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This is an open book, open notes examination. Cooperation with anyone other than the instructor is not allowed. Wireless devices are prohibited, calculators are allowed.

All proofs using mathematical induction must clearly state your basis step, your inductive assumption, the inductive step of the proof, and what you have finally proven. Any omissions will result in points being deducted.

I. Prove the following statement using mathematical induction:
$\mathbf{2+4+6+\ldots+2N=N^2+N}$ **for all integers** $\mathbf{N \geq 1}$

Answer:

1. Basis step: $N=1$ $\quad P(1) \equiv \quad 2\times 1 = 1^2+1$

$2N=2 \qquad N^2+N=1+1=2$

Therefore, for $N=1$, $\quad 2N=N^2+N$

2. Inductive Assumption: $\quad P(N) \equiv 2+4+6+\ldots+2N=N^2+N$ for $N \geq 1$

3. Inductive Step:
$2+4+6+\ldots+2N+2(N+1)=N^2+N+2(N+1)$

according to our inductive assumption.

$$N^2+N+2(N+1) = N^2+N+2N+2$$

$$= N^2+2N+1+N+1$$

$$= (N+1)^2+(N+1)$$

$$\sum_{i=1}^{N} i =$$

Therefore: $\quad 2+4+6+\ldots+2N+2(N+1)=(N+1)^2+(N+1)$

Therefore: $\quad P(N)\rightarrow P(N+1)$

4. Therefore: $\quad 2+4+6+\ldots+2N=N^2+N \qquad$ for all integers $N \geq 1$

Name:______________________________________ **Id:____________________**

II. Prove the following statement using mathematical induction:

$$\boldsymbol{\sum_{i=1}^{N-1} i(i+1) = \tfrac{N(N-1)(N+1)}{3}} \quad \textbf{for all integers } \boldsymbol{N \geq 2}$$

Answer:

1. Basis Step: $N = 2$

$$\sum_{i=1}^{N-1} i(i+1) = \sum_{i=1}^{2-1} i(i+1) = \sum_{i=1}^{1} i(i+1) = 2$$

$$\frac{N(N-1)(N+1)}{3} = \frac{2(2-1)(2+1)}{3} = \frac{6}{3} = 2$$

Therefore, for $N = 2$, $\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$

2. Inductive assumption: $P(N) \equiv \sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$

3. Inductive proof:

$$\sum_{i=1}^{(N+1)-1} i(i+1) = \sum_{i=1}^{N} i(i+1) = \sum_{i=1}^{N-1} i(i+1) + N(N+1)$$

$$\sum_{i=1}^{(N+1)-1} i(i+1) = \frac{N(N-1)(N+1)}{3} + N(N+1)$$ by our inductive assumption.

$$= \frac{N(N-1)(N+1)+3N(N+1)}{3} = \frac{N(N+1)}{3}\Big[(N-1)+3\Big]$$

$$= \frac{N(N+1)(N+2)}{3} = \frac{(N+1)\big[(N+1)-1\big]\big[(N+1)+1\big]}{3}$$

Therefore: $P(N) \rightarrow P(N+1)$

4. Therefore: $\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$ for all integers $N \geq 2$

Name:____________________________________ **Id:**__________________

III. Prove the following statement using *strong* induction:
(Proofs using mathematical induction will not be accepted.)
For any integer $N \geq 2$, if N is even then any sum of N odd integers is even.

Answer:

1. Basis step: $N = 2$
   Let $X = 2 \times K + 1$ and $Y = 2 \times J + 1$
   where I and J are integers so X and Y are odd.

$$X + Y = (2 \times K + 1) + (2 \times J + 1) = (2 \times K) + (2 \times J) + 2$$
$$= 2 \times \left[K + J + 1\right] \quad \text{so } X + Y \text{ is even.}$$

2. Inductive assumption:

$P(N) \equiv$ For all $I \leq N$, if I is even, $\sum_{i=1}^{I} X_i$ is even where all X_i are odd.

3. Inductive Proof:

We can split $\sum_{i=1}^{N+2} X_i$ into a sum of sums as:

$$\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{I} X_i + \sum_{i=I+1}^{J} X_i$$

where I and J are even and $I + J = N + 2$

By our inductive assumption, both $\sum_{i=1}^{I} X_i$ and $\sum_{i=I+1}^{J} X_i$ are even.

Therefore: $\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{I} X_i + \sum_{i=I+1}^{J} X_i$ is the sum of two even integers

so $\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{I} X_i + \sum_{i=I+1}^{J} X_i$ is even.

4. Therefore: For any integer $N \geq 2$, if N is even then any sum of N odd integers is even.

Name:__________________________________ **Id:**__________________

A simpler version 1:

According to the basis step, $P(2)$ is true.

According to the inductive assumption $P(N)$ is true.

Therefore: $\sum_{i=1}^{N+2} X_i = \sum_{i=1}^{2} X_i + \sum_{i=3}^{N+2} X_i$

Hence: $P(N+2)$ is true.

A simpler version 2:

According to the basis step, $P(2)$ is true.

Since $N+2$ is even we can split $\sum_{i=1}^{N+2} X_i$ into sum$s$ of pairs X_i and X_j.

Each sum a pair of odd integers is even, so the sum of the pairs is a sum of even integers. Therefore $\sum_{i=1}^{N+2} X_i$ is even.

