Simple Harmonic Motion

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Abstract

In this lab, my lab partner and I tested theories supporting translational and rotational simple harmonic motion. Principles of linear, simple harmonic motion guided my group to these more specific representations of systems. Testing the theory of Translational Simple Harmonic Motion involved comparing angular frequency of different hanging masses and different initial amplitudes. The oscillations witnessed were recorded using provided sensing equipment.

With the measurements taken, we calculated a standard angular frequency of 9.55 ± 0.02 rad/sec. For increased amplitude, our angular frequency was 2.09 ± 0.05 rad/sec. For increased amplitude and an additional 50 grams of hanging mass, the angular frequency was 6.28 ± 0.06 rad/sec. Angular frequency is dependent on mass, so theory agrees that the two values obtained from the second and third trial are different from each other. However, the two trial runs with the same mass, trial 1 and trial 3, should have produced the same result for the angular frequency following theory. This was clearly not the case and the error can most likely be attributed to human error in carrying out the experiment.

Afterwards, my group calculated a predicted value for angular frequency at 6.75 ± 0.06 rad/sec. This predicted value for angular frequency for the third trial run differs slightly, but is not within the range of uncertainty. Therefore error arose in a variety of possible ways. In this section of the lab, I'd label error to different measurements of initial amplitude between the sensor and the lab technician.

Upon the completion of the first lab section, my group was ready to begin the second lab section. In this section, my group tested the theory of Rotational Simple Harmonic Motion by applying torque to a rod of unknown metal. The rod was connected to a disk plate which performed the visible oscillations. Using data collected by another program in conjunction with a photo-gate sensor, I calculated a value for the moment of inertia of the solid cylindrical disk. This value measured 0.0397 ± 0.0003 kg*m^2.

Through derivation, our moment of inertia led to a suitable torsion constant. Using the relation of angular frequency to torsion constant and mass, I was able to calculate a torsion constant of 2.31 ± 0.02 m^4. The lab manual provided further insight into the specific metal we were using by including a formula for torsion modulus. The torsion modulus easily calculated from our results measured 2 x 10^{10} ± 2 x 10^{10} n/m^2. The lab manual again made life simple and provided a list of possible metal suspects. By comparing our value to the chart, our metal used could have been Lead, Magnesium, Aluminum, or Brass. However, it is hard to pinpoint one of these, because our uncertainty is so large. Out of the entire lab, the most error originated in this section. A simple one-percent error in the diameter propagated to a four-percent error in the torsional modulus. This can be labeled as the major contributing factor.

Introduction and Theory

-Translational Simple Harmonic Motion-

In the Translational Simple Harmonic Motion experiment, we will observe the effects of continuous, patterned motion along a single axis. First, it is key to assume the spring-system utilized is in an ideal state, while the system remains in a vacuum. These assumptions will make our derivations much simpler. The system used can be found in the following section, *Experimental Procedure*, below. As displayed in this figure, a mass hangs from our spring to its equilibrium position. When this mass is forced to a displacement, a patterned motion can be observed. This motion is translational simple harmonic motion. Proving the system has simple harmonic motion is as follows. Through the relations of forces in a spring, we can derive angular frequency.

$$F = -k(x-x_0) \quad \text{or} \quad F = m/a$$

Force (f) is traditionally represented as a mass (m) with acceleration (a). However, in a spring, Force is opposite the product of the spring constant (k) and its displacement ($x-x_0$).

$$-k(x-x_0) = m/a$$
$$a = (-k(x-x0))/m$$

Manipulation of these two equations yield the equality witnessed above. Acceleration is directly proportional to it spring constant and current displacement.

$$-k(x-x_0) = m(d^2x/dt^2)$$

This function can be integrated across time, and we receive a general solution.

$$x = x_m\sin(\omega t + \phi) + x_0$$

This solution describes the angular position (x) of our mass at any time (t). Angular position has a relationship between amplitude (x_m), its angular frequency (ω), the phase angle of the mass (ϕ), the initial offset (x_0), and time.

$$\omega = (k/m)^{1/2} = 2\pi/T$$

Through simple algebra, one can see the relationship of angular frequency with many other known variables. Two of the relations we haven't discussed concerns period (T) and frequency. Angular frequency is inversely proportional to period and proportional to frequency of the mass.

