Damped and Forced Oscillators

Jacob Alspaw
Department of Physics, Case Western Reserve University
Cleveland, OH 441067079

ABSTRACT

In the LCR lab, we were able to produce and witness the effects of damped oscillating circuits in addition to forced resonating circuits. Different combinations of inductors, capacitors, and resistors were used to evaluate a resonance frequency for different circuit formations. With the help of Logger Pro and a program provided by the Case Western Reserve department of physics, my lab partners and I were able to calculate experimental frequencies from our data, as can be seen in later tables, and then proceed to calculate theoretical values of frequency using Faraday's Law of Induction. In the later part of this lab, my lab partners and I configured a resonant circuit by applying a function generator as a power source. This data was easily extrapolated to a resonant plot to find a resonant frequency, ω_r , to have a value of 3510 ± 30 Hz and $\Delta\omega$ to have a value of 1000 ± 100 Hz. Ten data points were further used to determine a theoretical and experimental value for the Quality Factor, denoted as "Q", which were 8.9 ± 0.1 and 6.1 ± 0.6, respectively. The values of uncertainty were attributed to by the relatively high uncertainty from our equipment as our equations will illustrate below. Measurements were taken by an oscilloscope that can measure amplitude and period. These data points are transferred to origin to create the resonant graph. Sometimes the analytical fit that Origin employs can be "shaky", especially when your components do not have the exact working value that they are marked with. Discrepancies are often attributed to equipment errors, which was present in this lab, but may have also been the part of human error. It is difficult to measure the exact amplitude on the oscilloscope, but I assure that my lab partners and I often produce quality work. Therefore, keeping all things in mind, our uncertainty and final calculated experimental value make sense, but mostly did not agree with the supposed theoretical model.

INTRODUCTION & THEORY

Provided by Faraday's Law of Induction comes proof that a relationship exists between the current through an inductor and the EMF across it. From this theory, my lab partners and I can deduce the resonance frequency as a result of the specific LCR circuit configuration. The electronic oscillator frequency of any LCR circuit can be modeled with one equation:

$$\omega_r = \frac{1}{\sqrt{LC}}$$

In this equation, R is the magnitude of the resistor, L is the magnitude of the inductor, C is the magnitude of the capacitor, and ω is a value for frequency. In the LCR circuit used in this lab, the configuration will incorporate two buttons, a resistor, a capacitor, an inductor, and a battery voltage source as seen in the first below figure. This certain circuit design is classified as a damped mechanical oscillator that will produce data to measure the resonant frequency. These kinds of circuits abide by the following differential model:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

A solution for this model that has been solved for charge at a certain time is:

$$Q = Q_0 e^{-\frac{t}{\tau}}(\omega' t + \varphi)$$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R}{2L}^2}$$

$$\tau = L/R$$

The variable "Q_o" is the charge at time, t = 0 when the phase angle, , is equal to 90 degrees. R is the magnitude of the resistor, L is the magnitude of the inductor, C is the magnitude of the capacitor, ω is a value for frequency, and τ is the time constant. Then, to measure a frequency induced by the change in voltage, substitute charge values for voltages using the following equation:

$$Q = CV_C$$

$$Vc = \frac{(Q_0 e^{-\frac{t}{\tau}}(\omega' t + \varphi))}{C}$$

Q is still the capacitor's charge, the variable "Q_o" is the charge at time, t = 0 when the phase angle, , is equal to 90 degrees. R is the magnitude of the resistor, L is the magnitude of the inductor, C is the magnitude of the capacitor, ω is a value for frequency, and τ is the time constant.

A forced oscillator will also produce pertinent information about the circuit. The same circuit as above will be used, but the voltage source will be changed form a battery voltage source to a function generator. This will allow the experimenters to adjust the input wave's frequency of the voltage supply to allow for a wide set of measurements. Forced oscillating circuits follow a differential model similar to damped oscillating circuits, but with a few changes:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_{\,M}\cos(\omega t)$$

