Abstract Algebra - Assignment 2

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Definitions

A **group** is a triple $(G, \cdot, e)$ where G is a set, $\cdot$ is a binary operation on G and the indentity e, such that the following hold

a) **Closure**: For all $x, y \in G$ the result operation

$$x \cdot y \in G.$$

b) **Associativity**: For all $x, y, z \in G$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

c) **Indentity element**: There exists an identity element e such that for all $x \in G$

$$e \cdot x = x \cdot e = x.$$

d) **Inverse element**: For every $x \in G$ there exists $y \in G$ such that

$$x \cdot y = y \cdot x = e.$$

Semigroup: Only (a), (b) holds.

Monoid Only (a), (b), and) hold.

G is called **Abelian group** if the following also holds

d) **Commutativity**: For all $x, y \in G$ we have $x \cdot y = y \cdot x$.

Let G be a group and $\emptyset = H \subseteq G$ a subset of G. We say H is a **subgroup** of G, formally $H \subseteq G$, it is closed under the operation on G and satisfies all group axioms.

A subset H of a group G is a subgroup, if and only if the following hold

a) **Nonempty**: The subgroup is non-empty $H \neq \emptyset$.
b) **Closed under operation**: For all $a, b \in H$

$$a \cdot b \in H.$$

c) **Closed under inverse**: For all $a \in H$

$$a^{-1} \in H.$$

Alternatively

I) $H \neq \emptyset$
II) $a, b \in H$ implies $a \cdot b^{-1} \in H$.

For a group G and a subset $T \subseteq G$ define

$$\langle T \rangle := \bigcap \{H \subseteq G \mid T \subseteq H\},$$

the intersection of all subgroups that contain T.

- Group
- Subgroup
- Cayley table

Problem 1

a)

For a group G, show by induction, that

$$(a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{-1} = a_n^{-1} \cdot \ldots \cdot a_2^{-1} \cdot a_1^{-1}$$

for arbitrary $a_i \in G$, $i = 1, \ldots, n$ and $n \in \mathbb{N}$.

Trivial case $n = 1$: From *inverse element* axiom

$$a_1 \cdot a_1^{-1} = e.$$

Base case $n = 2$: From *closure* axiom $x = a_1 \cdot a_2 \in G$ then

$$\begin{aligned}
x \cdot x^{-1} &= e \\
(a_1 \cdot a_2) \cdot (a_1 \cdot a_2)^{-1} &= e \\
a_1 \cdot (a_2 \cdot (a_1 \cdot a_2)^{-1}) &= e \\
a_2 \cdot (a_1 \cdot a_2)^{-1} &= a_1^{-1} \cdot e \\
(a_1 \cdot a_2)^{-1} &= a_2^{-1} \cdot a_1^{-1} \cdot e \\
(a_1 \cdot a_2)^{-1} &= a_2^{-1} \cdot a_1^{-1}.
\end{aligned}$$

General case n: Using *closure* and *transitivity* axioms

$$x = a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot a_n = a_1 \cdot (a_2 \cdot \ldots \cdot (a_{n-1} \cdot a_n)) \in G.$$

Then using the same process as in the base case

$$(a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot a_n)^{-1} = a_n^{-1} \cdot a_{n-1}^{-1} \cdot \ldots \cdot a_2^{-1} \cdot a_1^{-1}.$$

b)

Write down the Caley table for the group $\mathbb{Z}_7^\times := \{1, 2, 3, 4, 5, 6\}$ with multiplication modulo 7.

$\times$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Julia code:

```
print(reshape([mod(i*j, 7) for i in 1:6 for j in 1:6],(6,6)))
```

Problem 2

Let G be a group, and let H and K be subgroups of G. Show that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

Subgroup axioms for H.

a) $H \neq \emptyset$
b) $h_1, h_2 \in H$ implies $h_1 \cdot h_2 \in H$
c) If $h \in H$ then $h^{-1} \in H$

Subgroup axioms for K.

a) $K \neq \emptyset$
b) $k_1, k_2 \in K$ implies $k_1 \cdot k_2 \in K$
c) If $k \in K$ then $k^{-1} \in K$

Lets test subgroup axioms for $H \cup K$. There are three cases in which the group $H \cup K$ **is a subgroup** of G. In all of these, the subgroup axioms of $H \cup K$ reduce to either subgroup axioms of H or K.

