## I. Introduction to the Binomial Theorem

A. Consider: $(A+B)^2$

1. $(A+B)^2 = (A+B) \times (A+B)$

2. $(A+B)^2 = \mathrm{A} \times \mathrm{A} + A \times B + B \times A + B \times B$

3. $(A+B)^2 = A^2 + 2AB + B^2$

B. Consider: $(A+B)^3$

1. $(A+B)^3 = (A+B) \times (A+B) \times (A+B)$

2. $$\begin{aligned}(A+B)^3 = A \times A \times A &+ A \times A \times B + A \times B \times A \\ &+ A \times B \times B + B \times A \times A \\ &+ B \times A \times B + B \times B \times A \\ &+ B \times B \times B\end{aligned}$$

3. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

## II. Introduction to the Binomial Theorem using $(A+B)^4$

A. Consider: $(A+B)^4$

1. $(A+B)^4 = \underbrace{(A+B)}_{1^{st}\ \text{Factor}} \times \underbrace{(A+B)}_{2^{nd}\ \text{Factor}} \times \underbrace{(A+B)}_{3^{rd}\ \text{Factor}} \times \underbrace{(A+B)}_{4^{t}\ \text{Factor}}$

2. $$\begin{aligned}(A+B)^4 = A \times A \times A \times A \ &+ A \times A \times A \times B \\ &+ A \times A \times B \times A \\ &+ A \times \mathrm{A} \times B \times B \\ &+ A \times B \times A \times A \\ &+ A \times B \times A \times B \\ &+ A \times B \times B \times A \\ &+ A \times B \times B \times B \\ &+ B \times A \times A \times A \\ &+ B \times A \times A \times B \\ &+ B \times A \times B \times A \\ &+ B \times A \times B \times B \\ &+ B \times B \times A \times A \\ &+ B \times B \times A \times B \\ &+ B \times B \times B \times A \\ &+ B \times B \times B \times B\end{aligned}$$

B. A Typical Term in the Expansion of $(A+B)^4$ is

$$A \times B \times A \times B$$

which is obtained by multiplying one of the two terms from the first factor times one of the two terms from the second factor times one of the two terms from the third factor times one of the two terms from the fourth factor as shown below:

$$\begin{array}{c} \quad\Downarrow \qquad\qquad\qquad \Downarrow \quad\quad \Downarrow \qquad\qquad\qquad \Downarrow \\ (A+B) \times (A+B) \times (A+B) \times (A+B) \end{array}$$

C. There are two possibilities, $A$ or $B$, for the selection of each term from one of the factors and four factors from which to make the selection.

D. Therefore: There are $2^4 = 16$ possibilities for creating a term in the expansion of $(A+B)^4$

E. Some terms can be combined

1. Consider all possible orderings of three $A$'s and one $B$, or $AAAB$, $AABA$, $ABAA$, and $BAAA$

    a. Each of these four terms can be combined as $A^3B$

    b. Therefore all of these terms are 'like' terms and appear in the summation that is the expansion of $(A+B)^4$ as $4A^3B$.

    c. Note: $\binom{4}{1} = 4$

2. Consider all possible orderings of two $A$'s and two $B$'s, or $AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$, and $BBAA$

    a. Each of these six terms can be combined as $A^2B^2$

    b. Again, all of these terms are 'like' terms and appear in the summation that is the expansion of $(A+B)^4$ as $6A^2B^2$.

