I. History of Logic I

A. Aristotle (385-322 B.C.) was the first person who is recorded as having investigated the special kind of deductive inference that has come to be denoted by the term *logic*.

B. His ***syllogisms***, nineteen of them in all, were intended to define all of the forms that correct arguments could take.

C. Example: A particular instance of one such syllogism, called *Darii*, is presented below:

All Ancient Greeks were perfect
Aristotle was an Ancient Greek
___
Therefore, Aristotle was perfect

1. The two sentences above the horizontal line are called the ***hypotheses*** of the syllogism.

2. The sentence below the line is the ***conclusion***.

3. If the hypothesis is true the conclusion must follow from it.

D. Syllogisms were intended only to specify the correct ***form*** of argument.

1. The actual contents of the sentences is immaterial to the logician.

2. The general form of Darii is:

All P's were Q
X was a P
___
Therefore, X was Q

E. <u>**Note:**</u> **This syllogism cannot be expressed in propositional logic**. Propositional logic does not provide a mechanism to state that all elements of a particular collection, i.e., Ancient Greek, have a particular property.

II. History of Logic II

A. Gottlieb Frege (1848-1925) was the first person to conceive and formulate the problem of reducing mathematics to logic.

B. In the process he originated what has now come to be known as *first-order logic,* or ***predicate logic.***

C. Predicate Logic deals with ***objects***, or elements of the universe of discourse, and ***predicates***, which specify relations between objects and properties of objects.

D. In propositional logic, the statement "Aristotle was an ancient Greek" is a proposition, it can have a truth value of either ***True*** or ***False***.

E. In predicate logic the proposition would be stated as:

Ancient_Greek(Aristotle)

1. *Ancient_Greek* is a predicate, which can have a truth value of either ***True*** or ***False***.

2. *Aristotle* is a object in a universe of discourse, i.e. all ancient Greeks.

3. The predicate *Ancient_Greek(Aristotle)* has truth value ***True*** because Aristotle is an ancient Greek.

4. The predicate *Ancient_Greek(Carl)* has truth value ***False*** because I am not an ancient Greek.

F. Predicate logic statements are of the form:

predicate_name(object of discourse)

1. The predicate name indicates a property, i.e., that of being an ancient Greek.

2. The object of discourse names an individual element that either possesses that property or not.

G. Predicate versus Propositional logic

1. In propositional logic one requires two statements, i.e.:

   a. $P \equiv$ Aristotle is an ancient Greek.
      $P \equiv True$

   b. $Q \equiv$ Carl is an ancient Greek.
      $Q \equiv False$

   to state whether or not two individuals, Carl and Aristotle, have the property of being an ancient Greek

2. In predicate logic one only need state the property once with two values for the associated objecty of discourse, or variable.

   a. *Ancient_Greek(Aristotle)* has truth value *True*.

   b. *Ancient_Greek(Carl)* has truth value *False*.

   c. In general the predicate can be stated as

      Ancient_Greek(X)

      with the truth value depending on the value of the variable X.

H. To implement the feature that was present in Aristotle's syllogisms but missing in propositional logic, i.e., statements about **all** of the objects in a domain or **any** object in a domain, we:

1. Separate the expression of the property of the object from the object possessing that property

2. Introduce the notion of a variable to represent an object.

3. Introduce the ability of a predicate to refer to one or many objects by means of *quantifiers*.

III. Predicates

A. Definition: A predicate is a proposition invoked with a parameter.

1. Like a proposition, a predicate must be a declarative statement with a truth value of either ***true*** ($\boldsymbol{T}$, $\boldsymbol{True}$) or ***false*** ($\boldsymbol{F}$, $\boldsymbol{False}$).

2. Unlike a proposition, a predicate involves a variable, and the truth value of the predicate depends on the value of the parameter variable.

3. Format: predicate-name(variable parameter 1, variable parameter 2, ...)

B. Example 1: $P(X) \equiv X\, is\, odd.$

1. $P(X)$ has truth value $\boldsymbol{True}$ if $X = 3$

2. $P(X)$ has truth value $\boldsymbol{False}$ if $X = 4$

C. Example 2: $Q(X,\, Y) \equiv X\, is\, older\, than\, Y$

1. $Q(X,\, Y)$ has truth value $\boldsymbol{True}$ if $X = John$ and $John$'s age is 55 and $Y = Mary$ and $Mary$'s age is 45.

