I. Reminders:

A. Product Rule: If a procedure can be broken down into a sequence of N tasks each of which can be completed in N_i ways then there are $\prod_{i=1}^{N} N_i$ ways to complete the procedure.

B. Summation Rule: If a procedure can be completed in any one of N_i ways and no $N_i = N_j$ then there are $\sum_{i=1}^{N} N_i$ ways to complete the procedure.

II. Discrete Probability

A. Restricted to cases of finitely many, equally likely outcomes.

B. Stated in the language of sets of outcomes of interest and sets of all outcomes.

C. Therefore stated in terms of **combinations** rather than permutations since sets have no order.

D. **Definition:** A ***sample space*** is the set of all possible outcomes of a random process or experiment.

E. **Definition:** An ***event*** E is a subset of a finite sample space S.

F. **Definition:** If S is a finite sample space in which all outcomes are equally likely and E is an event in S then the **probability** of an event E is:

$$p(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}$$

In the language of set theory we the above can be written as: $p(E) = \frac{|E|}{|S|}$

G. Important Qualifications:

1. ***equally likely outcomes*** The definition must be modified if this restriction is not met.
2. ***finite sample space*** The definition must be modified for cases of infinite sample spaces.

III. Axiomatic Definition

Let S be a sample space. A ***probability function*** P from the set of all events in S to the set of real numbers satisfies the following three axioms:

1. $0 \leq P(E) \leq 1$

2. $P(0) = 0$ and $P(S) = 1$

3. If E_1 and E_2 are disjoint ($E_1 \cap E_2 = \emptyset$) then the probability of the union of E_1 and E_2 (both events) is:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

IV. Example 1: Drawing Colored Balls from an Urn

A. Problem Description: An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is red?

B. Analysis:

1. The finite sample space consists of nine balls, any one of which is equally likely to be drawn from the urn.

2. The desired event, in this case, is drawing one of five red balls, any one of which is equally likely to be drawn from the urn.

3. The size of the sample space is then: $\mid S \mid \ = 9$

4. The size of the event space is: $|E \mid \ = 5$

5. Therefore: $p(E) = \ \mid E \mid / \mid S \mid \ = \frac{5}{9}$

6. If we replace the ball drawn each time we repeat the experiment of drawing a ball from the urn $p(E)$ will stay the same.

C. Variation on the Problem:

1. We could seek to draw multiple red balls and be concerned about the probability for so doing.

2. If we are drawing without replacement, and we draw a red ball the first time and do not replace it we then we are left with:

   $|\ S\ | = 8$ and $|\ E\ | = 4$

   a. Therefore: $p(E) = \frac{4}{8}$

   b. The changed probability value is a direct result of $|\ S\ |$ and $|E\ |$ changing.

3. If we draw a red ball the second time and do not replace it we then we are left with $|\ E\ | = 3$ and $|\ S\ | = 7$

   Therefore: $p(E) = \frac{3}{7}$

V. Example 2:

A. Problem Description:

In a certain lottery in which a player wins a large prize if he/she correctly guesses a four digit number that matches the four digit number chosen by some random number generator, such as agitated numbered ping-pong balls being sucked up into a tube. What is the probability of guessing the correct number?

Note: The requirement that the digits drawn form a specific number implies that the ping-pong balls are drawn in a particular order.

B. Analysis:

1. The total number of ways in which a single decimal digit can be chosen is ten, since we have digits 0 - 9.

2. Therefore, according to the product rule, the number of ways in which a four digit number can be chosen is 10^4, or $10,000$, the number of permutations of four positions.

3. Therefore: $|\ S\ | = 10^4 = 10,000$

4. Only one four digit number wins the prize, so $|\ E\ | = 1$

5. Therefore: $p(E) = 1/10^4 = 0.0001$

VI. Example 3: Variation on Example 2

A. Description: If a player in the lottery of Example 2 chooses a number that contains three digits that match the correct number, in both value and position, a smaller prize is awarded. What is the probability that a player will win the smaller prize?

B. Analysis:

1. The number of ways in which one could pick a number that differed from the winning number in all but the first digit is 9, since we may pick any digit but the correct one.

2. The same is true for the second, third, and fourth digits.

3. Therefore we have an event E_1 that can be done in $N_1 = 9$ different ways, an event E_2 that can be done in $N_2 = 9$ different ways, an event E_3 that can be done in $N_3 = 9$ different ways, and an event E_4 that can be done in $N_4 = 9$ different ways.

