0.1 From Examples to Precise Rules

Notations

- T_i denotes thread number i.
- $T_i \equiv T_j$ means both threads have same code.
- w_i^j is the j^{th} event in thread i which is a write.
- r_i^j is the j^{th} event in thread i which is a read.

A few definitions for our use

Definition 1. *Program Order(*po*) Total order between events in the same thread. Respects the execution order between events in the same thread.*

Definition 2. *Symmetric Memory Order (*smo*) A strict partal order between writes in a set of symmetric threads. Consider a set of symmetric threads $T_1 \equiv T_2 \equiv ... \equiv T_n$. Each of these threads have exactly one read event, and multiple write events, all to the same memory, say x.*

Then each write in the above threads are involved in a symmetric order, such that.

$$\forall i \in [0, n-1] \ . \ w_i^j \underset{smo}{\longrightarrow} w_{i+1}^j$$

Where j denotes the j^{th} event in any of the threads, which is a write.

Perhaps should put examples for the above defintion.

Definition 3. *Reads-From (*rf*) Binary relation that links a read to a write from which its value comes. Note that for our purpose, this relation is functional. For example, if a read r_i^j gets its read value from write w_k^l, then we have the relation.*

$$w_k^l \underset{rf}{\longrightarrow} r_i^j$$

Main Rule Using the above setup, our intention is to explore lesser execution graphs leveraging the symmetry that can result due to swapping of thread identities. For this, we enforce a restriction on possible $\overrightarrow{rf}$ relations that are to be considered *valid*. A valid $\overrightarrow{rf}$ relation is one that respects the following **irreflexivity constraints**.

$$smo; rf; po$$
$$smo; po; rf^{-1}$$

Perhaps consider labelling this rule

We can add here more as we go about to prove completeness.

Recall examples to show how our analysis through examples satisfy the above irreflexivity constraint.

0.2 Soundness of the rules above

To prove soundness, we first define the following:

Definition 4. *Implied Write Order(*iwo*) Binary relation between any two* distinct *writes of symmetric threads, derived through the following two sequential conposition:*

$$w_i^j; po; rf^{-1}; w_k^j$$
$$w_i^j; rf; po; w_k^j$$

Property 0.1. $w_i^j; po; rf^{-1}; w_k^j$ *corresponds to an implied write order between writes above the read((po before read)) and T_i's read being satisfied.*

Proof. Expanding the first sequential composition exposes the following relations involved.

$$w_i^j \underset{po}{\longrightarrow} r_i$$
$$r_i \underset{rf^{-1}}{\longrightarrow} w_k^j$$

By Def 1, w_i^j is above its read r_i. Because T_i and T_k are symmetric threads, we can infer $w_k^j \underset{po}{\longrightarrow} r_k$, thus verifying that w_k^j is also above its read. By Def 3, we can infer $w_k^j \underset{rf}{\longrightarrow} r_i$ implying T_i's read being satisfied. □

Property 0.2. $w_i^j; rf; po; w_k^j$ *corresponds to an implied write order between writes below the read (po after read) and T_k's read being satisfied.*

Proof. Expanding the first sequential composition exposes the following relations involved.

$$w_i^j \underset{rf}{\longrightarrow} r_k$$
$$r_k \underset{po}{\longrightarrow} w_k^j$$

By Def1, w_k^j is below its read r_k. Because T_i and T_k are symmetric threads, we can infer $r_i \underset{po}{\longrightarrow} w_i^j$, thus verifying that w_i^j is also below its read. By Def 3, we can infer T_k's read being satisfied. □

Property 0.3. *Simplified irreflexivity rule The irreflexivity constraint rule is equivalent to the following irreflexivity condition*

$$smo; iwo$$

Proof. Expanding for implied write order as per the definition, gives us the following two sequential compositions.

$$smo; w_i^j; po; rf^{-1}; w_k^j$$
$$smo; w_i^j; rf; po; w_k^j$$

If the above relations must be irreflexive, then so should the following:

$$w_k^j; smo; w_i^j; po; rf^{-1}$$
$$w_k^j; smo; w_i^j; rf; po$$

By Def 2, the above can be simplified to

$$smo; po; rf^{-1}$$
$$smo; rf; po$$

Thus, proving our property. $\square$

Property 0.4. *No write order is implied when a read reads from its own thread's write*

