I. Definitions of Terms

A. **Argument:** A sequence of propositions or predicates (statements) that end with a conclusion.

1. All but the final proposition are called ***premises***.

2. The final proposition is called the ***conclusion***.

3. An argument is valid if the truth of all of the premises implies that the conclusion is true.

B. Example 1:

1. Statement 1: If Prohibition succeeded in the 1920's then the war on drugs will succeed in the 2010's.

2. Statement 2: Prohibition did not succeed in the 1920's.

3. Conclusion: Therefore, the war on drugs will not succeed in the 2010's.

4. Reformulation:

   a. Proposition 1: P : Prohibition succeeded in the 1920's.

   b. Proposition 2: Q : The war on drugs will succeed in the 2010's.

   c. Argument:

$$\begin{array}{l} P \rightarrow Q \\ \underline{\neg P\quad} \\ \neg Q \end{array}$$

5. Alternative formulation: $((P \rightarrow Q) \wedge \neg P) \rightarrow \neg Q$

   a. The premises are: $(P \rightarrow Q) \wedge \neg P$

   b. The conclusion is: $\neg Q$

6. Truth Table:

P	Q	$\neg P$	$P \rightarrow Q$	$(P \rightarrow Q) \wedge \neg P$	$\neg Q$	$((P \rightarrow Q) \wedge \neg P) \rightarrow \neg Q$
T	T	F	T	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

7. Verification(?) Using Equivalences:

Compound Proposition	Rule
$((P \rightarrow Q) \wedge \neg P) \rightarrow \neg Q$	Premise
$((\neg P \vee Q) \wedge \neg P) \rightarrow \neg Q$	$p \rightarrow q \equiv \neg p \vee q$
$\neg((\neg P \vee Q) \wedge \neg P) \vee \neg Q$	$p \rightarrow q \equiv \neg p \vee q$
$((\neg\neg P \wedge \neg Q) \vee \neg\neg P) \vee \neg Q$	De Morgan's Law
$((P \wedge \neg Q) \vee P) \vee \neg Q$	Double Negation Law
$(P \wedge \neg Q) \vee P \vee \neg Q$	Associative Law
$(P \wedge \neg Q) \vee (P \vee \neg Q)$	Associative Law

8. **Note:**
   a. From the truth table it is seen that the premises are ***True*** in one case for which the conclusion, $\neg Q$, is ***False.***

   b. From the Equivalence table it is seen that the last equivalence is ***False*** when P is ***False*** and Q is ***True.***

9. Both methodologies demonstrate that the statement of the argument is not a tautology and the argument is, therefore, ***Invalid***.

C. Example 2:
1. Statement 1: If it is rainy then the pool will be closed.

2. Statement 2: It is rainy.

3. Conclusion: The pool will be closed.

4. Reformulation:
   a. Proposition 1: P : It is rainy.
   b. Proposition 2: Q : The pool will be closed.

   c. Argument:
   $$\begin{array}{l} P \rightarrow Q \\ \underline{P\qquad} \\ Q \end{array}$$

5. Alternative formulation: $((P \rightarrow Q) \wedge P) \rightarrow Q$
   a. The premises are: $(P \rightarrow Q) \wedge P$
   b. The conclusion is: Q

6. Truth Table:

P	Q	$P \rightarrow Q$	$(P \rightarrow Q) \wedge P$	$((P \rightarrow Q) \wedge P) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

7. Verification Using Equivalences:

Compound Proposition	Rule
$((P \rightarrow Q) \wedge P) \rightarrow Q$	Premise
$\neg((\neg P \vee Q) \wedge P) \vee Q$	$p \rightarrow q \equiv \neg p \vee q$
$(\neg(\neg P \wedge Q) \vee \neg P) \vee Q$	De Morgan's Law
$(\neg\neg P \vee \neg Q) \vee \neg P) \vee Q$	De Morgan's Law
$(P \vee \neg Q) \vee \neg P \vee Q$	$\neg(\neg p) \equiv p$
$P \vee \neg Q \vee \neg P \vee Q$	Associative Law
$P \vee \neg P \vee \neg Q \vee Q$	Associative Law
$(P \vee \neg P) \vee (\neg Q \vee Q)$	Associative Law
$T \vee T$	Negation Law
T	Domination Law

8. **Note:**
   a. From the truth table it is seen that the conclusion, Q, is ***True*** in all cases for which the premises are ***True.***
   b. From both the truth table and the demonstration of equivalence it is seen that the argument is a ***tautology***.