4. Since the event to be completed is E_1 or E_2 or E_3 or E_4 the Summation Rule applies. Therefore we have:

$$N_1 + N_2 + N_3 + N_4 = 9 + 9 + 9 + 9 = 36$$

ways to choose a number that has three digits that match the winning number and one that differs.

5. Therefore:

a. $\mid S \mid \; = 10^4$ (as before)

b. $\mid E \mid \; = 36$

c. So: $p(E) = 36/10^4 = 0.0036$

VII. Example 4: According to your text there are lotteries in which the prize is awarded for choosing a set of six digits out of the first n positive integers, where N is usually between 30 and 50. Assume that N is 40 and compute the probability that a person can pick the correct six numbers.

A. The number of ways to choose six numbers out of 40 is $C(40, 6)$ where:

$$C(40,6) = \frac{40!}{6! \times (40-36)!} = \frac{40!}{6! \times 4!} = 3,838,380$$

B. There is only one winning combination of six numbers so:

a. $|E| = 1$ and $|S| = 3,838,380$

b. The probability of winning a lottery such as this is

$$\frac{1}{3{,}838{,}380} \approx 0.00000026$$

IX. Example 5: Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro (or any other trio) each win a prize if each has entered the contest (the event E)?

A. The first event is E_1, having one of the trio win first prize in the drawing from 100 entries the name of the first prize winner. There are three ways in which one of the trio could win fist prize.

$$p(E_1) = \frac{3}{100} = 0.03$$

B. The second event is E_2, having one of the remaining two members of the trio win second prize in the drawing from the remaining 99 entries of the name of the second prize winner.

$$p(E_2) = \frac{2}{99} \approx 0.020202$$

C. The third event is E_3, having the remaining member of the trio win third prize in the drawing from the remaining 98 entries the name of the third prize winner.

$$p(E_3) = \frac{1}{98} \approx 0.010204$$

D. Each of these events is discrete since no one member of the trio can win more than one prize.

E. Therefore, since E_1, E_2, and E_3 must all occur, the product rule applies so we have:

$$p(E) = p(E_1) \times p(E_2) \times p(E_3) \approx 0.03 \times 0.020202 \times 0.010204$$

$$\approx 0.0000061$$

X. **Theorem 1:** If E is an event in a sample space S, the probability of the event $\overline{E}$, called the ***complement of E***, is given by:

$$p(\overline{E}) = 1 - p(E)$$

A. Proof:

1. All events that do not include a member of E are included in the set $S - E$.

2. Therefore: $p(\overline{E}) = \frac{|S-E|}{|S|} = \frac{|S|-|E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|}$

$$= 1 - p(E) \qquad \square$$

B. Example: A sequence of bits is randomly generated. What is the probability that at least one of these bits is 0 ?

1. Let E be the event that at least one of the 10 bits is 0.

2. Then: $\overline{E}$ is the event that none of the bits is 0, or that all of the bits are 1.

3. Then: $p(E) = 1 - p(\overline{E}) = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}$

C. **Note:** $p(E) + p(\overline{E}) = 1$

XI. Theorem 2: If E_1 and E_2 are events in a sample space S then

$$P(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

A. Proof:

1. The event whose probability we are computing is contained by the set of constituent events given by: $E_1 \cup E_2$

2. From set theory we have that:

   a. $| E_1 \cup E_2 | = | E_1 | + | E_2 | - | E_1 \cap E_2 |$

   b. Then:

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

B. Example: What is the probability that a positive integer selected at random from the set of positive integers

$$N = \{1, 2, 3, ..., 100\}$$

is divisible by either 2 or 5 but not both?

1. Set $E_1 \equiv$ the positive integer selected at random is divisible by 2.

2. Set $E_2 \equiv$ the positive integer selected at random is divisible by 5.

3. Then

   a. $E_1 \cup E_2 \equiv$ the positive integer selected at random is divisible by either 2 or 5.

   b. $E_1 \cap E_2 \equiv$ the positive integer selected at random is divisible by both 2 and 5, i.e., divisible by 10.

