derivative: $v_{CM} = \omega_{CM} R$ **- *OBVIOUSLY***

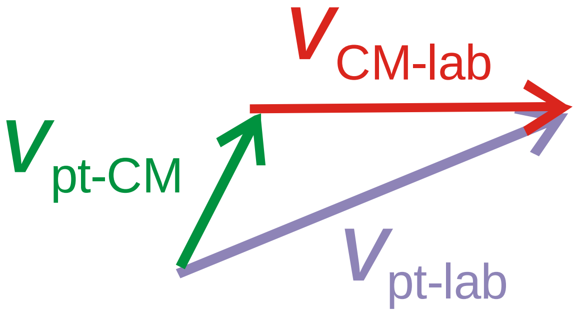


ROLLING WHEELS: $v_{CM} = \omega_{CM}R$

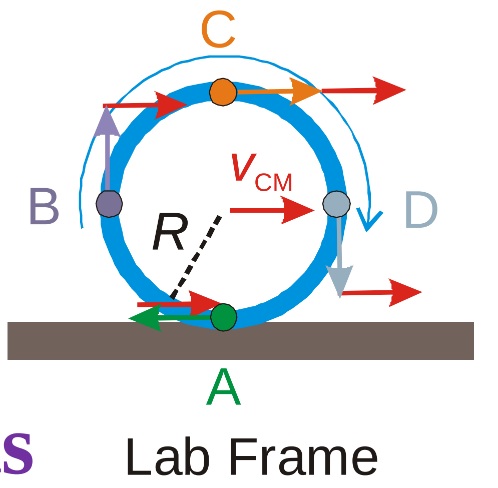
What are the velocities of points A, B, C & D in the figure
from the reference frame of someone sitting on the axle of the wheel?

- **Ignore the road, for now!**
- Every point on the wheel revolves about you at $\omega_{CM} \equiv \omega$.
- Every point on the rim has a velocity of magnitude $v_{tan} = \omega R = v_{CM}$.
  The direction of the velocity vector is different at each point; always tangent to the wheel in the direction of rotation.
- Point A is travelling backwards at v_{CM}.
  The road is also traveling backwards at v_{CM} from your perspective.
- Point B is traveling up,
  point C is traveling forward &
  point D is traveling down,
  all at $v_{tan} = \omega R$.





$$v_{\text{CM}} = \omega R$$

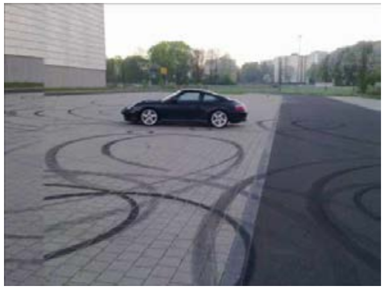


What are the velocities of points *A, B, C & D* for the rolling wheel from the point of view of the lab/earth/road frame of reference?

- The axle is moving to the right at $v_{\text{CM}} = \boldsymbol{v}_{\text{CM-lab}}$
- $\boldsymbol{v}_{\text{pt-lab}} = \boldsymbol{v}_{\text{pt-CM}} + \boldsymbol{v}_{\text{CM-lab}}$

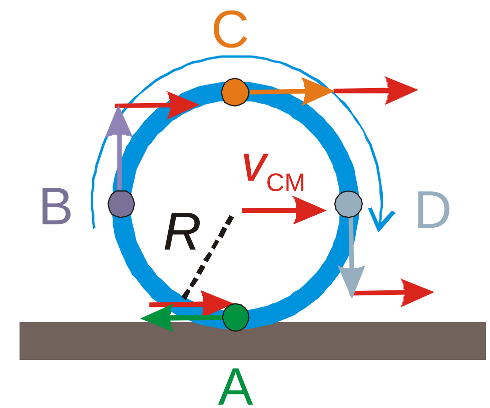
  $\Rightarrow$ add (*the vector*) $\boldsymbol{v}_{\text{CM-lab}}$ to all the velocities in the axle/CM frame of reference $\boldsymbol{v}_{\text{pt-CM}}$ (*previous slide*) to find the velocity of a point in the lab frame.

