PHYS 121 – SPRING 2015

Chapter 10: Systems of Particles, Momentum & Center of Mass

version 03/18/2015 ~ 86 slides

We completed this lecture on 3/16/2015, skipping the last several slides.

Get your clickers ready.

ANNOUNCEMENTS

I'm back!

Don't be frightened!

Released in 1980

ANNOUNCEMENTS

- **We're postponing Chapter 9 (*Gravitation*) until April, after we've covered angular motion more carefully.**
- **The next homework assignment is due Monday, March 16.**
  - ***Next week is spring break!***
  - ***Check for updates later today, after I see how far we get in lecture.***

Collect clicker BP's today – and last Monday?

Lake Erie →
tomorrow?

http://www.theofficialguidetospringbreak.com/wp-content/uploads/Lake-Bryan-Spring-Break-Country-Music-Lovers.jpg

ANNOUNCEMENTS: Midterm Course Evaluations

- **120 comments on helpful aspects of the class**
- **96 suggestions for improvements**

Case Western Reserve University
96:Midterm POC

Course: PHYS121[PHYS] : General Physics I - Mechanics

Instructor: Gary Chottiner *

1 - How would you rate the pace of the course?

Response Option	Weight	Frequency	Percentage
Very fast	(1)	13	9.29%
Rather fast	(2)	50	35.71%
Moderate	(3)	70	50%
Rather slow	(4)	6	4.29%
Very slow	(5)	1	0.71%
No response/not applicable	(0)	0	0%

Return Rate	Mean	STD	Median
140/257 (54.47%)	2.61	0.76	3.00

2 - How would you rate the work load of the course?

Response Option	Weight	Frequency	Percentage
Very heavy	(1)	6	4.26%
Rather heavy	(2)	38	26.95%
Moderate	(3)	95	67.38%
Rather light	(4)	2	1.42%
Very light	(5)	0	0%
No response/not applicable	(0)	0	0%

Return Rate	Mean	STD	Median
141/257 (54.86%)	2.66	0.58	3.00

ANNOUNCEMENTS

A	93
B	92
C	42
D	16
F	14

Midterm grades will be posted Monday, March 9.

- *50% exam #1 + 25% homework + 25% lab*
- *Strictly > 90% = A, 80% = B, 70% = C, 60% = D*
- *No bonus points or accommodations for missed homework, etc.*

GOOD NEWS: 67% of your course grade is yet to be determined!

BAD NEWS: Another 15% will be settled by Exam #2 on the Friday after Spring Break.

http://www.sherv.net/happy-emoticons.html

(*LINEAR or TRANSLATIONAL*) MOMENTUM

(*We'll encounter ANGULAR momentum later this semester*)

Newton's Second Law, Σ**F** = *m***a**, can be written in 1D as $F_{net} = m\frac{dv}{dt}$

If *m* is constant $F_{net} = \frac{d(mv)}{dt}$

The quantity *mv* is given the name ***momentum*** and the symbol *p*.

$$p \equiv mv$$

$$\Rightarrow \quad F_{net} = \frac{dp}{dt}$$

In higher dimensions, **F**, **v** and **p** must be treated as vectors.

$$\vec{p} = m\vec{v} \qquad \vec{F}_{net} = \frac{d\vec{p}}{dt}$$

The direction of an object's momentum = direction of its velocity.

LINEAR MOMENTUM

- For a system of particles or a macroscopic object:

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \qquad \vec{F} = \frac{d\vec{P}}{dt}$$

The switch to a capital **P** indicates momentum of a system or solid.

- Kinetic energy can be written in terms of momentum

$$K = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

- Solving problems using momentum is **often easier** than using **F** = *m***a**.
- Some problems **can't be solved** with **F** = *m***a** but can with **P.**
- Momentum is useful in quantum (*wave*) mechanics & relativity.
  - **F** = *m***a** is not applicable in much of modern physics.
  - It's hard to apply a force to a wave, but a wave does have momentum.

CONSERVATION OF MOMENTUM

If there are no external forces acting on objects in a system, then

$$\mathbf{F}_{\mathbf{ext}} = \mathbf{0}$$

$$\vec{F} = \frac{d\vec{P}}{dt} \quad \Rightarrow \quad \frac{d\vec{P}}{dt} = 0$$

$\Rightarrow$ **P is constant: $\mathbf{P}_{\boldsymbol{final}} = \mathbf{P}_{\boldsymbol{initial}}$**

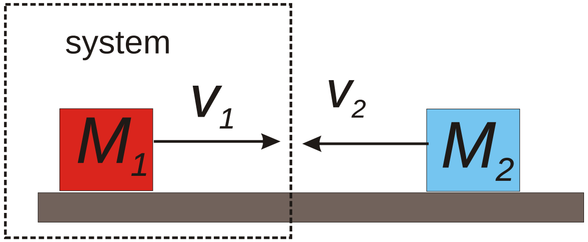
P is constant $\Leftrightarrow$ $\mathbf{F}_{\mathbf{ext}} = \mathbf{0}$

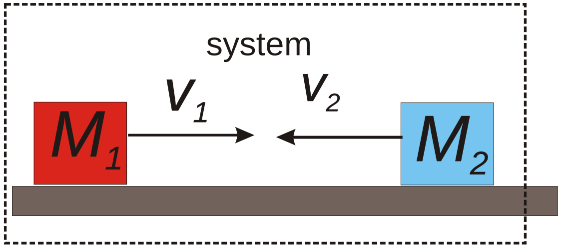
MOMENTUM IS CONSERVED

if the **net external** force acting on a system **is zero**.

*Don't forget the **magic words**.*

Define your system carefully to distinguish external from internal forces.





CONSERVATION OF MOMENTUM

Internal forces can't change the momentum of a system.

Newton's Third Law, *action-reaction*, *says*

$$\vec{F}_{12} = -\vec{F}_{21} \quad \underset{\text{during any interaction}}{\Rightarrow} \quad \frac{\Delta \vec{p}_1}{\Delta t} = -\frac{\Delta \vec{p}_2}{\Delta t} \quad \Rightarrow \quad \Delta \vec{p}_1 + \Delta \vec{p}_2 = 0$$

$$\Delta \vec{P} = \sum_i \Delta \vec{p}_i = 0$$

$\Rightarrow$ Ignore *internal forces* when using momentum methods.

LINEAR MOMENTUM

$\vec{F} = \dfrac{d\vec{P}}{dt}$ is correct even when the mass is changing, in which case

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = m\vec{a} + \vec{v}\frac{dm}{dt}$$

This is a more complete form of Newton's 2^{nd} Law.

An example of changing mass is a rocket, which expels mass from its tail in order to propel itself upward.

We'll consider rockets shortly, although Ohanian does not.

<u>CONSERVATION OF MOMENTUM</u>

<u>**CONSERVATION OF MOMENTUM**</u>

ranks with

<u>**CONSERVATION OF ENERGY**</u>

as a bedrock principle of physics.

You can use conservation of momentum *iff* there are <u>no external forces</u> <u>along some direction</u> of your <u>system</u>.

- **P** is a vector.
- Momentum is conserved along one direction even if there are forces acting in perpendicular directions.
- *Momentum is not conserved in those other directions.*

CULTURAL INTERLUDE

how physicists think

$$\frac{d\vec{P}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} \underset{\text{if m is constant}}{=} \frac{m\,d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

- We derived momentum from Newton's 2nd Law, *right to left* above.
- It's just as easy to derive Newton's 2nd from the definition of momentum.
- Where does Conservation of Momentum come from?

ANSWER: THE SYMMETRY OF EMPTY SPACE.

- If space is featureless & symmetric in some direction, say *x*,
  - $\Rightarrow$ There is no special origin for $x = 0$.
  - $\Rightarrow$ Every point *x* is equivalent to any other point *x*.
  - $\Rightarrow$ A choice of *x* for describing the position of an object is arbitrary.
  - $\Rightarrow$ If you shift your *x*-axis horizontally, no fundamental property of an object should change, including its momentum.

CULTURAL INTERLUDE

how physicists think

The momentum of an object in empty space is the same everywhere, for any choice of x.

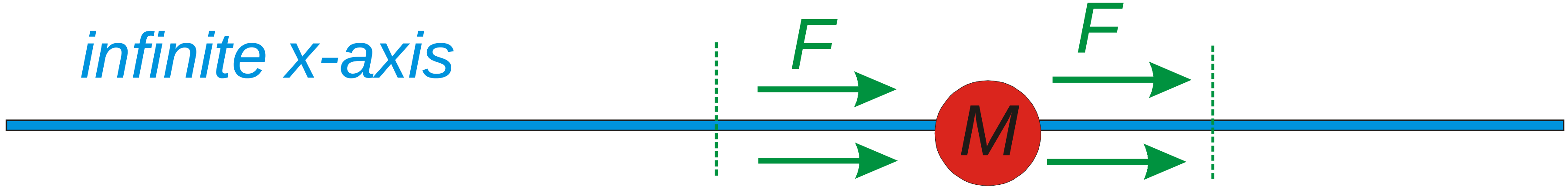
MOMENTUM IS CONSERVED

The presence of a force

breaks the symmetry

of empty space.

- There's now something special about part of space.
- Momentum is no longer conserved there.
- The value of x matters.



CULTURAL INTERLUDE: *Modern Physics*

Quantum mechanics introduces "*conjugate variables*" that show up in *Heisenberg's Uncertainty Principles*:

$\Delta p \Delta x \geq \frac{1}{2}\hbar$ and $\Delta E \Delta t \geq \frac{1}{2}\hbar$ where $\hbar$ is Planck's Constant

- Momentum is intertwined with position.

  $p(t)$ and $x(t)$ are equivalent ways to describe the motion of a particle, except for a constant x_o that probably doesn't matter.

- If there's no special value of position, $\Delta x \to \infty$, then the conjugate variable of position (*momentum*) is fixed, $\Delta p \to 0$.
- Energy E is intertwined with frequency & time. ($E = \hbar\omega \sim 1/time$)
- If there's no special value of time, $\Delta t \to \infty$, then the conjugate variable of time (*energy*) is fixed, $\Delta E \to 0$.

Don't worry if this doesn't make sense.

It's cultural for you, introduced in PHYS 221

& REQUIRED for physics majors.

Also, much of modern physics doesn't make sense!

- 5 options

$$\vec{F} = \frac{d\vec{P}}{dt} \quad \text{with} \quad \vec{P} = m\vec{v}$$

Two discs sit at rest on a frictionless horizontal surface. One disc has mass M while the other disc has mass $2M$. The same <u>constant</u> force F is applied to each disc for a <u>certain time</u> t.

Which disc gains the larger momentum over this time?

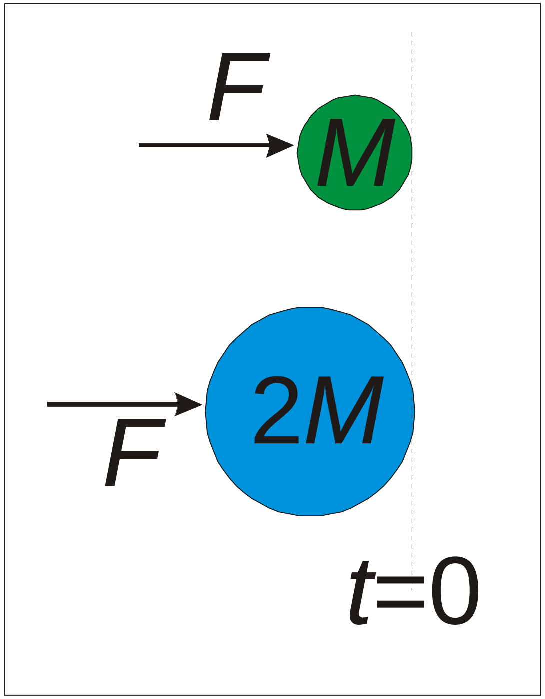
A. The disc with mass M.

B. The disc with mass $2M$.

C. They both gain the same amount of momentum.

D. It depends on the value of the applied force.

E. None of the above; quit trying to confuse me.



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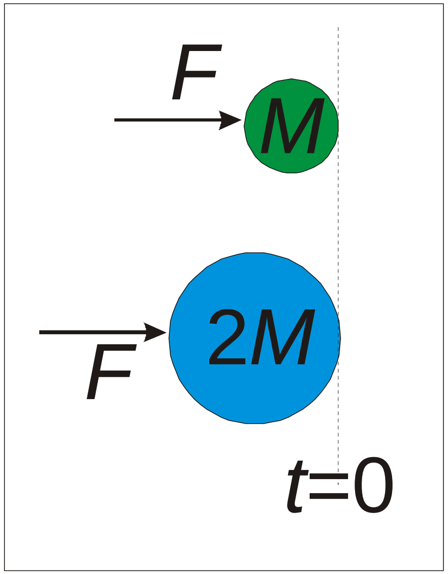

A. The disc with mass M.

B. The disc with mass $2M$.

C. They both gain the same amount of momentum.

D. It depends on the value of the applied force.

E. None of the above; quit trying to confuse me.

$\Delta p = F\Delta t$ and that's the same for both masses.

