I. Proof Example 1: An Existance Proof

Theorem: There exists a pair of consecutive integers such that one of these integers is a perfect square and the other a perfect cube.

A. Logical Statement:

There exists integers N and $N+1$ such that either:

$N = I^2$ and $N+1 = K^3$

or

$N = I^3$ and $N+1 = K^2$

where I and K are integers

B. This is an existance proof, i.e., one needs only(?) to conjure up a value of N for which the theorem holds to create a proof.

C. Starting with 1 and testing one quickly encounters $8 = 2^3$ and $9 = 3^2$.

D. Therefore: $N = 8 = 2^3$ and $N + 1 = 9 = 3^2$ $\square$

II. Proof Example 2: An Investigation and Proof of Observation

A. Definitions:

1. The ***quadratic mean*** of two real numbers X and Y is given by
$$\sqrt{\frac{X^2+Y^2}{2}}$$

2. The arithmetic mean of two real numbers X and Y is given by
$$\frac{X+Y}{2}$$

B. Investigate and formulate a conjecture regarding the relative sizes of the quadratic mean and the arithmetic mean.

III. Investigation #1 (easy):

A. $\sqrt{\frac{X^2+Y^2}{2}} \geq \frac{X+Y}{2}$ for $X \leq 0.0$, $Y \leq 0.0$

B. Proof: For $X \le 0.0,\quad X + Y = -|X| + Y < Y$

and $\frac{X^2+Y^2}{2} \ge \frac{X^2 - 2|X| \times Y + Y^2}{4} = \left(\frac{-|X|+Y}{2}\right)^2$

For $Y \le 0.0,\quad X + Y = X + -|Y| < X$

so $\frac{X^2+Y^2}{2} \ge \frac{X^2 - 2X \times |Y| + Y^2}{4} = \left(\frac{X+-|Y|}{2}\right)^2$

Therefore: $\sqrt{\frac{X^2+Y^2}{2}} \ge \sqrt{\left(\frac{X+Y}{2}\right)^2} = \frac{X+Y}{2}$

Note: For $X = 0$ we have:

$$\sqrt{\frac{X^2+Y^2}{2}} = \sqrt{\frac{Y^2}{2}} = \frac{Y}{\sqrt{2}} > \frac{Y}{2} = \frac{X+Y}{2}$$

and for $Y = 0$ we have:

$$\sqrt{\frac{X^2+Y^2}{2}} = \sqrt{\frac{X^2}{2}} = \frac{X}{\sqrt{2}} > \frac{X}{2} = \frac{X+Y}{2}$$

IV. Investigation #2 :

A. Investigation:

a. Java code:

```
package quadmean;

public class QuadMean
   {
   public static void main(String[] args)
      {
      double X   =   0.0;
      double Y   =   0.0;
      int    Lim =   10;

      final  double  One =   1.0;
      final  double  Two =   2.0;

      System.out.println("       X         Y    Quadratic Mean      Mean");
```

```
//      X, Y <= 1.0
      for(int i = Lim; i > 0; i--)
         {
         X = One / (double) i;
         for(int j = Lim; j > 0; j--)
            {
            Y = One / (double) j;
            double quadMean   =   Math.sqrt((X*X + Y*Y)/Two);
            double arithMean  =   (X + Y)/Two;

            if (arithMean > quadMean)
               System.out.println("Exception: arithMean = " + arithMean +
                     "and quardMean = " + quadMean);

            System.out.printf("%10.4f %10.4f %10.4f %15.4f\n", X, Y,
                           quadMean, arithMean);
            }
         }

//      X, Y >= 0.0
      for(int i = 0; i < Lim; i++)
         {
         X = (double) i;
         for(int j = 0; j < Lim; j++)
            {
            Y = (double) j;
            double quadMean   =   Math.sqrt((X*X + Y*Y)/Two);
            double arithMean  =   (X + Y)/Two;

            if (arithMean > quadMean)
               System.out.println("Exception: arithMean = " + arithMean +
                     "and quardMean = " + quadMean);

            System.out.printf("%10.2f %10.2f %10.2f %15.2f\n", X, Y,
                           quadMean, arithMean);
            }
         }
      }
   }
```

