

0.1 From Examples to Precise Rules

Notations

- T_i denotes thread number i .
- $T_i \equiv T_j$ means both threads have same code.
- w_i^j is the j^{th} event in thread i which is a write.
- r_i^j is the j^{th} event in thread i which is a read.

A few definitions for our use

Definition 1. *Program Order (po)* Total order between events in the same thread. Respects the execution order between events in the same thread.

Definition 2. *Symmetric Memory Order (smo)* A strict partial order between writes in a set of symmetric threads. Consider a set of symmetric threads $T_1 \equiv T_2 \equiv \dots \equiv T_n$. Each of these threads have exactly one read event, and multiple write events, all to the same memory, say x .

Then each write in the above threads are involved in a symmetric order, such that.

$$\forall i \in [0, n-1] . w_i^j \xrightarrow{smo} w_{i+1}^j$$

Where j denotes the j^{th} event in any of the threads, which is a write.

Perhaps should put examples for the above defintion.

Definition 3. *Reads-From (rf)* Binary relation that links a read to a write from which its value comes. Note that for our purpose, this relation is functional. For example, if a read r_i^j gets its read value from write w_k^l , then we have the relation.

$$w_k^l \xrightarrow{rf} r_i^j$$

Main Rule Using the above setup, our intention is to explore lesser execution graphs leveraging the symmetry that can result due to swapping of thread identities. For this, we enforce a restriction on possible \xrightarrow{rf} relations that are to be considered *valid*. A valid \xrightarrow{rf} relation is one that respects the following **irreflexivity constraints**.

$$\begin{aligned} & smo; rf; po \\ & smo; po; rf^{-1} \end{aligned}$$

Perhaps consider labelling this rule

We can add here more as we go about to prove completeness.

Recall examples to show how our analysis through examples satisfy the above irreflexivity constraint.

0.2 Soundness of the rules above

To prove soundness, we first define the following:

Definition 4. *Implied Write Order(iwo) Binary relation between any two distinct writes of symmetric threads, derived through the following two sequential composition:*

$$\begin{aligned} w_i^j; po; rf^{-1}; w_k^j \\ w_i^j; rf; po; w_k^j \end{aligned}$$

Property 0.1. $w_i^j; po; rf^{-1}; w_k^j$ corresponds to an implied write order between writes above the read((po before read)) and T_i 's read being satisfied.

Proof. Expanding the first sequential composition exposes the following relations involved.

$$\begin{aligned} w_i^j &\xrightarrow{po} r_i \\ r_i &\xrightarrow{rf^{-1}} w_k^j \end{aligned}$$

By Def 1, w_i^j is above its read r_i . Because T_i and T_k are symmetric threads, we can infer $w_k^j \xrightarrow{po} r_k$, thus verifying that w_k^j is also above its read. By Def 3, we can infer $w_k^j \xrightarrow{rf} r_i$ implying T_i 's read being satisfied. \square

Property 0.2. $w_i^j; rf; po; w_k^j$ corresponds to an implied write order between writes below the read (po after read) and T_k 's read being satisfied.

Proof. Expanding the first sequential composition exposes the following relations involved.

$$\begin{aligned} w_i^j &\xrightarrow{rf} r_k \\ r_k &\xrightarrow{po} w_k^j \end{aligned}$$

By Def1, w_k^j is below its read r_k . Because T_i and T_k are symmetric threads, we can infer $r_i \xrightarrow{po} w_i^j$, thus verifying that w_i^j is also below its read. By Def 3, we can infer T_k 's read being satisfied. \square

Property 0.3. *Simplified irreflexivity rule The irreflexivity constraint rule is equivalent to the following irreflexivity condition*

$$smo; iwo$$

Proof. Expanding for implied write order as per the definition, gives us the following two sequential compositions.

$$\begin{aligned} smo; w_i^j; po; rf^{-1}; w_k^j \\ smo; w_i^j; rf; po; w_k^j \end{aligned}$$

If the above relations must be irreflexive, then so should the following:

$$\begin{aligned} w_k^j; smo; w_i^j; po; rf^{-1} \\ w_k^j; smo; w_i^j; rf; po \end{aligned}$$

By Def 2, the above can be simplified to

$$\begin{array}{c} smo; po; rf^{-1} \\ smo; rf; po \end{array}$$

Thus, proving our property. \square

Property 0.4. *No write order is implied when a read reads from its own thread's write*

Proof. If the read is from its own thread's write, then we can infer that $i = k$ in both the sequential compositions. Hence

$$\begin{array}{c} w_i^j; rf; po; w_i^j \\ w_i^j; po; rf^{-1}; w_i^j \end{array}$$

which gives us $w_i^j \xrightarrow{iwo} w_i^j$. Since implied write orders are only between distinct writes, the property is proven. \square

Perhaps define implied write order as being irreflexive

Property 0.5. *Implied write orders between two symmetric threads are reversed when they are swapped.*

Proof. Considering first sequential composition, i.e. $w_i^j; rf; po; w_k^j$, expanding gives us the following binary relations involved:

$$\begin{array}{c} w_i^j \xrightarrow{rf} r_k \\ r_k \xrightarrow{po} w_k^j \end{array}$$

Swapping thread identities involves swapping the indices i and k for each event, thus giving us

$$\begin{array}{c} w_k^j \xrightarrow{rf} r_i \\ r_i \xrightarrow{po} w_i^j \end{array}$$

Through sequential composition of the above, we get $w_k^j; rf; po; w_i^j$, which by definition is $w_k^j \xrightarrow{iwo} w_i^j$.