Proof. If the read is from its own thread's write, then we can infer that $i = k$ in both the sequential compositions. Hence

$$w_i^j; rf; po; w_i^j$$
$$w_i^j; po; rf^{-1}; w_i^j$$

which gives us $w_i^j \xrightarrow[iwo]{} w_i^j$. Since implied write orders are only between distinct writes, the property is proven. $\square$

Perhaps define implied write order as being irreflexive

Property 0.5. *Implied write orders between two symmetric threads are reversed when they are swapped.*

Proof. Considering first sequential composition, i.e. $w_i^j; rf; po; w_k^j$, expanding gives us the following binary relations involved:

$$w_i^j \xrightarrow[rf]{} r_k$$
$$r_k \xrightarrow[po]{} w_k^j$$

Swapping thread identities involves swapping the indices i and k for each event, thus giving us

$$w_k^j \xrightarrow[rf]{} r_i$$
$$r_i \xrightarrow[po]{} w_i^j$$

Through sequential composition of the above, we get $w_k^j; rf; po; w_i^j$, which by definition is $w_k^j \xrightarrow[iwo]{} w_i^j$.

For the second sequential composition, i.e. $w_i^j; po; rf^{-1}; w_k^j$, expanding gives us the following binary relations involved:

$$w_i^j \xrightarrow[po]{} r_i$$
$$r_i \xrightarrow[rf^{-1}]{} w_k^j$$

Swapping thread identities T_i and T_k gives us the following relations

$$w_k^j \xrightarrow[po]{} r_k$$
$$r_k \xrightarrow[rf^{-1}]{} w_i^j$$

Whose sequential compostiion gives us $w_k^j; po; rf^{-1}; w_i^j$, which by defintion is $w_k^j \xrightarrow[iwo]{} w_i^j$. $\square$

Property 0.6. *There are at most two implied write orders between writes of two threads, with one between writes above the read event and one below.*

Proof. Consider two threads T_i and T_k. Suppose we have one implied write order between one of their writes, i.e.

$$w_i^j \underset{iwo}{\longrightarrow} w_k^j.$$

If the above is derived by $w_i^j; po; rf^{-1}; w_k^j$ then from Prop 0.1, we can infer the two writes are above the read and T_i' read being satisfied.

Now suppose we have an implied write order between another set of writes, i.e.

$$w_i^l \underset{iwo}{\longrightarrow} w_k^l.$$

If the above is derived by $w_i^l; po; rf^{-1}; w_k^l$, then by Prop 0.1 and Def 3, it violates $\underset{rf}{\longrightarrow}$ functionality. Hence it can only be derived by $w_i^l; rf; po; w_k^l$. From Prop 0.2, we can infer that the two writes are above the read and T_k's read being satisfied.

Suppose a third impleid write order exists of the form

$$w_i^n \underset{iwo}{\longrightarrow} w_k^n$$

If the above is derived by $w_i^n; po; rf^{-1}; w_k^n$, then by Prop 0.1 and Def 3, it violates $\underset{rf}{\longrightarrow}$ functionality. If the above is derived by $w_i^n; rf; po; w_k^n$, then by Prop 0.2 and Def 3, it violates $\underset{rf}{\longrightarrow}$ functionality.

Hence, we cannot have any more implied write orders between T_i and T_j, thus verifying our property.

Have not considered the from $w_k^l \underset{iwo}{\longrightarrow} w_i^l$, which I can eliminate through violation of coherence. □

Property 0.7. *Implied write orders between two threads either all respect irreflexivity condition by Prop 0.3 or they all do not.*

Proof. Consider our two threads T_i and T_k.

If each of their read reads from its own write, we have no implied write order established, thus maintaining the property.

If each read reads from another thread's (say T_m), write, we have no implied write order established between T_i and T_j's writes.. (I do not know how to complete the sentence.)

For cases where implied write orders are established between wrties of T_i and T_j, without loss of generality, let us consider one between writes above the read which respect Prop 0.3;

$$w_i^j \underset{iwo}{\longrightarrow} w_k^j \ \wedge \ w_i^j \underset{smo}{\longrightarrow} w_k^j$$

By Prop 0.1, the implied write order satisfies T_i's read by a reads-from relation with w_k^j.

The other set of implied write order, by Prop 0.7 can only be between writes below the read. Suppose we have such an order but not respecting Prop 0.3:

$$w_k^l \underset{iwo}{\longrightarrow} w_i^l \ \wedge \ w_i^l \underset{smo}{\longrightarrow} w_k^l$$

By Prop 0.2, this implies another $\underset{rf}{\longrightarrow}$ relation with T_i's read. By Def 3 this should not be allowed. Hence we can only have an implied write order respecting Prop 0.3.

$$w_i^l \underset{iwo}{\longrightarrow} w_k^l \ \wedge \ w_i^l \underset{smo}{\longrightarrow} w_k^l$$

The last argument seems incomplete. But perhaps compliant / non-compliant is a binary argument. So if one does not hold, the other should. There is no other option.