-Rotational Simple Harmonic Motion -

In the Rotational Simple Harmonic Motion experiment, we will observe continuous, patterned motion around a single axis. In this experiment, it is key to assume, when applying a force to initially rotate the rod, the disk remains locked in its vertical axis (the direction of the rod attached to the disk) and the disk itself is a solid cylinder. Manipulation of equations concerning torque will follow as means to produce a suitable equation representing the system.

$$\tau = -k\theta \quad \text{or} \quad \tau = Ia$$

Torque (τ) is generally defined as a rotational force around a defined axis. It is the force applied in this lab to rotate our disk. It is opposite, yet directly proportional to the angle the rod is twisted to (θ) and the torsion constant (k). The second representation of torque is represented in Newton's second law, concerning inertia of the rotating disc (I) and angular acceleration (α).

$$-k\theta = I\alpha$$
$$\alpha = d^2\theta/dt^2$$
Combining the two representations of torque results in the above equation where angular acceleration is the derivative of a change in the rod's angle over time.
$$-k\theta = I(d^2\theta/dt^2)$$
The knowledge of acceleration results in substitution. This equation can be integrated across time to synthesize the following general solution.
$$\theta = \theta_m \sin(\omega t + \phi)$$
The angular position (θ) is related to several constants: amplitude of oscillation (θ_m), the angular frequency (ω), the phase angle of operation (ϕ), and time.
$$\omega = (k/I)^{1/2} = 2\pi/T$$
As seen in translational simple harmonic motion, angular frequency is once again inversely related to period (T) and proportional to frequency. To receive the general solution, angular frequency must be the root of the torsion constant over the Inertia of the disc. The inertia of the rod is not included because the mass of the rod and its radius are much smaller than that of the disc's and therefore can be neglected.
$$I = \tfrac{1}{2}MR^2$$
$$\omega = (2k)/(MR^2)$$
Angular frequency can be related to the mass of the disc (M) and the radius of the disc (R) with further substitution of Inertia (I).
$$k = (nA^2)/(2\pi L)$$
Torsion constants differ from material to material and the shape of the rod. The relationship is predefined for this lab and can be witnessed above. The torsion constant is directly proportional to the torsion modulus (n), and the cross sectional area of the rod (A). The torsion constant is inversely proportional to the length (L) of the rod.

Experimental Procedure

-Translational Simple Harmonic Motion-

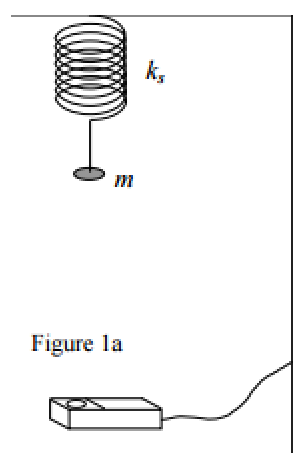


Figure 1a

To start this lab, the setup must first be completed. A diagram of the setup can be seen here. It comprises of a spring hanging from a stationary stand with an attached hanger at the spring's base. This hanger allows for other masses to be placed on it and remain stationary. At ground level, directly beneath the mass, lies a sensor to track the motion of the spring-mass system. Prior to arrival to lab, this setup had been completed.

From here, the lab technician must set up the sensor. A program should be included that will function with the sensor. This lab utilizes the *Logger Pro* program to take measurements. A file is included with this lab that adjusts the program's settings accordingly.

Once the program has been set, measurements of our equipment are necessary. The lab technician should mass the hanger with an accurate balance along with all additional masses that are to be hung on the hanger.

The technician should return to their station and do practice tests to their setup. My partner and I had applied a downward force to the hanging mass to assure it was securely attached to the spring. We also started data collection and moved an object over the sensor to assure it was recording amplitude and oscillation correctly. Once our lab setup was deemed fit for experimentation, we began recording data.

