General Relativity 2

1. $\nabla_\rho e_\nu = \nabla_{e_\rho} e_\nu = \Gamma^\mu{}_{\nu\rho} e_\mu$

Change of basis $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$

$\nabla_{e'_\rho} e'_\nu = \Gamma'^\mu{}_{\nu\rho} e'_\mu$

$$\nabla_{e'_\rho} e'_\nu = \nabla_{e'_\rho}((A^{-1})^\sigma{}_\nu e_\sigma) = (A^{-1})^\tau{}_\rho \nabla_\tau ((A^{-1})^\sigma{}_\nu e_\sigma)$$

$$= (A^{-1})^\tau{}_\rho (\nabla_\tau (A^{-1})^\sigma{}_\nu) e_\sigma + (A^{-1})^\tau{}_\rho (A^{-1})^\sigma{}_\nu \nabla_\tau e_\sigma$$

$$= (A^{-1})^\tau{}_\rho e_\sigma e_\tau((A^{-1})^\sigma{}_\nu) + (A^{-1})^\tau{}_\rho (A^{-1})^\sigma{}_\nu \Gamma^\mu{}_{\sigma\tau} e_\mu$$

$$\Rightarrow \Gamma'^\mu{}_{\nu\rho} e'_\mu = (A^{-1})^\tau{}_\rho e_\tau((A^{-1})^\sigma{}_\nu) e_\sigma + (A^{-1})^\tau{}_\rho (A^{-1})^\sigma{}_\nu \Gamma^\mu{}_{\sigma\tau} e_\mu$$

$$\Rightarrow \Gamma'^\mu{}_{\nu\rho} = A^\mu{}_\tau (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma^\tau{}_{\lambda\sigma} + A^\mu{}_\tau (A^{-1})^\sigma{}_\rho e_\sigma((A^{-1})^\tau{}_\nu)$$

by $e_\mu = A^\nu{}_\mu e'_\nu$ and extracting components.

2. Not torsion free $\Gamma^\mu{}_{[\nu\rho]} \neq 0$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$(T(X,Y))^\mu = (\nabla_X Y)^\mu - (\nabla_Y X)^\mu - ([X,Y])^\mu$$

$$= X^\nu e_\nu(Y^\mu) + \Gamma^\mu{}_{\rho\nu} Y^\rho X^\nu - Y^\nu e_\nu(X^\mu) - \Gamma^\mu{}_{\rho\nu} X^\rho Y^\nu$$
$$- X^\nu e_\nu(Y^\mu) + Y^\nu e_\nu(X^\mu)$$

$$= \Gamma^\mu{}_{\rho\nu} Y^\rho X^\nu - \Gamma^\mu{}_{\rho\nu} X^\rho Y^\nu = 2\Gamma^\mu{}_{[\rho\nu]} Y^\rho X^\nu$$

$(T(X,Y))^a$ is a vector field $\because$ $(\nabla_X Y)^a$, $(\nabla_Y X)^a$, $[X,Y]^a$ are v. fields

As X^a, Y^b are vector fields, $T^a{}_{bc}$ is a $(1,2)$ tensor field with

$$T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} = 2\Gamma^\mu{}_{[\rho\nu]} \qquad ((T(X,Y))^\mu = T^\mu{}_{\nu\rho} X^\nu Y^\rho)$$

$$2\nabla_{[a}\nabla_{b]} f = 2 f_{;[ab]} = f_{;ab} - f_{;ba} = (df)_{a;b} - (df)_{b;a} \quad \text{(covectors)}$$

$$2\nabla_{[\mu}\nabla_{\nu]} f = (df)_{\mu,\nu} - \Gamma^\rho{}_{\mu\nu}(df)_\rho - (df)_{\nu,\mu} + \Gamma^\rho{}_{\nu\mu}(df)_\rho$$

in a coordinate basis

$$= f_{,\mu\nu} - \Gamma^{\rho}_{\mu\nu} f_{,\rho} - f_{,\nu\mu} + \Gamma^{\rho}_{\nu\mu} f_{,\rho} \qquad (f_{,\mu\nu} = f_{,\nu\mu})$$

$$= -\Gamma^{\rho}_{\mu\nu} f_{,\rho} + \Gamma^{\rho}_{\nu\mu} f_{,\rho} = (-\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\nu\mu}) f_{,\rho}$$

$$\overset{?}{=} - T^{\rho}_{\mu\nu}\, f_{;\rho}$$

This is a tensor eqn, hence holds generally $2\nabla_{[a}\nabla_{b]} f = -T^{c}_{ab}\nabla_c f$.

