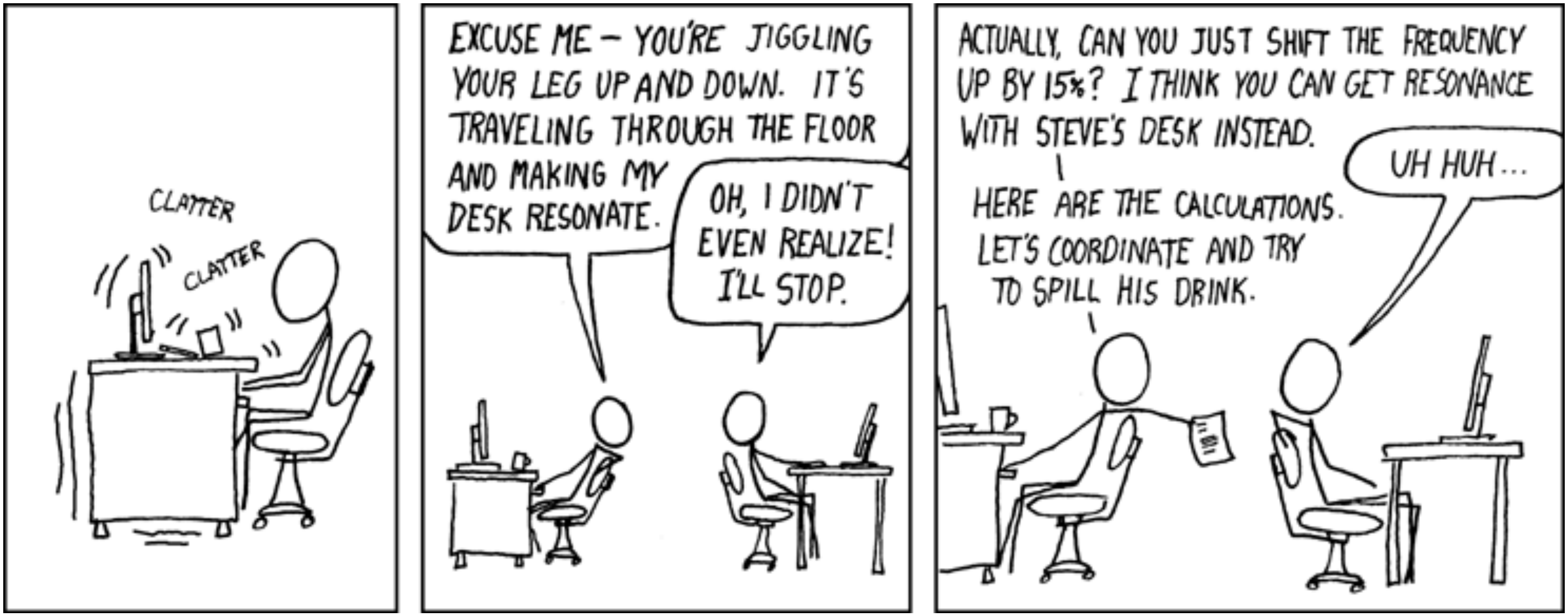
PHYS 121 – SPRING 2015

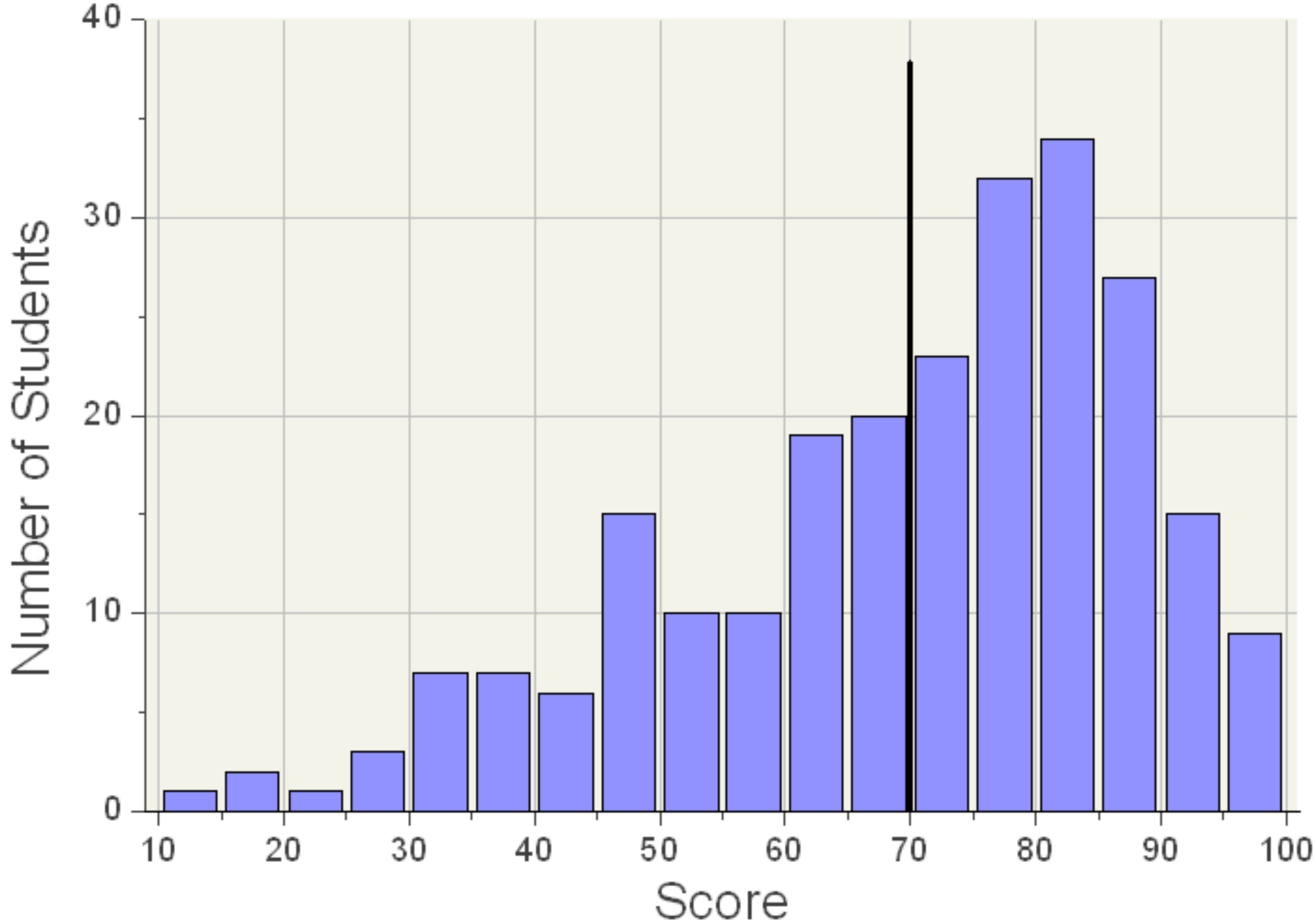


Chapter 15: Oscillations

Version 4/20/2015

Announcements

Exam 3 average was 69.7 ± 19%



Announcements

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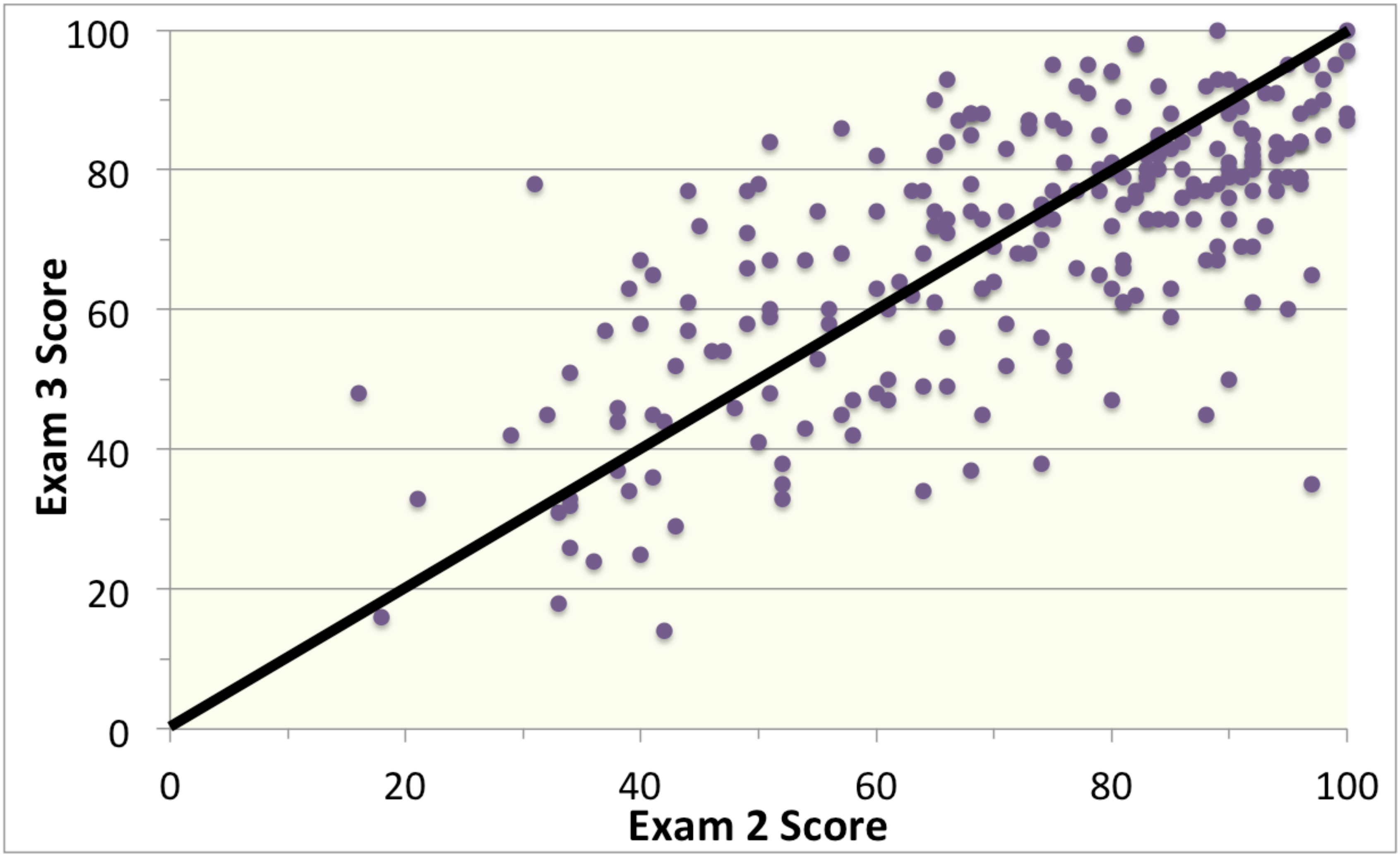
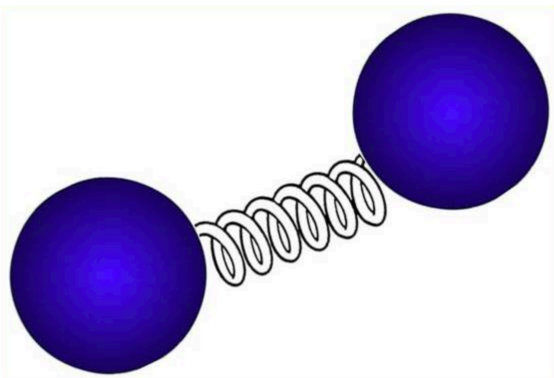