For the second sequential composition, i.e. $w_i^j; po; rf^{-1}; w_k^j$, expanding gives us the following binary relations involved:

$$\begin{array}{c} w_i^j \xrightarrow{po} r_i \\ r_i \xrightarrow{rf^{-1}} w_k^j \end{array}$$

Swapping thread identities T_i and T_k gives us the following relations

$$\begin{array}{c} w_k^j \xrightarrow{po} r_k \\ r_k \xrightarrow{rf^{-1}} w_i^j \end{array}$$

Whose sequential composition gives us $w_k^j; po; rf^{-1}; w_i^j$, which by definition is $w_k^j \xrightarrow{iwo} w_i^j$. \square

Property 0.6. *There are at most two implied write orders between writes of two threads, with one between writes above the read event and one below.*

Proof. Consider two threads T_i and T_k . Suppose we have one implied write order between one of their writes, i.e.

$$w_i^j \xrightarrow{iwo} w_k^j.$$

If the above is derived by $w_i^j; po; rf^{-1}; w_k^j$ then from Prop 0.1, we can infer the two writes are above the read and T_i 's read being satisfied.

Now suppose we have an implied write order between another set of writes, i.e.

$$w_i^l \xrightarrow{iwo} w_k^l.$$

If the above is derived by $w_i^l; po; rf^{-1}; w_k^l$, then by Prop 0.1 and Def 3, it violates \xrightarrow{rf} functionality. Hence it can only be derived by $w_i^l; rf; po; w_k^l$. From Prop 0.2, we can infer that the two writes are below the read and T_k 's read being satisfied.

Suppose a third implied write order exists of the form

$$w_i^n \xrightarrow{iwo} w_k^n$$

If the above is derived by $w_i^n; po; rf^{-1}; w_k^n$, then by Prop 0.1 and Def 3, it violates \xrightarrow{rf} functionality. If the above is derived by $w_i^n; rf; po; w_k^n$, then by Prop 0.2 and Def 3, it violates \xrightarrow{rf} functionality.

Hence, we cannot have any more implied write orders between T_i and T_j , thus verifying our property. \square

Property 0.7. *Implied write orders between two threads either all respect irreflexivity condition by Prop 0.3 or they all do not.*

Proof. Consider our two threads T_i and T_k .

For cases where implied write orders are established between writes of T_i and T_j , let us first consider one between writes above the read which respect Prop 0.3;

$$w_i^j \xrightarrow{iwo} w_k^j \wedge w_i^j \xrightarrow{smo} w_k^j$$

By Prop 0.1, the implied write order satisfies T_i 's read by a reads-from relation with w_k^j .

The other set of implied write order, by Prop 0.6 can only be between writes below the read. Suppose we have such an order but not respecting Prop 0.3:

$$w_k^l \xrightarrow{iwo} w_i^l \wedge w_i^l \xrightarrow{smo} w_k^l$$

By Prop 0.2, this implies another \xrightarrow{rf} relation with T_i 's read. By Def 3 this violates the functional property of \xrightarrow{rf} . Hence we can only have an implied write order respecting Prop 0.3.

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w_i^l \xrightarrow{smo} w_k^l$$

By symmetry, if the implied write order between writes above the read did not respect Prop 0.3, then so would the writes below the read. \square

Property 0.8. *Implied write orders are acyclic*

Proof. Suppose a cycle exists. Then without loss of generality, we can consider the cycle composed of 3 writes.

$$w_i^j \xrightarrow{iwo} w_k^j \wedge w_k^j \xrightarrow{iwo} w_l^j \wedge w_l^j \xrightarrow{iwo} w_i^j.$$

By Def 4 and Prop 0.1, 0.2, we can have just two cases; the set of writes involved in cycle are either above the read event or below.

If they are above the read, then we have the following relations that result in the cycle

$$\begin{aligned} w_i^j &\xrightarrow{po} r_i \wedge r_i \xrightarrow{rf^{-1}} w_k^j \\ w_k^j &\xrightarrow{po} r_k \wedge r_k \xrightarrow{rf^{-1}} w_l^j \\ w_l^j &\xrightarrow{po} r_l \wedge r_l \xrightarrow{rf^{-1}} w_i^j. \end{aligned}$$

The above relations violate $po \cup rf^{-1}$ acyclic rule, thus violating coherence.

If the writes are below the read, then we have the following relations that result in the above cycle.

$$\begin{aligned} w_i^j &\xrightarrow{po} r_i \wedge r_i \xrightarrow{rf} w_k^j \\ w_k^j &\xrightarrow{po} r_k \wedge r_k \xrightarrow{rf} w_l^j \\ w_l^j &\xrightarrow{po} r_l \wedge r_l \xrightarrow{rf} w_i^j. \end{aligned}$$

The above relations form a cycle thus violating $po \cup rf$ acyclic rule, thus violating coherence.

Because both cases violate coherence, we conclude that \xrightarrow{iwo} is acyclic. \square

Property 0.9. *Writes w_i above read in each thread can only have one relation of the form $w_i \xrightarrow{iwo} w_i$ but can have several of the form $w \xrightarrow{iwo} w_i$.*

Proof. Suppose a write w_i^j above a read has a relation with w_k^j such that $w_i^j \xrightarrow{iwo} w_k^j$. By Prop 0.1, we can infer T_i 's read has been satisfied. By Def 3 and our assumption of one read per thread, we cannot have any more implied write order relations of the form $w_i^j \xrightarrow{iwo} w^j$.

On the other hand, there can be many relations of the form $w_k^j \xrightarrow{iwo} w_i^j$ as k can be identity of any thread, whose read gets satisfied by the argument above. \square

Property 0.10. *Writes w_i below read in each thread can only have one relation of the form $w \xrightarrow{iwo} w_i$ but can have several of the form $w_i \xrightarrow{iwo} w$.*

Proof. Suppose a write w_i^j below the read has a relation with w_k^j such that $w_k^j \xrightarrow{iwo} w_i^j$. By Prop 0.2, we can infer that T_i 's read has been satisfied by write w_k^j . By Def 3 and our assumption of one read per thread, we cannot have any more implied write order relations of the form $w^j \xrightarrow{iwo} w_i^j$.

