I. Introduction to Logic

A. The rules of logic enable the distinction between valid and invalid arguments, both mathematical and other.

1. In general, Logic, as a discipline, is not concerned with the truth or falsity of any particular statement.

2. Logic is concerned only with the validity of certain techniques for manipulating statements, or propositions in a valid, consistent, manner to produce other valid statements.

3. Logic is concerned with the systematic formalization and cataloguing of valid methods of reasoning.

B. One major goal of this course is the acquisition of the ability to construct valid mathematical arguments, i.e., proofs.

II. Statements:

A. English Language Statements

1. Facts are expressed using declarative statements.

2. For simple cases, the statement of fact is either true or false.

3. In this class we will not consider the vast majority of English language statements that are:

   a. Partly true.

   b. True only under some vaguely specified or implied conditions.

   c. True only part of the time.

   d. The study of such statements merits a course, probably more than one course, in itself.

B. Example 1: **Ten is less than seven.**

1. This statement happens to be false, since in common arithmetic useage, ten is greater than, not less than, seven.

2. We would say, then, that the truth value of this statement is false.

3. This statement uses the terms ***Ten*** and ***seven*** to designate **constant** values.

C. Example 2: **She is very talented.**

1. The term ***She*** is a ***variable***.

2. The value of the variable is the name of the person whom the statement describes.

3. Possible values would be Jane, or Kristine, or Mary, or the name of any other talented female.

4. Since the value of the variable is not specified or, to use a Computer Science term, the variable is not ***bound***, the truth value of this statement is not defined.

5. By definition, therefore, the sentence "She is very talented." is not a statement.

6. If the the adjective 'she' is associated with a particular person, i.e., bound, the sentence "She is very talented." would be a statement since we would have a sentence that would be either true or false.

III. The Basic Building Blocks of Logic: Propositions/Statements

A. A proposition is a statement that has a truth value of either ***true*** or ***false*** with no other possible truth value.

B. Example 1: The Mississippi is a long river.
   1. Since the Mississippi is the longest river in the world the statement is true.

   2. Therefore the ***truth value*** of this statement is ***true*.**

   3. The truth value **<u>must</u>** be either ***true*** or ***false***.

   4. The truth value cannot be both true and false.

   5. The statement cannot have a truth value that is something other than true or false.

C. Example 2: There are life forms on other planets in the Universe.
   1. The truth value of this statement is either ***true*** or ***false***.

   2. At the present time the truth value is not known.

   3. But, since the truth value **<u>must</u>** be either ***true*** or ***false***, the statement is a proposition.

D. Some English (or other) language sentences are not propositions.
   1. Example A: Is the sky blue?
      a. No fact is asserted by this sentence.
      b. It has no truth value.

      c. It is not a proposition

   2. Example B: Do what I say.
      a. No fact is asserted by this sentence.
      b. It has no truth value.

      c. It is not a proposition.

E. A proposition, then, is a ***declarative*** sentence that states a fact.

1. Adam loves Eve.

   a. The sentence is an assertion that Adam is in love with Eve.

   b. If Adam is, in fact, in love with Eve the truth value of this proposition is ***true***.

   c. If not, then Adam is not in love with Eve and the truth value of the proposition if ***false***.

2. Paris is the capital of France.

   a. Assuming that the sentence designates the well known city of Paris and the equally well known country of France, it states that the city of Paris has been designated as the capital of the country France.

   b. If no major governmental action or revolution has designated another city to the capital of France, this proposition is true.

   c. Otherwise, the truth value of the proposition if false.

F. It is not necessary that the truth value be known for a declarative sentence to be a proposition.

1. Example: All crows are black.

   a. The crows that I have seen in my lifetime are black.

   b. Could there exist a bird that is classified as a crow that is not black?

   c. I don't know.

   d. Therefore I do not know the truth value of this statement.

   e. But: The truth value of the statement must be either true or false.

   e. Therefore the statement is a proposition.

2. Sneaky Solution: Define a crow as a bird that, regardless of its other qualities, is black.
   a. Therefore the statement "All crows are black." is true (by definition).

   b. If a bird is not black, it is not a crow.