3. Field tensor F_{ab}, $g^{ab}\nabla_a F_{bc} = 0$ $\quad \nabla_{[a} F_{bc]} = 0$

Equation of motion

$$u^b u^a{}_{;b} = \frac{q}{m} F^a{}_b u^b$$

Path is parameterised by τ, consider

$$\frac{d}{d\tau}(g_{ab} u^a u^b)$$

Use normal coordinates:

$$\frac{d}{d\tau}(g_{\mu\nu} u^\mu u^\nu) = \frac{dx^\sigma}{d\tau}\frac{\partial}{\partial x^\sigma}(g_{\mu\nu} u^\mu u^\nu)$$

$$= u^\sigma g_{\mu\nu}(u^\mu{}_{,\sigma} u^\nu + u^\mu u^\nu{}_{,\sigma}) \qquad \text{as } g_{\mu\nu,\sigma} = 0 \text{ at } p$$

$$= g_{\mu\nu} u^\nu (u^\sigma u^\mu{}_{,\sigma}) + g_{\mu\nu} u^\mu (u^\sigma u^\nu{}_{,\sigma})$$

$$= g_{\mu\nu} u^\nu \frac{q}{m} F^\mu{}_\sigma u^\sigma + g_{\mu\nu} u^\mu \frac{q}{m} F^\nu{}_\sigma u^\sigma \qquad \text{(relabel)}$$

$$= 2\frac{q}{m} F_{\mu\nu} u^\mu u^\nu = 0 \qquad \text{as } F_{\mu\nu} \text{ antisymmetric.}$$

This is a tensor eq $\Rightarrow$ holds in general $\square$

4. Can define $\mathcal{E} \equiv T_{ab} u^a u^b$, in the rest LIF of an observer $u^a = e_0^a$
and $\mathcal{E} = T_{ab} u^a u^b = T_{ab} e_0^a e_0^b = T_{00}$, the energy.

Hence, the weak energy condition $T_{ab} u^a u^b \geq 0$ $\forall$ timelike u^a can be interpreted as requiring $\mathcal{E} \geq 0$ (energy non-negative) for all observers (which must have timelike u^a).

Eigenvalues $T^a{}_b v^b = \lambda v^a$ eigenvectors provide ~~an orthonormal~~ a basis $\{e_0^a, e_\alpha^a\}$ s.t.

$$T^\mu{}_\nu e_0^\nu = -\rho\, e_0^\mu, \quad T^\mu{}_\nu e_\alpha^\nu = p_{(\alpha)} e_\alpha^\mu$$

or $T^\mu{}_\nu e_\alpha^\mu = p_{(\alpha)} e_\alpha^\mu$ with $p_{(0)} = -\rho$

A timelike vector

$u = \cancel{} \; u^\alpha e_\alpha$ where $\alpha = 0, 1, 2, 3$

$$g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu} u^\alpha e_\alpha^\mu u^\beta e_\beta^\nu = u^\alpha u^\beta g_{\mu\nu} e_\alpha^\mu e_\alpha^\nu < 0$$

Now

$$T_{\mu\nu} u^\mu u^\nu = g_{\mu\sigma} T^\sigma{}_\nu u^\mu u^\nu = g_{\mu\sigma} u^\mu (T^\sigma{}_\nu u^\nu)$$

$$= g_{\mu\sigma} u^\mu (T^\sigma{}_\nu u^\alpha e_\alpha^\nu) = g_{\mu\sigma} u^\mu u^\alpha (T^\sigma{}_\nu e_\alpha^\nu)$$

$$= g_{\mu\sigma} u^\mu u^\alpha p_{(\alpha)} e_\alpha^\sigma = p_{(\alpha)} u^\alpha u^\beta g_{\mu\sigma} e_\beta^\mu e_\alpha^\sigma$$