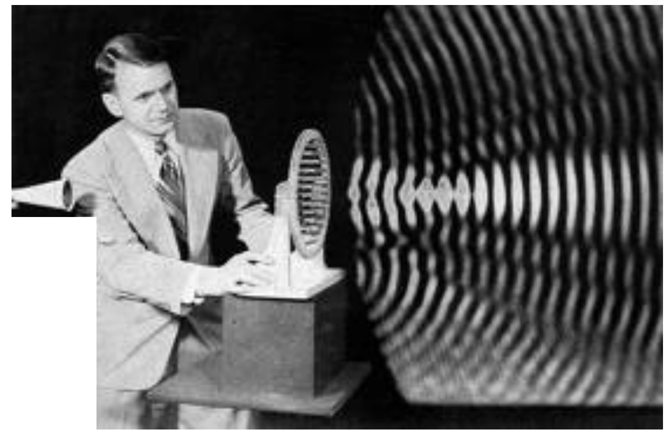
Announcements

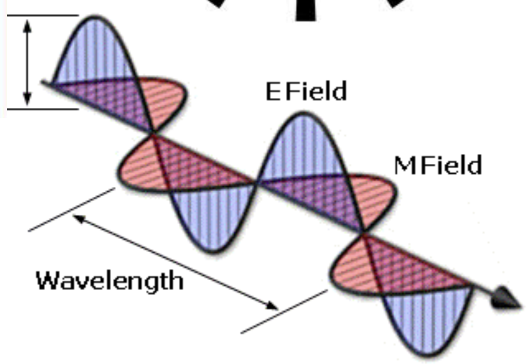
- Problem 2 was the hardest (falling rod). The N2L problem was much this time around.
- Two of the problems were similar to HW problems, and 1 was also covered in an SI session.
- The final exam is two weeks away! Monday, May 4th at 4pm. Place to be determined.

Oscillations

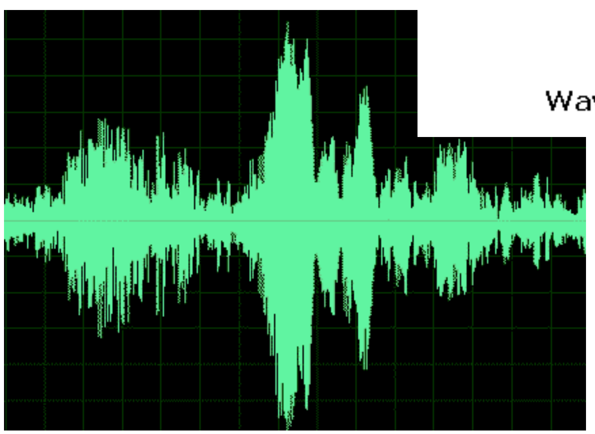
Everything around you is constantly oscillating, literally.

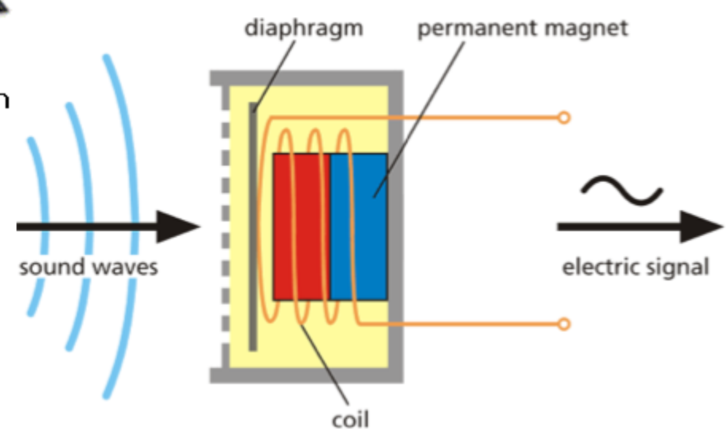




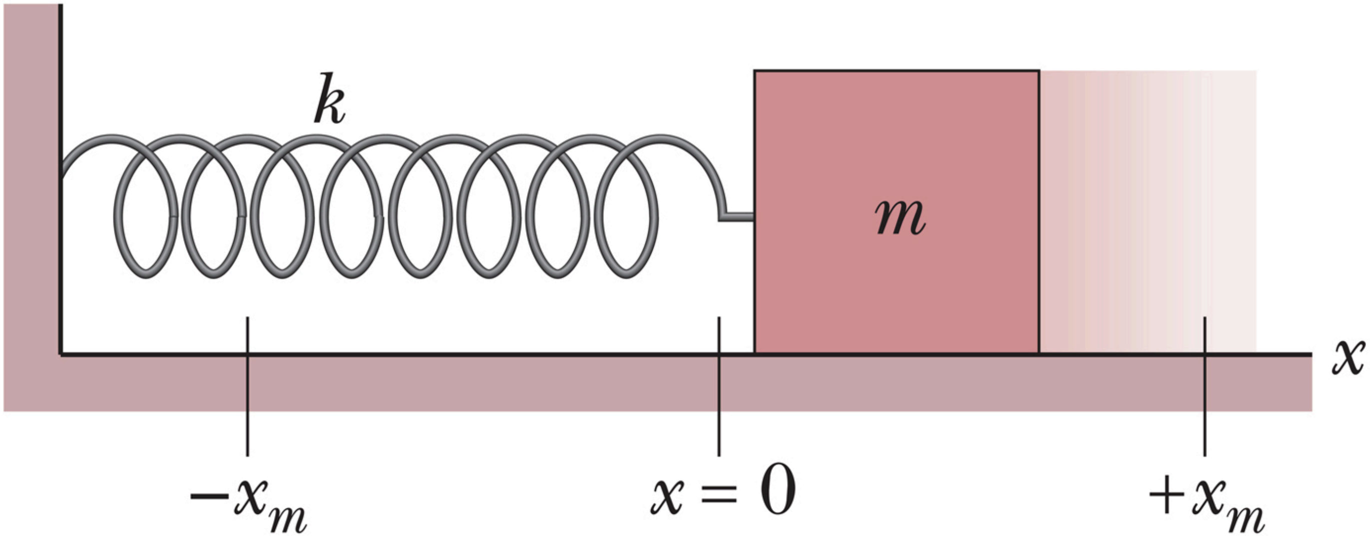


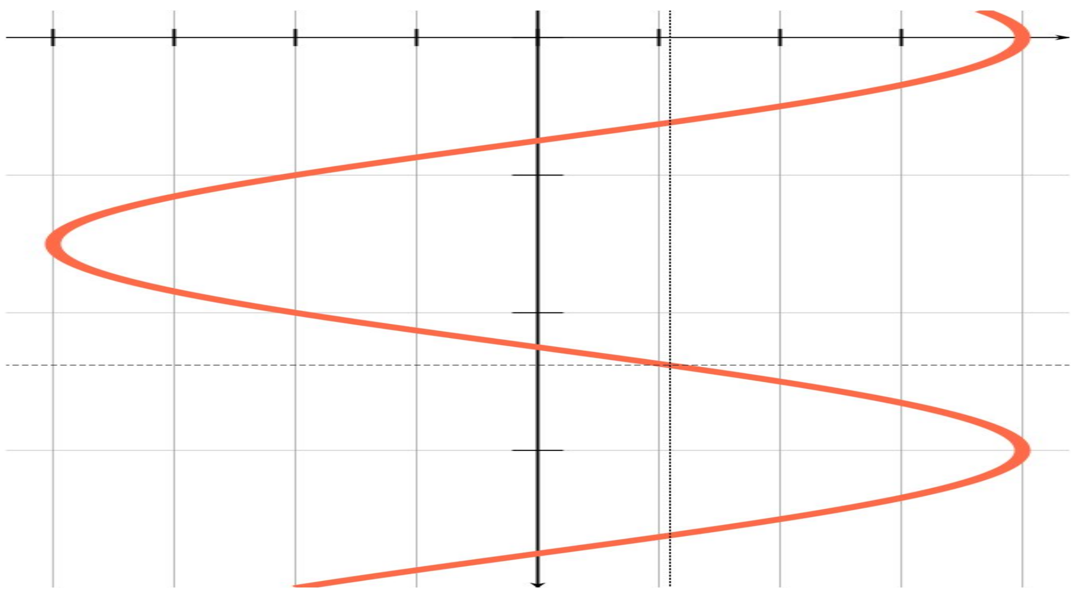
Waves of Electromagnetic Radiation





Motion of a Spring





Equation of Motion for Spring

Quantitatively

We want an equation for position as a function of time, *x(t)*

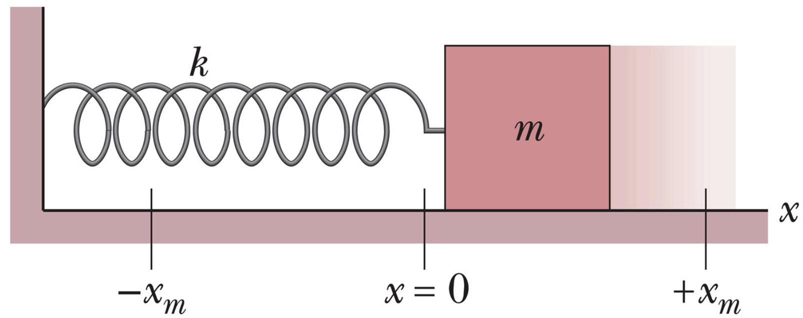
$$\sum \vec{F} = m\vec{a}$$

$$\sum F_x = ma_x$$

$$-kx = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$



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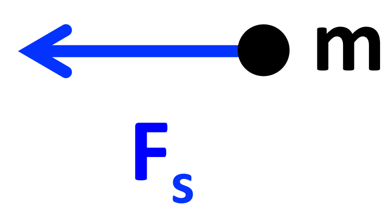


Differential Equations

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

We know everything in this equation except *x*, so in principle we can solve this for *x*.

The *best method* for solving equations with derivatives is to guess.

> We want a function, which when we takes its derivative twice, we get back the negative of the function.

$$\cos(t) \rightarrow -\sin(t) \rightarrow -\cos(t)$$

Guess: $\boldsymbol{x(t) = A\cos(\omega t + \delta)}$ A, ω, and δ are unknown constants.

Solution to Spring Equation

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

Plug ***x(t) = A cos(ωt + δ)*** into our differential equation and see what happens.

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \delta) = -\left(\frac{k}{m}\right)A\cos(\omega t + \delta)$$

Cosine terms cancel, and we are left with:

$$A\omega^2 = \frac{k}{m}A$$

$$\boxed{x(t) = A\cos\left(\sqrt{k/m}\,t + \delta\right)}$$

$$\Rightarrow\ \omega = \sqrt{\frac{k}{m}}$$

General solution for any spring.

A and *δ* depend on how you start it.

Simple Harmonic Motion

A particle whose motion is described by:

$$x(t) = A\cos\left(\omega t + \delta\right)$$

is said to be undergoing **simple harmonic motion**.