    c. Note: $\binom{4}{2} = 6$

3. Similarly:
   a. $ABBB,\ BABB,\ BBAB,$ and $BBBA$ appear in the expansion as $AB^3$ and $\binom{4}{3} = 4$

   b. $BBBB$ appears as $B^4$ and $\binom{4}{4} = 1$

   c. $AAAA$ appears as $A^4$ and $\binom{4}{0} = 1$

F. Finally:

$$(A+B)^4 = \binom{4}{0}A^4 + \binom{4}{1}A^3B + \binom{4}{2}A^2B^2 + \binom{4}{3}AB^3 + \binom{4}{4}B^4$$

$$= A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$$

**III. The Binomial Theorem:**
**Given any real numbers $A$ and $B$ and any non-negative integer $N$,**

$$(A+B)^N = \sum_{k=0}^{N} \binom{N}{k} A^{N-k}B^k$$

$$= A^N + \binom{N}{1}A^{N-1}B + \binom{N}{2}A^{N-2}B^2 + \ldots + \binom{N}{N-1}A^1B^{N-1} + B^N$$

**IV. The recursive definition of $A^N$ is:**

$$A^N = \begin{cases} 1 & \text{if } N = 0 \\ A \times A^{N-1} & \text{if } N > 0 \end{cases}$$

**with the special case of:** $0^0 = 1$

**V. Pascal's Formula:** **Let $N$ and $r$ be positive integers with $r < N$.**

**Then:** $\binom{N+1}{r} = \binom{N}{r-1} + \binom{N}{r}$

Proof:

1. $\binom{N}{r-1} + \binom{N}{r} = \frac{N!}{(r-1)!(N-r+1)!} + \frac{N!}{r!(N-r)!}$

$$= \frac{N!}{(r-1)!(N-r+1)!}\left(\frac{r}{r}\right) + \frac{N!}{r!(N-r)!}\left(\frac{N-r+1}{N-r+1}\right)$$

after multiplying the first term by $\frac{r}{r}$ and the second term by $\frac{N-r+1}{N-r+1}$ to create a common denominator.

2. Then:

$$\binom{N}{r-1} + \binom{N}{r} = \frac{N!r}{(N-r+1)!r(r-1)!} + \frac{NN!-N!r+N!}{(N-r+1)(N-r)!r!}$$

$$= \frac{N!r}{(N-r+1)!r!} + \frac{N!(N-r+1)}{(N-r+1)(N-r)!r!}$$

$$= \frac{N!r}{(N-r+1)!r!} + \frac{N!(N-r+1)}{(N-r+1)!r!}$$

$$= \frac{N!r-N!r+NN!+N!}{(N-r+1)!r!}$$

$$= \frac{NN!+N!}{(N-r+1)!r!} = \frac{N!(N+1)}{(N-r+1)!r!}$$

$$= \frac{(N+1)!}{r!(N+1-r)!} = \binom{N+1}{r}$$

## VI. The Binomial Theorem: Proof by Mathematical Induction

A. Theorem: Given any real numbers $A$ and $B$ and any non-negative integer $N$,

$$P(N) = \quad (A+B)^N \quad = \sum_{k=0}^{N} \binom{N}{k} A^{N-k} B^k$$

B. Basis step:

1. $P(0) = \quad (A+B)^0 \quad = \sum_{k=0}^{0} \binom{0}{k} A^{0-k} B^k$

2. $(A+B)^0 \quad = 1$

3. $\sum_{k=0}^{0} \binom{0}{k} A^{0-k} B^k = \binom{0}{0} A^{0-0} B^0 = \frac{0!}{0!(0-0)!} = 1$

4. Hence: $P(0)$ is true.

C. Inductive Assumption: $(A+B)^N = \sum_{k=0}^{N} \binom{N}{k} A^{N-k} B^k$

D. Inductive Proof:

1. $(A+B)^{N+1} = (A+B)(A+B)^N$

$$= (A+B) \sum_{k=0}^{N} \binom{N}{k} A^{N-k} B^k$$

by our inductive assumption.