On the other hand, there can be many relations of the form $w_i^j \xrightarrow{iwo} w_k^j$ as k can be identity of any thread, whose read gets satisfied by the argument above. \square

0.2.1 Soundness

To prove that our rules are sound, we need to show that for every execution graph that does not respect the irreflexivity constraint, there is a symmetric execution that does respect and hence covered by our rules.

By Prop 0.3, we can infer that an execution not respecting our irreflexivity constraint will have implied write orders which are "incorrect".

Perhaps put Incorrect implied write orders as a definition??

By Prop 0.5, we can swap thread identities to reverse implied write orders, thus "fixing" implied write orders, which then can make the resultant execution respect our rules. If we can show this process of "fixing" implied write orders is terminating, our soundness proof is complete.

We also fix the order in which we do such "fixing". We first address all implied write orders above the read and then go below. If any new implied write orders is introduced in the process, then we fix them as per the above order, i.e. if the new relation is above the read, we fix that next, contrast to any new order introduced below the read.

However, for each new implied write order introduced, we need to show that it wasn't "fixed" before, hence giving us guarantee that "fixing" is a terminating process.

However, before all this, we first need to ensure that implied write orders do not form a cycle, as "fixing" them may not terminate. By Prop 0.8, we have that \overrightarrow{iwo} is acyclic, hence they do not form a cycle.

Given this, we show that for one given set of equal writes, swapping thread identities will always result in the implied write orders respecting Prop 0.3 (Lemma 1). We then show that this holds in general given multiple sets of equal writes (Lemma 2).

Lemma 1. *For one given set of equal writes, every implied write order that is "incorrect", can be fixed and will remain fixed. (A more formal statement required perhaps?)*

Proof. Suppose we have one implied write order which is correct / fixed between writes w_i^j and w_k^j .

$$w_i^j \xrightarrow{iwo} w_k^j$$

We divide our proof into two cases, one for writes above the read and one for writes below.

Case1 : The writes are above the read

- Part 1: There exists an implied write order between w_i^j and some w_m^j

Because Prop 0.9 and $w_i^j \xrightarrow{iwo} w_k^j$, other implied write orders with w_i^j can be of the form:

$$w_m^j \xrightarrow{iwo} w_i^j$$

If this implied order is wrong, swapping thread identities will give us the following relations

$$w_i^j \xrightarrow{iwo} w_m^j \wedge w_m^j \xrightarrow{iwo} w_k^j$$

Here, there is no implied write order relation between w_i^j and w_k^j , hence we can consider them as remaining fixed. If the implied order between w_m^j and w_k^j is wrong, swapping it will result in the following relations

$$w_i^j \xrightarrow{iwo} w_k^j \wedge w_k^j \xrightarrow{iwo} w_m^j$$

Here, the implied write order between w_i^j and w_k^j remains the same. Thus, for this case, we can conclude that the relation remains "fixed".

- Part 2: There exists an implied write order between w_k^j and some w_m^j

By Prop 0.9, implied write orders with w_k^j can be of the forms:

$$\begin{aligned} w_k^j &\xrightarrow{iwo} w_m^j \\ w_m^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

The first form is symmetric to Case1, hence we only consider the second form.

If this implied write order is wrong, swapping thread identities will give us the following relations.

$$w_i^j \xrightarrow{iwo} w_m^j \wedge w_k^j \xrightarrow{iwo} w_m^j$$

Here, there is no implied write order relation between w_i^j and w_k^j , hence we can consider them as remaining fixed.

If the implied write order between w_m^j and w_i^j is wrong, swapping their threads will give us

$$w_m^j \xrightarrow{iwo} w_i^j \wedge w_k^j \xrightarrow{iwo} w_i^j$$

thus making our claim invalid. To show that this state is not possible, note firstly that from the initial configuration, we can infer by Def 2 and Prop 0.3:

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_k^j \xrightarrow{smo} w_m^j$$

Because *smo* is a total order w.r.t. one set of writes, we have by transitivity.

$$w_i^j \xrightarrow{smo} w_m^j$$

After swapping threads T_k and T_m , we get $w_i^j \xrightarrow{iwo} w_m^j$ which respects the irreflexivity constraint Prop 0.3. Hence, this implied write order is not wrong. Thus we cannot have the case which results in $w_k^j \xrightarrow{iwo} w_i^j$.

Case2: The writes are below the read

- Part 1: There exists an implied write order between w_i^j and some w_m^j

By Prop 0.10, implied write order between w_i^j and w_m^j can be of two forms

$$\begin{aligned} w_m^j &\xrightarrow{iwo} w_i^j \\ w_i^j &\xrightarrow{iwo} w_m^j \end{aligned}$$

Considering the first form, if it is wrong, swapping it will result in the following relations

$$w_i^j \xrightarrow{iwo} w_m^j \wedge w_m^j \xrightarrow{iwo} w_k^j$$

Here, there is no implied write order relation between w_i^j and w_k^j , hence we can consider them as remaining fixed.

If the implied write order between w_m^j and w_k^j is wrong, swapping their threads will give us

$$w_i^j \xrightarrow{iwo} w_k^j \wedge w_k^j \xrightarrow{iwo} w_m^j$$

Here, the implied write order between w_i^j and w_k^j remains the same, while we fixed the relations they had with w_m^j . Thus, for this case, we can conclude that the implied write order remains "fixed".