2. $Q(X,\, Y)$ has truth value $\boldsymbol{False}$ if $X = John$ and $John$'s age is 35 and $Y = Mary$ and $Mary$'s age is 55.

D. Example 3: $S(N_1,\, N_2,\, N_3,\, N_4) \equiv$ the list of numbers $N_1,\, N_2,\, N_3,\, N_4$ is sorted in ascending order.

1. $S(1, 3, 5, 9)$ has truth value $\boldsymbol{True}$

2. $S(5, 1, 3, 9)$ has truth value $\boldsymbol{False}$

E. A predicate P with N variables, i.e., $P(k_1,\, k_2,\, k_3, \ldots k_N)$, is known as an N-ary predicate.

IV. Quantifiers

A. Allow a predicate to apply to more than a single object in the domain of discourse.

B. Allow a predicate to apply to at least one unknown object in the domain of discourse.

C. The English words all, $some$, $none$, $many$, and *there exists* are examples of quantifiers.

D. The use of quantifiers implies a ***domain of discourse***.

V. The Universal Quantifier (∀)

A. The symbol ∀ is read *for all objects* or *for all*

B. $\forall x\ P(x)$ is read *for all objects x, P(x) is true*

C. We can translate our original syllogism as follows:

$$\forall x(Ancient\_Greek(x) \rightarrow Perfect(x))$$
$$\underline{Ancient\_Greek(Aristotle)}$$
$$Perfect(Aristotle)$$

D. Going further we say that in syllogistic logic:

1. *Perfect(Aristotle)* is a logical consequence of the statements
   $\forall x$ *(Ancient_Greek(x) $\Rightarrow$ Perfect(x))* and
   Ancient_Greek(Aristotle)

2. Therefore:

$$(\forall x(Ancient\_Greek(x) \Rightarrow Perfect(x))$$
$$\wedge\ Ancient\_Greek(Aristotle))$$
$$\Rightarrow Perfect(Aristotle)$$

E. Once again:

1. The symbol x represents a **variable**

2. The symbol $Aristotle$ represents an ***instantiation*** of that variable.

F. The sentence *Ancient_Greek(x)* has no discernible truth value since we do not know the value associated with x, or the ***instantiation*** of x.

G. The variable x in *Ancient_Greek(x)* is said to be a ***free*** variable.

H. The variable x in the sentence $\forall x$ *(Ancient_Greek(x)* → *Perfect(x))* is said to be a ***bound*** variable.

I. The sentence $\forall x P(x)$ can be assigned a truth value

1. $\forall x P(x)$ is true if $P(x)$ is true for every instantiation of x

2. $\forall x P(x)$ is false if $P(x)$ is false for at least one x in the domain of discourse.

3. Must consider the ***domain of discourse*** when evaluating quantified sentences.

4. In this case the domain of discourse would seem to be all humans, but it could be all living organisms.

VI. The Existential Quantifier (∃)

A. The existential quantifier, ∃, provides the ability to say that at least one element of the domain has a particular property without specifying the particular element.

B. English language equivalents

1. There is an X such that $P(X)$ has truth value $\boldsymbol{True}$.

2. There is at least one X such that $P(X)$ has truth value $\boldsymbol{True}$.

3. At least one object in our domain has the property of being an ancient Greek can be stated:

((∃ x)Ancient_Greek(x))

C. Once again, variables used with a quantifier are ***bound***.

1. The statement *Ancient_Greek(x)* has no truth value since x is ***free***, i.e., it has no variable associated with it.

2. *((∃ x) Ancient_Greek(x))* must be either $\boldsymbol{True}$ or $\boldsymbol{False}$ since x is now ***bound***.

3. *((∃ x) Ancient_Greek(x))* asserts that at least one object in our domain has the property of being an ancient Greek.

4. The formula has truth value **true** if and only if the domain of discourse includes an ancient Greek, such as Aristotle.