In basis $\{e_0, e_\alpha\}$, splitting time and space on α

$$u^\alpha u^\beta g_{\alpha\beta} < 0 \qquad \Rightarrow u^0 u^\beta g_{0\beta} + u^{\alpha'} u^\beta g_{\alpha'\beta} < 0$$

$$T_{\mu\nu} u^\mu u^\nu = p_{(\alpha)} u^\alpha u^\beta g_{\alpha\beta} \Rightarrow -\rho u^0 u^\beta g_{0\beta} + p_{(\alpha)} u^{\alpha'} u^\beta g_{\alpha'\beta}$$

$$\geq \rho u^{\alpha'} u^\beta g_{\alpha'\beta} + p_{(\alpha)} u^{\alpha'} u^\beta g_{\alpha'\beta} \quad \text{by above iff } \rho \geq 0$$

$$= (\rho + p_{(\alpha)}) u^{\alpha'} u^\beta g_{\alpha'\beta} = (\rho + p_{(\alpha)}) u^\alpha u^\beta g_{\alpha\beta} \quad \text{by def } p_{(0)} = -\rho$$

Claim: $\rho + \min_\alpha \{0, p_{(\alpha)}\} \geq 0 \quad (*)$

WEC $\Rightarrow$ $(*)$ Basis of eigenvectors T_{ab} symmetric $\Rightarrow$ orthonormal basis

$$\therefore \rho (u^0)^2 + \sum_\alpha p_{(\alpha)} (u^\alpha)^2 \geq 0$$

Take $u^0 = 1$, $u^i = 0 \ \forall i \Rightarrow \rho \geq 0$

$u^0 = 1$, $u^i = 1 - \varepsilon$ for some i, $\lim_{\varepsilon \to 0} \Rightarrow \rho + p_i \geq 0 \ \forall i$

$\therefore \rho + \min_\alpha \{0, p_{(\alpha)}\} \geq 0 \ \square$

$(*) \Rightarrow$ WEC: $T_{ab} u^a u^b = \rho (u^0)^2 + \sum_\alpha p_{(\alpha)} (u^\alpha)^2$

$$\geq \rho (u^0)^2 + \min(0, p_{(\alpha)}) \sum_\alpha (u^\alpha)^2$$

$$\geq [\rho + \min(0, p(\alpha))] \sum_\alpha (u^\alpha)^2 \quad \text{as } u^\alpha \text{ timelike}$$

$$\geq 0 \ \square$$

5. (a) Use normal coords: $\nabla^a T_{ab}$, $T_{ab} = (\nabla_a \Phi)(\nabla_b \Phi) - \frac{1}{2} g_{ab} [(\nabla^c \Phi)(\nabla_c \Phi) + m^2 \Phi^2]$

$$T_{\mu\nu;}{}^{\mu} = \Phi_{,\mu}{}^{\mu} \Phi_{,\nu} + \Phi_{,\mu} \Phi_{,\nu}{}^{\mu} - \frac{1}{2} g_{\mu\nu} \left[\Phi^{,\sigma\mu} \Phi_{,\sigma} + \Phi^{,\sigma} \Phi_{,\sigma}{}^{\mu} + 2m^2 \Phi \Phi^{,\mu} \right]$$

$$= \Phi_{,\mu}{}^{\mu} \Phi_{,\nu} + \Phi_{,\mu} \Phi_{,\nu}{}^{\mu} - \Phi^{,\sigma} \Phi_{,\sigma\nu} - m^2 \Phi \Phi_{,\nu}$$

$$= \Phi_{,\mu} \Phi_{,\nu}{}^{\mu} - \Phi^{,\sigma} \Phi_{,\sigma\nu}$$ by KG eq $\nabla^a \nabla_a \Phi - m^2 \Phi = 0$

$$= \Phi^{,\mu} \Phi_{,\nu\mu} - \Phi^{,\sigma} \Phi_{,\sigma\nu} = 0$$ by sym of $\Phi_{,\nu\mu}$