Considering the second form, if it is wrong, swapping it will result in the following relations

$$w_m^j \xrightarrow{iwo} w_i^j \wedge w_m^j \xrightarrow{iwo} w_k^j$$

If the implied write order between w_m^j and w_i^j is wrong, swapping their threads will give us

$$w_k^j \xrightarrow{iwo} w_m^j \wedge w_k^j \xrightarrow{iwo} w_i^j$$

thus making our claim invalid. To show that this state is not possible, note firstly that from the initial configuration, we can infer by Def 2 and Prop 0.3 (given relation between w_i^j and w_m^j is wrong):

$$w_m^j \xrightarrow{smo} w_i^j \wedge w_i^j \xrightarrow{smo} w_k^j$$

Because *smo* is a total order w.r.t. one set of writes, we have by transitivity.

$$w_m^j \xrightarrow{smo} w_k^j$$

After swapping threads T_i and T_m , we get $w_m^j \xrightarrow{iwo} w_k^j$ which respects the irreflexivity constraint Prop 0.3. Hence, this implied write order is not wrong. Thus we cannot have the case which results in $w_k^j \xrightarrow{iwo} w_i^j$.

- Part 2: There exists an implied write order between w_k^j and some w_m^j

By Prop 0.10 and $w_i^j \xrightarrow{iwo} w_k^j$, relation between w_k^j and w_m^j can only be of one form

$$w_k^j \xrightarrow{iwo} w_m^j$$

This case is symmetric to the first form of Part 1, hence this case is already proved.

Thus, for a given set of equal writes, once an implied write order is fixed, it remains fixed. \square

Lemma 2. *For a given set of equal writes whose implied write orders respect Prop 0.3, new implied write order introduced among them by fixing other sets, if wrong, can be fixed and will remain fixed. (A more formal statement required? Discuss with Viktor)*

Proof. Main part We go about the proof by considering a set of writes fixed and another set that needs fixing. We then show that on fixing an implied write order, any new implied write order to the fixed set may either be right, or if wrong, was not something fixed before, and hence can be fixed by lemma 1. Since we fixed the order in which we "fix" these orders, we have to only consider the following cases:

1. Write orders above read fixed , write orders above still need fixing. Let us assume that the fixed set **above the read is the set of writes** w^j . Let us consider two threads T_i and T_k whose writes **above the read** w_i^l and w_k^l is going to be fixed. Without loss of generality, let us consider the symmetric memory order between these two threads to be of the form $w_i \xrightarrow{smo} w_k$. By Prop 0.7, we then have the configuration as:

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_k^l \xrightarrow{iwo} w_i^l$$

We divide our analysis into two cases:

- There is an implied write order with w_i^j and some w^j

By Prop 0.9, implied write orders between the two events can be of the form

$$\begin{aligned} w^j &\xrightarrow{iwo} w_i^j \\ w_i^j &\xrightarrow{iwo} w^j \end{aligned}$$

For the first case, note that by Def 2 and Prop 0.3, we have

$$w * j \xrightarrow{smo} w_k^j$$

By Prop 0.5, on swapping T_i and T_k , we have the following relations

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w^j \xrightarrow{iwo} w_k^j$$

The above relations do not violate Prop 0.3, hence we can stop here.

For the second case, by Prop 0.5, on swapping T_i and T_k , we have the following relations

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w_k^j \xrightarrow{iwo} w^j$$

The new relation $w_k^j \xrightarrow{iwo} w^j$ may be wrong. To show that this relation was not fixed before, we consider cases where it could have been fixed. We then show that such a configuration cannot exist. For this, we consider another event x , and the following cases:

- Case where $w^j \xrightarrow{iwo} x \wedge x \xrightarrow{iwo} w_k^j$
- Case where $x \xrightarrow{iwo} w^j \wedge x \xrightarrow{iwo} w_k^j$
- Case where $w^j \xrightarrow{smo} x \wedge x \xrightarrow{iwo} w_k^j$
- Case where $x \xrightarrow{iwo} w^j \wedge x \xrightarrow{iwo} w_k^j$

- There is an implied write order with w_k^j and some w^j

By Prop 0.9, implied write orders between the two events can be of the form

$$w^j \xrightarrow{iwo} w_k^j$$

For the above case, by Prop 0.5, on swapping T_i and T_k , we have the following relations

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w^j \xrightarrow{iwo} w_i^j$$

The new relation $w^j \xrightarrow{iwo} w_i^j$ may be wrong. To show that this relation was not fixed before, we consider cases where it could have been fixed. We then show that such a configuration cannot exist. For this, we consider another event x , and the following cases:

- Case where $x \xrightarrow{iwo} w^j \wedge x \xrightarrow{iwo} w_i^j$
- Case where $x \xrightarrow{iwo} w^j \wedge w_i^j \xrightarrow{iwo} x$
- Case where $w^j \xrightarrow{smo} x \wedge x \xrightarrow{iwo} w_i^j$
- Case where $x \xrightarrow{smo} w^j \wedge x \xrightarrow{iwo} w_i^j$
- Case where $x \xrightarrow{smo} w^j \wedge w_i^j \xrightarrow{iwo} x$
- Case where $w^j \xrightarrow{smo} x \wedge w_i^j \xrightarrow{iwo} x$

2. Write orders above read fixed , write orders below still need fixing. Let us assume that the fixed set **above the read is the set of writes** w^j . Let us consider two threads T_i and T_k whose writes **below the read** w_i^l and w_k^l is going to be fixed. Without loss of generality, let us consider the symmetric memory order between these two threads to be of the form $w_i \xrightarrow{smo} w_k$. By Prop 0.7, we then have the configuration as:

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_k^l \xrightarrow{iwo} w_i^l$$

We divide our analysis into two cases

- There is an implied write order with w_i^j and some w^j

By Prop 0.9 and Prop 0.2, implied write orders between the two events can be of the form

$$w^j \xrightarrow{iwo} w_i^j$$

By Def 2 and Prop 0.3, we have

$$w^j \xrightarrow{smo} w_k^j$$

By Prop 0.5, on swapping T_i and T_k , we have

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w^j \xrightarrow{iwo} w_k^j$$

The above relations do not violate Prop 0.3, hence we can end this case here.

