0.1 From Examples to Precise Rules

Notations

- T_i denotes thread number i.
- $T_i \equiv T_j$ means both threads have same code.
- w_i^j is the j^{th} event in thread i which is a write.
- r_i^j is the j^{th} event in thread i which is a read.

A few definitions for our use

Definition 1. *Program Order(*po*) Total order between events in the same thread. Respects the execution order between events in the same thread.*

Definition 2. *Symmetric Memory Order (*smo*) A strict partal order between writes in a set of symmetric threads. Consider a set of symmetric threads $T_1 \equiv T_2 \equiv ... \equiv T_n$. Each of these threads have exactly one read event, and multiple write events, all to the same memory, say x.*

Then each write in the above threads are involved in a symmetric order, such that.

$$\forall i \in [0, n-1]\ .\ w_i^j \xrightarrow[smo]{} w_{i+1}^j$$

Where j denotes the j^{th} event in any of the threads, which is a write.

Perhaps should put examples for the above defintion.

Definition 3. *Reads-From (*rf*) Binary relation that links a read to a write from which its value comes. Note that for our purpose, this relation is functional. For example, if a read r_i^j gets its read value from write w_k^l, then we have the relation.*

$$w_k^l \xrightarrow[rf]{} r_i^j$$

Main Rule Using the above setup, our intention is to explore lesser execution graphs leveraging the symmetry that can result due to swapping of thread identities. For this, we enforce a restriction on possible $\xrightarrow[rf]{}$ relations that are to be considered *valid*. A valid $\xrightarrow[rf]{}$ relation is one that respects the following **irreflexivity constraints**.

$$smo; rf; po$$
$$smo; po; rf^{-1}$$

We can add here more as we go about to prove completeness.

Recall examples to show how our analysis through examples satisfy the above irreflexivity constraint.

0.2 Soundness of the rules above

To prove soundness, we first define the following:

Definition 4. *Implied Write Order(*iwo*) Binary relation between any two* distinct *writes, derived through the following two sequential conposition:*

$$w_i^j; po; rf^{-1}; w_k^j$$
$$w_i^j; rf; po; w_k^j$$

Property 1. *Simplified irreflexivity rule*

The irreflexivity constraint rule is equivalent to the following irreflexivity condition

$$smo; iwo \tag{1}$$

Proof. Expanding for implied write order as per the definition, gives us the following two sequential compositions.

$$smo; w_i^j; po; rf^{-1}; w_k^j \tag{2}$$
$$smo; w_i^j; rf; po; w_k^j \tag{3}$$

From the definiton of symmetric order, the above can be simplified to

$$smo; po; rf^{-1} \tag{4}$$
$$smo; rf; po \tag{5}$$

Hence, proving our property. □

Property 2. *No write order is implied when a read reads from its own thread's write*

Proof. If the read is from its own thread's write, then we can infer that $i = k$ in both the sequential compositions. Hence

$$w_i^j; rf; po; w_i^j$$
$$w_i^j; po; rf^{-1}; w_i^j$$

which gives us $w_i^j \xrightarrow[iwo]{} w_i^j$. Since implied write orders are only between distinct writes, the property is proven. □

Property 3. *Implied write orders between two symmetric threads are reversed when they are swapped.*

Proof. Considering first sequential composition, i.e. $w_i^j; rf; po; w_k^j$, expanding gives us the following binary relations involved:

$$w_i^j \xrightarrow[rf]{} r_k \tag{6}$$
$$r_k \xrightarrow[po]{} w_k^j \tag{7}$$

Swapping thread identities involves swapping the indices i and k for each event, thus giving us

$$w_k^j \xrightarrow[rf]{} r_i \tag{8}$$

$$r_i \xrightarrow[po]{} w_i^j \tag{9}$$

Through sequential composition of the above, we get $w_k^j; rf; po; w_i^j$, which by definition is $w_k^j \xrightarrow[iwo]{} w_i^j$.

