I. Introduction to "Strong" Induction

A. Strong Induction is a variant of the Principle of Mathematical Induction

B. A proof using strong induction uses the same basis step, i.e., that $P(1)$ must be true.

C. The inductive step assumes that $P(N)$ holds not only for some arbitrary N but for all $M < N$.

D. Strong Induction can be shown to be equivalent to mathematical induction as follows:

1. Let the proposition $Q(N)$ mean that $P(M)$ holds for all M, $0 \le M \le N$

2. Then $Q(N)$ is true for all N if and only if $P(N)$ is true for all N

3. Therefore a proof of $Q(N)$ by ordinary induction is equivalent to a proof of $P(N)$ by strong induction.

II. Strong Induction Example 1:

A. Theorem: Any integer $N \ge 1$ can be written as a product of primes and 1.

B. Basis step(s):

2 is a prime number.
$2 = 2 \times 1$ so 2 can be written as a product of primes and 1.

3 is a prime number.
$3 = 3 \times 1$ so 3 can be written as a product of primes and 1.

4 is not a prime number.
$4 = 2 \times 2$ so 4 can be written as a product of primes and 1.

C. Inductive Assumption: Any integer N can be written as a product of prime numbers.

D. Consider the integer $N+1$. Either $N+1$ is prime or is not prime.

1. If $N+1$ is prime then it can be written as

$$N+1=(N+1)\times 1$$

so $N+1$ is a product of primes and 1.

2. If $N+1$ is not prime then it can be written as the product of two integers I and K where:

$$I < N \quad \text{and} \quad K < N$$

By our inductive assumption $P(N)$, both I and K are the product of primes.

Since $N+1=I\times K,\ N+1$ is also a product of primes.

3. Therefore: $P(1)\equiv 1$ is a product of primes is true.

and

$$P(N)\Rightarrow P(N+1)$$

Therefore $P(N)$ is true for all N according to the Principle of Mathematical Induction.

III. Strong Induction Example 2:

A. Theorem: A positive integer $N\geq 2$ is either a prime number or a product of two or more prime numbers

B. Basis step: $P(2)$ is: 2 is prime.

If $N=2$ then N is prime by definition

C. Inductive Assumption: $P(k)$ for all $k\leq N$, i.e., either k is prime or k is the product of primes

D. Inductive step:

1. Assume that either k is prime or k is the product of primes, the inductive assumption.

2. Consider the case of $k = N + 1$

3. k is either prime or it is not

4. If k is prime we are done.

5. If $k = N + 1$ is not prime then:
   a. By definition $k = i*j$ where $i,\ j \leq N$

   b. By $P(N)$ both i and j are either prime or the product of primes

   c. Therefore, $k = N + 1$ must be the product of primes because $i,\ j \leq N$ are either primes or the products of

E. Therefore an integer N is either prime or the product of primes. □

IV. Strong Induction Example 3: A Jigsaw Puzzle

A. Problem Description

1. A jigsaw puzzle is assembled into the final version by successively joining pieces that fit together into blocks.

2. A *move* is made each time a piece is added to a block or each time two blocks are joined together.

3. Use strong induction to prove that no matter how the moves are implemented exactly $N - 1$ moves are required to assemble apuzzle with N pieces.

B. Basis Step: Assume that the puzzle has only one piece, or that $N = 1$.

1. A puzzle with only one pieces is already assembled so no further moves are required for assembly.

2. The number of moves K required for assembly is, therefore: $K = N - 1 = 0$

C. Additional Test: Assume that the puzzle has two pieces, or that $N = 2$.

1. The puzzle is then assembled by joining the two pieces together, a single move.

2. Therefore the number of moves K required to assemble the puzzle is $K = 1 = N - 1$.

D. Inductive Assumption: All puzzles with J pieces, $1 \leq J \leq N$ can be assembled in $J - 1$ moves.

E. Inductive Step:

1. Divide a puzzle with J pieces into two puzzles, one with X pieces and one with Y pieces.

2. $J = X + Y$

3. According to our inductive assumption the puzzle with X pieces can be assembled in $X - 1$ moves

4. According to our inductive assumption the puzzle with Y pieces can be assembled in $Y - 1$ moves.

5. Regardless of the values of X and Y the puzzle with J pieces can be completed with one additional move that joins the block with X pieces to that with Y pieces.

