Abstract Algebra - Assignment 4

Jaan Tollander de Balsch, 452056

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This report uses definitions from Gallian (2017), chapter *Normal subgroups and Factor Groups*

Problem 1

Problem 2

Problem 3

Let A and B be two arbitrary normal subgroups of group G. By normal subgroup test, the following conditions then hold:

1) $xAx^{-1} \subseteq A$ for all $x \in G$
2) $xBx^{-1} \subseteq B$ for all $x \in G$

(a)

Let $A \cap B$ be the intersection of the two normal subgroups of G. Then the intersection can also be shown to be a normal subgroup of G using the normal subgroup test

$$x(A \cap B)x^{-1} = (xAx^{-1}) \cap (xBx^{-1}) \subseteq A \cap B$$

for all $x \in G$.

(b)

Let $AB := \{a \cdot b \mid a \in A, b \in B\}$ be a product of two normal subgroups of G. Then the product can also be shown to be a normal subgroup of G using normal subgroup test

$$xABx^{-1} = (xAx^{-1})(xBx^{-1}) \subseteq AB, \quad x^{-1}x = e$$

for all $x \in G$.

Problem 4

Problem 5

Let D_8 denote the symmetry group of the square, meaning

$$D_8 = \{\varrho^i \sigma^i \mid i, j \in \mathbb{Z}, \quad \varrho^4 = e = \sigma^2, \quad \sigma\varrho = \varrho^3\sigma\}$$

(a)

Draw a Caley table for this group.

The distinct elements in D_8 are

$$D_8 = \{e, \varrho, \varrho^2, \varrho^3, \sigma, \varrho\sigma, \varrho^2\sigma, \varrho^3\sigma\}$$

The Caley table for this group can be formed from the product

$$D_8 \times D_8 = \begin{bmatrix} 1 & \varrho & \varrho^2 & \varrho^3 & \sigma & \varrho\sigma & \varrho^2\sigma & \varrho^3\sigma \\ \varrho & \varrho^2 & \varrho^3 & \varrho^4 & \varrho\sigma & \varrho^2\sigma & \varrho^3\sigma & \varrho^4\sigma \\ \varrho^2 & \varrho^3 & \varrho^4 & \varrho^5 & \varrho^2\sigma & \varrho^3\sigma & \varrho^4\sigma & \varrho^5\sigma \\ \varrho^3 & \varrho^4 & \varrho^5 & \varrho^6 & \varrho^3\sigma & \varrho^4\sigma & \varrho^5\sigma & \varrho^6\sigma \\ \sigma & \sigma\varrho & \sigma\varrho^2 & \sigma\varrho^3 & \sigma^2 & \sigma\varrho\sigma & \sigma\varrho^2\sigma & \sigma\varrho^3\sigma \\ \varrho\sigma & \varrho\sigma\varrho & \varrho\sigma\varrho^2 & \varrho\sigma\varrho^3 & \varrho\sigma^2 & \varrho\sigma\varrho\sigma & \varrho\sigma\varrho^2\sigma & \varrho\sigma\varrho^3\sigma \\ \varrho^2\sigma & \varrho^2\sigma\varrho & \varrho^2\sigma\varrho^2 & \varrho^2\sigma\varrho^3 & \varrho^2\sigma^2 & \varrho^2\sigma\varrho\sigma & \varrho^2\sigma\varrho^2\sigma & \varrho^2\sigma\varrho^3\sigma \\ \varrho^3\sigma & \varrho^3\sigma\varrho & \varrho^3\sigma\varrho^2 & \varrho^3\sigma\varrho^3 & \varrho^3\sigma^2 & \varrho^3\sigma\varrho\sigma & \varrho^3\sigma\varrho^2\sigma & \varrho^3\sigma\varrho^3\sigma \end{bmatrix}.$$

Here we'll be using 1 as the indentity because I used a computer algebra system for these computations. The product can be simplified by using the substitution rules

1) $\varrho^4 = e$
2) $\sigma^2 = e$
3) $\sigma\varrho = \varrho^3\sigma$
4) $\sigma\varrho^3 = \varrho\sigma$
5) $\varrho\sigma\varrho = \sigma$
6) $\sigma\varrho\sigma = \varrho^3$
7) $\sigma\varrho^2 = \varrho^2\sigma$

Then we'll have

$$D_8 \times D_8 = \begin{bmatrix} 1 & \varrho & \varrho^2 & \varrho^3 & \sigma & \varrho\sigma & \varrho^2\sigma & \varrho^3\sigma \\ \varrho & \varrho^2 & \varrho^3 & 1 & \varrho\sigma & \varrho^2\sigma & \varrho^3\sigma & \sigma \\ \varrho^2 & \varrho^3 & 1 & \varrho & \varrho^2\sigma & \varrho^3\sigma & \sigma & \varrho\sigma \\ \varrho^3 & 1 & \varrho & \varrho^2 & \varrho^3\sigma & \sigma & \varrho\sigma & \varrho^2\sigma \\ \sigma & \varrho^3\sigma & \varrho^2\sigma & \varrho\sigma & 1 & \varrho^3 & \varrho^2 & \varrho \\ \varrho\sigma & \sigma & \varrho^3\sigma & \varrho^2\sigma & \varrho & 1 & \varrho^3 & \varrho^2 \\ \varrho^2\sigma & \varrho\sigma & \sigma & \varrho^3\sigma & \varrho^2 & \varrho & 1 & \varrho^3 \\ \varrho^3\sigma & \varrho^2\sigma & \varrho\sigma & \sigma & \varrho^3 & \varrho^2 & \varrho & 1 \end{bmatrix}$$

(b)

Left H be the subgroup generated by ϱ in D_8. Show that H is normal in D_8.

