# Topics in Convex Optimisation

## 1.1

1. $\mathrm{conv}(S) = \{ x \in \mathbb{R}^n : \exists k \in \mathbb{N}_{\geq 1}, \lambda_1, \dots, \lambda_k \geq 0, s_1, \dots, s_k \in S$ s.t. $x = \sum_{i=1}^k \lambda_i s_i, \sum_{i=1}^k \lambda_i = 1 \}$

Proof:

$\Sigma := \{ \dots \}$

$\Sigma$ is convex and $S \subseteq \Sigma$ by definition $\Rightarrow$ $\mathrm{conv}(S) \subseteq \Sigma$

as $\mathrm{conv}(S) := \bigcap_{\substack{C \text{ convex} \\ S \subseteq C}} C$ and therefore $\mathrm{conv}(S) \subseteq C \ \forall C$.

Consider $C$ convex, $S \subseteq C$,

$x \in C \ \forall x \in \Sigma$ as $s_i \in S \subseteq C$ and $C$ convex

This holds for all $C$ $\Rightarrow$ $\Sigma \subseteq C \ \forall C$ convex, $S \subseteq C$.

Thus $\Sigma \subseteq \mathrm{conv}(S)$ as otherwise there would have to be a $C$ s.t. $\Sigma \not\subseteq C$.

$\therefore \ \mathrm{conv}(S) = \Sigma$ □

2. (Carathéodory theorem)

$x \in \mathrm{conv}(S)$ $\exists \lambda_i \geq 0$, $s_i \in S$, $i = 1, \dots, k$, $k \leq n+1$ s.t. $x = \sum_{i=1}^k \lambda_i s_i$, $\sum_{i=1}^k \lambda_i = 1$.

Proof:

Can always write

$x = \sum_{i=1}^k \lambda_i s_i$ where $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, $s_i \in S$ and $k \in \mathbb{N}_{\geq 1}$ by 1.1 (1).

If $k \leq n+1$, we are done.

$k \geq n+2$:

Consider $(s_i - s_k) \in \mathbb{R}^n$ for $i = 1, \dots, k-1$, as $k-1 \geq n+1$ these points are linearly dependent in $\mathbb{R}^n$

$\exists \mu_i, i = 1, \dots, k-1 : \sum_{i=1}^{k-1} \mu_i (s_i - s_k) = 0$ where not all $\mu_i$ are zero.

$\Rightarrow \sum_{i=1}^{k-1} \mu_i s_i = \left( \sum_{i=1}^{k-1} \mu_i \right) s_k$

Define

$\mu_k := -\sum_{i=1}^{k-1} \mu_i$ then $\sum_{i=1}^k \mu_i s_i = 0$ (affinely dependent)

and $\sum_{i=1}^k \mu_i = 0$ where not all $\mu_i$ are zero and thus at least one is +ve.

Write

$x = \sum_{i=1}^k \lambda_i s_i - \alpha \overbrace{\sum_{i=1}^k \mu_i s_i}^{=0} = \sum_{i=1}^k (\lambda_i - \alpha \mu_i) s_i$

Now take $\alpha = \min_{1 \leq i \leq k} \left\{ \frac{\lambda_i}{\mu_i} : \mu_i > 0 \right\} = \frac{\lambda_j}{\mu_j}$

ensuring $\lambda_i - \alpha \mu_i \geq 0 \ \forall i$ and

crucially $\lambda_j - \alpha \mu_j = 0$. $\therefore k \to k-1$ and by repeating the above find $k \leq n+1$. □

## 2.1 $K$ closed convex cone in $\mathbb{R}^n$

1. (i) $K$ has nonempty interior

   (ii) $\text{span}(K) = \mathbb{R}^n$

   (iii) For any $w \in \mathbb{R}^n \setminus \{0\}$ $\exists x \in K : \langle w, x \rangle \neq 0$

(i) $\Rightarrow$ (ii):

$x \in \text{int}(K)$ $\exists \varepsilon > 0$ : $y \in S$ $\forall y \in \mathbb{R}^n : \|y - x\|_2 < \varepsilon$

But $\text{span}(\{y - x\}) = \mathbb{R}^n$ as this is a n-ball. $\Rightarrow \text{span}(K) = \mathbb{R}^n$ $\square$

(ii) $\Rightarrow$ (iii):

$\text{span}(K) = \{x \in \mathbb{R}^n : x = \sum_i \mu_i k_i, \ k_i \in K\} = \mathbb{R}^n$

Hence any $w \in \mathbb{R}^n \setminus \{0\}$ can be written as $w = \sum_i \mu_i k_i$.