A solution for this model that has been solved for current at a certain time is:

$$I = I_m \cos(\omega t + \varphi)$$

$$I_m = \frac{V_m}{\sqrt{(\omega L - \frac{1}{\omega C})^2 + R^2}}$$

As the same as any other LRC circuit:

$$\omega = \sqrt{\frac{1}{LC}}$$

The production of a plot pitting I_m versus ω, will establish a resonance curve that illuminates the required input frequency in order to achieve the resonance's greatest amplitude. The quality of these circuits can be analyzed using a quality factor as a measurement system. The quality factor for this circuit design is:

$$Q = \frac{\omega_r L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

The above is a theoretical solution for the Q value of the circuit. The following equation is a method to find the Q value experimentally with analysis of the resonance curve.

$$Q = \frac{C\omega_r}{\Delta\omega}$$

In this equation, $\Delta\omega$ is the half-maximum along the full width of the resonance curve, ω_r , and c is a constant equal to $\sqrt{3}$.

PROCEDURE

To measure oscillation in this experiment, my lab partners and I implemented specific circuit configurations and utilized Logger Pro to record data. The damped oscillator portion of this experiment began with building a LCR circuit on a Pasco Board. This circuit incorporated a .022μF capacitor initially which was later changed for a .0047μF capacitor in the end of the damped oscillator section. The circuit always used a 0.47 H inductor which has a resistance of around 600 Ω and an uncertainty of ± 72 Ω. Combinations of different magnitude resistors, measuring 1 kΩ and 3 kΩ, were also used in the damped oscillating circuit. The circuit to the right was made out of these circuit components. The two probes are attached across the capacitor to record its voltage oscillation. This data was read, stored, and graphed by the Logger Pro program. Pressing the push-buttons to first charge and discharge the capacitor resulted in the needed oscillation. For each trial, there was a need for a different combination of resistor and capacitor, meaning there were four different trials that each used the same inductor, but a mix of other components. These runs are as follows:

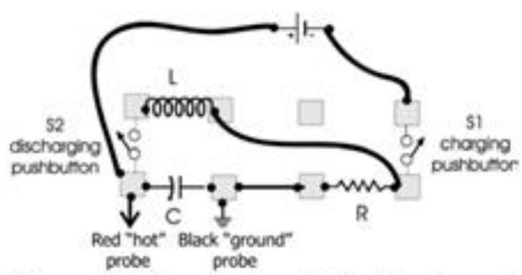

Figure 5: Circuit layout for the damped oscillator.

Physics Introductory labs: E&M Manual. Updated September 18, 2012

1. R = 0 kΩ C = 0.022 μF

2. R = 1 kΩ C = 0.022 μF
3. R = 3 kΩ C = 0.022 μF
4. R = 1 kΩ C = 0.0047 μF

These vales will need to be measured and their magnitudes recorded for the analysis of this experiment. Mine and my lab partners' values for this experiment can be found in a chart in the analysis below.

After the finish of each trial, a non-linear fit was applied to the data:

$$y = A * \exp\left(-\frac{t}{L}\right) * \sin(W * t - P)$$

In this equation, A is the initial voltage of the circuit, W is a circuit's frequency, L is the time constant of the circuit, and P is the phase angle to the solution.

The second major section of this lab is to find the resonant frequency of a circuit, which can prove difficult and time consuming. The experimenters must implement the circuit in the picture to the right. All circuit components are in series with one another and the voltage source has now been changed to an AC function generator that allows for the change of the input wave's frequency. The resistor for this experiment is supposed to be a magnitude of 10 Ω. Waveform measurements were taken with the help of an oscilloscope. The voltage settings were changed to almost maximum amplitude, and adjusted the scope's graph to 5 mV/DIV and 0.2 seconds/DIV. The digital multimeter was then set to measure current. Results from changing the frequency were collected and recorded by the Logger Pro program. Adjust the frequency until the maximum resonant amplitude is displayed; record the frequency value on the function generator. Adjust the function generator again until two values that differ 10% form the maximum are found. Then take ten more data points of frequency range with five points on either side of the maximum resonant amplitude's frequency value. Plot the resulting data in Origin and create a resonant graph that has an adjusted Lorentzian curve applied to the data set. The curve can be seen in the final attached graphed.