The subgroup generated by ϱ in D_8 is

$$H = \{1, \varrho, \varrho^2, \varrho^3\}.$$

It can be shown to be normal using the *Normal Subgroup Test*, i.e. showing that $xHx^{-1} \subseteq H$ for all x in D_8. We'll start by computing xHx. I will represent these group here as matrices where elements in H are iterate column-wise and the elements in D_8 are iterated row-wise, i.e. $\{\{xhx^{-1} \mid h \in H\} \mid x \in D_8\}$.

$$\begin{bmatrix} 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \sigma\varrho\sigma^{-1} & \sigma\varrho^2\sigma^{-1} & \sigma\varrho^3\sigma^{-1} \\ \varrho\sigma\,(\varrho\sigma)^{-1} & \varrho\sigma\varrho\,(\varrho\sigma)^{-1} & \varrho\sigma\varrho^2\,(\varrho\sigma)^{-1} & \varrho\sigma\varrho^3\,(\varrho\sigma)^{-1} \\ \varrho^2\sigma\,(\varrho^2\sigma)^{-1} & \varrho^2\sigma\varrho\,(\varrho^2\sigma)^{-1} & \varrho^2\sigma\varrho^2\,(\varrho^2\sigma)^{-1} & \varrho^2\sigma\varrho^3\,(\varrho^2\sigma)^{-1} \\ \varrho^3\sigma\,(\varrho^3\sigma)^{-1} & \varrho^3\sigma\varrho\,(\varrho^3\sigma)^{-1} & \varrho^3\sigma\varrho^2\,(\varrho^3\sigma)^{-1} & \varrho^3\sigma\varrho^3\,(\varrho^3\sigma)^{-1} \end{bmatrix}$$

The inverse elements of D_8 can be read from the Caley table above. By subtituting the inverses

1) $\sigma^{-1} = \sigma$
2) $(\varrho\sigma)^{-1} = \varrho\sigma$
3) $(\varrho^2\sigma)^{-1} = \varrho^2\sigma$
4) $(\varrho^3\sigma)^{-1} = \varrho^3\sigma$

$$\begin{bmatrix} 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \sigma\varrho\sigma & \sigma\varrho^2\sigma & \sigma\varrho^3\sigma \\ \varrho\sigma\varrho\sigma & \varrho\sigma\varrho^2\sigma & \varrho\sigma\varrho^3\sigma & \varrho\sigma\varrho^4\sigma \\ \varrho^2\sigma\varrho^2\sigma & \varrho^2\sigma\varrho^3\sigma & \varrho^2\sigma\varrho^4\sigma & \varrho^2\sigma\varrho^5\sigma \\ \varrho^3\sigma\varrho^3\sigma & \varrho^3\sigma\varrho^4\sigma & \varrho^3\sigma\varrho^5\sigma & \varrho^3\sigma\varrho^6\sigma \end{bmatrix}$$

Then by substituting the identities

1) $\varrho^4 = e$
2) $\sigma^2 = e$
3) $\sigma\varrho\sigma = \varrho^3$
4) $\sigma\varrho^2\sigma = \varrho^2$
5) $\sigma\varrho^3\sigma = \varrho$

$$\begin{bmatrix} 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho & \varrho^2 & \varrho^3 \\ 1 & \varrho^3 & \varrho^2 & \varrho \\ 1 & \varrho^3 & \varrho^2 & \varrho \\ 1 & \varrho^3 & \varrho^2 & \varrho \\ 1 & \varrho^3 & \varrho^2 & \varrho \end{bmatrix}$$

As can be read from the rows of this matrix, the group generated by the ϱ satisfies the normal subgroup test.

(c)

Show that $K := \{e, \varrho^2\}$ is a normal subgroup of D_8, too.

In the same way as in the section (b) we'll compute xKx^{-1}

$$\begin{bmatrix} 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \sigma\varrho^2\sigma^{-1} \\ \varrho\sigma\left(\varrho\sigma\right)^{-1} & \varrho\sigma\varrho^2\left(\varrho\sigma\right)^{-1} \\ \varrho^2\sigma\left(\varrho^2\sigma\right)^{-1} & \varrho^2\sigma\varrho^2\left(\varrho^2\sigma\right)^{-1} \\ \varrho^3\sigma\left(\varrho^3\sigma\right)^{-1} & \varrho^3\sigma\varrho^2\left(\varrho^3\sigma\right)^{-1} \end{bmatrix}$$

Once again, by subtituting the inverses and identities we get

$$\begin{bmatrix} 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \\ 1 & \varrho^2 \end{bmatrix}$$

From the rows we can read that xKx^{-1} satisfies the normal subgroup test.

(d)

Determine the size of D_8/K, and decide whether or not this factor group is cyclic.

The factor group is defined

$$G/H = \{aH \mid a \in G\}$$

Then entried for D_8/K can be read from the Caley table

$$\begin{bmatrix} 1 & \varrho^2 \\ \varrho & \varrho^3 \\ \varrho^2 & 1 \\ \varrho^3 & \varrho \\ \sigma & \varrho^2\sigma \\ \varrho\sigma & \varrho^3\sigma \\ \varrho^2\sigma & \sigma \\ \varrho^3\sigma & \varrho\sigma \end{bmatrix}$$

References

Gallian, J., 2017. *Contemporary Abstract Algebra.* 9th ed. Cengage Learning.