- There is an implied write order with w_k^j and some w^j By Prop 0.9 and Prop 0.2, implied write orders between the two events can be of the form

$$w_k^j \xrightarrow{iwo} w^j$$

$$w^j \xrightarrow{iwo} w_k^j$$

For the first case, by Def 2 and Prop 0.3, we have

$$w_i^j \xrightarrow{smo} w^j$$

By Prop 0.5, on swapping T_i and T_k , we have

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w_i^j \xrightarrow{iwo} w^j$$

The above relations do not violate Prop 0.3, hence we can end this case here.

For the second case, by Prop 0.5, on swapping, we have

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w^j \xrightarrow{iwo} w_i^j$$

The new relation $w^j \xrightarrow{iwo} w_i^j$ may be wrong. To show that this relation was not fixed before, we consider cases where it could have been fixed. We then show that such a configuration cannot exist. For this, we consider another event x , and the following cases:

- Case where $x \xrightarrow{iwo} w^j \wedge x \xrightarrow{iwo} w_i^j$
- Case where $w^j \xrightarrow{smo} x \wedge x \xrightarrow{iwo} w_i^j$
- Case where $x \xrightarrow{smo} w^j \wedge x \xrightarrow{iwo} w_i^j$

3. Write orders below read fixed , write orders below still need fixing. Let us assume that the fixed set **below the read is the set of writes** w^j . Let us consider two threads T_i and T_k whose writes **below the read** w_i^l and w_k^l is going to be fixed. Without loss of generality, let us consider the symmetric memory order between these two threads to be of the form $w_i \xrightarrow{smo} w_k$. By Prop 0.7, we then have the configuration as:

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_k^l \xrightarrow{iwo} w_i^l$$

We divide our analysis into two cases

- There is an implied write order with w_i^j and some w^j

By Prop 0.10 and Prop 0.2, implied write orders between the two events can be of the form

$$w_i^j \xrightarrow{iwo} w^j$$

By Prop 0.5, on swapping, we have

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w_k^j \xrightarrow{iwo} w^j$$

The new relation $w_k^j \xrightarrow{iwo} w^j$ may be wrong. To show that this relation was not fixed before, we consider cases where it could have been fixed. We then show that such a configuration cannot exist. For this, we consider another event x , and the following cases:

- Case where $w^j \xrightarrow{iwo} x \wedge x \xrightarrow{iwo} w_k^j$
- Case where $w^j \xrightarrow{iwo} x \wedge w_k^j \xrightarrow{iwo} x$
- Case where $w^j \xrightarrow{smo} x \wedge x \xrightarrow{iwo} w_k^j$
- Case where $x \xrightarrow{smo} w^j \wedge x \xrightarrow{iwo} w_k^j$

To show the above case cannot be possible cause of the order in which we prescribe implied write orders should be resolved.

- Case where $w^j \xrightarrow{smo} x \wedge w_k^j \xrightarrow{iwo} x$
- Case where $x \xrightarrow{smo} w^j \wedge w_k^j \xrightarrow{iwo} x$

- There is an implied write order with w_k^j and some w^j

By Prop 0.10 and Prop 0.2, implied write orders between the two events can be of the form

$$\begin{aligned} w_k^j &\xrightarrow{iwo} w^j \\ w^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

For the first case, by Def 2 and Prop 0.3, we have

$$w_i^j \xrightarrow{smo} w^j$$

By Prop 0.5, on swapping T_i and T_k , we have

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w_i^j \xrightarrow{iwo} w^j$$

The above relations do not violate Prop 0.3, hence we can end this case here.

For the second case, by Prop 0.5, on swapping, we have

$$w_i^l \xrightarrow{iwo} w_k^l \wedge w^j \xrightarrow{iwo} w_i^j$$

The new relation $w^j \xrightarrow{iwo} w_i^j$ may be wrong. To show that this relation was not fixed before, we consider cases where it could have been fixed. We then show that such a configuration cannot exist. For this, we consider another event x , and the following cases:

- Case where $x \xrightarrow{iwo} w^j \wedge w_i^j \xrightarrow{iwo} x$
- Case where $w^j \xrightarrow{iwo} x \wedge w_i^j \xrightarrow{iwo} x$
- Case where $x \xrightarrow{smo} w^j \wedge w_i^j \xrightarrow{iwo} x$
- Case where $w^j \xrightarrow{smo} x \wedge w_i^j \xrightarrow{iwo} x$

To show the above case cannot be possible cause of the order in which we prescribe implied write orders should be resolved.

We still will have another case for any new implied write order introduced, (Consider this cases in the end. We need to divide it into two cases.)

4. Write orders below read fixed , write orders above still need fixing. Let us assume that the fixed set **below the read** is the set of writes w^j . Let us consider two threads T_i and T_k whose writes **above the read** w_i^l and w_k^l is going to be fixed. Without loss of generality, let us consider the symmetric memory order between these two threads to be of the form $w_i \xrightarrow{smo} w_k$. By Prop 0.7, we then have the configuration as:

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_k^l \xrightarrow{iwo} w_i^l$$

We divide our analysis into two cases

- There is an implied write order with w_i^j and some w^j
- There is an implied write order with w_k^j and some w^j

□

Proof. Part 1 Suppose, for a given set of equal writes, say of the form w^j all the implied write orders are fixed. Let us consider them to be writes above the read. We consider two threads T_i and T_k , between which an implied write order is wrong. Let those writes be w_i^l and w_k^l .

Without loss of generality, let us consider the symmetric memory order between writes of T_i and T_k to be of the form $w_i \xrightarrow{smo} w_k$. Thus we have

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_i^l \xrightarrow{smo} w_k^l$$

We assume write order is wrong between w_i^l and w_k^l

$$w_k^l \xrightarrow{iwo} w_i^l$$

There cannot be an implied write order between w_i^j and w_k^j , by Prop 0.7.

