Local potential approximation

$$S[\varphi] = \int d^dx \ \tfrac{1}{2}(\partial\varphi)^2 + V(\varphi) \qquad \text{where} \quad V(\varphi) = \sum_{k\geq 1} \frac{\Lambda^{d-k(d-2)}}{(2k)!}\varphi^{2k}$$

$$\varphi = \phi + \chi \qquad [\Lambda - \delta\Lambda, \Lambda]$$

The action at scale Λ becomes

$$S[\phi+\chi] = S[\phi] + \int d^dx \ \tfrac{1}{2}(\partial\chi)^2 + \tfrac{1}{2}\chi^2 V''(\phi) + \tfrac{1}{3!}\chi^3 V'''(\phi)$$

where we've chosen ϕ s.t. $V'(\phi) = 0$

Path integral over range $[\Lambda - \delta\Lambda, \Lambda]$. Each loop integral for χ has the form

- $$\int d^dp \, (\quad) \simeq \Lambda^{d-1}\delta\Lambda \int_{S^{d-1}} d\Omega \, (\quad)$$

  $\uparrow$ denotes integral over a unit $(d-1)$ sphere in momentum space

Since each χ-loop integral comes with a factor of $\delta\Omega$, to leading order, we only need to worry about 1-loop diagrams.

A connected graph with E edges, L loops, and V_i vertices of χ-valency i (arbitrarily many ϕ) obeys Euler's identity

$$L - 1 = E - \sum_{i\in\mathbb{Z}^+} V_i$$

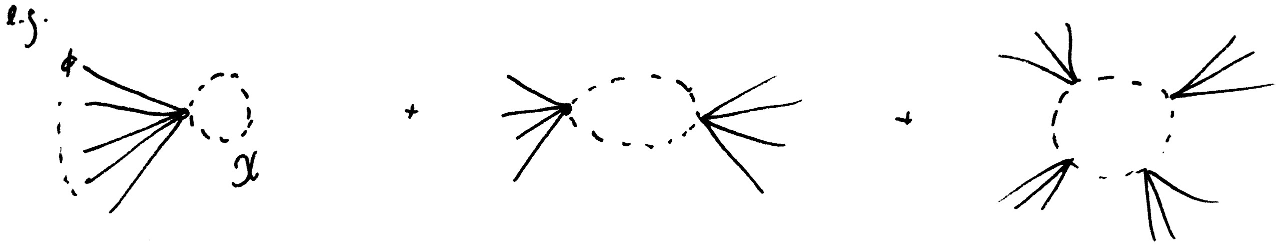
- Also, every edge contributes to 2 vertices, and each vertex of type i has i χ-lines, so

$$2E = \sum_i iV_i$$

Therefore
$$L = 1 + \underbrace{\sum_{i=2}^{\infty} \frac{(i-2)V_i}{2}}_{\geq 0 \text{ with equality iff } V_i = 0 \ \forall i\geq 3}$$

Hence for 1 loop diagrams, only need to consider vertices with 2 χ lines attached.

e.g.



We can thus truncate

$$S[\phi+\chi] = S[\phi] + \int d^dx \left[\tfrac{1}{2}(\partial\chi)^2 + \tfrac{1}{2}\chi^2 V''(\phi)\right] \quad \text{to 1-loop}$$

If we make the temporary simplifying assumption that ϕ is const, then in momentum space

$$S[\phi+\chi] - S[\phi] = \int_{\Lambda-\delta\Lambda \le |p| \le \Lambda} \frac{d^dp}{2(2\pi)^d}\, \hat{\chi}(-p)\left[p^2 + V''(\phi)\right]\hat{\chi}(p)$$

$$= \frac{\Lambda^{d-1}\delta\Lambda}{(2\pi)^d}\left[\Lambda^2 + V''(\phi)\right]\int_{S^{d-1}} d\Omega\, \hat{\chi}(-\Lambda\hat{p})\,\hat{\chi}(\Lambda\hat{p})$$

The χ path integral is finite if we work on a compact space, say T^d with side length L then $p_\mu = \frac{2\pi}{L}n_\mu$ and the gaussian integral over the $\hat{\chi}$ modes gives

$$e^{-\delta_\Lambda S} = \int \mathcal{D}\chi\, e^{-(S[\phi+\chi] - S[\phi])}$$

$$= C\left(\frac{\pi}{\Lambda^2 + V''(\phi)}\right)^{N/2}$$

← # of $\hat{\chi}$ modes in our shell of radius Λ thickness

$$\frac{N}{2} = \frac{1}{2}\underbrace{\mathrm{Vol}(S^{d-1})\,\Lambda^{d-1}}_{\text{area of the shell}}\underbrace{\delta\Lambda}_{\text{thickness}}\underbrace{\left(\frac{L}{2\pi}\right)^d}_{\text{volume of quantized momentum}} = a\,\Lambda^{d-1}\delta\Lambda\, L^d$$

where $a = \dfrac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d} = \dfrac{1}{(4\pi)^{d/2}\,\Gamma(d/2)}$

