Question 6 - Fake Coin Detection

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There are n gold coins, one of which is a fake. All gold coins have the same weight, and the fake coin has different weight. You are allowed to use a balance whether two subsets of coins have equal weights. Describe a strategy that uses $O(\log n)$ measurements of balance and find a fake coin.

This algorithms assumes that $n \geq 3$. In the case $n \in \{1, 2\}$ it is impossible to determine the fake coin using a balance. The recursion starts with $n_1 = n$ coins.

1) Split the stack of n_i coins into 3 *stacks* of equal size. The stacks will be of size $\lfloor n_i/3 \rfloor$. If the number of coins is not divisible by 3, there will be a *leftover stack* of 1 or 2 coins ($n \mod 3$).
2) Lets denote the weights of the three stacks by w_1, w_2 and w_3. In order to figure out which stack has the fake coin, we need to compare each of the stack with each other using the balance. There are four different scenarios:
   I) If $w_1 \neq w_2 \wedge w_1 \neq w_3 \wedge w_2 = w_3$ then the fake coin is in the *stack 1.*
   II) If $w_1 \neq w_2 \wedge w_1 = w_3 \wedge w_2 \neq w_3$ then the fake coin is in the *stack 2.*
   III) If $w_1 = w_2 \wedge w_1 \neq w_3 \wedge w_2 \neq w_3$ then the fake coin is in the *stack 3.*
   IV) If $w_1 = w_2 \wedge w_1 = w_3 \wedge w_2 = w_3$ then the fake coin is in the *leftover stack.*
3) Now we have scenarios:
   I) If fhe fake coin is in the *stack 1, 2 or 3.*
      a) If the size of the stack is 1 then this is the fake coin.
      b) Otherwise repeat the procedure from step 1 using the stack that has the fake coin.
   II) Otherwise the fake coin is in the *leftover stack.*
      a) If there is 1 coin in the leftover stack, this is the fake coin.
      b) If there is 2 coins in the leftover stack, take one coin from any of the stacks 1, 2 or 3, and compare the leftover coins against it. The leftover coin that has unequal weight compared to this coin is the fake coin.