Assume $\langle w, x \rangle = 0$ $\forall x \in K$, then

$$\langle w, k_j \rangle = 0 \quad \forall j \quad \text{where } k_j \text{ as above.}$$

$$\Rightarrow \mu_j \langle w, k_j \rangle = 0$$

$$\Rightarrow \langle w, \sum_j \mu_j k_j \rangle = 0 \quad \Rightarrow \quad \langle w, w \rangle = 0,$$

but assumed $w \neq 0$ #

$\Rightarrow \exists x \in K : \langle w, x \rangle \neq 0$ $\square$

$\neg$(ii) $\Rightarrow$ $\neg$(iii):

If $\text{span}(K) \neq \mathbb{R}^n$ $\exists y \in \mathbb{R}^n : \langle y | x \rangle = 0$ $\forall x \in \text{span}(K)$

$\Rightarrow \forall x \in K$ $\square$

⌈ Construct orthonormal basis on $\text{span}(K)$ $\hat{x}_1, \dots, \hat{x}_n$ then $\forall x \in \text{span}(K)$,

$x = \sum_i \alpha_i \hat{x}_i$ where $\alpha_i := \langle \hat{x}_i, x \rangle$,

then $y \notin \text{span}(K)$ has component

$y_\perp := y - \sum_i \alpha_i \hat{x}_i$ which $\langle y, \hat{x}_i \rangle = 0$ $\forall i$. ⌋

(iii) $\Rightarrow$ (i):

Assume $\text{span}(K) = \mathbb{R}^n$. Let $a_1, \dots, a_n$ be a basis of $\mathbb{R}^n$ contained in $K$. Let $x_0 = a_1 + \dots + a_n$

Claim: $x_0 \in \text{int}(K)$

Let $A = [a_1 | \dots | a_n] \in \mathbb{R}^{n \times n}$ invertible $N(x) = \|A^{-1} x\|_\infty$ a norm.

Let $B = \{x : N(x - x_0) \leq \frac{1}{2}\}$ ball around $x_0$. Show $B \subseteq K$

Let $x \in B$, This means that $x - x_0 = \sum_{i=1}^n \alpha_i a_i$ where $|\alpha_i| \leq \frac{1}{2}$. $\Rightarrow x = x_0 + \sum_i \alpha_i a_i = \sum_i \underbrace{(1+\alpha_i)}_{\geq 0} a_i \in K$ $\square$

2. $K$ is pointed iff $\text{int}(K^*) \neq \emptyset$

$K \cap (-K) = \{0\}$ $\qquad K^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \ \forall x \in K\}$

Proof:

$K$ closed convex cone $\Rightarrow$ $K^*$ closed convex cone

Suppose $x \neq 0$, $x \in K$ and $x \in -K$ $\Rightarrow$ $-x \in K$.

Then for $K^*$ require $y \in \mathbb{R}^n$ : $\langle y, x \rangle \geq 0$ and $\langle y, -x \rangle = -\langle y, x \rangle \geq 0$

$\Rightarrow \langle y, x \rangle = 0 \quad \forall y \in K^*$

Therefore $\text{int}(K^*) = \emptyset$ (as for any $y \in K^*$ no neighbouring elements in direction parallel to $x$).

$\therefore \text{int}(K^*) \neq \emptyset \Rightarrow K \cap (-K) = \{0\}$

Now if $\text{int}(K^*) = \emptyset$, $\exists w \in \mathbb{R}^n \setminus \{0\}$ s.t. $\nexists x \in K^* : \langle w, x \rangle \neq 0$

by (1) ~~the~~ equivalence relations. $\Rightarrow \exists w \in \mathbb{R}^n \setminus \{0\} : \langle w, x \rangle = 0 \ \forall x \in K^*$

Therefore, by theorem 2.2

$$K = (K^*)^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \ \forall x \in K^*\}$$

and $\mu w \in K \ \forall \mu \in \mathbb{R}$ but hence $w \in K$ and $w \in -K$ and $K$ is not pointed.