6. Therefore the total number of moves M needed to assemble a jigsaw puzzle with J pieces is M where:

$$\begin{aligned} M &= (X-1) + (Y-1) + 1 \\ &= X + Y - 1 - 1 + 1 \quad = \quad X + Y - 1 \\ &= J - 1 \end{aligned}$$

F. Therefore, no matter how the moves are implemented exactly $N - 1$ moves are required to assemble a puzzle with N pieces. QED

V. Strong Induction Example 4: A polynomial of degree $N \geq 1$ with real coefficients has at most N real zeroes, not all necessarily distinct.

A. Basis Step:

1. Polynomials of degree 1 are of the form: $P(x) = a_1 x + a_0$ with $a_1 \neq 0$.

2. The zero of $P(x) = a_1 x + a_0 = 0$ is $x = \frac{a_0}{a_1}$

3. So a polynomial of degree 1 with real coefficients unequal to zero has one real zero.

B. Inductive Assumption: All polynomials of degree $1 < k \leq N$ with real coefficients have at most k real zeroes, not all necessarily distinct.

C. Inductive Step:

1. Let: $P(x) = a_{k+1}x^{k+1} + a_k x^k + ... + a_1 x + a_0$

2. If $P(x)$ has no real zeroes then the theorem is true for this case.

3. If we assume that $P(x)$ has at least one real zero, c, we can write it as: $P(x) = (x-c)^t T(x)$

   where $T(x)$ is a polynomial of degree $(k+1) - t$ with real coefficients.

4. Case 1: If $t = k+1$ then :

   a. $T(x)$ is a constant.

   b. $P(x)$ has exactly $k+1$ zeroes.

   c. The theorem is proven for $t = k+1$

5. Case 2: If $t < k+1$ then:
   a. $T(x)$ is a polynomial of degree between 1 and k with real coefficients.

   b. By the inductive hypothesis, $T(x)$ has at most $(k+1)-t$ real zeroes.

   c. Every zero of $T(x)$ is a zero of $P(x)$ so all the zeroes of $T(x)$ must be counted as zeroes of $P(x)$.

   d. Therefore the real zeroes of $P(x)$ are the real zeroes of $T(x)$.

   e. c is also a zero of $P(x)$ which can be counted no more than t times.

   f. Therefore $P(x)$ has at most
   $$\Big([(k+1)-t]+t\Big) = k+1$$
   real zeroes.

   g. The theorem is proven for $t = k+1$

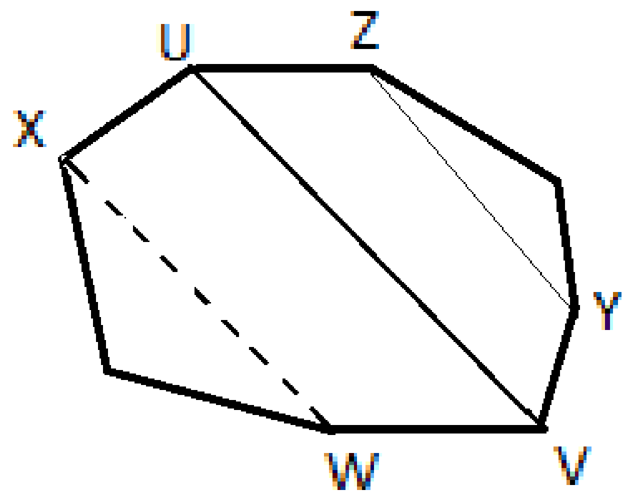
D. Therefore, by the Principle of Strong Mathematical Induction, a polynomial of degree $N \geq 1$ with real coefficients has at most N real zeroes, not all necessarily distinct.

VI. Computational Geometry Example

A. Theorem: Whenever non-intersecting diagonals are drawn inside a strictly convex polygon with N sides, at least two vertices of the polygon are not endpoints of any of the diagonals.

B. Definition: A ***convex polygon*** is a simple polygon with all interior angles less than or equal to 180 degrees. In a ***strictly convex polygon*** all interior angles are strictly less than 180 degrees.