To start recording data, the technician must pull the hanging mass a few centimeters below the spring's equilibrium position and release. However, a moment before releasing, the *Logger Pro* program must be started by clicking the start button. Once a multitude of sin waves have formed on the program's graphical interface, the program may be stopped. If the data does not produced a graphed sine wave, repeat the process.

After collecting the data in the *Logger Pro* program, I added a new calculation of angular position to the program: $x_m*\sin(\omega t + \phi) + x_0$. This equation ideally represents the harmonic oscillations displayed by the spring and hanging mass. The *Logger Pro* program will graph what this system's motion should have been if under ideal conditions.

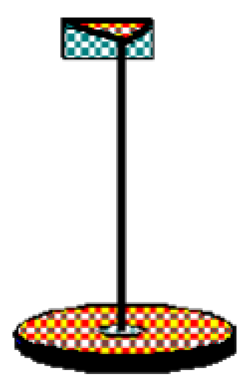
Then comes estimating values for unknowns (amplitude, angular frequency, frequency, phase angle, and initial offset) until a match to the graph of ideal motion, obtained from the extra calculation, is formed. This will result in a new equation where time is our only variable. Time's magnitude is reliable and easily tracked. The results should be transported to a separate program, referred to as *Origin*. *Origin* is a program whose abilities will be used to analyze statistics of our data, and provide a formal graph. The data should be again graphed, but in *Origin.* Print out this graph, including the statistical findings on the data. Record the angular velocity and its uncertainty in your lab notebook, along with all other statistics.

The same process (subtracting *Origin*) should be repeated for collecting data with a larger amplitude. Increasing the amplitude of oscillation can be completed by applying a greater downward force on the hanging mass. Record all values in the lab notebook.

For a third trial, an additional 50 grams to the hanging mass should be hung and an increased amplitude should be used. Repeat the process, once again subtracting *Origin*. Record all values in a lab notebook.

-Rotational Simple Harmonic Motion –

Like the Translational Simple Harmonic Motion section, the setup for this experiment was completed prior to class. The setup includes a large holding block on a table that will support the rod and disc. Hanging firmly from this block, there is the rod and disc component. The disc has been welded to the rod. On the rod lies a narrow protruding cylinder that aids in measurements. This part of the lab also comes with a file to adjust the *Logger Pro* program accordingly. At its equilibrium, the light sensor should be placed in such a way that the light-sensing gate reads blocked on the *Logger Pro* program. Assure the gate sensor is reporting blocked and unblocked correctly. Once done, setup has been complete.



The next step in the experimental process is to attempt practice runs of our experiment. Twist the disc to an angle no more than 45 degrees from its equilibrium position. A gently grasp of the rod may be necessary to assure it does not rotate off its vertical access. Once the technician feels confident in his approach, he or she may attempt a trial run.

Once confident, start collecting data on the *Logger Pro* program. Allow the program to collect anywhere above 15 periods. Once the minimum number of periods has been collected, stop the *Logger Pro* program. Transfer the data to the *Origin* program. Select a good set of data and record statistics calculated by the program. Form a graph from the data and print this graph along with the statistical findings. Record all data in the lab notebook.

Results and Analysis

-Translational Simple Harmonic Motion-

In this portion of the lab, I will perform an analysis of data collected in the translational simple harmonic motion section. My first value for angular velocity was 9.55 rad/sec. This trial had no additional mass hung from the hanger and included a normal displacement of the hanger from the equilibrium. The estimated uncertainty of this angular frequency is ±0.02 rad/sec as provided by the *Origin* program's statistics. This trial's graph can be found attached at the back of the report. My angular velocity for the second trial, which required no additional mass, but an increased amplitude, was 2.09 rad/sec. The angular velocity of the third trial was 6.28 rad/sec. Trial 3 was conducted with an increased amplitude and an additional 0.05 Kg mass. To calculate the spring constant (k) I used the following equations where ω is the angular frequency, m is the mass of the hanging objects, and T is the period of oscillation.

$$\omega = (k/m)^{1/2}$$

$$\omega = 2\pi/T$$

$$k = (m)(\omega^2) = (.05)(9.55)^2 = 4.56 \text{ N/m}$$

To calculate the uncertainty in the spring constant, I used the derivative method.