The argument is symmetric for the second sequential composition. □

Property 4. *There are at most two implied write orders between writes of two threads.*

Proof. Consider two threads T_i and T_k. Suppose we have one implied write order between one of their writes, i.e.

$$w_i^j \xrightarrow[iwo]{} w_k^j. \tag{10}$$

Expanding as per the first sequential composition gives us

$$w_i^j; po; rf^{-1}; w_k^j \tag{11}$$

which also indicates a $\xrightarrow[rf]{}$ with T_i's read and that the writes involved in the composition are above the respective reads.

Now suppose we have an implied write order between another set of writes, i.e.

$$w_i^l \xrightarrow[iwo]{} w_k^l. \tag{12}$$

Expanding as per the first sequential composition of implied write order is not possible as $\xrightarrow[rf]{}$ is functional. Hence, using the second we have

$$w_i^j; rf; po; w_k^j \tag{13}$$

which also indicates a $\xrightarrow[rf]{}$ with T_i's read that the writes involved in the composition are below the respective reads.

Since both reads are now involved in a $\xrightarrow[rf]{}$ relation, and since this relation is functional, we cannot have any more implied write orders between T_i and T_j, thus verifying our property.

Better written in contrast to previous argument. However, is it necessary to show by contradiction? □

Property 5. *Implied write ordes between two threads are either all compliant with $stck_{smo}$ or they are all not*

Proof. If each read reads from its own write, we have no implied write order established, thus maintaining the property.

For cases where implied write orders are established, without loss of generality, let us consider one between writes above the read are compliant with $stck_{smo}$:

$$w_i^j \xrightarrow[iwo]{} w_k^j \ \wedge \ w_i^j \xrightarrow[smo]{} w_k^j \tag{14}$$

The other set of implied write order, if established can only be between writes below the read. Suppose we have such an order but not compliant with $stck_{smo}$:

$$w_k^l \xrightarrow[iwo]{} w_i^l \ \wedge \ w_i^l \xrightarrow[smo]{} w_k^l \tag{15}$$

Upon expanding using the second sequential composition (because writes are below the read), we get

$$w_k^l; rf; po; w_i^l \tag{16}$$

But this implies another $\overrightarrow{rf}$ relation with T_i's read, which violates the functional property of it. Hence we can only have an implied write order compliant with $stck_{smo}$.

$$w_i^l \underset{iwo}{\longrightarrow} w_k^l \;\wedge\; w_i^l \underset{smo}{\longrightarrow} w_k^l \tag{17}$$

Not sure if we need to show that the compliant relation also holds as it only brings an rf realtion with T_k's read, which wanst established before.

The opposite case would make both the implied write orders requiring to not be compliant, thus by symmetry completing our proof. □

Property 6. *Implied write orders are acyclic*

Proof. Suppose a cycle exists. Then without loss of generality, we can consider the cycle composed of 3 writes.

$$w_i^j \underset{iwo}{\longrightarrow} w_k^j \;\wedge w_k^j \underset{iwo}{\longrightarrow} w_l^j \;\wedge w_l^j \underset{iwo}{\longrightarrow} w_i^j \tag{18}$$

If these writes are above the read, then we have the following relations that result in the above cycle.

$$w_i^j \underset{po}{\longrightarrow} r_i \;\wedge\; r_i \underset{rf^{-1}}{\longrightarrow} w_k^j \tag{19}$$
$$w_k^j \underset{po}{\longrightarrow} r_k \;\wedge\; r_k \underset{rf^{-1}}{\longrightarrow} w_l^j \tag{20}$$
$$w_l^j \underset{po}{\longrightarrow} r_l \;\wedge\; r_l \underset{rf^{-1}}{\longrightarrow} w_i^j \tag{21}$$

The above relations form a cycle thus violating $po \cup rf^{-1}$ acyclic rule for coherence and hence violating coherence.