2. Then:

$$(A+B)^{N+1} = A\sum_{k=0}^{N}\binom{N}{k}A^{N-k}B^k + B\sum_{k=0}^{N}\binom{N}{k}A^{N-k}B^k$$

$$= \sum_{k=0}^{N}\binom{N}{k}A^{N+1-k}B^k + \sum_{k=0}^{N}\binom{N}{k}A^{N-k}B^{k+1}$$

3. To get the powers of $A$ and $B$ to agree we change the second summation so that we can use $j = k+1$ as the index of summation. We have, then:

$$\binom{N}{k}A^{N-k}B^{k+1} = \binom{N}{j-1}A^{N-(j-1)}B^j = \binom{N}{j-1}A^{N+1-j}B^j$$

4. Therefore:

$$(A+B)^{N+1} = \sum_{k=0}^{N}\binom{N}{k}A^{N+1-k}B^k + \sum_{k=0}^{N}\binom{N}{k}A^{N-k}B^{k+1}$$

$$= \sum_{k=0}^{N}\binom{N}{k}A^{N+1-k}B^k + \sum_{j=1}^{N}\binom{N}{k-1}A^{N+1-k}B^k$$

5. Since $j$ is a dummy variable we can, for the sake of the simplicity for our upcomming algebraic manipulations, replace it with $k$, making sure that:
   a. We replace $j$ with $k$ in all places where $j$ occurs.

   b. We maintain the summation limits for $k$ that are in place for $j$.

6. We have, then:

$$(A+B)^{N+1} = \sum_{k=0}^{N} \binom{N}{k} A^{N+1-k} B^k + \sum_{k=1}^{N+1} \binom{N}{k-1} A^{N+1-k} B^k$$

7. To make the summation limits the same we :
   a. Extract the $0$th term

   b. Extract the $N$th term

   to get:

$$\begin{aligned}(A+B)^{N+1} &= \binom{N}{0} A^{N+1-0} B^0 + \sum_{k=1}^{N} \binom{N}{k} A^{N+1-k} B^k \\ &\quad + \sum_{k=1}^{N+1} \binom{N}{k-1} A^{N+1-k} B^k \\ &= \binom{N}{0} A^{N+1-0} B^0 \\ &\quad + \sum_{k=1}^{N} \left[\binom{N}{k} + \binom{N}{k-1}\right] A^{N+1-k} B^k \\ &\quad + \binom{N}{(N+1)-1} A^{N+1-(N+1)} B^{N+1}\end{aligned}$$

8. Recall:
   a. $A^0 = B^0 = 1$

   b. $\binom{N}{0} = \binom{N}{N} = 1$

9. Therefore: $\binom{N}{0} A^{N+1-0} B^0 = A^{N+1}$ and

   $$\binom{N}{(N+1)-1} A^{N+1-(N+1)} B^{N+1} = B^{N+1}$$

10. We then have:

    $$(A+B)^{N+1} = A^{N+1} + \sum_{k=1}^{N} \left[ \binom{N}{k} + \binom{N}{k-1} \right] A^{N+1-k} B^k + B^{N+1}$$

11. Pascal's formula states: $\left[ \binom{N}{k} + \binom{N}{k-1} \right] = \binom{N+1}{k}$

12. Therefore:

    $$(A+B)^{N+1} = A^{N+1} + \sum_{k=1}^{N} \binom{N+1}{k} A^{N+1-k} B^k + B^{N+1}$$

13. Since: $\binom{N+1}{0} = \binom{N+1}{N+1} = 1$

    we can include $A^{N+1}$ and $B^{N+1}$ in the summation to get:

    $$(A+B)^{N+1} = \sum_{k=0}^{N+1} \binom{N+1}{k} A^{(N+1)-k} B^k$$

    and we have proven $P(N) \rightarrow P(N+1)$

E. Therefore: $(A+B)^N = \sum_{k=0}^{N} \binom{N}{k} A^{N-k} B^k$

## V. The Binomial Theorem: Combinatorial Proof

A. Theorem: Given any real numbers $A$ and $B$ and any non-negative integer $N$,

$$(A+B)^N = \sum_{k=0}^{N} \binom{N}{k} A^{N-k} B^k$$

B. For $N = 0$:

1. $P(0) = \quad (A+B)^0 = \sum_{k=0}^{0} \binom{0}{k} A^{0-k} B^k$
2. $(A+B)^0 = 1$
3. $\sum_{k=0}^{0} \binom{0}{k} A^{0-k} B^k = \binom{0}{0} A^{0-0} B^0 = \frac{0!}{0!(0-0)!} = 1$
4. Hence: $P(0)$ is true.

