I. Special Compound Propositions

A. Tautology

1. Definition: A compound proposition that is always ***true***, regardless of the truth values of the component propositions is a tautology.

2. Note: No atomic sentence P can be a tautology. A given interpretation can make P ***true***, another make P ***false***.

3. Examples:

a. $P \vee \neg P$ Truth Table:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

b. $P \rightarrow (P \vee Q)$

Truth Table:

P	Q	$P \vee Q$	$P \rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

c. $\neg(P \wedge \neg P)$

Truth Table:

P	$\neg P$	$P \wedge \neg P$	$\neg(P \wedge \neg P)$
T	F	F	T
F	T	F	T

Note that the above example is simply the negation of the contradiction presented on the next page.

The negation of a contradiction is a tautology.

B. Contradiction

1. Definition: A compound proposition that is always ***false***, regardless of the truth values of the component propositions is a contradition.

2. Examples:

a. $P \wedge \neg P$ Truth Table:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

b. $P \leftrightarrow \neg P$

Truth Table:

P	$\neg P$	$P \rightarrow \neg P$	$\neg P \rightarrow P$	$(P \rightarrow \neg P) \wedge (\neg P \rightarrow P)$	$P \leftrightarrow \neg P$
T	F	F	T	F	F
F	T	T	F	F	F

C. Paradox: "*This sentence is false.*" is **NOT** a proposition.

1. If you assume that the content of the sentence is true the conclusion is that the truth value is false.

2. If you assume that the sentence is false the conclusion is that the truth value is true.

3. This example is called a paradox and is not a proposition, because it is neither true nor false.

II. Logical Equivalence

A. Defintion:

Two compound propositions which have identical truth tables are logically equivalent.

If the compound propositions X and Y are logically equivalent we designate this status by writing: $X \equiv Y$

B. Examples:

1. Two compound propositions $X = P \to Q$ and $Y = \neg Q \to \neg P$ are logically equivalent if the compound proposition $X \leftrightarrow Y$ is a tautology.

   Here $X \leftrightarrow Y = (P \to Q) \leftrightarrow (\neg Q \to \neg P)$

   Truth Table:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$	$(P \to Q) \leftrightarrow (\neg Q \to \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

2. Two tautologies are logically equivalent.

3. Two contradictions are logically equivalent.

C. Truth Tables

1. A common method for demonstrating logical equivalence is the truth table.

2. Example:
   The Associative Rule: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

P	Q	R	$P \vee Q$	$Q \vee R$	$(P \vee Q) \vee R$	$P \vee (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

III. Important Logical Equivalences

A. Equivalences Involving the Usual Compound Statements

Equivalence	Name
$P \wedge \boldsymbol{T} \equiv P$	Identity Laws
$P \wedge \boldsymbol{F} \equiv \boldsymbol{F}$	
$P \vee \boldsymbol{T} \equiv \boldsymbol{T}$	Domination Laws
$P \vee \boldsymbol{F} \equiv P$	
$P \vee P \equiv P$	Idempotent Laws
$P \wedge P \equiv P$	
$\neg(\neg P) \equiv P$	Double Negation Law
$P \vee Q \equiv Q \vee P$	Commutative Laws
$P \wedge Q \equiv Q \wedge P$	
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (Q \vee R)$	Distributive Laws
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (Q \wedge R)$	
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	
$P \vee (P \wedge Q) \equiv P$	Absorption Laws
$P \wedge (P \vee Q) \equiv P$	
$P \vee \neg P \equiv \boldsymbol{T}$	Negation Laws
$P \wedge \neg P \equiv \boldsymbol{F}$	

B. Logical Equivalences Involving Conditional Statements

1. $P \rightarrow Q \equiv \neg P \vee Q$
2. $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
3. $P \vee Q \equiv \neg P \rightarrow Q$
4. $P \wedge Q \equiv \neg(P \rightarrow \neg Q)$
5. $\neg(P \rightarrow Q) \equiv P \vee \neg Q$
6. $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$
7. $(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$
8. $(P \rightarrow Q) \vee (P \rightarrow R) \equiv P \rightarrow (Q \vee R)$
9. $(P \rightarrow R) \vee (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$

C. Logical Equivalences Involving Biconditional Statements

1. $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
2. $P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$
3. $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$
4. $\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$

IV. Logical Equivalences and Truth Tables

A. Problem: Show that $(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$ is a tautology

B. Truth Table Solution:
To save space let: $Y = (P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$

P	Q	$P \vee Q$	$\neg P$	R	$\neg P \vee R$	$(P \vee Q) \wedge (\neg P \vee R)$	$Q \vee R$	Y
T	T	T	F	T	T	T	T	T
T	T	T	F	F	F	F	T	T
T	F	T	F	T	T	T	T	T
T	F	T	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	T	T	F	T	T	T	T
F	F	F	T	T	T	F	T	T
F	F	F	T	F	T	F	F	T

1. The above truth table shows that; for all combinations of truth values for P, Q, and R; the compound proposition
$$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$$
has truth value ***True***.