Harmonic = periodic

Simple = single sine or cosine

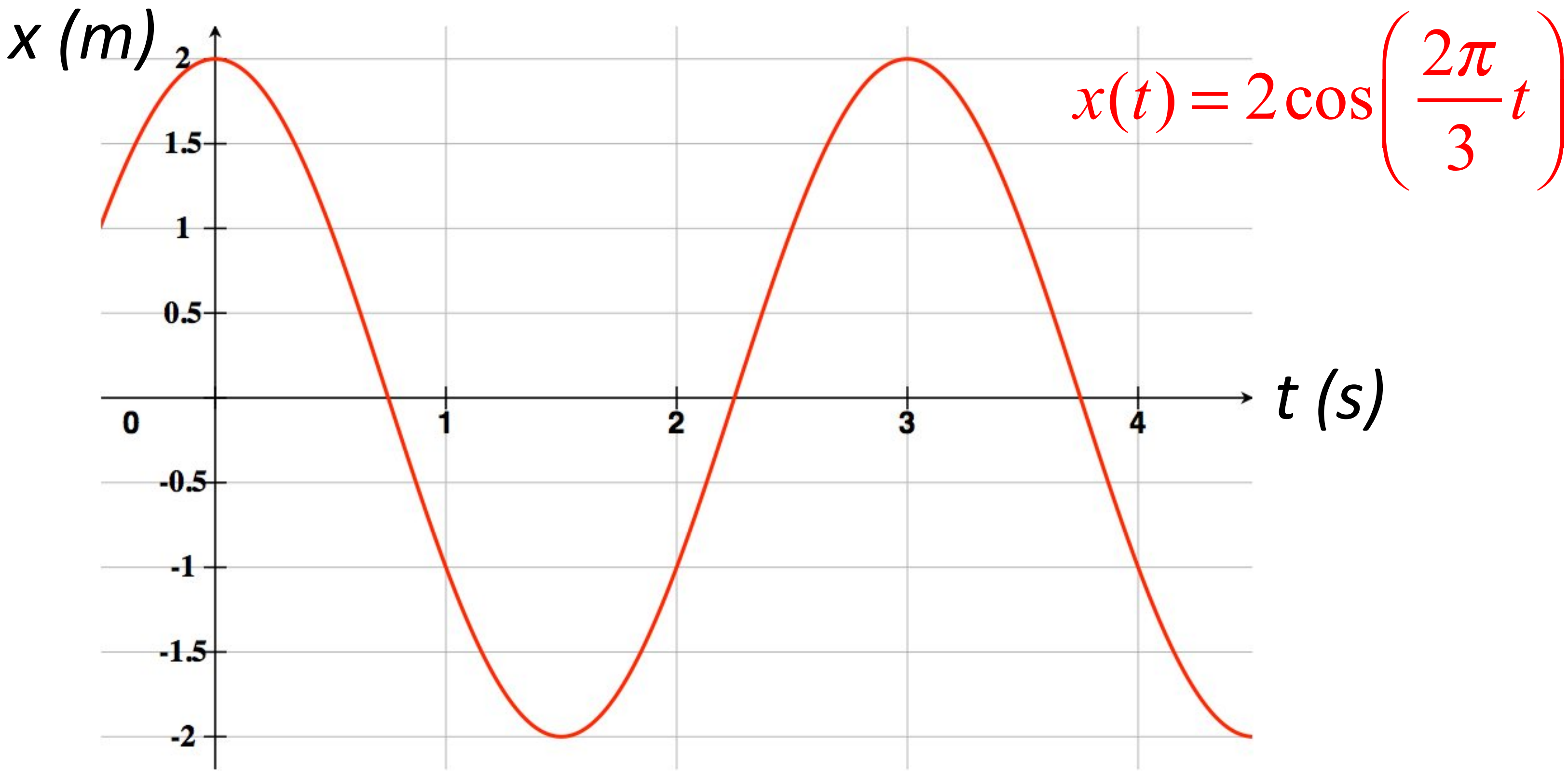
SHM always results when the total force on an object is a linear restoring force: $F \propto -x$

Amplitude: *A*

$$\boxed{x(t) = A\cos\left(\omega t + \delta\right)}$$

Maximum value of cosine, *regardless of ω, t, or δ,* is always 1.

➔ *Maximum x = A*



Angular Frequency: ω

$$\boxed{x(t) = A\cos(\omega t + \delta)}$$

The *Period* (*T*) of the motion is the time it takes for the mass to complete one cycle of motion.

$$x(t) = x(t+T)$$

$$A\cos(\omega t + \delta) = A\cos(\omega(t+T) + \delta)$$

The value of *cosine* repeats itself when the argument changes by *2π*.

$$[\omega(t+T)+\delta] - [\omega t + \delta] = 2\pi \qquad \Rightarrow \ \omega T = 2\pi$$

$$\boxed{T = \frac{2\pi}{\omega}} = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

Angular Frequency: ω (rad/s)

$$\boxed{x(t) = A\cos\left(\omega t + \delta\right)}$$

The *Period* (*T*) of the motion is the time it takes for the mass to complete one cycle of motion.

$$x(t) = x(t+T)$$

$$A\cos\left(\omega t + \delta\right) = A\cos\left(\omega(t+T) + \delta\right)$$

The value of *cosine* repeats itself when the argument changes by *2π*.

$$[\omega(\cancel{t} + T) + \cancel{\delta}] - [\cancel{\omega t} + \cancel{\delta}] = 2\pi$$

$$\Rightarrow \ \omega T = 2\pi$$

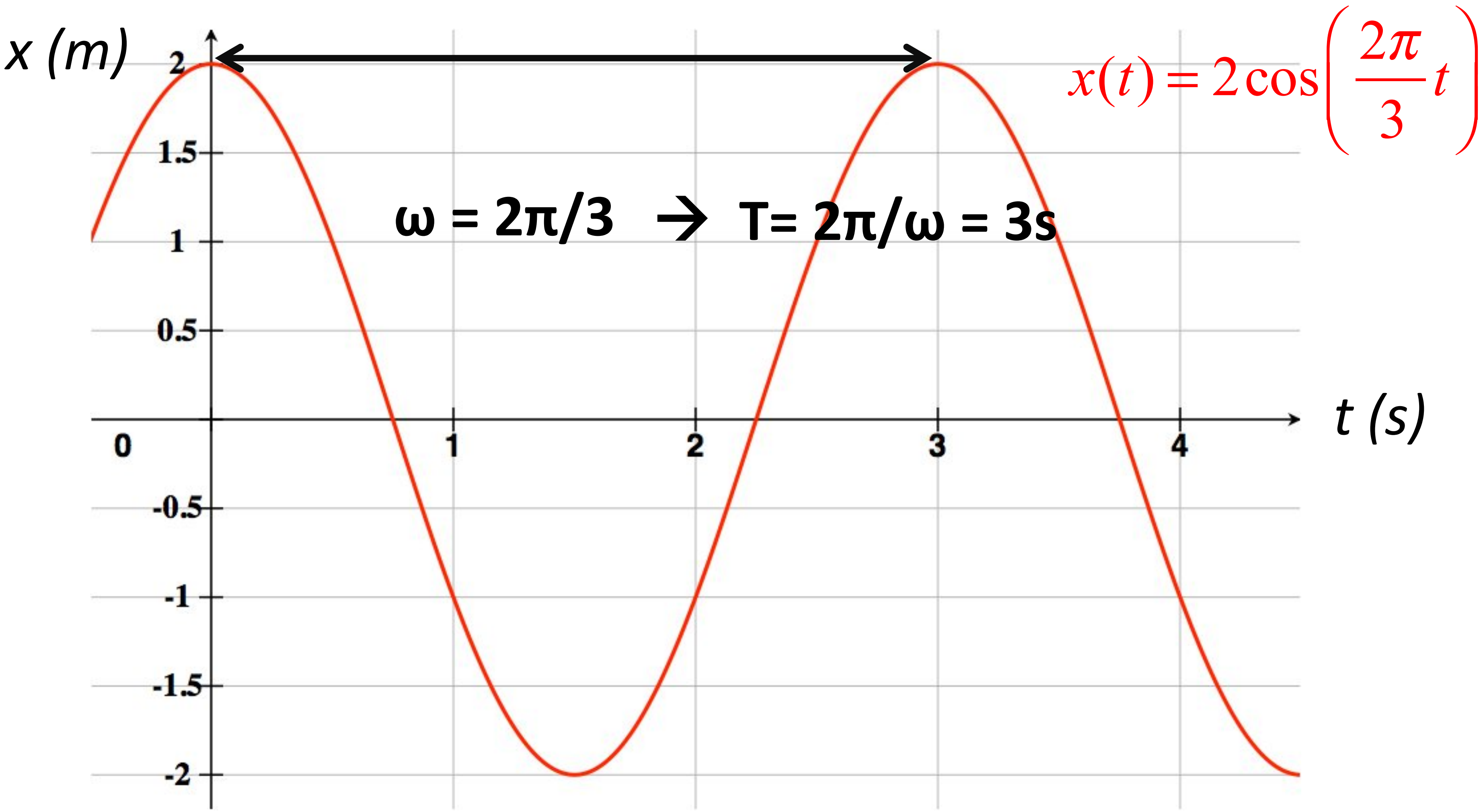
$$T = 2\pi\sqrt{\frac{m}{k}} \quad for\ spring$$