By symmetry, if the implied write order between writes above the read did not respect Prop 0.3, then so would the writes below the read. $\square$

Property 0.8. *Implied write orders are acyclic*

Proof. Suppose a cycle exists. Then without loss of generality, we can consider the cycle composed of 3 writes.

$$w_i^j \xrightarrow[iwo]{} w_k^j \wedge w_k^j \xrightarrow[iwo]{} w_l^j \wedge w_l^j \xrightarrow[iwo]{} w_i^j.$$

By Def 4 and Prop 0.1, 0.2, we can have just two cases; the set of writes involved in cycle are either above the read event or below.

If they are above the read, then we have the following relations that result in the cycle

$$\begin{aligned} w_i^j \xrightarrow[po]{} r_i \ \wedge \ r_i \xrightarrow[rf^{-1}]{} w_k^j \\ w_k^j \xrightarrow[po]{} r_k \ \wedge \ r_k \xrightarrow[rf^{-1}]{} w_l^j \\ w_l^j \xrightarrow[po]{} r_l \ \wedge \ r_l \xrightarrow[rf^{-1}]{} w_i^j. \end{aligned}$$

The above relations violate $po \cup rf^{-1}$ acyclic rule, thus violating coherence.

If the writes are below the read, then we have the following relations that result in the above cycle.

$$\begin{aligned} w_i^j \xrightarrow[po]{} r_i \ \wedge \ r_i \xrightarrow[rf]{} w_k^j \\ w_k^j \xrightarrow[po]{} r_k \ \wedge \ r_k \xrightarrow[rf]{} w_l^j \\ w_l^j \xrightarrow[po]{} r_l \ \wedge \ r_l \xrightarrow[rf]{} w_i^j \end{aligned}$$

The above relations form a cycle thus violating $po \cup rf$ acyclic rule, thus violating coherence.

Because both cases violate coherence, we conclude that $\xrightarrow[iwo]{}$ is acylic. $\square$

Property 0.9. *Writes w_i above read in each thread can only have one relation of the form $w_i \xrightarrow[iwo]{} w$ but can have several of the form $w \xrightarrow[iwo]{} w_i$.*

Proof. Suppose a write w_i^j above a read has a relation with w_k^j such that $w_i^j \xrightarrow[iwo]{} w_k^j$. By Prop 0.1, we can infer T_i's read has been satisfied. By Def 3 and our assumption of one read per thread, we cannot have any more implied write order relations of the form $w_i^j \xrightarrow[iwo]{} w^j$.

On the other hand, there can be many relations of the form $w_k^j \xrightarrow[iwo]{} w_i^j$ as k can be identity of any thread, whose read gets satisfied by the argument above. $\square$

Property 0.10. *Writes w_i below read in each thread can only have one relation of the form $w \xrightarrow[iwo]{} w_i$ but can have several of the form $w_i \xrightarrow[iwo]{} w$.*

Proof. Suppose a write w_i^j below the read has a relation with w_k^j such that $w_k^j \xrightarrow[iwo]{} w_i^j$. By Prop 0.2, we can infer that T_i's read has been satisfied by write w_k^j. By Def 3 and our assumption of one read per thread, we cannot have any more implied write order relations of the form $w^j \xrightarrow[iwo]{} w_i^j$.

On the other hand, there can be many relations of the form $w_i^j \xrightarrow[iwo]{} w_k^j$ as k can be identity of any thread, whose read gets satisfied by the argument above. $\square$

0.2.1 Soundness

To prove that our rules our sound, we show that for every implied write order such that $smo; iwo$ is reflexive, by Prop 0.5 we can reverse them by swapping thread identities, thus respectiing our irreflexivity constraint.

However, we must ensure that once write orders are "fixed" in this fashion, they remain fixed, i.e. the relation cannot appear again to be wrong. For this, we first need to ensure that implied write orders do not form a cycle. By Prop 0.8, this is indeed the case.

Given they are acyclic, we need to show that once fixed by Prop 0.5, they remain fixed. Lastly, we need to show that this holds in general given multiple sets of equal writes that needs fixing.