E. Example 1: $\exists x\, Q(x)$ where: $Q(x) \equiv (x^2 = 16)$

and: The domain is the integers.

1. The truth value of $Q(x)$ is underterminable since the value of x is not specified.

2. The truth value of $\exists x\ Q(x)$ is $\boldsymbol{True}$ since $x = 4 \rightarrow x^2 = 16$

F. Note: The existential quantifier is equivalent to an unlimited disjunction. In Example 1, above, $\exists x\, Q(x)$ can be translated as:

$Q(1) \vee\ Q(2)\ \vee Q(3) \vee\ Q(4)\ \vee Q(5) \vee\ Q(6)$ ....

where the value of the variable x is extended to cover all integers.

G. Example 2: $\exists x\, Q(x)$ where: $Q(x) \equiv (x^2 = 10)$

and: The domain is the integers.

1. The truth value of $Q(x)$ is underterminable since the value of x is not specified.

2. The truth value of $\exists x Q(x)$ is $\boldsymbol{False}$ since there is no integer N such that $N^2 = 10$

VII. The Uniqueness Quantifier

1. The existenial quantifier $(\exists)$ and the universal quantifier are the most often used quantifiers.

2. **But:** There is really no limit to the number of quantifiers that can be defined.
   a. $\exists_2(x)\ P(x) \equiv$ There exists exactly 2 values of x for which $P(x)$ has truth value ***True***.

   b. $\forall_{10}(x)\ P(x) \equiv$ For all sets of 10 values of x, $P(x)$ is true.

3. The Uniqueness Quantifier: $\exists_1(x)\ P(x) \equiv \exists!(x)\ P(x)$ is the most often used of these "other" quantifiers.

4. Most, if not all, of the other quantifiers can be expressed in terms of the existential and universal quantifiers and negation so we will restrict our coverage to those.

VIII. Relationship Between the Universal Quantifier and the Existential Quantifier

A. The sentence: $\forall x\ P(x)$ is false if there is a single instantiation of x for which $P(x)$ is $\boldsymbol{False}$.

B. Under these circumstances the statement: $\neg(\forall x\ P(x))$ is true.

C. Therefore: $\neg(\forall x P(x))$ is true in exactly the same situations for which $\exists x\ \neg P(x)$ is $\boldsymbol{True}$.

D. If: $\exists x \neg P(x)$ is $\boldsymbol{True}$, then: $\forall x\ P(x)$ is $\boldsymbol{False}$.

E. Therefore: $\neg(\forall x P(x)) \equiv \exists x\ \neg P(x)$

F. If ∃x A(x) is true then:
   1. ∀x ¬A(x) must be false since if there exists an object x that has property A then it is not true that all objects in the domain do not have property.
   2. We can state this as: $\exists x\ A(x) \equiv\ \neg(\forall x\, \neg A(x))$

IX. Empty Domains

A. The implicit assumption in most quantified statements is that the domain of discourse is not empty.

B. An empty domain requires that $\exists x P(x)$ and $\exists x \neg P(x)$ both be $\boldsymbol{False}$ since there is no value of x for which $P(x)$ can be $\boldsymbol{True}$ and no value of x for which $\neg P(x)$ can be $\boldsymbol{True}$.

C. A proof regarding an empty domain has no value.

VIII. Restricted Domains

A. Domains can be specified as being parts of other domains.

B. In the previous examples we have used domains specified as "*the integers*" or "*the real numbers*" which implicitly include all elements of the described domain.

C. The predicate $\forall x P(x)$ where $P(x) \equiv (x\%3 = 0)$

1. is $\boldsymbol{False}$ if the domain is the integers

2. is $\boldsymbol{True}$ if the domain is every third integer starting with 3

IX. Precedence of Quantifiers

A. Quantifiers have precedence over any and all of the propositional connectives.

B. Example 1: $\forall x P(x) \rightarrow Q(x)$

1. Should be read as: $(\forall x P(x)) \rightarrow Q(x)$

   and **NOT** read as $(\forall x P(x) \rightarrow Q(x))$

2. Problem: In $(\forall x P(x))$ x is a bound variable.
   In $\rightarrow Q(x)$ x is free

C. Example 2: $\forall x P(x) \vee Q(x)$

1. Should be read as: $(\forall x P(x)) \vee Q(x)$

   and **NOT** read as $(\forall x P(x) \vee Q(x))$

2. Problem: In $(\forall x P(x))$ x is a bound variable.
   In $\vee\ Q(x)$ x is free

D. Example 3: $\forall x\, \neg P(x) \neq \neg\forall x\, P(x)$

1. $\forall x\, \neg P(x)$ should be read as:
   $P(x)$ is $\boldsymbol{False}$ for every element x in the domain.

