I. Introduction to Sequences

A. Definition: A ***sequence*** is a function f mapping elements of a domain I that is a subset of the integers to a set S, or a ***sequence*** $\equiv f : I \to S$.

The elements of S are images of the elements of the domain I. The image of the nth element of I is denoted by a_n, a ***term*** of the sequence, so $f(n) = a_n$.

Note: A sequence is listed as $\{s_n\}$, as was a set, even though the elements of a sequence are ordered.

B. Examples:

1. Consider: $f : Z^+ \to S$ where $f(n) = 2^{n-1}$

The sequence is $S = \{2^0,\ 2^1,\ 2^2,\ 2^3, ..., 2^{n-1}\}$

$= \{1,\ 2,\ 4,\ 8, ..., 2^{n-1}\}$

where: $f(n) = a_n = 2^{n-1}$

2. Consider: $f : Z^+ \to S$ where: $f(1) = 10$
$f(n) = a_n = a_{n-1} - 3$

The sequence is $S = \{10,\ 7,\ 4,\ 1,\ -2,\ -5,\ -8,\ ...\}$

where: $f(n) = a_n = a_{n-1} - 3$

3. Consider: $f : Z^+ \to S$ where: $f(n) = 3^n - 2^n$

The sequence is
$S = \{(3-2),\ (3^2 - 2^2),\ (3^3 - 2^3),\ (3^4 - 2^4),\ ...\}$

$= \{1,\ 5,\ 19,\ 65,\ ...\}$

where: $f(n) = a_n = 3^n - 2^n$

II. Special Sequences

A. *Geometric Progression*

1. Definition: A ***geometric progression*** is a sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

where a is called the ***initial term*** and r the ***common ratio*** and both $a,\ r \in \Re$.

2. Example 1:

The geometric progression is $S = \{b_n = (-1)^n\}$

where $S = \{1,\ -1,\ 1,\ -1,\ 1,\ \ldots\ (-1)^n\}$

with $f(n) = a_n = (-1)^n$

3. Example 2:

The geometric progression is $S = \{c_n = 2 \times 5^n\}$

where $S = \{2,\ 10,\ 50,\ 150,\ 1250,\ \ldots\}$

with $f(n) = a_n = 2 \times 5^n$

B. *Arithmetic Progression*

1. Definition: An ***arithmetic progression*** is a sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d\ldots,\ a+nd,\ \ldots$$

where a is called the ***initial term*** and d the ***common difference*** and both $a,\ d \in \Re$.

2. Example 1:

The arithmetic progression is $S = \{s_n = 3n + 4\}$

where: $S = \{4,\ 7,\ 10,\ 13,\ \ldots\}$

3. Example 2:

The arithmetic progression is $S = \{s_n = -n + 4\}$

where: $S = \{4,\ 3,\ 2,\ 1,\ 0,\ -1,\ -2,\ \ldots\}$

III. Recurrence Relations

A. A ***recurrence relation*** is a mapping that defines each term of the sequence in terms of the preceding term or terms.

1. Example 1: Consider the sequence S created by defining the first term as $s_0 = 10$ and with each succeeding term given by the recurrence relation $s_n = 2 \times s_{n-1} + 5$

$$S = \{10,\ 15,\ 25,\ 25,\ ...,\ 2 \times s_{n-1} + 5, ...\}$$

2. Example 2: A very well known sequence is the Fibonacci Sequence in which the first term is $f_0 = 0$ and the second term is $f_1 = 1$ and with the remaining terms defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$

$$F = \{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ ...\ f_n = f_{n-1} + f_{n-2},\ ...\}$$

3. Example 3: A sequence of factorials can be defined using a recurrence relation as:

$$f_0 = 1 \qquad \text{and} \qquad f_1 = 1$$

$$\text{and} \quad f_n = n \times f_{n-1}$$

$$F = \{1,\ 1,\ 2,\ 6,\ 24,\ 120,\ 720,\ ...\ f_n = n \times f_{n-1}, ...\}$$

B. **Formal Definition:**

A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n using previous element of the sequence a_i where $i < n$. Normally the first term, a_0 must be stated explicitly.

