Temperature Interpolation in District Heating Networks

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October 30, 2025

Abstract

We study the problem of temperature interpolation in the district heating context. The problem is formulated in a metric graph setting, the flow of thermal energy is modelled as governed by a stochastic advection-diffusion equation along the edges, and the states of the discretised model are estimated with Kalman filtering.

1 Introduction

District heating systems (DHS) transfer thermal energy to demand points in a network of water pipes. The problem of the temperature field interpolation in such networks is relevant to e.g. optimal heating control (Hannula et al., 2025) or operational efficiency. Following Bolin et al. (2022), we model the network as a metric graph and the transport of thermal energy with a stochastic partial differential equation (SPDE). Then Kalman filtering is utilized to estimate the temperature profile over time from sparse measurements.

2 Model

We adapt the notation and definitions given by Bolin et al. (2022) and let the DHS be represented by the connected compact metric graph $\Gamma = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ a set of edges. Each edge $e \in \mathcal{E}$ is associated with the interval $\Omega_e = [0, \ell_e] \subset \mathbb{R}$ where $\ell_e \geq 0$ is the length of an edge. A function $f : \Gamma \to \mathbb{R}$ is identified with the collection of real-valued functions $\{f_e\}_{e\in\mathcal{E}}$ s.t. $f|_e = f_e$. Linear operators $T = \oplus_{e\in\mathcal{E}} T_e : \Gamma \to \Gamma$ applied to f are applied element-wise to each f_e and the product fg is identified with $\{f_e g_e\}_{e\in\mathcal{E}}$ for all $f, g : \Gamma \to \mathbb{R}$. We focus on the space of real-valued functions

$$H^1(\Gamma) := \{\{f_e\}_{e\in\mathcal{E}} : f_e(\cdot, t) \in H^1(0, \ell_e)\, \forall t \geq 0, f \text{ satisfies vertex conditions}\}, \tag{1}$$

where H^1 is the Sobolev space of order 1, and equip $H^1(\Gamma)$ with the inner product

$$\langle f, g\rangle_\Gamma = \sum_{e\in\mathcal{E}} \langle f_e, g_e\rangle_e = \sum_{e\in\mathcal{E}} \int_0^{\ell_e} f_e g_e \, dx. \tag{2}$$

The following stochastic advection-diffusion equation defined on Γ for all $t \geq 0$ is considered:

$$\partial_t u(x,t) + v(t)\partial_x u(x,t) - D\partial_{xx} u(x,t) + \alpha u(x,t) = f(x,t) + \varepsilon(x,t), \tag{3}$$

where $v_e(t), D_e, \alpha_e$ are constant for each $e \in \mathcal{E}$ and $t \geq 0$, f is a forcing function and ε a stochastic term on Γ. Next the discretised form of (3) is derived adapting the methods presented by Clarotto et al. (2022) and Bolin et al. (2022). Let $\phi \in H^1(\Gamma)$ be a test function. The variational formulation of (3) is given by

$$\langle \phi, \partial_t u\rangle_\Gamma + \langle \phi, v\partial_x u\rangle_\Gamma - \langle \phi, D\partial_{xx} u\rangle_\Gamma + \langle \phi, \alpha u\rangle_\Gamma = \langle \phi, f\rangle_\Gamma + \langle \phi, \varepsilon\rangle_\Gamma. \tag{4}$$

Let $\mathcal{E}_v$ be the set of edges incident to $v \in \mathcal{V}$. Let $\partial_e f_e(w,t)$ denote the outward directional derivative of f on node w and edge e. We assume the vertex conditions

$$\sum_{e\in\mathcal{E}_w} v_e(t)u_e(w,t) - D_e\partial_e u_e(w,t) = 0, \quad \forall w \in \mathcal{V} \qquad \text{(flux condition)} \tag{5}$$

$$u_e(w,t) = c_w(t), \quad \forall w \in \mathcal{V}, \forall e \in \mathcal{E}_w \qquad \text{(same value on shared nodes)} \tag{6}$$

on $H^1(\Gamma)$. Then, expanding the inner product, and integrating by parts, (4) becomes

$$\sum_{e\in\mathcal{E}} \langle \phi_e, \partial_t u_e\rangle_e + \langle \phi_e, v_e\partial_x u_e\rangle_e + \langle \partial_x\phi_e, D_e\partial_x u_e\rangle_e + \langle \phi_e, \alpha_e u_e\rangle_e = \langle \phi, f\rangle_\Gamma + \langle \phi, \varepsilon\rangle_\Gamma. \tag{7}$$