Lemma 1. *For a given set of equal writes, once an implied write order is fixed, it remains fixed. (A more formal statement required perhaps?)*

Proof. Suppose we have one implied write order which is correct / fixed between writes w_i^j and w_k^j.

$$w_i^j \xrightarrow[iwo]{} w_k^j$$

We divide our proof into two cases, one for writes above the read and one for writes below.

Case1 : The writes are above the read

- Part 1: There exists an implied write order between w_i^j and some w_m^j

  Because Prop 0.9 and $w_i^j \xrightarrow[iwo]{} w_k^j$, other implied write orders with w_i^j can be of the form:

  $$w_m^j \xrightarrow[iwo]{} w_i^j$$

  If this implied order is wrong, swapping thread identities will give us the following relations

  $$w_i^j \xrightarrow[iwo]{} w_m^j \ \wedge \ w_m^j \xrightarrow[iwo]{} w_k^j$$

  Here, there is no implied write order relation between w_i^j and w_k^j, hence we can consider them as remaining fixed. If the implied order between w_m^j and w_k^j is wrong, swapping it will result in the following relations

  $$w_i^j \xrightarrow[iwo]{} w_k^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_m^j$$

  Here, the implied write order between w_i^j and w_k^j remains the same. Thus, for this case, we can conclude that the relation remains "fixed".

- Part 2: There exists an implied write order between w_k^j and some w_m^j

  By Prop 0.9, implied write orders with w_k^j can be of the forms:

  $$w_k^j \xrightarrow[iwo]{} w_m^j$$
  $$w_m^j \xrightarrow[iwo]{} w_k^j$$

  The first form is symmetric to Case1, hence we only consider the second form.

If this implied write order is wrong, swapping thread identities will give us the following relations.

$$w_i^j \xrightarrow[iwo]{} w_m^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_m^j$$

Here, there is no implied write order relation between w_i^j and w_k^j, hence we can consider them as remaining fixed.

If the implied write order between w_m^j and w_i^j is wrong, swapping their threads will give us

$$w_m^j \xrightarrow[iwo]{} w_i^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_i^j$$

thus making our claim invalid. To show that this state is not possible, note firstly that from the initial configuration, we can infer by Def 2 and Prop 0.3:

$$w_i^j \xrightarrow[smo]{} w_k^j \ \wedge \ w_k^j \xrightarrow[smo]{} w_m^j$$

Because *smo* is a total order w.r.t. one set of writes, we have by transitivity.

$$w_i^j \xrightarrow[smo]{} w_m^j$$

After swapping threads T_k and T_m, we get $w_i^j \xrightarrow[iwo]{} w_m^j$ which respects the irreflexivity constraint Prop 0.3. Hence, this implied write order is not wrong. Thus we cannot have the case which results in $w_k^j \xrightarrow[iwo]{} w_i^j$.

Case2: The writes are below the read

- Part 1: There exists an implied write order between w_i^j and some w_m^j

  By Prop 0.10, implied write order between w_i^j and w_m^j can be of two forms

$$w_m^j \xrightarrow[iwo]{} w_i^j$$
$$w_i^j \xrightarrow[iwo]{} w_m^j$$

  Considering the first form, if it is wrong, swapping it will result in the following relations

$$w_i^j \xrightarrow[iwo]{} w_m^j \ \wedge \ w_m^j \xrightarrow[iwo]{} w_k^j$$

  Here, there is no implied write order relation between w_i^j and w_k^j, hence we can consider them as remaining fixed.

  If the implied write order between w_m^j and w_k^j is wrong, swapping their threads will give us

$$w_i^j \xrightarrow[iwo]{} w_k^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_m^j$$

  Here, the implied write order between w_i^j and w_k^j remains the same, while we fixed the relations they had with w_m^j. Thus, for this case, we can conclude that the implied write order remains "fixed".

  Considering the second form, if it is wrong, swapping it will result in the following relations

$$w_m^j \xrightarrow[iwo]{} w_i^j \ \wedge \ w_m^j \xrightarrow[iwo]{} w_k^j$$

If the implied write order between w^j_m and w^j_i is wrong, swapping their threads will give us

$$w^j_k \underset{iwo}{\longrightarrow} w^j_m \ \wedge \ w^j_k \underset{iwo}{\longrightarrow} w^j_i$$

thus making our claim invalid. To show that this state is not possible, note firstly that from the initial configuration, we can infer by Def 2 and Prop 0.3 (given relation between w^j_i and w^j_m is wrong):

$$w^j_m \underset{smo}{\longrightarrow} w^j_i \ \wedge \ w^j_i \underset{smo}{\longrightarrow} w^j_k$$

Because *smo* is a total order w.r.t. one set of writes, we have by transitivity.

$$w^j_m \underset{smo}{\longrightarrow} w^j_k$$

After swapping threads T_i and T_m, we get $w^j_m \underset{iwo}{\longrightarrow} w^j_k$ which respects the irreflexivity constraint Prop 0.3. Hence, this implied write order is not wrong. Thus we cannot have the case which results in $w^j_k \underset{iwo}{\longrightarrow} w^j_i$.

