Lorentz Gauge $\quad \partial_\mu A^\mu = 0$

$$\partial_\mu \partial^\mu A^\nu = 0 \qquad (\ddagger)$$

Construct a theory where $(\ddagger)$ arises from $\mathcal{L}$ directly.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + \partial^\nu(\partial_\mu A^\mu) = 0 \quad \Rightarrow \quad \partial_\mu \partial^\mu A^\nu = 0$$

Work with new $\mathcal{L}$, impose $\partial_\mu A^\mu = 0$ later at the op level.

[In general, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$ $\alpha \in \mathbb{R}$ const $\quad$ $\alpha = 1$: Feynman gauge, $\alpha = 0$: Landau gauge]

Our new theory has no gauge symmetry. But now both A_0 and $\underline{A}$ are dynamical:

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu \quad , \quad \pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial^i A^0 - \dot{A}^i$$

Now apply the comm relations

$$[A_\mu(\underline{x}), A_\nu(\underline{y})] = [\pi^\mu(\underline{x}), \pi^\nu(\underline{y})] = 0$$

$$[A_\mu(\underline{x}), \pi^\nu(\underline{y})] = i\delta^3(\underline{x}-\underline{y})\, \delta_\mu^{\ \nu}$$

$$\Rightarrow [A_\mu(\underline{x}), \pi_\nu(\underline{y})] = i\delta^3(\underline{x}-\underline{y})\, \eta_{\mu\nu}$$

[In Heisenberg picture, using $\pi^\mu = -\dot{A}^\mu + \ldots$ this becomes $[A_\mu(\underline{x},t), \dot{A}_\nu(\underline{y},t)] = -i\eta_{\mu\nu}\delta^3(\underline{x}-\underline{y})$

Using the familiar expansion

$$A_\mu(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\underline{p}|}} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(\underline{p}) \left[a_{\underline{p}}^\lambda e^{i\underline{p}\cdot\underline{x}} + a_{\underline{p}}^{\lambda\dagger} e^{-i\underline{p}\cdot\underline{x}} \right]$$

$$\pi^\nu(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} (i) \sqrt{\frac{|\underline{p}|}{2}} \sum_{\lambda=0}^{3} (\varepsilon^{(\lambda)}(\underline{p}))^\nu \left[a_{\underline{p}}^\lambda e^{i\underline{p}\cdot\underline{x}} - a_{\underline{p}}^{\lambda\dagger} e^{-i\underline{p}\cdot\underline{x}} \right]$$

(+ rather than −, so that $\pi^\mu = -\dot{A}^\mu + \ldots$ in H-picture)

4 polarization vectors $\varepsilon^{(\lambda=0,1,2,3)}$. Pick ε^0 to be timelike and $\varepsilon^{1,2,3}$ to be spacelike.
Choose normalization $\varepsilon^{(\lambda)}_\mu \varepsilon^{(\lambda')\mu} = \eta^{\lambda\lambda'}$ and ε^3 is chosen to be longitudinal, whereas $\varepsilon^{1,2}$ are transverse, i.e. $\varepsilon^{1,2}_\mu p^\mu = 0$.

For example, when $p^\mu \propto (1,0,0,1)$: $\varepsilon^{(0)}_\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\varepsilon^{(1)}_\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\varepsilon^{(2)}_\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\varepsilon^{(3)}_\mu = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

For other momenta, get $\varepsilon^{(\lambda)}$ with a suitable Lorentz boost $\Lambda^\nu{}_\mu$

$$[A_\mu(\underline{x}), \pi_\nu(\underline{y})] = i\eta_{\mu\nu}\delta^3(\underline{x}-\underline{y}) \iff [a_{\underline{p}}^\lambda, a_{\underline{q}}^{\lambda'}] = [a_{\underline{p}}^{\lambda\dagger}, a_{\underline{q}}^{\lambda'\dagger}] = 0$$

$$[a_{\underline{p}}^\lambda, a_{\underline{q}}^{\lambda'\dagger}] = -\eta^{\lambda\lambda'}(2\pi)^3\delta^3(\underline{p}-\underline{q})$$

For $\lambda = 1,2,3$, we have the familiar comm relations with a + sign.

Def^n Lorentz inv't vacuum $|0\rangle$ s.t. $a_{\underline{p}}^\lambda |0\rangle = 0$.

One part^c states $|p,\lambda\rangle = a_p^{\lambda\dagger}|0\rangle$. For $\lambda = 1,2,3$ this makes sense, but for $\lambda = 0$, get states with -ve norm:

$$\langle p,\lambda=0|q,\lambda=0\rangle = \langle 0|a_p^0 a_q^{0\dagger}|0\rangle = -(2\pi)^3\delta^3(p-q) \quad \text{WTF?}$$

Hilbert space with -ve norm => -ve probabilities.

Constraint eq $\partial_\mu A^\mu = 0$ reduces # of polarisation states:

① $\partial_\mu A^\mu = 0$ is required but does not work, as $\pi^0 = -\partial_\mu A^\mu$ so comm relations could not be obeyed.

② Impose the condition on the Hilbert space rather than the ops.
Split the Hilbert space into good states and bad states that somehow decouple (the -ve norm states).
Physical states: maybe $\partial_\mu A^\mu|\psi\rangle = 0$ – (*) on all physical states, but this is too strong

Decompose $A_\mu(\underline{x}) = A_\mu^+(\underline{x}) + A_\mu^-(\underline{x})$ with

$$A_\mu^+(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|p|}}\,\varepsilon_\mu^{(\lambda)}\,a_p^\lambda\,e^{-ip\cdot x}$$

$$A_\mu^-(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|p|}}\,\varepsilon_\mu^{(\lambda)}\,a_p^{\lambda\dagger}\,e^{ip\cdot x}$$

Then $A_\mu^+|0\rangle = 0$ but $\partial A_\mu^-|0\rangle \neq 0$ so not even vacuum $|0\rangle$ is physical.

③ Physical states $\partial^\mu A_\mu^+(\underline{x})|\psi\rangle = 0$ – (†)

$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0$ so $\partial_\mu A^\mu$ has vanishing matrix elements between physical states

(†) is known as the Gupta – Bleuler condition. The linearity of (†) means that physical states span Hilbert space $\mathcal{H}_{phys}$.

$\mathcal{H}_{phys}$: decompose $|\psi\rangle$ in the Fock space into

$|\psi\rangle = |\psi_T\rangle|\phi\rangle$ — $|\psi_T\rangle$ transverse photons, created by $a_k^{1,2\dagger}$; $|\phi\rangle$ timelike/longitudinal photons, created by $a_k^{0/3\dagger}$

$\partial_\mu A^{\mu+}|\phi\rangle = 0$ becomes $(a_k^3 - a_k^0)|\phi\rangle = 0$ – (**)

In general, $|\phi\rangle$ will be linear combinations of states containing combinations of long. + timelike photons.

$$|\phi\rangle = \sum_{n=0}^{\infty}|\phi_n\rangle \leftarrow \text{contains } n \text{ t-like or long. } \gamma\text{'s, each obeys (**)}$$