G. A sentence can have more than one truth value is NOT a proposition.
1. Example: $X + 1 = 2$

2. If $X = 1$ the sentence has truth value true.

3. If $X \neq 1$ the sentence has truth value false.

IV. Proposition: A Formal Definition

A. A proposition is a declarative sentence that has a truth value of either ***true*** (usually denoted by either T or 1) or ***false*** (usually denoted by either F or 0).

B. The sentence $X + 2 = 2$
1. Is not a proposition if the value of X is unknown.

2. Is a proposition (with truth value T, or 1, or ***true***) if the value of X is known to be 0.

3. Is a proposition (with truth value F, or 0, or ***false***) if the value of X is known to be 2.

C. The sentence "*This sentence is false.*" is **NOT** a proposition.
1. If you assume that the content of the sentence is true the conclusion is that the truth value is false.

2. If you assume that the sentence is false the conclusion is that the truth value is true.

3. This example is called a paradox and is not a proposition, because it is neither true nor false.

V. Representation of Propositions

A. When dealing with the rules involving propositions, propositions are represented as letters.

B. In your text the letters normally used are p, q, r, s, ...

C. In these notes the letters normally used are P, Q, R, S, ...

V. Negation: An Operation Performed on a Single Proposition

A. If P is a proposition then $\neg P$ is a proposition.

B. If P has truth value T, or ***true***, then $\neg P$ has truth value F, or ***false***.

C. If P has truth value F, or ***false***, then $\neg P$ has truth value T, or ***true***.

D. Example 1:

P : Amanda's personal computer has at least 32GB of RAM.

$\neg P$: It is not true that Amanda's personal computer has at least 32GB of RAM.

or

$\neg P$: Amanda's personal computer has less than 32GB of RAM.

E. Example 2:

P : You are all learning propositional logic.

$\neg P$: It is not true that all of you are learning propositional logic.

or

$\neg P$: At least one of you is not learning propositional logic.

F. Example 3:

P : The Mississippi is the longest river in the world.

$\neg P$: It is not true that the Mississippi is the longest river in the world.

or

$\neg P$: A river that is longer than the Mississippi has just been discovered.

VI. Truth Tables: A Graphical Method of Displaying Truth Values.

A. The truth table for negation is:

P	$\neg P$
T	F
F	T

B. All possible truth values for the proposition P (***true*** and ***false***) are displayed in the column headed by P.

C. All possible truth values for the proposition $\neg P$ are displayed in the column headed by $\neg P$

D. The corresponding values are given in the rows.

1. The row with a value of T in the column headed by proposition P contains an F in the column headed by proposition $\neg P$.

2. The row with a value of F in the column headed by proposition P contains an T in the column headed by proposition $\neg P$.

3. So: When P is ***true***, $\neg P$ is ***false*** and when P is ***false***, $\neg P$ is ***true***.

E. Consider: Double Negation

1. If P is ***true***, then $\neg P$ is ***false***.

2. Then $\neg\neg P$ must be $\neg$***false***, or ***true***

3. Truth Table:

P	$\neg P$	$\neg\neg P$
T	F	T
F	T	F

4. Hence: $P \equiv \neg\neg P$ or P ***is equivalent to*** $\neg\neg P$ because P and $\neg\neg P$ have the same truth values.

VI. Conjunction: An Operation Performed on Two Propositions

A. Definition: Let P and Q be propositions. The conjunction of P and Q, denoted by $P \wedge Q$ is true when both P and Q have truth value ***true*** and is ***false*** otherwise.

B. Truth Table Definition:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that the truth table contains all possible combinations of truth values for P and Q.

C. The conjunction of P and Q is often referred to as "$P\ and\ Q$"

D. Examples:

1. John is in love with Mary and Joe is in love with Lousia.
   a. P : John is in love with Mary.

   b. Q : Joe is in love with Mary.

   c. $P \wedge Q$: John is in love with Mary and Joe is in love with Lousia.

   d. $P \wedge Q$ is true only if both P and Q are true.

   e. If John does not love Mary, i.e., if P is false, then $P \wedge Q$ is false.

   f. If Joe does not love Louisa, i.e., if Q is false, then $P \wedge Q$ is false.