- Part 2: There exists an implied write order between w^j_k and some w^j_m

  By Prop 0.10 and $w^j_i \underset{iwo}{\longrightarrow} w^j_k$, relation between w^j_k and w^j_m can only be of one form

  $$w^j_k \underset{iwo}{\longrightarrow} w^j_m$$

  This case is symmetric to the first form of Part 1, hence this case is already proved.

Thus, for a given set of equal writes, once an implied write order is fixed, it remains fixed. □

Lemma 2. *For a given set of equal writes whose implied write orders respect Prop 0.3, new implied write order introduced among them by fixing other sets, if wrong, can be fixed and will remain fixed. (A more formal statement required? Discuss with Viktor)*

Proof. Part 1 Suppose, for a given set of equal writes, say of the form w^j all the implied write orders are fixed. Let us consider them to be wrties above the read. We consider two threads T_i and T_k, between which an implied write order is wrong. Let those writes be w^l_i and w^l_k.

Without loss of generality, let us consider the symmetric memory order between writes of T_i and T_k to be of the form $w_i \underset{smo}{\longrightarrow} w_k$. Thus we have

$$w^j_i \underset{smo}{\longrightarrow} w^j_k \ \wedge \ w^l_i \underset{smo}{\longrightarrow} w^l_k$$

We assume write order is wrong between w^l_i and w^l_k

$$w^l_k \underset{iwo}{\longrightarrow} w^l_i$$

There cannot be an implied write order between w^j_i and w^j_k, by Prop 0.7.

Case1: w^l is below the read.

- Part1: There exists an implied write order between another write w^j and w_i^j.

  By 0.9 and $w_k^l \xrightarrow[iwo]{} w_i^l$, thus satisfying T_i's read, w_i^j can only be involved in implied write orders of the form

  $$w^j \xrightarrow[iwo]{} w_i^j$$

  From our assumption of implied write orders among w^j are fixed already, we have

  $$w^j \xrightarrow[smo]{} w_i^j$$

  By Def 2, we then have $w^j \xrightarrow[smo]{} w_k^j$.

  By Prop 0.5, swapping T_i and T_k to fix $w_k^l \xrightarrow[iwo]{} w_i^j$, gives us the following new relations.

  $$w_i^l \xrightarrow[iwo]{} w_k^j$$
  $$w^j \xrightarrow[iwo]{} w_k^j$$

  Both these relations repsect our irreflexivity constraint Prop 0.3. Thus, concluding this part.

- Part2: There exists an implied write order between another write w^j and w_k^j.

  By Prop 0.9, w_k^j can be involved in implied write orders of the form

  $$w_k^j \xrightarrow[iwo]{} w^j$$
  $$w^j \xrightarrow[iwo]{} w_k^j$$

  The first is symmetric to Part1, hence we only consider the second form.

  By Prop 0.5, on swapping T_i and T_k, we get the following new relations:

  $$w_i^l \xrightarrow[iwo]{} w_k^j$$
  $$w^j \xrightarrow[iwo]{} w_i^j$$

  The second one may not be compliant to the irreflexivity condition Prop 0.3, hence can be a new implied relation which is wrong. To show that this can be fixed and will remain fixed, we need to show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j.

  Consider the original configuration we have

  $$w_i^j \xrightarrow[smo]{} w_k^j \ \wedge \ w^j \xrightarrow[iwo]{} w_k^j$$

  For $w^j \xrightarrow[iwo]{} w_k^j$ to have been there before, there must be a write x, say, such that

  $$x \xrightarrow[iwo]{} w_i^j \ \wedge \ x \xrightarrow[iwo]{} w^j$$

  By Prop 0.9, such an x cannot exist, Hence, such a relation could have not been there before, and hence was not fixed before.

Now that this new relation exists, fixing it, will keep it remain fixed, by our first part of proof (label them please.

Case2: w^l is above the read.

- Part1: There exists an implied write order between another write w^j and w_i^j.