Case1: w^l is below the read.

- Part1: There exists an implied write order between another write w^j and w_i^j .

By 0.9 and $w_k^l \xrightarrow{iwo} w_i^l$, thus satisfying T_i 's read, w_i^j can only be involved in implied write orders of the form

$$w^j \xrightarrow{iwo} w_i^j$$

From our assumption of implied write orders among w^j are fixed already, we have

$$w^j \xrightarrow{smo} w_i^j$$

By Def 2, we then have $w^j \xrightarrow{smo} w_k^j$.

By Prop 0.5, swapping T_i and T_k to fix $w_k^l \xrightarrow{iwo} w_i^l$, gives us the following new relations.

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

Both these relations respect our irreflexivity constraint Prop 0.3. Thus, concluding this part.

- Part2: There exists an implied write order between another write w^j and w_k^j .

By Prop 0.9, w_k^j can be involved in implied write orders of the form

$$\begin{aligned} w_k^j &\xrightarrow{iwo} w^j \\ w^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

The first is symmetric to Part1, hence we only consider the second form.

By Prop 0.5, on swapping T_i and T_k , we get the following new relations:

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w^j &\xrightarrow{iwo} w_i^j \end{aligned}$$

The second one may not be compliant to the irreflexivity condition Prop 0.3, hence can be a new implied relation which is wrong. To show that this can be fixed and will remain fixed, we need to show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j .

Consider the original configuration we have

$$w_i^j \xrightarrow{smo} w_k^j \wedge w^j \xrightarrow{iwo} w_k^j$$

For $w^j \xrightarrow{iwo} w_k^j$ to have been there before, we can have the following cases

1. There must be a write x , say, such that

$$x \xrightarrow{iwo} w_i^j \wedge x \xrightarrow{iwo} w^j$$

By Prop 0.9, such an x cannot exist, Hence, such a relation could have not been there before, and hence was not fixed before.

2. There must be a write x such that

$$x \xrightarrow{iwo} w_i^j \wedge x \xrightarrow{smo} w^j$$

There could have been an instance where T_x and T were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w' respectively. Then by Def 2, we can infer

$$x' \xrightarrow{smo} w' \wedge x' \xrightarrow{iwo} w'$$

Suppose they are writes below the read. By Prop 0.1 and 0.9, T 's read has already been satisfied. By Prop 0.10 and 0.2, we can infer that $x' \xrightarrow{iwo} w'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes above the read. The set of relations result in $po \cup rf^{-1}$ cycle which violates coherence.

3. There must be a write x such that

$$x \xrightarrow{iwo} w_i^j \wedge w^j \xrightarrow{smo} x$$

There could have been an instance where T_x and T were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w' respectively.

Then by Def 2, we can infer

$$w' \xrightarrow{smo} x' \wedge w' \xrightarrow{iwo} x'$$

Suppose they are writes below the read. By Prop 0.1 and 0.9, T_x 's read has already been satisfied. By Prop 0.10 and 0.2, we can infer that $w' \xrightarrow{iwo} x'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes above the read. The set of relations result in $po \cup rf^{-1}$ cycle which violates coherence.

4. There must be a write x such that

$$x \xrightarrow{sno} w_i^j \wedge x \xrightarrow{iwo} w^j$$

There could have been an instance where T_x and T_i were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w'_i respectively.

Then by Def 2, we can infer

$$x' \xrightarrow{sno} w'_i \wedge x' \xrightarrow{iwo} w'_i$$

Suppose they are writes below the read. By Prop 0.1 and 0.9, T_i 's read has already been satisfied. By Prop 0.10 and 0.2, we can infer that $x' \xrightarrow{iwo} w'_i$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes above the read. The set of relations result in $po \cup rf^{-1}$ cycle which violates coherence.

5. There must be a write x such that

$$w_i^j \xrightarrow{sno} x \wedge x \xrightarrow{iwo} w^j$$

There could have been an instance where T_x and T_i were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w'_i respectively.

Then by Def 2, we can infer

$$w'_i \xrightarrow{sno} x' \wedge w'_i \xrightarrow{iwo} x'$$

Suppose they are writes below the read. By Prop 0.1 and 0.9, T_x 's read has already been satisfied. By Prop 0.10 and 0.2, we can infer that $w'_i \xrightarrow{iwo} x'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes above the read. The set of relations result in $po \cup rf^{-1}$ cycle which violates coherence.

For this case, we can also have an x such that it only has an sno with one of the two events and an implied write order with the other. As swapping them might be needed to fix some other implied write order.

Now that this new relation exists, fixing it, will keep it remain fixed, by our first part of proof (label them please).

Case2: w^l is above the read.

- Part1: There exists an implied write order between another write w^j and w_i^j .

Since T_i 's read is still free, w_i^j can have implied write orders with w^j of the form

$$\begin{aligned} w^j &\xrightarrow{iwo} w_i^j \\ w_i^j &\xrightarrow{iwo} w^j \end{aligned}$$

For the first form of relation, note that we also have by Prop 0.3 and by Def 2,

$$w^j \xrightarrow{smo} w_i^j \wedge w^j \xrightarrow{smo} w_k^j$$

By Prop 0.5 on swapping two threads T_i and T_k , we have

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^j \\ w^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

Both relations respect our irreflexivity constraint. Hence, maintaining that new implied write order relations with w^j are not wrong.

