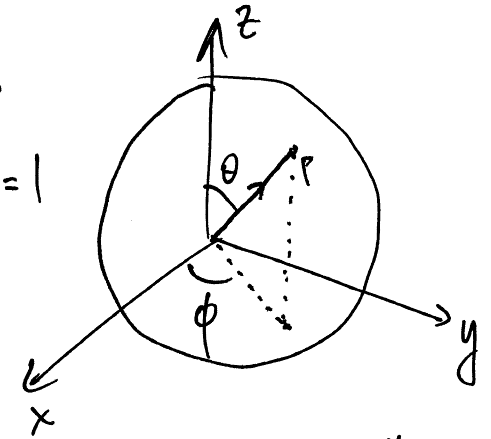
Manifolds

$p \in S_1^2$

$S^2 \in \mathbb{R}^3$

$x^2+y^2+z^2=1$

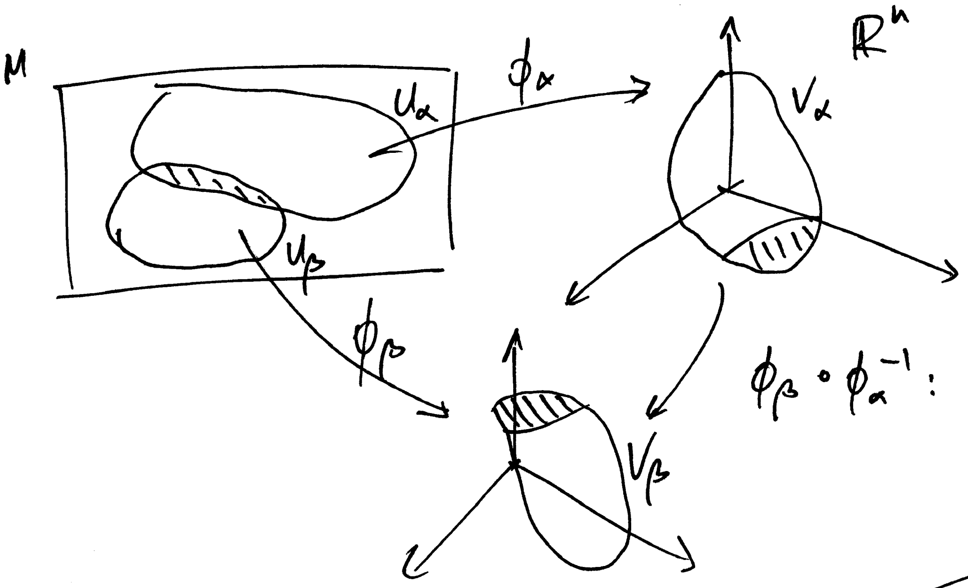


(θ, ϕ) polar coordinates S_1^2.

Not valid if $\theta = 0$ or π (what would ϕ be?)

Defn An n-dimensional smooth manifold M is a set with a collection of open sets U_α, $\alpha = 1, 2, \ldots$ called charts such that

- U_α cover M



$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta)$
$\longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$

There exist maps

$\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$

for each α such that are smooth maps from $\mathbb{R}^n \to \mathbb{R}^n$.

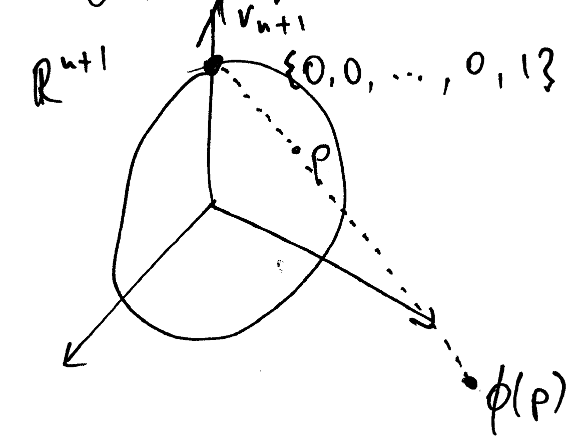
So manifolds are topological spaces with an additional structure which allows local calculus.

