CS-E4500 Problem Set 4

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This report uses algorithms from Gathen and Gerhard (2013), chapters 5.1-5.4 and 10.1-10.3.

Problem 1

Let F be a finite field. In this case $F = \mathbb{Z}_p = \{0, 1, ..., p\}$ for p prime. Let $\varphi_0 \in F$ be a **secret** that we'll split into s **shares** such that **knowledge** of any k shares enables recovery of the secret.

1) Let $\xi_1, \xi_2, ..., \xi_s \in F$ be **distinct** and **nonzero**.
2) Select elements $\varphi_1, \varphi_2, ..., \varphi_{k-1}$ independently and uniformly at random.
3) Let $f = \varphi_0 + \varphi_1 x + \varphi_2 x^2 + ... + \varphi_{k-1} x^{k-1} \in F[x]$.
4) For $j = 1, 2, ..., s$ share j is the pair $(\xi_j, f(\xi_j)) \in F^2$.

The secret can be **recovered** by interpolating the k shares back into polynomial f and evaluation the polynomial at $f(\xi_0) = f(0) = \varphi_0$. We'll use Lagrange interpolation as the interpolating algorithm.

Implementation in Python code

```
from sympy import *
init_printing("mathjax")
import random
```

```
a = 0
b = 99
p = nextprime(b)
a, b, p
```

$$(0, \quad 99, \quad 101)$$

```
Z_p = list(range(p+1))
secret = random.randint(a, b)
shares = 10
knowledge = 5
secret,shares,knowledge
```

$$(42, \quad 10, \quad 5)$$

Step 1

```
xi = random.sample(Z_p[1:], shares)
xi
```

$$[3, \quad 51, \quad 85, \quad 33, \quad 7, \quad 35, \quad 81, \quad 2, \quad 1, \quad 26]$$

Step 2

```
phi = random.choises(Z_p, k=knowledge-1)
phi
```

$$[17, \quad 42, \quad 50, \quad 9]$$

Step 3

```
x = symbols('x', integer=True, positive=True)
f = Poly(reversed([secret]+phi), x)
f
```

$$\operatorname{Poly}\left(9x^4 + 50x^3 + 42x^2 + 17x + 42, x, domain = \mathbb{Z}\right)$$

Step 4

```
f_xi = [f(x)%p for x in xi]
f_xi
```

$$[25, \quad 11, \quad 30, \quad 55, \quad 73, \quad 2, \quad 42, \quad 81, \quad 59, \quad 53]$$

```
pairs = list(zip(xi, f_xi))
pairs
```

$$[(3, \quad 25), \quad (51, \quad 11), \quad (85, \quad 30), \quad (33, \quad 55), \quad (7, \quad 73), \quad (35, \quad 2), \quad (81, \quad 42), \quad (2, \quad 81), \quad (1, \quad 59$$

Recovering the secret

```
data = random.sample(pairs, knowledge)
data
```

$$[(81, \quad 42), \quad (3, \quad 25), \quad (1, \quad 59), \quad (7, \quad 73), \quad (85, \quad 30)]$$

```
xi2, f_xi2 = list(zip(*data))
xi2, f_xi2
```

$$((81, \quad 3, \quad 1, \quad 7, \quad 85), \quad (42, \quad 25, \quad 59, \quad 73, \quad 30))$$

Source: https://stackoverflow.com/questions/4798654/modular-multiplicative-inverse-function-in-python/4798776

```
def egcd(a, b):
    if a == 0:
        return (b, 0, 1)
    else:
        g, y, x = egcd(b % a, a)
        return (g, x - (b // a) * y, y)

def modinv(a, m):
    g, x, y = egcd(a, m)
    if g != 1:
        raise Exception('modular inverse does not exist')
    else:
        return x % m
```

```
def lagrange_interpolation(u, v):
    assert len(u) == len(v)
    n = len(u)
    P = 0
    for i in range(n):
        s = v[i]
        for j in range(n):
            if i == j:
                continue
            s *= (x - u[j])*modinv((u[i]-u[j])%p, p)
        P += s
    return poly(expand(P, modulus=p))
```

```
P = lagrange_interpolation(xi2, f_xi2)
P
```

$$\mathrm{Poly}\left(9x^4 + 50x^3 + 42x^2 + 17x + 42, x, domain = \mathbb{Z}\right)$$

Lets verify that our interpolation polynomial is correct.

```
f==P
```

```
True
```

Now we can recover the secret and verify that it is correct.

```
P(0)
```

```
P(0) == secret
```

```
True
```

Problem 2

Let R be a ring and let $q, r \in R[x]$ with $a = qb + r$ and $\deg r < \deg b$ where b monic.

(a)

Show that for all $\xi \in R$ and $f \in R[x]$ we have $f(\xi) = f \operatorname{rem} (x - \xi)$.

