Baltic Way 2019 training: functional equations

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1. *(BW 2010)* Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2) + f(xy) = f(x)f(y) + yf(x) + xf(x+y)$$
for all $x, y \in \mathbb{R}$.

2. *(BW 2012)* Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(x+y) = f(x-y) + f(f(1-xy))$$
holds for all real numbers x and y.

3. *(Hong Kong 2020 TST 2 Q4)* Find all real-valued functions f defined on the set of real numbers such that
$$f(f(x)+y) + f(x+f(y)) = 2f(xf(y))$$
for any real numbers x and y.

4. *(SL 2011)* Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy
$$g(f(x+y)) = f(x) + (2x+y)g(y)$$
for all real numbers x and y.

5. *(BW 2011)* Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that, for all integers x and y, the following holds:
$$f(f(x) - y) = f(y) - f(f(x))$$
Show that f is bounded.

6. *(Tournament of Towns 1983)* Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(f(n)) = 3n$ for all natural numbers n. Find $f(2001)$.

7. *(APMO 2011)* Determine all functions $f : \mathbb{R} \to \mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:
   (a) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.
   (b) For every pair of real numbers x and y, $f(xf(y)) + yf(x) = xf(y) + f(xy)$ is satisfied.

8. *(SL 2013)* Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions
$$f(x)f(y) \geq f(xy) \quad \text{and} \quad f(x+y) \geq f(x) + f(y)$$
for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.