VIII. Disjunction: Another Operation Performed on Two Propositions

A. Definition: Let P and Q be propositions. The disjunction of P and Q, denoted by $P \vee Q$, is true when either or both P and Q have truth value ***true*** and is ***false*** otherwise.

B. Truth Table Definition:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

C. The disjunction of P and Q is often referred to as "$P\ or\ Q$"

D. Examples:

1. John is in love with Mary or Louisa.
   a. P : John is in love with Mary.

   b. Q : John is in love with Louisa.

   c. $P \vee Q$: John is in love with Mary or John is in love with Lousia.

   d. $P \vee Q$ is true if
      i. P is true, i.e., if John loves Mary.

      ii. Q is true, i.e., if John loves Louisa.

      iii. Both P and Q are true.

   e. $P \vee Q$ is false only if John does not love Mary and John does not love Louisa, i.e., if P is false Q is false.

XI. Negation of Conjunction and Disjunction:

A. Negation of Conjunction

1. Truth Table

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

2. The truth values of $\neg(P \wedge Q)$ are identical to those of $\neg P \vee \neg Q$

3. Hence: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

or

$\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$

B. Negation of Disjunction

1. Truth Table

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

2. The truth values of $\neg(P \vee Q)$ are identical to those of $\neg P \wedge \neg Q$

3. Hence: $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

or

$\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$

XII. Exclusive OR: $P \oplus Q$

A. In some arguments it is convenient to use an operation on two propositions P and Q which has truth value ***true*** only when exactly one of the two propositions has truth value ***true*** and has truth value **false** otherwise.

B. This operation is known as the ***exclusive OR*** or $P \oplus Q$

C. Truth Table:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

D. Note: $P \oplus Q \equiv (\neg P \wedge Q) \vee (P \wedge \neg Q)$ as shown in the following truth table:

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge Q$	$P \wedge \neg Q$	$(\neg P \wedge Q) \vee (P \wedge \neg Q)$	$P \oplus Q$
T	T	F	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	T	F	T	F	T	T
F	F	T	T	F	F	F	F

XIII. Conditional Statements: $P \rightarrow Q$

A. If P and Q are propositions, then the conditional statement $P \rightarrow Q$ is the statement if P, then Q.

B. Expressions that designate a conditional statement $P \rightarrow Q$

1. if P then Q
2. P implies Q
3. if P, Q
4. Other examples are given on page 6 of your text.

C. English Language Examples:

1. If the clouds blow this way we will have rain.

2. If you try to pat that dog he will bite you.

3. If I am elected I will lower taxes.

D. Truth Table

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

E. Note that $P \to Q$ has truth value ***false*** only when P is true and Q is false.

1. Hence $P \to Q$ is ***true*** when P is ***false***.

2. This departs from the traditional meaning of implication, of the
 if (premise) then conclusion.

3. The Java statement: if (A == B) Z = A + B;

 causes the assignment statement Z = A + B to be executed ***only*** if the value stored in the variable A is equal to the value stored in the variable B.

 Otherwise the statement Z = A + B is not executed.

4. The traditional meaning of implication, or of if-then statements, is given as ***modus ponens***, to be covered later.

F. An equivalent statement: $\neg P \vee Q$

1. Truth Table:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

2. Since $P \rightarrow Q$ and $\neg P \vee Q$ have the same truth values we say, once again, that

$P \rightarrow Q$ ***is equivalent to*** $\neg P \vee Q$

or:

$$P \rightarrow Q \equiv \neg P \vee Q$$

XIV. BiConditional/BiImplication Statements: $P \leftrightarrow Q$

A. If P and Q are propositions the dispositional statement $P \leftrightarrow Q$ is the proposition P if and only if Q.

B. Truth Table:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

C. The proposition $P \leftrightarrow Q$ has truth value ***true*** if and only if P and Q have the same truth value.

D. $P \leftrightarrow Q$ is logically equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$
which is often written as: $(P \rightarrow Q) \wedge (P \leftarrow Q)$

E. Truth Table:

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

XV. Truth Tables for Compound Propositions

A. The logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), $\leftrightarrow$ (biimplication), and $\neg$ (negation) can be used in statements including any number of propositions.