C. A convex polygon, with 8 sides, with three non-intersecting diagonals and with two vertices that are not endpoints of any of the diagonals is shown below:



1. The diagonal UV has three vertices other than U or V (the end points) above it and three vertices other than U or V below it.

2. The diagonal WX has five vertices other than X or W above it and one vertex other than X or W below it.

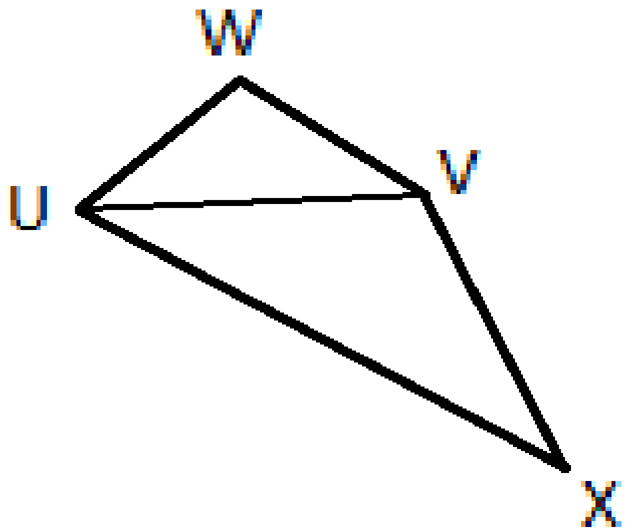
3. The diagonal YZ has one vertex other than Y or Z above it and five vertices other than Y or Z below it.

4. We note that the theorem is not true for $N = 3$ because a three-sided polygon is a triangle and there can be no diagonals.

D. Problem: Prove $P(N)$ using strong induction for $N \geq 4$

E. Basis step: $P(4) \equiv$ When non-intersecting diagonals are drawn inside a strictly convex polygon with four sides two non-adjacent vertices of the polygon are not endpoints of any of the diagonals.

1. Consider the polygon shown below with 4 sides.



2. The diagonal UV separates the original polygon into two sub-polygons with three sides and leaves two non-adjacent vertices, W and X, that are not endpoints of the diagonal.

3. The same statement could be made if the diagonal used was WX.

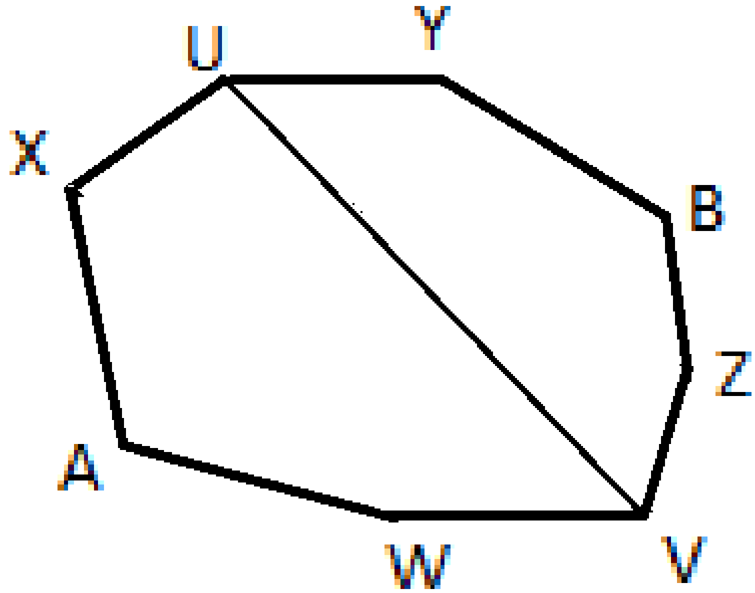
4. Note that U and V cannot be counted because the diagonal makes them adjacent, i.e., because they are endpoints of the diagonal.

E. Inductive Hypothesis:
$P(N) \equiv$ When non-intersecting diagonals are drawn inside a strictly convex polygon with J sides, $4 \leq J \leq N$, at least two non-adjacent vertices of the polygon are not endpoints of any of the diagonals.

F. Inductive Step:

1. Consider a polygon with $N + 1$ sides

2. The diagonal $U - V$ in the polygon shown below divides the polygon into two sub-polygons, each one of which can have a set of non-intersecting diagonals.



