Problem Set 2

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H2.1

We have languages A and B in **NP**, which means that nondeterministic Turing machines can decide them in polynomial time. We will denote these Turing machines using N_A and N_B.

Union

Input $x \in A \cup B$ implies $x \in A$ or $x \in B$. We define nondeterministic Turing machine N as follows:

$N(x)$: if $x \in A$ return $N_A(x)$ else if $x \in B$ return $N_B(x)$

Then, for all inputs $x \in A \cup B$, x is decided by N in polynomial time. Thus language $A \cup B$ is in **NP**.

Intersection

Since $A \cap B \subseteq A$, the language $A \cap B$ is also decided by N_A in polynomial time. Therefore, it is in **NP**.

H2.2

Need to show that **NP** $\subseteq$ **PSPACE** and **PSPACE** $\subseteq$ **EXP**.

Nondeterministic Turing machine operating in time polynomial time n^k can write at most a string of length n^k, therefore, requiring at most polynomial n^k amount of space.

However, language decided by Turing machine in polynomial space can take more than polynomial time by nondeterministic Turing machine since we can reuse space.

Therefore we have **NP** $\subseteq$ **PSPACE**.

For some Turing machine operating in n^k space, there are $|\Sigma|^{n^k}$ possible strings. Therefore, the maximum amount of time it can take some Turing machine to decide L is exponential $2^{p(n)}$ for some polynomial $p(n)$, which means going through every string.

A Turing machine operating in exponential time can write strings that take more than polynomial space.

Therefore we have **PSPACE** $\subseteq$ **EXP**.

H2.3

Reduce language halting H to language A where

- $H = \{M; x \mid M \text{ halts on } x\}$
- $A = \{M \mid M \text{ accepts strings of length } |x| \leq 7\text{, and only those}\}$.

We use similar proof as the example of reducing halting to total:

Given input $M; x$, consider a machine M_x that works as follows:

$M_x(y)$: **if** $|y| \leq 7$ and $M(x)$ halts **then** *yes* **else** $\nearrow$.

Define a reduction mapping $t(M : x) = M_x$, which hard codes the input x to the machine code of M resulting in new code.

Now $M; x \in H$ iff M halts on input x iff M_x accepts string of length ≤ 7 iff $M_x \in A$.