Let $a = f$ and $b = (x - \xi)$ then

$$\begin{aligned} f(x) &= q(x) \cdot (x - \xi) + r(x) \\ f(\xi) &= q(\xi) \cdot (\xi - \xi) + r(\xi) \\ f(\xi) &= r(\xi) \\ f(\xi) &= f \operatorname{rem} (x - \xi). \end{aligned}$$

(b)

Let $a, b, c \in R[x]$, with b and c monic, and suppose that c divides b. Show that $a \operatorname{rem} c = (a \operatorname{rem} b) \operatorname{rem} c$.

We have equalities $a = q_1 b + r_1$ and $b = q_2 c + r_2$ where the remainders are

$$r_1 = a \operatorname{rem} b$$

and $r_2 = 0$ since c divides b.

The remainder r_1 can also be written in this form $r_1 = q'c + r'$ where $\deg r' < \deg c$. Then we have a remainder

$$r' = r_1 \operatorname{rem} c = (a \operatorname{rem} b) \operatorname{rem} c.$$

We also have

$$\begin{aligned} a &= q_1 b + r_1 \\ a &= q_1 (q_2 c) + (q'c + r') \\ a &= (q_1 q_2 + q')c + r'. \end{aligned}$$

Then the remainder r' has equivalency

$$r' = a \operatorname{rem} c.$$

By combining the equivalencies for the remainder r' we have

$$a \operatorname{rem} c = (a \operatorname{rem} b) \operatorname{rem} c.$$

Problem 3

Let R be a ring, let $\xi_0, \xi_1, ..., \xi_{e-1} \in R$, and $\lambda_0, \lambda_1, ..., \lambda_{e-1} \in R$ be given as input. The form of the Lagrange interpolation polynomial suggests that one should first seek to construct the coefficients of the polynomial

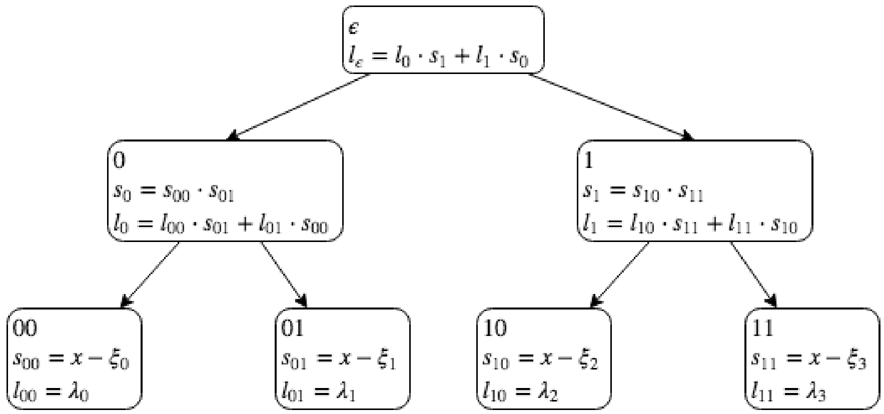
$$L_e(x) = \sum_{i=0}^{e-1} \lambda_i \prod_{j=0 j \neq i}^{e-1} (x - \xi_j) \in R.$$

Show that we can compute the coefficients of L in $O(M(e) \log e)$ operations in R. You may assume that $e = 2^k$ for a nonnegative integer k. Here $M(e) = e \log e \log \log e$.

The common substructure within the polynomials is evident if we expand the Lagrange polynomials with $e \in \{1, 2, 4\}$

$$\begin{aligned}
L_1(x) =& \lambda_0 \\
L_2(x) =& \lambda_0(x - \xi_1) + \lambda_1(x - \xi_0) \\
L_4(x) =& \lambda_0(x - \xi_1)(x - \xi_2)(x - \xi_3) + \\
& \lambda_1(x - \xi_0)(x - \xi_2)(x - \xi_3) + \\
& \lambda_2(x - \xi_0)(x - \xi_1)(x - \xi_3) + \\
& \lambda_3(x - \xi_0)(x - \xi_1)(x - \xi_2) \\
=& (\lambda_0(x - \xi_1) + \lambda_1(x - \xi_0)) \cdot (x - \xi_2)(x - \xi_3) + \\
& (\lambda_2(x - \xi_3) + \lambda_3(x - \xi_2)) \cdot (x - \xi_0)(x - \xi_1).
\end{aligned}$$

As can be seen, L_1 is the base case (leaf nodes) and L_2 (non-leaf nodes) form the rule for the recursive case. Then L_4 can be computed using these rules as seen on the binary tree representation.



The **generalized form**: Let the depths of the nodes in the binary tree starting from root be $i = 0, 1, .., k$ where $k = \log e$. We'll denote the nodes with binary string $\{0,1\}^i = \{\{\varepsilon\}, \{0,1\}, \{00, 01, 10, 11\}, ...\}$ for depths $i = 0, 1, 2, ...$ There are binary 2^i strings per at depth i, i.e. number of nodes at depth i.

