Quantum Field Theory
Example sheet 3

1. $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ $\quad , \{\gamma^a, \gamma^b\} = \{\gamma^b, \gamma^a\}$

$$\{\gamma^0, \gamma^0\} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} + \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} = 0$$

$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$= -(\{\sigma^i, \sigma^j\}) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -2\delta^{ij}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \{\gamma^a, \gamma^b\} = 2\eta^{ab}\,\mathbb{1}$

$(\gamma')^a = U\gamma^a U^\dagger \qquad (\gamma')^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$(\gamma'^a)U = U\gamma^a \quad$ Try $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, from γ^0

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = \begin{pmatrix} B & A \\ D & C \end{pmatrix}$$

$\Rightarrow A = B \quad -C = D$

Claim

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Indeed,

$UU^\dagger = 1 \quad , (\gamma')^a = U\gamma^a U^\dagger$

$$UU^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U\gamma^0 U^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$U\gamma^i U^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sigma^i & \sigma^i \\ -\sigma^i & \sigma^i \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \square$$

2. $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$

$$[\gamma^a\gamma^b,\gamma^c\gamma^d] = \gamma^a\gamma^b\gamma^c\gamma^d - \gamma^c\gamma^d\gamma^a\gamma^b$$

$$= \gamma^a\gamma^b\gamma^c\gamma^d - \gamma^c\{\gamma^d,\gamma^a\}\gamma^b + \gamma^c\gamma^a\gamma^d\gamma^b$$

$$= \gamma^a\gamma^b\gamma^c\gamma^d - 2\eta^{ad}\gamma^c\gamma^b + \gamma^c\gamma^a\{\gamma^d,\gamma^b\} - \gamma^c\gamma^a\gamma^b\gamma^d$$

$$= \gamma^a\gamma^b\gamma^c\gamma^d - 2\eta^{ad}\gamma^c\gamma^b + 2\eta^{bd}\gamma^c\gamma^a - \{\gamma^c,\gamma^a\}\gamma^b\gamma^d + \gamma^a\gamma^c\gamma^b\gamma^d$$

$$= \gamma^a\gamma^b\gamma^c\gamma^d - 2\eta^{ad}\gamma^c\gamma^b + 2\eta^{bd}\gamma^c\gamma^a - 2\eta^{ac}\gamma^b\gamma^d + \gamma^a\{\gamma^c,\gamma^b\}\gamma^d - \gamma^a\gamma^b\gamma^c\gamma^d$$

$$= -2\eta^{ac}\gamma^b\gamma^d + 2\eta^{bc}\gamma^a\gamma^d + 2\eta^{bd}\gamma^c\gamma^a - 2\eta^{ad}\gamma^c\gamma^b \quad \square$$

$$S^{ab} \equiv \frac{1}{4}[\gamma^a,\gamma^b] = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) = \frac{1}{4}(\gamma^a\gamma^b - \{\gamma^b,\gamma^a\} + \gamma^a\gamma^b)$$

$$= \frac{1}{2}(\gamma^a\gamma^b - \eta^{ab})$$

$$[S^{ab},S^{cd}] = \frac{1}{4}[\gamma^a\gamma^b - \eta^{ab},\gamma^c\gamma^d - \eta^{cd}] = \frac{1}{4}[\gamma^a\gamma^b,\gamma^c\gamma^d] =$$

$$= \frac{1}{2}(\eta^{bc}\gamma^a\gamma^d - \eta^{ac}\gamma^b\gamma^d + \eta^{bd}\gamma^c\gamma^a - \eta^{ad}\gamma^c\gamma^b)$$

$$= \eta^{bc}(S^{ad} + \tfrac{1}{2}\eta^{ad}) - \eta^{ac}(S^{bd} + \tfrac{1}{2}\eta^{bd}) + \eta^{bd}(S^{ca} + \tfrac{1}{2}\eta^{ac}) - \eta^{ad}(S^{cb} + \tfrac{1}{2}\eta^{bc})$$

$$= \eta^{bc}S^{ad} - \eta^{ac}S^{bd} + \eta^{bd}S^{ca} - \eta^{ad}S^{cb}$$

This is the Lie algebra structure of the Lorentz group.