http://videoclipsimage.agaclip.com/0OFnzvIhg3f-_-porsche-996-4s-donuts.jpg



$$v_{\text{pt-lab}} \quad v_{\text{pt-CM}} + v_{\text{CM-lab}}$$

$$v_{\text{A-lab}} = -v_{\text{CM}} + v_{\text{CM}} = 0$$



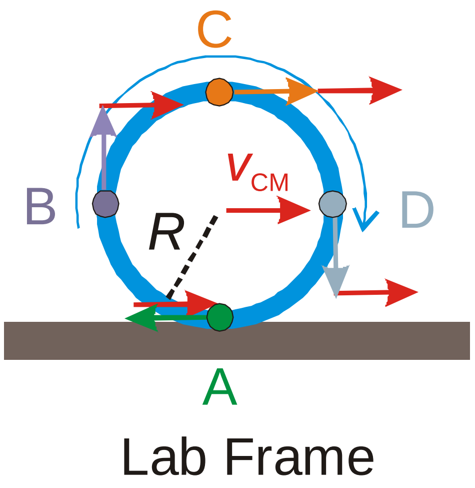
Lab Frame

Point A of the wheel is at rest in the lab frame!

- *It's at rest only 'instantaneously' = for a very short time.*
  - It moves *up & down* before this time, not horizontally.
  - This is necessary if the wheel rolls WITHOUT SLIPPING!
  - **Static friction**, not kinetic friction, describes rolling objects.
- If the wheel **slips/slides** along a surface as in a skid, kinetic friction applies.
  - Point A & the surface of the road move horizontally *wrt* each other.
- $\mu_k \leq \mu_s \Rightarrow$ you have less traction during a skid. Skids are hard to control.
  - You generally WANT static friction when you drive!
  - Unless you are playing around, creating 'donut' skid marks.
- If something slips & rolls simultaneously, you have a much harder problem!
  - You need to solve this problem to get good at billiards or driving in snow.
  - You don't need to solve it for PHYS 121!

$$\boldsymbol{v}_{\text{pt-lab}} = \boldsymbol{v}_{\text{pt-CM}} + \boldsymbol{v}_{\text{CM-lab}}$$

$$\boldsymbol{v}_{\text{C-lab}} = +\boldsymbol{v}_{\text{CM}} + \boldsymbol{v}_{\text{CM}} = 2\boldsymbol{v}_{\text{CM}}$$



- Point C at the top of the wheel has v = $2v_{\text{CM}}$

  Point C is moving twice as fast as the CM!

- Points B & D are moving at 45° angles based on summing $\boldsymbol{v}_{\text{CM}}$ up or down with $\boldsymbol{v}_{\text{CM}}$ to the right.

ROTATING *about* WHAT?

- We've assumed the wheel is rotating about its CM.
- But point "**A**" is *instantaneously* stationary & one can treat the system as pure rotation about this point *at any given instant.*

$$\Rightarrow\ v_{CM} = \omega_A R$$

- $v_{CM\text{-}A} = -v_{A\text{-}CM} \Rightarrow \boldsymbol{\omega_A = \omega_{CM}}$ (*plus sign because* $\boldsymbol{r}_{CM\text{-}A} = -\boldsymbol{r}_{A\text{-}CM}$)
- Note that $I_A = I_{CM} + MR^2$ (*parallel axis theorem*)
- You can use either axis; they give the same result.

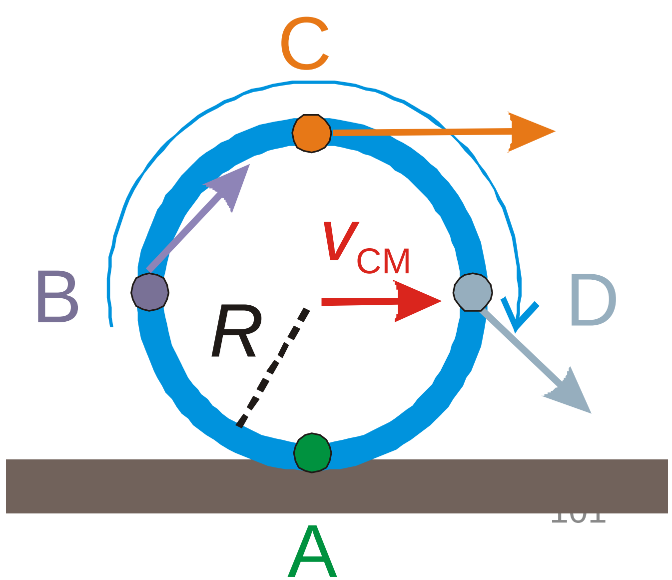
CM: $K_{total} = \frac{1}{2}Mv^2_{CM} + \frac{1}{2}I_{CM}\omega^2_{CM}$

Pt. A: $K_{total} = \frac{1}{2}I_A\omega^2_A$

$= \frac{1}{2}(I_{CM} + MR^2)\ \omega^2_{CM}$

$= \frac{1}{2}MR^2\omega^2_{CM} + \frac{1}{2}I_{CM}\omega^2_{CM}$

$= \frac{1}{2}Mv^2_{CM} + \frac{1}{2}I_{CM}\omega^2_{CM}$



Choose wisely; which is easier?