2. Hence, the compound proposition
$$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$$
is a tautology.

3. The truth table method is straight-forward:
   a. Create all possible combinations of truth values for the atomic propositions involved.

   b. Generate the truth value for each compound component proposition from the truth values of the atomic propositions used in the compound proposition.

   c. Repeat step b. until a truth value for the final compound proposition is computed.

4. In this case:
   a. All combinations of truth values for P, Q, and R are specified.

   b. The truth values for $\neg P$ corresponding to each truth value for P are computed.

   c. The truth values for $(P \vee Q)$, $(\neg P \vee R)$, and $(Q \vee R)$ corresponding to each truth value for P, Q, and R are computed.

   d. The truth values for $(P \vee Q) \wedge (\neg P \vee R)$ corresponding to each truth value for P, Q, and R are computed.

   e. The truth values for $(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R)$ corresponding to each truth value for P, Q, and R are computed.

   f. The final truth values computed in step (e) are all ***True***, demonstrating that $(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R)$ is a tautology.

C. Solution Using Equivalences:

1. Strategy:
   a. We are seeking to show that
      $$(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R)$$
      is a tautology.

   b. Hence we want to reduce
      $$(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R)$$
      into a form such as:
      i. $X \vee \boldsymbol{T} \equiv \boldsymbol{T}$ Identity Law

      or

      ii. $X \vee \neg X \equiv \boldsymbol{T}$ Negation Law

2. Begin with the initial compound proposition:

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise

3. Convert the implication into a form that produces more negations in the compound proposition.

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \to Q \equiv \neg P \vee Q$

4. Apply De Morgan's Law

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \to Q \equiv \neg P \vee Q$
$\Big[\neg(P \vee Q) \vee \neg(\neg P \vee R)\Big] \vee (Q \vee R)$	De Morgan's Law

5. Apply Associative Law to simplify:

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \to Q \equiv \neg P \vee Q$
$\Big[\neg(P \vee Q) \vee \neg(\neg P \vee R)\Big] \vee (Q \vee R)$	De Morgan's Law
$\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee Q \vee R$	Associative Law

6. Apply De Morgan's and Double Negation Laws:

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \to Q \equiv \neg P \vee Q$
$\Big[\neg(P \vee Q) \vee \neg(\neg P \vee R)\Big] \vee (Q \vee R)$	De Morgan's Law
$\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee Q \vee R$	Associative Law
$(\neg P \wedge \neg Q) \vee (\neg\neg P \wedge \neg R) \vee Q \vee R$	De Morgan's Law
$(\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee Q \vee R$	Double Negation Law

7. Apply Associative Law to rearrange propositions.

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \rightarrow Q \equiv \neg P \vee Q$
$\Big[\neg(P \vee Q) \vee \neg(\neg P \vee R)\Big] \vee (Q \vee R)$	De Morgan's Law
$\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee Q \vee R$	Associative Law
$(\neg P \wedge \neg Q) \vee (\neg\neg P \wedge \neg R) \vee Q \vee R$	De Morgan's Law
$(\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee Q \vee R$	Double Negation Law
$Q \vee (\neg P \wedge \neg Q) \vee R \vee (P \wedge \neg R)$	Associative Law

8. Apply Distributive Law to create instances of $(X \vee \neg X)$ and, then, the Negation and Domination Laws to simplify.

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \rightarrow Q \equiv \neg P \vee Q$
$\Big[\neg(P \vee Q) \vee \neg(\neg P \vee R)\Big] \vee (Q \vee R)$	De Morgan's Law
$\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee Q \vee R$	Associative Law
$(\neg P \wedge \neg Q) \vee (\neg\neg P \wedge \neg R) \vee Q \vee R$	De Morgan's Law
$(\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee Q \vee R$	Double Negation Law
$Q \vee (\neg P \wedge \neg Q) \vee R \vee (P \wedge \neg R)$	Associative Law
$\Big[(Q \vee \neg P) \wedge (Q \vee \neg Q)\Big] \vee \Big[(R \vee P) \wedge (R \vee \neg R)\Big]$	Distributive Law
$\Big[(Q \vee \neg P) \wedge \boldsymbol{T}\Big] \vee \Big[(R \vee P) \wedge \boldsymbol{T}\Big]$	Negation Law
$(Q \vee \neg P) \vee (R \vee P)$	Domination Law