If these writes are below the read, then we have the following relations that result in the above cycle.

$$w_i^j \underset{po}{\longrightarrow} r_i \;\wedge\; r_i \underset{rf}{\longrightarrow} w_k^j \tag{22}$$
$$w_k^j \underset{po}{\longrightarrow} r_k \;\wedge\; r_k \underset{rf}{\longrightarrow} w_l^j \tag{23}$$
$$w_l^j \underset{po}{\longrightarrow} r_l \;\wedge\; r_l \underset{rf}{\longrightarrow} w_i^j \tag{24}$$

The above relations form a cycle thus violating $po \cup rf$ acyclic rule for coherence and hence violating coherence.

Because both cases violate coherence, we conclude that $\underset{iwo}{\longrightarrow}$ must be acylic.

A bit better written compared to the previous proofs □

Need to also prove the properties of incmoing or outgoing write orders.

0.2.1 Soundness

To prove that our rules our sound, we show that for every implied write order such that $smo; iwo$ is reflexive, we can swap the corresponding thread identities to reverse the write order between them, thus respectiing our irreflexivity constraint. However, we must ensure that once write orders are "fixed" in this fashion, they remain fixed, i.e. the relation cannot appear again to be wrong. Once we show this, we also need to show that this holds in general given multiple sets of equal writes.

Part1 For a given set of equal writes, once an implied write order is fixed, it remains fixed. (A more formal statement required)

Proof. Suppose we have one implied write order which is correct / fixed between writes w_i^j and w_k^j.

$$w_i^j \xrightarrow[iwo]{} w_k^j \tag{25}$$

Without loss of generality, suppose these writes are above the read. We then divide our concern into two parts, one with implied write orders with w_i^j which are wrong and the second with those of w_k^j.

Case1: Because w_i^j is a write above read and $w_i^j \xrightarrow[iwo]{} w_k^j$, other implied write orders with w_i^j can be of the form:

$$w_m^j \xrightarrow[iwo]{} w_i^j \tag{26}$$

If this implied order is wrong, swapping thread identities will give us the following relations

$$w_i^j \xrightarrow[iwo]{} w_m^j \ \wedge \ w_m^j \xrightarrow[iwo]{} w_k^j \tag{27}$$

Here, there is no implied write order relation between w_i^j and w_k^j, hence we can consider them as remaining fixed. If the implied order between w_m^j and w_k^j is wrong, swapping it will result in the following relations

$$w_i^j \xrightarrow[iwo]{} w_k^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_m^j \tag{28}$$

Here, the implied write order between w_i^j and w_k^j remains the same. Thus, for this case, we can conclude that the relation remains "fixed".

Case2: Other implied write orders with w_k^j can be of the forms:

$$w_m^j \xrightarrow[iwo]{} w_k^j \tag{29}$$

$$w_k^j \xrightarrow[iwo]{} w_m^j \tag{30}$$

The first form is symmetric to our Case1, hence we only consider the second form.

If this implied write order is wrong, swapping thread identities will give us the following relations.

$$w_i^j \xrightarrow[iwo]{} w_m^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_m^j \tag{31}$$

Here, there is no implied write order relation between w_i^j and w_k^j, hence we can consider them as remaining fixed.

But if the implied write order between w_m^j and w_i^j is wrong, swapping them will give us

$$w_m^j \xrightarrow[iwo]{} w_i^j \ \wedge \ w_k^j \xrightarrow[iwo]{} w_i^j \tag{32}$$

thus making our claim invalid. To show that this state is not possible, note firstly that from the initial configuration, we can infer if $w_m^j \xrightarrow[iwo]{} w_k^j$ is wrong:

$$w_i^j \xrightarrow[smo]{} w_k^j \ \wedge \ w_k^j \xrightarrow[smo]{} w_m^j \tag{33}$$

Because *smo* is a total order w.r.t. one set of writes, we have by transitivity.

$$w_i^j \xrightarrow[smo]{} w_m^j \tag{34}$$

After swapping threads T_k and T_m, we get $w_i^j \xrightarrow[iwo]{} w_m^j$ which respects the irreflexivity constraint $smo; iwo$ scyclic. Hence, this implied write order is not wrong. Thus we cannot have the case which results in $w_k^j \xrightarrow[iwo]{} w_i^j$.