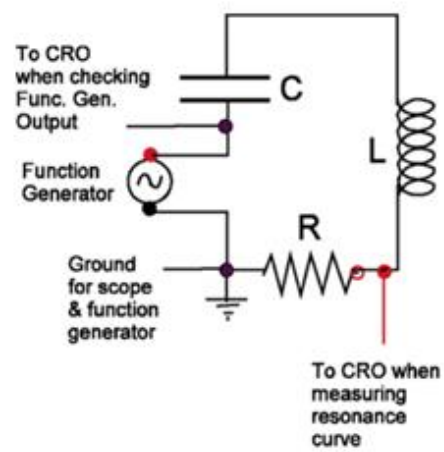

Figure 3: Forced, damped oscillator.

Physics Introductory labs: E&M Manual. Updated September 18, 2012

ANALYSIS

For the first section of this lab, we had conducted for trials of different combinations of resistors and capacitors. The inductor stayed constant throughout the experiment at 599 ± 2 Ω resistance and 478 ± 1 mH inductance. The resistance of the inductor must be included in the calculations because the resistance in the long strand of wire contributes to the circuit's total resistance. The trials asked for a 1 kΩ resistor, a 3 kΩ resistor, a 22 nF capacitor, and a 4.7 nF capacitor. The actual measured values for these components are 991 ± 2 kΩ, 3030 ± 2 kΩ, 22.6 ± 0.1 nF, and 4.1 ± 0.1 nF, respectively.

Data from Logger-Pro Non-linear Fit

Trial	Resistance (Ω)	Capacitance (nF)	Max Amplitude (V)	Frequency (Hz)	Time Constant (ΩF)	Phase
1	599 ± 2	22.6 ± 0.1	10.4 ± 0.1	9628 ± 9	0.00153 ± 2e-5	60.26 ± 0.01
2	1590 ± 2	22.6 ± 0.1	10.03 ± 0.07	9580 ± 20	0.000573 ± 5e-6	-6.293 ± 0.006
3	3629 ± 2	22.6 ± 0.1	10.3 ± 0.1	8800 ± 30	0.000275 ± 0.000003	-31.570 ± 0.005
4	1590 ± 2	4.1 ± 0.1	10.32 ± 0.1	21470 ± 50	0.00059 ± 0.00001	15.33 ± 0.04

The above table is a representation of our experimental results. This data is later used in comparison with the theoretical values seen below.

Theoretical Data from Measured Values:

Trial	Resistance (Ω)	Capacitance (nF)	Time Constant(ΩF)	Frequency(Hz)
1	599 ± 2	22.6 ± 0.1	0.015960 ± 0.000007	9602 ± 2
2	1590 ± 2	22.6 ± 0.1	0.000601 ± 0.000002	9483 ± 2
3	3629 ± 2	22.6 ± 0.1	0.0002634 ± 0.000006	8877 ± 1
4	1590 ± 2	4.1 ± 0.1	0.000601 ± 0.000002	22530 ± 10

For trials 1 - 3, the theoretical values for time constants, including their uncertainties, did not overlap with the experimental data that was achieved in this lab. These differences will be further discussed in the conclusion while we talk about general sources of error. The values in trial 4 have uncertainties that overlap, and therefore our results are said to be valid. For the frequency measurements, only trial 3 produced an agreement between experimental and theoretical values. Again, the discrepancies for the other trials will be discussed at a later time.

The exact method to calculate the time constant of the four trials is to use the following equation:

$$\tau = \frac{2L}{R}$$

The uncertainty in values from this equation can be found by applying the derivative method. Find the uncertainty in the time constants by first solving for the contribution of uncertainty from individual parts of the equation and then add these values in quadrature.

$$\delta_{\tau,L} = \left|\frac{2}{r}\right| * \delta_L$$
$$\delta_{\tau,R} = \left|\frac{2L}{R^2}\right| * \delta_R$$
$$\delta_\tau = \sqrt{\delta_{\tau,L}^2 + \delta_{\tau,R}^2}$$

Afterwards, calculating frequency can be done by using one of the equations outlined in the Introduction & Theory Section which has been further simplified:

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R}{2L}^2} = \sqrt{\frac{2 - R^2C}{2LC}}$$