It is, however true that the lighter mass will be moving faster after this time, exactly twice as fast since it has half the mass and $p = mv$.

- 5 options

A large truck collides head-on with a small sports car (***NOT Dr. C's Porsche!***). The change in momentum of the car is Δp_{car} and the change in momentum of the truck is Δp_{truck}.

Ignoring road friction and the crunching sounds of metal & bone,

which of the following statements is correct?

A. $|\Delta p_{truck}|$ is larger than $|\Delta p_{car}|$

B. $|\Delta p_{car}|$ is larger than $|\Delta p_{truck}|$

C. $|\Delta p_{car}| = |\Delta p_{truck}|$

D. The answer depends on the speed of the truck.

E. The answer depends on whether energy is conserved during the collision.

- 5 options

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D. The answer depends on the speed of the truck.

E. The answer depends on whether energy is conserved during the collision.

There are no external forces in the horizontal direction on the system of car + truck.

$\Rightarrow$ Overall momentum is conserved.

$\Rightarrow$ The change in momentum for the car is equal and opposite in sign to the change in momentum for the truck.

- 4 options

You’re driving down a narrow, one-way street at 35 mph when you notice a car identical to yours coming straight towards you at 35 mph. You have only two options: hitting the other car head on or swerving into a massive concrete wall, also head on. **Your best option, to minimize the force you experience, is to**

A. hit the other car

B. hit the wall

C. hit either one—it makes no difference

D. pay more attention in class

- 4 options

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A. hit the other car

B. hit the wall

C. hit either one—it makes no difference, except perhaps to your insurance company

D. pay more attention in class

Your change in momentum is the same in either case, you start with v = 35 mph and end up with v = 0.

There's no change in the time it takes to stop either. Imagine a thin sheet of steel between the cars. It stays in place, just like the wall, if the cars collide.

So $F_{avg} = \Delta p/\Delta t$ is the same.

- 6 options

A 1-kg block falls 5 m from rest before hitting the ground. When it hits, what force, F, does the rock exert on the ground? (*Assume $g \sim 10\ m/s^2$*)

A. 0.2 N
B. 5 N
C. 50 N
D. 100 N
E. The force can't be determined.
F. The force can be determined, but I can't figure it out.

- 6 options

A 1-kg block falls 5 m from rest before hitting the ground. When it hits, what force, *F*, does the rock exert on the ground?

(*Assume g ~ 10 m/s*^{2})

A. 0.2 N

B. 5 N

C. 50 N

D. 100 N

E. The force can't be determined.

F. The force can be determined, but I can't figure it out.

The force $F = \Delta p/\Delta t$ but we aren't given Δt, how long the collision with the earth takes.

CENTER of MASS

beyond the spherical cow!

We've been treating objects (*except massive ropes*) as particles but it's time to handle things more realistically.

- Consider a system made up of lots of particles.
- We can analyze such a system with methods we've developed for single particles using the concept of

CENTER of MASS

CENTER of MASS, CM

- CM = point in space associated with an object.
- Objects move* as though all their mass is at their CM.
- Objects behave* as if external forces act at their CM.

$\Rightarrow$ **You can replace an object by its CM for many purposes.**

The proof will be provided later.

*In terms of translational motion.
This does not apply to rotational motion,
which we will study later.

POINT PARTICLES

For N particles numbered from *i =1 to i = N*

the *position* of the *center of mass* is given by the *weighted average* of the positions of the particles.

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M_{total}} \qquad \text{with} \qquad M_{total} = \sum_{i=1}^{N} m_i$$

$\vec{r}_{CM}$ has vector components

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}} \qquad y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{M_{total}} \qquad z_{CM} = \frac{\sum_{i=1}^{N} m_i z_i}{M_{total}}$$

POINT PARTICLES in a 2D UNIVERSE

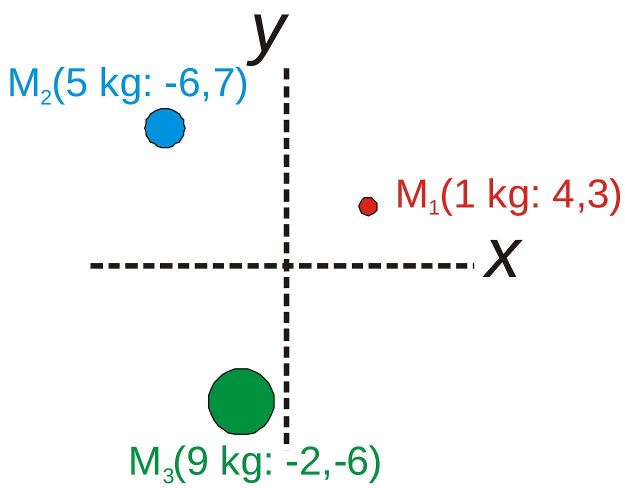
Three point particles have masses & positions given by

$M_1 = 1$ kg $\quad x_1 = 4$ m $\quad y_1 = 3$ m

$M_2 = 5$ kg $\quad x_2 = -6$ m $\quad y_2 = 7$ m

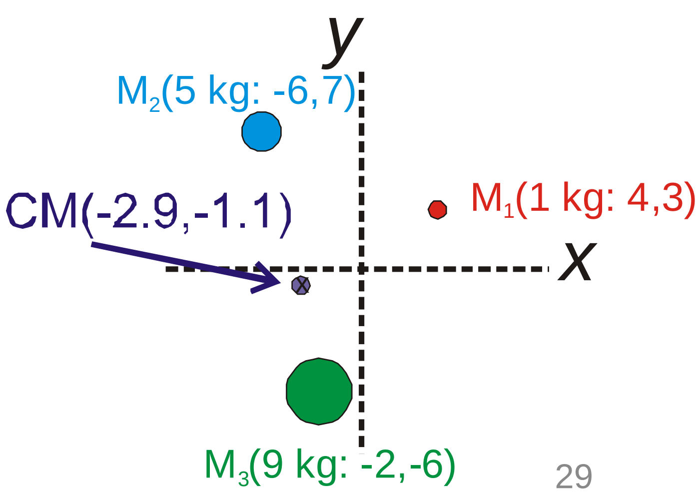
$M_3 = 9$ kg $\quad x_3 = -2$ m $\quad y_3 = -6$ m

Find the position of the CM.



$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{(1kg)(4m)+(5kg)(-6m)+(9kg)(-2m)}{1kg+5kg+9kg} = -2.9m$$

$$y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i} = \frac{(1kg)(3m)+(5kg)(7m)+(9kg)(-6m)}{1kg+5kg+9kg} = -1.1m$$



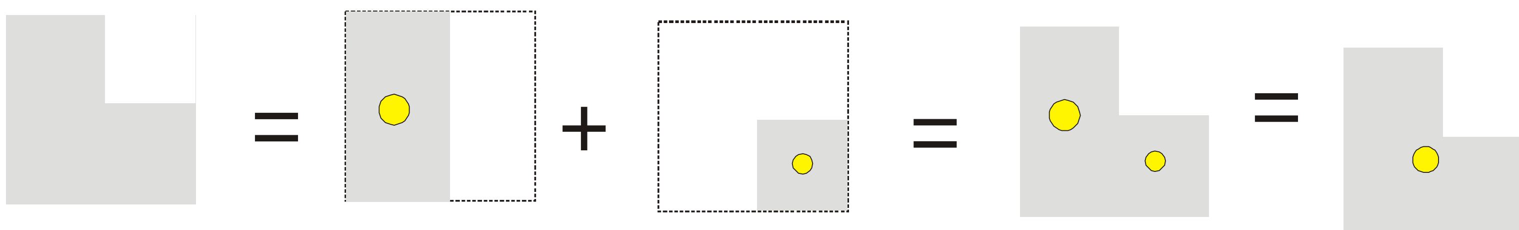
CM ADDITION

CM's of pieces of an object sum to an overall CM, just like particles.

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M_{total}}$$

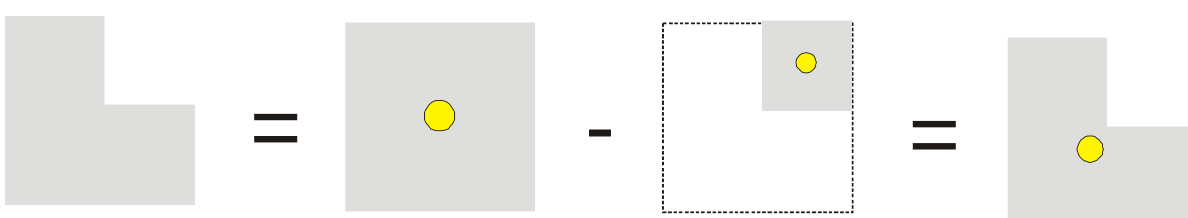
The proof is on the next slide.

You can also SUBTRACT pieces to find the CM.



CM? = $M_1 r_1$ + $M_2 r_2$

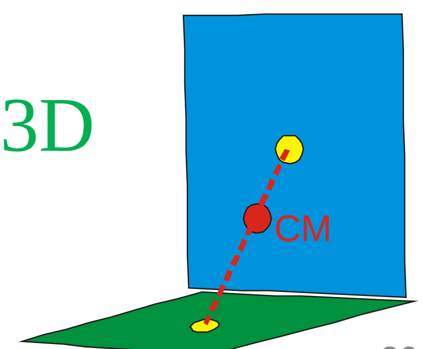
$$r_{CM} = \frac{M_1 r_1 + M_2 r_2}{M_1 + M_2}$$



CM? = $M_1 r_1$ - $M_2 r_2$

$$r_{CM} = \frac{M_1 r_1 - M_2 r_2}{M_1 + M_2}$$

3D



CM ADDITION *JUSTIFICATION*

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}}$$

Split an object that has N pieces into two objects,

one with pieces #1 to #*s* ($s < N$), mass m_1 & $\mathbf{r}_1 = \mathbf{r}_{CM1}$

and the other with pieces #(s+1) to #N, mass m_2 & $\mathbf{r}_2 = \mathbf{r}_{CM2}$.

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{s} m_i \vec{r}_i + \sum_{i=s+1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{s} m_i + \sum_{i=s+1}^{N} m_i} = \frac{\sum_{i=1}^{s} m_i \vec{r}_i + \sum_{i=s+1}^{N} m_i \vec{r}_i}{m_1 + m_2}$$

In the next line, we just multiply by the 1st term by m_1 / m_1 & the 2nd term by m_2 / m_2

$$\vec{r}_{CM} = \left(\frac{m_1}{m_1 + m_2}\right)\left(\frac{\sum_{i=1}^{s} m_i \vec{r}_i}{m_1}\right) + \left(\frac{m_2}{m_1 + m_2}\right)\left(\frac{\sum_{i=s+1}^{N} m_i \vec{r}_i}{m_2}\right)$$

$$\vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \text{QED}$$

We made it to slide 31 on Friday, March 6.

PHYS 121 – SPRING 2015

Chapter 10: Systems of Particles, Momentum & Center of Mass

version 03/16/2015 ~ 86 slides

We made it to slide #31 on 3/6/2015

Get your clickers ready.

ANNOUNCEMENTS

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ANNOUNCEMENTS

BAD NEWS: Friday = Exam #2

Wednesday = review, responding to questions asked by the class.

Exam #2 from spring 2014 is posted on Blackboard, under

COURSE DOCUMENTS > EXAMS > PRACTICE EXAMS

A tentative formula sheet for Exam #2 is posted on Blackboard.