A sequence is called a ***solution*** of a recurrence relation if all of its terms satisfy the recurrence relation.

C. Example Recurrence Relation Problem:

Find a recurrence relation for the balance $B(k)$ owed at the end of k months on loan of amount A with an anual interest rate of r% assuming that a payment P is made each month.

1. Since the annual interest rate is r the monthly interest rate is $\frac{r}{12}$

.2. The initial amount owed in month 0 is the amount of the loan itself, or $B(0) = A$.

3. Each month the amount of the load is reduced by the amount of the payment P but increased by the interest that is accumulated each month.

4. Therefore, to compute the amount remaining for each month one must subtract the amount of the payment P that was not used to pay interest from the balance of the previous month.

5. Then the amount remaining at each month k is:

$$B(k) = B(k-1) - \left(P - \tfrac{r}{12} \times B(k-1)\right)$$

or

$$B(k) = B(k-1) \times \left(1 + \tfrac{r}{12}\right) - P$$

6. If the initial loan was for $A = \$100,000$, the monthly payment is $\$1,000$, and the annual interest rate is 10% the sequence defining the amount owed is:

$B = \{\$100,000,\ 99833.33,\ 99665.28,\ 99495.82,\ 99324.95, \ldots$

with $B(k) = 0.8333 \times (B(k-1) - \$1000.00)$

7. Under these conditions the loan will be paid off in 18 years and 8 months with a refund of $\$93.87$ (see Java program on following page).

8. Java Program Code:

```
package loanpaymnt;
import  java.util.Scanner;
public class LoanPaymnt
   {
   public static void main(String[] args)
      {
      Scanner input = new Scanner(System.in);

      System.out.print("Enter initial amout: ");
      double  Amount  =   input.nextDouble();
      System.out.print("Enter monthly payment amount: ");
      double  Payment =   input.nextDouble();

      System.out.print("Enter annual interest rate as a percent: ");
      double  Rate    =   input.nextDouble();

      long    Months  =   0;
      double  One =   1.0;
      double  Twelve = 12.0;
      double  monthlyPct  =   Rate /(Twelve * 100.0);

      System.out.printf("Monthly interest is %10.2f\n", monthlyPct);
      System.out.println("Month      Remainder   Amout to Interest");
      double  remainingAmt    =   Amount;

      while(remainingAmt > 0.0)
         {
         Months++;
         double  intrAmt =   remainingAmt * monthlyPct;
         remainingAmt = (One + monthlyPct) * remainingAmt - Payment;

         if (Months % 12 == 0)
            System.out.printf("%5d %14.2f %10.2f\n",
                        Months, remainingAmt, intrAmt);
         }
      System.out.printf("Loan payed off in %5d months\n", Months);
      }
   }
```

9. Java Program Output:

```
run:
Enter initial amout: 100000
Enter monthly payment amount: 1000
Enter annual interest rate as a percent: 10
Monthly interest is     0.01
Month    Remainder  Amout to Interest
  12     97905.74    817.40
  24     95592.18    798.28
  36     93036.36    777.16
  48     90212.92    753.83
  60     87093.82    728.05
  72     83648.11    699.57
  84     79841.60    668.11
  96     75636.49    633.36
 108     70991.05    594.97
 120     65859.17    552.56
 132     60189.92    505.70
 144     53927.02    453.94
 156     47008.32    396.76
 168     39365.13    333.60
 180     30921.61    263.81
 192     21593.94    186.73
 204     11289.54    101.57
 216       -93.87      7.49
Loan payed off in  216 months
BUILD SUCCESSFUL (total time: 16 seconds)
```