Examples 1. $\mathbb{R}^n$. Trivial mfd with only one open set

2. $S^n \subset \mathbb{R}^{n+1}$, $\{ \underline{r} \in \mathbb{R}^{n+1} : |\underline{r}| = 1 \}$

Define open sets U_1, U_2. $U_1 = S^n / \{0, 0, \ldots, 0, 1\}$
$U_2 = S^n / \{0, 0, \ldots, 0, -1\}$

Stereographic projection



On U_1, $\phi_1 : U_1 \to \mathbb{R}^n$

$$\phi_1(\underline{v}) = \left(\frac{v_1}{1-v_{n+1}}, \frac{v_2}{1-v_{n+1}}, \dots, \frac{v_n}{1-v_{n+1}} \right)$$

$$= (x_1, x_2, \dots, x_n)$$

On U_2, $\phi_2 : U_2 \to \mathbb{R}^n$

$$\phi_2(\underline{v}) = \left(\frac{v_1}{1+v_{n+1}}, \frac{v_2}{1+v_{n+1}}, \dots, \frac{v_n}{1+v_{n+1}} \right)$$

$$= (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$$

$$\underbrace{\frac{v_k}{1+v_{n+1}}}_{\tilde{x}_k} = \frac{1-v_{n+1}}{1+v_{n+1}} \cdot \underbrace{\frac{v_k}{1-v_{n+1}}}_{x_k}, \quad k = 1, \dots, n$$

$$x_1^2 + x_2^2 + \dots + x_n^2 = \frac{v_1^2 + v_2^2 + \dots + v_n^2}{(1-v_{n+1})^2} = \frac{1-v_{n+1}^2}{(1-v_{n+1})^2} = \frac{1+v_{n+1}}{1-v_{n+1}}$$

$\tilde{x}_k = \dfrac{x_k}{x_1^2 + x_2^2 + \dots + x_n^2}$, so $\phi_2 \circ \phi_1^{-1}(x_1, \dots, x_n) = \left(\dfrac{x_1}{x_1^2 + \dots + x_n^2}, \dots, \dfrac{x_n}{x_1^2 + \dots + x_n^2} \right)$

smooth maps on $\phi_1(U_1 \cap U_2)$

3. Cartesian product of two manifolds is a manifold.

4. <u>Surface</u> $f_1, f_2, \dots, f_k : \mathbb{R}^N \to \mathbb{R}$

$$M_f = \{ \underline{v} \in \mathbb{R}^N : f_1(\underline{v}) = f_2(\underline{v}) = \dots = f_k(\underline{v}) = 0 \}.$$

is a manifold of dimension $N-k$ if the matrix

$$M_{i\alpha} = \frac{\partial f_i}{\partial x^\alpha} \text{ has maximal rank.}$$

e.g. $S^{N-1} \subset \mathbb{R}^N$, $f_1 = 1 - |\underline{v}|^2$

Theorem (Whitney)

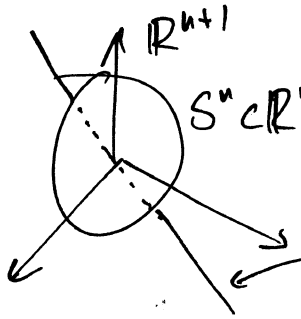
Every n-dim^n smooth manifold M can be embedded as a surface in $\mathbb{R}^N$ where $N \leq 2n+1$

5. Real projective space $\mathbb{R}P^n = \overset{\mathbb{R}^{n+1}/\{0\}}{\mathbb{R}^{n+1}}/\sim$ (n-dim mfd)

$$\underbrace{[X_1, X_2, \ldots, X_{n+1}]}_{\text{homogeneous coordinates}} \sim c[X_1, X_2, \ldots, X_{n+1}], \quad c \in \mathbb{R}^*$$

$$U_\alpha = \{[X] \in \mathbb{R}^{n+1} : X_\alpha \neq 0\}, \quad \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n, \quad \phi_\alpha[X] = \left(\frac{X_1}{X_\alpha}, \ldots, \frac{X_{\alpha-1}}{X_\alpha}, \frac{X_{\alpha+1}}{X_\alpha}, \ldots, \frac{X_{n+1}}{X_\alpha}\right)$$

<u>Exercise</u> check that $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth.



<u>Def</u> A map between smooth manifolds

$$f : M \to \tilde{M}, \quad \dim(M) = n, \quad \dim(\tilde{M}) = \tilde{n}$$

is called smooth if $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^{\tilde{n}}$.