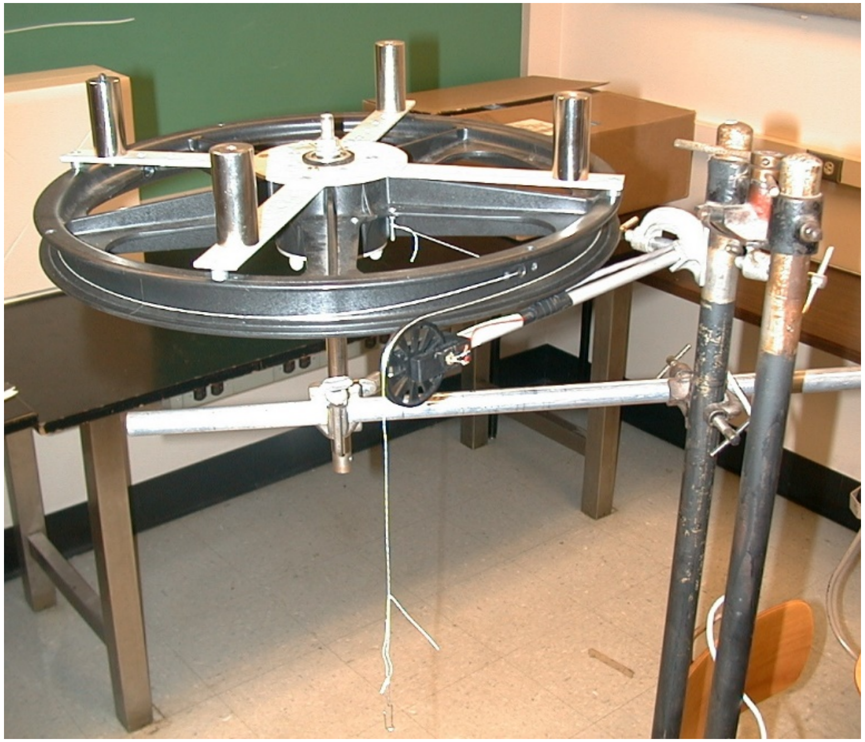
The homework due today on Chapter 8 is fair game for the exam but Chapter 10 problems (*CM & momentum*) are not.

Bonus Points were collected for class the Monday (*183 students*) & Friday (*82 students*) before Spring Break.

<u>Lab #4: Rotational Kinetic Energy</u>

March 18 - 26

You will use the principle of conservation of energy to determine the moment of inertia of a system of four identical masses symmetrically located on the circumference of a wheel which is rotating about its axis.



Tomorrow, Tuesday, March 17 is Saint Patrick's Day

I'll be calling on people wearing green!

REMINDER:

Momentum

$$\vec{p} = m\vec{v}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

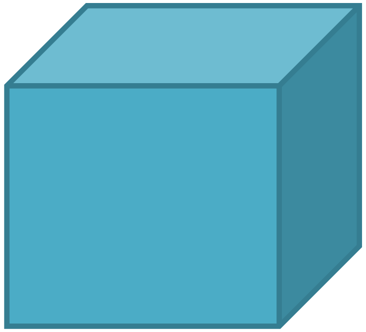
Conservation of Momentum:

If $\mathbf{F}_{\text{external}} = 0$, **P is constant**

Center of Mass

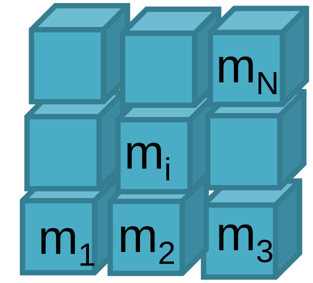
for particles

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}}$$



CM CALCULATIONS

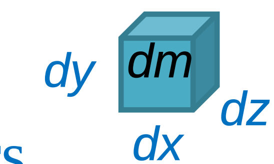
for solid objects rather than particles



Conceptually, break the object into *many* (***many*** $\equiv N$) small chunks of mass ~ particles $\Delta m \to m_i$ and add them.

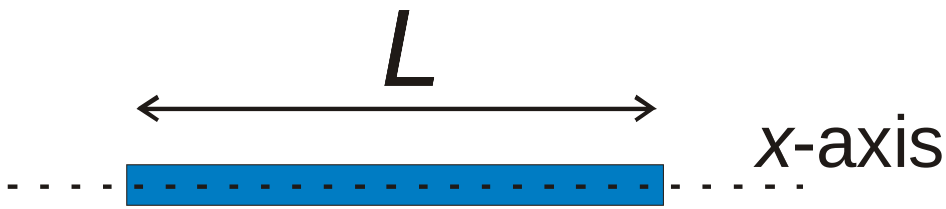
$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}} \qquad y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{M_{total}} \qquad z_{CM} = \frac{\sum_{i=1}^{N} m_i z_i}{M_{total}} \quad \text{with} \quad M_{total} = \sum_{i=1}^{N} m_i$$

Actually, break the object into an *infinite #* of *infinitesimal* chunks *dm* and replace the sums with integrals, $\sum m_i \to \int dm$ with *dm* expressed as a function of *xyz* & *dxdydz* and the limits of integration set by the object's borders.



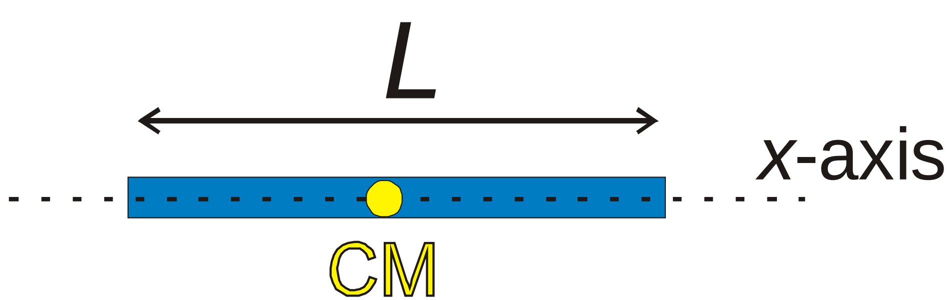
$$x_{CM} = \frac{\int x\,dm}{M_{total}} \qquad y_{CM} = \frac{\int y\,dm}{M_{total}} \qquad z_{CM} = \frac{\int z\,dm}{M_{total}} \qquad \text{with} \qquad M_{total} = \int dm$$

CM of a UNIFORM ROD



Where is the CM of a uniform rod of mass M and length L?

- Hopefully you can guess the answer; it's in the middle.
- This demonstrates the **power of symmetry** in physics.
- And if you did guess the answer, it shows that somewhere deep inside you is the **soul of a physicist**.



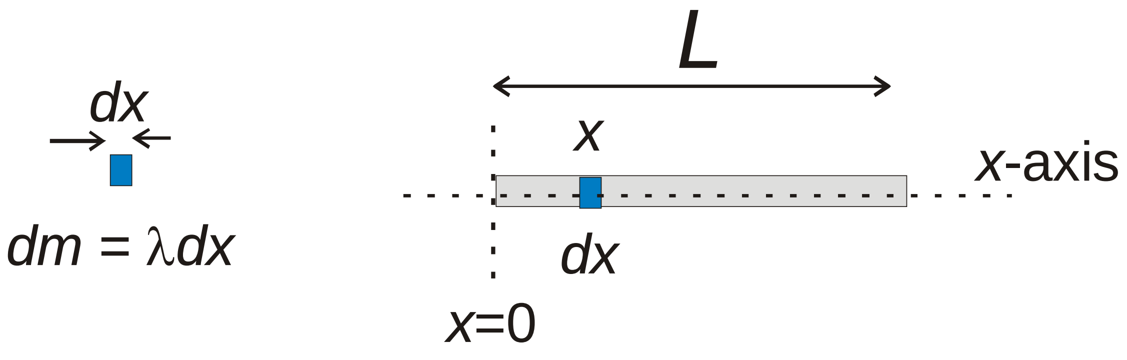
BRUTE FORCE METHOD

no culture needed, just math

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}}$$

Break the rod into infinitesimal pieces of length *dx*.

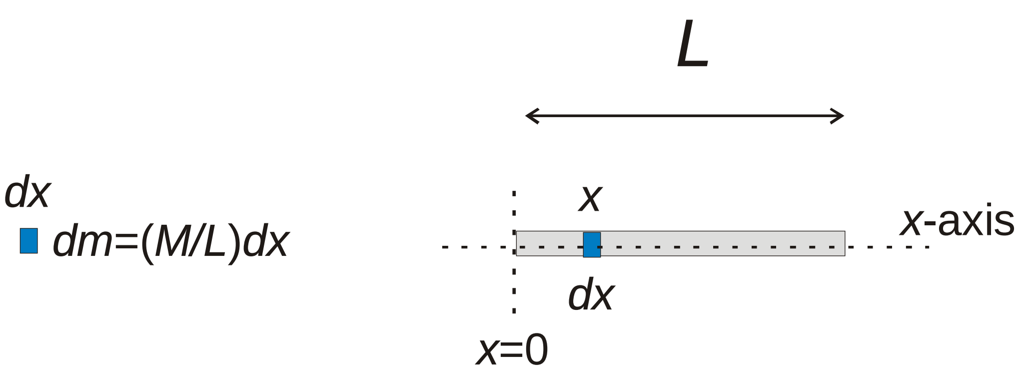
- The mass of each infinitesimal piece is $dm = \lambda dx$
  - the Greek letter *lamda* λ is the *linear mass density*
- λ = overall mass / overall length = M/L ~ ρ = *mass/volume*
- Some people prefer to think of $dm = M(dx/L)$ instead of $dm = \lambda dx = (M/L)dx$ but they are equivalent, for now.
- You can set your origin anywhere but choose wisely.



BRUTE FORCE METHOD

no culture needed, just math

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}}$$



$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}} = \frac{\int_{x=0}^{x=L} x\,dm}{M_{total}}$$

dm must be converted to a dx form.

You mustn't have hidden factors of x in your integrals.

$$x_{CM} = \frac{\int_{x=0}^{x=L} x(\lambda dx)}{M_{total}} = \left(\frac{\lambda}{M_{total}}\right)\left(\frac{x^2}{2}\right)_{x=0}^{x=L} = \left(\frac{\frac{M_{total}}{L}}{M_{total}}\right)\left(\frac{L^2}{2}\right) = \frac{L}{2}$$

The symmetry argument was easier!

But symmetry may not work for more complex shapes.

SYMMETRY ARGUMENTS

SOPHISTICATED (*cultured*) EXPLANATION

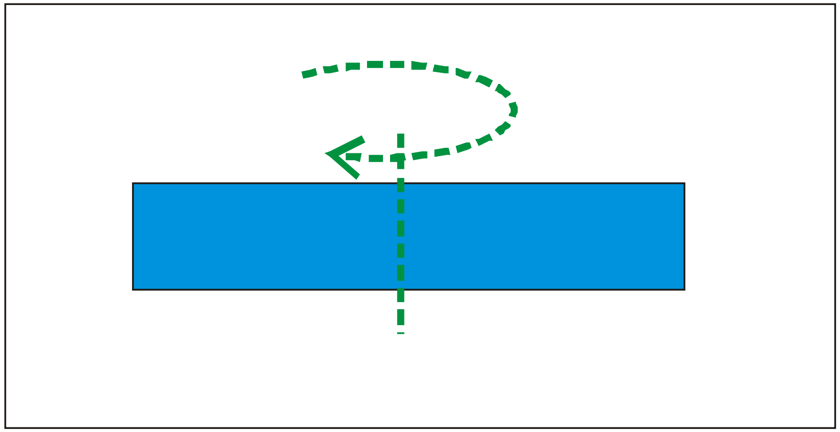
If an object is the same (*is fundamentally indistinguishable*) when rotated, reflected in a mirror, *etc*.

the CM must be the same too.

Consider the symmetry of the uniform rod in the figure.

If you rotate it 180° about the axis shown (*or reflect it in a mirror through the middle*), **you can't know** that anything has changed,

including the CM.



SYMMETRY ARGUMENTS

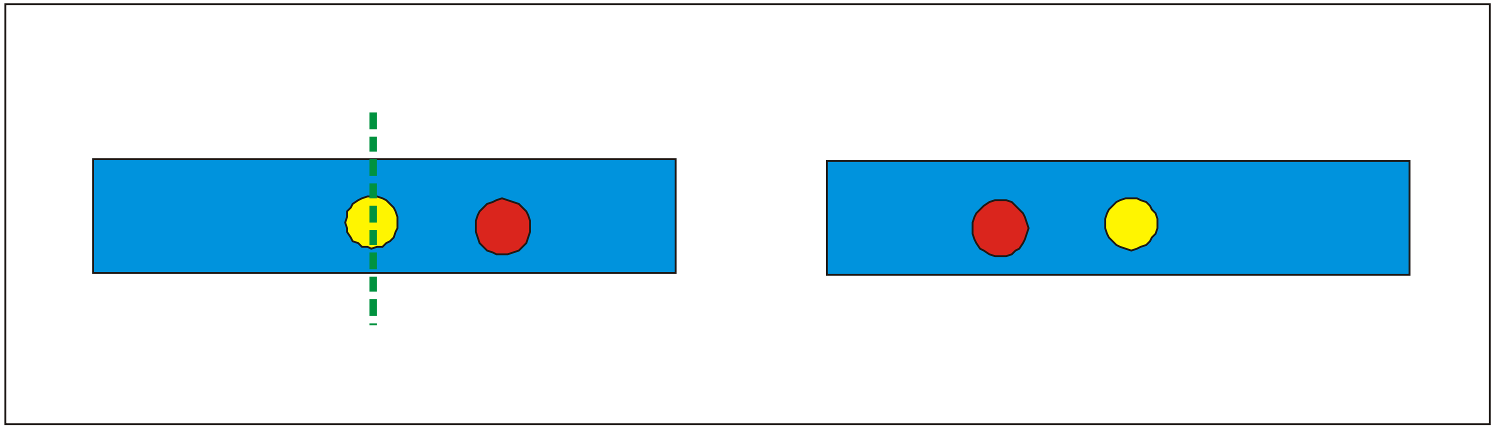
Which of the points in the figure below could possibly be the CM?