IV. Special Integer Sequences

A. Problem: Given a few terms of a sequence, how does one construct the sequence?

B. Answer 1: Compare the known terms with examples of common sequences and try for a fit.

C. Tool: Examples of Useful Sequences:

$nth\, term$	1	2	3	4	5	6	7	8	9	10
n^2	1	4	9	16	25	36	49	64	81	100
n^3	1	8	27	64	125	216	343	512	729	1000
n^4	1	16	81	256	625	1296	2401	4096	6561	10000
2^n	2	4	8	16	32	64	128	256	512	1024
3^n	3	9	27	81	243	729	2187	6561	19683	59049
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
f_n	1	1	2	3	5	8	13	21	34	55

D. Example 1: Consider the following sequence S:

$$S = \{5,\ 5,\ 10,\ 15,\ 40,\ 65,\ ...\}$$

1. It would seem that each term in the series is the sum of the previous two terms.

2. This observation suggests that S is somehow related to the Fibonacci sequence.

3. Comparing the terms with those of the Fibonacci Sequence f_n in the table above suggests that

$$s_n = 5 \times f_n$$

V. Summations

A. The summations to be considered here are the summations of terms in a sequence.

B. The notation used to describe the sum of the terms

$a_m,\ a_{m+1},\ a_{m+2}, \ldots ,\ a_n$ is: $\sum_{k=m}^{n} a_k$

where:

1. The variable j is known as the ***index of summation***.
2. The expression $k = m$ in $\sum_{k=m}^{n} a_k$ specifies the starting point of the summation or the ***lower limit*** of k.
3. The value n in $\sum_{k=m}^{n} a_k$ specifies the ***upper limit*** of k.

C. Example: $$\sum_{k=0}^{4} k! = 0! + 1! + 2! + 3! + 4!$$
$$= 1 + 1 + 2 + 6 + 24 = 34$$

VI. Proven Results of Summations

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \quad (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1} \quad (r \neq 1)$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k \quad \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1} \quad \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

VII. Problem Examples

A. Example 1: Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$

where $a_0,\ a_1,\ a_2,\ \ldots a_n$ is the sequence of real numbers.

1. The first few terms of this summantion are:
$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots$

$$= -a_0 + (a_1 - a_1) + (a_2 - a_2) + a_3 + \ldots$$
$$= a_3 - a_0 + \ldots$$

2. The subtrahend of each term cancels the addend of the previous term as in:
$$(a_2 - a_1) + (a_3 - a_2) = -a_1 + (a_2 - a_2) + a_3$$
3. Therefore, regardless of the value of n, the only two terms remaining in a summation are $-a_0$ and a_n, making the value of the summation equal to $a_n - a_0$.

B. Example 2: Use the result $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$

and the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

to compute the value of $\sum_{k=1}^{n}\frac{1}{k(k+1)}$

1. The first few terms of the summation are, using the identity given:

$$(\tfrac{1}{1} - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{2}) + (\tfrac{1}{4} - \tfrac{1}{3}) + \ldots$$

$$= \tfrac{1}{1} + (\tfrac{1}{2} - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{3}) + \; - \tfrac{1}{4} + \ldots$$

$$= \tfrac{1}{1} - \tfrac{1}{4} + \ldots$$

2. Once again, regardless of the value of n, the only two terms remaining in a summation are $\frac{1}{1} = 1$, the first, and $-\frac{1}{4}$, the last, making the value of the summation equal to $1 - \frac{1}{n}$

C. Iterative Summation Solution to the recurrence relation for the balance $B(k)$ owed at the end of k months on loan of amount A with an anual interest rate of r% assuming that a payment P is made each month (Page 155).

1. Recall: $B(k) = B(k-1) \times \left(1 + \frac{r}{12}\right) - P$

where:

a. $B(0) = A$ the original amount borrowed.

b. P $=$ the monthly payment amount.

c. $k =$ the number of months since payments started.