where the boundary terms vanish due to the choice of vertex conditions. For each $e \in \mathcal{E}$, let $\{\psi_k\}_{k=1}^N \subset H^1(\Gamma)$ be a finite collection of basis functions and let u^h be defined by

$$u^h = \sum_{k=1}^N u_k^h \psi_k, \quad u_e^h(x,t) = \sum_{j\in J_e} u_j^h|_e(t)\psi_j|_e(x) \tag{8}$$

where J_e indexes the basis functions supported on $e \in \mathcal{E}$. Choosing the test functions $\phi_j \in H^1(\Gamma)$, Equations (7) and (8) lead to

$$M\partial_t u^h + vAu^h + DKu^h + \alpha Mu^h = f^h + \varepsilon^h, \tag{9}$$

where $M_{jk} = \langle \phi_j, \psi_k\rangle_\Gamma$, $A_{jk} = \langle \phi_j, \partial_x\psi_k\rangle_\Gamma$, $K_{jk} = \langle \partial_x\phi_j, \partial_x\psi_k\rangle_\Gamma$, $f_j^h = \langle \phi_j, f\rangle_\Gamma$, and $\varepsilon_j^h = \langle \phi_j, \varepsilon\rangle_\Gamma$. To discretise over the time domain, we integrate over $[t_k, t_{k+1}]$, where $t_{k+1} - t_k = \Delta t$, and employ the implicit Euler scheme to get

$$M(x_{k+1} - x_k) + \Delta t L_k x_{k+1} = \Delta t f_k + \int_{t_k}^{t_{k+1}} \varepsilon^h \, dt, \tag{10}$$

where $x_0 \in \mathbb{R}^N$ is given and $L_k = vA_k + DK_k + \alpha M_k$. To account for the stochastic term, we make the assumption

$$C_\varepsilon(x,y,t,s) = \delta(t-s)C_\varepsilon(x,y), \tag{11}$$

i.e. that the covariance of ε^h is (formally) white in time and exhibits spatial correlation. Then, assuming zero mean for the stochastic term, we have

$$(C_\varepsilon^h)_{ij} = \mathrm{Cov}(\varepsilon_i^h, \varepsilon_j^h) = \mathbb{E}[\varepsilon_i^h\varepsilon_j^h] = \int_\Gamma\int_\Gamma \phi_i(x)\phi_j(y)C_\epsilon(x,y)\,dxdy. \tag{12}$$

Thus, letting $q_k = \int_{t_k}^{t_{k+1}} \varepsilon^h(t)\,dt$, we get

$$\mathrm{Cov}(q_i, q_j) = \delta_{ij}\Delta t C_\varepsilon^h. \tag{13}$$

Following the SPDE approach of Lindgren et al. (2011) and Bolin et al. (2022), we model C_ε^h as the solution to the SPDE

$$(\kappa^2 - \Delta_\Gamma)f = W_\Gamma \tag{14}$$

on Γ, where $\Delta_\Gamma := \oplus_{e\in\mathcal{E}}\Delta_e$ is the edge-wise Laplace operator, $\kappa^2 \in \mathbb{R}$, and W_Γ is a Gaussian white noise on Γ. The solution to (14) is obtained by finite element (FE) discretisation. Thus the final discretised model is given by

$$x_{k+1} = E_k M x_k + \Delta t E_k f_k + E_k q_k, \quad q_k \sim \mathcal{N}(0, \Delta t C_\varepsilon^h) \tag{15}$$

where $E_k(M + \Delta t L_k)^{-1}$ and C_ε^h is the discretised Matérn covariance on Γ.

3 Methods

We form the global FE matrices $M, A, K \in \mathbb{R}^{n\times n}$ with the streamline upwind Petrov-Galerkin (SUPG) stabilization method (Kuzmin and Hämäläinen, 2014) using linear basis functions, solve for $Q_\varepsilon^h = (C_\varepsilon^h)^{-1}$ via the SPDE approximation method (Lindgren et al., 2011), and apply Kalman filtering to the system

$$x_{k+1} = E_k M x_k + \Delta t E_k f_k + E_k q_k, \qquad q_k \sim \mathcal{N}(0, \Delta t C_\varepsilon^h) \tag{16}$$

$$y_{k+1} = H_{k+1} x_{k+1} + r_{k+1}, \qquad r_{k+1} \sim \mathcal{N}(0, R_{k+1}) \tag{17}$$

where $H_{k+1} \in \mathbb{R}^{m\times n}$ is the operator mapping observations to the FE nodes.

