Deriving the Lane-Emden Equation

Starting with the hydrostatic equation

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

multiply through by $\frac{r^2}{\rho}$ then take a derivative w.r.t r

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{d}{dr}(-GM) = -G\frac{dM}{dr} = -4\pi G r^2 \rho$$

** $\frac{dM}{dr} = 4\pi r^2 \rho$ is the mass continuity Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \left(= -\nabla^2 \Phi\right)$$

Poisson's Equation.

Let's replace the polytropic model: $P = K\rho^{1+\frac{1}{n}}$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dK\rho^{1+\frac{1}{n}}}{dr}\right) = -4\pi G\rho$$

let us define the density in terms of a dimensionless variable θ

$\rightarrow \rho = \rho_c \theta^n$ where $\rho_c = \rho(0)$

$\Rightarrow P = K \rho_c^{1+\frac{1}{n}} \theta^{n+1} = P_c \theta^{1+n}$

$\Rightarrow P_c = K \rho_c^{1+\frac{1}{n}}$

[Replacing in $\nabla^2 \Phi$]

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 K}{\rho_c \theta^n}\frac{d}{dr}\left(\rho_c^{1+\frac{1}{n}}\theta^{n+1}\right)\right) = -4\pi G \rho_c \theta^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 K \rho_c^{1+\frac{1}{n}}}{\rho_c \theta^n}(n+1)\theta^n \frac{d\theta}{dr}\right) = -4\pi G \rho_c \theta^n$$

$$\frac{(n+1) K \rho_c^{1/n-1}}{4\pi G r^2}\frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\theta^n$$

to further make the equation dimensionless

let $r = r_n \xi$ with $r_n^2 = \frac{(n+1) K \rho_c^{\frac{1}{n}-1}}{4\pi G}$

$$\frac{\cancel{r_n^2}}{\cancel{r_n^2}\xi^2}\frac{d}{\cancel{r_n} d\xi}\left(\cancel{r_n^2}\xi^2 \cdot \frac{d\theta}{\cancel{r_n} d\xi}\right) = -\theta^n$$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n} \Rightarrow \text{Lane-Emden Equation}$$