3. $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$, $(\gamma^a)^2 = \frac{1}{2}\{\gamma^a,\gamma^a\} = \eta^{aa}$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\quad \cancel{(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3}$

$$\{\gamma^5,\gamma^a\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^a + i\gamma^a\gamma^0\gamma^1\gamma^2\gamma^3 = \pm i\Big(2\gamma^a\gamma^a\prod_b\gamma^b - \eta^{aa}\prod_b\gamma^b\Big) = 0$$

(a) $\mathrm{Tr}(\gamma^5\gamma^a\gamma^5) = -\mathrm{Tr}(\gamma^a\gamma^5\gamma^5)$ using $\{\gamma^5,\gamma^a\} = 0$

$= \mathrm{Tr}(\gamma^a\gamma^5\gamma^5)$ using cyclicity of Tr

$\Rightarrow \mathrm{Tr}(\gamma^5\gamma^a\gamma^5) = 0$ Use $(\gamma^5)^2 = 1$ from (d) $\Rightarrow$ $\underline{\mathrm{Tr}\,\gamma^a = 0}$

(b) $\mathrm{Tr}(\gamma^a\gamma^b) = \frac{1}{2}\mathrm{Tr}(\gamma^a\gamma^b) + \frac{1}{2}\mathrm{Tr}(\gamma^b\gamma^a)$ by cyclicity

$= \frac{1}{2}\mathrm{Tr}(\{\gamma^a,\gamma^b\}) = \mathrm{Tr}\,\eta^{ab} = 4\eta^{ab}$ as 4 dims

$\cancel{(c)\ \mathrm{Tr}(\gamma^a\gamma^b\gamma^c) = \mathrm{Tr}(\{\gamma^a,\gamma^b\}\gamma^c - \gamma^b\gamma^a\gamma^c)}$

$\cancel{= \mathrm{Tr}(\{\gamma^a,\gamma^b\}\gamma^c - \gamma^b\{\gamma^a,\gamma^c\} + \gamma^b\gamma^c\gamma^a)}$

$\cancel{= 2\eta^{ab}\mathrm{Tr}(\gamma^c) - 2\eta^{ac}\mathrm{Tr}(\gamma^b) + \mathrm{Tr}(\gamma^b\gamma^c\gamma^a)}$

(c) $\mathrm{Tr}(\gamma^a\gamma^b\gamma^c) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^a\gamma^b\gamma^c)$ using $(\gamma^5)^2 = 1$ from (d)

$= -\mathrm{Tr}(\gamma^5\gamma^a\gamma^5\gamma^b\gamma^c) = \mathrm{Tr}(\gamma^5\gamma^a\gamma^b\gamma^5\gamma^c)$

$= -\mathrm{Tr}(\gamma^5\gamma^a\gamma^b\gamma^c\gamma^5)$ using $\{\gamma^5,\gamma^a\} = 0$

$= -\mathrm{Tr}(\gamma^5\gamma^5\gamma^a\gamma^b\gamma^c)$ by cyclicity

$\Rightarrow \underline{\mathrm{Tr}(\gamma^a\gamma^b\gamma^c) = 0}$

(d) $(\gamma^5)^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^5\gamma^3 = i\gamma^0\gamma^1\gamma^5\gamma^2\gamma^3$

$= -i\gamma^0\gamma^5\gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3$

$= -\gamma^1\gamma^2\gamma^1\gamma^3\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^1\gamma^2\gamma^3\gamma^3 = -\gamma^1\gamma^2\gamma^1\gamma^2$

$= \gamma^1\gamma^1\gamma^2\gamma^2 = -\gamma^1\gamma^1 = 1$

(e) $\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3) = -\mathrm{Tr}(i\gamma^0\gamma^1\gamma^3\gamma^2) = \mathrm{Tr}(i\gamma^0\gamma^3\gamma^1\gamma^2)$

$= -\mathrm{Tr}(i\gamma^3\gamma^0\gamma^1\gamma^2) = -\mathrm{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3) = -\mathrm{Tr}(\gamma^5) = 0$

(f) $\not{p}\not{q} = p_a\gamma^a q_b\gamma^b = p_a q_b\{\gamma^a,\gamma^b\} - p_a q_b\gamma^b\gamma^a = 2\eta^{ab}p_a q_b - q_b\gamma^b p_a\gamma^a$

$= 2p\cdot q - \not{q}\not{p}$

$\not{p}\not{q} = \frac{1}{2}(p_a q_b\gamma^a\gamma^b) + \frac{1}{2}(2\eta^{ab}p_a q_b - q_b\gamma^b p_a\gamma^a) = p\cdot q + \frac{1}{2}p_a q_b[\gamma^a,\gamma^b]$