$$\delta_{k,\omega} = 2m\omega * \delta_\omega = 2(.05)(9.55) * (0.02) = 0.0191 \text{ n/m}$$

$$\delta_{k,m} = \omega^2 * \delta_m = (9.55)^2 * (0.001) = 0.0912 \text{ n/m}$$

$$\delta_k = (\delta_{k,\omega}{}^2 + \delta_{k,m}{}^2)^{1/2} = 0.09 \text{ n/m}$$

The uncertainty in angular frequency was given by the *Origin* program. The uncertainty in the mass is set to the uncertainty of the electronic balance used to measure it. These calculations result in a final spring constant of 4.56 ± 0.09 n/m. To find the angular frequency of oscillations with increased mass (ω_p) at 0.100 kg I used the spring constant and the derivations below, where

$$\omega_p = (k/m)^{0.5} = ((4.56)/(.1))^{1/2} = 6.753 \text{ rad/sec}$$

$$\delta_{\omega p,k} = (0.5/m)(k/m)^{-1/2} * \delta_k = (.5/.1)(4.56/.1)^{-1/2} * (.09) = 0.067\text{rad/sec}$$

$$\delta_{\omega p,m} = (-k/2)(m^{-2})(k/m)^{-1/2} * \delta_m = (-2.28)(.1^{-2})(45.6)^{-1/2} * (.001) = -0.0338 \text{ rad/sec}$$

$$\delta_{\omega p} = (\delta_{wp,k}{}^2 + \delta_{wp,m}{}^2)^{1/2} = .06 \text{ rad/sec}$$

The uncertainty in the mass is set to the same uncertainty of the electronic scale used. These calculations result in a final value of the angular frequency with increased mass as 6.75 ± 0.06 rad/sec.

-Rotational Simple Harmonic Motion –

In this portion of the lab, I will perform an analysis of data collected in the rotational simple harmonic motion section. A histogram containing the measured periods of oscillations can be found attached to the back of this lab report. From the *Origin* program, I was able to obtain the values concerning certain statistics: the minimum, the maximum, the sum, the mean, the median, the standard deviation, and the standard error. I then applied a Gaussian Curve to my bar data set. This curve has the predefined equation of $Max*exp(-(x-T)^2/(2*\sigma^2))$, where Max is the maximum value, exp is a representation of constant e, x is a position, T is period, and σ is standard deviation. This equation was given in the lab manual. Our collected data was used in the *Origin* program to plot the formula below.

Gaussian Curve = $0.792*e\text{^}(-(x-0.716)^2/(2(0.032)^2)))$

Our calculated and graphed Gaussian curve fit the data fit the data fairly well. The downhill slope of the curve much less fits.

Next, as the lab manual dictates, I assumed the plate was a solid cylinder and, therefore, used a suiting formula to calculate the moment of Inertia (I_p) below.

$$I = ½ MR^2$$

$$I = ½(4.7)(.13)^2 = .0397 \text{ kg*m}^2$$

I is the moment of inertia, M is the provided mass of the disc, and its radius (R). To calculate the uncertainty in the moment of inertia, I used the derivative method.

$$\delta_{I,M} = ½(R^2) * \delta_M = (0.5)(.13^2) * (.001) = 8.45 \times 10^{-6} \text{ kg * m}^2$$

$$\delta_{I,R} = (M)(R) * \delta_R = (4.7)(.13) * (.0005) = 3.06 \times 10^{-4} \text{ kg * m}^2$$

$$\delta_I = (\delta_{I,M}{}^2 + \delta_{I,R}{}^2)^{1/2} = 0.0003 \text{ kg * m2}$$

The uncertainty in the mass is set to the same uncertainty of the electronic balance used to mass the disc. The uncertainty in the radius of the disc is determined by the technician's faith in their measurements. I thought I could measure accurately within one millimeter. These calculations resulted in an inertial value of 0.0397 ± 0.0003 kg*m^2. Using this value and the following formula, it was simple to calculate the Torsion constant (k).

