Problem Set 5

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H5.1

We show that MAX CUT is **NP**-complete for simple graphs by reduction from NAESAT to MAX CUT. A simple graph does not have any multi-edges or self-loops.

We have NAESAT formula ϕ with n variables $\{x_1, ..., x_n\}$ and m clauses $\{C_1, ..., C_m\}$.

1) Literals x_i and $\neg x_i$ in ϕ cannot be in the same truth assignment.
2) Let $\alpha \vee \beta \vee \gamma$ be a clause in ϕ. Then, literals α, β, γ cannot all be in the same truth assignment.

We will construct a simple graph $G_\phi = (V, E)$ as follows.

We create the set of vertices V such that for each literal $x_1, ...x_n, \neg x_1, ..., \neg x_n$ we create $3m$ the literals, one for each possible location in the set of clauses.

$$V = \left\{ \begin{array}{c} x_{1,1}, ..., x_{1,3m}, \\ ..., \\ x_{n,1}, ..., x_{n,3m}, \\ \neg x_{1,1}, ..., \neg x_{1,3m}, \\ ..., \\ \neg x_{n,1}, ..., \neg x_{n,3m} \end{array} \right\}.$$

We create the set of edges E such that for each $j = 1, ..., m$ where jth clause is $C_j = \alpha_j \vee \beta_j \vee \gamma_j$ and

- $\alpha_j \in \{x_{i,3j-2}, \neg x_{i,3j-2} \mid i = 1, ..., n\}$,
- $\beta_j \in \{x_{i,3j-1}, \neg x_{i,3j-1} \mid i = 1, ..., n\}$,
- $\gamma_j \in \{x_{i,3j}, \neg x_{i,3j} \mid i = 1, ..., n\}$.

1) Add edges $(\alpha_j, \neg\alpha_j)$, $(\beta_j, \neg\beta_j)$ and $(\gamma_j, \neg\gamma_j)$ to enforce that a literal and its negation are not in the same truth assignment.
2) Add edges (α_j, β_j), (α_j, γ_j) and (β_j, γ_j) to enforce that not all literals are in the same truth assignment.

Now a cut $(S, V - S)$ of size $5m$ in G_ϕ corresponds to a truth assignment satisfying ϕ in the sense of NAESAT.

H5.2

We show that CLIQUE COVER is **NP**-complete by reduction from GRAPH COLORING to CLIQUE COVER.

Given simple graph $G = (V, E)$ its complement graph is $H = (V, E')$ where $E' = (V \times V) \setminus E$, that is, vertices are adjacent in H if an only if they are not adjacent in G.

Deciding graph coloring of at most K colors in G reduces to deciding clique cover of at most K cliques in H. Each clique corresponds to unique color, and all vertices in a clique have the same color.

Proof:

1) If $(u, v) \in E$, vertices u and v must have different colors in G. Then $(u, v) \notin E'$, vertices u and v must belong to different cliques in H.
2) If $(u, v) \notin E$, vertices u and v can have the same colors in G. Then $(u, v) \in E'$, vertices u and v can belong to the same clique in H.

H5.3

Set cover problem

- SET COVER
- INSTANCE: A family $F = \{S_1, ..., S_n\}$ of subsets of a finite set U and an integer B.
- QUESTION: Is there a subfamily of $\leq B$ sets in F whose union is U?

Formalization of the mention all problem.

- MENTION ALL
- INSTANCE: Families $F' = \{S'_1, ..., S'_n\}$ and $F'' = \{S''_1, ..., S''_n\}$ of subsets of a finite set U.

- QUESTION: Is there subfamily $\{S_1, ..., S_n\}$ where $S_i \in \{S'_i, S''_i\}$ for all $i = 1, ..., n$ whose union is U?

Generalization of MENTION ALL

- MENTION k-ALL
- INSTANCE: Families $F^1 = \{S^1_1, ..., S^1_n\}, ..., F^k = \{S^k_1, ..., S^k_n\}$ of subsets of a finite set U where $k \geq 2$ is integer.
- QUESTION: Is there subfamily $\{S_1, ..., S_n\}$ where $S_i \in \{S^1_i, ..., S^k_i\}$ for all $i = 1, ..., n$ whose union is U?

Reduction from SET COVER to MENTION k-ALL.

We can set $F^1 = ... = F^B = F$ to reduce *set cover* to *mention k-all*.

Reduction from MENTION k-ALL to MENTION ALL.

If the recursive step of *mention all*, we set the elements in the subset families as follows. For all $i = 1, ..., n$:

- $S'_i = S^1_1 \cup ... \cup S_i^{\lfloor k/2 \rfloor}$
- $S''_i = S_i^{\lfloor k/2 \rfloor + 1} \cup ... \cup S^k_i$

After the iteration we have solution $S_i \in \{S'_i, S''_i\}$ if it exists. Each recursive step splits the subsets forming the union.

We repeat the recursive step if solution exists until each subset S_i is of of the subsets $\{S^1_i, ..., S^k_i\}$. It takes $\log_2 k$ iterations.