1. The yellow point in the middle doesn't move under rotation.
2. The red point moves from one side to the other side.

Symmetry $\rightarrow$ you **CAN'T KNOW** if the rod has rotated.

$\Rightarrow$the yellow point is the CM

since you CAN measure the position of the CM.



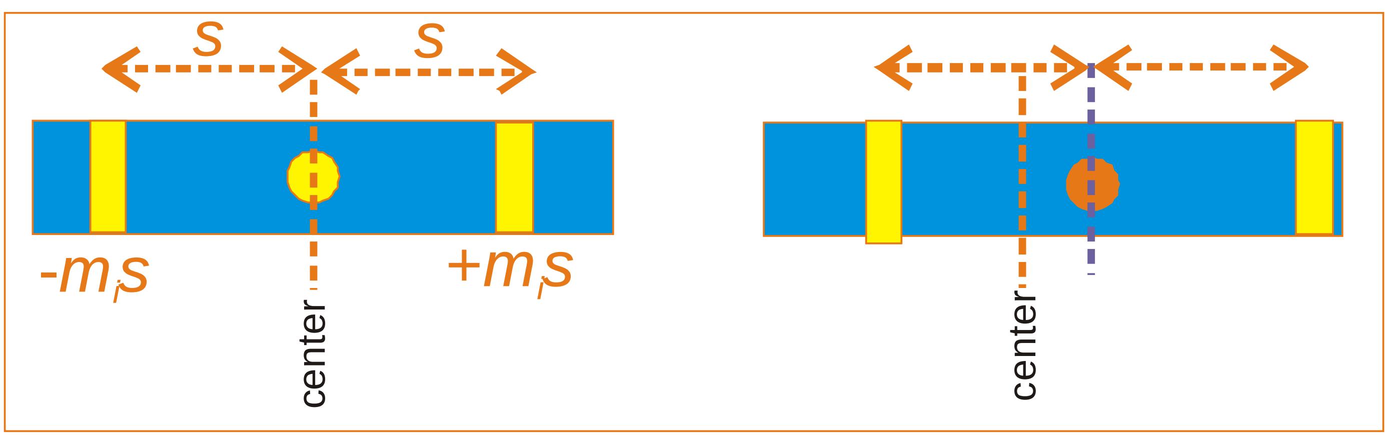
SYMMETRY ARGUMENTS II

LESS SOPHISTICATED (*but fine*) EXPLANATION

Every *dm* on one side of the CM at $\mathbf{r}_i$ must have an *evil twin* on the opposite side at $-\mathbf{r}_i$ that cancels it when you do the sum to determine $\mathbf{r}_{CM}$ relative to that position.

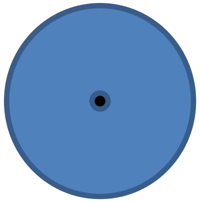
The yellow slices on opposite sides of the yellow dot cancel.

If you try to use the red dot as the CM, there's mass left over on the left when you do the sum.

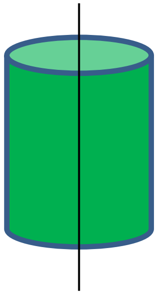


$$\vec{r}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M_{total}}$$

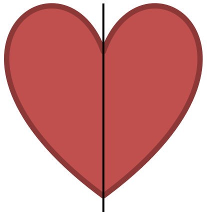
SYMMETRY in CM CALCULATIONS

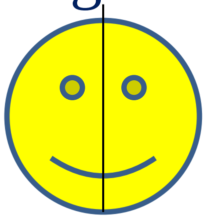


- If an object is symmetric about some point in space (*spherical symmetry*), the CM is at that point.
  - The CM of a solid or hollow sphere lies at the center of the sphere.



- If an object is symmetric about some axis (*rotational symmetry*), the CM lies somewhere along that axis.
  - The CM of a circle, hoop, disk, donut, ice cream cone, *etc*. lies somewhere on its axis.
- If an object is symmetric on either side of some plane, the CM lies somewhere in that plane.
  - The CM of the person sitting next to you lies somewhere along a vertical plane that cuts him/her in half side to side, bisecting the nose, belly-button and other body parts.

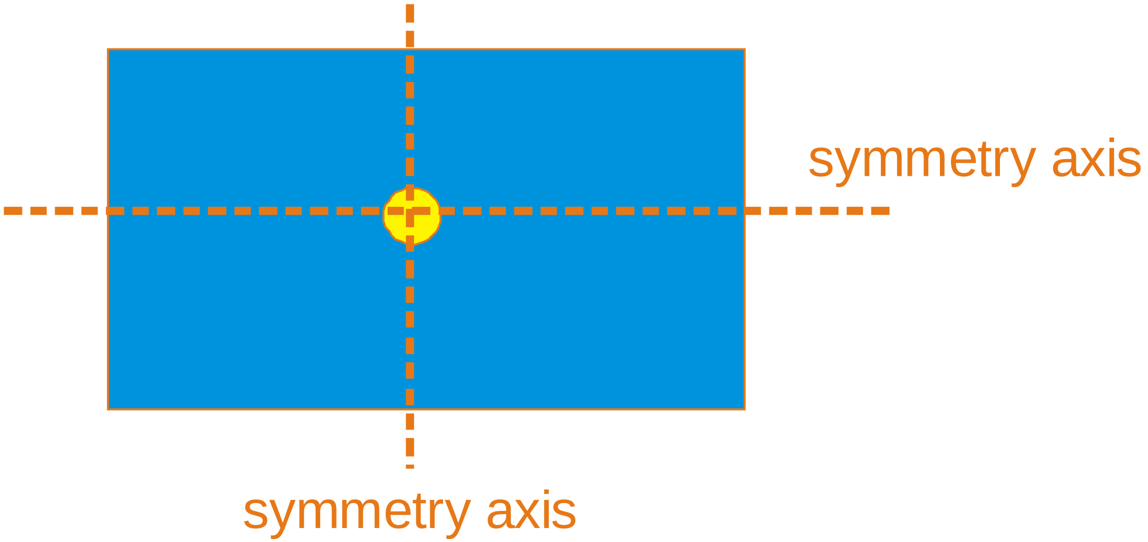




SYMMETRIES

If an object has multiple symmetries, the CM must fit all of them.

The CM of a square or rectangle or cube, *etc*. *m*ust be at its center.



NON-UNIFORM ROD

Let's break a symmetry!

A rod of length L has a linear mass density

(λ = *lambda* = *mass/length*)

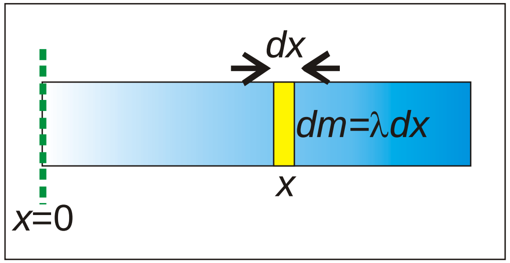
that **increases with distance** from its left end as

$$\lambda = \alpha x$$

where α is a constant and x is measured from the left end of the rod.

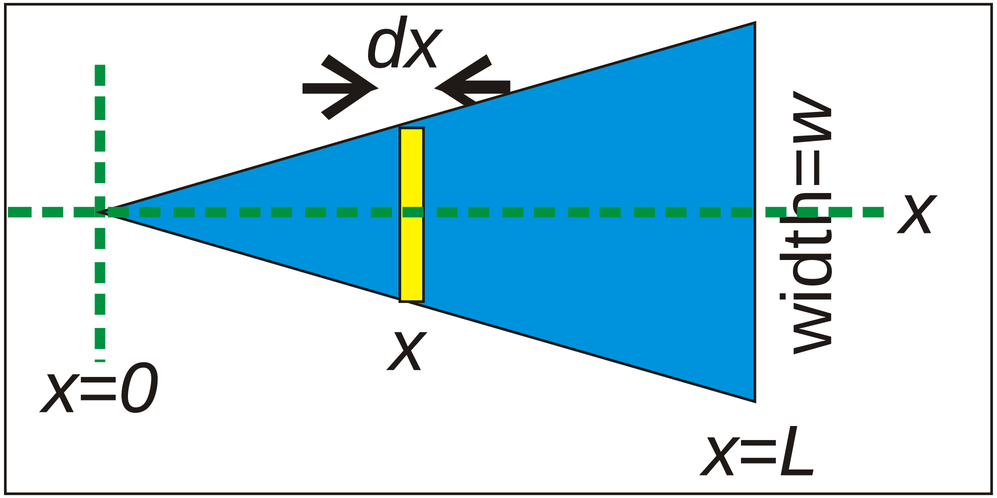
Find the position of the CM.

$$x_{CM} = \frac{\int x\,dm}{\int dm} = \frac{\int_{x=0}^{x=L} x(\lambda dx)}{\int_{x=0}^{x=L} (\lambda dx)} = \frac{\int_{x=0}^{x=L} x(\alpha x)\,dx}{\int_{x=0}^{x=L} (\alpha x)\,dx} = \frac{\left.\frac{x^3}{3}\right|_0^L}{\left.\frac{x^2}{2}\right|_0^L} = \frac{\frac{L^3}{3}}{\frac{L^2}{2}} = \frac{2}{3}L$$



Note that this compares to x_{CM} = ½ L for a uniform rod.

WEDGE



What's the CM for the flat wedge shown above?

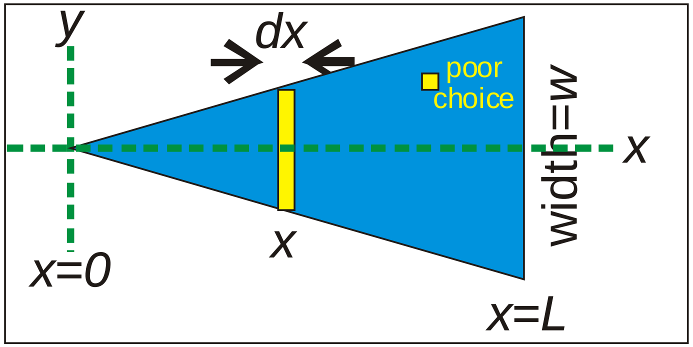
It has a length L, a width W at its base and an overall mass M.

Symmetry $\rightarrow$ CM is somewhere along the x-axis.

$$x_{CM} = \frac{\int x\,dm}{M_{total}}$$

Note the wise choice of coordinate system and origin.

WEDGE



$$x_{CM} = \frac{\int x\,dm}{M_{total}}$$

- Use mass element *dm* = strips of width *dx* a distance *x* from the origin.
- Every part of each strip is at the same *x* $\Rightarrow$ don't need *dm* ~ *dxdy*.

- *Surface mass density:* σ = *sigma*

  = mass/surface area

$$\sigma = \frac{M}{\left(\frac{LW}{2}\right)} = \frac{2M}{LW}$$

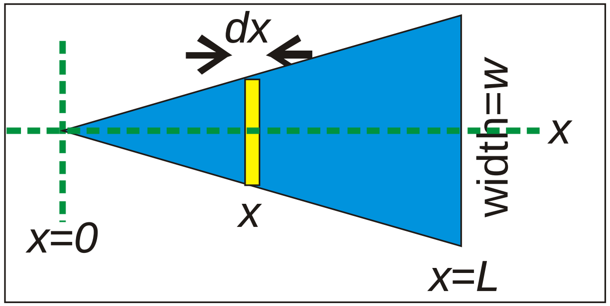
- Each infinitesimally wide strip has a height $2y = 2[(W/2)/L]x = (W/L)x$ and a surface area

$$da = \left(\frac{W}{L}x\right)dx$$

- Each strip has mass

$$dm = \sigma da = \sigma\left(\frac{W}{L}x\right)dx$$

WEDGE



$$x_{CM}=\frac{\int x\,dm}{M_{total}}=\frac{\int_{x=0}^{x=L}x\left(\sigma\frac{W}{L}x\,dx\right)}{M}=\frac{\frac{\sigma W}{L}\int_{x=0}^{x=L}x^2dx}{M}=\frac{\frac{\sigma W}{L}\frac{x^3}{3}\Big|_0^L}{M}$$

$$=\frac{\frac{\sigma}{L}W\frac{L^3}{3}}{M}=\frac{\left(\frac{2M}{LW}\right)\frac{WL^2}{3}}{M}$$

$$x_{CM}=\frac{2L}{3}$$

It should make sense that $x_{CM} > L/2$

since there's more mass to the right of $x = L/2$.