4. There are 50 even integers between 1 and 100 so we have that:

$$p(E_1) = \frac{50}{100} = 0.5$$

5. There are 20 integers that are divisible by 5 between 1 and 100 so we have that: $p(E_2) = \frac{20}{100} = 0.2$

6. There are 10 integers between 1 and 100 that are divisible by 10 so we have that: $p(E_1 \cap E_2) = \frac{10}{100} = 0.1$

7. Then: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

$$= 0.5 + 0.2 - 0.1 = 0.6$$

8. Note that:
   a. If $E_1 \equiv$ the integer is even then $p(E_1) = \frac{50}{100} = 0.5$

   b. If $E_2 \equiv$ the integer is odd then $p(E_2) = \frac{50}{100} = 0.5$

   c. Since E_1 and E_2 cover the entire sample space we have $E_1 \cup E_2 = S$ so $p(E_1 \cup E_2) = p(S) = 1.0$

   d. Since E_1 and E_2 are disjoint $p(E_1 \cap E_2) = 0$

XII. Assigning Probabilities

A. If S is a sample space with a finite or countable number of outcomes and we assign a probability $p(s)$ to each outcome s then the following conditions must be met:
   1. $0 \le p(s) \le 1$ for each $s \in S$

   2. $\sum\limits_{s \in S} p(s) = 1$

B. If there are N possible outcomes the above two conditions can be rewritten as:
   1. $0 \le p(x_i) \le 1$ for $i = 1,\ 2,\ 3,\ ...,\ N$

   2. $\sum\limits_{i=1}^{N} p(x_i) = 1$

C. The function p from the set of all outcomes of the sample space S is called a ***probability distribution***.

D. **Example 1:**

1. A ***fair die*** is one which, when tossed, has the same probability of coming to rest with a particular side up as it does with any other side.

2. Therefore the face labeled 3 appears with the same probability as the faces labeled $1,\ 2,\ 4,\ 5,$ and 6

3. Therefore: $p(1) = p(2) = p(3) = p(4) = p(5) = p(6)$

4. Since: $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$
   we must have: $p(i) = \frac{1}{6}$

5. This is an example of a ***uniform distribution***.

6. Therefore the probability of rolling a 3 is $\frac{1}{6}$

7. The probability of rolling one 3 in two rolls is, according to the summation rule, $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$

E. Definition: If S is a set of N elements the ***uniform distribution*** assigns the probability $\frac{1}{N}$ to each event $E \in S$.

XIII. Summation Rule for Probabilities

A. Definition: The probability of the event E is the sum of the probabilities of the outcomes of E, or $p(E) = \sum_{s \in E} p(s)$

B. **Example 2:**

1. Suppose that a die is loaded, or biased, so that the face labeled 3 appears twice as often as any other. The other five outcomes are equally likely.

2. Then: $p(1) = p(2) = p(4) = p(5) = p(6) = x$

   and $p(3) = 2x$

3. We must have:

$$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$$

4. Therefore: $x + x + 2x + x + x + x = 1$ and $x = \frac{1}{7}$

5. Therefore: $p(1) = p(2) = p(4) = p(5) = p(6) = \frac{1}{7}$

   and $p(3) = \frac{2}{7}$

6. What is the probability that the outcome of a roll will be an odd value?
   a. We wish to find the probability of the event $E = \{1, 3, 5\}$.

   b. $p(E) = p(1) + p(3) + p(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}$

XIV. Generalized Summation Rule for Probabilities

A. **Theorem:** If E_1, E_2, ..., E_N is a sequence of pairwise disjoint events is a sample space S, then:

$$P\left(\bigcup_i E_i\right) = \sum_i p(E_i)$$

B. Proof by Strong Induction

1. Basis Step: $N = 2$

   For two events, E_1 and E_2, we have, from Theorem 2 on page 449 of your text and page 280 of you notes:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

   Since the events E_1 and E_2 are, in this case, pair-wise disjoint, w have that: $E_1 \cap E_2 = \emptyset$ and $p(E_1 \cap E_2) = 0$

   Therefore, for $N = 2$, $p(E_1 \cup E_2) = p(E_1) + p(E_2)$

2. Inductive Assumption: $P\left(\bigcup_i^N E_i\right) = \sum_{i=1}^{N} p(E_i)$

   for all i, $2 \leq i \leq N$

3. Inductive Step:

a. $$P\left(\bigcup_i^{N+1} E_i\right) = P\left(\left(\bigcup_i^{N} E_i\right) \cup E_{N+1}\right)$$

If we treat $\bigcup_i^N E_i$ as a single event we have:

b. $$P\left(\bigcup_i^{N+1} E_i\right) = P\left(\bigcup_i^{N} E_i\right) \cup P(E_{N+1})$$ from our basis step.

$$= P\left(\bigcup_i^{N} E_i\right) + P(E_{N+1})$$

$$= \sum_{i=1}^{N} p(E_i) + P(E_{N+1})$$ from our inductive assumption.

$$= \sum_{i=1}^{N+1} p(E_i)$$

Therefore: $P(N) \rightarrow P(N+1)$

4. Therefore: $P\left(\bigcup_i E_i\right) = \sum_i p(E_i)$ $\square$

XV. Conditional Probability

A. Example Problem:

Imagine a couple with two children, eachof which is equally likely to be a boy or a girl. Now suppose that you are given the information that one is a boy. What is the probability that the other is a boy?

