Weight of $sl(2)_\alpha$

$$R(h^\alpha)v = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v \quad \Rightarrow \quad \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \quad \forall \lambda \in S_R \quad \alpha \in \Phi$$

$$h^\alpha = \frac{2}{(\alpha,\alpha)} H^\alpha = \frac{2}{(\alpha,\alpha)} (K^{-1})_{ij} \alpha^i H^j$$

$$R(h^\alpha)_v = \frac{2}{(\alpha,\alpha)} (K^{-1})_{ij} \alpha^i R(H^j) v = \frac{2}{(\alpha,\alpha)} (K^{-1})_{ij} \alpha^i \lambda^j v = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v \quad \forall v \in V_\lambda$$

Weight of $R_\alpha \in \mathbb{Z}$

$$\Rightarrow \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \begin{matrix} \forall \lambda \in S_R \\ \alpha \in \Phi \end{matrix} \quad (*)$$

<u>Root and Weight lattice</u>

From iv) of simple roots

$$\beta \in \Phi \Rightarrow \beta = \sum_{i=1}^{r} \beta^i \alpha_{(i)} \qquad \beta^i \in \mathbb{Z} \quad i=1,\dots,r$$

Hence all roots lie in <u>root lattice</u>

$$\mathcal{L}[g] := \operatorname{span}_{\mathbb{Z}} \{ \alpha_{(i)} \quad i=1,\dots,r \}$$

simple co-roots

$$\check{\alpha}_{(i)} = \frac{2\alpha(i)}{(\alpha_{(i)},\alpha_{(i)})} \qquad i=1,\dots,r$$

coroot lattice

$$\check{\mathcal{L}}[g] = \operatorname{span}_{\mathbb{Z}} \{ \check{\alpha}_{(i)} \quad i=1,\dots,r \}$$

The <u>weight lattice</u> $\mathcal{L}_W[g]$ is <u>dual</u> to co-root lattice

$$\mathcal{L}_W[g] = \check{\mathcal{L}}^*[g] = \{ \lambda \in h_{\mathbb{R}}^* : (\lambda,\mu) \in \mathbb{Z} \quad \forall \mu \in \check{\mathcal{L}}[g] \}$$

$$\lambda \in \mathcal{L}_W[g] \Rightarrow (\lambda, \check{\alpha}_{(i)}) = \frac{2(\lambda,\alpha_{(i)})}{(\alpha_{(i)},\alpha_{(i)})} \in \mathbb{Z}$$

$(*) \Rightarrow$ All weights $\lambda \in S_R$ are in $\mathcal{L}_W[g]$

Given the basis

$B = \{ \check{\alpha}_{(i)} \quad i=1,\dots,r \}$ for $\check{\mathcal{L}}[g]$

define dual basis $B^* = \{ w_{(i)}, \quad i=1,\dots,r \}$ for $\mathcal{L}_W[g]$ by

$$(\check{\alpha}_{(i)}, w_{(j)}) = \frac{2(\alpha_{(i)}, w_{(j)})}{(\alpha_{(i)},\alpha_{(i)})} = \delta_{ij} \quad (+)$$ <u>$w_{(i)}$ are fundamental weights</u>

As simple roots span $h_{\mathbb{R}}^{*}$, can write

$$w_{(i)} = \sum_{j=1}^{r} B_{ij}\, \alpha_{(j)} \qquad B_{ij} \in \mathbb{R} \qquad i,j = 1, \dots, r$$

In (†),

$$\sum_{k=1}^{r} \frac{2(\alpha_{(i)}, \alpha_{(k)})}{(\alpha_{(i)}, \alpha_{(i)})} B_{jk} = \delta_j^{\,i} \quad \Rightarrow \quad \sum_{k=1}^{r} B_{jk} A^{ki} = \delta_j^{\,i}$$