Thus, for a given set of equal writes, once an implied write order is fixed, it remains fixed. □

Part2 For a given set of writes, any new implied write order introduced is new and, if wrong, it can be fixed and will remain fixed. (A more formal statement required)

Proof. Suppose, for a given set of equal writes, say of the form w^j all the implied write orders are fixed. Without loss of generality, let us consider them to be wrties above the read. We consider two threads T_i and T_k, between which an implied write order is wrong. Let those writes be w_i^l and w_k^l.

Once again, without loss of generality, let us consider the symmetric memory order between writes of T_i and T_k to be of the form $w_i \xrightarrow[smo]{} w_k$. Thus we have

$$w_i^j \xrightarrow[smo]{} w_k^j \ \wedge \ w_i^l \xrightarrow[smo]{} w_k^l \tag{35}$$

Because we assume write order is wrong between w_i^l and w_k^l, by property x (forgot the number), and by our assumptions that implied write orders among w^j are fixed, we have

$$w_k^l \xrightarrow[iwo]{} w_i^l \tag{36}$$

There cannot be an implied write order between w_i^j and w_k^j, as if there were, then it would be the fixed one, and by property y (forgot number), that between w_i^l and w_k^l would also be the fixed one.

Case1: w^l is below the read.

Part1: There exists an implied write order between another write w^j and w_i^j.

Because w^l is below the read and $w_k^l \xrightarrow[iwo]{} w_i^l$, w_i^j can only be involved in implied write orders of the form

$$w^j \xrightarrow[iwo]{} w_i^j \tag{37}$$

From this and the fact that impleid write orders of w^j are fixed already, we have

$$w^j \xrightarrow[smo]{} w_i^j \tag{38}$$

By transitivity, we then have $w^j \xrightarrow[smo]{} w_k^j$.

If we swap T_i and T_k to fix $w_k^l \xrightarrow[iwo]{} w_i^j$, we have the following new relations.

$$w_i^l \xrightarrow[iwo]{} w_k^j \tag{39}$$

$$w^j \xrightarrow[iwo]{} w_k^j \tag{40}$$

Both these relations repsect our irreflexivity constraint. Hence, maintaining that new implied write order relations with w^j are not wrong.

Part2: There exists an implied write order between another write w^j and w_k^j.

w_k^j can be involved in implied write orders of the form

$$w_k^j \xrightarrow[iwo]{} w^j \tag{41}$$

$$w^j \xrightarrow[iwo]{} w_k^j \tag{42}$$

The first is symmetric to Part1, hence we only consider the second form.

On swapping T_i and T_k, we get the following new relations:

$$w_i^l \xrightarrow[iwo]{} w_k^j \tag{43}$$

$$w^j \xrightarrow[iwo]{} w_i^j \tag{44}$$

The second form may not be compliant to the irreflexivity condition, hence can be a new implied relation which is wrong. To show that this is new and could have not occurred before while swapping threads to fix implied write orders of w^j, consider the original configuration we have

$$w_i^j \xrightarrow[smo]{} w_k^j \ \wedge \ w^j \xrightarrow[iwo]{} w_k^j \tag{45}$$

For $w^j \xrightarrow[iwo]{} w_k^j$ to have been there before, there must be a write x, say, such that

$$x \xrightarrow[iwo]{} w_i^j \ \wedge \ x \xrightarrow[iwo]{} w^j \tag{46}$$

So that while swapping to reverse relation between x and w^j or x and w_i^j, we can get our new relation. But such an x cannot exist, as by property z (forgot number), writes above read, can only have one relation of the form $x \xrightarrow[iwo]{} y$ where y is another equal write. Hence, such a relation could have not been there before, and hence was not fixed.

Now that this new relation exists, fixing it, will keep it remain fixed, by our first part of proof (label them please).

Case2: w^l is above the read.

□