2. $\neg\forall x\, P(x)$ should be read as:
   There exists at least one element x, and possibly all elements x, in the domain for which $P(x)$ is $\boldsymbol{False}$ or $\exists x\, \neg P(x)$

XI. De Morgan's Laws for Quantifiers

Stmt	Negation	Equivalent	Negation is True	Negation is False
$\exists x\, P(x)$	$\neg\, \exists x\, P(x)$	$\forall x\, \neg P(x)$	$P(x)$ is $\boldsymbol{False}$ for every x	$P(a)$ is $\boldsymbol{True}$
$\forall x P(x)$	$\neg\, \forall x P(x)$	$\exists x \neg P(x)$	$P(a)$ is $\boldsymbol{False}$	$P(x)$ is $\boldsymbol{True}$ for every x

XII. Higher Order Logics

A. The quantifiers of first-order logic allow quantification only over variables that represent objects, or elements of the domain of discourse, thereby limiting the number of correct argument forms that can be expressed.

B. Second-order logic allows variable predicates and quantification over predicates.

C. Third-order logic allows quantification over functionals, and so on, until one reaches general type theory.

XIII. Tranlating Statements in English Into Statements in Predicate Logic

A. Not a single psychologist attended the convention.

1. Predicates
   a. $Psychologist(x) \equiv x$ is a psychologist.

   b. $AttendedTheConvention(x) \equiv x$ attended the convention.
2. Statement 1:
   $\neg\exists x\ (Psychologist(x) \wedge AttendedTheConvention(x))$

3. Statement 2:
   $\forall x\ (Psychologist(x) \rightarrow \neg AttendedTheConvention(x))$

B. It is not the case that every Girl Scout sells cookies.

1. Predicates:
   a. $GirlScout(x) \equiv x$ is a Girl Scout.

   b. $SellsCookies(x) \equiv x$ sells cookies.

2. Statement 1: $\neg\forall x\ (GirlScout(x) \rightarrow SellsCookies(x))$

3. Statement 2: $\exists x\ (GirlScout(x) \wedge \neg SellsCookies(x))$

C. If Elisabeth is a historian then some women are historians.

1. Predicates:
   a. $Woman(Elisabeth) \equiv$ Elizabeth is a woman.

   b. $Woman(x) \equiv x$ is a woman.

   c. $Historian(Elisabeth) \equiv$ Elizabeth is a historian.

   d. $Historian(x) \equiv x$ is a historian.

2. Statement:

$$(Woman(Elisabeth) \wedge Historian(Elisabeth)) \rightarrow \exists x\ (Woman(x) \wedge Historian(x))$$

D. Only snakes and lizards thrive in the desert.

1. Predicates
   a. $Snake(x) \quad \equiv \quad x$ is a snake.

   b. $Lizard(x) \quad \equiv \quad x$ is a lizard.

   c. $ThrivesInDesert(x) \quad \equiv \quad x$ thrives in the desert.

2. Statement:
   $\neg\exists x\ (ThrivesInDesert(x) \leftrightarrow Snake(x) \vee Lizard(x))$

E. If some cellists are music directors, then some orchestras are properly led.

1. Predicates:
   a. $Cellist(x) \quad \equiv \quad x$ is a cellist.

   b. $MusicDirector(x) \quad \equiv \quad x$ is a music director.

   c. $Orchestra(x) \quad \equiv \quad x$ is an orchestra.

   d. $ProperlyLed(x) \quad \equiv \quad x$ is properly led.