$$\omega = 2\pi/T = (k/I)^{1/2}$$

$$\omega = 2\pi/T = 2\pi/(.716) = 8.775 \text{ rad/sec}$$

$$\delta_\omega = \delta_T = 0.008 \text{ rad/sec}$$

Where ω is the rotational frequency, k is the torsion constant, I is the moment of inertia, and T is the period. The uncertainty in the period and angular frequency was given by the *Origin* program. The calculated angular frequency is 8.775 ± 0.008 rad/sec.

$$k = I\omega^2 = (.0397)(8.775^2) = 2.310 \text{ m}^4$$

To calculate the uncertainty in the torsion constant, I used the derivative method.

$$\delta_{k,I} = (\omega^2) * \delta_I = (8.775^2) * (.0003) = 0.0231 \text{ m}^4$$

$$\delta_{k,\omega} = 2(I)(\omega) * \delta_\omega = 2(.0397)(8.775) * (.008) = 0.0056 \text{ m}^4$$

$$\delta_k = (\delta_{k,I}{}^2 + \delta_{k,\omega}{}^2)^{1/2} = 0.02 \text{ m}^4$$

These calculations result in a torsion constant of $2.31 \pm 0.02\ m^4$. From here, we must calculate the torsion modulus. I manipulated a lab manual-provided equation and substituted with values calculated above. In this equation, torsion constant (k) is directly proportional to torsion modulus (n) and cross-sectional area (A), while it is inversely proportional to length of the rod (L).

$$k = (nA^2)/(2\pi L)$$

$$n = 2\pi Lk/(A^2) \quad \text{where } A = \pi r^2$$

$$n = ((2\pi)(1)(2.31))/((\pi(.003)^2)^2) = 1.815 \times 10^{10}\ n/m^2$$

$$\delta_{n,k} = 2\pi L/(A^2) * \delta_k = (2\pi(1)/((\pi(.003)^2)^2) * (.02) = 1.572 \times 10^8\ n/m^2$$

$$\delta_{n,L} = 2\pi k/(A^2) * \delta_L = 2\pi(2.31)/((\pi(.003)^2)^2) * (.005) = 9.078 \times 10^7\ n/m^2$$

$$\delta_A = 2\pi r * \delta_r = 2\pi(.003) * (.001) = 1.885 \times 10^{-5}\ m^2$$

$$\delta_{n,A} = -4\pi Lk/(A^3) * \delta_A = 4\pi(1)(-2.31)/((\pi(.003)^2)^3) * (\delta_A) = 2.421 \times 10^{10}\ n/m^2$$

$$\delta_n = (\delta_{n,k}{}^2 + \delta_{n,L}{}^2 + \delta_{n,A}{}^2)^{1/2} = 1.210 \times 10^{11}\ n/m^2$$

The uncertainty in length and radius of the rod are determined by the technician's faith in the accuracy of their measurements. These calculations result in a torsion modulus of $2 \times 10^{10} \pm 2 \times 10^{10}\ n/m^2$. To show that an error in the diameter will compound and produce an error four times over in the torsion modulus, I must derive further. If the radius is 3 mm, then the diameter of the rod is 6 mm. One percent of 6 mm is 0.00006 m.

$$\delta_A = 2\pi r * \delta_r = 2\pi(.003) * (.00006) = 1.131 \times 10^{-6}\ m^2$$

$$\delta_{n,A} = -4\pi Lk/(A^3) * \delta_A = 4\pi(1)(-2.31)/((\pi(.003)^2)^3) * (\delta_A) = 7.262 \times 10^8\ n/m^2$$

$$\% = \delta_{n,A}/n = (7.262 \times 10^8)/(1.815 \times 10^{10}) = 4.001\%$$

The calculation above shows how a slight misjudgment in measurements can have a vast trickledown effect. Based on our results for torsion modulus and its large uncertainty, we can compare our value with those of the table. Our value leads to matches with one of several metals: Lead, Magnesium, Aluminum, and Brass.