Associate each **leaf node** $v \in \{0,1\}^k$

$$\begin{aligned} s_v =& x - \xi_v \\ l_v =& \lambda_v. \end{aligned}$$

In each **non-leaf** node $u = \{0,1\}^{k'}$ where $0 \leq k' < k$ the algorithm does the following computations

$$\begin{aligned} s_u =& s_{u0} \cdot s_{u1} \\ l_u =& l_{u0} \cdot s_{u1} + l_{u1} \cdot s_{u0}. \end{aligned}$$

The algorithm will terminate once the **root** node ε is reached. Then $l_\varepsilon = L(x)$. There is no need to calculate s_ε. Since $n = \deg s_u > \deg l_u$ for all $u \in \{0,1\}^k$ the total computational complexity in each node is $O(n \log n)$ using **fast polynomial multiplication**.

Total computational complexity from the multiplication operations can be calculated multiplying the number of nodes m_i with the complexity of multiplying polynomials with maximum degree of n_i at depth i and summing over the total depth of the binary tree $i = 0, 1, ..., k$. At depth i the binary tree has $m_i = 2^i$

nodes and polynomials have maximum degree of $n_i = 2^{\log e - i}$.

$$\begin{aligned}
&\sum_{i=0}^{\log e} m_i \cdot O(n_i \log n_i) \\
=&\sum_{i=0}^{\log e} 2^i \cdot O(2^{\log e - i} \log 2^{\log e - i}) \\
=&O\left(\sum_{i=0}^{\log e} 2^i \cdot 2^{\log e - i} \log 2^{\log e - i}\right) \\
=&O\left(\sum_{i=0}^{\log e} e(\log e - i)\right) \\
=&O\left(e \sum_{i=0}^{\log e} i\right) \\
=&O\left(e \cdot \frac{\log e(\log e + 1)}{2}\right) \\
=&O(e(\log e)^2).
\end{aligned}$$

(Not sure where the $\log\log e$ term should come from.)

Problem 4

Let R be a ring and let $\xi_0, \xi_1, ..., \xi_{e-1} \in R$ and $\eta_0, \eta_1, ..., \eta_{e-1} \in R$ such that $\xi_i - \xi_j$ is a unit in R for all $0 \leq i < j \leq e-1$. Shot that we can compute the coefficients of the Lagrange interpolation polynomial

$$L(x) = \sum_{i=0}^{e-1} \left(\eta_i \prod_{j=0 j \neq i}^{e-1} (\xi_i - \xi_j)^{-1}\right) \prod_{j=0 j \neq i}^{e-1} (x - \xi_j) \in R[x]$$

that satifies $L(\xi_i) = \eta_i$ for all $i = 0, 1, ..., e-1$ in $O(M(e)\log e)$ operations in R. You may assume that $e = 2^k$ for a nonnegative integer k.

1.

Let $f(x)$ be a polynomial

$$f(x) = \sum_{i=0}^{e-1} \lambda_i \prod_{j=0 j \neq i}^{e-1} (x - \xi_j),$$

where $\lambda_i = 1$ for all $i = 0, 1, ..., e-1$. Then it's coefficients can be calculated using the algorithm from problem 3 in $O(M(e)\log e)$ operations.

2.

Using batch evaluation on the polynomial f in points ξ_k where $k = 0, 1, ..., e-1$ we have

$$\begin{aligned} f(\xi_k) &= \sum_{i=0}^{e-1} \prod_{j=0, j\neq i}^{e-1} (\xi_k - \xi_j) \\ &= \prod_{j=0, j\neq k}^{e-1} (\xi_k - \xi_j). \end{aligned}$$

Batch evaluation can be done in $O(M(e)\log e)$ operations. The inverses of these values are the terms inside the Lagrange interpolation polynomial

$$\begin{aligned} f(\xi_k)^{-1} &= \left(\prod_{j=0, j\neq k}^{e-1} (\xi_k - \xi_j) \right)^{-1} \\ &= \prod_{j=0, j\neq k}^{e-1} (\xi_k - \xi_j)^{-1}. \end{aligned}$$

Computing the inverses can be done in $e \log e$ using fast multiplication.

3.

Using the results above, the Lagranges interpolation polynomial takes form

$$L(x) = \sum_{i=0}^{e-1} \lambda_i \prod_{j=0, j\neq i}^{e-1} (x - \xi_j)$$

where $\lambda_i = \eta_i f(\xi_i)^{-1}$ for all $i = 0, 1, ..., e-1$. Then it's coefficients can be calculated using the algorithm from problem 3 in $O(M(e)\log e)$ operations.

4.

Therefore the total amount of operations is $O(M(e)\log e)$.

References

Gathen, Joachim von zur, and Jurgen Gerhard. 2013. *Modern Computer Algebra.* 3rd ed. New York, NY, USA: Cambridge University Press.