C. Then, from each ith element in our list we select d_{ii} and create a real number $r_{n+1} = d_{11}d_{22}d_{33}d_{44}....$

with the restriction that each d_{ii} be set equal to a_{ii} where $a_{ii} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $a_{ii} \neq d_{ii}$

D. Therefore, after assuming that our list of real numbers includes all $r,\ 0 \leq r \leq 1$, we have contructed an $x \in r$ which is not included in our list.

E. Therefore $x \notin R^*$ and R^* does not include all $r,\ 0 \leq r \leq 1$

F. If we set $R_1^* = R^* \cup \{x\}$ we can repeat the process with the same result.

F. Therefore our assumption that we can generate a complete list of all real numbers $r,\ 0 \leq r \leq 1$, has led to a contradiction.

G. Therefore, the set of all real numbers $R^* = \{r \,|\, 0 \leq r \leq 1\}$ is uncountable.

H. Since $R^* \subset R$ we have shown that R has an uncountable subset.

I. Therefore, since $|R^*| \leq |R|$, R is uncountable.

XI. Schroeder-Bernstein Theorem

A. Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

B. Restatement: If there is a one-to-one function $f : A \rightarrow B$ between the elements of A and those of B and a one-to-one function $g : B \rightarrow A$ between the elements of B and those of A then there is a one-to-one correspondence between the elements of A and B.

XI. Recall: Definitions of Open and Closed Intervals

A. Definitions: Intervals for real numbers are defined as follows for the real numbers a and b with $a \leq b$:

1. $[a,\, b] = \{x \,|\, a \leq x \leq b\}$ is called the ***closed*** interval between a and b.
2. $[a,\, b) = \{x \,|\, a \leq x < b\}$
3. $(a,\, b] = \{x \,|\, a < x \leq b\}$
4. $(a,\, b) = \{x \,|\, a < x < b\}$ is called the ***open*** interval between a and b.

B. Note:

1. $(a,\, b) \subset [a,\, b]$, $[a,\, b) \subset [a,\, b]$, and $(a,\, b] \subset [a,\, b]$
2. $(a,\, b) \subset (a,\, b]$ and $[a,\, b)$

XIII. Schroder-Bernstein Theorem Application 1:

A. Theorem: $|(0,\, 1)| = |(0,\, 1]|$

B. Proof:

1. $(0,\, 1) \subset (0,\, 1]$ by definition.
2. Therefore: $f(x) = x$ is a one-to-one correspondence from $(0,\, 1)$ to $(0,\, 1]$
3. The function $g(x) = \frac{x}{2}$ is one-to-one and maps $(0,\, 1]$ to $(0,\, \frac{1}{2}] \subset (0,\, 1)$
4. Since we have found one-to-one functions from $(0,\, 1)$ to $(0,\, 1]$ and from $(0,\, 1]$ to $(0,\, 1)$ we have, according to the Schroder-Bernstein Theorem, proven that $|(0,\, 1)| = |(0,\, 1]|$

XIV. Schroder-Bernstein Theorem Application 2:

A. Show that $(0,\ 1)$ and R have the same cardinality.

B. According to the Schroeder-Bernstein Theorem we can demonstrate this result by finding one-to-one functions $f : (0,\ 1) \rightarrow\ R$ and $g : R \rightarrow (0,\ 1)$

1. This result would establish a one-to-one correspondence between $(0,\ 1)$ and R.

2. The one-to-one correspondence between $(0,\ 1)$ and R requires that $|(0,\ 1)| = |R|$

C. A one-to-one function $f : (0,\ 1) \rightarrow\ R$ is $f(x) = x$ since $f(x) = f(y)$ requires that $x = y$.

D. The arctangent function has:

1. A domain all of real numbers

2. A range is from $\frac{\pi}{2}$ radians to $-\ \frac{\pi}{2}$ radians and it is one-to-one for this range.

3. Therefore $g(x) = \frac{2 \times arctan(x)}{\pi}$ has a range of $(0,\ 1)$

E. Therefore, we have one-to-one functions $f : (0,\ 1) \rightarrow\ R$ and $g : R \rightarrow (0,\ 1)$

F. Then, according to the Schroeder-Bernstein Theorem, $|(0,\ 1)| = |R|$

XIV. Computable Functions

A. Definition: A function is ***computable*** if there is a computer program in some programming language that, given the parameters in the domain, finds the corresponding values in the range or codomain. If a function is not computable it is said to be ***uncomputable***.

B. Definition of a computer program: A computer program P is a string of $0'$s and $1'$s (bits) of finite length.

1. $P = \{p_i \mid p_i = 0 \lor p_i = 1\}$

2. $|P| = N$ where N is some finite integer

3. Therefore any computer program can be represented by a bit string of some finite length.

4. Therefore any program corresponds to some binary integer of finite length.

C. Therefore each computer program P_i can be represented by a countable set of 0's and 1's.

D. We have shown that union of a countable number of countable sets is countable (Example Problem 3), or $\bigcup_{i=1}^{n} P_i$ is countable.

E. Since:

1. Each computable function corresponds to a computer program

   and

2. The set of computer programs is countable

   then

3. The number of computable functions is countable.

XV. Uncomputable Functions

A. Example of Uncountability:

1. Theorem A: If A and B are sets, and A is uncountable, and $A \subseteq B$, then B is uncountable.

2. Proof by contradiction:
   a. Assume that B is countable.

   b. Since B is countable the elements of B can be listed as a countable sequence $b_1,\ b_2,\ b_3,\ ...b_n$.

   c. Since A is a subset of B, all of the elements of A are elements of B.

   d. Therefore the sequence $\{b_n\}$ contains the all of the elements of B that are elements of A.

   e. The elements of the sequence $\{b_n\}$ are countable.

   f. Since $\{b_n\}$ contains all of the elements of A, A must be countable.

   g. But A is uncountable, generating a contradiction.

3. Therefore, if A and B are sets, and A is uncountable, and $A \subseteq B$, then B is uncountable.

B. As stated previously, in XIV, a computer program can be represented as a positive binary integer.

C. Theorem B: There are uncountably many functions from the set of positive integers to to the set
$$D = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}.$$

1. We have already demonstrated that the set of real numbers between 0 and 1 is uncountable.

2. Let us associate to each real number in this range (including 0 but excluding 1) a function from the set of positive integers to the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} as follows:
   a. If x is a real number whose decimal representation is 0.d1d2d3 . . . (with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associate to x the function whose rule is given by $f(n) = d_n$

   b. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set D.

   c. Two different real numbers must have dfferent decimal representations, so the corresponding functions are dfferent. A few functions are left out, because of forbidding representations such as 0.239999 . . ..

   d. Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable.

   e. But the set of all such functions has at least this cardinality, so, therefore, it must be uncountable, from Theorem A, above.