For the second form, while we swap threads T_i and T_k , we get the following new relations (correct his)

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^j \\ w_k^j &\xrightarrow{iwo} w^j \end{aligned}$$

The second relation may not be compliant to the irreflexivity condition, hence can be a new implied relation which is wrong. To show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j , consider the original configuration we have

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_i^j \xrightarrow{iwo} w^j$$

For $w^j \xrightarrow{iwo} w_k^j$ to have been there before, we need an event x to connect, hence by possible implied write orders one can have with w^j and w_k^j ,

1. We need an x such that one of the conditions below hold

$$\begin{aligned} w^j &\xrightarrow{iwo} x \wedge x \xrightarrow{iwo} w_k^j \\ x &\xrightarrow{iwo} w^j \wedge x \xrightarrow{iwo} w_k^j \end{aligned}$$

The first condition violates coherence ($po \cup rf_{-1}$ acyclic), while the second condition cannot hold as such an event x cannot exist due to property x (number the properties).

2. We need an x such that

$$x \xrightarrow{iwo} w_k^j \wedge x \xrightarrow{smo} w^j$$

There could have been an instance where T_x and T were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w' respectively. Then by Def 2, we can infer

$$x' \xrightarrow{smo} w' \wedge x' \xrightarrow{iwo} w'$$

Suppose they are writes above the read. By Prop 0.1 and 0.9, T_x 's read has already been satisfied. By Prop 0.9 and 0.1, we can infer that $x' \xrightarrow{iwo} w'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes below the read. The set of relations result in $po \cup rf^{-1}$ cycle which violates coherence.

Thus, such a relation could have not been there before, and hence was not fixed.

Write the above argument better please.

- Part2: There exists an implied write order between another write w^j and w_k^j .

Since T_k 's read is already established in a reads-from relation, by Prop 0.9, w_k^j can have implied write orders with some w^j only of the form

$$w^j \xrightarrow{iwo} w_k^j$$

By Prop 0.5, swapping thread identities T_i and T_k will give us the following relations

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w^j &\xrightarrow{iwo} w_i^j \end{aligned}$$

The second relation may not be compliant to the irreflexivity condition, hence can be a new implied relation which is wrong. To show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j , consider the original configuration we have

$$w_i^j \xrightarrow{smo} w_k^j \wedge w^j \xrightarrow{iwo} w_k^j$$

For $w^j \xrightarrow{iwo} w_i^j$ to have been there before, we need an event x to connect, hence by possible implied write orders one can have with w^j and w_k^j . By Prop 0.9, w^j can only have relations of the form $w \xrightarrow{iwo} w^j$. Thusm we have the possible two forms of relations that could exist:

$$\begin{aligned} x &\xrightarrow{iwo} w^j \wedge w_i^j \xrightarrow{iwo} x \\ x &\xrightarrow{iwo} w^j \wedge x \xrightarrow{iwo} w_i^j \end{aligned}$$

The second form cannot exist due to Prop 0.9. While the first form could exist, it violates coherence ($po \cup rf_{-1}$ acyclic).

Thus, such a relation could have not been there before, and hence was not fixed.

Although the soundness proof is written completely, it requires proper formatting and references to the properties and definitions we wrote above. This will make it concise and a lot of symmetric arguments can be avoided. \square

Proof. Part 2 Suppose, for a given set of equal writes, say of the form w^j all the implied write orders are fixed. Let us consider them to be wrties below the read. We consider two threads T_i and T_k , between which an implied write order is wrong. Let those writes be w_i^l and w_k^l .

Without loss of generality, let us consider the symmetric memory order between writes of T_i and T_k to be of the form $w_i \xrightarrow{smo} w_k$. Thus we have

$$w_i^j \xrightarrow{smo} w_k^j \wedge w_i^l \xrightarrow{smo} w_k^l$$

We also have

$$w_k^l \xrightarrow{iwo} w_i^l$$

There cannot be an implied write order between w_i^j and w_k^j , by Prop 0.7.

Case 1: w^l is above the read.

- Part 1: There exists an implied write order between another write w^j and w_i^j .

By Prop 0.10, implied write orders between some w^j and w_i^j can only be of the form:

$$w^j \xrightarrow{iwo} w_i^j w_i^j \xrightarrow{iwo} w^j$$

For the first form of relation, note that we also have by Prop 0.3 and by Def 2,

$$w^j \xrightarrow{smo} w_i^j \wedge w^j \xrightarrow{smo} w_k^j$$

By Prop 0.5 on swapping threads T_i and T_j , we have

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

Both these relations respect our irreflexivity constraint Prop 0.3, hence we can conclude this case.

For the second form of relation, note that after swapping we have

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w_k^j &\xrightarrow{iwo} w^j \end{aligned}$$

The latter relation may not respect our irreflexivity constraint, and hence can be wrong. To show that this implied write order could not have existed and hence fixed before, we observe the original configuration

$$w_i^j \xrightarrow{iwo} w^j \wedge w_i^j \xrightarrow{smo} w_k^j$$

To have had $w_k^j \xrightarrow{iwo} w^j$ before, we have the following cases to address: 1. There must have been an event x such that

$$w^j \xrightarrow{iwo} x \wedge w_k^j \xrightarrow{iwo} x$$

By Prop 0.10, such an event cannot exist.

2. There must be an x such that

$$w_k^j \xrightarrow{iwo} x \wedge w^j \xrightarrow{smo} x$$

There could have been an instance where T_x and T were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w' respectively. Then by Def 2, we can infer

$$w' \xrightarrow{smo} x' \wedge w' \xrightarrow{iwo} x'$$

Suppose they are writes above the read. By Prop 0.2 and 0.10, T 's read has already been satisfied. By Prop 0.9 and 0.1, we can infer that $w' \xrightarrow{iwo} x'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes below the read. The set of relations result in $po \cup rf$ cycle which violates coherence.

