The Lane-Emden Equation will admit 3 analytical Solutions

1^{st} Solution : sphere of constant density

$\Rightarrow \rho(r) = \rho_c \Rightarrow n = 0$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta_0}{d\xi}\right) = -1$$

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta_0}{d\xi}\right) = -\xi^2$$

integrating $\rightarrow \xi^2 \frac{d\theta_0}{d\xi} = -\frac{\xi^3}{3} + C$

$$\frac{d\theta_0}{d\xi} = -\frac{1}{3}\xi + \frac{C}{\xi^2}$$

integrating again

$$\theta_0 = -\frac{1}{6}\xi^2 - \frac{C}{\xi} + B.$$

From the boundary conditions $\left.\begin{matrix}\theta(0) = 1 \rightarrow \\ \theta'(0) = 0 \rightarrow\end{matrix}\right\{\begin{matrix}C = 0 \\ B = 1\end{matrix}$

$$\Rightarrow \boxed{\theta_0 = 1 - \frac{1}{6}\xi^2}$$

2nd Solution $n = 1$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta_1}{d\xi}\right) = -\theta_1$$

$$\frac{d^2\theta_1}{d\xi^2} + \frac{2}{\xi}\frac{d\theta_1}{d\xi} + \theta_1 = 0$$

Assuming a power series solution

$$\theta_1 = \sum_n c_n \xi^n \rightarrow \begin{cases} \dfrac{d\theta_1}{d\xi} = \sum_n n c_n \xi^{n-1} \\ \dfrac{d^2\theta_1}{d\xi^2} = \sum_n n\cdot(n-1) c_n \xi^{n-2} \end{cases}$$

replacing in the differential equation

$$\sum_n n\cdot(n-1) c_n \xi^{n-2} + \frac{2}{\xi}\sum_n n c_n \xi^{n-1} + \sum_n c_n \xi^n = 0$$

$$\sum_n n c_n \xi^{n-2}(n+1) + \sum_n c_n \xi^n = 0$$

reindexing the 1^{st} sum $n \rightarrow n+2$

$$\sum_n \left[(n+3)(n+2)c_{n+2} + c_n\right]\xi^n = 0$$

for the summation to be zero, the term

$$c_n + (n+2)(n+3)\,c_{n+2} = 0$$

$$\Rightarrow c_{n+2} = \frac{-c_n}{(n+2)(n+3)}$$

a few terms of the recursion:

$$n=0\,;\quad c_2 = \frac{-c_0}{6} = -\frac{c_0}{3!}$$

$$n=1\,;\quad c_3 = \frac{-c_1}{4\cdot 3} = -2\frac{c_1}{4!}$$

$$n=2\,;\quad c_4 = -\frac{c_2}{5\cdot 4} = +\frac{c_0}{5!}$$

$$n=3\,;\quad c_5 = -\frac{c_3}{6\cdot 5} = 2\frac{c_1}{6!}$$

$$\vdots$$

$$\rightarrow \theta_1(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \cdots$$

$$= c_0 + c_1\xi - \frac{c_0}{3!}\xi^2 - \frac{2c_1}{4!}\xi^3 + \frac{c_0}{5!}\xi^4 + \frac{2}{6!}c_1\xi^5 + \cdots$$

$$= c_0\left(1 - \frac{\xi^2}{3!} + \frac{\xi^4}{5!} \cdots\right) + 2c_1\left(\frac{\xi}{2!} - \frac{\xi^3}{4!} + \frac{\xi^5}{6!} \cdots\right)$$

$$\xi\,\theta_1(\xi) = c_0\left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} \cdots\right) - 2c_1\left(-\frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} \cdots\right)$$

$$\xi\theta_1 = C_0 \sin\xi - 2C_1\cos\xi + 2C_1$$

$$\Rightarrow \theta_1 = C_0\frac{\sin\xi}{\xi} - 2C_1\frac{\cos\xi}{\xi} + \frac{2C_1}{\xi}$$

from the boundary conditions:

$\theta_1(0) = 1$

$$C_0 \lim_{\xi\to 0} \overbrace{\frac{\sin\xi}{\xi}}^{\to 1} + 2C_1 \lim_{\xi\to 0} \overbrace{\left(\frac{1-\cos\xi}{\xi}\right)}^{\to 0} = 1$$

$$\Rightarrow C_0 = 1$$

$$\left.\frac{d\theta_1}{d\xi}\right|_{\xi=0} = 0 = \left[\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2} + 2C_1\frac{\sin\xi}{\xi} + 2C_1\frac{\cos\xi}{\xi^2} - \frac{2C_1}{\xi^2}\right]_0 = 0$$

to avoid the divergence at 0, $C_1 \equiv 0$

$$\Rightarrow \theta_1(\xi) = \frac{\sin\xi}{\xi}$$

please note that this model requires multiple transforms

3^{rd} Solution : Plummer Sphere $n=5$

~~(Isothermal sphere)~~

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_s}{d\xi^2}\right) = -\theta_s^5$$

1^{st} Apply the Emden transformation:

$$\theta = Azx^{\bar{\omega}} \quad \text{with } \bar{\omega} = \frac{2}{n-1} \rightarrow n\bar{\omega} = 2+\bar{\omega}$$

$$\cancel{\frac{d\theta}{dx} = Az\bar{\omega}x^{\bar{\omega}-1}\,dx^{\bar{\omega}}} \qquad \frac{d\theta}{dx} = A\left[x^{\bar{\omega}}\frac{dz}{dx} + \bar{\omega}zx^{\bar{\omega}-1}\right]$$

$$\frac{d^2\theta}{dx^2} = A\left[x^{\bar{\omega}}\frac{d^2z}{dx^2} + \bar{\omega}x^{\bar{\omega}-1}\frac{dz}{dx} + \bar{\omega}x^{\bar{\omega}-1}\frac{dz}{dx} + \bar{\omega}(\bar{\omega}-1)zx^{\bar{\omega}-2}\right]$$

$$= A\left[x^{\bar{\omega}}\frac{d^2z}{dx^2} + 2\bar{\omega}x^{\bar{\omega}-1}\frac{dz}{dx} + \bar{\omega}(\bar{\omega}-1)x^{\bar{\omega}-2}z\right]$$

The transform introduced by Lord Kelvin:

$$x \rightarrow \frac{1}{\xi} \; ; \; \frac{d}{d\xi} = -x^2\frac{d}{dx}$$

the Lane-Emden Equation becomes: $X^4\frac{d\theta}{dx^2} = -\theta^5$

$$Ax^4\left(x^{\bar\omega}\frac{d^2z}{dx^2} + 2\bar\omega x^{\bar\omega-1}\frac{dz}{dX} + \bar\omega(\bar\omega-1)x^{\bar\omega-2}z = -A^5x^{5\bar\omega}z^5\right.$$

$$x^{\bar\omega+4}\frac{d^2z}{dx^2} + 2\bar\omega x^{\bar\omega+3}\frac{dz}{dx} + \bar\omega(\bar\omega-1)zx^{\bar\omega+2} + A^4x^{5\bar\omega}z^5 = 0$$

$$x^{5\bar\omega+2}\frac{d^2z}{dx^2} + 2\bar\omega x^{5\bar\omega+1}\frac{dz}{dx} + \bar\omega(\bar\omega-1)zx^{5\bar\omega} + A^4x^{5\bar\omega}z^5 = 0$$

$$x^2\frac{d^2z}{dx^2} + 2\bar\omega x\frac{dz}{dx} + \bar\omega(\bar\omega-1)z + A^4z^5 = 0$$

let $x = e^t$ & $\theta = \left(\frac{1}{2}e^t\right)^{\frac{1}{2}} z$

$dx = e^t dt$

$$\to \frac{dz}{dx} = e^{-t}\frac{dz}{dt}; \quad \frac{d^2z}{dx^2} = e^{-2t}\left[\frac{d^2z}{dt^2} - \frac{dz}{dt}\right]$$

$$\Rightarrow \frac{d^2z}{dt^2} + (2\bar\omega-1)\cdot\frac{dz}{dt} + \bar\omega(\bar\omega-1) + A^4z^5 = 0$$