1) If $H = K$ then $H \cup K = H = K$. Reduces to the subgroup axioms of H or K.
2) If $K \subseteq H$ then $H \cup K = H$. Reduced to the subgroup axioms of H.
3) If $H \subseteq K$ then $H \cup K = K$. Reduces to the subgroup axioms of K.

In the fourth case, $H \subsetneq K$ and $K \subsetneq H$, $H \cup K$ **is not a subgroup** of G.

Proof: This proof provides a counterexample that $H \cup K$ is not **closed under operation** if $H \subsetneq K$ and $K \subsetneq H$ which implies that $H \cup K$ **is not a subgroup** of G.

1) Let $a, b \in H \cup K$ such that $a \in H$, $a \notin K$, $b \in K$, and $b \notin H$
2) Implies that $a \cdot b \in H \cup K$
3) Implies that $a \cdot b \in H$ or $a \cdot b \in K$
4) If $a \cdot b \in H$ then $a, b \in H$ or if $a \cdot b \in K$ then $a, b \in K$
5) These are in conflict with the assumptions and therefore $a \cdot b \notin H \cup K$ and $H \cup K$ is not a subgroup of G.

Problem 3

Let G be a group and let H be a subset of G.

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a

If H is finite and non-empty, then H is a subgroup of G if and only if H is closed under the operation on G.

The axiom for being non-empty $H \neq \emptyset$ is already satisfied. The existence of inverse element can be proven using the existence of identity element, which is always included in the subgroup. Lets assume that H is **closed under operation**, which means that all elements $a \in H$ are products $a = b \cdot c$ under the operation on elements $b, c \in H$.

TODO: Cayley table, finite, For all $a \in H$ there exists $b \in H$ such that

$$a \cdot b = ea^{-1} = b.$$

b

Let the group G be $(\mathbb{Z}, +, 0)$. Then infinite subset $H = (\mathbb{N}, +, 0)$ of G is closed under the operation $+$ on G, but **not a subgroup** of G because its not **closed under inverse**, i.e. $a \in H$ does not imply $a^{-1} \in H$.

Problem 4

Let G be a group. For $g \in G$ define the *centralizer* $C_G(g) := \{x \in G \mid x \cdot g = g \cdot x\}$ of g in G, and prove that $C_G(g)$ is a subgroup for every g.

Nonempty: For all g,

$$C_G(g) \neq \emptyset.$$

Proof: For every g if $x = g$ then

$$x \cdot g = g \cdot g = g \cdot x$$

therefore the centralizer always contains atleast the element g.

Closure under operation: For all $a, b \in C_G(g)$

$$a \cdot b \in C_G(g)$$

Proof: Let $a, b \in C_G(g)$ then $a \cdot g = g \cdot a$ and $b \cdot g = g \cdot b$. It can be shown that $a \cdot b$ belongs to $C_G(g)$ if it satisfies the property $x \cdot g = g \cdot x$.

$$\begin{aligned}
(a \cdot b) \cdot g &= a \cdot (b \cdot g) \\
&= a \cdot (g \cdot b) \\
&= (a \cdot g) \cdot b \\
&= (g \cdot a) \cdot b \\
&= g \cdot (a \cdot b).
\end{aligned}$$

Closure under inverse: For all $a \in C_G(g)$ there exists $a^{-1} \in C_G(g)$

Proof: Let $a \in C_G(g)$ then $a \cdot g = g \cdot a$. It can shown that a^{-1} belongs to $C_G(g)$ if it satisfies the property $x \cdot g = g \cdot x$.

$$\begin{aligned}
a^{-1} \cdot g &= g \cdot a^{-1} \\
a^{-1} \cdot g \cdot a &= g \\
g \cdot a &= a \cdot g \\
a \cdot g &= g \cdot a.
\end{aligned}$$

Problem 5