Tensor eq => holds generally □

(b) $T_{ab} = \frac{1}{4\pi} \left(F_{ac} F_b{}^c - \frac{1}{4} F^{cd} F_{cd} g_{ab} \right)$ (Levi-Civita connection)

$$\nabla^a T_{ab} = \frac{1}{4\pi} \left(F_b{}^c \cancel{\nabla^a F_{ac}} + F_{ac} \nabla^a F_b{}^c - \frac{1}{2} g_{ab} F^{cd} \nabla^a F_{cd} \right)$$

0 by

$$= \frac{1}{4\pi} \left(F_{ac} \nabla^a F_b{}^c - \frac{1}{2} F^{cd} \nabla_b F_{cd} \right)$$

Use $\nabla_{[a} F_{bc]} = \frac{1}{6} \left(\nabla_a F_{bc} - \nabla_a F_{cb} + \nabla_b F_{ca} - \nabla_b F_{ac} + \nabla_c F_{ab} - \nabla_c F_{ba} \right)$

$$= \frac{1}{3} \left(\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} \right)$$ by antisymmetry of F_{ab}

$$= 0$$

$$\cancel{\Rightarrow \nabla^a T_{ab} = \frac{1}{4\pi} \left(F^{ac} \nabla_b F_{ca} + F^{ab} \nabla_c F_{ab} + \frac{1}{2} F^{cd} \nabla_b F_{cd} \right)}$$

$$\cancel{= \frac{1}{4\pi} \left(F^{ac} \nabla_c F_{ab} - \frac{1}{2} F^{ac} \nabla_b F_{ac} \right)}$$

$$\Rightarrow \nabla^a T_{ab} = \frac{1}{4\pi} \left(F^{ac} \nabla_a F_{bc} + \frac{1}{2} F^{cd} \nabla_c F_{db} + \frac{1}{2} F^{cd} \nabla_d F_{bc} \right)$$

$$= \frac{1}{8\pi} \left(F^{ac} \nabla_a F_{bc} + F^{cd} \nabla_d F_{bc} \right) = 0$$

(c) $T_{ab} = (\rho + p) u_a u_b + p g_{ab}$ $\quad \nabla^a T_{ab} = 0$

$$\Rightarrow (\nabla^a \rho + \nabla^a p) u_a u_b + (\rho + p)(\nabla^a u_a) u_b + (\rho + p) u_a \nabla^a u_b + \nabla^a p \, g_{ab} = 0$$

$$\cancel{u^b (\nabla^b \rho + \nabla^b p) u_a u_b + (\rho + p)(\nabla^b u_b) u_a + (\rho + p) u_b \nabla^b u_a + \nabla^b p \, g_{ab} = 0}$$

6. $\nabla_X Y = 0$ and $\nabla_X X = 0$ as geodesic

$|X|^2 \equiv g(X,X)$

$\nabla_X g(X,X) = g(\nabla_X X, X) + g(X, \nabla_X X) = 0$ as Levi-Civita connection

magnitude $|X|$ constant

$|Y|^2 \equiv g(Y,Y)$

$\nabla_X g(Y,Y) = g(\nabla_X Y, Y) + g(Y, \nabla_X Y) = 0$

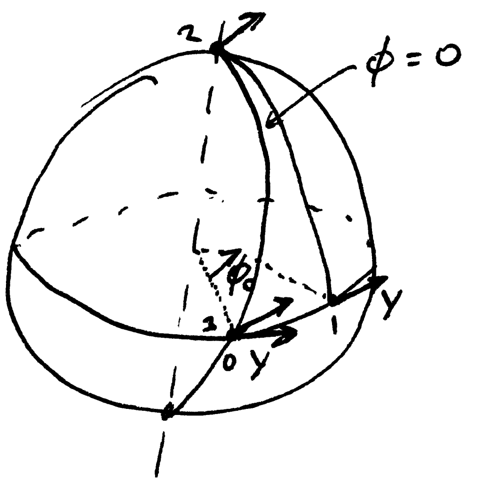
magnitude $|Y|$ constant

Angle is defined on a Riemannian manifold

$\cos\theta \equiv \frac{g(X,Y)}{|X|\,|Y|}$ $\qquad \nabla_X \cos\theta = \frac{\nabla_X g(X,Y)}{|X|\,|Y|}$ as $|X|, |Y|$ const

