$X = \frac{1}{p!} X_{\mu_1 \dots \mu_p} f^{\mu_1} \wedge \dots \wedge f^{\mu_p}$ $\{e^a_\mu\}$ orthonormal $g_{ab} e^a_\mu e^b_\nu = \eta_{\mu\nu}$ dual basis $\{e^\mu_a\}$

$$g_{ab} = \eta_{\mu\nu} e^\mu_a e^\nu_b$$

$(\omega^\mu{}_\nu)_a = e^\mu_b \nabla_a e^b_\nu$ $\qquad (\omega^\mu{}_\nu)_a = \Gamma^\mu_{\nu\rho} e^\rho_a$

$\omega_{\mu\nu} = -\omega_{\nu\mu}$ $\qquad de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$

$de^\mu = -(\omega^\mu{}_\nu)_\rho e^\rho \wedge e^\nu = (\omega^\mu{}_{[\nu})_{\rho]} e^\nu \wedge e^\rho$

$(de^\mu)_{\nu\rho} = 2(\omega^\mu{}_{[\nu})_{\rho]}$ $\qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \Rightarrow (\omega_{\mu\nu})_\rho = (\omega_{\mu[\nu})_{\rho]} + (\omega_{\nu[\rho})_{\mu]} - (\omega_{\rho[\mu})_{\nu]}$

Ex Schwarzschild $\quad ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$f = \sqrt{1 - \frac{2M}{r}}$ $\qquad e^0 = f dt \qquad e^2 = r d\theta$

$\qquad e^1 = f^{-1} dr \qquad e^3 = r \sin\theta \, d\phi$

$de^0 = df \wedge dt + f \, \cancel{d(dt)}^{\,0} = f' dr \wedge dt = f' e^1 \wedge e^0$

$de^1 = -f^{-2} f' dr \wedge dr = 0$

$de^2 = dr \wedge d\theta = \frac{f}{r} e^1 \wedge e^2$

$de^3 = \sin\theta \, dr \wedge d\phi + r\cos\theta \, d\theta \wedge d\phi = \frac{f}{r} e^1 \wedge e^3 + \frac{1}{r} \cot\theta \, e^2 \wedge e^3$

$\omega^0{}_1 = f' e^0 \qquad \omega_{01} = -f' e^0 \qquad \omega_{10} = f' e^0 \qquad \omega^1{}_0 = f' e^0 \qquad \omega^1{}_0 \wedge e^0 = 0$

$\omega^2{}_1 = \frac{f}{r} e^2 \qquad \omega^1{}_2 = -\frac{f}{r} e^2 \qquad \omega^1{}_2 \wedge e^2 = 0$

$\omega^3{}_1 = \frac{f}{r} e^3 \qquad \omega^3{}_2 = \frac{1}{r} \cot\theta \, e^3 \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}.$

$\nabla_\nu Y^\mu = e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho = \partial_\nu Y^\mu + (\omega^\mu{}_\rho)_\nu Y^\rho$

$e_\nu^\alpha \partial_\alpha$ coord basis

Curvature 2-forms

Def^n The curvature 2-forms are $\Theta^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} e^\rho \wedge e^\sigma$

$\Theta_{\mu\nu} = -\Theta_{\nu\mu}$

Lemma $\Theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$

Ex Schwarzschild $\quad \omega^0{}_1 = f' e^0$

$d\omega^0{}_1 = f' de^0 + f'' dr \wedge e^0 = f'^2 e^1 \wedge e^0 + f f'' e^1 \wedge e^0$

$\omega^0{}_\rho \wedge \omega^\rho{}_1 = \omega^0{}_1 \wedge \omega^1{}_1 = 0$

non-zero only if $\rho = 0$ $\qquad \omega_{11} = 0$

$\therefore \Theta_{01} = -\Theta^0{}_1 = \overbrace{(f f'' + f'^2)}^{\frac{1}{2}(f^2)''} e^0 \wedge e^1 = -\frac{2M}{r^3} e^0 \wedge e^1$

$R_{0101} = -R_{0110} = -\frac{2M}{r^3}$ other $R_{01\rho\sigma}$ vanish

Volume form

Def^n A manifold of dimension n is orientable if it admits an orientation: a smooth, nowhere vanishing n-form $\varepsilon_{a_1 \ldots a_n}$.

Orientations $\varepsilon, \varepsilon'$ are equivalent if $\varepsilon' = f\varepsilon$ $f : M \to \mathbb{R}$ is +ve.

X n-form can write $X = f\varepsilon$ X orientation if $f \neq 0$.

$\therefore \exists$ 2 inequivalent orientations $f > 0$ or $f < 0$.

Def^n Coord chart x^μ is right-handed w.r.t. ε iff

$\varepsilon = f(x)\, dx^1 \wedge \ldots dx^n$ with $f > 0$ (LH if $f < 0$)

Def^n On an oriented manifold with metric g, the volume form is defined by

$\varepsilon_{1 \ldots n} = \sqrt{|g|}$ $g = \det g_{\mu\nu}$ in a RH coord chart.

Ex Show that 1. this is independent of choice of RH coord chart; 2. $\varepsilon^{12\ldots n} = \pm \frac{1}{\sqrt{|g|}}$ + Riemannian − Lorentzian

Lemma $$\varepsilon^{a_1 \ldots a_p c_{p+1} \ldots c_n} \varepsilon_{b_1 \ldots b_p c_{p+1} \ldots c_n} = \pm p!(n-p)!\, \delta^{a_1}_{[b_1} \delta^{a_2}_{b_2} \cdots \delta^{a_p}_{b_p]} \quad (*)$$

Def^n Hodge dual of p-form X is $(n-p)$-form

$$(*X)_{a_1 \ldots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \ldots a_{n-p} b_1 \ldots b_p} X^{b_1 \ldots b_p}$$

Lemma p-form $*(*X) = \pm(-1)^{p(n-p)} X$ using $(*)$

$$(*d*X)_{a_1 \ldots a_{p-1}} = \pm(-1)^{p(n-p)} \nabla^b X_{a_1 \ldots a_{p-1} b}$$

Examples 1. 3D Euclidean space $\nabla f = df$ $\operatorname{div} X = *d(*X)$ $\operatorname{curl} X = *dX$

2. $\nabla^a F_{ab} = -4\pi j_b$ $\nabla_{[a} F_{bc]} = 0$

current density

can write as $d*F = -4\pi *j$ $dF = 0 \Rightarrow F = dA$ locally

$\Rightarrow d*j = 0 \Rightarrow \nabla^a j_a = 0$ conserved current