$\therefore K \cap (-K) = \{0\} \Rightarrow \text{int}(K^*) \neq \emptyset \quad \square$

3. $y \in \text{int}(K^*)$ iff $\langle y, x \rangle > 0 \ \forall x \in K \setminus \{0\}$

Proof:

Consider $B = \{z \in \mathbb{R}^n : \|z - y\|_2 < \varepsilon\}$ for some $\varepsilon \in \mathbb{R}_{>0}$

as $y \in \text{int}(K^*)$ $\exists \varepsilon > 0$ : $B \subseteq K^*$.

$\Rightarrow \langle z, x \rangle \geq 0 \quad \forall z \in B \ (z \in K^*) \quad \forall x \in K$

$$\langle (z-y) + y, x \rangle \geq 0 \Rightarrow \langle y, x \rangle \geq -\langle (z-y), x \rangle \quad \forall z \in B, x \in K$$

Using Cauchy-Schwarz inequality

$$\langle z-y, x \rangle^2 \leq \|z-y\|_2^2 \|x\|_2^2$$

$$\Rightarrow |\langle z-y, x \rangle| \leq \|z-y\|_2 \|x\|_2 < \varepsilon \|x\|_2$$

As $\text{span}\{z-y\} = \mathbb{R}^n$ (interior of $n$-ball),

$\forall x \in K \setminus \{0\}$ $\exists z : \langle z-y, x \rangle < 0$

$\Rightarrow \langle y, x \rangle > 0 \quad \forall x \in K \setminus \{0\}$.

3. $y \in \mathrm{int}(K^*)$ iff $\langle y, x \rangle > 0 \quad \forall x \in K \setminus \{0\}$

Let $y \in \mathrm{int}(K^*)$. $\exists\, \varepsilon > 0 : \quad y - \varepsilon x \in K^*$ (since $y \in \mathrm{int}(K^*)$)

$$\Rightarrow \langle y - \varepsilon x, x \rangle \geq 0 \Rightarrow \langle y, x \rangle \geq \varepsilon \langle x, x \rangle > 0 \quad \square$$

Let $y \in K^*$ s.t. $\langle y, x \rangle > 0 \ \forall x \in K \setminus \{0\}$.

~~$C = \{x \in K : \langle y, x \rangle = 1\}$ C is compact (closed and bounded).~~

Define $\varepsilon = \min\limits_{\substack{x \in K \\ \|x\| = 1}} \langle y, x \rangle > 0$ by our assumption on $y$

We will show that the Ball $B(y, \varepsilon)$ is contained in $K^*$

Let $r$ s.t. $\|r\| \leq \varepsilon$. For any $x \in K$ we have

$$\langle y + r, \tfrac{x}{\|x\|} \rangle = \langle y, \tfrac{x}{\|x\|} \rangle + \langle r, \tfrac{x}{\|x\|} \rangle \overset{C.S.}{\geq} \varepsilon - \|r\| \geq 0$$

This shows that $\langle y + r, x \rangle \geq 0 \ \forall x \in K$ which means that $y + r \in K^*$. $\square$

## 2.2 Minkowski theorem for closed convex pointed cones

$K$ is a closed pointed convex cone in $\mathbb{R}^n$. Then $K$ is the conical hull of its extreme rays.

$$K = \{ x \in \mathbb{R}^n : \exists k \in \mathbb{N}_{\geq 1}, s_1, \dots, s_k \in S, \lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0} : x = \sum_{i=1}^{k} \lambda_i s_i \}$$

for $S = \{\text{extreme rays of } K\}$.

1. Assume $\exists y \in \mathbb{R}^n : \langle y, x \rangle > 0 \ \forall x \in K \setminus \{0\}$.

Proof:

$$C := \{ x \in K : \langle y, x \rangle = 1 \}$$

$C$ is a convex set as when $x, z \in C$ then

$$\langle \lambda x + (1-\lambda) z, y \rangle = \lambda \langle x, y \rangle + (1-\lambda) \langle z, y \rangle = 1 \Rightarrow \lambda x + (1-\lambda) z \in C.$$

$C$ is closed as $K$ is closed and $\langle y, x \rangle = 1$ defines a closed surface in $\mathbb{R}^n$.