$$Period \ \boxed{T = \frac{2\pi}{\omega}} \ = \frac{1}{f}$$

f = "frequency" in cycles/sec, or Hz

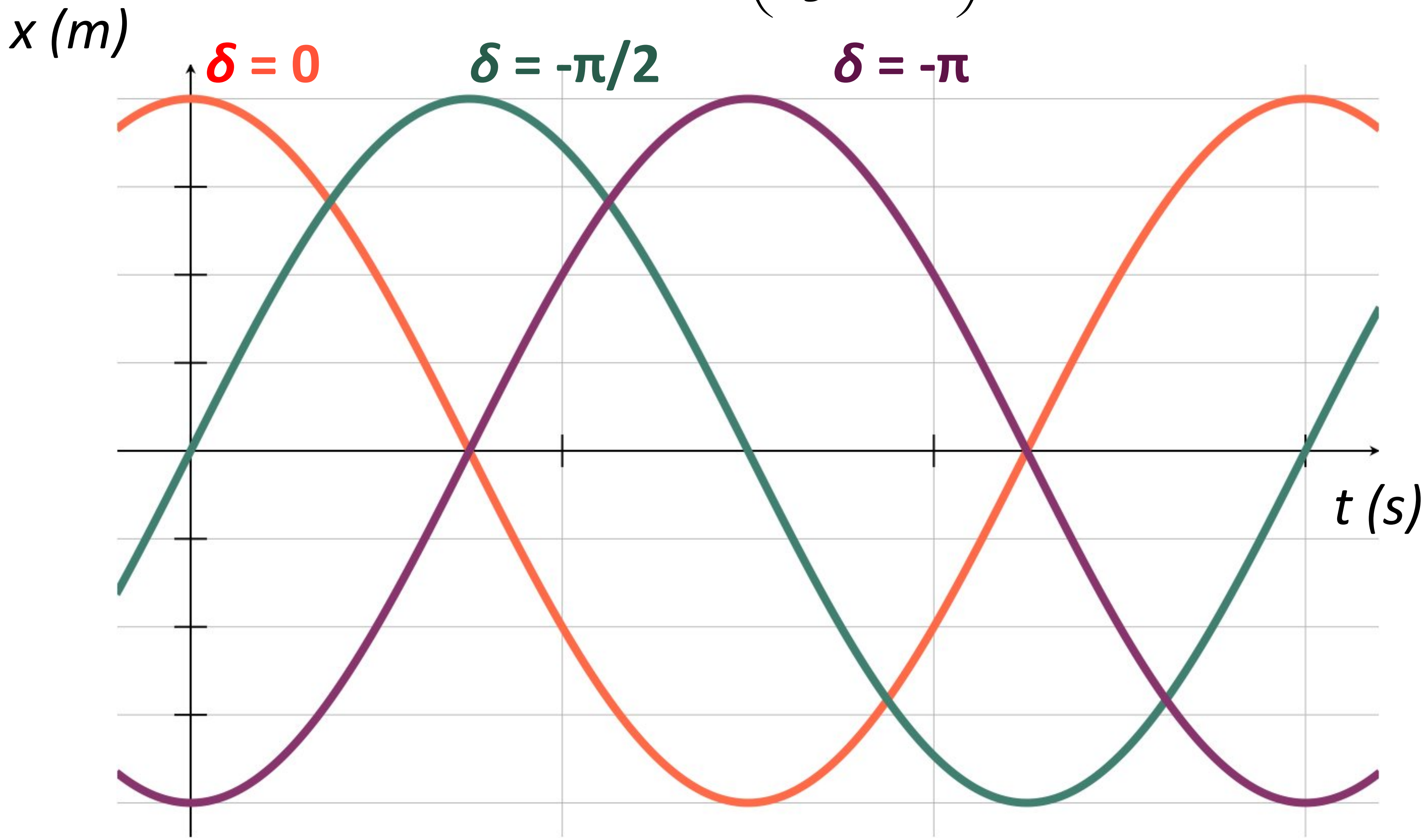
Angular Frequency: ω

$$x(t) = A\cos\left(\omega t + \delta\right)$$



Phase: δ

$$x(t) = 2\cos\left(\frac{2\pi}{3}t + \delta\right)$$



Phase: δ

$$x(t) = 2\cos\left(\frac{2\pi}{3}t + \delta\right)$$

δ is the horizontal shift of the curve in radians.

The particle completes one cycle of motion every T seconds (one Period). But the cosine completes one cycle when the argument changes by 2π radians.

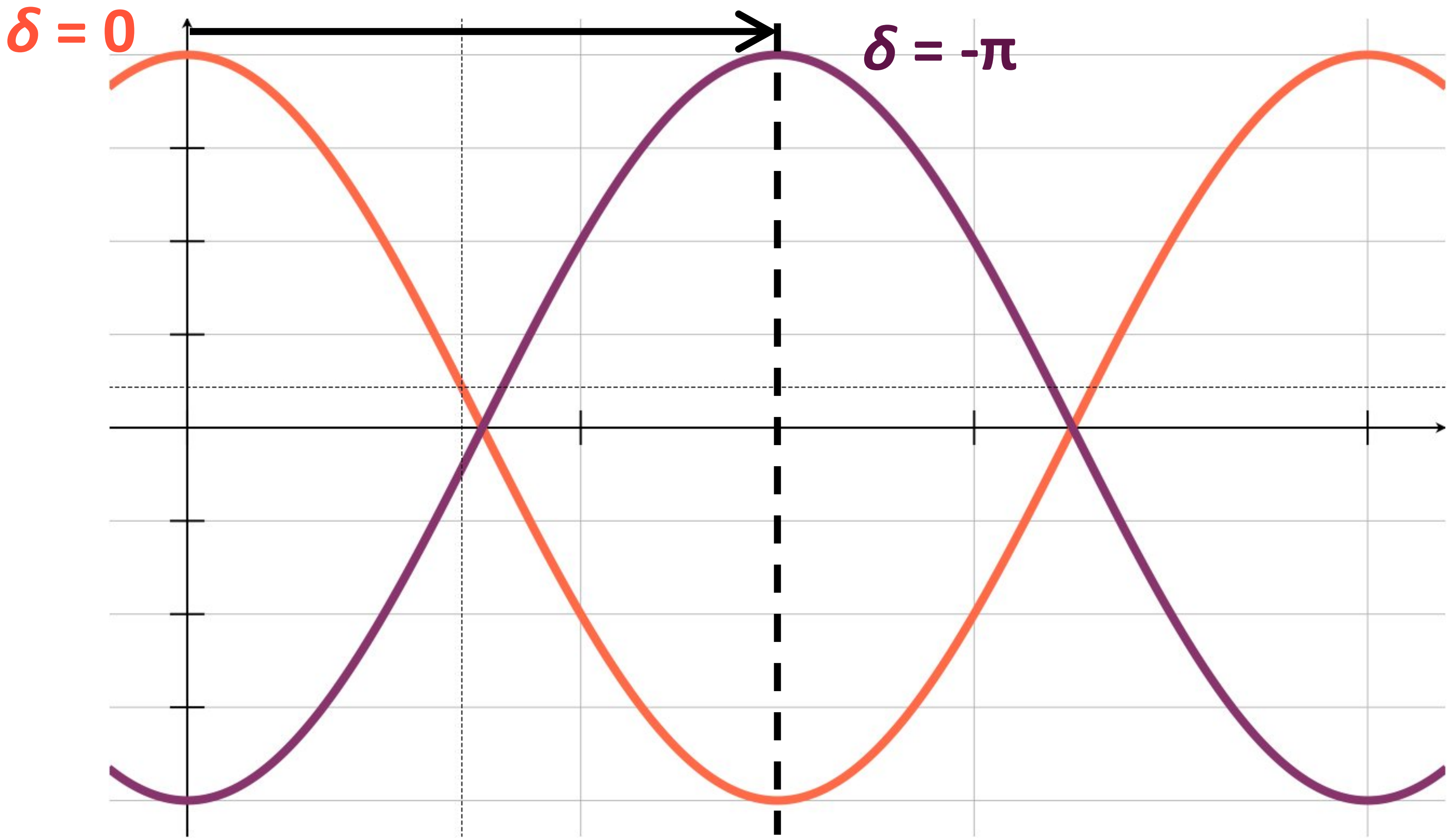
If $\delta = -\pi$, then the curve is shifted by half a period.

Minus means it is shifted to the right.

Phase: δ

$$x(t) = 2\cos\left(\frac{2\pi}{3}t + \delta\right)$$

If ϕ = -π, then the curve is shifted by half a period.



Initial Conditions

$$\boxed{x(t) = A\cos(\omega t + \delta)}$$

This equations has 2 unknowns, A and δ.

Thus, to get a complete solution you will need at least 2 pieces of information

This is usually the position *x* at some time, and the velocity *v* at some time.

SHM Example

Suppose you are a given mass (0.50 kg) attached to a spring with spring constant 4.0 N/m.

- What is ω?

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\ N/m}{0.50\ kg}} = 2.828\ rad/s$$

What is the period?

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.828\ rad/s} = 2.221\ s$$

SHM Example

You are told the mass starts (at $t=0$) at position $x_i=1.2m$ with a velocity $v_i=3.0m/s$.

- What the position of the mass as a function of time?

$$x(t) = A\cos(\omega t + \delta) \quad \rightarrow \quad x(0) = x_i = A\cos(\delta)$$

$$v(t) = -A\omega\sin(\omega t + \delta) \quad \rightarrow \quad v(0) = v_i = -A\omega\sin(\delta)$$

Solve for A and δ:

$$\frac{v_i}{x_i} = \frac{-A\omega\sin(\delta)}{A\cos(\delta)} = -\omega\tan(\delta)$$

SHM Example

You are told the mass starts (at $t=0$) at position $x_i=1.2m$ with a velocity $v_i=3.0m/s$.

- What the position of the mass as a function of time?

$$\frac{v_i}{x_i} = -\omega\tan(\delta) \qquad \delta = \tan^{-1}\left(\frac{-3.0}{1.2\cdot 2.8}\right) = -0.2304\pi$$

Plug δ back in to one of our initial conditions to get A

$$x_i = A\cos(\delta) \qquad A = \frac{x_i}{\cos(\delta)} = \frac{1.2}{\cos(-0.2304\pi)} = 1.602$$

$$\Rightarrow \boxed{x(t) = 1.6\cos\left(2.8t - 0.23\pi\right)}$$

SHM Example

$$\boxed{x(t) = 1.6\cos\left(2.8t - 0.23\pi\right)}$$

- What is the max velocity of the mass?

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t - \delta) \qquad v_{\max} = A\omega$$

$$= 1.60 \cdot 2.83 = \boxed{4.53\ m/s}$$

- What is the max acceleration?

$$a(t) = \frac{dv}{dt} = A\omega^2\cos(\omega t - \delta) \qquad a_{\max} = A\omega^2$$

$$= 1.60 \cdot 2.83^2 = \boxed{12.8\,m/s^2}$$

Another Example

Suppose you are a given mass (0.50 kg) attached to a spring with unknown spring constant. You measure its period to be 10s and max. speed to be 2.0m/s.

- What is the spring constant?

$$\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad k = \omega^2 m$$

We do not know ω but we do know the period:

$$k = \left(\frac{2\pi}{T}\right)^2 m = \left(\frac{2\pi}{10s}\right)^2 \cdot 0.5kg = 0.198\, N/m$$

Another Example

Suppose you are a given mass (0.50 kg) attached to a spring with unknown spring constant. You measure its period to be 10s and max. speed to be 2.0m/s.

- What is max. position of the mass?

$$v(t) = \frac{dx(t)}{dt} = -A\omega\cos(\omega t + \delta) \qquad |v_{\max}| = A\omega$$

$$A = \frac{v_{max}}{\omega} = \frac{2.0\ m/s}{2\pi/10s} = 3.18\ m$$

Another Example

Suppose you are a given mass (0.50 kg) attached to a spring with unknown spring constant. You measure its period to be 10s and max. speed to be 2.0m/s.

- What is max. force on the mass?

$$a(t) = \frac{d^2x(t)}{dt^2} = A\omega^2 cos(\omega t + \delta) \qquad |a_{\max}| = A\omega^2$$

$$F_{max} = ma_{max} = mA\omega^2 = Ak = \left(3.2\ m\right)\left(0.20\ N/m\right) = 0.64N$$

Another Example

Suppose you are a given mass (0.50 kg) attached to a spring with unknown spring constant. You measure its period to be 10s and max. speed to be 2.0m/s.

- If the particle started at *x=1.2m*. What was its position at 4.0s?

$$x(0) = 1.2 = A\cos(\delta) \qquad \delta = cos^{-1}\left(\frac{1.2}{3.2}\right) = 1.186\, rad$$

$$x(t) = 3.2\cos(0.632t + 1.186) \qquad x(4) = -2.7\, m$$

Energy in SHM

What is the energy of a mass moving with simple harmonic motion?