Conclusion

-Translational Simple Harmonic Motion-

In the first section of the experiment, I tested theory supporting translational simple harmonic motion through the use of other known principles. To test translational harmonic motion, I experimented with various weights of hanging mass on a vertically fixated spring and measured its oscillations when moved from its equilibrium. Additionally, I changed the amplitude of oscillation and noted more change. I observed and graphed the value of the first trial's angular frequency, $\omega = 9.55 \pm 0.02$ rad/sec. The second trial's increased amplitude resulted in an angular frequency of 2.09 rad/sec, while the third trials increased amplitude and increased hanging mass resulted in an angular frequency of 6.28 rad/sec. These results make sense in terms of the theory. The one practice run we had done for trial one, whose angular frequency was 9.42 ± 0.03 rad/sec, does not agree within the uncertainty of the other run.

Even though these two runs for the first trial are minutely different, there is still error contributing. I see the largest error arising from estimating the amplitude. The amplitude was hard to measure. Based off the position of the hanging mass, my partner and I found the easiest way to measure the amplitude was to hold a ruler stationary in air. I'm guessing this ruler did not remain perfectly stationary. Another error is air resistance. Any resistance against the spring's movement is dampening the harmonic motion. In other terms, it's no longer simple harmonic motion. Air resistance is very tiny though.

Then I calculated a predicted value for angular frequency, $\omega_p = 6.75 \pm 0.06$ rad/sec, from the best data set of multiple runs, and then compared that to the angular frequency of trial 3 with increased amplitude and increased mass, $\omega_p = 6.28$ rad/sec. These values differed moderately most likely from the same experimental errors above: human error and dampening.

There are many ways to improve this experiment. First off, old, outdated equipment is not going to be as accurate and precise as newer models. Replacing this old equipment would be helpful. Another way to improve this experiment is to obtain a more precise way to measure displacement of the hanging spring when starting the experiment. An awkward setup meant one person had to hold the meter-stick and pull down the mass while another technician started the lab. It'd be nice to split these jobs between three people.

-Rotational Simple Harmonic Motion –

In the second section of the experiment, I tested theory supporting rotational simple harmonic motion through the use of other known principles. This part of the experiment included applying torque to an aluminum rod and disc system. The oscillations produced, when the induced torque stopped, was measured and recorded. The average period of this oscillation was 0.716 seconds with an uncertainty of 0.008 seconds. I then proceeded to calculate a value for the moment of inertia of a presumed solid cylindrical disc at the base of the rod. This disc is said to have a moment of inertia equal to half the product of its mass and radius squared. Using measured values and derived equations, the disc has a moment of inertia of $.0397 \pm 0.0003$ kg*m^2. This value led to derivation of the torsion constant with the relationship of angular frequency to the constant and mass. The system used at my particular station used a rod with torsion constant equal to 2.31 ± 0.02 m^4. At this point, the lab provided an equation relation the torsion constant to the torsional modulus, the cross sectional area of the rod, and its length. The value calculated for the torsional modulus of the used metal was $2 \times 10^{10} \pm 2 \times 10^{10}$ n/m^2. This value for torsional modulus included four of the listed metals: Lead, Magnesium, Aluminum, and Brass. These all fall within the range of uncertainty. Our uncertainty is so extremely high compared to the calculated value most entirely due to the uncertainty of the area. As we had calculated, a single percent error in the diameter results in four percent error for torsional modulus. Our uncertainty was 100% of the calculated value of the torsional constant. Using the 1:4 ratio, it is likely the diameter measurement had an error hovering near 25%.

There are many potential sources of error in this section of the lab. To list some, there is unaccounted friction in the system, the rod could not possibly be stopped from veering of it vertical axis, and human error in measurements. All three of these are monumental in this section's outcome. Dampening reduces energy in the system, reducing the period of oscillation. When the system swayed, there couldn't have been accurate measurement from the photo-gate sensor. And lastly, human error could have resulted in a 100% error as calculated above, from measurements of the diameter of the rod.

Ways to improve this experiment are as follows. The most cost effective solution that made this lab so difficult would be to find a way to stop the rod from swaying off its vertical axis. If there is a method with minimal friction, it should be included into the lab manual. Another solution is new equipment. The first rod we had, was obviously old. It couldn't even go through five oscillations before coming to a complete halt. Old equipment should be replaced. These steps will ensure accuracy and precision.

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References

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