$= p\cdot q + 2S^{ab}p_a q_b$ as $S^{ab} = \frac{1}{4}[\gamma^a,\gamma^b]$

(g) $\mathrm{Tr}(\not{p}\not{q}) = \mathrm{Tr}(p\cdot q + 2S^{ab}p_a q_b) = 4p\cdot q + 2p_a q_b\,\mathrm{Tr}(S^{ab}) = 4p\cdot q$

as $\mathrm{Tr}(S^{ab}) = \frac{1}{4}\mathrm{Tr}(\gamma^a\gamma^b - \gamma^b\gamma^a) = \frac{1}{2}(\eta^{ab} - \eta^{ba}) = 0$

(h) $\mathrm{Tr}(\not{p}_1\cdots\not{p}_n) = p_{a_1}\cdots p_{a_n}\mathrm{Tr}(\gamma^5\gamma^5\gamma^{a_1}\cdots\gamma^{a_n}) = -p_{a_1}\cdots p_{a_n}\mathrm{Tr}(\gamma^5\gamma^{a_1}\cdots\gamma^{a_n}\gamma^5)$

$= -\mathrm{Tr}(\not{p}_1\cdots\not{p}_n) = 0$ for n odd

(i) $\mathrm{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = p_{1,a}\, p_{2,b}\, p_{3,c}\, p_{4,d}\, \mathrm{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d)$

$\mathrm{Tr}(\gamma^a\gamma^b\gamma^c\gamma^d) = 2\eta^{ab}\,\mathrm{Tr}(\gamma^c\gamma^d) - \mathrm{Tr}(\gamma^b\gamma^a\gamma^c\gamma^d)$

$= 8\eta^{ab}\eta^{cd} - 8\eta^{ac}\eta^{bd} + \mathrm{Tr}(\gamma^b\gamma^c\gamma^a\gamma^d)$

$= 8\eta^{ab}\eta^{cd} - 8\eta^{ac}\eta^{bd} + 8\eta^{ad}\eta^{bc} - \mathrm{Tr}(\gamma^b\gamma^c\gamma^d\gamma^a)$

$\Rightarrow \mathrm{Tr}(\gamma^a\gamma^b\gamma^c\gamma^d) = 4\eta^{ab}\eta^{cd} + 4\eta^{ad}\eta^{bc} - 4\eta^{ac}\eta^{bd}$

$\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right]$

(j) $\mathrm{Tr}(\gamma^5 \not{p}_1\not{p}_2) = -\mathrm{Tr}(\not{p}_1\gamma^5\not{p}_2) = -\mathrm{Tr}(\gamma^5\not{p}_2\not{p}_1) =$

using (k)

$= -\frac{1}{2}\mathrm{Tr}(\gamma^5\not{p}_1\gamma_a\not{p}_2\gamma^a) = -\frac{1}{2}\mathrm{Tr}(\gamma^a\gamma^5\not{p}_1\gamma_a\not{p}_2) = \frac{1}{2}\mathrm{Tr}(\gamma^5\gamma^a\not{p}_1\gamma_a\not{p}_2) = -\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2) = 0$

(k) $\gamma_a\not{p}\gamma^a = \gamma_a p_b\gamma^b\gamma^a = \gamma_a p_b\{\gamma^b,\gamma^a\} - \gamma_a\gamma^a p_b\gamma^b$

$= 2\eta^{ab}\gamma_a p_b - \eta_{ac}\gamma^a\gamma^c p_b\gamma^b$

$= 2\eta^{ab}\gamma_a p_b - \frac{1}{2}\eta_{ac}(\gamma^a\gamma^c + \gamma^c\gamma^a)p_b\gamma^b$

$= 2\not{p} - \frac{1}{2}\eta_{ac}2\eta^{ac}\not{p} = 2\not{p} - \delta^a_a\not{p} = -2\not{p}$ $(\gamma^a\gamma_a = 4)$

(l) $\gamma_a\not{p}_1\not{p}_2\gamma^a$

$= \gamma_a p_{1b}\gamma^b p_{2c}\{\gamma^c,\gamma^a\} - \gamma_a p_{1b}\gamma^b\gamma^a p_{2c}\gamma^c$ ← by (k)