C. For $N > 0$

1. The expression $(A+B)^N$ can be expanded into the sum of products of $N$ letters where each letter is either $A$ or $B$.

2. Example:

$$\begin{aligned}(A+B)^3 &= A^3 + 3A^2B + 3AB^2 + B^3 \\ &= A\times A\times A + A\times A\times B + A\times A\times B \\ &\quad + A\times A\times B + A\times B\times B \\ &\quad + A\times B\times B + A\times B\times B \\ &\quad + B\times B\times B\end{aligned}$$

2. For each $k = 0, 1, 2, 3, ..., N$ the product

$$A^{N-k}B^k = A\times A\times ...\times A\times B\times B\times B\times ....\times B$$

a, Has $(N-k)$ $A$'s and $k$ $B$'s

b. Occurs as a term in the sum the same number of times as there are orderings of $(N-k)$ $A$'s and $k$ $B$'s.

c. This number is $\binom{N}{k}$, the number of ways to choose $k$ positions into which to place a $B$, with the remaining positions being filled by $A$'s.

3. Therefore, when like terms are combined, the coefficient of $A^{N-k}B^k$ in the sum is $\binom{N}{k}$.

4. Therefore: $(A+B)^N = \sum_{k=0}^{N} \binom{N}{k} A^{N-k}B^k$

## VI. Applications of the Binomial Theorem

A. Prove that: $2^N = \sum_{k=0}^{N} \binom{N}{k} = \binom{N}{0} + \binom{N}{1} + \binom{N}{2} + ... + \binom{N}{N}$

for all integers $N \geq 0$

1. Note that: $2 = 1 + 1$ so that $2^N = (1+1)^N$

2. We apply the binomial theorem to the product of a sum to get:

$$2^N = \sum_{k=0}^{N} \binom{N}{k} 1^{N-k} \times 1^k = \sum_{k=0}^{N} \binom{N}{k} 1 \times 1 = \sum_{k=0}^{N} \binom{N}{k}$$

B. Express the sum $\sum_{k=0}^{N} \binom{N}{k} 9^k$ in closed form, i.e., without using a summation.

1. $\sum_{k=0}^{N} \binom{N}{k} 9^k = \sum_{k=0}^{N} \binom{N}{k} 1^{N-k} 9^k$

2. If we let $A = 1$ and $B = 9$ we have:

$$\sum_{k=0}^{N} \binom{N}{k} 9^k = \sum_{k=0}^{N} \binom{N}{k} A^{N-k}B^k$$

3. But: $\sum_{k=0}^{N} \binom{N}{k} A^{N-k}B^k = (A+B)^N$

4. Therefore: $\sum_{k=0}^{N} \binom{N}{k} 9^k = \sum_{k=0}^{N} \binom{N}{k} 1^{N-k} 9^k = (1+9)^N = 10^N$

so: $\sum_{k=0}^{N} \binom{N}{k} 9^k = 10^N$

C. Prove that: $\sum_{k=0}^{N}(-1)^k\binom{N}{k} = 0$

1. $0 = ((-1)+1)$ so $0^N = ((-1)+1)^N$

2. Therefore: $0 = 0^N = \sum_{k=0}^{N}\binom{N}{k}(-1)^k \times 1^{N-k}$

$$= \sum_{k=0}^{N}\binom{N}{k}(-1)^k \times 1$$

$$= \sum_{k=0}^{N}\binom{N}{k}(-1)^k$$

3. Therefore: $0 = \sum_{k=0}^{N}\binom{N}{k}(-1)^k$