Problem Set 10

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H10.1

Vertex Cover

- **Instance**: An undirected graph $G = (V, E)$ and an integer B.
- **Question**: Is there a subset $C \subseteq V$ with $|C| \leq B$ such that for every $(i, j) \in E$, either $i \in C$ or $j \in C$?

Given a graph, is there a set of vertices of size at most given threshold such each edge has atleast one end point in the set?

Directed Cycle Cover

- **Instance**: A directed graph $G = (V, E)$ and an integer K.
- **Question**: Is there a subset $F \subseteq E$ with $|F| \leq K$ such that any directed cycle in the graph G contains at least one arc (directed edge) from set F?

Given a directed graph, is there a set of arcs of size at most given threshold that need to be removed to make it acyclic?

Directed Cycle Cover is in **NP**. We can check the certificate in polynomial time as follows.

Proof: Given a directed graph $G = (V, E)$ and subset $F \subseteq E$ with $|F| \leq K$, check if graph $G' = (V, E \setminus F)$ is acyclic using depth first search which runs in polynomial time $O(|V| + |E \setminus F|) = O(|V| + |E|)$.

We show that Directed Cycle Cover is **NP**-hard by logspace reduction from Vertex Cover.

1) Let the undirected graph $G = (V, E)$ and integer B be an instance of vertex cover.

FIXME: solution is not correct

2) Then, we transform it into an instance of directed cycle cover $G' = (V, E')$ where for each edge $(i, j) \in E$, there we create a cycle by creating arcs $(i, j) \in E'$ and $(j, i) \in E'$. The tranformation can be computed in logspace.
3) Since the arcs form a cycle, Directed Cycle Cover is forced to choose one of them.
4) Let $F \subseteq E'$ with $|F| \leq B$ be a solution for directed cycle cover. Then, the solution for vertex cover is $C = \{i \mid (i, j) \in F\}$ with $|C| \leq B$.

Proof: F is directed cycle cover $\Leftrightarrow$ C is vertex cover

$\Rightarrow$:

$\Leftarrow$:

H10.2

H10.3