Hence: $P(N+2)$ is true.

Name:________________________________ **Id:**________________

IV. Explain your answers to the following counting problems clearly.

Suppose there are three roads from city A to city B and five roads from city B to city C.

a. In how many ways is it possible to travel from city A to city C via city B?

Answer: Task $T1$: Traveling from city A to city B.

Since there are three roads from city A to city B there are three ways to perform this task.

Task $T2$: Traveling from city B to city C.

Since there are five roads from city B to city C there are five ways to perform this task.

According to the product rule there are $3 \times 5 = 15$ ways to perform $T1$ and $T2$ in sequence.

b. How many round-trip routes are there from city A to B to C and back to B and then A?

Answer: Task $T1$ and Task $T2$: Same as above.

Task $T3$: Traveling back from city B to city C.

Since there are five roads from city C to city B there are five ways to perform this task.

Task $T4$: Traveling from city B to city A.

Since there are three roads from city B to city A there are three ways to perform this task.

According to the product rule there are $3 \times 5 \times 5 \times \ 3 = 225$ ways to perform $T1$, $T2$, $T3$, and $T4$ in sequence.

Name:________________________________ Id:________________

c. **How many different routes are there from city $\boldsymbol{A}$ to $\boldsymbol{B}$ to $\boldsymbol{C}$ and back to $\boldsymbol{B}$ and then $\boldsymbol{A}$ in which no road is traversed twice?**

Answer: Task $T1$ and Task $T2$: Same as above.

Task $T3$: Traveling back from city C to city B.

There are five roads from city C to city B but we used one in performing task $T2$ so we must choose one of the four remaining.

Task $T4$: Traveling from city B to city A.

Since there are three roads from city B to city A but we used one in performing task $T1$ so we must choose one of the two remaining.

According to the product rule there are $3 \times 5 \times 4 \times 2 = 120$ ways to perform $T1$, $T2$, $T3$, and $T4$ in sequence.

Reference values for Problem V:

$C(15,8) = 6435$	$C(12,8) = 495$	$C(10,8) = 45$
$C(15,7) = 6435$	$C(12,7) = 792$	$C(10,7) = 120$
$C(15,6) = 5005$	$C(12,6) = 924$	$C(10,6) = 210$
$C(15,5) = 3003$	$C(12,5) = 792$	$C(10,5) = 252$
$C(15,4) = 1365$	$C(12,4) = 495$	$C(10,4) = 210$
$C(15,3) = 455$	$C(12,3) = 220$	$C(10,3) = 120$
$C(15,2) = 105$	$C(12,2) = 66$	$C(10,2) = 45$
$C(15,1) = 15$	$C(12,1) = 12$	$C(10,1) = 10$

Name:___________________________________ Id:__________________

V. Suppose a department is composed of 10 men and 15 women. In how many ways can you form a committee with six members if the committee must have more women than men?

Answer:
For women to have a majority there must be at least four women.

The number of ways to form the committee is, then, by the summation rule, the sum of:

a. The number of ways to choose four women plus two men.
b. The number of ways to choose five women plus one man.
c. The number of ways to choose six women and no men.

Let:
$W_1 =$ the number of ways to choose four women out of $15 = C(15,\ 4)$
$M_1 =$ the number of ways to choose two men out of $10 = C(10,\ 2)$
Then the number of ways to form a committee of four women and two men is, according to the product rule, $W_1 \times M_1 = C(15,\ 4) \times C(10,\ 2)$

$W_2 =$ the number of ways to choose five women out of $15 = C(15, 5)$
$M_2 =$ the number of ways to choose one man out of $10 = C(10,\ 1)$
Then the number of ways to form a committee of five women and one men is, according to the product rule, $W_2 \times M_2 = C(15,\ 5) \times C(10,\ 1)$

$W_3 =$ the number of ways to choose six women out of $15 = C(15, 6)$

Then the number of ways to form a committee of six with more women than men is, according to the summation rule, N where (using the table on the previous page):

$$N = C(15, 4) \times C(10,\ 2) + C(15, 5) \times C(10,\ 1) + C(15, 6)$$

$$= 1365 \times 45 + 3003 \times 10 + 5005 = 61425 + 30030 + 5005$$

$$= 96, 460$$

Name:____________________________________ **Id:**__________________

Second Answers: (Sneaky)

Question 1: We proved in class that: $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

Therefore: $2 \times \sum_{i=1}^{N} i = \sum_{i=1}^{N} 2 \times i = N(N+1) = N^2 + N$

The above is a correct proof.

Question 2: We proved in class that: $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

$$\sum_{i=1}^{N-1} i(i+1) = \sum_{i=1}^{N-1} (i^2 + i) = \sum_{i=1}^{N-1} i^2 + \sum_{i=1}^{N-1} i$$

$\sum_{i=1}^{N-1} i^2 = \frac{(N-1)N(2N-1)}{6}$ and $\sum_{i=1}^{N-1} i = \frac{(N-1)N}{2}$

according to the above.

so: $\sum_{i=1}^{N-1} i(i+1) = \frac{(N-1)N(2N-1)}{6} + \frac{(N-1)N}{2}$

$$= \frac{N(N-1)(5N-4)}{6}$$

which is not the desired result.