2. Iterative Solution

a. $B(1) = B(0) \times \left(1 + \frac{r}{12}\right) - P$

b. $B(2) = B(1) \times \left(1 + \frac{r}{12}\right) - P$

$$= \left(B(0) \times \left(1 + \frac{r}{12}\right) - P\right) \times \left(1 + \frac{r}{12}\right) - P$$

$$= B(0) \times \left(1 + \frac{r}{12}\right)^2 - P \times \left(1 + \frac{r}{12}\right) - P$$

c. $B(3) = B(2) \times \left(1 + \frac{r}{12}\right) - P$

$$= \left(B(0) \times \left(1 + \frac{r}{12}\right)^2 - P \times \left(1 + \frac{r}{12}\right) - P\right) \times \left(1 + \frac{r}{12}\right) - P$$

$$= B(0) \times \left(1 + \frac{r}{12}\right)^3 - P \times \left(1 + \frac{r}{12}\right)^2 - P \times \left(1 + \frac{r}{12}\right) - P$$

d. Proceeding onward the result would seem to be:

$$B(k) = B(0) \times \left(1 + \frac{r}{12}\right)^k - P \times \sum_{i=0}^{k-1} \left(1 + \frac{r}{12}\right)^i$$

e. We now use the Summation Identity

$$\sum_{k=0}^{n} ax^k \quad (r \neq 0) = \frac{ax^{n+1} - a}{x-1} \quad (x \neq 1)$$

with $a = 1$ and $x = 1 + \frac{r}{12}$

so that:

$$\sum_{i=0}^{k-1} \left(1 + \frac{r}{12}\right)^i = \frac{\left(1+\frac{r}{12}\right)^k - 1}{1 + \frac{r}{12} - 1} = \frac{\left(1+\frac{r}{12}\right)^k - 1}{\frac{r}{12}}$$

f. Then:

$$B(k) = B(0) \times \left(1 + \frac{r}{12}\right)^k - P\left(\frac{\left(1+\frac{r}{12}\right)^k - 1}{\frac{r}{12}}\right)$$

$$= B(0) \times \left(1 + \frac{r}{12}\right)^k - \frac{12 \times P\left(1+\frac{r}{12}\right)^k}{r} + \frac{12 \times P}{r}$$

$$= \left(1 + \frac{r}{12}\right)^k \times \left(B(0) - \frac{12 \times P}{r}\right) + \frac{12 \times P}{r}$$

D. Algebraic Solution to the Problem of finding the number of months before loan of Page 155 is paid off.

1. We must solve the equation above for the value of k such that $B(k) = 0$, or:

$$B(k) = 0 = \left(1 + \frac{r}{12}\right)^k \times \left(B(0) - \frac{12 \times P}{r}\right) + \frac{12 \times P}{r}$$

a. $$\left(1 + \frac{r}{12}\right)^k \times \left(B(0) - \frac{12 \times P}{r}\right) = -\frac{12 \times P}{r}$$

b. $$\left(1 + \frac{r}{12}\right)^k = -\frac{12 \times P}{r} \div \left(B(0) - \frac{12 \times P}{r}\right)$$

$$= -\frac{12 \times P}{r} \times \left(\frac{r}{r \times B(0) - 12 \times P}\right)$$

$$= \frac{-12 \times P}{r \times B(0) - 12 \times P}$$

c. Taking the logarithm of both sides gives:

$$k \times log\left(1 + \frac{r}{12}\right) = log\left(\frac{-12 \times P}{r \times B(0) - 12 \times P}\right)$$

or: $$k = \frac{log\left(\frac{-12 \times P}{r \times B(0) - 12 \times P}\right)}{log\left(1+\frac{r}{12}\right)}$$

d. For the example given above:

$$k = \frac{log\left(\frac{-12\times P}{r\times B(0)-12\times P}\right)}{log\left(1+\frac{r}{12}\right)} = \frac{log\left(\frac{-12000}{-11000}\right)}{log\frac{12.01}{12}}$$

$$= \frac{log\left(6.0\right)}{log(1.0083333333333333)} = \frac{0.7781512503836436}{0.0036041242688252}$$

$$= 215.90577692185983$$

which agrees with the interative solution to the recurrence relation of 216 months with a refund of \$93.87 from the last payment, of which \$7.49 was devoted to interest.