4 options

A solid disk rolls without slipping on a horizontal surface so that its center moves to the right with speed 5.0 m/s.

The point A is the uppermost point on the disk while the point B is on the rim at the same height as the CM but to its left. Point C is where the disk touches the ground.

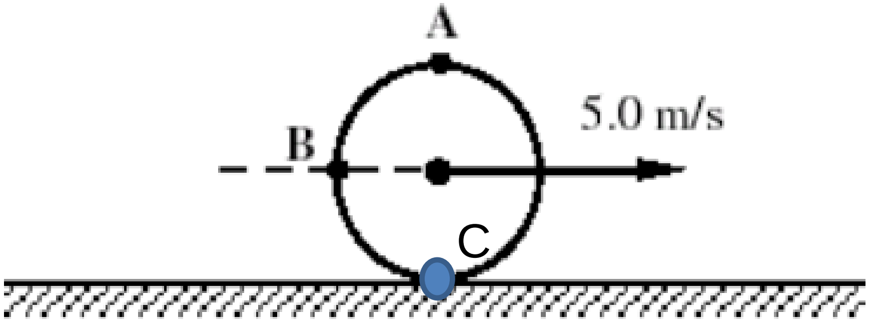
What are the *instantaneous* speeds of points A, B and C respectively *wrt* the ground?

A. 0.0 m/s, 10.0 m/s, 5.0 m/s

B. 5.0 m/s, 5.0 m/s, 5.0 m/s

C. 10.0 m/s, 5.0 m/s, 0.0 m/s

D. 10.0 m/s, 7.1 m/s, 0.0 m/s



4 options

A solid disk rolls without slipping on a horizontal surface so that its center moves to the right with speed 5.0 m/s.

The point A is the uppermost point on the disk while the point B is on the rim at the same height as the CM but to its left.

Point C is not shown but is where the disk touches the ground.

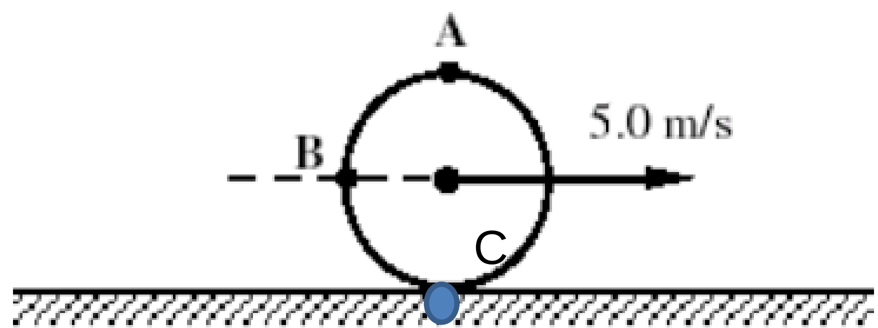
What are the *instantaneous* speeds of points A, C and B respectively *wrt* the ground?

A. 0.0 m/s, 10.0 m/s, 5.0 m/s

B. 5.0 m/s, 5.0 m/s, 5.0 m/s

C. 10.0 m/s, 5.0 m/s, 0.0 m/s

D. 10.0 m/s, 7.1 m/s, 0.0 m/s



You need to add the velocity of each *wrt* the CM to the velocity of the CM.

B has two perpendicular components, 5 m/s up and 5 m/s to the right.

Pythagorean's Theorem tells us these sum to 7.1 m/s.

Pont C is at rest while point A moves at $2v_{CM}$.

ROLLING HOOPS, DISCS & SPHERES

$K_{total} = K_{rotation} + K_{translation}$

$$K_{total} = ½I_{CM}\omega^2 + ½Mv^2_{CM}$$

We proved this for an object that can rotate AND move translationally at the same time.

The only difference now is a connection between these two modes of motion.

$$\boldsymbol{v}_{CM} = \omega R$$

$$K_{total} = ½I_{CM}(v_{CM}/R)^2 + ½Mv^2_{CM}$$

RACE DOWN AN INCLINE

4 options

Which travels down an incline faster – a rolling wheel or a frictionless block?

(*You aren't told the mass or size of either.*)

A. the rolling wheel

B. the frictionless block

C. It's a tie.

D. I need to know more about the objects

PLACE YOUR BETS!