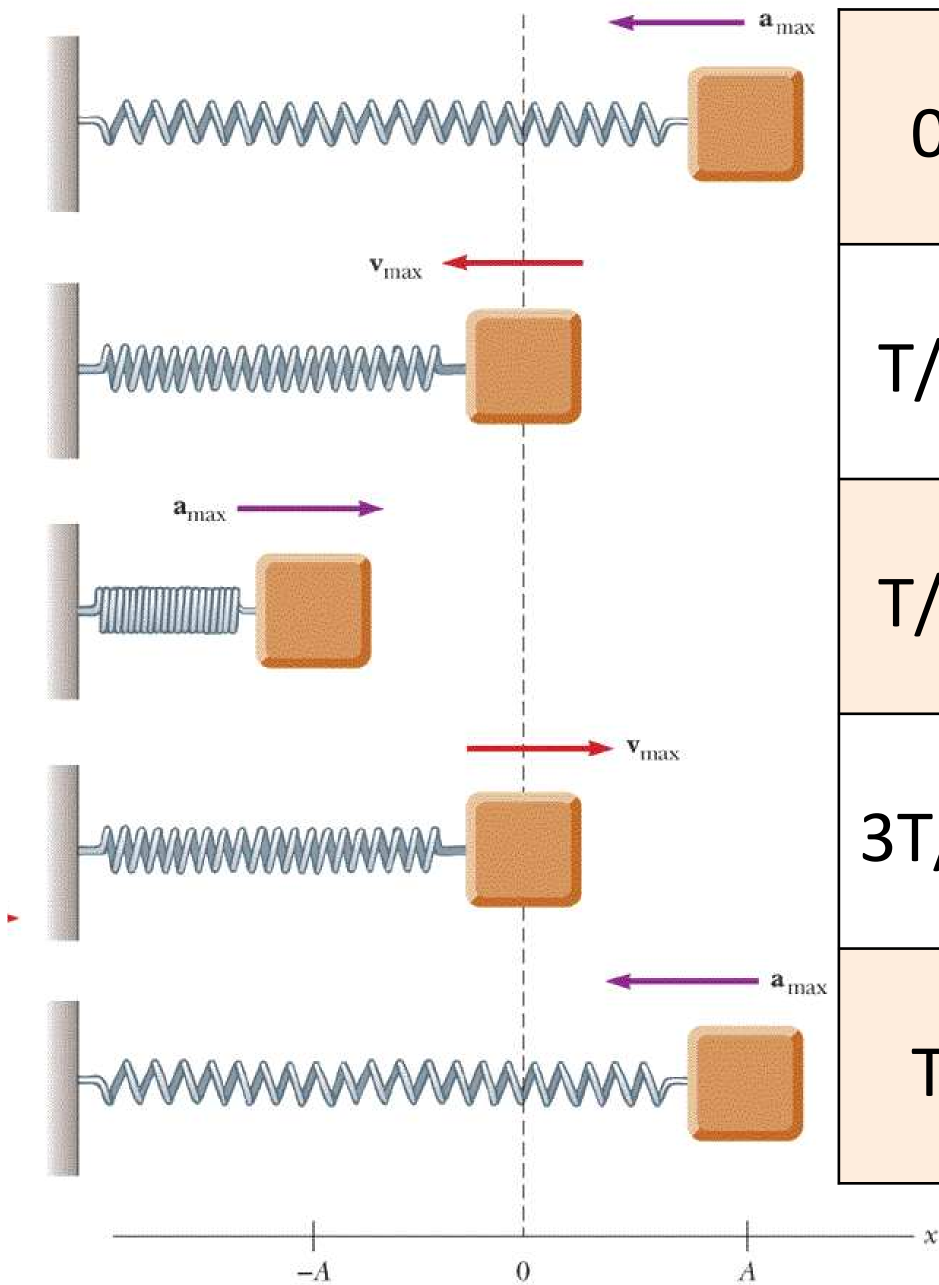
$$E = U + K$$

At zero position, the particle has maximum speed and spring is unstretched

$$E = 0 + K = \frac{1}{2}mv_{max}^2 \quad = \frac{1}{2}m\omega^2 A^2$$

$$\boxed{E = \frac{1}{2}kA^2}$$

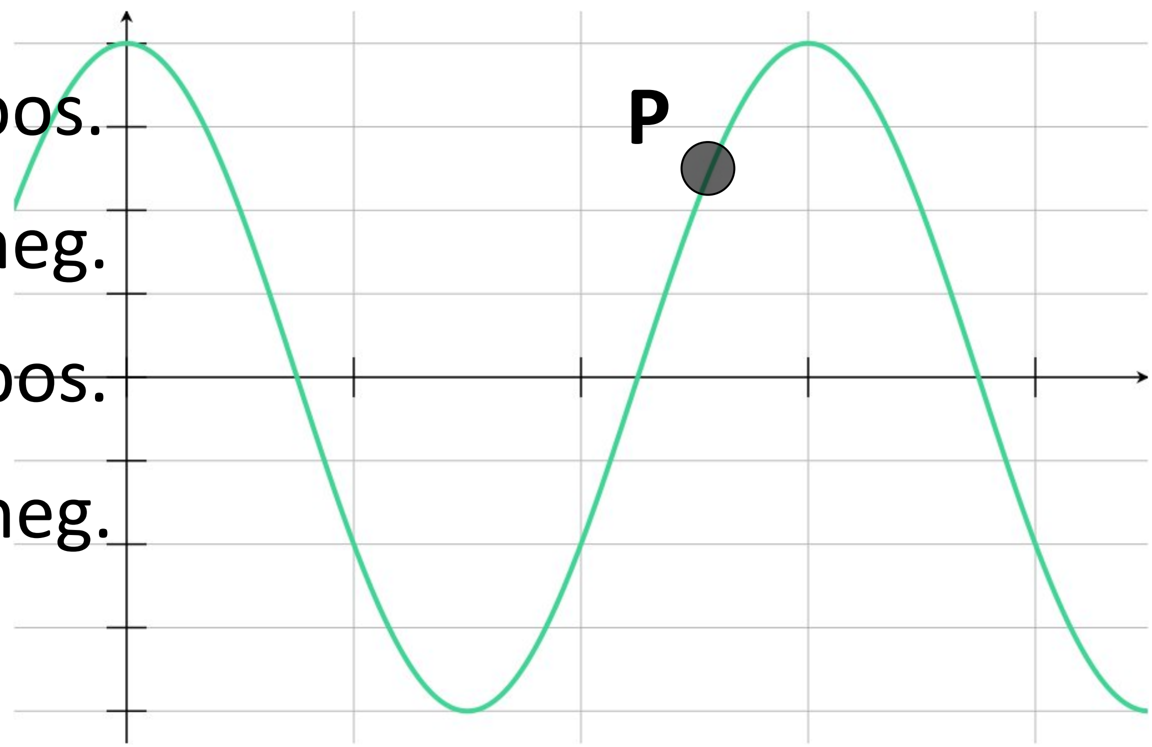
Energy is proportional to the amplitude squared!



t	x	v	a	KE	PE
0	A	0	$-A\omega^2$	0	$\frac{1}{2}kA^2$
T/4	0	$-A\omega$	0	$\frac{1}{2}kA^2$	0
T/2	-A	0	$A\omega^2$	0	$\frac{1}{2}kA^2$
3T/4	0	$A\omega$	0	$\frac{1}{2}kA^2$	0
T	A	0	$-A\omega^2$	0	$\frac{1}{2}kA^2$

Clickers

A mass attached to a spring oscillates back and forth as indicated in the position vs. time plot shown below. What is true about the velocity and acceleration at point P?



a. vel. is pos., acc. is pos.

b. vel. is pos., acc. is neg.

c. vel. is neg., acc. is pos.

d. vel. is neg., acc. is neg.

e. None of the above

Clickers

A mass attached to a spring oscillates back and forth as indicated in the position vs. time plot shown below. What is true about the velocity and acceleration at point P?

a. vel. is pos., acc. is pos.
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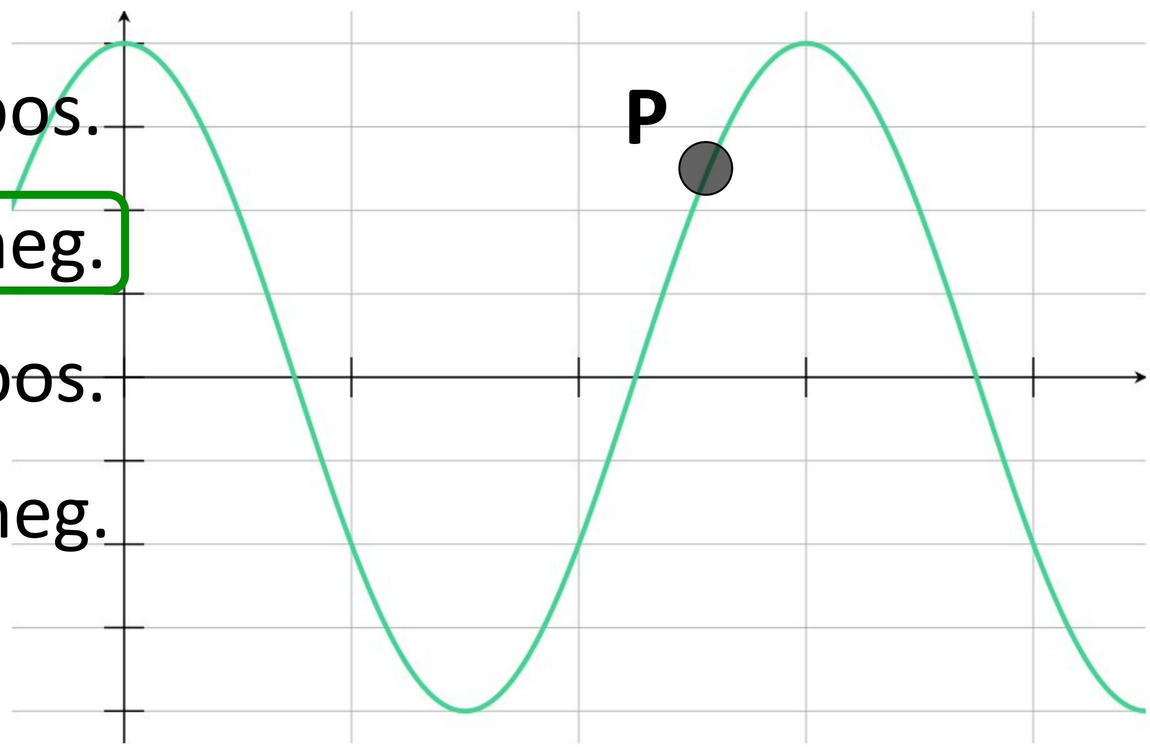
c. vel. is neg., acc. is pos.
d. vel. is neg., acc. is neg.
e. None of the above



Clickers

A mass on a spring undergoes simple harmonic motion of amplitude *A=3m.* Through what total distance does the particle move during one complete cycle of the motion?

a. 1.5m

b. 3m

c. 6m

d. 12m

Clickers

A mass on a spring undergoes simple harmonic motion of amplitude *A=3m.* Through what total distance does the particle move during one complete cycle of the motion?

a. 1.5m

b. 3m

c. 6m

d. 12m

If A=3m, then the max and min positions are at +3m and -3m. One cycle means it goes from:

Max to Zero

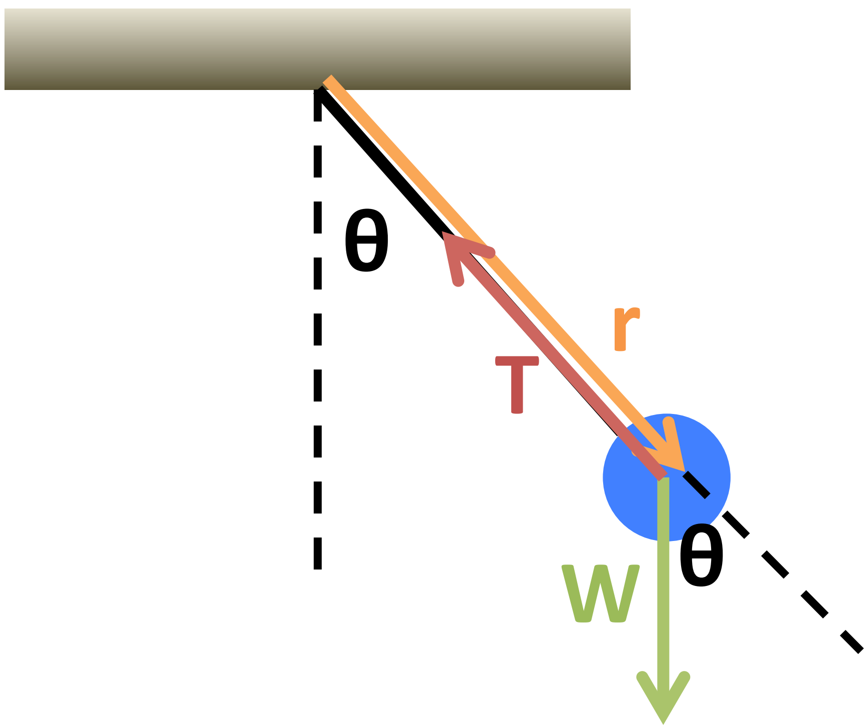
Zero to Min

Min to Zero

Zero to Max

which is a total distance of 4A = 12m.