The uncertainty for these values can be found after applying the derivative method to this equation. The result can be seen below:

$$\delta_{\omega',R} = \frac{-R}{L\sqrt{\frac{2 - R^2C}{2LC}}} * \delta_R$$

$$\delta_{\omega',C} = \left(-\frac{1}{LC^2}\right)^{-1/2}$$

$$\delta_{\omega',L} = \left(\frac{R^2C-2}{2L^2C}\right)^{-1/2}$$

$$\delta_{\omega'} = \sqrt{\delta_{\omega',R}^2 + \delta_{\omega',C}^2 + \delta_{\omega',L}^2}$$

The following is a set of ten data points that reflect a Lorentzian curve as a result of the procedure from the second part of lab. These are data points taken directly from the oscilloscope. The measured values for the circuit components were a 10.5 ± 0.1 Ω resistor, accompanied by another 599 ± 2 Ω resistance form the inductor, 478 ± 1 mH inductance, and a 4.1 ± 0.1 nF capacitor.

Frequency (kHz)	Amplitude (mV)	Normalized Amplitude
9200 ± 200	6 ± 2	0.1
4800 ± 200	18 ± 2	0.3
4100 ± 200	30 ± 2	0.5
3750 ± 200	48 ± 2	0.8
3400 ± 200	60 ± 2	1
3200 ± 200	48 ± 2	0.8
3000 ± 200	30 ± 2	0.5
2900 ± 200	25 ± 2	0.42
2600 ± 200	18 ± 2	0.3
1400	6	0.1

The above data was then put into origin and had a Lorentizian curve applied to it. This supplied us with more values that described our data set. Referring to our graph of normalized amplitude plotted against frequency, which is the last attached graph, we found ω_r to have a value of 3510 ± 30 Hz and $\Delta\omega$ to have a value of 1000 ± 100 Hz. Using these two values, we then calculated the quality factor for our experiment.

$$Q = \frac{C\omega_r}{\Delta\omega} = \frac{\sqrt{3} * 3510}{1000} = 6.079$$

The uncertainty in this result was acquired by applying the derivative method to the quality factor equation:

$$\delta_{Q,\omega_r} = \left|\frac{C}{\Delta\omega}\right| \delta_{\omega_r} = \left|\frac{\sqrt{3}}{1000}\right| * 30 = 0.0520$$

$$\delta_{Q,\Delta\omega} = \left|\frac{C\omega_r}{\Delta\omega^2}\right| \delta_{\Delta\omega} = \left|\frac{\sqrt{3}*3510}{1000^2}\right| * 100 = 0.6079$$

$$\delta_Q = \sqrt{\delta_{Q,\omega_r}^2 + \delta_{Q,\Delta\omega}^2} = \sqrt{0.0027 + 0.3696} = 0.6102$$

For the quality factor of this lab, my lab partners and I would like to report a experimental quality factor value of 6.1 ± 0.6. From this point, we continued our analysis and calculated the supposed solution to the quality factor, Q, using the theoretical approach. Remember to keep in mind that the total resistance of the circuit, R, incorporates both the resistance from the resistor and the resistance form the inductor.

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \left(\frac{1}{10.5 + 599}\sqrt{\frac{0.478}{4.1e{-}9}}\right) = 17.7153$$

The uncertainty in this value will be calculated using the derivative method:

$$\delta_{Q,R} = \left|\frac{1}{R^2}\sqrt{\frac{L}{C}}\right| * \delta_R = \left|\frac{1}{(10.5 + 599)^2}\sqrt{\frac{0.478}{4.1e{-}9}}\right| * 2 = 0.0291$$

$$\delta_{Q,L} = \left|\frac{1}{2RC\sqrt{\frac{L}{C}}}\right| * \delta_L = \left|\frac{1}{2(10.5 + 599)(4.1e - 9)\sqrt{\frac{0.478}{4.1e - 9}}}\right| * \ 0.001 = 0.0185$$

$$\delta_{Q,C} = \left|\frac{L}{2RC^2\sqrt{\frac{L}{C}}}\right| * \delta_C = \left|\frac{0.478}{2(10.5 + 599)(4.1e{-}9)^2\sqrt{\frac{0.478}{4.1e{-}9}}}\right| * 0.1e{-}9 = 0.2160$$

$$\delta_Q = \sqrt{\delta_{Q,R}^2 + \ \delta_{Q,L}^2 + \ \delta_{Q,C}^2} = 0.2188$$

Our theoretical value for the quality factor, Q, is calculated to be 17.7 ± 0.2. If we redo the calculations to include the resistance of the function generator that is supposedly around 600Ω, then we find a new theoretical Q value that is 8.9 ± 0.1. This value is much closer to our experimental quality factor value, but there is discrepancy between the theoretical value and the experimental value that will be discussed in the conclusion.

