Question 3 - Growth Bounds of Fibonacci Sequence

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The Fibonacci numbers $F_0, F_1, F_2, ...$ are defined by the rule

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, n \geq 2.$$

In this problem we will confirm that this sequence grows exponentially fast and obtain some bounds on its growth.

1) Use induction to prove that $F_n \geq 2^{0.5n}$ for $n \geq 6$.
2) Find a constant $c < 1$ such that $F_n \leq 2^{cn}$ for all positive integer n. Show that your answer is correct.
3) What is the largest c you can find for which $F_n = \Omega(2^{cn})$

Lets prove that $F_n \geq 2^{0.5n}$ for $n \geq 6$ using induction.

1) Base case $n = 6$:

$$F_6 = 8 \geq 2^{0.5 \cdot 6}$$

2) Step case n:

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &\geq 2^{0.5 \cdot (n-1)} + 2^{0.5 \cdot (n-2)} \\ &= 2^{0.5n} \cdot (2^{0.5 \cdot (-1)} + 2^{0.5 \cdot (-2)}) \\ &\geq 2^{0.5n} \end{aligned}$$

3) Step case $n + 1$:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\geq 2^{0.5n} + 2^{0.5 \cdot (n-1)} \\ &= 2^{0.5n} \cdot (1 + 2^{0.5 \cdot (-1)}) \\ &\geq 2^{0.5 \cdot n} \cdot 2^{0.5} \\ &= 2^{0.5 \cdot (n+1)} \end{aligned}$$

Lets assume that for some positive real numbers α and c and a real number n_0:

$$F_n \leq c\alpha^n \, \text{for all} \, n \geq n_0.$$

Using the Fibonacci recurrence relation we can now form new inequality

$$c\alpha^{n-1} + c\alpha^{n-2} \leq c\alpha^n.$$

Divided by $c\alpha^{n-2} > 0$ we obtain

$$\alpha + 1 \leq \alpha^2 \alpha^2 - \alpha - 1 \geq 0$$

By solving the positive roots of the second order equation

$$\alpha \geq \frac{1+\sqrt{5}}{2}.$$

The base case $F_1 = 1 \leq c\alpha^1$ is true for example by choosing $c = 1$ therefore the induction holds true. (1)

Similar analysis can be done from below $F_n \geq c\alpha^n$ which will obtain

$$\alpha \leq \frac{1+\sqrt{5}}{2}.$$

The base case $F_1 = 1 \geq c\alpha^1$ is true for example by choosing $c = 1/3$ therefore the induction holds true.

These two case give the Fibonacci sequence tight bound

$$F_n = \Omega(\varphi^n)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The answer to questions 2 and 3:

$$(2^c)^n = \varphi^n \Rightarrow c = \log_2 \varphi$$

References

1. ShreevatsaR. Prove upper bound (big o) for fibonacci's sequence? (2014) Available at: https://math.stackexchange.com/q/674766 [Accessed October 1, 2018]