$= 2\not{p}_2\not{p}_1 + 2\not{p}_1\not{p}_2 \underset{\text{by (f)}}{=} 2(2p_1\cdot p_2) = 4p_1\cdot p_2$

(m) $\gamma_a\not{p}_1\not{p}_2\not{p}_3\gamma^a = \gamma_a\not{p}_1\not{p}_2 p_{3b}\{\gamma^b,\gamma^a\} - \gamma_a\not{p}_1\not{p}_2\gamma^a\not{p}_3$

$= 2\not{p}_3\not{p}_1\not{p}_2 - 4p_1\cdot p_2\not{p}_3$ by (l)

$= -2\not{p}_3(2p_1\cdot p_2 - \not{p}_1\not{p}_2) = -2\not{p}_3\not{p}_2\not{p}_1$ by (f)

(n) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = \frac{1}{4}\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4\gamma_a\gamma^a) = -\frac{1}{4}\mathrm{Tr}(\gamma^5\gamma^a\not{p}_1\not{p}_2\not{p}_3\not{p}_4\gamma_a)$

$= -\frac{1}{2}\mathrm{Tr}(\gamma^5\not{p}_2\not{p}_3\not{p}_4\not{p}_1) - \frac{1}{2}\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_4\not{p}_3\not{p}_2)$ using (m) for 2nd term

Antisymmetric under swapping by (f), suggests $\propto \epsilon_{abcd}\,p_1^a p_2^b p_3^c p_4^d$, when $p_1 = \begin{pmatrix}0\\0\\0\\p_1\end{pmatrix}$, $p_2 = \begin{pmatrix}0\\0\\p_2\\0\end{pmatrix}$, ...

$\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\,\epsilon_{abcd}\,p_1^a p_2^b p_3^c p_4^d$ must hold generally.

4. $u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}$, $v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ -\sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}$

$\sigma^\mu = (1, \underline{\sigma})$, $\bar\sigma^\mu = (1, -\underline{\sigma})$ $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$ $r,s \in \{1,2\}$

$$u^r(p)^\dagger \cdot u^s(p) = (\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad \xi^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}$$

$$= (\xi^r)^\dagger p\cdot\sigma\, \xi^s + (\xi^r)^\dagger p\cdot\bar\sigma\, \xi^s = (\xi^r)^\dagger (p_0 + \underline{p}\cdot\underline{\sigma} + p_0 - \underline{p}\cdot\sigma)\xi^s$$

$$= 2p_0 (\xi^r)^\dagger \xi^s = 2p_0\delta^{rs}$$

$$\bar u^r(p)\cdot u^s(p) = (\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad \xi^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}$$

$$= \xi^{r\dagger}\left(\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} + \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)}\right)\xi^s$$

$$= \xi^{r\dagger}\left(2\sqrt{(p_0 + \underline{p}\cdot\underline{\sigma})(p_0 - \underline{p}\cdot\underline{\sigma})}\right)\xi^s = 2\xi^{r\dagger}\sqrt{p_0^2 - (\underline{p}\cdot\underline{\sigma})^2}\,\xi^s = 2m\delta^{rs}$$

$\Big\lceil (\underline{p}\cdot\underline{\sigma})^2 = |\underline{p}|^2$ as $(\underline{p}\cdot\underline{\sigma})^2 = \sum_{a,b} p_a p_b \sigma^a\sigma^b = \frac{1}{2}\sum_{a,b} p_a p_b \{\sigma^a, \sigma^b\} = \sum_{a,b} p_a p_b \delta^{ab} = |\underline{p}|^2 \Big\rfloor$

$$v^r(p)^\dagger \cdot v^s(p) = (\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad -\xi^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ -\sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix} = 2p_0\delta^{rs}$$

$$\bar v^r(p)\cdot v^s(p) = (\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad -\xi^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ -\sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}$$

$$= -2m\delta^{rs}$$

$$\bar u^s(p)\cdot v^r(p) = (\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad \xi^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ -\sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix} = 0$$

$$u^s(p)^\dagger \cdot v^r(-p) = (\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad \xi^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} \sqrt{p'\cdot\sigma}\ \xi^s \\ -\sqrt{p'\cdot\bar\sigma}\ \xi^s \end{pmatrix}$$

$$= \xi^{r\dagger}\left(\sqrt{(p\cdot\sigma)(p'\cdot\sigma)} - \sqrt{(p\cdot\bar\sigma)(p'\cdot\bar\sigma)}\right)\xi^s$$

$$= \xi^{r\dagger}\left(\sqrt{(p_0 + \underline{p}\cdot\underline{\sigma})(p_0 - \underline{p}\cdot\underline{\sigma})} - \sqrt{(p_0 - \underline{p}\cdot\underline{\sigma})(p_0 + \underline{p}\cdot\underline{\sigma})}\right)\xi^s = 0$$