$= \frac{g(\nabla_X X, Y) + g(X, \nabla_X Y)}{|X|\,|Y|} = 0$

θ conserved



Changed by ϕ_0.

Magnitude is constant. ~~as all curves are geodesics.~~

0. Parallel to equator at $\phi = 0$, $\theta = \pi/2$
1. Parallel to equator at $\phi = \phi_0$, $\theta = \pi/2$
   $\perp$ to line $\phi = \phi_0$
2. $\perp$ to line $\phi = \phi_0$ at ~~ϕ~~ $\theta = 0$
   $\measuredangle\, \phi_0$ to line $\phi = 0$
3. $\measuredangle\, \phi_0$ to line $\phi = 0$ at $\phi = 0$, $\theta = \pi/2$.

7. $\nabla_c \nabla_d \eta_a - \nabla_d \nabla_c \eta_a \overset{!}{=} -R^b{}_{acd}\, \eta_b$

Use normal coordinates:

$$\nabla_\rho \nabla_\sigma \eta_\mu - \nabla_\sigma \nabla_\rho \eta_\mu = \nabla_\rho(\eta_{\mu,\sigma} - \Gamma^\lambda_{\mu\sigma}\eta_\lambda) - \nabla_\sigma(\eta_{\mu,\rho} - \Gamma^\lambda_{\mu\rho}\eta_\lambda)$$

$$= \eta_{\mu,\sigma\rho} - \Gamma^\lambda_{\mu\sigma,\rho}\eta_\lambda - \Gamma^\tau_{\mu\rho}(\eta_{\tau,\sigma} - \Gamma^\lambda_{\tau\sigma}\eta_\lambda) - \Gamma^\tau_{\sigma\rho}(\eta_{\mu,\tau} - \Gamma^\lambda_{\mu\tau}\eta_\lambda)$$

$$- \eta_{\mu,\rho\sigma} + \Gamma^\lambda_{\mu\rho,\sigma}\eta_\lambda + \dots + \dots = -\Gamma^\lambda_{\mu\sigma,\rho}\eta_\lambda + \Gamma^\lambda_{\mu\rho,\sigma}\eta_\lambda$$

as $\Gamma = 0$ in normal coordinates at p

$= -R^\nu{}_{\mu\rho\sigma}\,\eta_\nu$ $\qquad$ Both sides are tensors $\Rightarrow$ holds in any basis

Choice of p arbitrary $\Rightarrow$ global

8. $R^a{}_{b[cd;e]} = 0$ Bianchi identity

$$R^a{}_{bcd;e} - R^a{}_{bdc;e} + R^a{}_{bde;c} - R^a{}_{bed;c} + R^a{}_{bec;d} - R^a{}_{bce;d} = 0$$

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0 \quad \text{as} \quad R^a{}_{b(cd)} = 0$$

Contract a,c (multiply by $\delta^c{}_a$)

$$R^a{}_{bad;e} + R^a{}_{bde;a} + R^a{}_{bea;d} = R_{bd;e} + R^a{}_{bde;a} + R^a{}_{bea;d} = 0$$

by definition $R_{ab} = R^c{}_{acb}$.