This is the same answer as the non-uniform rod $\lambda = \alpha x$ example because it's the same problem!

The mass at a distance *x* from the *y*-axis increases linearly with *x*.

ICE CREAM CONE

(*with a flat top*)

Consider a 3D cone of height *H* and radius *R*

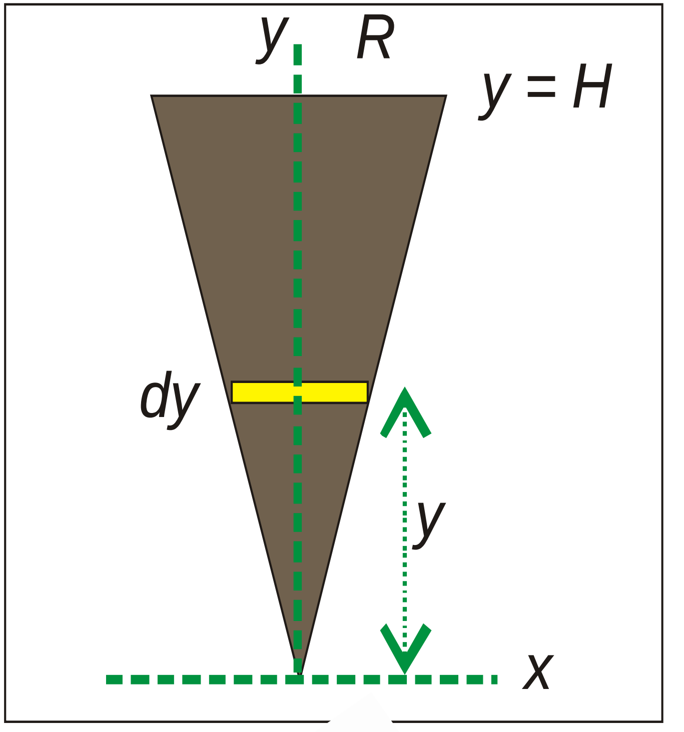
filled just to the top with

CHOCOLATE

ice cream of density ρ

(ρ = *rho* = *mass/volume*)

Symmetry ⇒ the CM lies somewhere along the *y*-axis.



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ICE CREAM CONE

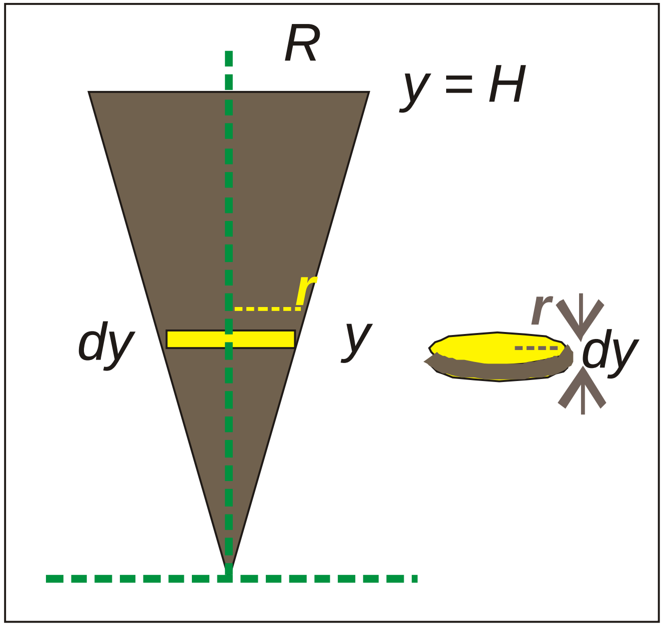
Elements of mass *dm* are

CHOCOLATE

discs of area πr^2

with $r = Ry/H$

and thickness *dy*



$$y_{CM} = \frac{\int y\,dm}{M_{total}} = \frac{\int_{y=0}^{y=H} y\left(\rho\pi r^2 dy\right)}{\int_{y=0}^{y=H} \left(\rho\pi r^2 dy\right)}$$

$$= \frac{\int_{y=0}^{y=H} y\left(\rho\pi\left(\frac{Ry}{H}\right)^2 dy\right)}{\int_{y=0}^{y=H} \left(\rho\pi\left(\frac{Ry}{H}\right)^2 dy\right)} = \frac{\int_{y=0}^{y=H} y^3 dy}{\int_{y=0}^{y=H} y^2 dy} = \frac{\frac{H^4}{4}}{\frac{H^3}{3}}$$

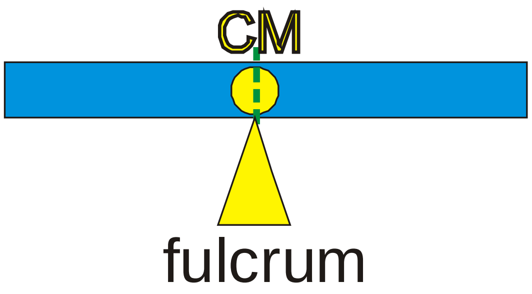
$$y_{CM} = \frac{3}{4}H$$

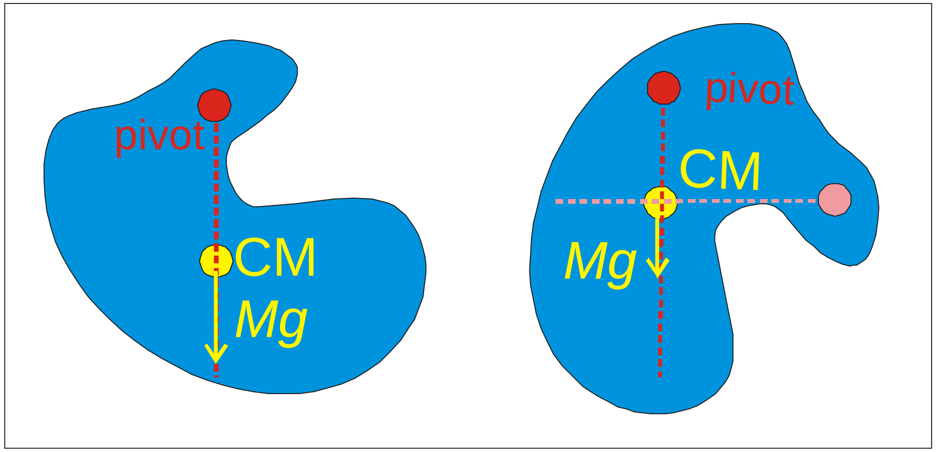
MEASURING THE CM OF ARBITRARY SHAPES

External forces, such as gravity, act at the CM of an object.

$\rightarrow$ gravity aligns the CM at or directly below a pivot point.

- For a 1D object, the CM = balance point.
- For 2D & 3D objects, the CM is stable directly below any pivot point so vertical lines drawn below 2 or 3 pivot points intersect at the CM.





Alternately, find a point in an object that moves like a particle, perhaps in projectile motion.

- 5 options

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}}$$

Where is the CM (x, y) of the system of 3 identical balls shown in the figure?

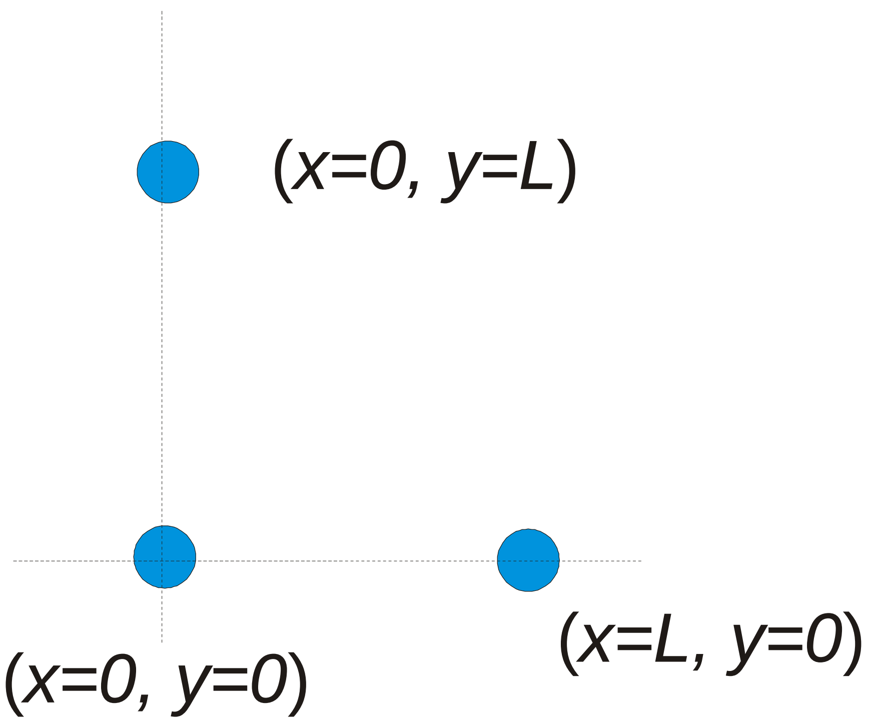
A. $L/2$, $L/2$

B. $L/3$, $L/3$

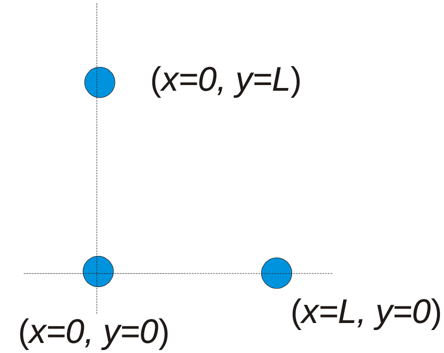
C. $L/2^{1/2}$, $L/2^{1/2}$

D. 0, 0

E. somewhere else



- 5 options



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A. $L/2$, $L/2$

B. $L/3$, $L/3$

C. $L/2^{1/2}$, $L/2^{1/2}$

D. 0, 0

E.Somewhere else

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}$$

$$\boldsymbol{x_{CM} = (m \cdot 0 + m \cdot 0 + m \cdot L)/(m+m+m)}$$

$$\boldsymbol{y_{CM} = (m \cdot 0 + m \cdot L + m \cdot 0)/(m+m+m)}$$

 - 4 options

Can an object’s Center of Mass ever lie outside of the material from which the object is constructed?

A. No, don’t be silly.

B. Yes it can!

C. The center of mass is not defined outside the object.

D. I have no idea what you are talking about.

 - 4 options

Can an object's Center of Mass ever lie outside of the material from which the object is constructed?

A. No, don't be silly.

B. Yes it can!

C. The center of mass is not defined outside the object.

D. I have no idea what you are talking about.

Consider a hoop or donut, where the CM lies at the center, or a U-shaped object or boomerang.

F=ma & CM

Given $\mathbf{r}_{CM}$ take the time derivative (*twice*) $\vec{r}_{CM} = \dfrac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}$

$$\vec{v}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{v}_i}{M_{total}} \qquad \vec{a}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{a}_i}{M_{total}}$$

The equation for a_{cm} can be rewritten as $\sum_{i=1}^{N} m_i \vec{a}_i = M_{total} \vec{a}_{CM}$

$\sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_i$ = sum of ***external*** forces acting on each individual mass, m_i in our system = *net force* = $\mathbf{F}_{\text{external}}$

Internal forces cancel, $\mathbf{F}_{12} = -\,\mathbf{F}_{21}$

Instead of worrying about the effect an external force has on each m_i we can use $\sum_{i=1}^{N} m_i \vec{a}_i = M_{total} \vec{a}_{CM}$ to write

$$\vec{F}_{external} = M_{total} \vec{a}_{CM}$$

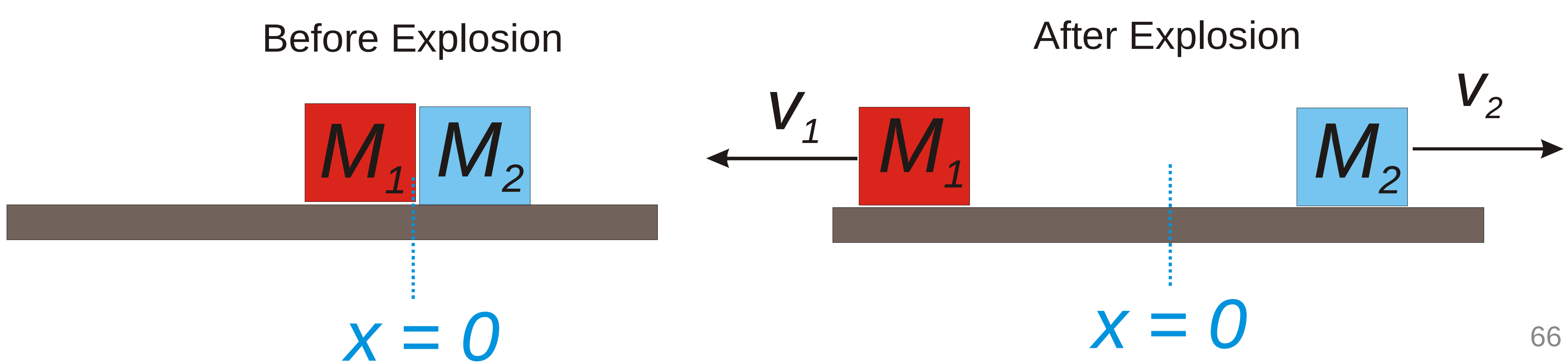
and treat the system like a single mass M_{total} at the position of its CM.

