Question 2 - Complexity of Recurrence Relation

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2018-09-30

Let $T(n)$ be a functions such that

$$T(n) \leq T(n/5) + T(3n/10) + O(n)$$

and $T(j) = 1$ for all $j < 10$. Prove that $T(n) = O(n)$.

From the *Introduction to algorithms* (1), chapter 4.3 *The substitution method for solving recurrences.*

$T(n) = O(n)$ if and only if there exists a positive real number c and a real number n_0 such that $T(n) \leq cn$ for all $n \geq n_0$. If we assume that $T(n) = O(n)$ is true then there exists positive real numbers c_1 c_2 and c_3 such that

$$\begin{aligned} T(n) &\leq T(n/5) + T(3n/10) + O(n) \\ &\leq c_1(n/5) + c_2(3n/10) + c_3 n \\ &= (1/5c_1 + 3/10c_2 + c_3)n \\ &\leq cn. \end{aligned}$$

We have substituted the inequalities

$$T(n/5) \leq c_1(n/5)$$

and

$$T(3n/10) \leq c_2(3n/10)$$

and the constant c is chosen such that

$$c \geq (1/5c_1 + 3/10c_2 + c_3).$$

We need to also check the base case for the inductive proof.

$$T(1) = 1 \leq c \cdot 1$$

For which c can be chosen to be larger than 1 for the claim to hold true.

References

1. Cormen TH, Leiserson CE, Rivest RL, Stein C. *Introduction to algorithms.* MIT press (2009).