9. This argument is ***Valid***.

II. Common Rules of Inference for Propositional Logic

Rule of Inference	Tautology	Name
P	$(P \wedge (P \to Q)) \to P$	Modus ponens
$P \to Q$		
$\therefore P$		
$\neg Q$	$(\neg Q \wedge (P \to Q)) \to \neg P$	Modus tollens
$P \to Q$		
$\therefore \neg P$		
$P \to Q$	$((P \to Q) \wedge (Q \to R)) \to (P \to R)$	Hypothetical Syllogism
$Q \to R$		
$\therefore P \to R$		
$P \vee Q$	$((P \vee Q) \wedge \neg P) \to Q$	Disjunctive Syllogism
$\neg P$		
$\therefore Q$		
P	$P \to (P \vee Q)$	Addition
$\therefore P \vee Q$		
$P \wedge Q$	$(P \vee Q) \to P$	Simplification
$\therefore P$		
P	$((P) \wedge (Q)) \to (P \wedge Q)$	Conjunction
Q		
$\therefore P \wedge Q$		
$P \vee Q$	$((P \vee Q) \wedge (\neg P \vee R)) \to (Q \vee R)$	Resolution
$\neg P \vee R$		
$\therefore Q \vee R$		

Note 1: These are not the only inference rules. They are the simplest and most often used. We will be proving others, probably less general and more complicated, during this course.

Note 2: The Resolution Inference Rule is the one used by Prolog

III. Precaution: A Valid Argument Can Lead to an Incorrect Conclusion if One or More of Its Premises is $False$.

IV. Argument Example

A. English Argument:

It is either hotter than 100° today or the pollution is dangerous. It is less than 100° outside today. Therefore, the pollution is dangerous.

B. Logical Statement:

Premise 1: It is either hotter than 100° today or the pollution is dangerous.
Premise 2: It is less than 100° outside today.
Conclusion: The pollution is dangerous.

C. Conversion to Propositional Logic:

1. $P \equiv$ It is hotter than 100° today.

2. $Q \equiv$ The pollution is dangerous.

3. Argument:
$$\begin{array}{l} P \vee Q \\ \underline{\neg P \quad} \\ Q \end{array}$$

4. Tautological Form: $((P \vee Q) \wedge \neg P) \rightarrow Q$

5. This is the valid argument form ***Disjunctive Syllogism*** (page 72).

D. Equivalence Relation Demonstration of Tautology

Compound Proposition	Rule
$((P \vee Q) \wedge \neg P) \rightarrow Q$	Premise
$\neg((P \vee Q) \wedge \neg P) \vee Q$	$p \rightarrow q \equiv \neg p \vee q$
$(\neg(P \vee Q) \vee \neg\neg P) \vee Q$	De Morgan's Law
$\neg(P \vee Q) \vee \neg\neg P \vee Q$	Associative Law
$\neg(P \vee Q) \vee P \vee Q$	$\neg(\neg p) \equiv p$
$\neg(P \vee Q) \vee (P \vee Q)$	Associative Law
T	$p \vee \neg p \equiv T$

V. Rule of Inference Creation Example:

A. Consider the English Language statements as Propositions and Premises for an Argument:

1. $P \equiv$ Randy works hard.

2. $Q \equiv$ Randy is a dull boy.

3. $R \equiv$ Randy will get the job.

4. $P \to Q \equiv$ If Randy works hard then Randy is a dull boy.

5. $Q \to \neg R \equiv$ If Randy is a dull boy then Randy will not get the job.