E. Algebraic Solution to the Problem of finding the required payment if the loan is to be paid off in T months.

a. $$\left(1+\frac{r}{12}\right)^T = \frac{-12\times P}{r\times B(0)-12\times P}$$

b. $$\left(r\times B(0) - 12\times P\right)\times\left(1+\frac{r}{12}\right)^T = -12\times P$$

c. $$\left(r\times B(0)\times\left(1+\frac{r}{12}\right)^T\right) - 12\times P\times\left(1+\frac{r}{12}\right)^T = -12\times P$$

d. $$r\times B(0)\times\left(1+\frac{r}{12}\right)^T = 12\times P\times\left(1+\frac{r}{12}\right)^T - 12\times P$$

$$= 12\times P\times\left(\left(1+\frac{r}{12}\right)^T - 1\right)$$

e. $$P = \frac{r\times B(0)\times\left(1+\frac{r}{12}\right)^T}{12\times\left(\left(1+\frac{r}{12}\right)^T-1\right)}$$

f. Java Code to Calculate Solution

```
package payamttmon;
import java.util.Scanner;

public class PayAmtTMon
   {
   public static void main(String[] args)
      {
      Scanner input = new Scanner(System.in);

      System.out.print("Enter initial amout: ");
      double  Amount  =   input.nextDouble();

      System.out.print("Enter annual interest rate as a percent: ");
      double  prcntRate    =   input.nextDouble();
      double  Rate     =   prcntRate/(double) 100.0;

      System.out.print("Enter number of months before loan is paid off: ");
      long    Months  =   input.nextLong();

      double  powerT  =   powIntT(Rate, Months);
      double payMent  =   (Rate * Amount * powerT) /
                        ((double) 12.0 *(powerT - (double) 1.0));

      System.out.printf("The payment required to pay off a loan of %10.2f",
            Amount);
      System.out.printf("\n in %7d months is %10.2f\n", Months, payMent);
      }

   public static double powIntT(double   Rate, long numMonths)
      {
      double  Factor  =   (double) 1.0 + Rate/(double) 12.0;
      double  Product = (double) 1.0;

      for (long i=1; i<=numMonths; i++) Product = Product * Factor;

      return Product;
      }
   }
```

f. Java Code Output

```
run:
Enter initial amout: 100000
Enter annual interest rate as a percent: 10
Enter number of months before loan is paid off: 216
The payment required to pay off a loan of  100000.00
 in     216  months is     999.84
```

VIII. Cardinality of Sets

A. The cardinality of a finite set is the number of unique elements in that set.

B. For infinite sets the definition of cardinality provides only a relative measure of size.

C. Two sets A and B have the same cardinality if and only if there is a one-to-one correspondence between the elements of A and the elements of B. When A and B have the same cardinality we write: $|A| = |B|$

IX. Countable Sets

A. Definitions:

1. A set A that is either finite or has the same cardinality as the set of postive integers Z^+ is called ***countable***.

2. A set that is not countable is called ***uncountable***.

3. When an infinite set S is countable the cardinality of S is denoted by $\aleph_0$ (pronounced aleph null), or $|S| = \aleph_0$

B. Example Problem 1: Show that the set of all integers is countable.

To show that the set of all integers Z is countable we must demonstrate a one-to-once correspondence between the elements of Z and the elements of Z^+.

1. One way of doing this would be to start at 0 and alternately associate positive and negative integers with consecutive positive integers.

