Feynman Rules for ϕ^4

$H_{int} = \frac{\lambda}{4!}\phi^4$

Single interaction vertex  $\sim(-i\lambda)$

Scattering $\phi\phi \to \phi\phi$ at $\mathcal{O}(\lambda)$ has a single contribution

$\frac{-i\lambda}{4!}\langle p_1', p_2' | :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): | p_1, p_2\rangle$, $4!$ ways of pairing ϕ to outer states

For some diagrams, get extra combinatoric factors, symmetry factors (typically 2 or 4).

The Yukawa Potential

In the non-relativistic limit, calculation of $\psi\psi \to \psi\psi$ gives a potential

$$U(\underline{r}) = -\frac{\lambda^2 e^{-mr}}{4\pi r} \quad , \quad \lambda \equiv \frac{g}{2\mu} \text{ non-relativistic normalization}$$

force is attractive
(also for $\psi\bar{\psi}$)

NB, For ϕ^4, $i\mathcal{A} = -i\lambda \Rightarrow U(\underline{r}) = -\lambda\int\frac{d^3p}{(2\pi)^3}e^{i\underline{p}\cdot\underline{r}} = -\lambda\,\delta^3(\underline{r})$

Green's Functions

S-matrix elements are physical, but can compute more elementary correlation f^ns.

Let $|\Omega\rangle$ be the vacuum of the interacting theory ($|0\rangle$ is the vacuum of the free theory).

$H|\Omega\rangle = 0$ fixes normalisation of H (and $\underline{p}|\Omega\rangle = 0$)

$\langle\Omega|\Omega\rangle = 1$. Define $G^{(n)}(x_1,\dots,x_n) = \langle\Omega|T\,\phi_H(x_1)\dots\phi(x_n)|\Omega\rangle$.

Green's f^n Heisenberg

Compute $G^{(n)}$ using Feynman diagrams: Interaction

$$G^{(n)}(x_1,\dots,x_n) = \frac{\langle 0|T\,\phi_{1I}\cdots\phi_{nI}\,S|0\rangle}{\langle 0|S|0\rangle} \quad \text{where } \phi_i = \phi(x_i)$$

Take $t_1 > t_2 > \dots > t_n$, then T is trivial, num of RHS:

$$\langle 0|U_I(\infty,t_1)\phi_{1I}\,U(t_1,t_2)\phi_{2I}\cdots\phi_{nI}U_I(t_n,-\infty)|0\rangle$$

$$= \langle 0|U_I(\infty,t_1)\phi_{1H}\cdots\phi_{nH}\,U_I(t_n,-\infty)|0\rangle$$

For outer U_I, consider arbitrary state $|\Psi\rangle$,

$$\langle\Psi|U_I(t,-\infty)|0\rangle = \langle\Psi|U(t,-\infty)|0\rangle \quad \text{because } H_0|0\rangle = 0$$

Insert complete set of states, energy e'states of $H_0 + H_{int}$

$$\langle \Psi | U(t,-\infty) | 0 \rangle = \langle \psi | U(t,-\infty) \Big[|\Omega\rangle\langle\Omega| + \sum_{n\neq 0} |n\rangle\langle n| \Big] |0\rangle$$

$$= \langle \Psi | \Omega \rangle \langle \Omega | 0 \rangle + \underbrace{\lim_{t' \to \infty} \sum_{n>0} e^{iE_n(t'-t)} \langle \Psi | n \rangle \langle n | 0 \rangle}_{=0}$$

because of Riemann–Lebesgue lemma: $\forall$ well-behaved f^{ns} $f(x)$, $\lim_{\mu \to \infty} \int_a^b f(x) e^{i\mu x} dx = 0$.

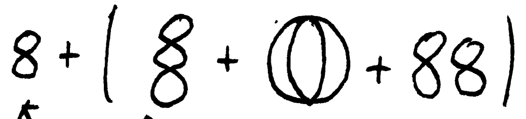
Works as $|n\rangle$ part of a continuum, i.e. $\lim_{t' \to \infty} \langle \Psi | U(t,t') | 0 \rangle = \langle \Psi | \Omega \rangle \langle \Omega | 0 \rangle$.

Then,

$$\text{RHS} = \frac{\langle 0|\Omega\rangle\langle\Omega| T \phi_{1H} \cdots \phi_{nH} |\Omega\rangle\langle\Omega|0\rangle}{\langle 0|\Omega\rangle\langle\Omega|\Omega\rangle\langle\Omega|0\rangle} \qquad \square$$

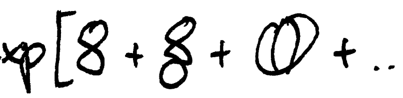
Connected Diagrams and Vacuum Bubbles

Can compute both $\langle 0 | T \phi_I(x_1) \ldots \phi_I(x_n) S | 0 \rangle$ and $\langle 0 | S | 0 \rangle$. Denominator has a simple interpretation.

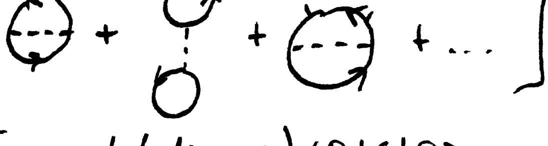
ϕ^4 theory: $\langle 0 | S | 0 \rangle = 1 +$  $+ \ldots$

vacuum bubbles

The combinatoric factors turn out s.t. sums to exp:

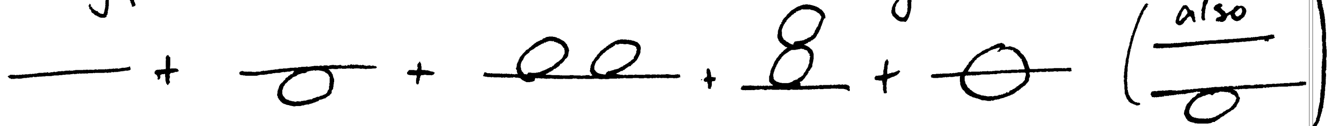
$\langle 0 | S | 0 \rangle = \exp\Big[$  $+ \ldots \Big] = \exp\big[\text{all distinct vacuum bubbles}\big]$

always get these

Scalar Yukawa theory: $\langle 0 | S | 0 \rangle = \exp\Big[$  $+ \ldots \Big]$

Can show $\langle 0 | T \phi_{1I} \cdots \phi_{nI} S | 0 \rangle = \left(\sum \text{connected diagrams} \right) \langle 0 | S | 0 \rangle$

connected $\equiv$ every part is connected to at least 1 external leg

e.g. ϕ^4 

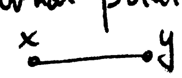
also

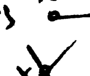
Thus $\langle \Omega | T \phi_{H1} \cdots \phi_{Hn} | \Omega \rangle = \sum$ connected Feynman diagrams

Here, Feynman diagrams depend on x_i, $i \in \{1, \ldots, n\}$ (in S-matrix elements integrated them)

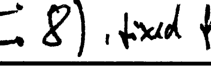
Adapt Feynman rules for momentum space to compute $G^{(n)}(x_1, \ldots, x_n)$ directly:

Draw n external points $x_1, \ldots, x_n$. Connect with

(a) propagators 

(b) vertices  $= -i\lambda \int d^4x$

e.g. $\langle \Omega | T \phi_H(x_1) \ldots \phi_H(x_n) | \Omega \rangle =$  $+ 3$ perm $+ \ldots$

but don't include , fixed from shift $|0\rangle$ to $|\Omega\rangle$