Multiply by g^{bd}, and $g^{ab} R_{ab} = R$

$$R_{;e} + R^{ab}{}_{be;a} + R^{ab}{}_{ea;b} = 0$$

$$R_{;e} - R^a{}_{e;a} - R^b{}_{e;b} = 0 \quad \text{by} \quad R^a{}_{b(cd)} = 0, \; R_{(ab)cd} = 0$$

$$\Rightarrow \nabla^a R_{ab} - \frac{1}{2} \nabla_b R = 0 \quad \text{Contracted Bianchi identity}$$

$$\nabla^a G_{ab} = G_{ab;}{}^a = G^a{}_{b;a} = 0$$

10. Have $R^{a}{}_{b(cd)} = 0$. $R_{(ab)cd} = 0$ ①

$R^{a}{}_{[bcd]} = 0$ ③

$R_{abcd} = R_{cdab}$ ②

Hence $\cancel{n^4} \cancel{\frac{n^2}{2!}} \cancel{\frac{n^3}{3!}}$ ①: Each means that have $\binom{n}{2}$ independent components for a pair of indices (0 if equal; otherwise order does not give new comps)

$\Rightarrow \binom{n}{2}^2$ independent (non-zero) components

②: Consider these values as a matrix R_{KL} where K label pairs of ab and L pairs cd.

This matrix is symmetric $\Rightarrow \binom{n}{2} \times \binom{n}{2}$ symmetric matrix

$$\sum_{i=0}^{\binom{n}{2}} i = \frac{\binom{n}{2}\left(\binom{n}{2}+1\right)}{2}$$ independent components (upper triangle + diag)

$$= \frac{\left(\frac{n(n-1)}{2}\right)\left(\left(\frac{n(n-1)}{2}\right)+1\right)}{2} = \frac{n(n-1)(n^2-n+2)}{8}$$

$$\cancel{= \frac{n(n-1)(n^2-n+2)}{8}}$$

③: No new constraints if any pair equal (satisfied by other conditions).
Swapping any two also gives no new constraints hence $\binom{n}{4}$ constraints.

$$\therefore \text{components} = \frac{n(n-1)(n^2-n+2)}{8} - \frac{n(n-1)(n-2)(n-3)}{24}$$

$$= \frac{3(n^4 - n^3 + 2n^2 - n^3 + n^2 - 2n) - (n^4 - 6n^3 + 11n^2 - 6n)}{24}$$

$$= \frac{2n^4 - 2n^2}{24} = \frac{1}{12} n^2 (n^2-1)$$

In 2D, 1 components, show this form satisfies (hence unique as only 1 component.)

$R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc})$ satisfies symmetries

$R_{bd} = g^{ac} R_{abcd} = \frac{1}{2} R g_{bd}$

$R = g^{bd} R_{bd} = R$ as required.

$G_{ab} = \frac{1}{2} R g_{ab} - \frac{1}{2} R g_{ab} = 0 \Rightarrow T_{ab} = 0$ everywhere by Einstein.

Pure vacuum universe

11. $C_{abcd} = R_{abcd} + \alpha(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) + \beta R(g_{ac}g_{bd} - g_{ad}g_{bc})$

$$C_{ab(cd)} = R_{ab(cd)} + \frac{\alpha}{2}(\cancel{R_{ac}g_{bd}} + \cancel{R_{ad}g_{bc}} + \cancel{R_{bd}g_{ac}} + \cancel{R_{bc}g_{ad}} - \cancel{R_{ad}g_{bc}} - \cancel{R_{ac}g_{bd}} - \cancel{R_{bc}g_{ad}} - \cancel{R_{bd}g_{ac}}) + \frac{\beta}{2}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ad}g_{bc} - g_{ac}g_{bd}) = 0$$

$C_{(ab)cd} = \dots = 0$

$$C_{abcd} \overset{?}{=} C_{cdab} = R_{cdab} + \alpha(R_{ca}g_{db} + R_{db}g_{ca} - R_{cb}g_{da} - R_{da}g_{cb}) + \beta R(g_{bd}g_{ac} - g_{cb}g_{da}) = C_{abcd}.$$