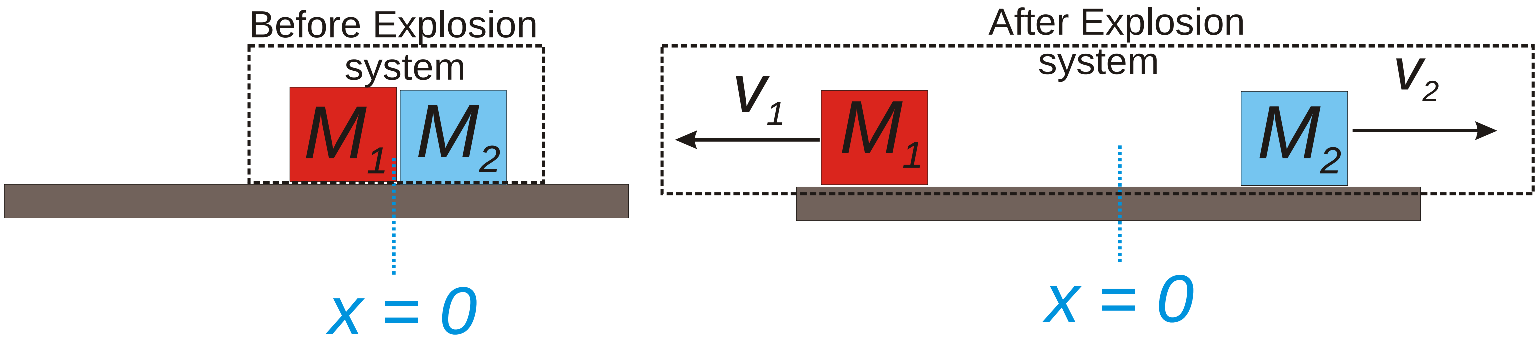
MOTION of the CM: EXPLOSIONS

Consider an object made up of

- a red part plus a blue part,
- initially stationary at $x = 0$,
- that are about to be separated because of an internal explosion.
- The parts have mass M_1 and M_2.
- The explosion causes them to fly apart at speeds v_1 and v_2.

What are these speeds, compared to each other?





- The CM before the explosion is obviously at $x = 0$ because both parts (= *system*) are there.

(*Remember that we're treating the pieces as point masses*.)

Before the explosion $$x_{CM} = \frac{(M_1 \bullet 0 + M_2 \bullet 0)}{(M_1 + M_2)} = 0$$

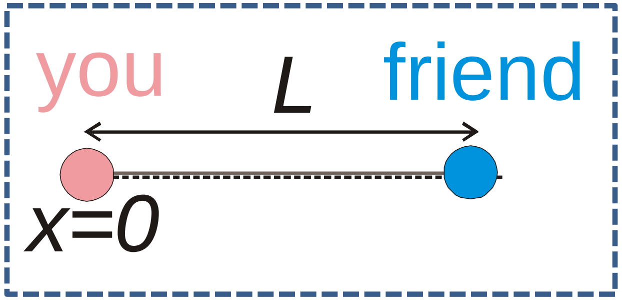
- There are no <u>external</u> forces $\Rightarrow$ CM remains at $x = 0$.

$$x_{CM} = \frac{(M_1 x_1 + M_2 x_2)}{(M_1 + M_2)} = \text{constant} = 0$$

$\Rightarrow M_1 x_1 + M_2 x_2 = 0$ at any time after the explosion

- After the explosion $x_1 = -(M_2/M_1)\, x_2$
- Taking the time derivative $\boldsymbol{v_1 = -(M_2/M_1)v_2}$

MAKING AN EXAMPLE of YOU

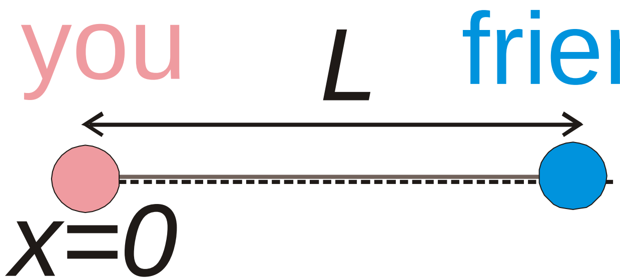


You (*mass* M_y) and a friend (*mass* M_f) are on frictionless ice separated by L. You each hold one end of an ideal, massless rope and pull until you meet.

Where will that be?

- Use an x-axis aligned with the rope.
  - You can pick $x = 0$ wherever you wish and still get the right answer
  - but one wise choice is to set $x = 0$ at your location.
  - Other wise choices are $x = 0$ at the CM, wherever that is, or at your friend's position.
  - Any other option is a poor choice since it will require more effort.
- There are no external forces in the x-direction on the system: you + friend.

  $\Rightarrow$ the CM of this system will not move.
- When you and your friend meet, it will be at the CM, so all we need to do is to calculate the position of the CM with respect to our $x = 0$.



$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}}$$

Let

i = 1 represents you with $m_1 = M_y$ at $x_i = 0$

i = 2 is your friend with $m_2 = M_f$ at $x_i = L$

Plugging into the equation for x_{CM}

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M_{total}} = \frac{M_y 0 + M_f L}{M_y + M_f} = \left(\frac{M_f}{M_y + M_f}\right) L$$

Does this answer make sense?

✓ If you and your friend have equal masses you meet halfway at $x = ½L$.

✓ If your friend is much lighter, you meet near you at $x_{CM} \sim 0$.

✓ If your friend is much heavier, you meet near him/her at $x_{CM} \sim L$.

- 4 options

You decide to play possum and only pretend to pull on the rope, letting your friend do all the work while you simply hold on. Assume that you have roughly equal masses.

Where will you meet now?

A. Still in the middle.
B. Closer to your friend who is doing all the pulling.
C. Closer to you since you aren't doing any pulling.
D. You'll never meet; your friend will finally get fed up with your laziness and leave. You'll go home and cry.

- 4 options

You decide to play possum and only pretend to pull on the rope, letting your friend do all the work while you simply hold on. Assume that you have roughly equal masses.

Where will you meet now?

A. Still in the middle.

B. Closer to your friend who is doing all the pulling.

C. Closer to you since you aren't doing any pulling.

D. You'll never meet; your friend will finally get fed up with your laziness and leave. You'll go home and cry.

There are still no *external* forces in this system so the CM doesn't change.

Also, the tension in an ideal rope is the same at both ends even if only one person "does all the pulling". You have to exert the same force as your friend does, just to hold onto the rope.

⇒ Your friend might not realize that you're lazy!

- 6 options

A 1-kg rock is suspended by a massless string from the left end of a 1 meter ruler. The ruler is balanced when a fulcrum is placed at the 0.25 meter mark, ¼ of the distance from its left end.

What is the mass of the ruler?

A. 0.25 kg

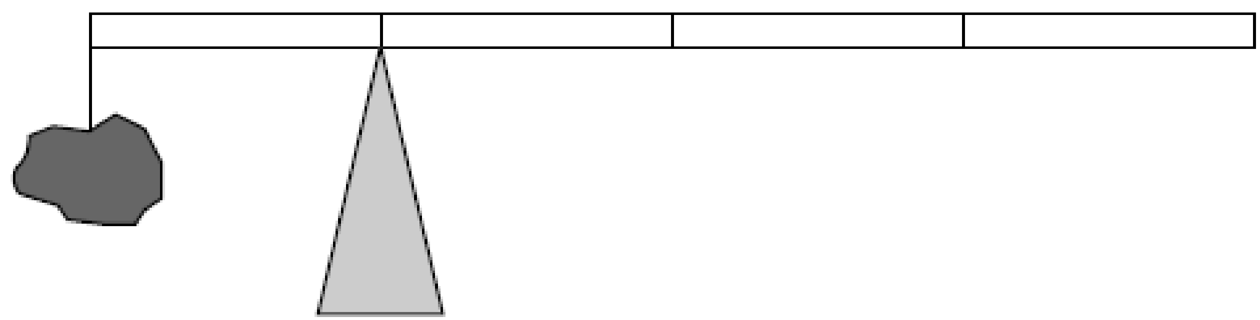
B. 0.5 kg

C. 1 kg

D. 2 kg

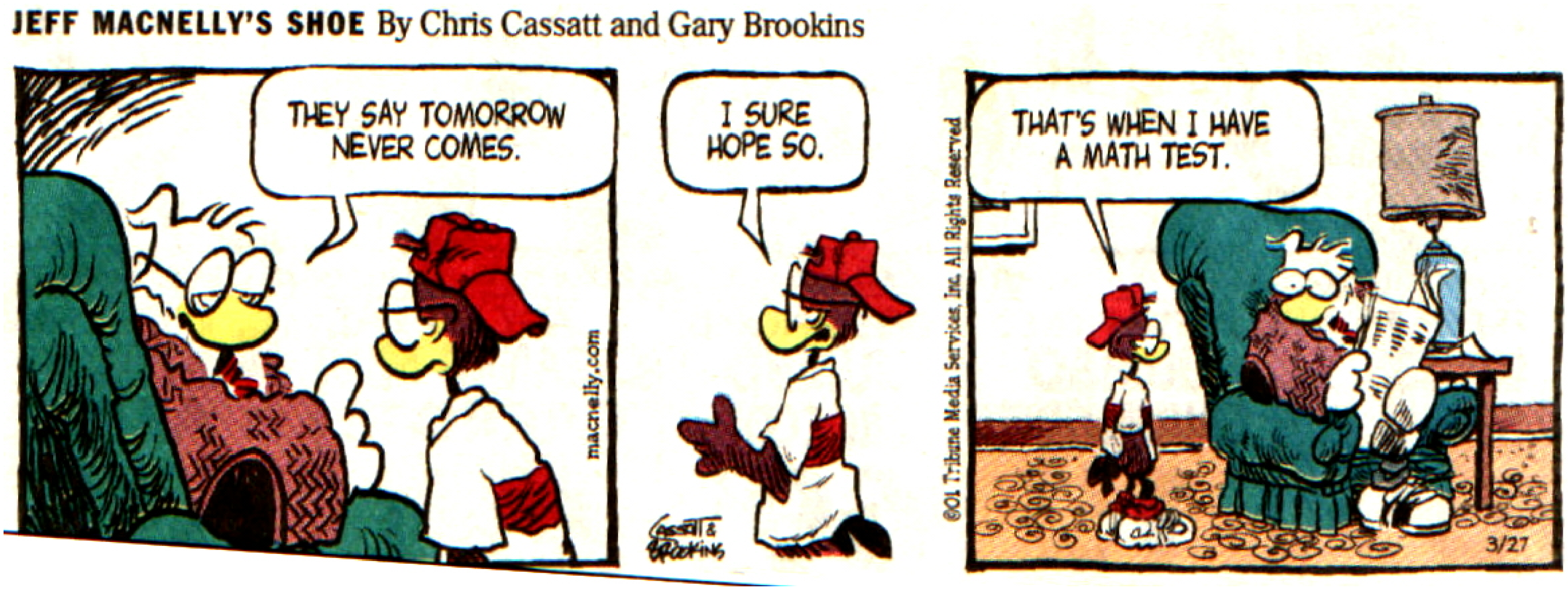
E. 4 kg

F. It can't be determined without more information.



We made it to slide #75 on Monday, March 16.