B. One way of determining the truth value of these compound statements is by constructing a truth table.

C. Example: $(P \to Q) \vee (\neg Q \wedge R)$

1. Truth Table:

P	Q	R	$P \to Q$	$\neg Q$	$\neg Q \wedge R$	$(P \to Q) \vee (\neg Q \wedge R)$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

D. Note: The truth table must contain all possible combinations of truth values for the atomic propositions contained in the compound proposition represented.

1. In the example above, there are three atomic propositions; P, Q, and R; each of which may have two values, T, or F.

2. Three variables each of which may have two values gives a total of $2^3 = 8$ possible combinations.

3. In general, K variables, each of which may assume N values, produce N^K possible combinations.

   (We will prove this when we discuss mathematical induction.)

4. To hold all possible combinations, the preceding truth table has eight rows.

5. Each possible combination of truth values is known as an ***interpretation.***

XVI. Logical Propositions Stated in Natural Language (English)

A. Consider the situation described by the following statements:

1. Berries are ripe along the trail, but grizzly bears have not been seen in the area.

2. Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

3. If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

4. It is not safe to hike on the trail, but grizzly bears have not been seen in the area and berries along the trail are ripe.

5. For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.
6. Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

B. Before stating any of these sentences as compound propositions one must, first, specify the individual propositions that are being used. In this case the following propositions would seem to be sufficient to compose all of the sentences:

1. $P \equiv$ Grizzly bears have been seen in the area.
2. $Q \equiv$ Hiking is safe on the trail.
3. $R \equiv$ Berries are ripe along the trail.

C. Sentence 1: Berries are ripe along the trail, but grizzly bears have not been seen in the area.

1. Berries are ripe along the trail $\equiv$ R
2. grizzly bears have not been seen in the area. $\equiv$ $\neg P$
3. In this case, the word "but" is being used and conjunction.
4. In logical symbols, Sentence 1 $\equiv$ $R \wedge \neg P$

D. Sentence 2: Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

1. Grizzly bears have not been seen in the area $\equiv$ $\neg P$
2. hiking on the trail is safe $\equiv$ Q
3. berries are ripe along the trail. $\equiv$ R
4. In logical symbols, Sentence 2 $\equiv$ $\neg P \wedge Q \wedge R$

E. Sentence 3: If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

1. Restatement: If berries are ripe along the trail then hiking is safe if and only if grizzly bears have not been seen in the area.

2. If berries are ripe along the trail then $\equiv$ $R \rightarrow$

3. hiking is safe if and only if grizzly bears have not been seen in the area. $\equiv$ $Q \leftrightarrow \neg P$

4. In logical symbols, Sentence 3 $\equiv$ $R \rightarrow (Q \leftrightarrow \neg P)$

F. Sentence 4: It is not safe to hike on the trail, but grizzly bears have not been seen in the area and berries along the trail are ripe.

1. It is not safe to hike on the trail $\equiv$ $\neg Q$

2. grizzly bears have not been seen in the area $\equiv$ $\neg P$

3. berries along the trail are ripe. $\equiv$ R

4. In logical symbols, Sentence 4 $\equiv$ $\neg Q \wedge \neg P \wedge R$

G. Sentence 5: For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.

1. Restatement: If hiking on the trail is safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.

2. The interesting English language statement:

   A is necessary but not sufficient for B

   can be analyzed as follows:

   a. A is necessary for B $\equiv$ $B \to A$

   b. A is not sufficient for B $\equiv$ $\neg(A \to B)$

   c. Equating "but" with "and" gives us:

   A is necessary but not sufficient for B
   $$\equiv \quad (B \to A) \wedge \neg(A \to B)$$

3. Second restatement:

   It is necessary that berries not be ripe along the trail and for grizzly bears not to have been seen in the area for hiking on the trail to be safe

   and

   It is not sufficient that that berries not be ripe along the trail and for grizzly bears not to have been seen in the area for hiking on the trail to be safe.

4. It is necessary that berries not be ripe along the trail and for grizzly bears not to have been seen in the area for hiking on the trail to be safe
   $$\equiv \quad (\neg R \wedge \neg P) \to Q$$

5. It is not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area for hiking on the trail to be safe.
   $$\equiv \quad \neg\Big[Q \to (\neg R \wedge \neg P)\Big]$$

6. Final result: $\Big[(\neg R \wedge \neg P) \to Q\Big] \wedge \neg\Big[Q \to (\neg R \wedge \neg P)\Big]$

H. Sentence 6: Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

1. Restatement: If grizzly bears have been seen in the area and berries are ripe along the trail hiking is not safe on the trail.