3. By the inductive hypothesis each of these two sub-polygons created from the original by the separating diagonal has J sides, $3 \leq J \leq N$, and at least one vertex that is not the endpoint of the separating diagonal.

4. Therefore the original polygon with $N + 1$ sides, split into the two sub-polygons by the dividing diagonal, must have at least two vertices that are not the end-points of the diagonal.

G. Therefore the theorem is proven by Strong Induction.

VII. Strong Induction Example 6: Proving the Property of a Sequence

A. Problem:

1. Let $S_0,\ S_1, \dots,\ S_N$ be the sequence defined by specifying that

   a. $S_0 = 0$

   b. $S_1 = 4$

   c. $S_k = 6a_{k-1} - 5a_{k-2}$ for all integers $k \geq 2$

2. Let the property possessed by all terms in the sequence $P(N)$ be:

$$S_n = 5^N - 1$$

3. The claim to be proven is that all terms of the sequence satisfy the property $P(N)$.

B. Note that, since the two previous terms of the sequence are needed to specify the kth term:

1. The assumption that $P(N)$ holds is not sufficient to prove $P(N+1)$.

2. Therefore we must use strong induction with the inductive assumption that $P(k)$ holds for $1 \leq k \leq N$

C. Proof:

1. Basis Step:

   a. $S_0 = 0 = 1 - 1 = 5^0 - 1$

   b. $S_1 = 4 = 5 - 1 = 5^1 - 1$

   c. $S_2 = 6a_{2-1} - 5a_{2-2} = 6a_1 - 5a_0$

$$= 6 \times 4 - 5 \times 0 = 24$$

$$= 25 - 1 = 5^2 - 1$$

2. Inductive Assumption:
   $S_i = 5^i - 1$ for all integers i such that $0 \leq i \leq k$

3. Inductive Proof:
   $S_{k+1} = 6S_k - 5S_{k-1}$ by the definition of the terms of the sequence.
   $= 6(5^k - 1) - 5(5^{k-1} - 1)$ by the inductive assumption.

   $= 6 \times 5^k - 6 - 5 \times 5^{k-1} + 5$

   $= 6 \times 5^k - 5^k - 1$

   $= (6 - 1) \times 5^k - 1$

   $= 5 \times 5^k - 1$

   $= 5^{k+1} - 1$

D. Therefore, the property $P(N) \equiv S_n = 5^N - 1$ is possessed by all terms of the sequence defined by
   a. $S_0 = 0$

   b. $S_1 = 4$

   c. $S_k = 6a_{k-1} - 5a_{k-2}$

IX. The Well Ordering Property

A. The well ordering property (Axiom 4 in Appendix 1 of your text) states that every nonempty subset of the positive integers has a least element.

B. The well ordering property is the basis for the validity of mathematical induction.

C. Demonstration (Proof by Contradiction) of the validity of Mathematical Induction:

1. Suppose that we know that:
   a. $P(1)$ is true. ***and***
   b. For all positive integers k, $P(k) \to P(k+1)$.

2. Assume that there exists at least one positive integer j such that $P(j)$ is false.

3. Then there exists a non-empty set S of positive integers $S = \{..., j, ...\}$for which $P(n)$ is false.

4. By the well ordering property we know that S must have a least element which we will denote by m.

5. Since $P(1)$ is true we know that $m > 1$ and because m is positive we must have $m - 1 > 0$.

7. Since $m - 1 < m$ we must have that $m - 1 \notin S$

8. Therefore $P(m-1)$ must be true.

9. But, since for all positive integers k, $P(k) \to P(k+1)$, we have that $P(m-1) \to P(m)$

10. Therefore the assumption that there exists at least one positive integer j such that $P(j)$ is false has led to:
    a. The requirement that there is a set of integers S, with $|S| \geq 1,$ for which the principle of mathematical induction does not hold.

    b. The conclusion that if m is the least element of the set S containing j, then $P(m)$ is true.

    c. So the set S of positive integers for which $P(j)$ is false contains an element m for which $P(m)$ is true.

11. The validity of the Principle of Mathematical Induction has been demonstrated.