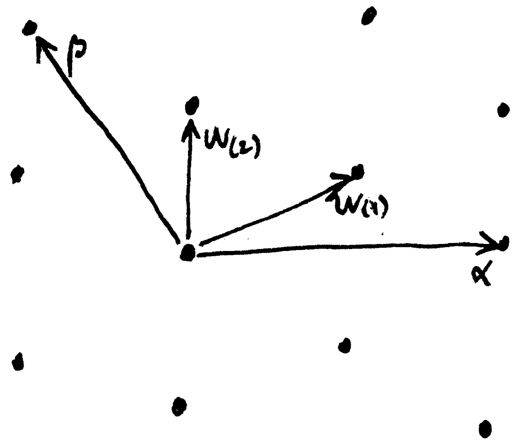
Cartan matrix

Equivalently,

$$\alpha_{(i)} = \sum_{j=1}^{r} A^{ij} w_{(j)}$$

<u>Example</u> $g = A_2$ $\qquad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$\alpha = \alpha_{(1)} = 2w_{(1)} - w_{(2)}$$
$$\beta = \alpha_{(2)} = -w_{(1)} + 2w_{(2)}$$
$$\Rightarrow \quad w_{(1)} = \tfrac{1}{3}(2\alpha + \beta)$$
$$w_{(2)} = \tfrac{1}{3}(\alpha + 2\beta)$$



for any weight
$\lambda \in S_R \subset \mathcal{L}_W[g]$
$\Rightarrow \lambda = \sum_{i=1}^{r} \lambda^i w_{(i)} \qquad \lambda^i \in \mathbb{Z} \qquad i = 1, \dots, r$

$\{\lambda^i\}$ Dynkin labels of weight λ

<u>Highest weight repns</u>

Every f.d. irrep R of g has a highest weight

$$\Lambda = \sum_{i=1}^{r} \Lambda^i w_{(i)} \in S_R \qquad \Lambda^i \in \mathbb{Z} \qquad \Lambda^i \geq 0$$

Eigenvector

$$v_\Lambda \in V, \quad R(h^i)\, v_\Lambda = \Lambda^i v_\Lambda \qquad i = 1, \dots, r$$

is annihilated by

$$R(E^\alpha)\, v_\Lambda = 0 \qquad \forall\, \alpha \in \Phi_+$$

integers Λ^i are Dynkin labels of R.

Remaining weights $\lambda \in S_R$ are generated by acting with lowering operators

$$R(E^{-\alpha}) \qquad \alpha \in \Phi_+$$

$v \in V_\lambda$ then

$$R(E^\alpha)\, v \in \begin{cases} V_{\lambda+\alpha} & \text{if } \lambda+\alpha \in S_R \\ 0 & \text{otherwise} \end{cases}$$

Useful result:

For any f.d. repn of $\mathfrak{g}$

if $\lambda = \sum_{i=1}^{r} \lambda^i w_{(i)} \in S_R$

$\Rightarrow \lambda - m_{(i)}\alpha_{(i)} \in S_R$ for $m_{(i)} \in \mathbb{Z}$ $\quad 0 \le m_{(i)} \le \lambda_i$

Process terminates when all Dynkin labels are negative.

— $\mathfrak{g} = A_2$ fundamental repn R_f has dynkin labels $(1, 0)$

$\Rightarrow \Lambda = w_{(1)}$

- $\Lambda = w_{(1)} \in S_f \Rightarrow \Lambda - \alpha_{(1)} = w_{(1)} - (2w_{(1)} - w_{(2)}) = -w_{(1)} + w_{(2)} \in S_f$
- $\lambda = -w_{(1)} + w_{(2)} \in S_f \Rightarrow \lambda - \alpha_{(2)} = -w_{(1)} + w_{(2)} - (2w_{(2)} - w_{(1)}) = -w_{(2)} \in S_f$

$S_f = \{ w_{(1)}, -w_{(1)} + w_{(2)}, -w_{(2)} \}$.