Simple Pendulum



$$\sum \vec{\tau} = I\vec{\alpha}$$

$$\sum \vec{\tau} = \left(\cancelto{0}{\vec{r} \times \vec{T}}\right) + \left(\vec{r} \times \overrightarrow{F_g}\right)$$

$$\tau_{net} = Lmg\sin(\theta)$$

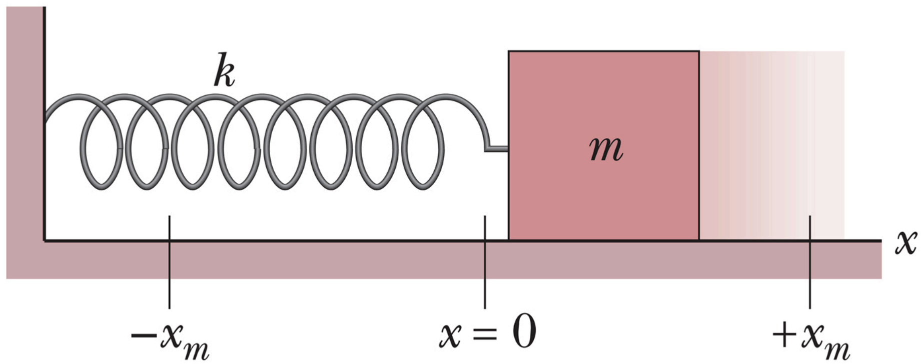
$$Lmg\sin(\theta) = I\alpha = mL^2 \cdot -\frac{d^2\theta}{dt^2}$$

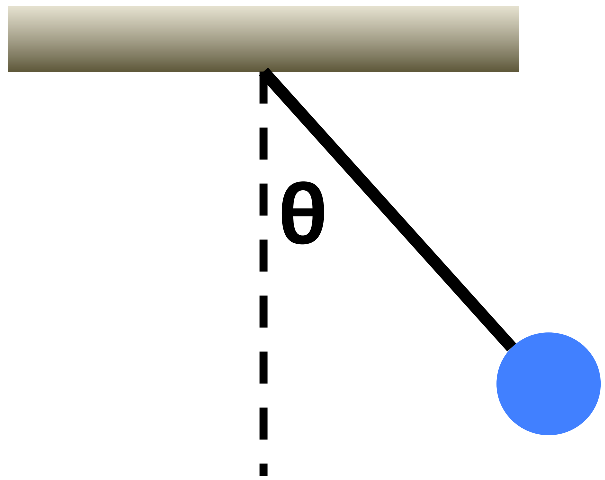
In general, this does not have a neat solution. So we say that we will keep the angle small.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$

$$\sin(\theta) \approx \theta \qquad \rightarrow \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

Pendulum vs. Spring





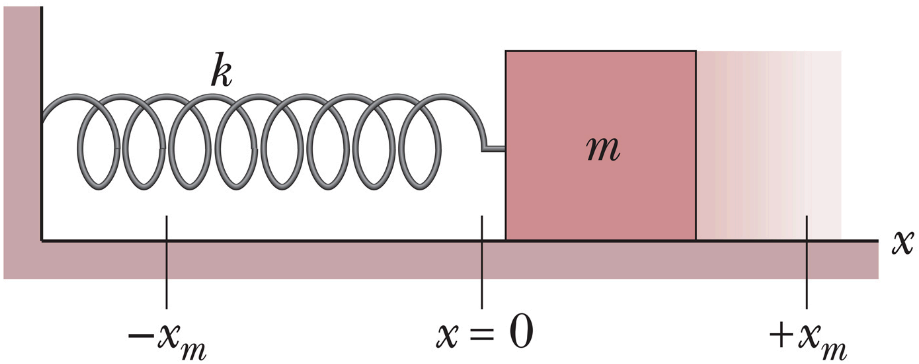
$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

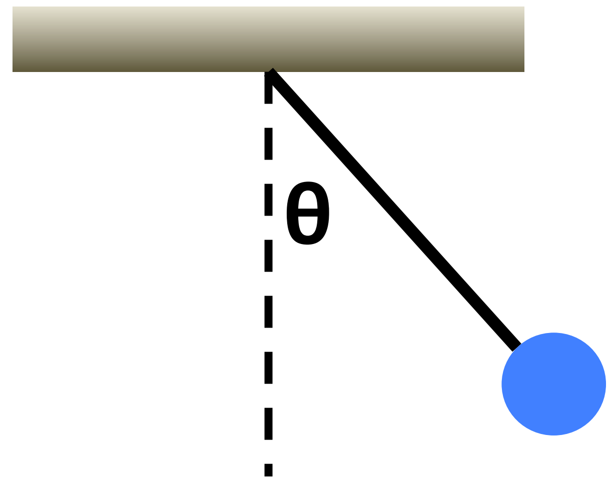
$$\frac{d^2\theta}{dt^2} = -\left(\frac{g}{L}\right)\theta$$

$$x(t) = A\cos\left(\sqrt{k/m}\,t + \delta\right)$$

$$\theta(t) = \theta_{max}\cos\left(\sqrt{g/L}\,t + \delta\right)$$

Pendulum vs. Spring





$$x(t) = A\cos\left(\sqrt{k/m}\,t + \delta\right)$$

$$\theta(t) = \theta_{max}\cos\left(\sqrt{g/L}\,t + \delta\right)$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Period of oscillations is independent of the amplitude!

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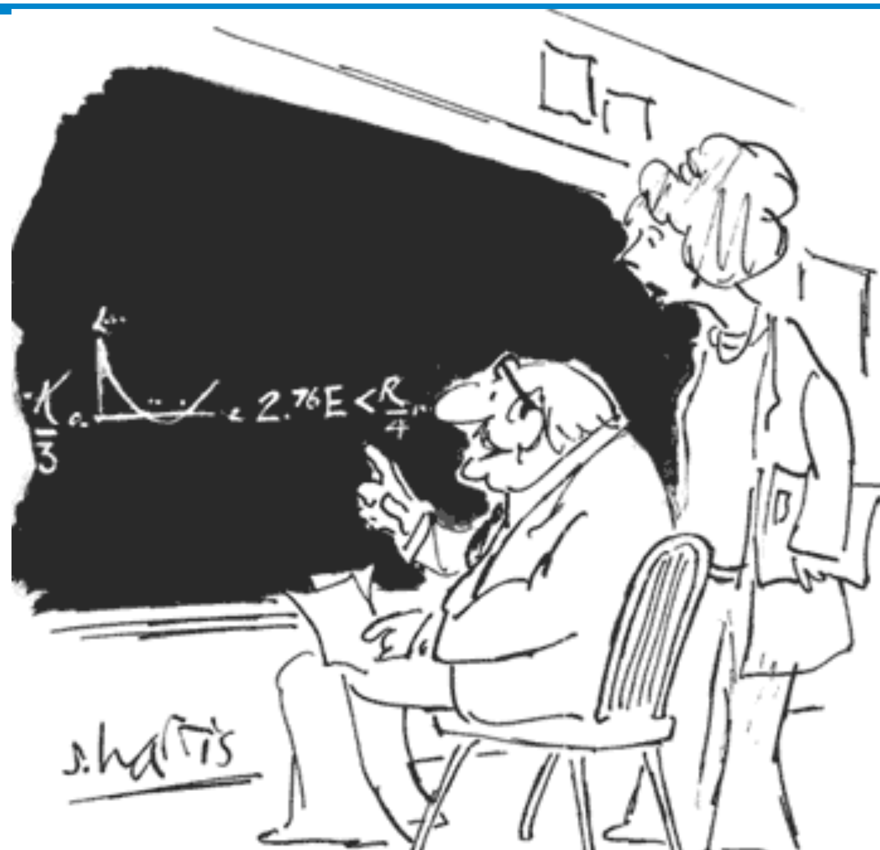


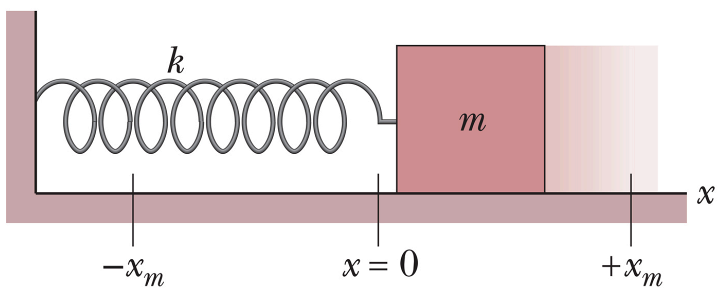
"The beauty of this is that it is only of theoretical importance, and there is no way it can be of any practical use whatsoever."