B. Argument:

1. P Premise

2. $P \to Q$ Premise

3. Q Modus Ponens: $p \wedge (p \to q) \to q$

4. $Q \to \neg R$ Premise

5. $\neg R$ Modus Ponens: $p \wedge (p \to q) \to q$

6. Conclusion: Randy will not get the job.

C. Note: We have created a new rule of inference:

$$\begin{array}{l} P \\ P \to Q \\ \underline{Q \to \neg R} \\ \neg R \end{array}$$

or, in tautological form:

$$P \wedge (P \to Q) \wedge (Q \to \neg R) \to \neg R$$

D. In Truth Table Form:

To save space, let $S \equiv P \wedge (P \to Q) \wedge (Q \to \neg R) \to \neg R$

P	Q	R	$\neg R$	$P \to Q$	$Q \to \neg R$	$P \wedge (P \to Q) \wedge (Q \to \neg R)$	S
T	T	T	F	T	F	F	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	T	F	T	F	T
F	T	T	F	T	F	F	T
F	T	F	T	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	T	T	T	F	T

so $P \wedge (P \to Q) \wedge (Q \to \neg R) \to \neg R$

is a tautology, as is required of a valid argument form.

VI. Examples

A. Example 1: If you do every problem in this book, then you will learn Discrete Mathematics. You learned Discrete Mathematics. Therefore, you did every problem in the book.

1. Propositional Statement of Problem:

Proposition: $P \equiv$ You did every problem in the book

Premise: $Q \equiv$ You learned Discrete Mathematics.

Premise: $P \to Q \equiv$ If you do every problem in this book then you will learn Discrete Mathematics.

Inference: $((P \to Q) \wedge Q) \to P$
If you do every problem in this book then you will learn Discrete Mathematics and you learned Discrete Mathematics so you did every problem in this book.

Conclusion: $P \equiv$ You did every problem in this book.

2. Truth Table:

P	Q	$P \to Q$	$(P \to Q) \wedge Q$	$((P \to Q) \wedge Q) \to P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

a. Inference Method is not a Tautology.
b. Inference is not valid.

B. Example 2: Modified Problem Statement of Example 1.
If and only if you do every problem in this book, then you will learn Discrete Mathematics. You learned Discrete Mathematics. Therefore, you did every problem in the book.

1. Propositional Statement of the Modified Problem:
Proposition: P $\equiv$ You did every problem in this book.

Premise: Q $\equiv$ You learned Discrete Mathematics.

Inference Rule: $P \leftrightarrow Q$
If and only if you do every problem in this book, then you will learn Discrete Mathematics.

Argument: $((P \leftrightarrow Q) \wedge Q) \to P$

Conclusion: P $\equiv$ You did every problem in this book.

2. Truth Table:

P	Q	$P \leftrightarrow Q$	$(P \leftrightarrow Q) \wedge Q$	$((P \leftrightarrow Q) \wedge Q) \to P$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

a. Modified Inference Method is a Tautology.
b. Inference is valid.

C. Example 3: If you do not do every problem in this book, then you will not learn Discrete Mathematics. You did not do every problem in the book. Therefore, you did not learn Discrete Mathematics.

1. Propositional Statement of Problem:

   Premise: $\neg P \equiv$ You did not do every problem in this book.

   Inference: $\neg P \rightarrow \neg Q$ If you do not do every problem in this book then you will not learn Discrete Mathematics

   Conclusion: $\neg Q \equiv$ You did not learn Discrete Mathematics.

   Argument: $((\neg P \rightarrow \neg Q) \wedge \neg P) \rightarrow \neg Q$
   If you do not do every problem in this book then you will not learn Discrete Mathematics and you did not do every problem in the book so you did not learn Discrete Mathematics.

2. Truth Table:

P	$\neg P$	Q	$\neg Q$	$\neg P \rightarrow \neg Q$	$(\neg P \rightarrow \neg Q) \wedge \neg P$	$((\neg P \rightarrow \neg Q) \wedge \neg P) \rightarrow \neg Q$
T	F	T	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	F	F	T
F	T	F	T	T	T	T

   a. Inference Method is a Tautology.

   b. Inference is valid.

3. This result should not have been unexpected since if one rewords the argument as follows:

   Premise: $P \equiv$ You did not do every problem in this book.

   Inference: $P \rightarrow Q$

   Conclusion: $Q \equiv$ You did not learn Discrete Mathematics.

Argument: $((P \to Q) \land P) \to Q$
If you do not do every problem in this book, then you will not learn Discrete Mathematics.

The argument is an example of modus ponens.

