Soundness of the tree test for validity of arguments in SL

Soundness is the property of "no false positives". If the tree test SAYS an argument is valid, the argument actually IS valid. Let us mark that off as a theorem.

SOUNDNESS THEOREM FOR VALIDITY IN SL: If the tree for an argument in SL closes, then that argument is valid.

The first step toward a proof is to turn this around. I mean, notice it is equivalent to saying: If an argument has a counterexample, then the tree for it does not close.

In fact we will show something about trees in general, not only trees for arguments. Make sure you understand why this is what we need:

FIRST LEMMA: If a given row of the truth table makes a certain sentence True, and we apply the tree rule to that sentence, then that same row will make at least one of the resulting branches True.

To prove this lemma we check it for each kind of sentence:

1-2. If the sentence is atomic or negated atomic, there is no tree rule.

3a. If the sentence is a conjunction ($\mathscr{P}$ & $\mathscr{Q}$), applying the rule gives one branch, with $\mathscr{P}$ and $\mathscr{Q}$ on it (actually, one extension to every branch that is currently open under the sentence). Clearly every truth table row that makes ($\mathscr{P}$ & $\mathscr{Q}$) True, also makes $\mathscr{P}$ and $\mathscr{Q}$ both True.

3b. If the sentence is a negated conjunction ~($\mathscr{P}$ & $\mathscr{Q}$), applying the rule gives two branches (actually, two branches under every branch that is currently open under the sentence), one with ~$\mathscr{P}$ on it, and one with ~$\mathscr{Q}$. But if a row of the truth table makes the sentence ~($\mathscr{P}$ & $\mathscr{Q}$) True, it must make one of $\mathscr{P}$ and $\mathscr{Q}$ False, and so it either makes the branch with ~$\mathscr{P}$ on it True, or the one with ~$\mathscr{Q}$.

I leave it to you to finish the cases ($\mathscr{P} \vee \mathscr{Q}$), ~($\mathscr{P} \vee \mathscr{Q}$), ($\mathscr{P} \supset \mathscr{Q}$), ~($\mathscr{P} \supset \mathscr{Q}$), ($\mathscr{P} \equiv \mathscr{Q}$), ~($\mathscr{P} \equiv \mathscr{Q}$). Do one or two of these for yourself, in full, as exercise.

SECOND LEMMA: If some row of the truth table makes all the sentences on a certain branch True, then after applying any truth tree rule to a sentence on that branch at least one of the resulting branches remains open. More: that same row will make all the sentences on at least one of the resulting branches True.

I leave it to you to prove this lemma.