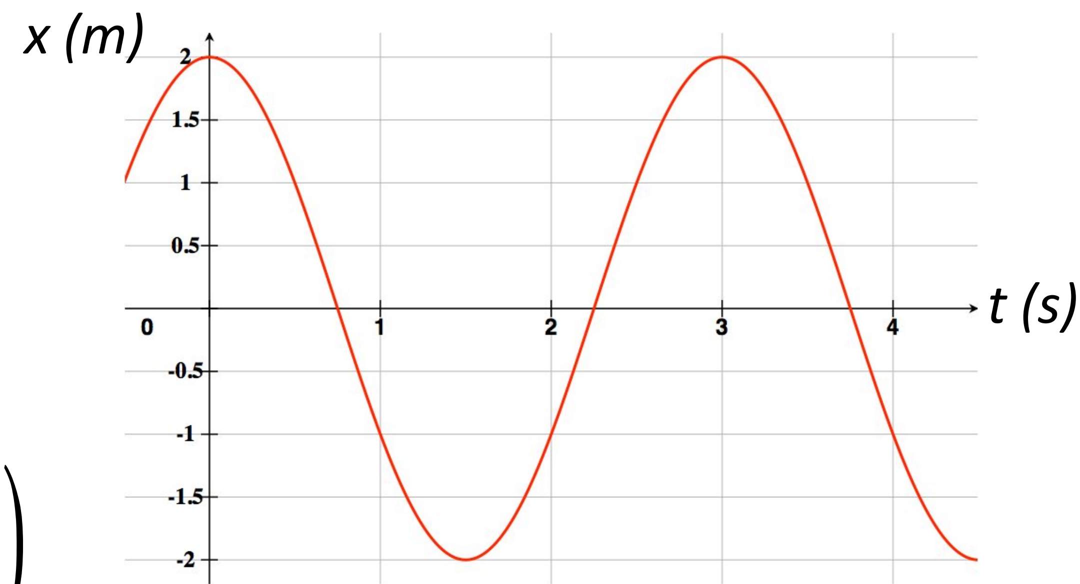
Chapter 15: Oscillations

Version 4/22/2015

Simple Harmonic Motion

Mass/Spring

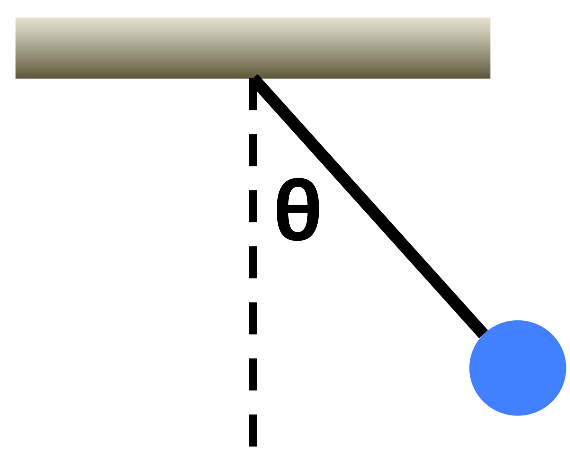




$$x(t) = A\cos\left(\sqrt{k/m}\,t + \delta\right)$$

$$F = -kx$$

Pendulum

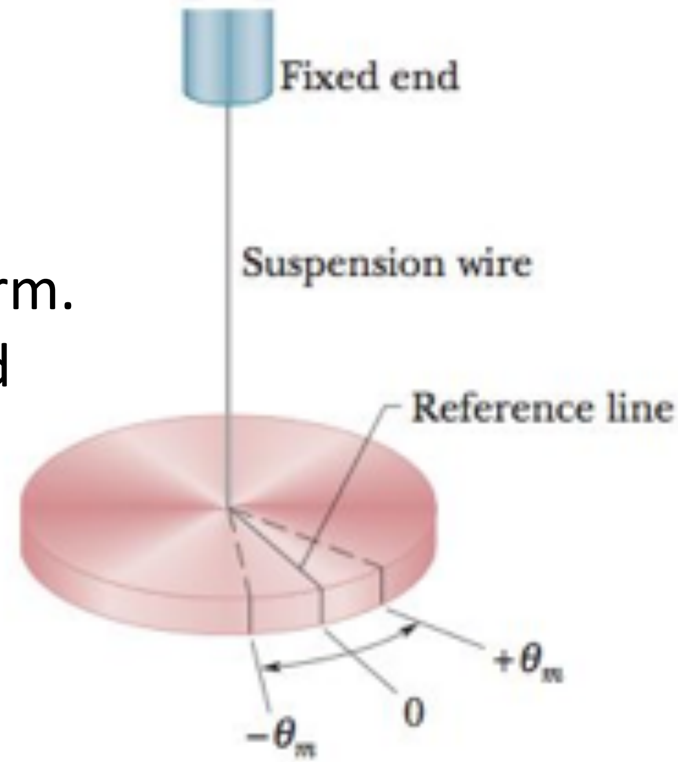


$$\tau \approx -Lmg\theta$$

$$\theta(t) = \theta_{max}\cos\left(\sqrt{g/L}\,t + \delta\right)$$

Torsion Pendulum

$$\sum \vec{\tau} = I\vec{\alpha} \qquad \tau_{net} = -\kappa\theta$$



We say that the twisting of the wire produces a torque on the disc with this form. This is true for most things like springs and wires as long as the angle is small.

$$I\alpha = -I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

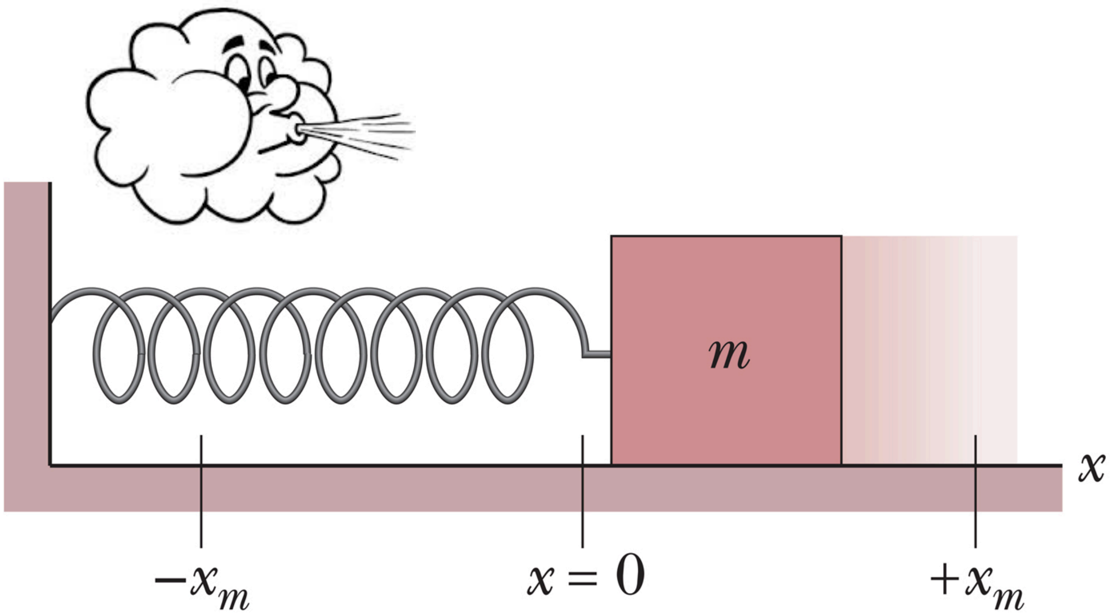
No small angle approximation is needed, unless we twist the support so much it starts to deform.

$$\rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \qquad T = 2\pi\sqrt{\frac{I}{\kappa}}$$

A torsional pendulum with linear oscillations as well:

[Wilberforce Pendulum - YouTube]

Damped Oscillations



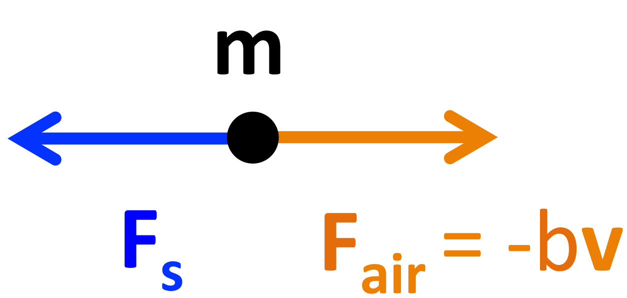
$$\sum \vec{F} = m\vec{a}$$

$$\sum F_x = ma_x$$

$$-kx - bv = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

FBD:



$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x - \left(\frac{b}{m}\right)\frac{dx}{dt}$$

Damped Oscillations

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x - \left(\frac{b}{m}\right)\frac{dx}{dt}$$

Another differential equation...

We still guess a solution, but it is probably not obvious what form for *x(t)* to guess.

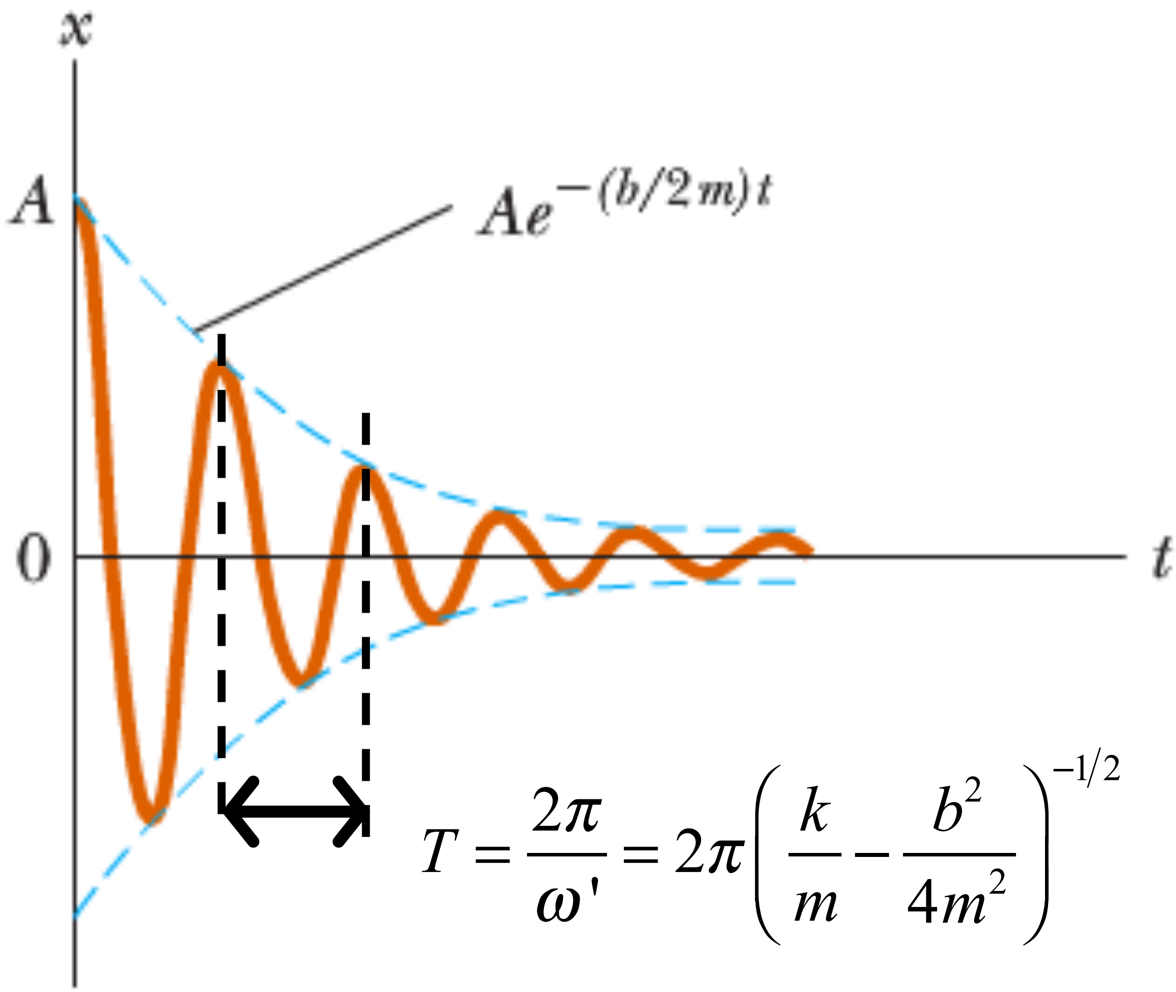
If we limit this to cases where $(b/2m)^2 < (k/m)$, then the solution is:

$$\boxed{x(t) = Ae^{-\left(b/2m\right)t}\cos(\omega' t + \delta)}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Damped Oscillations

$$x(t) = Ae^{-\left(b/2m\right)t}\cos(\omega' t + \phi)$$



$$T = \frac{2\pi}{\omega'} = 2\pi\left(\frac{k}{m} - \frac{b^2}{4m^2}\right)^{-1/2}$$