D. Example 4: If you do every problem in this book, then you will learn Discrete Mathematics. You did not do every problem in the book. Therefore, you did not learn Discrete Mathematics.

1. Propositional Statement of Problem:
Premise: $\neg P \equiv$ You did not do every problem in this book.

Inference: $P \to Q$

Conclusion: $Q \equiv$ You did not learn Discrete Mathematics.

Argument: $((P \to Q) \land \neg P) \to \neg Q$
If you do not do every problem in this book, then you will not learn Discrete Mathematics.

2. Truth Table:

P	$\neg P$	Q	$\neg Q$	$P \to Q$	$(P \to Q) \land \neg P$	$((P \to Q) \land \neg P) \to \neg Q$
T	F	T	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	T	F	T	T	T	T

a. Inference Method is not a Tautology.
b. Inference is not valid.

3. This type of invalid reasoning has been named
The Fallacy of Denying the Hypothesis

from its denial of the hypothesis in the implication:

If you do every problem in this book, then you will learn Discrete Mathematics.

VII. Propositional Inference Rules for Predicate Logic

A. The inference rules for propositional logic apply to statements in predicate logic.

B. No quantifiers: Example: Modus Ponens

$$((P(a) \to Q(b)) \land P(a)) \to Q(b)$$

1. Valid in predicate logic just as it was for propositional logic.

2. The predicate parameters must be constants.

C. Partly Quantified: Example: Modus Ponens

$$(\forall x(P(x) \to Q(x)) \land P(a)) \to Q(a)$$

1. The inference $P(x) \to Q(x)$ holds for all values of x.

2. a is a possible value for the variable x, i.e., a is an element of the domain of $\forall x(P(x) \to Q(x))$.

3. We assert as a premise that $P(a)$ has truth value $\boldsymbol{True}$.

4. Therefore: $(\forall x(P(x) \to Q(x)) \land P(a)) \to Q(a)$

D. Completely Quantified: Example: Modus Ponens

$$(\forall x(P(x) \to Q(x)) \land \forall x\ P(x)) \to \forall x\ Q(x)$$

1. Premises:
   a. $\forall x(P(x) \to Q(x))$ has truth value $True$ so the inference $P(x) \to Q(x)$ holds for all values of x in the domain.

   b. $\forall x(P(x)$

2. Conclusion: $(\forall x(P(x) \to Q(x)) \land \forall x P(x)) \to \forall x\ Q(x)$

VIII. Inference Rules for Predicate Logic

A. $\underline{\forall x\ P(x)}$ Universal Instantiation

$\therefore P(c)$

1. Premises:
   a. $\forall x P(x)$ states that $P(x)$ has truth value $\boldsymbol{True}$ for all elements x in the domain of discourse.

   b. c is an element of the domain of discourse.

2. Therefore: $P(c)$ has truth value $\boldsymbol{True}$.

3. Example: $\forall x P(x)$ $\equiv$ All tigers have stripes.
   <u>c is a tiger.</u>
   c has stripes.

B. $\underline{\quad P(c)}$ for an arbitrary c Universal Generalization

$\therefore \forall x\ P(x)$

1. Premises:
   a. $P(c)$ has truth value $\boldsymbol{True}$.

   b. c is an ***arbitrary*** element of the domain of discourse.
      i. We can make no assumption regarding c that is not true for all elements of the domain.

      ii. Adding unwarranted assumptions concerning c, e.g., qualifying c to make it a specific element of the domain of discourse will result in an incorrect proof.

2. Therefore: $\forall x\ P(x)$

3. Example:
   a. I am told that this random large cat-like striped animal that I have just seen is a tiger.

   b. Upon seeing another large cat-like striped animal I conclude that it is a tiger.

C. $\underline{\exists x\, P(x)}$ for some element c Existential Instantiation
$\therefore P(c)$

1. Premise: There exists some arbitrary element x of the domain of discourse for which $P(x)$ has truth value $\boldsymbol{True}$.

2. Therefore: c is the name of that element.

D. $\underline{P(c)}$ for some element c Existential Generalization
$\therefore \exists x\, P(x)$

1. Premise: $P(c)$ has truth value $True$ where c is a specifice element of the domain of discourse for $P(x)$.

2. Therefore: There exists some element x of the domain of discourse, namely $x = c$, for which $P(x)$ has truth value $\boldsymbol{True}$.