2. If grizzly bears have been seen in the area and berries are ripe along the trail $\equiv$ $(P \wedge R) \rightarrow$

3. hiking is not safe on the trail. $\equiv$ $\neg Q$

4. Final result: $(P \wedge R) \rightarrow \neg Q$

XVII. Precedence of Logical Operators

A. Up until the order in which operations are to be performed has been specified using parentheses.

B. $(P \rightarrow Q) \vee (\neg Q \wedge R)$ is understood to mean:
The disjunction of the propositions $P \rightarrow Q$ and $\neg Q \wedge R$

1. The truth value of $P \rightarrow Q$ is, then, disjoined with the truth value of $\neg Q \wedge R$ to obtain the truth value of the disjunction.

2. We do **NOT** compute the disjunction of the proposition Q in $P \rightarrow Q$ with the proposition $\neg Q$ in the proposition $\neg Q \wedge R$

3. $P \rightarrow Q$ is understood to mean the proposition P implies the proposition Q

   The truth value of $P \rightarrow Q$ is computed and only then used in the computation of the truth value of $(P \rightarrow Q) \vee (\neg Q \wedge R)$

4. $\neg Q \wedge R$ is understood to mean the conjunction of the negation of proposition Q and the proposition R.

   The truth value of $\neg Q \wedge R$ is computed and only then used in the computation of the truth value of $(P \rightarrow Q) \vee (\neg Q \wedge R)$

5. The truth value of the proposition $\neg Q$ in the compound proposition $\neg Q \wedge R$ is understood to mean the negation of the proposition Q.

   The truth value of $\neg Q$ is computed and only then used in the computation of the truth value of $\neg Q \wedge R$.

C. Parentheses, although very convenient, are not absolutely necessary.

1. Logical operations must be executed in a pre-defined order, known as precedence.

2. Precedence Table of Logical Operators:

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

3. Therefore: Perform negation of single propositions before any other operation.
   a. $\neg Q \wedge R$ is the negation of the proposition Q conjoined with the proposition R.
   b. $\neg Q \wedge R$ is NOT the negation of the disjunction of the propositions Q and R.

XVIII. Logical Operators in Hardware

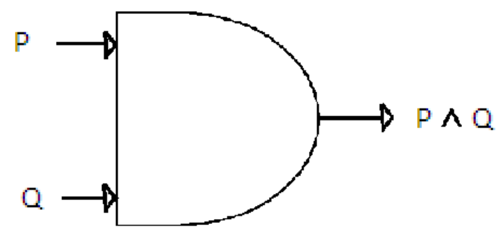
A. Each of the logical operations discussed previously (among others) can be implemented in computer hardware.

B. Since binary computers implement bit manipulations, the truth value ***true*** corresponds to the value $1B$ (the bit is turned on) and the truth value false corresponds to the value $0B$ (the bit is turned off).

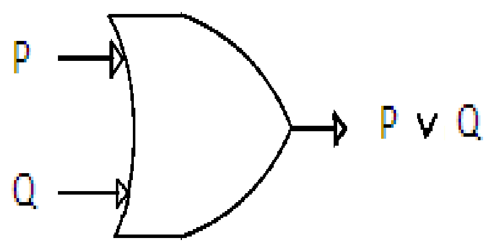
C. Hardware Representation of Truth Values:

1. ***true*** corresponds to a switch that is turned on while ***false*** corresponds to a switch that is turned off.
2. ***true*** corresponds to a voltage of $+5$ volts while ***false*** corresponds to a voltage of -5 volts.

