XIII Th: There are $N!$ Permutations of a List S of N Distinct Elements

I. $P(N) \equiv N!$ permutations can be constructed from a list S of N elements.

II. Basis Step(s):

A. $N = 1$ $S_1 = [a]$ Number of permutations $= 1 = N!$

$N = 2$ $S_1 = [a,\, b]$ Number of permutations $= 2 = N!$
$S_2 = [b,\, a]$

$N = 3$ $S_1 = [a,\, b,\, c]$ Number of permutations = 6
$S_2 = [a,\, c,\, b]$ $6 = N!$
$S_3 = [b,\, a,\, c]$
$S_4 = [b,\, c,\, a]$
$S_5 = [c,\, a,\, b]$
$S_6 = [c,\, b,\, a]$

B. Inductive Step:

1. Assume that a list S with N distinct elements has $N!$ permutations.

2. We insert a new element x, which is distinguishable from any of the existing elements of S, in one of the $N!$ permutations of of S.

3. x can be inserted:
   a. As the first element, i.e., before any existing element of S, creating one new list.
   b. After the last element of S creating one more new list.
   c. Between any two existing elements of S, creating $N - 1$ new lists.

4. For each of the existing permutations of S the addition of a new element has created
   $1 + 1 + (N - 1) = N + 1$ new permutations.

5. The total number of permutations of a list S with $N + 1$ elements is, then: $(N + 1) \times N! = (N + 1)!$

C. So: $P(1)$ is true and $P(N) \Rightarrow P(N + 1)$
Therefore $P(N)$ is true for all N.

XIV: Th: DeMorgan's Law: $\boldsymbol{\overline{\bigcup_{j=1}^{n} A_j} = \bigcap_{j=1}^{n} \overline{A_j}}$

A. Basis step: $n = 2$ $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ DeMorgan's Law for two sets.

Recall: $x \in A_1 \cup A_2 \rightarrow x \in A_1 \vee x \in A_2$

Therefore: $x \in \overline{A_1 \cup A_2} \rightarrow \neg\left(x \in A_1 \vee x \in A_2\right)$

$$\equiv x \notin A_1 \wedge x \notin A_2$$

$$\equiv x \in \overline{A_1} \wedge x \in \overline{A_2}$$

$$\equiv x \in \overline{A_1} \cap \overline{A_2}$$

B. Inductive step:

Assume: $\overline{\bigcup_{j=1}^{n} A_j} = \bigcap_{j=1}^{n} \overline{A_j}$

Then: $\overline{\bigcup_{j=1}^{n+1} A_j} = \overline{\bigcup_{j=1}^{n} A_j \cup A_{n+1}}$ by the definition of union

So: $\overline{\bigcup_{j=1}^{n+1} A_j} = \overline{\bigcup_{j=1}^{n} A_j} \cap \overline{A_{n+1}}$ by DeMorgan's law for two sets.

Here, one set is $\bigcup_{j=1}^{n} A_j$ and the other is A_{n+1}

Then: $\overline{\bigcup_{j=1}^{n+1} A_j} = \bigcap_{j=1}^{n} \overline{A_j} \cap \overline{A_{n+1}}$ according to our inductive assumption.

and: $\overline{\bigcup_{j=1}^{n+1} A_j} = \bigcap_{j=1}^{n+1} \overline{A_j}$ by the associative law for disjunction

□

XV: Detecting a Counterfeit Coin

A. Problem Description

1. You are given a collection of coins with the proviso that one coin is a counterfeit.

2. Your scale is a simple balance scale that detects only that the object is one pan is lighter, heavier, or the same weight as the object in the other pan.

3. The counterfeit coin is lighter than any of the others.

4. Prove by induction that 3^N coins is the largest number for which the counterfeit coin can be detected in N weighings.

B. Basis Step:

1. For three coins, one weighing is sufficient.
   a. If the two coins, one in each pan of the balance, are of equal weight, the remaining coin is the counterfeit.

   b. Otherwise, the lighter coin is the counterfeit.

   c. $3 = 3^1$
2. For nine coins, two weighings are sufficient.
   a. If two coins on the balance are of equal weight, one of the remaining two coins is the counterfeit and one more weighing will detect the lighter coin which is the counterfeit.

   b. Otherwise, one weighing will detect the lighter coin which is the counterfeit.

   c. $9 = 3^2$

C. Inductive Assumption: 3^N coins is the largest number of coins for which a single counterfeit coin can be detected in N weighings.

D. Inductive Step:

1. Consider 3^{N+1} coins.

2. $3^{N+1} = 3 \times 3^N$

3. Therefore we can divide our set of 3^{N+1} coins into three sets of 3^N coins.

4. One weighing is required to determine which of the three sets contains the counterfeit coin.
   a. If, on the first weighing with 3^N coins on each pan of the scales, one side is lighter than the other, the lighter side contains the counterfeit coin.

   b. If both sides balance, the remaining set of 3^N coins contains the counterfeit coin.

5. According to our inductive assumption we can isolate the counterfeit coin from a set of 3^N coins in N weighings.

6. Since it requires at most one additional weighing to isolate the counterfeit coin from a set of 3^{N+1} coins the isolation will require at most $N+1$ weighings.

E. Test: Suppose that we have a set of $3^{N+1}+1$ coins.

1. We can place two sets of 3^N coins on each pan.
   a. If the two sets are of equal weight there will be more than 3^N coins remaining to be tested.

   b. If the two sets are of unequal weight we can test the lighter set of 3^N coins in N weighings.

2. For the case of equal weights we can divide the set of 3^N+1 into two sets of 3^{N-1} coins and one set of $3^{N-1}+1$ coins and repeat step E.1.

3. Proceeding in this way for a total of N repetitions leaves us with four coins.
    a. One more weighing is not guaranteed to isolate the counterfeit coin, two may be required.

    b. Therefore $(N+1)+1$ weighings may be required.

4. Therefore a set of $3^{N+1}+1$ coins may require more than $N+1$ weighings to determine the counterfeit coin.

F. Therefore: 3^N coins is the largest number of coins for which the counterfeit coin can be reliably detected in N weighings on a two pan balance.