Symmetries and currents

(i) Space-time translations

$$x^\mu \to x^\mu + \varepsilon^\mu$$

$$\delta\psi = \varepsilon^\mu \partial_\mu \psi \qquad \delta\bar{\psi} = \varepsilon^\mu \partial_\mu \bar{\psi}$$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$S = \frac{1}{2}\int d^4x \left[\bar{\psi}(i\overleftarrow{\partial\!\!\!/} - m)\psi + \bar{\psi}(i\overrightarrow{\partial\!\!\!/} - m)\psi\right]$$

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}$$

or $$T^{\mu\nu} = \frac{i}{2}\bar{\psi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\psi - \eta^{\mu\nu}\mathcal{L}$$

Since we get a conserved current when eom are obeyed, so we can impose them. Our eom here is $(i\partial\!\!\!/ - m)\psi = 0$

$\Rightarrow$ in $T^{\mu\nu}$, $\mathcal{L} \to 0$ i.e. $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi$

e.g. $$E = \int T^{00}d^3x = \int d^3x\, i\bar{\psi}\gamma^0\partial^0\psi = \int d^3x\, \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

$$\underline{P}^i = \int d^3x\, T^{0i} = \int d^3x\, i\psi^\dagger\partial^i\psi$$

(ii) Lorentz transformations $$\psi^\alpha \to S[\Lambda]^\alpha{}_\beta\, \psi^\beta\overbrace{(x^\mu - \omega^\mu{}_\nu x^\nu)}^{\text{arg of }\psi}$$

$$\Rightarrow \delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu\partial_\mu\psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta$$

where $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$

but $(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu \Rightarrow \omega^{\mu\nu} = \Omega^{\mu\nu}$ from above

$$\Rightarrow \delta\psi^\alpha = -\omega^{\mu\nu}\left[x_\nu\partial_\mu\psi^\alpha - \frac{1}{2}(S_{\mu\nu})^\alpha{}_\beta\psi^\beta\right]$$

similarly

$$\delta\bar{\psi}_\alpha = -\omega^{\mu\nu}\left[x_\nu\partial_\mu\bar{\psi}_\alpha + \frac{1}{2}\bar{\psi}_\beta(S_{\mu\nu})^\beta{}_\alpha\right]$$

$\uparrow S[\Lambda]^{-1}$

We can write this as

$$j^\mu = -\omega^{\rho\sigma}\left[i\bar{\psi}\gamma^\mu(x_\sigma\partial_\rho\psi - \frac{1}{2}S_{\rho\sigma}\psi)\right] + \cancel{\omega^\mu{}_\nu x^\nu\mathcal{L}} \;\; \text{eom} = 0$$

i.e. $$(\mathcal{J}^\mu)^{\rho\sigma} = x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma} - \underbrace{i\bar{\psi}\gamma^\mu S^{\rho\sigma}\psi}_{\text{gives spin } \frac{1}{2} \text{ after quantisation}}$$

e.g. $$(\mathcal{J}^0)^{ij} = -i\bar{\psi}S^{ij}\psi = \frac{1}{2}\varepsilon^{ijk}\psi^\dagger\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\psi$$ is angular momentum op

(iii) Internal symmetries $\psi \to e^{i\alpha}\psi \Rightarrow \delta\psi = i\alpha\psi$

$$j_V^\mu = \bar\psi\gamma^\mu\psi$$

check

$$\partial_\mu j_V^\mu = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu(\partial_\mu\psi)$$

$$= im\bar\psi\psi - im\bar\psi\psi = 0 \quad \text{using} \begin{cases} i\partial\!\!\!/\psi = m\psi \\ i\bar\psi\overleftarrow{\partial\!\!\!/} = -m\bar\psi \end{cases}$$

The conserved quantity is

$$Q = \int d^3x\, j_V^0 = \int d^3x\, \bar\psi\gamma^0\psi = \int d^3x\, \psi^\dagger\psi \leftarrow \text{electric charge or particle \#}$$

(iv) Axial symmetry. When $m=0$, $\mathcal{L}$ admits another internal symmetry. Rotating LH/RH fermions in opposite directions (such symmetries are called chiral).

$$\psi \to e^{i\alpha\gamma^5}\psi \qquad \bar\psi \to \bar\psi e^{i\alpha\gamma^5}$$

gives the conserved axial current $j_A^\mu = \bar\psi\gamma^\mu\gamma^5\psi$ - an axial vector, only conserved when $m=0$:

$$\partial_\mu j_A^\mu = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi = 2im\bar\psi\gamma^5\psi$$

It turns out that the classical axial symmetry does not survive quantisation. — it's anomalous.