1. All four equally likely combinations, with the oldest child on the left, are $BB,\ BG,\ GB,\ GG$

2. Hence: $S = \{BB,\ BG,\ GB,\ GG\}$ is the sample space.

3. If $X \equiv$ At least one child is a boy $\equiv \{BB,\ BG,\ GB\}$.
   we have that: $P(X) = \frac{|X|}{|S|} = \frac{3}{4}$

4. If $Y \equiv$ Two of the children are boys $\equiv \{BB\}$
   we have that: $P(Y) = \frac{|Y|}{|S|} = \frac{1}{4}$

5. The set $\{BB,\ BG,\ GB\}$ is now the sample space for the second event Z, i.e., that, given that the first child is a boy, the second child is also a boy. All of the events in this sample space are equally likely.

6. $Z = \{BB\}$ so: $P(Z) = \frac{|Z|}{|S|} = \frac{1}{3}$

6. $\frac{P(\text{at least one child is a boy and the other child is also a boy})}{P(\text{at least one child is a boy})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

B. Definition: If A and B are events in a sample space S and $P(A) \neq 0$ the conditional probability of B given A, denoted by $P(B|A)$, is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

C. Example 2:

1. Description:
   a. Assume that bit strings of length four are being generated by a true random number generator.

   b. The probability that a '1' will be generated is, then, equal to the probability that a '0' will be generated and both probabilities are equal to 0.5.

   c. What is the probability that a bit string contains two consecutive zeroes given that the first bit is a zero?

2. Analysis:

a. Let F be the event that the first bit is '0' and E be the event that the bit string has two consecutive zeroes. We are seeking, then:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

b. The total number of bit strings of length four is $2^4 = 16$, as shown below:

0	$=$	0000	8	$=$	1000
1	$=$	0001	9	$=$	1001
2	$=$	0010	10	$=$	1010
3	$=$	0011	11	$=$	1011
4	$=$	0100	12	$=$	1100
5	$=$	0101	13	$=$	1101
6	$=$	0110	14	$=$	1110
7	$=$	0111	15	$=$	1111

c. The number of bit strings of length four with the first bit equal to a 0 is $2^3 = 8$, or

$$F = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}$$

d. The set of bit strings of length four with two consecutive zero's is:

$$E = \{0000,\ 0001,\ 0010,\ 0011,\ 0100,\ 1000,\ 1001,\ 1100\}$$

e. Then: $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$

f. Then:

$\mid E \cap F \mid\ = 5$ and $\mid F \mid\ = 8$ and

$\mid S \mid\ = 2^4 = 16$

g. Therefore: $P(F) = \frac{8}{16}$ and $P(E \cap F) = \frac{5}{16}$

h. Therefore: $p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{5}{16} / \frac{8}{16} = \frac{5}{8}$

XVI. Independent Events

A. Discussion:

We generally assume that, when we flip a coin one or more times, the probability of a head turning up remains unchanged. That is, regardless of how many heads we have flipped previously, the probability of a head turning up on the next flip is still $\frac{1}{2}$ if we are flipping a true coin. Actually, even if it is theoretically possible, after flipping a coin n times and getting n heads normally leads us to believe that the coin is biased. Events such as this are termed **independent**.

B. Definition: The events E and F are said to be **independent** if and only if $P(E \cap F) = P(E)P(F)$

C. Example:

1. In the previous example of bits strings of length four there were $2^4 = 16$ possible bit strings, all of which are equally likely.

2. If $F \equiv$ the event that the bit string begin with a '1' then
$F = \{1111, 1110, 1101, 1100, 1011, 1001, 1010, 1000\}$

3. If $E \equiv$ the event that a bit string of length four contains an even number of 1's then:
$E = \{1111, 1100, 1001, 1010, 0011, 0110, 0101, 0000\}$

4. We then have that: $E \cap F = \{1111, 1100, 1001, 1010\}$

$\mid F \mid \ = 8$ $\mid E \mid \ = 8$ $\mid E \cap F \mid \ = 4$

$\mid S \mid \ = 16$

5. Therefore: $P(F) = \frac{8}{16} = \frac{1}{2}$ and $P(E) = \frac{8}{16} = \frac{1}{2}$

and $p(E \cap F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

6. Hence: $P(E \cap F) = P(E)P(F)$

7. Therefore E and F are independent events, by definition.