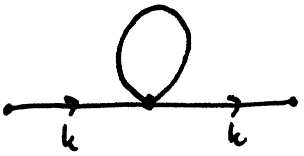


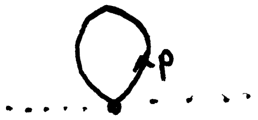
For most diagrams, regularising with a cut-off $|p| \leq \Lambda_0$ is awkward, and in gauge theories it's (at least naively) incompatible with gauge invariance. $\psi(x) \to e^{i\lambda(x)}\psi(x)$

For these reasons, it is often convenient to regularize perturbatively using dimensional regularisation: we analytically continue the results of loop integrals in d. Unlike imposing a cut-off, dim. reg. only makes sense perturbatively – it provides a way to obtain finite loop integrals but it does not give any def^n of a regularized path integral measure.

Consider again $S[\phi] = \int d^dx \, \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$. In d dim, $[\phi] = \frac{d-2}{2}$, so $[\lambda] = 4-d$. We thus write

$\lambda = \mu^{4-d} g(\mu)$ in terms of some (arbitrary) mass scale μ, where $g(\mu)$ is dim.less.

We have

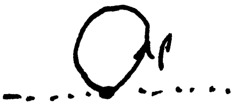


$$= \frac{1}{2} g \mu^{4-d} \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+m^2)} = \frac{g\mu^{4-d}}{2(2\pi)^d} \mathrm{Vol}(S^{d-1}) \int_0^\infty \frac{p^{d-1}\,dp}{p^2+m^2}$$

We have $\mathrm{Vol}(S^{d-1}) = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$

$$\mu^{4-d}\int_0^\infty \frac{p^{d-1}\,dp}{p^2+m^2} = \frac{1}{2}\mu^{4-d}\int_0^\infty \frac{(p^2)^{d/2-1}\,d(p^2)}{p^2+m^2} = \frac{m^2}{2}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)}$$

Combining the factors



$$= \frac{g m^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma(1-d/2)$$

We set $d = 4-\epsilon$ and use the asymptotic formula $\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma + O(\epsilon)$ we find

$$\overset{d\to 4}{\sim} -\frac{g m^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon)\right]$$

Remark This diverges as $\epsilon \to 0$. The pole in $\frac{1}{\epsilon}$ reflects the divergence of this loop integral as $\Lambda_0 \to \infty$ in the cut-off regularization. We need to obtain a finite limit as $d \to 4$ by tuning our initial couplings $g(\mu)$. We do this by including counterterms.

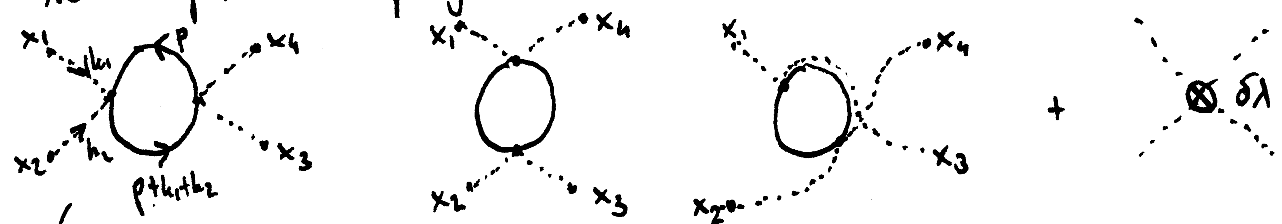
To fix the finite part of the remaining contrib^n, we choose a regularization scheme. We can again use on-shell renormalization. However, in dim reg other renormalization schemes are often more convenient.

1) Minimal Subtraction (MS) - choose $\delta m^2 = \frac{-gm^2}{16\pi^2\epsilon}$ so as to cancel just the pole

2) Modified Minimal Subtraction ($\overline{MS}$) - choose $\delta m^2 = \frac{-gm^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln(4\pi)\right]$ to remove some pesky constants

Renormalization of the ϕ^4 coupling

At 1-loop, the coupling receives contributions from



$$\frac{g^2\mu^{4-d}}{2}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2}\,\frac{1}{(p+k_1+k_2)^2+m^2}$$

This diagram knows about k_1, k_2, so as well as contributing to ϕ^4 coupling, it also contributes to $(\partial\phi)^2\phi^2$ etc. Only the ϕ^4 contribution is divergent in $d=4$ (Check!).

This contribution is k_i-independent

$$\to \frac{g^2\mu^{4-d}}{2(2\pi)^d}\int\frac{d^dp}{(p^2+m^2)^2} = \frac{1}{2}\,\frac{g^2}{(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma(2-d/2)$$

There are three channels, and to zeroth order in the k_i they're all equivalent.

Thus, altogether we have $O(\hbar)$ contributions to ϕ^4 of

$$-\delta\lambda + \frac{3g^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma(2-d/2) \overset{d\to4}{\sim} -\delta\lambda + \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2} + O(\epsilon)\right)$$

Therefore, in the $\overline{MS}$ scheme, we choose $\delta\lambda = \frac{3g^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \log 4\pi\right]$ and so we get an order $\hbar$ shift in the ϕ^4 coupling of $\frac{3g^2}{32\pi^2}\log\left(\frac{\mu^2}{m^2}\right)$. If we vary the scale μ at which we defined our original theory, then we should vary our initial value of $g(\mu)$ as

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} = \frac{3g^2}{16\pi^2} > 0$$

Since $\beta(g) > 0$, the coupling λ which was marginal in $d=4$ classically, is in fact marginally irrelevant (We found exactly the same β f^n using the LPA).