2. Statement 1:
   $\exists x\ (Cellist(x) \wedge MusicDirector(x)) \rightarrow$
   $\quad \exists y\ (Orchestra(y) \wedge ProperlyLed(y))$

F. All novels are interesting if and only if some of Steinbeck's novels are not romances.

1. Predicates:
   a. $Novel(x) \quad \equiv \quad x$ is a novel.
   b. $Romance(x) \quad \equiv \quad x$ is a romance.
   c. $Interesting(x) \quad \equiv \quad x$ is interesting.
   d. $AuthoredBySteinbeck(x) \quad \equiv \quad x$ was authored by Steinbeck.
2. Statement:
   $\exists x\ (Novel(x) \rightarrow Interesting(x)) \leftrightarrow$
   $\quad \exists y\ (Novel(y) \wedge$
   $\quad\quad AuthoredBySteinbeck(y) \wedge$
   $\quad\quad \neg Romance(y))$

XIV. Programming in Logic - Prolog

A. Prolog is a general purpose logic programming language associated with artificial intelligence and computational linguistics.

B. Prolog is based on first-order predicate logic.

C. Prolog is a ***declarative*** programming language.

1. Most programming languages are ***procedural*** programming languages, in which the programmer specifies a recipe for solving a problem.

2. In a declarative language like Prolog the programmer describes the problem in the language of formal logic and the problem is solved using an algorithm (the ***resolution method***) built in to the language.

D. History:

1. Prolog was first conceived by a group including Alain Colmerauer in Marseille, France, in the early 1970s.

2. The first Prolog system was developed in 1972 by Colmerauer with Philippe Roussel.

E. Example Problem
A fruit vendor has three boxes of fruit, all labeled incorrectly.

1. Box 'a' is labeled 'apples', box 'b' is labeled 'oranges', box 'c', 'bananas'.

2. Each box does, in fact, contain one of the three fruits just listed, but not that fruit displayed on the label.

3. The vendor offers you a wager.
   a. If you can guess the contents of each box you get a box of fruit for no charge.

   b. Otherwise, you pay double for your chosen box.

4. You agree on the condition that you get to see the contents of one box. The vendor agrees, and opens the box labeled 'oranges'. It contains apples.

5. Question: What is the fruit in each of the other boxes?

F. Programming the Problem in Prolog

1. Specify the predicates (with a constant parameter) that denote a box:

```
box(a).
box(b).
box(c).
```

These Prolog statements state that the predicate box has truth value $\boldsymbol{True}$ for each of the constant values a, b, and c and $\boldsymbol{False}$ for any other.

2. Specify the predicates that define the label attached to each box:

```
label(a, apples).
label(b, oranges).
label(c, bananas).
```

These Prolog statements state that the predicate $label$ has truth value $\boldsymbol{True}$ for each the pairs of the constant parameters (a, $apples$), (b, $oranges$), and (c, $bannas$) and $\boldsymbol{False}$ for any other.

3. Specify the values that denote a fruit:

```
fruit(apples).
fruit(oranges).
fruit(bananas).
```

These Prolog statements state that the predicate $fruit$ has truth value $\boldsymbol{True}$ for each of the constant values $apples$, $oranges$, and $bananas$ and $\boldsymbol{False}$ for any other.

4. State the rules for determining which fruit is in which box.
   a. We know that box 'b' contains apples because we looked.

      In Prolog that fact is stated: contains(b, apples).

   b. For the other boxes we state:

```
contains(Box, Contents)            :-
        box(Box),            /*    Box is a box          */
        Box \= b,            /*    This rule should not be
                                   used for box 'b'.     */
        fruit(Contents),     /*    Contents must be fruit*/
        Contents \= apples,        /*    If it is not box 'b',
                                         the contents can't
                                         be apples.  */
        label(Box, X),             /*    Box must have a
                                         label          */
        Contents \= X.             /*    The fruit in the
                                         box is not that on
                                         the label      */
```

   c. In English, this Prolog predicates states:
      The values of the variables Box and $Contents$ must satisfy the following constraints:
      i. For some value of Box the predicate box must be $True$.
      ii. The value of the variable Box cannot be b.

      iii. For some value of $Contents$ the predicate $fruit$ must be $True$.
      iv. For some values of the variables Box and X the predicate $label$ must be $True$.

      v. The values assigned to the variables $Contents$ and X cannot be the same.

      vi. The values assigned to the variables, an assignment that satifies all of the above conditions, will be returned.