PHYS 121 – SPRING 2015



Chapter 10: Systems of Particles, Momentum & Center of Mass

version 03/18/2015 ~ 86 slides

We made it to slide #75 on 3/16/2015

ANNOUNCEMENTS

- **There IS a homework assignment due next Monday.**
  - The assignment was posted two days ago.
  - The problems are from chapter 10, momentum and center of mass.
  - We essentially finished chapter 10 in lecture on Monday.
  - The entire lecture file is posted on Blackboard, although we did not cover the last several slides in class.
  - We do have two days of cancelled class to make up!

ANNOUNCEMENTS

- **Exam #2 from spring 2014 is posted on Blackboard, under**

  COURSE DOCUMENTS > EXAMS > PRACTICE EXAMS
- **A tentative formula sheet for Exam #2 is posted on Blackboard.**
- **The exam covers chapters 5 through 8.**
  - Chapter 10 problems (*CM & momentum*) are not included.
  - Chapter 5, Newton's Laws of Motion, was partially covered on Exam #1.
  - If you can handle Chapter 6 material on Further Applications of Newton's Laws, there's no need to explicitly review Chapter 5.
- The style of the exam is the same as before; three homework-style problems + a couple of clicker-type concept questions.
- There are no numerical values for the three homework-style problems $\Rightarrow$ a calculator should not be necessary. (*This should save you time plugging in #.*)
- When you enter Strosacker, pick up a **BLUE BOOK** and sit in a aisle seat or 3, 5, 7, *etc.* seats in from an aisle. *We'll need about 30 students in the balcony.*
- Remember to bring a pencil and eraser; we don't have spares!

  Use a pen at your own risk!

KEY CONCEPTS FOR EXAM #2

The exam cannot cover every important topic in chapters 5 – 8!

But you need to be prepared for problems on every important topic.

- Friction (*with* $\Sigma F = ma$)
- Drag forces & terminal velocity
- Uniform Circular Motion
- Work
- Kinetic Energy
- Potential Energy (*springs* & *gravity*)
- Conservation of Energy
- Power

- 6 options

A 1-kg rock is suspended by a massless string from the left end of a 1 meter ruler. The ruler is balanced when a fulcrum is placed at the 0.25 meter mark, ¼ of the distance from its left end.

What is the mass of the ruler?

A. 0.25 kg

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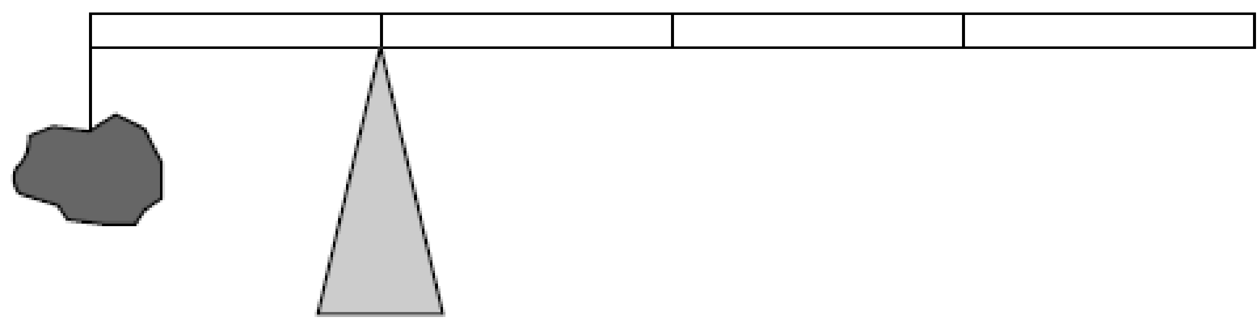
C. 1 kg

D. 2 kg

E. 4 kg

F. It can’t be determined without more information.



- 6 options

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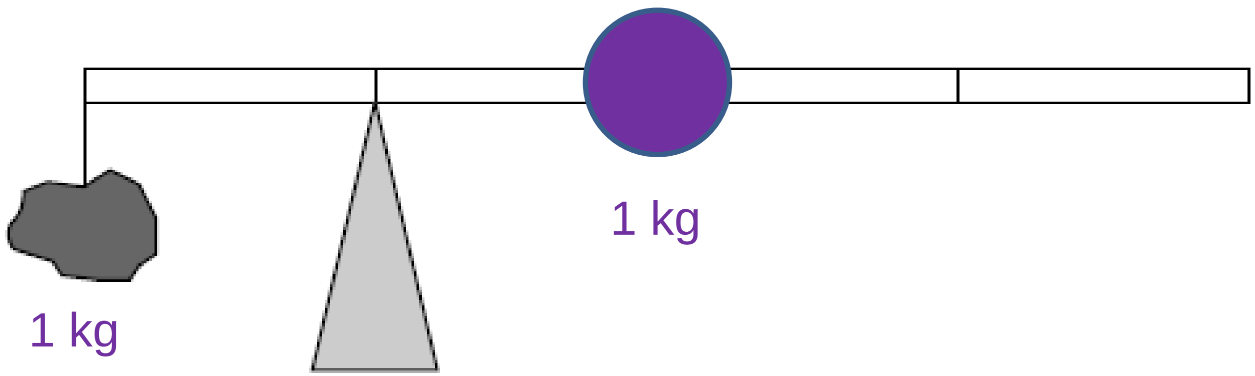


A. 0.25 kg

B. 0.5 kg

C. 1 kg

D. 2 kg

E. 4 kg

F. It can’t be determined without more information.

All of the ruler’s mass might as well be at its center of mass at the 0.5-m mark, the first mark to the right of the fulcrum.

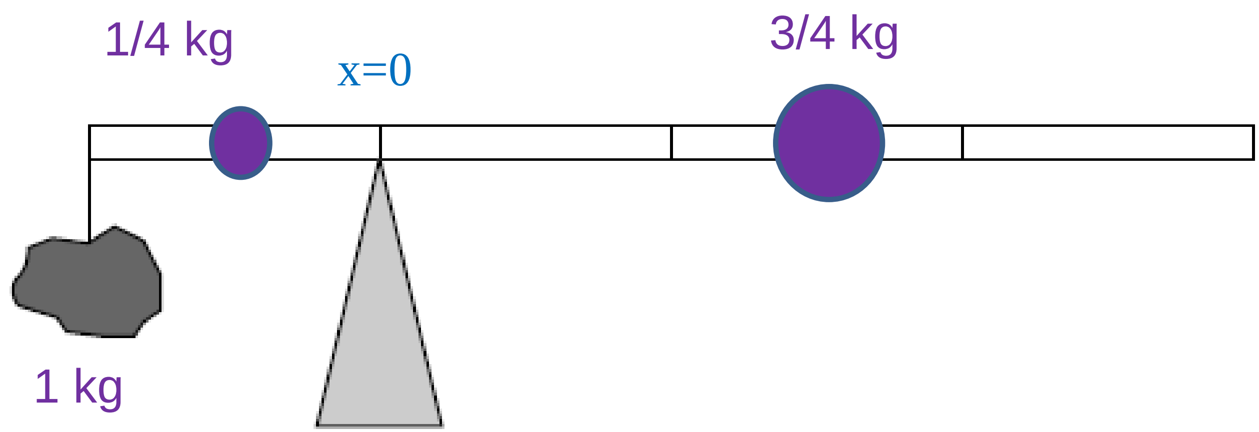
The fulcrum is midway between the two masses, and the system is balanced only if the total mass on the right (*the ruler’s CM*) is also 1 kg.

- 6 options

A 1-kg rock is suspended by a massless string from the left end of a 1 meter ruler. The ruler is balanced when a fulcrum is placed at the 0.25 meter mark, ¼ of the distance from its left end.

What is the mass of the ruler?

A.1 kg



Some people are skeptical of the previous explanation.

The HARD WAY to understand this, until we introduce *torque*, is illustrated above, with 1.25 kg on the left and 0.75 kg on the right.

This drawing separates the CM of the left & right sides of the meter stick.

As drawn above, picking $x = 0$ at the CM/fulcrum;

LEFT SIDE: (1 kg)(1/4 m) + (1/4 kg)(1/8 m) = 9/32 kg-m

RIGHT SIDE: (3/4 kg)(3/8 m) = 9/32 kg-m

- 5 options

The CM of a system constructed from two separate random objects with arbitrary shapes, masses and positions is:

A. Always outside both objects
B. Equidistant from the CM of each object
C. Always on a straight line drawn between the CM's of the two objects
D. Always inside one of the objects and outside the other object
E. Impossible to characterize without more information

- 5 options

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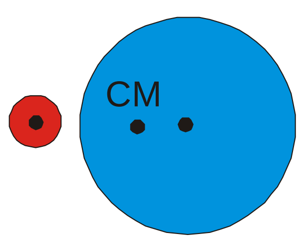
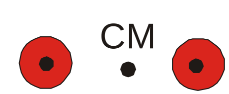
This comes from our CM addition rule.

Call that straight line your x-axis. From the definition of CM, we know the $y_{CM} = (1/M)\Sigma m_i y_i = 0$ for the CM of each object.

Some examples of why A, B, & C are wrong are shown below.



A & B are wrong



D is wrong

- 6 options

Suppose A in the figure below is a **uniform blueberry pie** and has a piece cut out of it as shown.

Which of the following can be found without doing an integral?

A. The CM of A

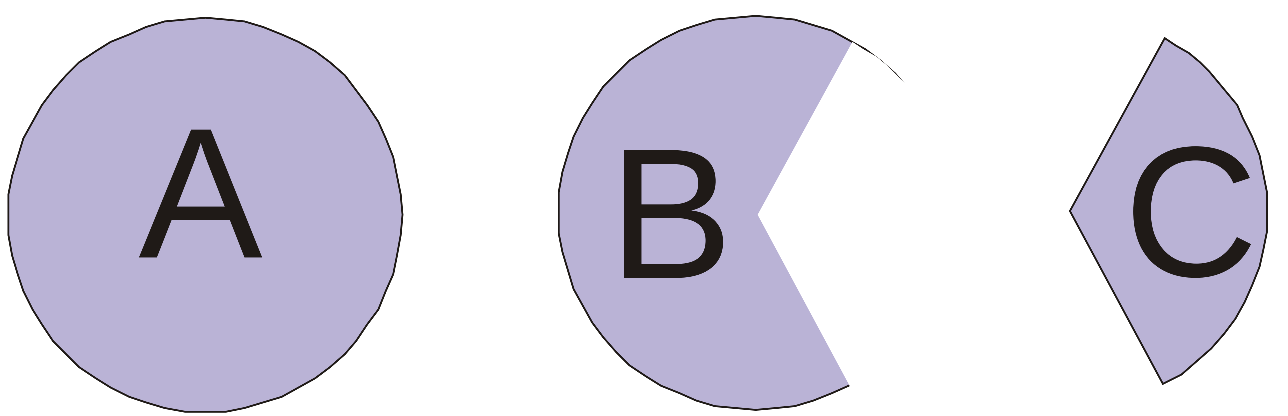
B. The CM of B

C. The CM of C

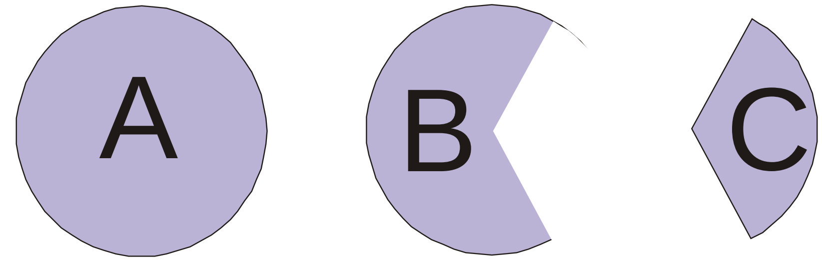
D. The CM of A & B but not C

E. The CM of A & C but not B

F. They can all be calculated without an integral.



- 5 options



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C. The CM of C

D. The CM of A & B but not C

E. The CM of A & C but not B

F. They can all be calculated without an integral.

We know A's CM by simple symmetry; it's at the center of the circle.

To find the CM of B or C we need to know the sector angle of the cut and perform a non-trivial calculation, integrating small elements of the piece. **You'll do this for homework!**

If you did know the CM of either B or C, the other CM would be easy to find using our CM addition rule.

 - 4 options

The CM of a baseball bat is (*obviously, I hope*) closer to the heavier, wider end of the bat.

