- Every $\mathfrak{g}$ (complex simple) has a real form $\mathfrak{g}_{\mathbb{R}}$ of compact type

- $\mathfrak{g}_{\mathbb{R}} = \mathcal{L}(G)$ where G is a compact Lie group

$\mathfrak{g}$	G
A_n	$SU(2n)$
B_n	$SO(2n+1)$
C_n	$Sp(2n)$
D_n	$SO(2n)$

- Each irrep R_Λ of $\mathfrak{g}$ is an irrep of $\mathfrak{g}_{\mathbb{R}}$ and irrep of G $\quad D_\Lambda = \mathrm{Exp}[R_\Lambda]$

Further R_Λ is unitary

$$D_\Lambda(g)\, D_\Lambda(g)^\dagger = \mathbb{1} \qquad \forall g \in G$$

$$\Rightarrow R_\Lambda(X)^\dagger = -R_\Lambda(X) \qquad \forall X \in \mathfrak{g}_{\mathbb{R}}$$

Gauge Invariance

gauge symmetry = redundancy in description of the system

EM
$$\vec{E} = -\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{E}$ and $\vec{B}$ invariant under

$$\Phi \to \Phi + \frac{\partial \alpha}{\partial t}$$
$$\vec{A} \to \vec{A} + \vec{\nabla}\alpha$$

relativistic

$$A^\mu = \Phi \quad \mu = 0$$
$$\quad A_i \quad \mu = 1,2,3 = i$$

gauge transformation

$$a_\mu \longrightarrow a_\mu + \partial_\mu \alpha$$

field strength tensor

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

Lagrangian

$$\mathcal{L}_{EM} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu}$$

quantisation $\rightsquigarrow$ free massless spin one particle

$A_\mu = -i a_\mu \in i\mathbb{R}$ imaginary $\in$ ~~$\mathcal{L}(U(1))$~~ $\mathcal{L}(U(1))$

<u>compact scalar field</u> $\phi : \mathbb{R}^{3,1} \longrightarrow \mathbb{C}$

$$\mathcal{L}_\phi = \partial_\mu \phi^* \partial^\mu \phi - W(\phi^* \phi)$$

$U(1)$ global symmetry $\quad \phi \to g\phi$

$\phi^* \to g^{-1}\phi^* \qquad g = \exp(i\delta) \in U(1)$

$\delta \in [0, 2\pi)$

infinitesimal transformation

$g = \exp(\varepsilon X) \qquad \varepsilon \ll 1$

$\approx 1 + \varepsilon X \qquad X \in \mathcal{L}(U(1)) \in i\mathbb{R}$

$\phi \to \phi + \delta_X \phi \qquad \delta_X \phi = \varepsilon X \phi$

$(\delta_X \phi)^* = -\varepsilon \underset{\in i\mathbb{R}}{X} \phi^*$

$\delta_X \mathcal{L}_\phi = 0 \Rightarrow$ conserved charge (Noether's theorem)

To couple the scalar to EM, <u>gauge</u> the $U(1)$ global symmetry

$g : \mathbb{R}^{3,1} \longrightarrow U(1)$

$\phi \to g(x)\phi$

$\phi^* \to g(x)^{-1}\phi^*$

Under infinitesimal change

$\delta_X \phi = \varepsilon X \phi$

$\Rightarrow \delta_X(\partial_\mu \phi) = \partial_\mu(\delta_X \phi) = \varepsilon \partial_\mu X \phi + \varepsilon X \delta\phi$

$\Rightarrow \mathcal{L}_\phi$ no longer invariant

Restore the invariant with covariant derivative

$D_\mu = \partial_\mu + A_\mu \qquad A_\mu : \mathbb{R}^{3,1} \longrightarrow \mathcal{L}(U(1)) =: \mathbb{R}$

U(1) gauge field

$$A_\mu \to A_\mu + \delta_X A_\mu \qquad \delta_X A_\mu = -\varepsilon \partial_\mu X \qquad (24)$$

$$\delta_X(D_\mu \phi) = \delta_X(\partial_\mu \phi + A_\mu \phi)$$

$$= \partial_\mu(\delta_X \phi) + A_\mu(\delta_X \phi) \cancel{\phi} - \varepsilon \partial_\mu X$$

$$= \varepsilon X \partial_\mu \phi + \varepsilon X A_\mu \phi$$

$$= \varepsilon X D_\mu \phi$$

$\Rightarrow$ Kinetic term $(D_\mu \phi)^*(D^\mu \phi)$ is gauge invariant

$$\mathcal{L} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) - W(\phi^* \phi)$$

quantisation $\longrightarrow$ scalar "electrons" interacting with photons

<u>Symmetry based on Lie Group ~~g~~ G</u>

- Choose some repn D of ~~g~~ G, dim $= N$

  repn space $V \approx \mathbb{C}^N$

  standard inner product

  $$(u, v) = \vec{u}^\dagger \vec{v} \qquad \forall \vec{u}, \vec{v} \in V$$

- Scalar field takes value in V

  $$\phi : \mathbb{R}^{3,1} \to V$$

- Scalar lagrangian

  $$\mathcal{L}_\phi = (\partial_\mu \phi, \partial^\mu \phi) - W[(\phi, \phi)]$$

- If D unitary

  $$D(g)^\dagger D(g) = \mathbb{1} \quad \forall g \in G$$

  $\mathcal{L}_\phi$ is invariant under a global symmetry transformation

  $$\phi \to D(g)\phi \qquad \forall g \in G$$

  Near the identity

  $$g = \operatorname{Exp}(\varepsilon X)$$

  $$D(g) = \operatorname{Exp}(\varepsilon R(X)) \in \operatorname{Mat}_N(\mathbb{C})$$

  antihermitian matrix

  $R : \mathcal{L}(G) \longrightarrow \operatorname{Mat}_N(\mathbb{C})$ defines a unitary rep of $\mathcal{L}(G)$

For $\varepsilon \ll 1$

$$D(g) \simeq 1_N + \varepsilon R(x) + O(\varepsilon^2)$$

infinitesimal symmetry transformation

$$\phi \to \phi + \delta_x \phi \qquad \delta_x \phi = \varepsilon R(x) \phi \quad \in V$$

Now gauge. symmetry transformation labelled by

$$X : \mathbb{R}^{3,1} \longrightarrow \mathcal{L}(G)$$

$$\delta_X \phi = \varepsilon R(X(x)) \phi \quad \in V$$