$C_{a[bcd]} = \dots = 0$

Set $$C^a{}_{bad} = R^a{}_{bad} + \alpha(R^a{}_a g_{bd} + R_{bd}g^a{}_a - R^a{}_d g_{ba} - R_{ba}g^a{}_d) + \beta R(g^a{}_a g_{bd} - g^a{}_d g_{ba})$$

$$= R_{bd} + \alpha(Rg_{bd} + nR_{bd} - R_{bd} - R_{bd}) + \beta R(ng_{bd} - g_{bd})$$

$$= (1 + \alpha n - 2\alpha)R_{bd} + (\alpha + \beta n - \beta)Rg_{bd} = 0$$

Choose $1 + \alpha n - 2\alpha = 0$, $\alpha + \beta n - \beta = 0$ $\Rightarrow$ $\alpha = -\frac{1}{n-2}$, $\beta = -\frac{\alpha}{n-1}$ $\quad n \neq 2$

$n=2$: $C^a{}_{bad} = R_{bd} + (\cancel{2\alpha}+\beta)Rg_{bd}$

$n=3$: $C^a{}_{bad} = 0$ $\quad \alpha = -1$, $\beta = \frac{1}{2}$

C_{abcd} vanishes in $n=2$ if set $\alpha = 0$, $\beta = -\frac{1}{2}$ by result of problem 10.

$n=3$: $C_{abcd} = R_{abcd} - (R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) + \cancel{\beta}\frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc})$

R_{abcd} has 6 independent components. Symmetry of R_{ab} $\Rightarrow$ up to 6 independent components of R_{ab}

$C_{abcd} = 0$ if $R_{abcd} = (R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) - \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc})$

$\Rightarrow R_{bd} = g^{ac}R_{abcd} = (Rg_{bd} + 3R_{bd} - R_{bd} - R_{bd}) - \frac{1}{2}R(3g_{bd} - g_{bd})$

$= R_{bd}$ as required.

$n=4$: R_{abcd} has 20 independent components

R_{ab} can have 10

C_{abcd} is equiv to R_{abcd}. $- C^a{}_{bad} = 0 = 20 - 10$ cons

C_{abcd} represents gravitational degrees of freedom.

In vacuum, $R_{ab} - \frac{1}{2}Rg_{ab} = 0$ $\qquad \alpha = -\frac{1}{2}, \beta = \frac{1}{6}$

$$C_{abcd} = R_{abcd} - \frac{1}{4}R(g_{ac}g_{bd} + g_{bd}g_{ac} - g_{ad}g_{bc} - g_{bc}g_{ad}) + \frac{1}{6}R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$= R_{abcd} - \frac{1}{3}R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$\nabla^a C_{abcd} = \nabla^a R_{abcd} - \frac{1}{3}(g_{ac}g_{bd} - g_{ad}g_{bc})\nabla^a R$$

$= \cancel{\nabla^a R_{abcd} - \frac{1}{3}(g_{ac}g_{bd} - g_{ad}g_{bc})\nabla^a R}$

$\cancel{\nabla^a C_{ab} = \nabla^a R_{ab} - \frac{1}{2}\nabla_b R}$

$$= R_{abcd;}{}^{a} - \frac{1}{3}(g_{ac}g_{bd} - g_{ad}g_{bc})R^{;a}$$

$$= -R^a{}_{bda;c} - R^a{}_{bac;d} - \frac{1}{3}(g_{ac}g_{bd} - g_{ad}g_{bc})R^{;a}$$

$$= R_{bd;c} - R_{bc;d} - \frac{1}{3}(\dots)R^{;a} = \frac{1}{2}g_{bd}R_{;c} - \frac{1}{2}g_{bc}R_{;d} - \frac{1}{3}(\dots)R^{;a}$$

Vacuum $R_{ab} - \frac{1}{2}Rg_{ab} = 0$

$$= \frac{1}{2}g_{bd}R_{;c} - \frac{1}{2}g_{bc}R_{;d}$$
$$- \frac{1}{3}g_{bd}R_{;c} + \frac{1}{3}g_{bc}R_{;d}$$
$$= \frac{1}{6}g_{bd}R_{;c} - \frac{1}{6}g_{bc}R_{;d}$$
$$= \nabla^a C_{ab[cd]}$$
$$= 0 \text{ as } C_{ab[cd]} = 0$$