3. There must be an x such that

$$w_k^j \xrightarrow{iwo} x \wedge x \xrightarrow{smo} w^j$$

There could have been an instance where T_x and T were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w' respectively. Then by Def 2, we can infer

$$x' \xrightarrow{smo} w' \wedge x' \xrightarrow{iwo} w'$$

Suppose they are writes above the read. By Prop 0.2 and 0.10, T_x 's read has already been satisfied. By Prop 0.9 and 0.1, we can infer that $x' \xrightarrow{iwo} w'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes below the read. The set of relations result in $po \cup rf$ cycle which violates coherence.

4. There must be an x such that

$$w_k^j \xrightarrow{smo} x \wedge w^j \xrightarrow{iwo} x$$

There could have been an instance where T_x and T_k were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w'_k respectively. Then by Def 2, we can infer

$$w'_k \xrightarrow{smo} x' \wedge w'_k \xrightarrow{iwo} x'$$

Suppose they are writes above the read. By Prop 0.2 and 0.10, T_k 's read has already been satisfied. By Prop 0.9 and 0.1, we can infer that $w'_k \xrightarrow{iwo} x'$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes below the read. The set of relations result in $po \cup rf$ cycle which violates coherence.

5. There must be an x such that

$$x \xrightarrow{smo} w_k^j \wedge w^j \xrightarrow{iwo} x$$

There could have been an instance where T_x and T_k were swapped, to fix some other implied write order between writes. Let us call those writes to be x' and w'_k respectively. Then by Def 2, we can infer

$$x' \xrightarrow{smo} w'_k \wedge x' \xrightarrow{iwo} w'_k$$

Suppose they are writes above the read. By Prop 0.2 and 0.10, T_x 's read has already been satisfied. By Prop 0.9 and 0.1, we can infer that $x' \xrightarrow{iwo} w'_k$ cannot exist. Hence, such an x cannot exist.

Suppose they are writes below the read. The set of relations result in $po \cup rf$ cycle which violates coherence.

- Part 2: There exists an implied write order between another write w^j and w_k^j .

By Prop 0.10 and Prop 0.1, implied write orders between w^j and w_k^j can only be of the form

$$w_k^j \xrightarrow{iwo} w^j$$

For the first form of relation, note that we also have by Prop 0.3 and by Def 2,

$$w_k^j \xrightarrow{smo} w^j \wedge w_i^j \xrightarrow{smo} w^j$$

By Prop 0.5 on swapping threads T_i and T_j , we have

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w_i^j &\xrightarrow{iwo} w^j \end{aligned}$$

Both these relations respect our irreflexivity constraint Prop 0.3, hence we can conclude this case.

Case 2: w^l is below the read.

- Part 1: There exists an implied write order between another write w^j and w_i^j . By Prop 0.10 and Prop 0.2, implied write order between w^j and w_i^j can be of the form

$$w_i^j \xrightarrow{iwo} w^j$$

By Prop 0.3, on swapping, we have the following relations

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w_k^j &\xrightarrow{iwo} w^j \end{aligned}$$

The latter relation may not respect our irreflexivity constraint. To show that this relation could not have been there before and thus can be fixed by Lemma 1, consider the original configuration

$$w_i^j \xrightarrow{iwo} w^j \wedge w_i^j \xrightarrow{smo} w_k^j$$

For $w_k^j \xrightarrow{iwo} w^j$ to have been there before, we need an event x to connect, having implied write orders with both. By Prop 0.10, w^j can only have relations of the form $w^j \xrightarrow{iwo} w$. Thus we have the possible two forms of relations that could exist:

$$\begin{aligned} w^j &\xrightarrow{iwo} x \wedge x \xrightarrow{iwo} w_k^j \\ w^j &\xrightarrow{iwo} x \wedge w_k^j \xrightarrow{iwo} x \end{aligned}$$

The first set of relations violate coherence ($po \cup rf$ acyclic), while the second set of relations is not possible as due to Prop 0.10, such an event x cannot exist.

Thus, the implied write order if exists after swapping, is new and could not have occurred before and hence, by Lemma 1, we can fix it and will remain fixed.

- Part 2: There exists an implied write order between another write w^j and w_k^j .

By Prop 0.10, implied write orders between w^j and w_k^j can be of the form

$$\begin{aligned} w_k^j &\xrightarrow{iwo} w^j \\ w^j &\xrightarrow{iwo} w_k^j \end{aligned}$$

Taking the first form, by Prop 0.3 and Def 2, we have

$$w_k^j \xrightarrow{smo} w^j \wedge w_i^j \xrightarrow{smo} w^j$$

By Prop 0.5, on swapping, we have

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w_i^j &\xrightarrow{iwo} w^j \end{aligned}$$

None of the above relations violate the irreflexivity constraint Prop 0.3. Hence, for this case, we are done.

For the second form, on swapping we have

$$\begin{aligned} w_i^l &\xrightarrow{iwo} w_k^l \\ w^j &\xrightarrow{iwo} w_i^j \end{aligned}$$

The latter relation may not respect our irreflexivity constraint. To show that this relation could not have been there before and thus can be fixed by Lemma 1, consider the original configuration

$$w_i^j \xrightarrow{smo} w_k^j \wedge w^j \xrightarrow{iwo} w_k^j$$

For $w^j \xrightarrow{iwo} w_i^j$ to have been there before, we need an event x to connect, having implied write orders with both. By Prop 0.10, w_i^j can only have relations of the form $w_i^j \xrightarrow{iwo} w$. Thus we have the possible two forms of relations that could exist:

$$\begin{aligned} w_i^j &\xrightarrow{iwo} x \wedge x \xrightarrow{iwo} w^j \\ w_i^j &\xrightarrow{iwo} x \wedge w^j \xrightarrow{iwo} x \end{aligned}$$

The first set of relations violate coherence, ($po \cup rf$ acyclic). While the second one, by Prop 0.10, such an x cannot exist. Hence, our relation is new and could not have occurred before, and thus can be fixed by Lemma 1 and remain fixed.

□