9. Apply Associative Law three times to simplify and rearrange propositions.

Proposition	Equivalence
$(P \vee \ ) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise
$\neg\left[(P \vee Q) \wedge (\neg P \vee R)\right] \vee (Q \vee R)$	$P \to Q \equiv \neg P \vee Q$
$\left[\neg(P \vee Q) \vee \neg(\neg P \vee R)\right] \vee (Q \vee R)$	De Morgan's Law
$\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee Q \vee R$	Associative Law
$(\neg P \wedge \neg Q) \vee (\neg\neg P \wedge \neg R) \vee Q \vee R$	De Morgan's Law
$(\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee Q \vee R$	Double Negation Law
$Q \vee (\neg P \wedge \neg Q) \vee R \vee (P \wedge \neg R)$	Associative Law
$\left[(Q \vee \neg P) \wedge (Q \vee \neg Q)\right] \vee \left[(R \vee P) \wedge (R \vee \neg R)\right]$	Distributive Law
$\left[(Q \vee \neg P) \wedge \boldsymbol{T}\right] \vee \left[(R \vee P) \wedge \boldsymbol{T}\right]$	Negation Law
$(Q \vee \neg P) \vee (R \vee P)$	Domination Law
$Q \vee \neg P \vee R \vee P$	Associative Law
$R \vee Q \vee \neg P \vee P$	Associative Law
$(R \vee Q) \vee (\neg P \vee P)$	Associative Law

10. Finally, apply the Identity Law and the Domination Law to get the final table shown on the following page.
    a. The final truth value is $\boldsymbol{T}$, for ***True***.

    b. This value demonstates that the initial compound proposition, i.e., $(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$, has truth value ***True*** regardless of the truth values of the individual atomic propositions that compose it.

    c. Therefore: $(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R) \equiv \boldsymbol{T}$

    d. Therefore:

       $(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$ is a tautology.

Proposition	Equivalence
$(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$	Initial Premise
$\neg\Big[(P \vee Q) \wedge (\neg P \vee R)\Big] \vee (Q \vee R)$	$P \to Q \equiv \neg P \vee Q$
$\Big[\neg(P \vee Q) \vee \neg(\neg P \vee R)\Big] \vee (Q \vee R)$	De Morgan's Law
$\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee Q \vee R$	Associative Law
$(\neg P \wedge \neg Q) \vee (\neg\neg P \wedge \neg R) \vee Q \vee R$	De Morgan's Law
$(\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee Q \vee R$	Double Negation Law
$Q \vee (\neg P \wedge \neg Q) \vee R \vee (P \wedge \neg R)$	Associative Law
$\Big[(Q \vee \neg P) \wedge (Q \vee \neg Q)\Big] \vee \Big[(R \vee P) \wedge (R \vee \neg R)\Big]$	Distributive Law
$\Big[(Q \vee \neg P) \wedge \boldsymbol{T}\Big] \vee \Big[(R \vee P) \wedge \boldsymbol{T}\Big]$	Negation Law
$(Q \vee \neg P) \vee (R \vee P)$	Domination Law
$Q \vee \neg P \vee R \vee P$	Associative Law
$R \vee Q \vee \neg P \vee P$	Associative Law
$(R \vee Q) \vee (\neg P \vee P)$	Associative Law
$(R \vee Q) \vee T$	Identity Law
T	Domination Law

V. Comparison of Solution Methodologies:

A. Truth Tables

1. N propositions in a compound proposition requires 2^N rows in the truth table to represent all possible truth values for the compound proposition.

2. $(P \to Q) \wedge (R \vee S) \wedge (T \oplus U)$ requires $2^8 = 64$ rows.

3. Logic is straightforward - Computers can be programmed to process truth tables.

B. Equivalence Solutions

1. Solution length variable.

2. Number of statements equal to number of substitutions used.

VI. Satisfiability

A. Definition: A compound proposition is ***satisfiable*** if there is at least one assignment of truth values to its atomic propositions that give the compound proposition a truth value of ***True***.

B. Example 1: $X \equiv (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$

1. Truth Table:

P	$\neg P$	Q	$\neg Q$	$(P \vee \neg Q)$	$(\neg P \vee Q)$	$(\neg P \vee \neg Q)$	X
T	F	T	F	T	T	F	F
T	F	F	T	T	F	T	F
F	T	T	F	F	T	T	F
F	T	F	T	T	T	T	T

2. The interpretation: $P = F$ and $Q = F$ results is a truth value of ***T***, or ***True***, for the compound proposition

$$X \equiv (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$$

C. Example 2: $(P \leftrightarrow Q) \wedge (\neg P \leftrightarrow Q)$

1. Truth Table:

P	$\neg P$	Q	$(P \leftrightarrow Q)$	$(\neg P \leftrightarrow Q)$	$(P \leftrightarrow Q) \wedge (\neg P \leftrightarrow Q)$
T	F	T	T	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	T	F	F

2. No interpretation generates a truth value of ***True*** for the compound proposition $(P \leftrightarrow Q) \wedge (\neg P \leftrightarrow Q)$

3. Hence: $(P \leftrightarrow Q) \wedge (\neg P \leftrightarrow Q)$ is unsatisfiable.