XII. Compositions of Functions

A. Definition: If $g : A \to B$ and $f : B \to C$ then the ***composition*** of f and g, denoted by $f \circ g$, is defined by:

$$(f \circ g)(a) = f(g(a))$$

B. Alternative statement 1:

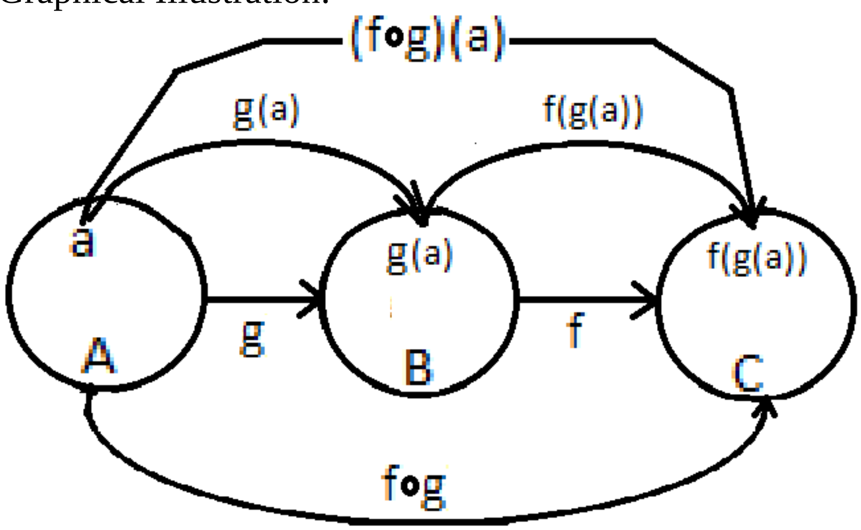
$(f \circ g)(a)$ is the function that:

1. Maps $a \in A$ to $b \in B$ where b is specified by $g(a) = b$

2. Maps $b \in B$ to $c \in C$ where c is specified by $f(b) = c$

C. Alternative statement 2: $(f \circ g)(a) = f(g(a) = b) = f(b) = c$

D. Graphical Illustration:



E. **Note 1:** $(f \circ g)(a) \neq (g \circ f)(a)$

1. Consider: $f(x) = x + 1$ and $g(x) = x^2$

2. $(f \circ g)(a) = f(g(a)) = f(g(x) = x^2) = x^2 + 1$

3. $(g \circ f)(a) = g(f(a)) = g(f(x) = x + 1) = (x + 1)^2$

F. **Note 2:** $(f \circ f^{-1})(b) = f(f^{-1}(b)) = \iota_A$

If $f(a) = b$ then $f^{-1}(b) = a$

Then: $f(f^{-1}(b)) = f(f^{-1}(b) = a) = f(a) = b$

And: $f^{-1}(f(a)) = f^{-1}(f(a) = b) = f^{-1}(b) = a$

Therefore: $(f \circ f^{-1})(b) = f(f^{-1}(b)) = \iota_B$

and: $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = \iota_A$

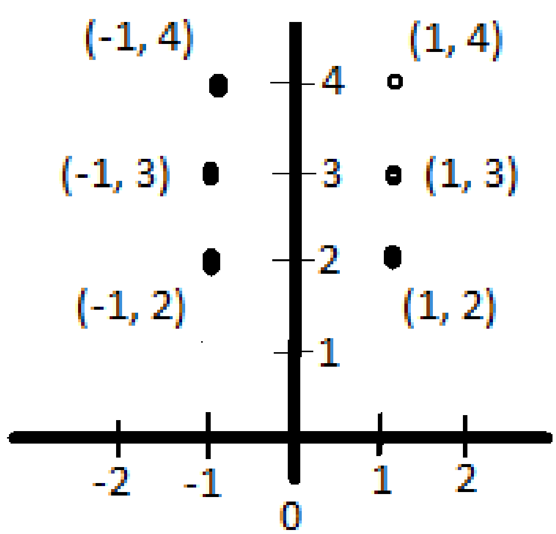
XIII. Graphs of Functions

A. Definition: The graph of a function $f : A \to B$ is the set S (possibly infinite) of ordered pairs $(a,\ b)$ where $a \in A$ and $b \in B$ and $f(a) = b$.

Therefore: $S = \{(a.\, b) \mid (a \in A) \wedge (b \in B) \wedge (f(a) = b)\}$

B. Example:

1. Let $A = \{-1,\ 0,\ 1\}$ and $B = \{2,\ 3,\ 4\}$
2. Define $f : A \to B$ as $f(a) = a + 3 = b$
3. Then: $S = \{(-1,\ 2),\ (0,\ 3),\ (1,\ 4)\}$
4. The graph is:



XIV. Floor and Ceiling Functions

A. Floor Function

1. Definition: The floor function assigns to some real number r the largest integer z that is less than or equal to x.

   The value of the floor function at r is usually denoted by $\lfloor r \rfloor = z$

2. If $\Re = \{r \mid r$ is a real number$\}$ and
   $Z = \{z \mid z$ is an integer$\}$

   then $\lfloor\ \rfloor : \Re \rightarrow Z$
   and $\lfloor r \rfloor = z,\ z$ is the largest integer less than r

3. Typically the floor function is used to round down a floating point number to the largest integer less than the floating point number.