5. $$\sum_s u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\sigma} \quad \xi^{s\dagger}\sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\cancel{\sum_s \left(\sqrt{p\cdot\sigma}(\xi^s)^2\sqrt{p\cdot\bar\sigma}\right) \sum_s \begin{pmatrix} (p\cdot\sigma) & \sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} \\ \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)} & (p\cdot\bar\sigma) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\,\sum_s \xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sum_s \xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\,\sum_s \xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\,\sum_s \xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} \qquad \text{as} \quad \sum_s \xi^s\xi^{s\dagger} = \mathbb{1}_2$$

$$= \not{p} + m$$

$$\sum_s v^s(p)\,\bar{v}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\sigma} \quad -\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix} = \not{p} - m$$

6. $$\{\psi(\underline{x}), \psi(\underline{y})\} = \sum_{r,s} \int \frac{d^3p\,d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \Big[\overbrace{\{b_p^s, b_q^r\}}^{=0} u^s(p) u^r(q)\, e^{i(p\cdot x + q\cdot y)}$$
$$+ \overbrace{\{c_p^s, c_q^r\}}^{=0} v^s(p) v^r(q)\, e^{-i(p\cdot \underline{x} + q\cdot y)} \Big] = 0$$

$$\{\psi^\dagger(\underline{x}), \psi^\dagger(y)\} = 0$$

$$\{\psi(\underline{x}), \psi^\dagger(y)\} = \sum_{r,s} \int \frac{d^3p\,d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \Big[\{b_p^s, b_q^{r\dagger}\} u^s(p) u^r(q)^\dagger e^{i(p\cdot \underline{x} - q\cdot y)}$$
$$+ \{c_p^s, c_q^{r\dagger}\} v^s(p) v^r(q)^\dagger e^{-i(p\cdot\underline{x} - q\cdot y)} \Big]$$

$$= \sum_{r,s} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big[\delta^{rs} u^s(p) u^r(p)^\dagger e^{ip\cdot(\underline{x}-y)} + \delta^{rs} v^s(p) v^r(p)^\dagger e^{-ip\cdot(\underline{x}-y)} \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big[(\not{p}+m)\gamma^0 e^{ip\cdot(\underline{x}-y)} + \underbrace{(\not{p}-m)\gamma^0 e^{-ip\cdot(\underline{x}-y)}}_{\text{take } -p \text{ as } \int_{-\infty}^{\infty}} \Big] \qquad \text{using } (\gamma^0)^2 = 1$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big[(p_0\gamma^0 + p_i\gamma^i + m)\gamma^0 e^{ip\cdot(\underline{x}-y)} + (p_0\gamma^0 - p_i\gamma^i - m)\gamma^0 e^{+ip\cdot(\underline{x}-y)} \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(2p_0(\gamma^0)^2 e^{ip\cdot(\underline{x}-y)} \right) = \delta^{(3)}(\underline{x}-y) \qquad \text{as } E_p = p_0 .$$

7. $$H = \int d^3x\, \bar\psi(-i\gamma^i\partial_i + m)\psi$$

$$= \int \frac{d^3p\, d^3q\, d^3x}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \sum_{r,s} \left[b_p^{s\dagger} u^s(p)^\dagger e^{-ip\cdot x} + c_p^s v^s(p)^\dagger e^{ip\cdot x} \right]\gamma^0$$

$$\Big[\underbrace{\left(-i\gamma^i (\cancel{+}iq_i) + m\right)}_{-\gamma^i q_i + m} b_q^r u^r(q) e^{iq\cdot x} + \underbrace{\left(-i\gamma^i(+iq_i) + m\right)}_{\gamma^i q_i + m} c_q^{r\dagger} v^r(q) e^{-iq\cdot x} \Big]$$