5. State the query, or the statement that, when the variables are instantiated, will contain the solution, i.e., that:
   a. Box 'a' holds fruit 'X'.

   b. Box 'b' holds fruit 'Y'.

   c. Box 'c' holds fruit 'Z'.

   d. No two boxes contain the same fruit.

      ```
      boxholds(a, X, b, Y, c, Z) :-   contains(a, X),
                                      contains(b, Y),
                                      contains(c, Z),
                                      X \= Y, X \= Z, Y \=Z.
      ```

   e. In English this predicate states:
      i. The variables X, Y, and Z have each been assigned a value that is one of the fruits.

      ii. Any value assigned to one of the variables X, Y, or Z cannot have been assigned to another.

6. Query: boxholds(a, X, b, Y, c, Z)?

   Asks Prolog to assign values of fruits to the variables X, Y, and Z, where the constant a designates box a, the constant b designates box b, and the constant c designates box c.

7. Program Output: boxholds(a,bananas,b,apples,c,oranges)

   Prolog has assigned to the variables the (in this case) the single sequence of values that satisfies the predicates used to define the problem.

C. Complete Program

```
/*      First, we tell Prolog what the boxes are.                         */
box(a).
box(b).
box(c).

/*      Next, we tell Prolog how each box is labeled.                     */
label(a, apples).
label(b, oranges).
label(c, bananas).

/*      Then we define the types of fruit that the program must deal with.*/
fruit(apples).
fruit(oranges).
fruit(bananas).

/*      Now we define the rules for determining which fruit is in which box. */
contains(b, apples).        /*      We know box 'b' contains apples because
                                    we looked.                              */
contains(Box, Contents)  :-
        box(Box),           /*      Box is a box                            */
        Box \= b,           /*      This rule should not be used for box
                                    'b'.                                    */
        fruit(Contents),    /*      Contents must be fruit                  */
        Contents \= apples, /*      If it is not box 'b', the contents can't
                                    be apples.                              */
        label(Box, X),      /*      Box must have a label                   */
        Contents \= X.      /*      The fruit in the box is not that on the
                                    label                                   */
/*      The main rule, or predicate, simply feeds to the rule 'contains' the
        initial conditions, i.e.., box 'a' holds fruit 'X', box 'b', 'Y', and box 'c',
        'Z' and that no two boxes contain the same fruit.                   */
boxholds(a, X, b, Y, c, Z)  :-      contains(a, X),
                                    contains(b, Y),
                                    contains(c, Z),
                                    X \= Y, X \= Z, Y \=Z.
```

XV. Nested Quantifiers

A. Definition: Quantifiers are referred to as ***nested quantifiers*** when one quantifier is within the scope of another.

B. Example:

1. Assumption: The domain is the real numbers.

2. Predicate with free variables: $P(x,\, y) \equiv x + y = 0$

3. Quantified predicate: $\forall x \,\exists y \, P(x,\, y)$

4. Alternate formulation:
   a. $P(x,\, y) \equiv x + y = 0$

   b. $Q(x) \equiv \exists y \, P(x,\, y)$

   c. $\forall x \, Q(x)$

5. Stated in English (either version):
   For all real numbers x there exists a real number y such that $x + y = 0$
   a. $y = \, - x$ is a real number.
   b. $x + y = \; x + \, - x = 0$
   c. Therefore, the expression $\forall x \,\exists y \, P(x,\, y)$ has truth value $\boldsymbol{True}$.