RACE DOWN AN INCLINE

3 options

Which gets down an incline faster – a rolling wheel or a frictionless block? You aren't told the mass or size of either

A. the rolling wheel

B. the frictionless block

C. It's a tie

D. I need to know more about the objects

When they've traveled a vertical distance $h = L\sin\theta$:

The block's speed v is given by $\boldsymbol{mgh = \tfrac{1}{2}mv^2} \rightarrow \boldsymbol{v = (2gh)^{1/2}}$

The wheel's speed v is given by $\boldsymbol{mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \rightarrow v = (2gh - I\omega^2/M)^{1/2}}$

Thanks to the negative sign in front of the second term, the frictionless block is moving faster at every moment and reaches the bottom first.

We can use this same calculation for other rolling objects.

If all else is equal, the object with the larger moment of inertia I gains more rotational kinetic energy and less translational kinetic energy & velocity.

The story might change if there IS friction for the block!

Friction could stop the block altogether while the wheel rolls merrily down the incline.

The wheel **REQUIRES** friction in order to roll!

Why do vehicle manufacturers try to make wheels in particular as light as possible?

$$W = \Delta K = \Delta K_{rotation} + \Delta K_{translation}$$

A rolling wheel includes BOTH terms.

It takes more work (*& energy & power*) to accelerate a wheel compared to non-rotating parts of a vehicle.

For example, if the wheel is a hoop of mass M and radius R, $I = MR^2$

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 = \frac{1}{2}(MR^2)(v/R)^2 + \frac{1}{2}Mv^2 = Mv^2$$

$\Rightarrow$ The wheel has TWICE as much kinetic energy as an equivalent nonrotating mass.

$\Rightarrow$ It takes **twice** as much work to accelerate it to a certain velocity.

$\Rightarrow$ Shaving mass from a wheel ~ shaving twice as much mass elsewhere.

That's why serious (*wealthy*) cyclists may pay \$2000 for a carbon fiber bicycle wheelset, 1362 g = 3 lb. for both front + rear wheels, including hubs & spokes.

http://www.excelsports.com/main.asp?page=8&descri endorCode=SHIM&major=1&minor=37

LOOPS *with a FRICTIONLESS BLOCK*

Remember finding v at the top of the loop?

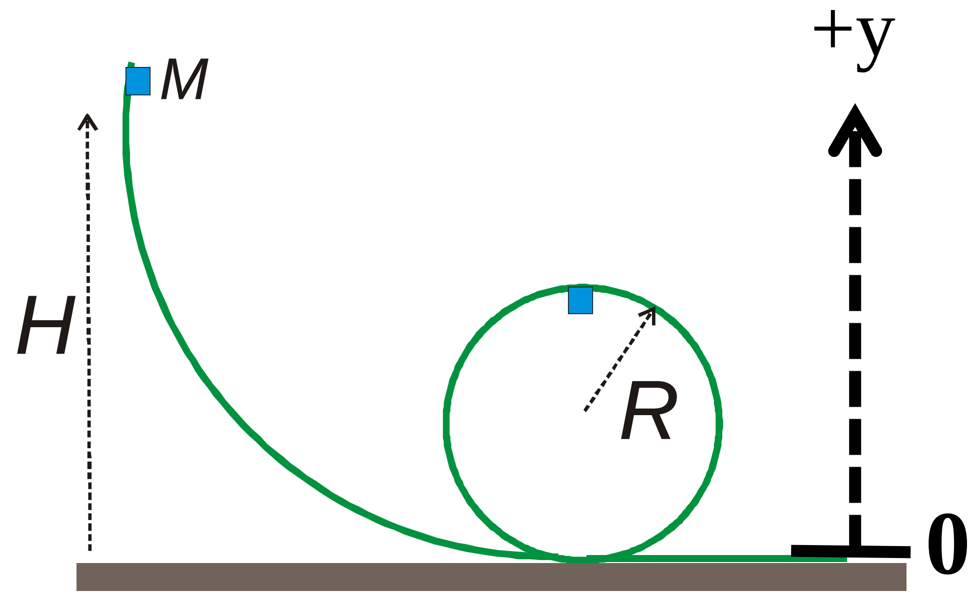
$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$0 + mgH = \frac{1}{2} m v_f^2 + mg(2R)$$