[Recall $(\gamma^\mu p_\mu - m)u(p) = 0 \Rightarrow (-\gamma^i q_i + m)u^r(q) = \gamma^0 q_0\, u^r(q)$

$(\gamma^\mu p_\mu + m)v(p) = 0 \Rightarrow (\gamma^i q_i + m)v^r(q) = -\gamma^0 q_0\, v^r(q)$]

$$= \int \frac{d^3p\, d^3q\, d^3x}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \sum_{r,s} \left[b_p^{s\dagger} u^s(p)^\dagger e^{-ip\cdot x} + c_p^s v^s(p)^\dagger e^{ip\cdot x} \right]\gamma^0$$

$$\gamma^0 q_0 \left[b_q^r u^r(q) e^{iq\cdot x} - c_q^{r\dagger} v^r(q) e^{-iq\cdot x} \right]$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{q_0}{2\sqrt{E_p E_q}} \sum_{r,s} \Big[b_p^{s\dagger} b_q^r u^s(p)^\dagger u^r(q)\, \delta^{(3)}(p-q)$$

$$+ c_p^s b_q^r v^s(p)^\dagger u^r(q)\, \delta^{(3)}(p+q) - b_p^{s\dagger} c_q^{r\dagger} u^s(p)^\dagger v^r(q)\, \delta^{(3)}(p+q)$$

$$- c_p^s c_q^{r\dagger} v^s(p)^\dagger v^r(q)\, \delta^{(3)}(p-q) \Big]$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{2} \sum_{r,s} \Big[b_q^{s\dagger} b_q^r u^s(q)^\dagger u^r(q) + c_{-q}^s b_q^r v^s(-q)^\dagger u^r(q)$$

$$- b_{-q}^{s\dagger} c_q^{r\dagger} u^s(-q)^\dagger v^r(q) - c_q^s c_q^{r\dagger} v^s(q)^\dagger v^r(q) \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[b_p^{s\dagger} b_p^s\, p_0 - c_p^s c_p^{s\dagger}\, p_0 \right] \quad \text{as } u^s(p)^\dagger \cdot v^r(-p) = 0$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left[b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s - (2\pi)^3 \delta^{(3)}(0) \right]$$

$$\longrightarrow \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left[b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s \right] \quad \text{after normal ordering}$$

8. $[b^r_p, b^{s\dagger}_q] = (2\pi)^3 \delta_{rs} \delta^{(3)}(p-q)$, $[c^r_p, c^{s\dagger}_q] = -(2\pi)^3 \delta_{rs} \delta^{(3)}(p-q)$

$\Rightarrow [\psi(\underline{x}), \psi(\underline{y})] = [\psi^\dagger(\underline{x}), \psi^\dagger(\underline{y})] = 0$

$$[\psi(\underline{x}), \psi^\dagger(\underline{y})] = \sum_{r,s} \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \Big([b^s_p, b^{r\dagger}_q] u^s(p) u^r(q)^\dagger e^{i(p\cdot x - q\cdot y)}$$

$$+ [c^{s\dagger}_p, c^r_q] v^s(p) v^r(q)^\dagger e^{-i(p\cdot x - q\cdot y)} \Big)$$

this ordering requires − sign in operator commutator

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(u^s(p) u^s(p)^\dagger e^{ip\cdot(x-y)} + v^s(p) v^r(p)^\dagger e^{-ip\cdot(x-y)} \right)$$

again using identities for u, v

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left((\not{p}+m)\gamma^0 e^{ip\cdot(x-y)} + (\not{p}-m)\gamma^0 e^{-ip\cdot(x-y)} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left((p_0\gamma^0 + p_i\gamma^i + m)\gamma^0 e^{ip\cdot(x-y)} + (p_0\gamma^0 - p_i\gamma^i - m)\gamma^0 e^{ip\cdot(x-y)} \right)$$

$$= \delta^{(3)}(\underline{x}-\underline{y})$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_s [b^{s\dagger}_p b^s_p - c^s_p c^{s\dagger}_p] \quad \text{as previously}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \sum_s [b^{s\dagger}_p b^s_p - c^{s\dagger}_p c^s_p - (2\pi)^3 \delta^{(3)}(0)]$$

due to commutation relation

$$\rightarrow \int \frac{d^3p}{(2\pi)^3} E_p \sum_s [b^{s\dagger}_p b^s_p - c^{s\dagger}_p c^s_p] \quad \text{after normal ordering}$$