$C$ is bounded

From Minkowski theorem for compact convex sets

$$C = \operatorname{conv}(\operatorname{ext}(C))$$

If $s \in S$ is an extreme ray of $K$, then $\exists \lambda \in \mathbb{R}_{\geq 0} : \lambda s \in C$

$$\left( \langle \lambda s, y \rangle = 1 \Rightarrow \lambda = (\underbrace{\langle s, y \rangle}_{>0 \text{ by assumption}})^{-1} \right)$$

$\lambda s$ must also be an extreme point of $C$:

$$\lambda s = \alpha x + (1-\alpha) z \quad \text{where } 0 < \alpha < 1 \text{ and } x, z \neq \lambda s$$

$$\Rightarrow \alpha x, (1-\alpha) z \in \mathbb{R}_+ s$$

but then $x, z = \lambda s$ as ray intersects $C$ uniquely #

By relabelling $\lambda s \to s$ (as ray is $\mathbb{R}_+ s$)

$$\Rightarrow C = \{ x \in \mathbb{R}^n : \exists k \in \mathbb{N}_{\geq 1}, \lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}, s_1, \dots, s_k \in S \ \text{s.t.}\ x = \sum_{i=1}^{k} \lambda_i s_i, \ \sum_{i=1}^{k} \lambda_i = 1 \}$$

Indeed, $\langle y, x \rangle = \sum_i \lambda_i \langle y, s_i \rangle = \sum_i \lambda_i = 1$.

But setting $\langle y, x \rangle = k \in \mathbb{R}_+$ get similar expressions for $C_k = \{ \dots, \sum_i \lambda_i = k \}$.

Now $K = \bigcup_{k \in \mathbb{R}_+} C_k \cup \{0\} = \{ \dots \}$ the required expression.

2. $y \in \mathbb{R}^n : \langle y, x \rangle > 0 \ \forall x \in K \setminus \{0\}$ exists as this is true for $y \in \operatorname{int}(K^*)$ and $\operatorname{int}(K^*) \neq \emptyset$ as $K$ pointed. □

2.3

2. $Q^3 = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ : \|x\|_2 \le t\}$

Convex: follows from convexity of $\|\cdot\|_2$

Closedness: follows from continuity of $\|\cdot\|_2$

Pointed: obvious

Non empty interior: e.g. $(0,0,1) \in \text{int}(K)$

$(Q^3)^* = Q^3$

$Q^3 \subseteq (Q^3)^*$ : Let $(a,b) \in Q^3$. We want to show that $\langle a,x\rangle + bt \ge 0 \ \forall (x,t) \in Q^3$

$$\langle a,x\rangle + bt \ge -\|a\|\|x\| + bt \quad \text{by Cauchy-Schwarz}$$
$$\ge -bt + bt \ge 0$$

$(Q^3)^* \subseteq Q^3$ : Let $(a,b) \in (Q^3)^*$. We want to show that $(a,b) \in Q^3$, i.e. $\|a\| < b$

Let $(x,t) = (-a, \|a\|) \in Q^3$

Hence $\langle a,x\rangle + bt \ge 0$

i.e. $-\|a\|^2 + b\|a\| \ge 0 \Rightarrow \|a\| \le b$.

Extreme rays: $S_x = \{\lambda(x,1) : \lambda \ge 0\}$ $\|x\| = 1$

Let's show that $S_x$ is an extreme ray of $Q^3$

~~...~~ $a, b \in Q^3$ s.t. $a+b = \lambda(x,1)$

Then show that $a, b \in S_x$

- If $\|x\| < t$, then $(x,t) = \underbrace{(x, \|x\|)}_{\in Q^3} + \underbrace{(0, t-\|x\|)}_{\in Q^3}$ $\Rightarrow$ shows that any ray spanned by $(x,1)$ where $\|x\| < 1$ is not extreme