(H) $[c_p^r, c_q^{s\dagger}] = -(2\pi)^3\delta^{rs}\delta^3(p-q)$

$$\mathcal{H} = \bar\psi(-i\gamma^i\partial_i + m)\psi \qquad H = \int d^3x\,\mathcal{H}$$

Plug expansion of ψ into:

$$(-i\gamma^i\partial_i + m)\psi = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Big[b_p^s(-\gamma^i p_i + m)u_p^s e^{ip\cdot x} + c_p^{s\dagger}(\gamma^i p_i + m)v_p^s e^{-ip\cdot x}\Big]$$

$$(\underline{x}\cdot\underline{p} = \sum_i x^i p^i = -x^i p_i)$$

use $(p\!\!\!/ - m)u_p = 0 \Rightarrow (-\gamma^i p_i + m)u_p^s = \gamma^0 p_0 u_p^s$

$(p\!\!\!/ + m)v_p = 0 \Rightarrow (\gamma^i p_i + m)v_p^s = -\gamma^0 p_0 v_p^s$

so

$$(-i\gamma^i\partial_i + m)\psi = \int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\,\gamma^0\Big[b_p^s u_p^s e^{ip\cdot x} - c_p^{s\dagger}v_p^s e^{-ip\cdot x}\Big]$$

$\vdots$

$$\Rightarrow H = \int\frac{d^3p}{(2\pi)^3}E_p\left(b_p^{s\dagger}b_p^s - c_p^s c_p^{s\dagger}\right) = \int\frac{d^3p}{(2\pi)^3}E_p\Big[b_p^{s\dagger}b_p^s - c_p^{s\dagger}c_p^s + \overbrace{(2\pi)^3\delta^3(0)}^{\text{N.O.}}\Big]$$

so H is not bounded from below $\Rightarrow$ quantum theory makes no sense.

Fermionic Quantization

For spin 0 fields, recall $[a_p^\dagger, a_q^\dagger] = 0$

$$\Rightarrow a_p^\dagger a_q^\dagger |0\rangle = |p, q\rangle = |q, p\rangle$$

To impose fermionic stats, we must impose anti-commutator relations $\{A, B\} = AB + BA$.

$$\{\psi_\alpha(\underline{x}), \psi_\beta(\underline{y})\} = \{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)\} = 0$$

and $\{\psi_\alpha(\underline{x}), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta}\, \delta^3(\underline{x} - \underline{y})$

Claim $\left.\begin{array}{l} \{b_p^r, b_q^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^3(p-q) \\ \{c_p^r, c_q^{s\dagger}\} = \ldots \end{array}\right\}$ all other relations vanishing

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3} E_p \left[b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s - \cancel{(2\pi)^3 \delta^3(0)} \right] \quad \text{N.O.}$$

Fermi-Dirac Stats

We define the vacuum as in the bosonic case

$$b_p^s |0\rangle = c_p^s |0\rangle = 0$$

Although b and c satisfy $\{\cdot, \cdot\}$ relations, the H has $[\cdot, \cdot]$ relations with them.

$$[H, b_p^{r\dagger}] = H b_p^{r\dagger} - b_p^{r\dagger} H = \int \frac{d^3q}{(2\pi)^3} E_q \left[(b_q^{s\dagger} b_q^s + c_q^{s\dagger} c_q^s) b_p^{r\dagger} - b_p^{r\dagger} (b_q^{s\dagger} b_q^s + c_q^{s\dagger} c_q^s) \right]$$

$$= E_p b_p^{r\dagger}.$$

$$[H, b_p^r] = -E_p b_p^r$$

Summary "the bosonic QFT, with spinor indices and minus signs"

- Heisenberg picture expansion of $\psi, \bar{\psi}$,
- Feynman prop $S_F = \langle 0 | T \psi \bar{\psi} | 0 \rangle = \begin{cases} \langle 0 | \psi \bar{\psi} | 0 \rangle \\ -\langle 0 | \psi \bar{\psi} | 0 \rangle \end{cases}$ explain

$$S_F = i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\not{p} - m}{p^2 - m^2 + i\epsilon}$$

- Wick's theorem $\psi\bar{\psi} \equiv T\psi\bar{\psi} - :\psi\bar{\psi}: = S_F$
- Yukawa theory
- Feynman rules

examples: $\psi\psi \to \psi\psi$, $\psi\bar{\psi} \to \phi\phi$, $\psi\bar{\psi} \to \psi\bar{\psi}$ w/ explanation of minus & spinor indices