2. The list and the correspondence would be:

Z^+	0	1	2	3	4	5	6	7	8	9	10	11	12
Z	0	-1	1	-2	2	-3	3	-4	4	-5	5	-6	6

C. Alternative Solutions to Problem 1: Show that the set of all integers is countable.

1. Consider the functions :

   a. $f(n) = \frac{n}{2}$ for even n

   b. $g(n) = \frac{-(n-1)}{2}$ for odd n.

2. $f(n) = \frac{n}{2}$ generates the list:

$$f(2) = 1,\ f(4) = 2,\ f(6) = 3,\ f(8) = 4,\ ....$$

   and associates each consecutive positive integer with a single even element of Z^+.

3. $g(n) = \frac{-(n-1)}{2}$ generates the list:

$$g(1) = 0,\ g(3) = -1,\ g(5) = -2,\ g(7) = -3, ...$$

   and associates each consecutive negative integer with a single odd element of Z^+.

4. Taken together $f(n)$ and $g(n)$ form a one-to-one and onto mapping between Z and Z^+, demonstrating that Z is countable.

C. Example Problem 2: Determine if the set $A \times Z^+$ is countable, where $A = \{2, 3\}$.

1. Starting with $1 \in Z^+$ we have that $A \times Z^+ = \{\{2, 1\}, \{3, 1\}, \{2, 2\}, \{3, 2\}, ...\}$

2. Therefore, for each $i \in Z^+$ we have two subsets of $A \times Z^+$, namely $\{2, i\}$ and $\{3, i\}$.
3. We can list these subset in order as follows:
   $1.$ $\{2, 1\}$ $2.$ $\{3, 1\}$ $3.$ $\{2, 2\}$
   $4.$ $\{3, 2\}$ $5.$ $\{2, 3\}$ $6.$ $\{3, 3\}$ ...

4. We have, then, established a $1 - 1$ correspondence between the elements of $A \times Z^+$ and the elements of Z^+.
5. Therefore, $A \times Z^+$ is countable.

D. Example Problem 3: Show that the union of a countable number of countable sets is countable.

1. We are trying to prove that $\bigcup_{i=1}^{n} A_i$ is countable when all of the A_i are countable.
   a. We are assuming that all of the A_i are disjoint and countable.
   b. If they are not disjoint, we replace each A_i with $A_i - A_j$, $\forall j \neq i$

2. Since each of the A_i are countable, finite or infinite, we can list the elements of that A_i as $A_{i1},\ A_{i2},\ A_{i3},\ A_{i4},\ \ldots$

3. Since each of the A_i are countable, finite or infinite, we can also state that $|A_i| = j$ where j is some integer.

4. We can then list the elements of $\bigcup_{i=1}^{n} A_i$ as
   $A_{11},\ A_{12},\ A_{13}, ...,\ A_{1j},\ A_{21},\ A_{22},\ A_{23},\ ...,\ A_{2k}, ...,$
   $A_{i1},\ A_{i2},\ A_{i3},\ ..., A_{im},\ ..., A_{nq}$
   where $|A_1| = j,\ |A_2| = k,\ ...,\ |A_i| = m,$ ... and $|A_n| = q$

5. Since we can create a list including each element of $\bigcup_{i=1}^{n} A_i$ we can assign consecutive increasing integers to each position in the list.

6. Therefore: $\bigcup_{i=1}^{n} A_i$ is a countable set.

D. Corollary to the solution to previous problem:

1. Since we can then list the elements of $\bigcup_{i=1}^{n} A_i$, each one of which is countable, as

   $A_{11},\ A_{12},\ A_{13}, ...,\ A_{1j},\ A_{21},\ A_{22},\ A_{23},\ ...,\ A_{2k}, ...,\ A_{i1},\ A_{i2},\ A_{i3},\ ..., A_{im},\ ..., A_{nq}$

   and we can assign increasing integers to each position in the list we know that $\bigcup_{i=1}^{n} A_i$ is a countable set.