Damped Oscillations

$$x(t) = Ae^{-\left(b/2m\right)t}\cos(\omega' t + \phi) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

1. This looks like regular undamped SHM, but with an amplitude that decays in time, as we might expect.

2. If *b* is zero, then we get back the same equations as for undamped SHM.

3. Energy of the oscillator is not constant, but decays in time roughly as:

$$E \approx \frac{1}{2}k(Amplitude)^2 \approx \frac{1}{2}kA^2 e^{-\left(b/m\right)t}$$

Quality Factor: Q

$$Q = \frac{\sqrt{km}}{b}$$

- A measure of how "damped" the system is:
- High Q → Low Damping
- Dimensionless (no units)

Using $E = \frac{1}{2}kA^2 e^{-\left(b/m\right)t}$

Can show the change in energy over one cycle is

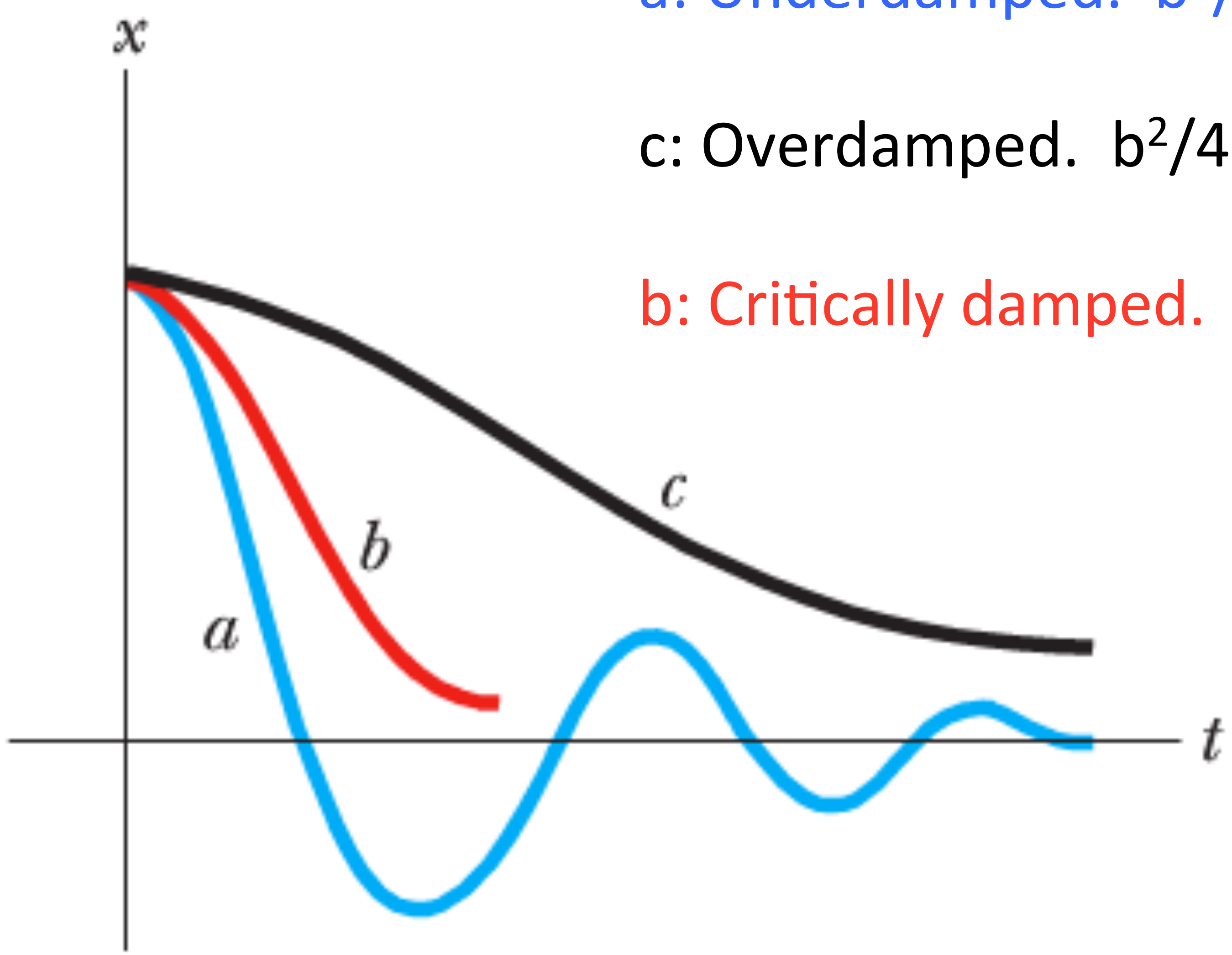
$\Delta E_{1\to 2} \approx \frac{2\pi}{Q} E_1$ High Q → Low energy loss

Damped Oscillations

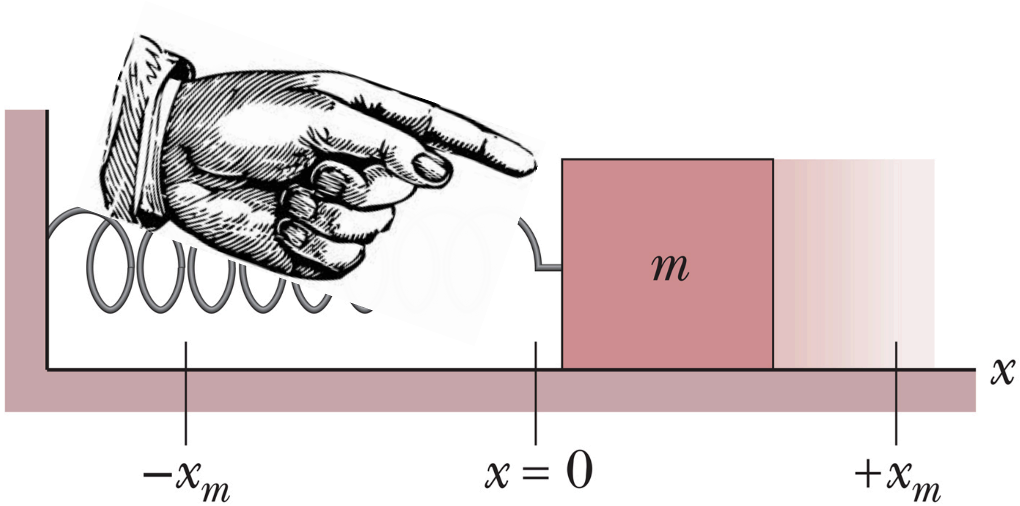
a: Underdamped. $b^2/4m^2 < k/m$

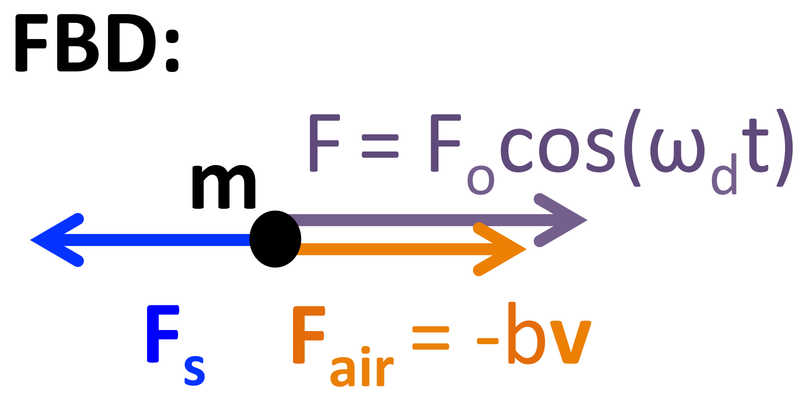
c: Overdamped. $b^2/4m^2 > k/m$

b: Critically damped. $b^2/4m^2 = k/m$



Forced Oscillations





$$\sum F_x = ma_x \qquad -kx + -bv + F_o\cos(\omega_d t) = m\frac{d^2x}{dt^2}$$

Note that there are 2 frequencies in this system:

$$\omega_d \;\; and \;\; \omega_o = \sqrt{k/m}$$

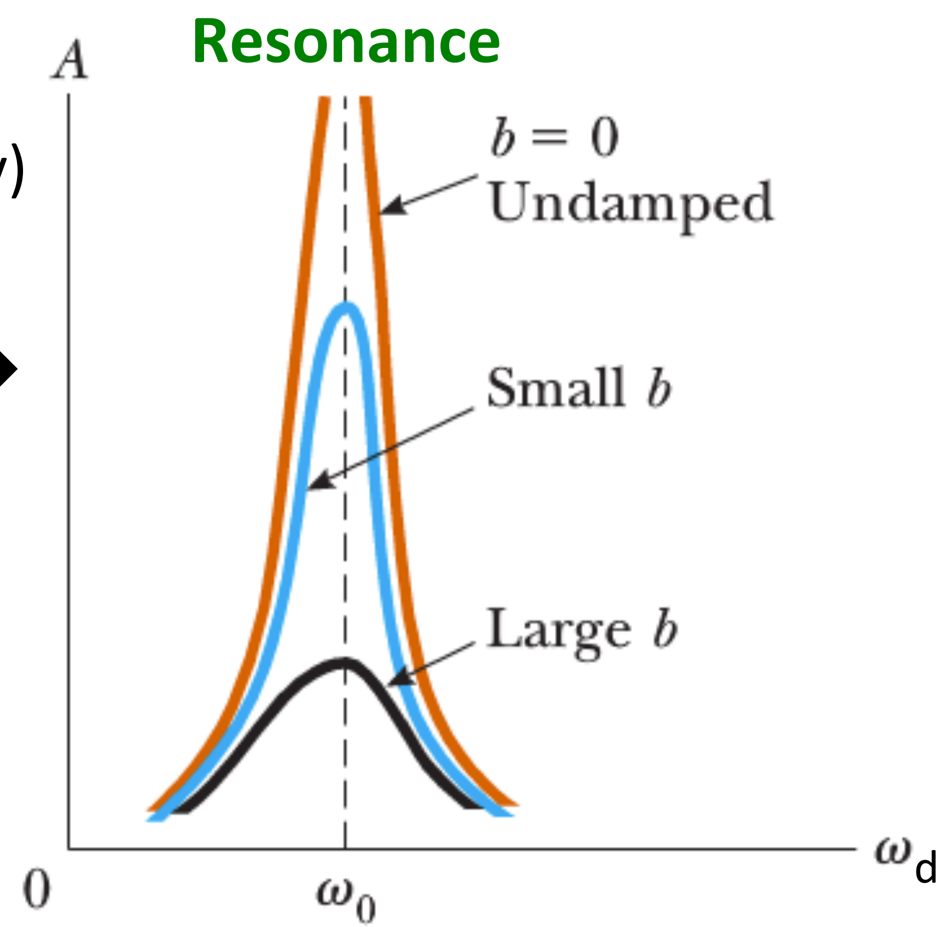
Forced Oscillations

If the external force has been applied for a long time, then *x(t)* for the mass will be:

$$x(t) = A\cos\left(\omega_d t + \delta\right)$$

(ω_d is the external driving frequency)

Amplitude depends on the external driving frequency ➔



If we drive the spring at the its "natural" frequency $\omega_o = \sqrt{k/m}$, then the amplitude of oscillation becomes large.

Amplitude ≈ Q

Tacoma Narrows Bridge

http://www.youtube.com/watch?v=j-zczJXSxnw