4 Numerical Examples

The method is applied to a test network shown in Figure 1 where the edge weights correspond to lengths. The Julia programming language is used for the implementation. The water is assumed to be supplied at node a. Measurements are obtained at nodes c and d and the temperature is estimated in the edges of the network. For the experiment, we assume constant velocities $v_{ab} = 10, v_{bc} = 8, v_{bd} = 7$, constant temperatures $u(c,t) = 10, u(d,t) = 8$, constant external forcing (ambient temperature) $f(x,t) = \alpha 5$, measurement variance $R_k = 0.01I$, and the Matérn parameters $\alpha_m = 2, \nu_m = 3/2, \ell_m = 5, \sigma_m = 1$. Figure 2 (a) displays the estimated temperature on each time step. In Figure 2 (b) we see how the uncertainty increases the farther away we move from the measurements. Varying the experiment parameters highlights the ill-posedness of the problem. To demonstrate this, 2 (c) displays the estimates with different ℓ_m parameters. We see how increasing ℓ_m can lead to unphysical temperature estimates lowering toward the upstream nodes. On the other hand, the reactivity parameter α has a regularizing effect, where higher values of α lead to physically more sensible estimates.

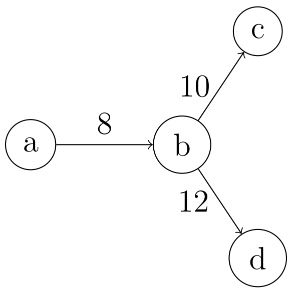
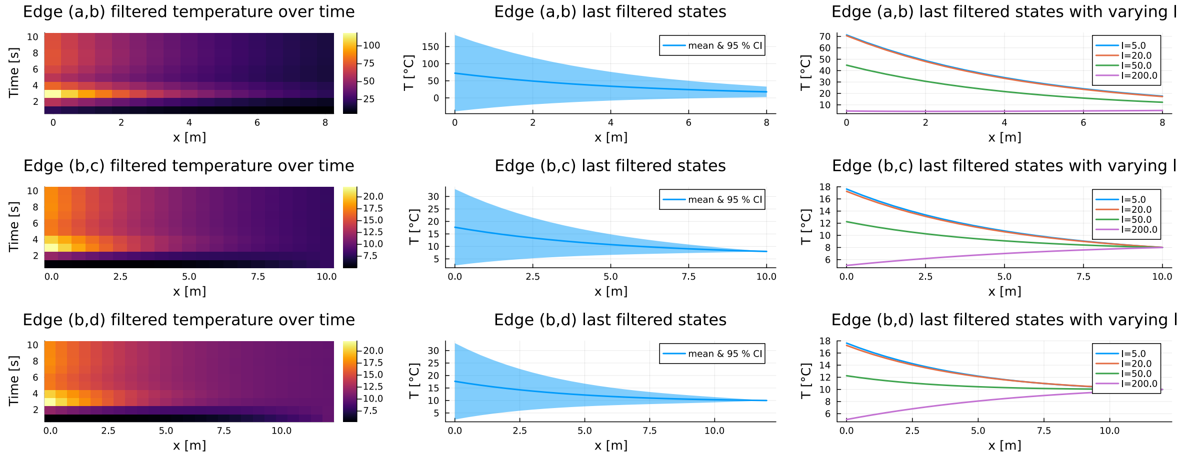


Figure 1: A test network with edge lengths.



(a) Filtered temperature field. (b) State estimate with credible intervals. (c) State estimates with varying Matérn correlation length.

Figure 2: The filtered temperature states over space and time (a), the last filtered states with 95% credible intervals (b), and the estimates with varying Matérn parameter ℓ_m (c).

5 Conclusions

We studied the problem of temperature interpolation in the district heating context. We formulated the problem in a metric graph setting, modelled the flow of thermal energy as governed by a stochastic advection-diffusion equation along the edges, and estimated the states of the discretised model with Kalman filtering. We demonstrated one of the challenges related to estimation by varying the Matérn covariance correlation length parameter. There are several possible avenues for future work including parameter estimation, studying the effect of the covariance model on the estimates and the regularity of the problem in more detail, expanding the flow model to include pressure, and testing the model with real data.

References

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