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I. Prove the following statement using mathematical induction:

$$\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3} \quad \textbf{for all integers } N \geq 2$$

A. Review of the steps in a proof by mathematical induction

1. Basis Step
   a. Demonstrate that the theorem is correct for a specified integer value.

   b. The integer value chosen should be the smallest value for which the theorem is true.

2. Inductive Assumption:
   a. Assume that the theorem is true for some unspecified integer value N.

   b. Note that you need not prove it to be true for N.

3. Prove that your assumption that the theorem is true for N leads to the conclusion that it is true for $N+1$.

B. Basis Step: $N = 2$

$$\sum_{i=1}^{N-1} i(i+1) = \sum_{i=1}^{2-1} i(i+1) = \sum_{i=1}^{1} i(i+1) = 2$$

$$\frac{N(N-1)(N+1)}{3} = \frac{2(2-1)(2+1)}{3} = \frac{6}{3} = 2$$

Therefore, for $N = 2$, $\displaystyle\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$

C. Inductive assumption: $\displaystyle P(N) \equiv \sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$

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D. Inductive proof:

1. Using the rules of arithmetic we can say:

$$\sum_{i=1}^{(N+1)-1} i(i+1) = \sum_{i=1}^{N} i(i+1) = \sum_{i=1}^{N-1} i(i+1) + N(N+1)$$

This step says that the sum of N terms is equal to the sume of $N-1$ terms plus the Nth term.

2. $\sum_{i=1}^{(N+1)-1} i(i+1) = \frac{N(N-1)(N+1)}{3} + N(N+1)$ by our inductive assumption.

   a. We have used our inductive assumption to substitute a value for the sum of N terms.

   b. The remainder of the steps are algebraic manipulation:

$$\begin{aligned}\sum_{i=1}^{(N+1)-1} i(i+1) &= \frac{N(N-1)(N+1)+3N(N+1)}{3} \\ &= \frac{N(N+1)}{3}\Big[(N-1)+3\Big] \\ &= \frac{N(N+1)(N+2)}{3} \\ &= \frac{(N+1)\big[(N+1)-1\big]\big[(N+1)+1\big]}{3}\end{aligned}$$

   c. So: $\sum_{i=1}^{(N+1)-1} i(i+1) = \frac{(N+1)\big[(N+1)-1\big]\big[(N+1)+1\big]}{3}$

   d. Therefore: $P(N) \to P(N+1)$

E. Therefore: $\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$ for all integers $N \geq 2$ according to a proof by mathematical induction.

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II. Incorrect Answer:

A. Assume: $$\sum_{i=1}^{(N+1)-1} i(i+1) = \frac{(N+1)\left[(N+1)-1\right]\left[(N+1)+1)\right]}{3}$$

B. Then: $$\sum_{i=1}^{(N+1)-1} i(i+1) - \sum_{i=N+1}^{(N+1)} i(i+1) = \sum_{i=1}^{N-1} i(i+1)$$

C. and: $$\sum_{i=1}^{(N+1)-1} i(i+1) - (N+1)\Big[(N+1)+1\Big] = \sum_{i=1}^{N-1} i(i+1)$$

D. Therefore:

$$\frac{(N+1)\left[(N+1)-1\right]\left[(N+1)+1)\right]}{3} - (N+1)\Big[(N+1)+1\Big]$$

$$= \frac{N(N-1)(N+1)}{3}$$

E. Doing the algebra gives us: $$\frac{N(N-1)(N+1)}{3} = \frac{N(N-1)(N+1)}{3}$$

F. Therefore, because our assumption has led to a correct result, we must have: $$\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$$

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III. Prove the following statement using *strong* induction:
(Proofs using mathematical induction will not be accepted.)
For any integer $N \geq 2$, if N is even then any sum of N odd integers is even.

A. My Answer:

1. Basis step: $N = 2$

Let $X = 2 \times K + 1$ and $Y = 2 \times J + 1$

where I and J are integers so X and Y are odd.

$$\begin{aligned} X + Y &= (2 \times K + 1) + (2 \times J + 1) \\ &= (2 \times K) + (2 \times J) + 2 \\ &= 2 \times \left[K + J + 1\right] \end{aligned}$$ so $X + Y$ is even.

2. Inductive assumption:

$P(N) \equiv$ For all $I \leq N$, if I is even $\sum_{i=1}^{I} X_i$ is even where all X_i are odd.

3. Inductive Proof:

a. We can split $\sum_{i=1}^{N+2} X_i$ into a sum of sums as:

$$\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{I} X_i + \sum_{i=I+1}^{J} X_i$$

where I and J are even and $I + J = N + 2$

b. By our inductive assumption, both of the sums $\sum_{i=1}^{I} X_i$ and $\sum_{i=I+1}^{J} X_i$ produce even integers.

c. Therefore: $\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{I} X_i + \sum_{i=I+1}^{J} X_i$ is the sum of two even integers

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d. Therefore: $\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{I} X_i + \sum_{i=I+1}^{J} X_i$ is even.

e. So: $\forall N(P(N)) \to P(N+2)$

$4.$ Therefore: For any integer $N \geq 2$, if N is even then any sum of N odd integers is even.

II. A simpler version 1:

According to the basis step, $P(2)$ is true.

According to the inductive assumption $P(N)$ is true.

Therefore: $\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{2} X_i + \sum_{i=3}^{N+2} X_i$

Hence: $P(N+2)$ is true.

A simpler version 2:

According to the basis step, $P(2)$ is true.

Since $N+2$ is even we can split $\sum_{i=1}^{N+2} X_i$ into sum$s$ of pairs X_i and X_j. Each sum a pair of odd integers is even, so the sum of the pairs is a sum of even integers. Therefore $\sum_{i=1}^{N+2} X_i$ is even.

Hence: $P(N+2)$ is true.