XV: Detecting a Counterfeit Coin

A. Problem Description

1. You are given a collection of coins with the proviso that one coin is a counterfeit.

2. Your scale is a simple balance scale that detects only that the object in one pan is lighter, heavier, or the same weight as the object in the other pan.

3. The counterfeit coin is lighter than any of the others.

4. Prove by induction that N weighings are sufficient to detect one counterfeit coin among 3^N coins.

B. Basis Step:

1. For three coins, one weighing is sufficient.
   a. If the two coins, one in each pan of the balance, are of equal weight, the remaining coin is the counterfeit. Otherwise, the lighter coin is the counterfeit.

   b. 3^N coins $\equiv 3^1$ coins $\equiv 1$ weighing

2. For four coins two weighings are required.

   If two coins are placed in each pan and one pan is lighter one additional weighing is sufficent to detect the counterfeit in the lighter pan.

3. For nine coins, two weighings can be necessary.
   a. If two coins on the balance are of equal weight, one of the remaining two coins is the counterfeit and one more weighing will detect the lighter coin which is the counterfeit.

   b. Otherwise, one weighing will detect the lighter coin which is the counterfeit.

   c. 3^N coins $\equiv 3^2$ coins $\equiv 9$ coins $\equiv 2$ weighings

4. For ten coins, three weighings may be necessary.

   If three coins on the balance are of equal weight, four coins remain, requiring two additional weighings. Otherwise, one additional weighing is required to determine which of the lighter three is counterfeit.

C. Inductive Assumption: N weighings are sufficient to detect a single counterfeit coin among 3^N coins.

D. Inductive Step:

1. Consider 3^{N+1} coins.

2. $3^{N+1} = 3 \times 3^N$

3. Therefore we can divide our set of 3^{N+1} coins into three sets of 3^N coins.

4. One weighing is required to determine which of the three sets contains the counterfeit coin.
   a. If, on the first weighing with 3^N coins on each pan of the scales, one side is lighter than the other, the lighter side contains the counterfeit coin.

   b. If both sides balance, the remaining set of 3^N coins contains the counterfeit coin.

5. According to our inductive assumption we can isolate the counterfeit coin from a set of 3^N coins in N weighings.

6. Since it requires at most one additional weighing to isolate the counterfeit coin from a set of 3^{N+1} coins the isolation will require at most $N + 1$ weighings.

E. Test: Suppose that we have a set of $3^{N+1}+1$ coins.

1. We can place two sets of 3^N coins on each pan.
   a. If the two sets are of equal weight there will be more than 3^N coins remaining to be tested.

   b. If the two sets are of unequal weight we can test the lighter set of 3^N coins in N weighings.

2. For the case of equal weights we can divide the set of 3^N+1 into two sets of 3^{N-1} coins and one set of $3^{N-1}+1$ coins and repeat step E.1.

3. Proceeding in this way for a total of N repetitions leaves us with four coins.
   a. One more weighing is not guaranteed to isolate the counterfeit coin, two may be required.

   b. Therefore $(N+1)+1$ weighings may be required.

4. Therefore a set of $3^{N+1}+1$ coins may require more than $N+1$ weighings to determine the counterfeit coin.

F. Therefore: N weighings are sufficient to detect a single counterfeit coin among 3^N coins.

XVI: Detecting a Counterfeit Coin II

A. Problem Description

1. You are given a collection of coins with the proviso that one coin is a counterfeit.

2. Your scale is a simple balance scale that detects only that the object in one pan is lighter, heavier, or the same weight as the object in the other pan.

3. The counterfeit coin is lighter than any of the others.

4. Prove by induction that N weighings may not be sufficient to detect a single counterfeit coin among 3^N+k coins, where k is some integer such that $3^N<3^N+k<3^{N+1}$.

B. Basis Step:

1. For three coins, 3^1, one weighing is sufficient.
   a. If the two coins, one in each pan of the balance, are of equal weight, the remaining coin is the counterfeit. Otherwise, the lighter coin is the counterfeit.

   b. 3^N coins $\equiv 3^1$ coins $\equiv 1$ weighing

2. For four coins two weighings are required.
   a. If two coins are placed in each pan and one pan is lighter one additional weighing is sufficent to detect the counterfeit in the lighter pan.

   b. $3^N + 1$ coins $\equiv 3^1 + 1$ coins $\equiv 2$ weighings

   c. Therefore: Number of weighings $> N$ when the number of coins is greater than 3^N.

3. For nine coins, two weighings can be necessary.
   a. If the six coins on the balance (three on each pan) are of equal weight, one of the remaining three coins is the counterfeit and one more weighing will detect the lighter coin which is the counterfeit.

   b. Otherwise, one additional weighing will detect the lighter coin which is the counterfeit.

   c. 3^N coins $\equiv 3^2$ coins $\equiv 9$ coins $\equiv 2$ weighings

4. For ten coins, three weighings may be necessary.
   a. If three coins on the balance are of equal weight, four coins remain, requiring two additional weighings. Otherwise, one additional weighing is required to determine which of the lighter three is counterfeit.

   b. $3^N + 1$ coins $\equiv 3^2 + 1$ coins $\equiv 10$ coins $\equiv 3$ weighings
   Therefore the number of weighings is greater than N when the number of coins is $3^N + 1$.

C. Inductive Assumption: N weighings may not be sufficient to detect a single counterfeit coin among a maximum of $3^N + 1$ coins.

D. Inductive Step:

1. Consider $3^{N+1} + k$ coins.

2. $3^{N+1} + k = 3 \times 3^N + k$

3. Therefore we can divide our set of $3^{N+1} + k$ coins into two sets of 3^N coins and one set of $3^N + k$ coins.

4. The worst case is that in which the two sets of 3^N coins are of equal weight, requiring that the counterfeit coin be contained in the third set of $3^N + k$ coins. One weighing determines this state.

5. Our inductive assumption states that N weighings may not be sufficient to isolate the counterfeit coin from $3^N + k$ coins.

6. We used one weighing to isolate the set of $3^N + k$ coins that contained the countefeit.

7. Our inductive assumption states that N weighings may not be sufficient to isolate the counterfeit coin from $3^N + k$ coins.

8. Therefore we have demonstrated that N weighings may not be sufficient to isolate the counterfeit coin from $3^{N+1} + k$ coins.

E. Therefore: N weighings may not be sufficient to detect a single counterfeit coin among a maximum of $3^N + k$ coins.