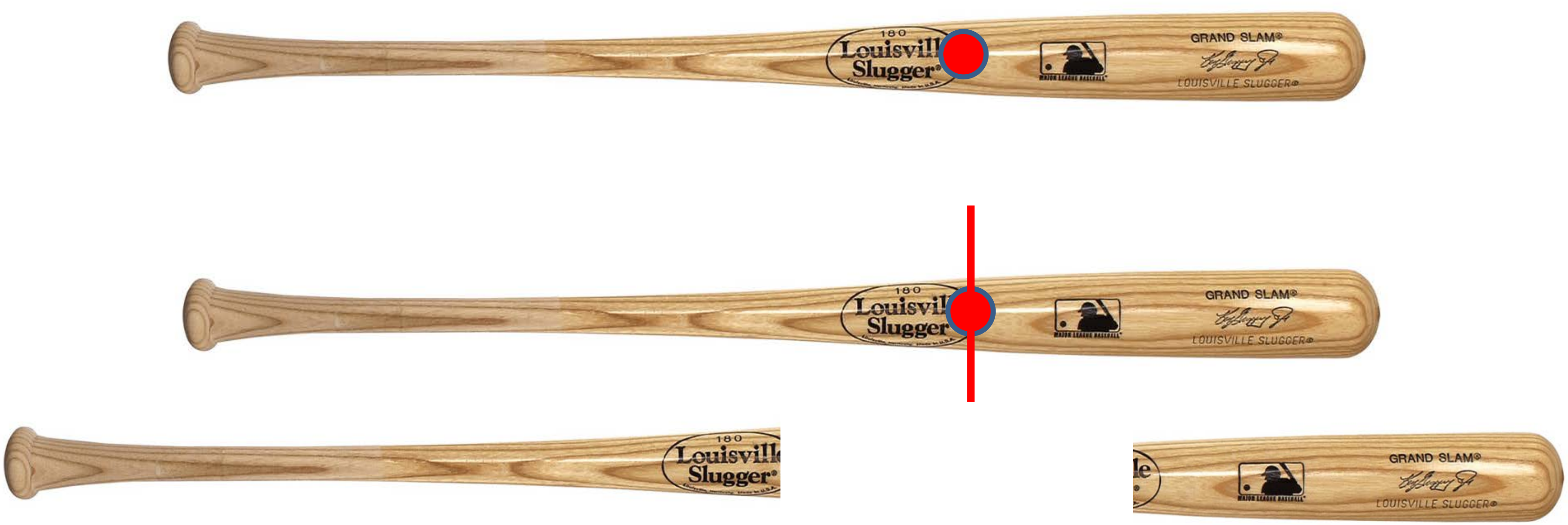
If you cut the bat in half through the CM, as shown, how do the weights of the two halves compare?

A. The handle half is heavier.

B. The handle half is lighter.

C. The two halves have the same mass.

D. You can't tell which half is heavier.



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D. You can't tell which half is heavier.

The CM of each half is close to the center of that half.

Since the handle half is longer, its CM is further from the original CM of the entire bat compared to the CM of the thicker end.

$x_{CM} = \Sigma m_i x_i \Rightarrow$ that longer distance is associated with less mass.

http://www.woodshopusa.com/assets/louisville-slu

5 options

An ant of mass *m* starts from rest and walks from the left end to the right end of a plank of length *L* that has the same mass *m* as the ant. The plank is resting on a **frictionless** tabletop as shown, initially protruding exactly halfway off the table. (*Note that the plank slides under the ant.*)

How far can the ant walk before falling off the plank or table?

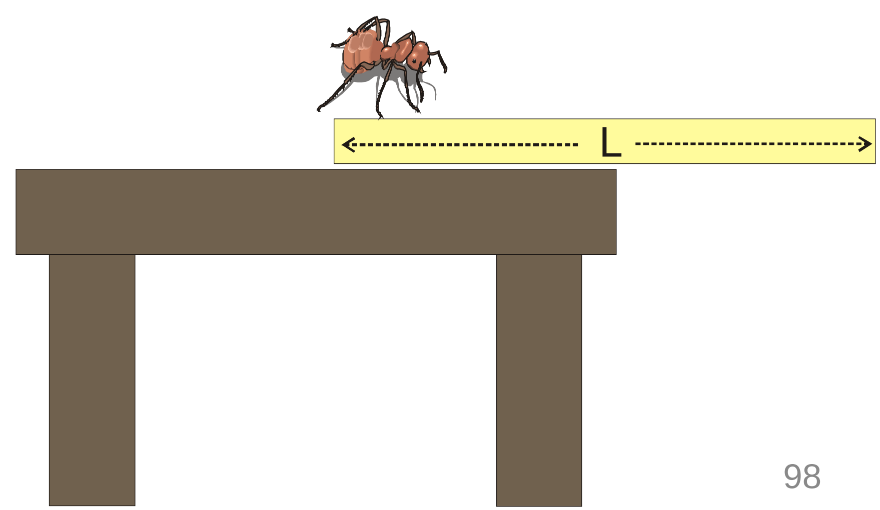
A. To the edge of the table

B. *L*/4 to the left of the edge

C. *L*/4 past the edge of the table

D. As soon as the ant starts walking the plank tips over

E. None of the above



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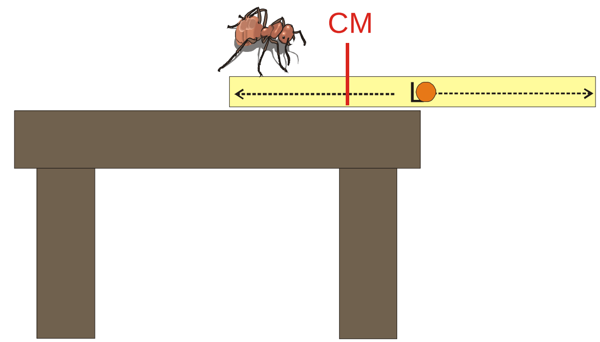
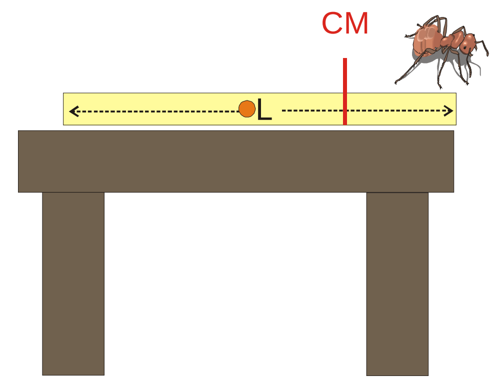




A. To the edge of the table

B. *L*/4 to the left of the edge

C. *L*/4 past the edge of the table

D. As soon as the ant starts walking the plank tips over

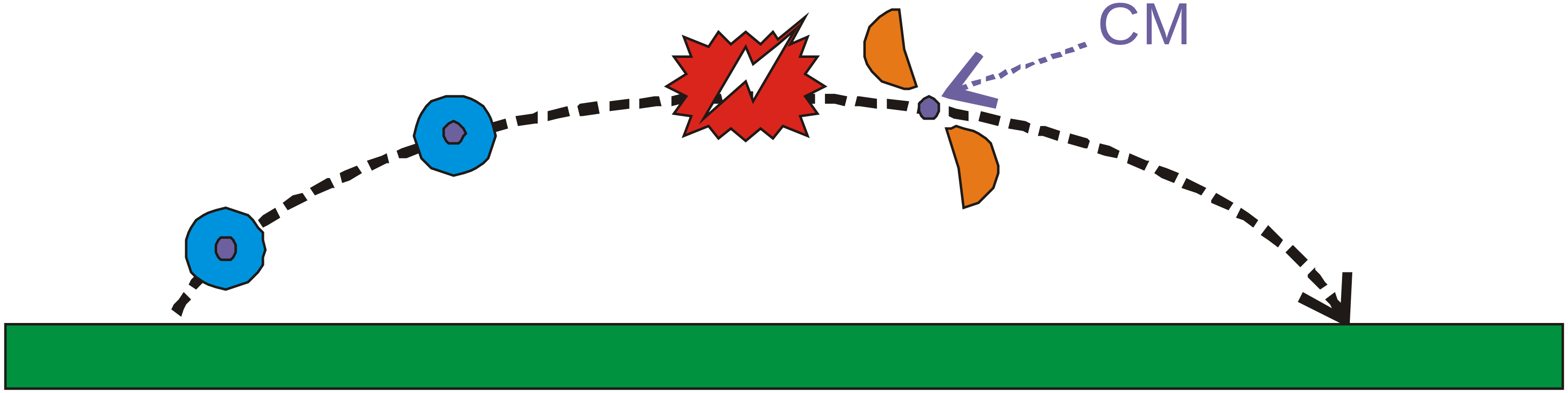
E. None of the above

- The CM of the ant-plank system must stay fixed.
- This CM is originally *L*/4 to the left of the right edge of the table, midway between the CM of the ant and the CM of the board.
- As the ant walks from one end of the plank to the other, the plank slides left until its right edge coincides with the right edge of the table.
- The CM of the plank is *L*/2 to the left of the edge when the ant is at the edge, so the total CM remains a distance *L*/4 left of the edge.

PROJECTILES THAT EXPLODE IN MID-AIR

A projectile moves as if the force of gravity acts on its CM, even if the projectile breaks into pieces while in the air.

- If you know the motion of all but one piece with respect to the CM, you know the path of that remaining piece.



EXAMPLE – EXPLODING PROJECTILES

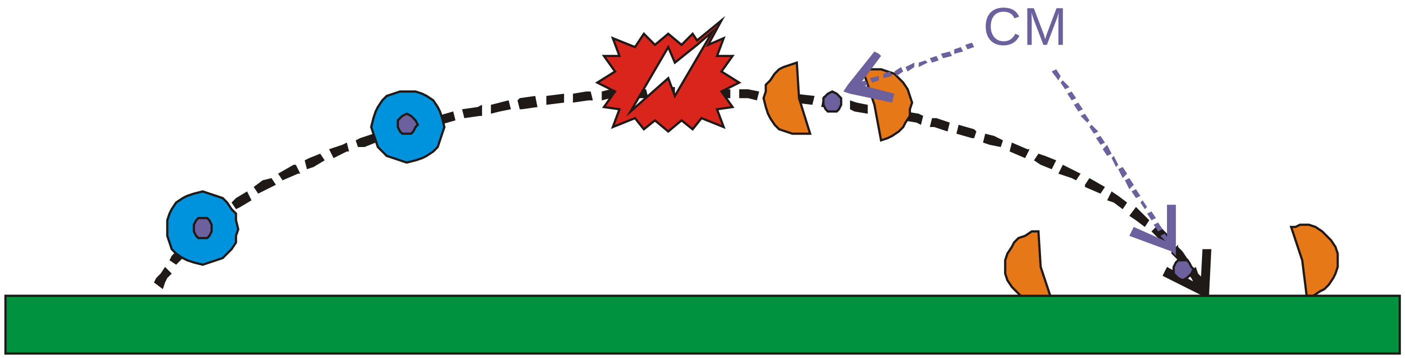
A projectile shot into the air at speed v_o and angle θ explodes into two pieces at the peak of its trajectory. The pieces have equal mass.

One piece is projected horizontally forward by the explosion and lands a distance *d* further than expected (*if the projectile hadn't exploded in the air*). Where does the other piece land?

A distance d less than expected

because the CM has to 'land' in the same spot as with no explosion

Given the special condition that only the horizontal components of velocity change, the time in the air remains the same for both pieces.



EXAMPLE – EXPLODING PROJECTILES

What would change if the explosion had instead:

1. Launched one piece up and the other down?

   If one piece is a distance H below the CM at any given time, the other piece is a distance H above the CM at that time (*until the lower piece hits the ground*).

2. Occurred somewhere besides the peak of the trajectory?

   If one piece is a distance $\mathbf{r}$ from the CM or moving away from it at velocity $\mathbf{v}$, the other piece is at $-\mathbf{r}$ or moving away at $-\mathbf{v}$

3. Created two pieces whose mass was not equal?

   Instead of the pieces being equidistant from the CM, you have to weight their position and velocity by their contribution to the total mass.

EXAMPLE – EXPLODING PROJECTILES

What would change if the explosion had instead:

4. Created multiple pieces?

   The math is more annoying but the concepts associated with the CM are the same.

It's sometimes wise to use a **Center of Mass reference frame**

since the motion of the CM is comparatively simple.

- Translate motion from the lab frame to the CM frame.

$$v_{objects\text{-}CM} = v_{objects\text{-}LAB} + v_{LAB\text{-}CM} = v_{objects\text{-}LAB} - v_{CM\text{-}LAB}$$

- Analyze the motion in the CM frame. (*where $v_{CM\text{-}LAB}$ might = 0 or constant*)
- Translate the motion back to the lab frame.

$$v_{objects\text{-}LAB} = v_{objects\text{-}CM} + v_{CM\text{-}LAB}$$

http://legacy.co.mohave.az.us/WIC/images/theEnd.gif