4. Output

X	Y	Quadratic Mean	Mean
0.1000	0.1000	0.1000	0.1000
0.1000	0.1111	0.1057	0.1056
	.		
0.1111	0.1667	0.1416	0.1389
	.		
0.1429	0.1111	0.1280	0.1270
	.		
0.1667	0.1667	0.1667	0.1667
	.		
0.2500	0.1111	0.1934	0.1806
	.		
0.2500	0.2500	0.2500	0.2500
	.		
0.3333	0.1250	0.2517	0.2292
	.		
0.3333	0.5000	0.4249	0.4167
0.3333	1.0000	0.7454	0.6667
0.5000	0.1000	0.3606	0.3000
	.		
1.0000	1.0000	1.0000	1.0000
0.00	0.00	0.00	0.00
	.		
1.00	8.00	5.70	4.50
	.		
2.00	9.00	6.52	5.50
	.		
4.00	5.00	4.53	4.50
	.		
6.00	6.00	6.00	6.00
	.		
7.00	7.00	7.00	7.00
	.		
9.00	0.00	6.36	4.50
	.		
9.00	8.00	8.51	8.50
9.00	9.00	9.00	9.00

Done

B. Conjecture 2:

The quadratic mean of X and Y , or $\sqrt{\frac{X^2+Y^2}{2}}$, is greater than or equal to the arithmetic mean of X and Y, or $\frac{X+Y}{2}$.

C. Proof:

$$\left(\sqrt{\frac{X^2+Y^2}{2}}\right)^2 \quad = \frac{X^2+Y^2}{2}$$

$$\left(\frac{X+Y}{2}\right)^2 = \frac{X^2+2\times X\times Y+Y^2}{4} = \frac{1}{2}\times\left(\frac{X^2+Y^2}{2}\right)+\frac{X\times Y}{2}$$

If: $X = Y$ $\quad X \times Y = X^2 = Y^2$

If: $X > Y$ $\quad X \times Y < X^2$

If: $X < Y$ $\quad X \times Y < Y^2$

Therefore: $\frac{X\times Y}{2} \leq \frac{Z^2}{2}$ where $Z = max(X,\, Y)$

Therefore: $\frac{X\times Y}{2} \leq \frac{X^2+Y^2}{2}$

Therefore: $\left(\frac{X+Y}{2}\right)^2 = \frac{1}{2}\times\left(\frac{X^2+Y^2}{2}\right)+\frac{X\times Y}{2} \leq \frac{X^2+Y^2}{2}$

Therefore: $\left(\frac{X+Y}{2}\right)^2 \leq \frac{X^2+Y^2}{2} \leq \left(\sqrt{\frac{X^2+Y^2}{2}}\right)^2$

Therefore: $\frac{X+Y}{2} \leq \sqrt{\frac{X^2+Y^2}{2}}$

Therefore The quadratic mean of X and Y , or $\sqrt{\frac{X^2+Y^2}{2}}$, is greater than or equal to the arithmetic mean of X and Y, or $\frac{X+Y}{2}$. $\square$

V. Investigation #3 : By Guess and by Golly

A. Problem: Prove or disprove that if you have an 8 gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, that you can accumulate 4 gallons of water in one of the jugs by successively pouring water from one jug into another jug.

B. Solution 1:

8 gal	5 gal	3 gal	Action
0	5	3	Empty 8 gal jug into other two
3	5	0	Empty 3 gal jug into the 8 gal jug
3	2	3	Fill 3 gal jug from 5 gal jug
6	2	0	Empty 3 gal jug into 8 gal jug
6	0	2	Empty 5 gal jug into 3 gal jug
1	5	2	Empty 8 gal jug into 5 gal jug
0	5	2	Empty 8 gal jug
2	5	0	Empty 3 gal jug into 8 gal jug
2	2	3	Empty 5 gal jug into 3 gal jug
4	0	3	Empty 5 gal jug into 8 gal jug

C. Solution 2:

8 gal	5 gal	3 gal	Action
3	5	0	Fill 5 gal jug from 8 gal jug
3	2	3	Fill 3 gal jug from 5 gal jug
6	2	0	Empty 3 gal jug into 8 gal jug
6	2	0	Empty 3 gal jug into 8 gal jug
6	0	2	Empty 5 gal jug into 3 gal jug
1	5	2	Fill 5 gal jug from 8 gal jug
0	4	3	Empty 5 gal jug into 3 gal jug