D. The hardware symbol for the conjunction of P and Q is:



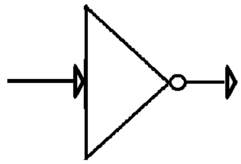
E. The hardware symbol for the disjunction of P and Q is:



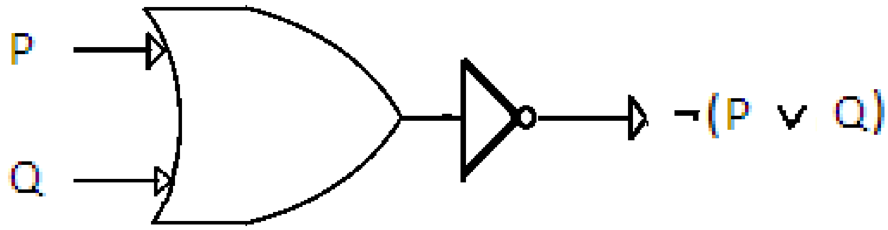
F. In both of the above the binary input for P, either $'1'B$ (true) or $'0'B$ (false), enters via the upper arrow (an electrical connection) while the the binary input for Q, either $'1'B$ (true) or $'0'B$ (false), enters via the upper arrow.

G. The output for either $P \wedge Q$ or $P \vee Q$, either $'1'B$ (true) or $'0'B$ (false) emerges from the right-most arrow.

H. The hardware symbol for the negation of P, or $\neg P$, is:



so $\neg(P \vee Q)$ would be represented in hardware as:

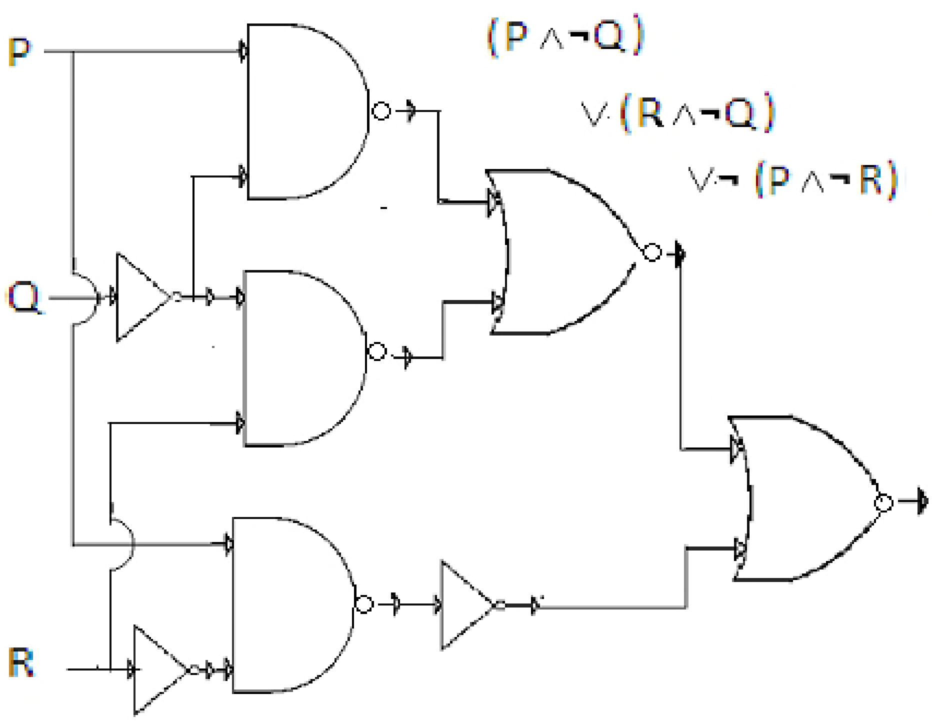


XIX. Representing Compound Logical Expressions in Hardware

A. Consider the logical expression:

$$(P \wedge \neg Q) \vee (R \wedge \neg Q) \vee \neg(P \wedge \neg R)$$

B. This expression can be represented in hardware as shown below:



C. The left-most column of hardware gates represents, from top-to-bottom:
1. $P \wedge \neg Q$
2. $R \wedge \neg Q$
3. $P \wedge \neg R$

D. The center column of hardware gates represents, from top-to-bottom:
1. $(P \wedge \neg Q) \vee (R \wedge \neg Q)$
2. $\neg(P \wedge \neg R)$

E. The right-most column, a single gate, represents:
$((P \wedge \neg Q) \vee (R \wedge \neg Q)) \vee \neg(P \wedge \neg R)$