CONCLUSION

For the first section of this lab, my lab partners and I found that the combination of resistors, capacitors, and single inductor in each trial produced a damped sin function. When evaluating the validity of the experiments results, we can say that the values from our Logger Pro program data came very close to the theoretical values calculated using our measurements, but the experimental data and theoretical data did not agree within their uncertainties. There is an exception to the third trial for calculating the time constant. It was the only very accurate and valid data obtained in the first section. For all of these cases, the frequencies are more accurately represented by the theoretical values. Because of the equation equating the time constant to the quotient of magnitude of the circuits total resistance and twice the circuit's inductance, my lab partners and I concluded an increase in resistance directly causes an increase time constant, so a decrease in frequency. When increasing the resistor's magnitude from 0 kΩ to 3 kΩ over trials 1-3, there was also a decrease in frequency which supports our group's claim. Unlike the relationship of resistance and frequency, capacitance is concluded to be indirectly proportional to the frequency. A higher frequency should result from a lower capacitance in the circuit based off the same set of equations. Our lab results again supported our conclusion as the trend towards a smaller capacitance from trials two and four produced a much larger frequency. Looking at the equation to calculate time constants for these four trials, the time constant's magnitude should remain unchanged when switching capacitors as capacitance is left out of the equation. This is evident in our results from trials two and four. As a result of the discrepancies in this section's data from experimental values to theoretical values, it can be noted that my lab group and I did not collect our data with utmost care and concern. The data was close, but there was still error that may have stemmed from the Logger Pro programs automatic non-linear fit or not adjusting the uncertainty of our DMM measurements to an appropriate value that would encompass the tool's uncertainty properly. Other error possibilities could have stemmed from errors in the push buttons because they were extremely difficult to push making discharging the circuit difficult, too. To increase this labs efficiency, the CWRU physics lab should no longer employ outdated equipment such as the PASCO boards. The boards often times are dilapidated and have poor soldering on their undersides. I would recommend the use of breadboards which are much more proper and much more commonly used in real-world application.

The second half of this lab was purposed to find the optimal resonance frequency of a specific circuit that will produce the largest maximum amplitude. This optimal frequency was 3400 ± 200 Hz which produced an amplitude of 60 ± 2 mV. The ten values taken from this section were read from an old oscilloscope where they were recorded in our lab notebooks and then transferred to the Origin program. From an Origin applied Lorentzian fit, our resonance curve produced values for ω_r at 3510 ± 30 Hz and $\Delta\omega$ at 1000 ± 100 Hz. Using these two values, my lab partners and I proceeded to calculate an experimental value for quality factor, Q, at 6.1 ± 0.6. The theoretical value for quality factor was 8.9 ± 0.1. The resonance curve was close to supporting the theoretical values, but did not agree within either's uncertainty. Therefore, it can be noted this section's data is not completely valid as error was present. These values do not agree within their uncertainty and I believe it is because we made substantial error in measuring the resistance of the function generator. The function generator probably had much higher resistance than 600 Ω which would result in a lower theoretical quality factor that might match the experimental value. Other sources of error could have again risen from old and worn equipment. The oscilloscopes do not have the modern capability of measuring amplitude for the user or frequency. The user must do this themselves which allows for user error. I am sure we did not measure the amplitude to complete accuracy. To improve upon the second section of this experiment, I would again recommend updating the old equipment for newer models, or to move the lab to the Glennan circuits lab, so we may use that equipment.

ACKNOWLEDGEMENTS

I would like to thank Mark Miller, Eric Black, the Case Western Reserve University Department of Physics, and the lab teaching assistant Amol Ratnaparkhe for their large involvement in obtaining the necessary experimental data. My gratitude for their kind help in the completion of this experiment.

REFERENCES

1) Christenson, Cory, D. Driscoll, *Physics 122 Lab Manual*, CWRU Bookstore, Fall 2015.