4. Example:
   You are planning a couples bridge party with two couples playing at each table. You have N chairs. How many couples can you invite?

   Each bridge table requires four chairs. The maximum number of tables that you can support with N chairs is $T = \lfloor \frac{N}{4} \rfloor$
   The maximum number of couples that you can invite is, then, $2 \times T$.

B. Ceiling Function

1. Definition: The ceiling function assigns to some real number r the smallest integer z that is less than or equal to x.

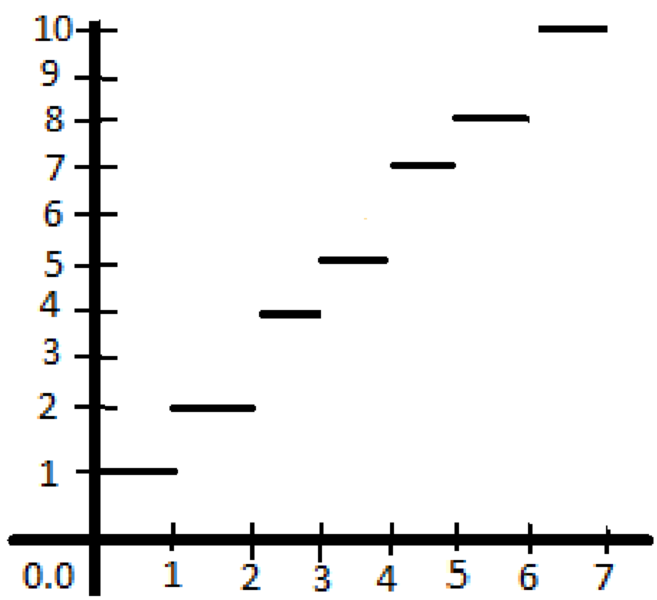
   The value of the ceiling function at r is usually denoted by $\lceil r \rceil = z$

2. If $\Re = \{r \mid r$ is a real number$\}$ and
   $Z = \{z \mid z$ is an integer$\}$

   then $\lceil\ \rceil : \Re \rightarrow Z$
   and $\lceil r \rceil = z,\ z$ is the smallest integer greater than r

3. Typically the ceiling function is used to round up a floating point number to the smallest integer greater than the floating point number.

4. Example:
Draw a graph of the function $f(r) = \lceil r \rceil + \lfloor \frac{r}{2} \rfloor$ where $f : \Re \rightarrow \Re$



XV. Factorial Function

A. Definition: $f : N \rightarrow Z^+$ where $f(n) = n!$

and: $n! = 1 \times 2 \times 3 \times 4 \times ... \times (N-1) \times N$

B. Example: Given an ordered trio of integers, e.g., $(1, 3, 5)$, the number of possible orderings is given by $3! = 6$

1. $(1, 3, 5)$ 4. $(1, 5, 3)$

2. $(3, 1, 5)$ 5. $(5, 1, 3)$

3. $(3, 5, 1)$ 6. $(5, 3, 1)$

C. In general (to be proven later), the number of possible orderings of an ordered N-tuple is N.

XVI. Partial Functions

A. Definition: A ***partial function*** $f : A \to B$ is an assignment of some, not all, elements of the domain A to a unique element b of B, the codomain.

The elements of A that are assigned to a unique element b of B belong to a subset of A known as the ***domain of definition*** of f.

When the domain of definition of f is equal to the domain A then f is known as a ***total function***.

B. Example 1: $f : Z \to \Re$ where $f = \frac{1}{n}$

1. The quotient $\frac{1}{n}$ is undefined for $n = 0$ and there is no real number $r \in \Re$ that represents the quotient $\frac{1}{0}$.

2. If the domain is changed to either Z^+ or Z^- to exclude 0 then f becomes a total function.

3. If we define f as: $f : Z \to \Re \cup \{\mu\}$ where:
   a. $f(n) \in \Re$ when $f(n) = \frac{1}{n}$ belongs to the domain of definition for $\frac{1}{n}$

   b. $f(n) = \mu$ if $f(n)$ is undefined at n.

   then f becomes a total function.

C. Example 2: $f : Z \times Z \to Q$ where $f(m, n) = \frac{m}{n}$

1. The quotient $\frac{m}{n}$ is undefined for $n = 0$ and there is no real number $r \in \Re$ that represents the quotient $\frac{m}{0}$.

2. If the domain is changed to either Z^+ or Z^- to exclude 0 then f becomes a total function.