2. We know that: $\left|\bigcup_{i=1}^{n} A_i\right| = j + k + ... + m + ... + q$

3. Therefore, the cardinality of a countable set $\bigcup_{i=1}^{n} A_i$, an integer, can be represented as a sum of the cardinalities of countable sets.

4. Therefore, a set such as $\bigcup_{i=1}^{n} A_i$ whose cardinality can be represented as the sum of the cardinalities of countable sets must also be be a countable set.

X. Uncountable Sets

Classic Example: The set of all real numbers is uncountable.

A. We assume that the real numbers are countable and create a countable list of all distinct real numbers $r,\ 0 \le r \le 1$ as shown below:

1. $r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}...$
2. $r_2 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}...$
3. $r_3 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}...$
4. $r_4 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}...$

.

.

.

$n.$ $r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}...$

where $d_{ij} \in \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$

B. Let $R^* = \{r \mid 0 \le r \le 1\}$ as constructed above.

C. Then, from each ith element in our list we select d_{ii} and create a real number $r_{n+1} = d_{11}d_{22}d_{33}d_{44}....$

with the restriction that if $d_{ii} \neq 4$ then $d_{ii} = 4$ and if $d_{ii} = 4$ then $d_{ii} = 5$.

D. Therefore, after assuming that our list of real numbers includes all $r,\ 0 \leq r \leq 1$, we have contructed an x which is not included in our list.

E. Therefore $x \notin R^*$ and R^* does not include all $r,\ 0 \leq r \leq 1$

F. If we set $R_1^* = R^* \cup \{x\}$ we can repeat the process with the same result.

F. Therefore our assumption that we can generate a complete list of all real numbers $r,\ 0 \leq r \leq 1$, has led to a contradiction.

G. Therefore, the set of all real numbers $R^* = \{r \,|\, 0 \leq r \leq 1\}$ is uncountable.

H. Since $R^* \subset R$ we have shown that R has an uncountable subset.

I. Therefore, since $|R^*| \leq |R|$, R is uncountable.

XI. Schroeder-Bernstein Theorem

A. Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

B. Restatement: If there are one-to-one functions $f : A \to B$ between the elements of A and those of B and $g : B \to A$ then there is a one-to-one correspondence between A and B.

XII. Recall: Definitions of Open and Closed Intervals

A. Definitions: Intervals for real numbers are defined as follows for the real numbers a and b with $a \leq b$:

1. $[a,\, b] = \{x \,|\, a \leq x \leq b\}$ is called the ***closed*** interval between a and b.

2. $[a,\, b) = \{x \,|\, a \leq x < b\}$

3. $(a,\, b] = \{x \,|\, a < x \leq b\}$

4. $(a,\, b) = \{x \,|\, a < x < b\}$ is called the ***open*** interval between a and b.

B. Note:

1. $(a,\, b) \subset [a,\, b]$, $[a,\, b) \subset [a,\, b]$, and $(a,\, b] \subset [a,\, b]$

2. $(a,\, b) \subset (a,\, b]$ and $[a,\, b)$

XIII. Application of the Schroder-Bernstein Theorem

A. Theorem: $|(0,\, 1)| = |(0,\, 1]|$

B. Proof:

1. $(0,\, 1) \subset (0,\, 1]$

2. Therefore: $f(x) = x$ is a one-to-one correspondence from $(0,\, 1)$ to $(0,\, 1]$

3. The function $g(x) = \frac{x}{2}$ is one-to-one and maps $(0,\, 1]$ to $(0,\, \frac{1}{2}] \subset (0,\, 1)$

4. Since we have found one-to-one functions from $(0,\, 1)$ to $(0,\, 1]$ and from $(0,\, 1]$ to $(0,\, 1)$ we have, according to the Schroder-Bernstein Theorem, proven that $|(0,\, 1)| = |(0,\, 1]|$