C. The Order of the Quantifiers is Important

1. In the case of $\forall x \,\exists y \, P(x,\, y)$ the expression is $\boldsymbol{True}$.

2. If we reverse the order of the quantifiers we get:
   $\exists y \;\forall x \, P(x,\, y)$

3. Stated in English the version with the order of the quantifiers reversed is:
   There exists a single real number y such that for all real numbers x, $x + y = 0$

4. Since there is no such real number, the truth value of $\exists y \;\forall x \, P(x,\, y)$ is $\boldsymbol{False}$.

XVI. Negating Nested Quantifiers

A. Example 1:

1. Statement: $\neg\,\exists y\,\exists x\; P(x,\,y)$

2. English language equivalents:
   a. It is not true that there exists values of the variables x and y such that $P(x,\,y)$ has truth value $\boldsymbol{True}$.
   b. It is not true that there exists a value of the variable x such that $P(x,\,y)$ has truth value $\boldsymbol{True}$ for some value of y.
   c. It is not true that there exists a value of the variable y such that $P(x,\,y)$ has truth value $\boldsymbol{True}$ for some value of x.

3. De Morgan's First Law states: $\neg\,\exists y\;\; P(y) \equiv\; \forall y\;\neg P(y)$

4. Applying De Morgan's Law gives:
$$\neg\,\exists y\,\exists x\; P(x,\,y) \equiv\; \forall y\;\neg\exists x\; P(x,\,y)$$

5. Applying De Morgan's Law once again gives:
$$\neg\,\exists y\,\exists x\; P(x,\,y) \equiv\; \forall y\,\forall x\;\neg P(x,\,y)$$

B. Example 2:

1. Statement: $\neg\,\forall x\;\exists y\;\;(P(x,\,y) \to Q(x,\,y))$

2. Applying De Morgan's Second Law gives:
$$\exists x\;\;\neg\exists y\;\;(P(x,\,y) \to Q(x,\,y))$$

3. Applying De Morgan's First Law gives:
$$\exists x\;\forall y\;\;\neg(P(x,\,y) \to Q(x,\,y))$$

4. Since $P(x,\,y) \to Q(x,\,y) \equiv \neg P(x,\,y) \vee Q(x,\,y)$
   we have: $\exists x\;\forall y\;\;\neg(\neg P(x,\,y) \vee Q(x,\,y))$

   or: $\exists x\;\forall y\,(\neg\neg P(x,\,y) \wedge \neg Q(x,\,y))$

   or: $\exists x\;\forall y\,(P(x,\,y) \wedge \neg Q(x,\,y))$

XVII. Some Miscellaneous Problems

A. Express the statement that there is a positive integer that is not the sum of three squares.

1. In this case the domain of discourse will be the integers.

2. $\exists x \, \exists w \, \exists y \, \exists z \, (x > 0 \ \wedge x \neq w^2 + y^2 + z^2)$

B. Express the statement that the product of two negative integers is positive:

$$\forall x \, \forall y \, (((x < 0) \wedge (y < 0)) \rightarrow (x \times y > 0))$$

C. Express the statement that the difference between two negative integers is not necessarily negative.

1. Translation: There exist two integers $x < 0$ and $y < 0$ such that $x - y > 0$

2. Expression: $\exists x \ \exists y \, ((x < 0 \ \wedge y < 0) \ \wedge (x - y > 0\,))$

D. Translate into an English statement expressing a mathematical fact the predicate logic statement regarding the real numbers:

$$\exists x \, \forall y \, (x + y = y)$$

Translation: For all real numbers y there is a real number x such that the sum of x and y is equal to y.

This statement expresses the existance of 0.0.

E. Find a common domain for the variables x, y, and z for which the statement $\forall x \, \forall y \, ((x \neq y) \rightarrow \ \forall z \, ((z = x) \ \vee (z = y)))$

is true and another domain for which it is false.

1. A domain for which the statement is true is: $D = \{1, \, 2\}$
   a. Consider: $x = 1$ and $y = 2$

   b. Clearly: $1 \neq 2$

   c. Then, for all $z \in D$, i.e, 1 or 2, either $z = 1$ or $z = 2$.

2. A domain for which the statement is false is: $D = \{1,\ 2, 3\}$
   a. Consider: $x = 1$ and $y = 2$

   b. Clearly: $1 \neq 2$

   c. Then, for all $z \in D$ includes 3 and $3 \neq 1$ and $3 \neq 2$ so the statement $\forall z\,((z = x) \vee (z = y))$ is false.