  Since T_i's read is still free, w_i^j can have implied write orders with w^j of the form

  $$w^j \xrightarrow[iwo]{} w_i^j$$
  $$w_i^j \xrightarrow[iwo]{} w^j$$

  For the first form of relation, note that we also have by Prop 0.3 and by Def 2,

  $$w^j \xrightarrow[smo]{} w_i^j \;\wedge\; w^j \xrightarrow[smo]{} w_k^j$$

  By Prop 0.5 on swapping two threads T_i and T_k, we have

  $$w_i^l \xrightarrow[iwo]{} w_k^j$$
  $$w^j \xrightarrow[iwo]{} w_k^j$$

  Both relations respect our irreflexivity constraint. Hence, maintaining that new implied write order relations with w^j are not wrong.

  For the second form, while we swap threads T_i and T_k, we get the following new relations (correct his)

  $$w_i^l \xrightarrow[iwo]{} w_k^j$$
  $$w_k^j \xrightarrow[iwo]{} w^j$$

  The second relation may not be compliant to the irreflexivity condition, hence can be a new implied relation which is wrong. To show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j, consider the original configuration we have

  $$w_i^j \xrightarrow[smo]{} w_k^j \;\wedge\; w_i^j \xrightarrow[iwo]{} w^j$$

  For $w^j \xrightarrow[iwo]{} w_k^j$ to have been there before, we need an event x to connect, hence by possible implied write orders one can have with w^j and w_k^j, we need an x such that one of the conditions below hold

  $$w^j \xrightarrow[iwo]{} x \;\wedge\; x \xrightarrow[iwo]{} w_k^j$$
  $$x \xrightarrow[iwo]{} w^j \;\wedge\; x \xrightarrow[iwo]{} w_k^j$$

  The first condition violates coherence ($po \cup rf_{-1}$ acylcic), while the second condition cannot hold as such an event x cannot exist due to property x (number the properties).

  Thus, such a relation could have not been there before, and hence was not fixed.

  Write the above argument better please.

- Part2: There exists an implied write order between another write w^j and w^j_k.

  Since T_k's read is already established in a reads-from relation, by Prop 0.9, w^j_k can have implied write orders with some w^j only of the form

  $$w^j \underset{iwo}{\longrightarrow} w^j_k$$

  By Prop 0.5, swapping thread identities T_i and T_k will give us the following relations

  $$w^l_i \underset{iwo}{\longrightarrow} w^l_k$$
  $$w^j \underset{iwo}{\longrightarrow} w^j_i$$

  The second relation may not be compliant to the irreflexivity condition, hence can be a new implied relation which is wrong. To show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j, consider the original configuration we have

  $$w^j_i \underset{smo}{\longrightarrow} w^j_k \;\wedge\; w^j \underset{iwo}{\longrightarrow} w^j_k$$

  For $w^j \underset{iwo}{\longrightarrow} w^j_i$ to have been there before, we need an event x to connect, hence by possible implied write orders one can have with w^j and w^j_k. By Prop 0.9, w^j can only have relations of the form $w \underset{iwo}{\longrightarrow} w^j$. Thusm we have the possible two forms of relations that could exist:

  $$x \underset{iwo}{\longrightarrow} w^j \;\wedge\; w^j_i \underset{iwo}{\longrightarrow} x$$
  $$x \underset{iwo}{\longrightarrow} w^j \;\wedge\; x \underset{iwo}{\longrightarrow} w^j_i$$

  The second form cannot exist due to Prop 0.9. While the first form could exist, it violates coherence ($po \cup rf_{-1}$ acylcic).

  Thus, such a relation could have not been there before, and hence was not fixed.

Although the soundness proof is written completely, it requires proper formatting and references to the properties and definitions we wrote above. This will make it concicse and a lot of symmetric arguments can be avoided. □

Proof. Part 2 Suppose, for a given set of equal writes, say of the form w^j all the implied write orders are fixed. Let us consider them to be wrties below the read. We consider two threads T_i and T_k, between which an implied write order is wrong. Let those writes be w^l_i and w^l_k.

Without loss of generality, let us consider the symmetric memory order between writes of T_i and T_k to be of the form $w_i \underset{smo}{\longrightarrow} w_k$. Thus we have

$$w^j_i \underset{smo}{\longrightarrow} w^j_k \;\wedge\; w^l_i \underset{smo}{\longrightarrow} w^l_k$$

We also have

$$w^l_k \underset{iwo}{\longrightarrow} w^l_i$$

□