Integrating out χ leads to a change in the effective action

$$\frac{\delta_\Lambda S^{eff}}{\delta\Lambda} = a\log\left[\Lambda^2 + V''(\phi)\right]\Lambda^{d-1}L^d$$

This diverges as $L \to \infty$; the divergence can be traced to our simplifying assumption that ϕ is everywhere const.

More generally, we have

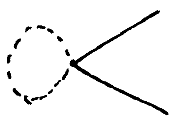
$$\frac{\delta S^{eff}}{\delta\Lambda}[\phi] = a\,\Lambda^{d-1}\int d^dx\,\log\left(\Lambda^2 + V''(\phi)\right)$$

The χ integral has modified the potential seen by ϕ. We have

$$\Lambda\frac{dg_{2k}}{d\Lambda} = \left[k(d-2)-d\right]g_{2k} - a\Lambda^{k(d-2)}\underbrace{\frac{\partial^{2k}}{\partial\phi^{2k}}\ln\left[\Lambda^2 + V''(\phi)\right]}_{\text{extract the coefficient of } \phi^{2k}}\Bigg|_{\phi=0}$$

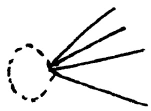
e.g. $k=1$

$$\Lambda\frac{dg_2}{d\Lambda} = -2g_2 - \frac{ag_4}{1+g_2}$$



e.g. $k=2$

$$\Lambda\frac{dg_4}{d\Lambda} = (d-4)g_4 - \underset{1}{\frac{ag_4}{1+g_2}} + \underset{2}{\frac{3ag_4^2}{(1+g_2)^2}}$$



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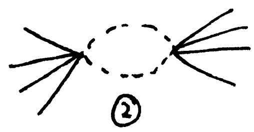

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e.g. $k=3$

$$\Lambda\frac{dg_6}{d\Lambda} = (2d-6)g_6 - \frac{ag_8}{1+g_2} + \frac{15ag_4g_6}{(1+g_2)^2} + \frac{30ag_4^2}{(1+g_2)^3}$$

Rmks

1) The 1^{st} term on RHS is the classical behaviour of the dimensionless coupling – nothing to do with cutoff. The remaining terms are quantum corrections ($\propto \hbar$) and each comes from specific Feynman diagrams

2) $g_2 = \frac{m^2}{\Lambda^2}$ is the mass of Φ in units of Λ. This is a relevant coupling (perturbatively), so increases at $\Lambda \to 0$. Consequently at scale $\Lambda \ll m$ the quantum corrections to theses β-f^{ns} are strongly suppressed

The Gaussian critical point

There is a (trivial) critical point with $g_{2k}^* = 0$ $\forall k \geq 2$ since there are no vertices.

In a neighbourhood of this c.p. we can expand the beta functions to the lowest order in the couplings. $\delta g = g - g^*$. We have

$$\beta_{2k} = \Lambda \frac{\partial g_{2k}}{\partial \Lambda} = [k(d-2) - d] g_{2k} - a g_{2k+2}$$

Thus the matrix B_{ij} is upper triangular, hence its eigenvalues are just

$$k(d-2) - d = \underbrace{2k - 4}_{\text{for } d=4}$$

Vertices ϕ^{2k} with $k \geq 3$ are thus irrelevant. $(2k-4>0) \to$ if we turn them on at some scale, they become irrelevent in IR.

The mass term g_2 is relevent, so even a small mass becomes increasingly important in the IR.

The interesting term g_4 is marginal to leading order. To next non-trivial order, we have

$$\Lambda \frac{dg_4}{d\Lambda} = 2a g_4^2 + O(g_4^2 g_2) \quad \text{, neglecting } g_6 \ldots$$

Thus to this order

$$\frac{1}{g_4} = C - \frac{3}{16\pi^2} \ln \Lambda$$

since want $g_4 > 0$

$$\Rightarrow g_4 = \frac{16\pi^2}{3} \Big/ \ln\left(\frac{\mu}{\Lambda}\right) \quad \text{for some scale } \mu > \Lambda$$

This g_4 runs only logarithmically in Λ and is marginally irrelevant