IX. Example 1:

1. Problem Statement in English:
Linda, a student in this class, owns a red convertible.
Everyone who owns a red convertible has gotten at least one speeding ticket.

Therefore: Someone in this class has gotten a speeding ticket.

2. Statements:
   a. $StudentInThisClass(X) \equiv$ X is a student in this class.

   b. $OwnsARedConvertible(X) \equiv$ X owns a red convertible.

   c. $GottenASpeedingTicket(X) \equiv$ X has gotten a speeding ticket.

   d. $\forall X\, (OwnsARedConvertible(X) \rightarrow GottenASpeedingTicket(X)$

3. Conclusion:
$\exists X\, (StudentInThisClass(X) \wedge GottenASpeedingTicket(X))$

4. Argument

a. $StudentInThisClass(\text{Linda})$ Premise

b. $OwnsARedConvertible(\text{Linda})$ Premise

c. $\forall X\ (OwnsARedConvertible(X) \rightarrow GottenASpeedingTicket(X))$ Premise

d. $\forall X\ (OwnsARedConvertible(X) \rightarrow GottenASpeedingTicket(X))$ Modus Ponens
$\wedge$
$\underline{OwnsARedConvertible(\text{Linda})}$
$GottenASpeedingTicket(\text{Linda})$

e. $GottenASpeedingTicket(\text{Linda})$ Conclusion of Modus Ponens

f. $StudentInThisClass(\text{Linda})$ Conjunction of (a) and (e)
$\wedge$
$GottenASpeedingTicket(\text{Linda})$

g. $StudentInThisClass(\text{Linda})$ Existential Generalization
$\wedge$
$\underline{GottenASpeedingTicket(\text{Linda})}$
$\exists X\ (StudentInThisClass(X) \wedge GottenASpeedingTicket(X))$

5. Conclusion: $\exists X\ (StudentInThisClass(X) \wedge GottenASpeedingTicket(X))$

X. Example 2:

1. Problem Statement in English:
   Doug, a student in this class, knows how to write programs in JAVA.

   Everyone who knows how to write programs in JAVA can get a high-paying job.

   Therefore, someone in this class can get a high-paying job.

2. Statements:
   a. $StudentInThisClass(X)$ $\equiv$ X is a student in this class.

   b. $KnowsHowToWritePogramsIn\,JAVA(X)$ $\equiv$ X knows how to write programs in JAVA.

   c. $CanGetAHighPayingJob(X)$ $\equiv$ X can get a high paying job.

   d. $\forall x\,(KnowsHowToWritePogramsIn\,JAVA(X) \rightarrow CanGetAHighPayingJob(X))$

3. Conclusion: $\exists x(CanGetAHighPayingJob(x) \wedge StudentInThisClass(x))$

4. Argument:
   a. $StudentInThisClass(\text{Doug})$ Premise

   b. $KnowsHowToWritePogramsIn\,JAVA(\text{Doug})$ Premise

   c. $\forall x\,(KnowsHowToWritePogramsIn\,JAVA(X) \rightarrow CanGetAHighPayingJob(X))$ Premise

d. $(\forall x\,(KnowsHowToWritePogramsIn\,JAVA(X) \rightarrow$
$CanGetAHighPayingJob(X))$
$\wedge$
$KnowsHowToWritePogramsIn\,JAVA(\text{Doug}))$
Conjunction of Premises

d. Modus Ponens

$$\frac{(\forall x\,(KnowsHowToWritePogramsIn\,JAVA(X) \rightarrow CanGetAHighPayingJob(X)) \wedge KnowsHowToWritePogramsIn\,JAVA(\text{Doug}))}{CanGetAHighPayingJob(\text{Doug})}$$

e. $CanGetAHighPayingJob(\text{Doug})$ Conclusion of Modus Ponens

f. Existential Generalization

$$\frac{CanGetAHighPayingJob(\text{Doug}) \wedge StudentInThisClass(\text{Doug})}{\exists x(CanGetAHighPayingJob(x) \wedge StudentInThisClass(x))}$$

g. Conclusion:
$\exists x(CanGetAHighPayingJob(x) \wedge StudentInThisClass(x))$