$$\frac{1}{2} m v_f^2 = mg(H - 2R)$$

$$\boxed{v_f = \sqrt{2g(H - 2R)}}$$

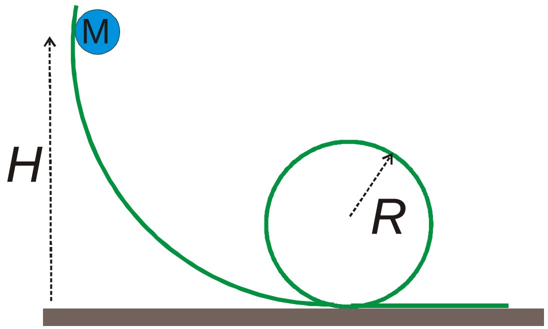


Need $v_f^2/R > g$ for $N > 0$

$$\frac{2g\left(H - 2R\right)}{R} > g$$

$$2H > R + 4R$$

$$H > 2.5R = 1.25D$$



LOOPS

with a solid sphere of radius *r*

What if the frictionless block → rolling ball with $I = (2/5)Mr^2$?

Some of the potential energy of gravity is 'diverted' into

$$K_{\text{rotational}} = \tfrac{1}{2} I_{\text{sphere}}\omega^2 = \tfrac{1}{2}[(2/5)Mr^2](v/r)^2 = (1/5)Mv^2$$

We still need a minimum $F_{\text{radial}} = Mg = Mv^2_{2R}/R$ at $h = 2R$

rewrite this as: $v^2_{2R} = gR$

Conservation of Energy: $U_H = K_{\text{translational-}2R} + K_{\text{rotational-}2R} + U_{2R}$

$$MgH = \tfrac{1}{2}Mv^2_{2R} + (1/5)Mv_{2R}^2 + Mg(2R)$$

Plugging in our minimum $v^2_{2R} = gR$ for $N > 0$

$$MgH = \tfrac{1}{2}M(gR) + (1/5)M(gR) + Mg(2R)$$

$H = 2.7\,R = 1.35\,D$ *(instead of 1.25 D earlier)*

Note that the radius *r* of the ball is irrelevant, as long as it's less than *R*!

I LIED: r also changes $U_{gravity} = mg(2R)$ term → mg(2R-r)

MOMENT OF INERTIA RACE

3 options

If a hoop and a disc, each of mass *M* and radius *R*, roll without slipping down an incline, which reaches the bottom first?

A. Hoop

B. Disc

C. Tie – *they reach the bottom at the same time*

PLACE YOUR BETS!

DEMOS

MOMENT OF INERTIA RACE

hoop *vs.*disc

https://www.youtube.com/watch?v=6gF6ajOcp_o&list=PL_g3d400fWB9i9NxFcTYvUqNeeuL0xoyr

MOMENT OF INERTIA RACE

3 options

If a hoop and a disc, each of mass *M* and radius *R*, roll without slipping down an incline, which reaches the bottom first?

A. Hoop

B. Disc

C. Tie – *they reach the bottom at the same time*

$I = \frac{1}{2} MR^2$ for the disc but MR^2 for the hoop.

The disc has a smaller moment inertia than a hoop of the same radius and mass, so less of the potential energy of gravity goes into kinetic energy associated with rolling and more into its translational motion, *i.e.* it travels faster down the incline.

The speed v is given by $\boldsymbol{mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2} \quad \rightarrow \quad \boldsymbol{v = (2gh - I\omega^2/M)^{1/2}}$

Note that a factor of v is hidden in $\boldsymbol{\omega = v/R}$ but this doesn't change our analysis since R is the same for both objects.

5 options

Which of the following is <u>slowest</u> down an incline?

The block slides w/o friction while the other objects roll w/o slipping.

M and *R* are the same wherever they appear below.

A. $I_{\text{frictionless block}} = 0$ *although its mass is M*

B. $I_{\text{hoop}} = MR^2$

C. $I_{\text{disc}} = \frac{1}{2}MR^2$

D. $I_{\text{solid sphere}} = 2/5\ MR^2$

E. $I_{\text{hollow sphere}} = 2/3\ MR^2$

5 options

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C. $I_{\text{disc}} = \frac{1}{2}MR^2$

D. $I_{\text{solid sphere}} = 2/5\ MR^2$.

E. $I_{\text{hollow sphere}} = 2/3\ MR^2$

The hoop is slowest; it has the largest moment of inertia and will have the largest kinetic energy of rotation when it does reach the bottom of the incline.

http://1.bp.blogspot.com/-V86-q3dVaiQ/UiB8j7ZygRI/AAAAAAAABlo/mSy8NESvMT0/s200/the-end-200px.jpg