** ~~from~~ the singular solution of the Kepler Equation has the form $\theta = ax^{\bar\omega}$
Replacing in original Lane-Emden Equation
we get $a^{n-1} = \bar\omega(1-\bar\omega)$

$$\Rightarrow A^{n-1} = \bar\omega(1-\bar\omega) = A^4$$

$$\frac{d^2z}{dt^2} + (2\bar{\omega}-1)\frac{dz}{dt} + \bar{\omega}(\bar{\omega}-1)(z-z^n) = 0$$

replacing $\bar{\omega} = \frac{2}{n-1}$ in $n=5 \Rightarrow \bar{\omega} = \frac{2}{4} = \frac{1}{2}$

$$\frac{d^2z}{dt^2} + \frac{1}{2}\left(-\frac{1}{2}\right)(z-z^5) = 0$$

$$\frac{d^2z}{dt^2} - \frac{z}{4}(1-z^4) = 0$$

$$\frac{dz}{dt}\cdot\frac{d^2z}{dt^2} = \frac{z}{4}(1-z^4)\frac{dz}{dt}$$

$$\times \Rightarrow uu' = \frac{1}{2}(u^2)'$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\frac{dz}{dt}\right)^2 = \frac{z}{4}(1-z^4)\frac{dz}{dt}$$

$$\frac{d}{dt}\left(\frac{dz}{dt}\right)^2 = \frac{z}{2}(1-z^4)\frac{dz}{dt} = \frac{1}{2}\left[z-z^5\right]\frac{dz}{dt}$$

$$\left(\frac{dz}{dt}\right)^2 = \frac{z^2}{4} - \frac{z^6}{12} + cst$$

$$\Rightarrow \frac{dz}{dt} = \pm\left[cst + \frac{z^2}{4} - \frac{z^6}{12}\right]^{1/2}$$

$$dt = \cancel{\frac{\pm dz}{\frac{z}{2}(1-}} \frac{\pm dz}{(cst + \frac{z^2}{4} - \frac{z^6}{12})^{1/2}}$$

to avoid elliptical solutions, cst = 0

$$\Rightarrow dt = \frac{\pm dz}{\frac{z}{2}(1 - \frac{z^4}{3})^{1/2}}$$

let $\frac{1}{3} z^4 = \sin^2 \xi \rightarrow \frac{4}{3} z^3 dz = 2 \sin\xi \cos\xi \, d\xi$

$$\frac{4}{3z} dz = 6 \cot \xi \, d\xi$$

$$\Rightarrow dt = \pm \frac{\cot \xi \, d\xi}{\cos \xi} = \csc \xi \, d\xi$$

$$\Rightarrow t = \pm \ln(\csc \xi + \cot \xi) + C$$

$$\underbrace{\csc \xi + \cot \xi}_{\cot(\xi/2)} = c_1 e^{\pm t}$$

$$\Rightarrow \tan \xi/2 = \frac{1}{c_1} e^{\pm t} = \frac{\sin \xi}{1 + \cos \xi}$$

$$\rightarrow \sin^2 \xi = (1+\cos\xi)^2 \tan^2 \xi/2 = \frac{4 \tan^2 \xi/2}{(1 + \tan^2 \xi/2)^2}$$

$$\Rightarrow z^4 = \frac{12 \tan^2 \xi/2}{(1+\tan^2 \xi/2)^2}$$

$$z = \pm \left(\frac{12 \tan^2 \xi/2}{(1+\tan^2 \xi/2} \right)^{1/4}$$

$$\theta = \left(\tfrac{1}{2} e^{t}\right)^{1/2} z$$

$$\theta_s = \left(\frac{3/c_1^2}{(1 + \frac{1}{c_1^2} \cancel{\tan} \xi^2)^2} \right)^{1/4} = \frac{3^{1/4}/c_1^{1/2}}{(1 + \frac{1}{c_1^2}\xi^2)^{1/2}}$$

from the bounday conditions

$$\theta_s(0) = 1$$

$$1 = \frac{3^{\frac{1}{4}}}{c_1^{1/2}} \Rightarrow c_1 = \sqrt{3}$$

$$\Rightarrow \theta_s(\xi) = \frac{1}{(1+\frac{\xi^2}{3})^{1/2}}$$