□

12. $\nabla^e \nabla_e R_{abcd} = \nabla^e R_{abcd;e}$

$$= -\nabla^e R_{abde;c} - \nabla^e R_{abec;d}$$

$$= -R_{abde;c}{}^{e} - R_{abec;d}{}^{e}$$

Use normal coords

$$= -R_{abde,}{}^{e}{}_{c} - R_{abec,}{}^{e}{}_{d}$$

$$= +R_{abe}{}^{e}{}_{,dc} + R_{abed,}{}^{e}{}_{c} + R_{abce,}{}^{e}{}_{d} + R_{abe}{}^{e}{}_{,cd}$$

$$= 2R_{abe}{}^{e}{}_{,dc} - R_{abdc,}{}^{e}{}_{e}$$

$$= 2R^e{}_{bea,dc} + 2R^e{}_{aeb,dc} - R_{abdc,}{}^{e}{}_{e}$$

$$= -2R_{ba,dc} + 2R_{ab,dc} - R_{abdc,}{}^{e}{}_{e} =$$

12. $\nabla^e \nabla_e R_{abcd} \doteq 2R_{aeaf} R_b{}^e{}_c{}^f - 2R_{aecf} R_b{}^e{}_d{}^f - R_{abef} R_{cd}{}^{ef}$

$\nabla^e R_{abcd;e} = -\nabla^e R_{abde;c} - \nabla^e R_{abec;d}$

In normal coords

$\nabla^\mu \nabla_\mu R_{\nu\sigma\rho\tau} = -R_{\nu\sigma\tau}{}^\mu{}_{;\rho\mu} - R_{\nu\sigma}{}^\mu{}_{\rho;\tau\mu}$

$= -R_{\nu\sigma\tau}{}^\mu{}_{;\mu\rho} - R_{\nu\sigma}{}^\mu{}_{\rho;\mu\tau}$

$= R_\tau{}^\mu{}_{\mu\nu;\sigma\rho} + R_\tau{}^\mu{}_{\sigma\mu;\nu\rho} + R^\mu{}_{\rho\sigma\mu;\nu\tau} + R^\mu{}_{\rho\mu\nu;\sigma\tau}$

$= -R_{\tau\nu;\sigma\rho} + R_{\tau\sigma;\nu\rho} - R_{\rho\sigma;\nu\tau} + R_{\rho\nu;\sigma\tau}$

Vacuum $R_{ab} - \frac{1}{2} R g_{ab} = 0$

∇^e(Bianchi), $\quad$ Ricci $(\nabla^e \nabla_c - \nabla_c \nabla^e) R_{abde}$

∇^e(Bianchi) $= \nabla^e \nabla_e R_{abcd} + \nabla^e \nabla_c R_{abde} + \nabla^e \nabla_d R_{abec} = 0$

Use $\nabla^a R_{abcd} = 0$ in vacuum

$\nabla^e \nabla_e R_{abcd} + (\nabla^e \nabla_c - \nabla_c \nabla^e) R_{abde} + (\nabla^e \nabla_d - \nabla_d \nabla^e) R_{abec} = 0$

$\nabla^e \nabla_e R_{abcd} + R^f{}_a{}^e{}_c R_{fbde} + R^f{}_b{}^e{}_c R_{afde} + R^f{}_d{}^e{}_c R_{abfe} + \cancel{R^f{}_e{}^e{}_c} R_{abdf}$ = 0 vacuum

$+ R^f{}_a{}^e{}_d R_{fbec} + R^f{}_b{}^e{}_d R_{afec} + 0 + R^f{}_c{}^e{}_d R_{abef}$

$= \nabla^e \nabla_e R_{abcd} + 2R_{aecf} R_b{}^e{}_d{}^f + 2 R_{aedf} R_b{}^e{}_c{}^f$

$-R_{abef}\left([R^f{}_d{}^e{}_c - R^f{}_c{}^e{}_d] = -R_c{}^f{}_d{}^e - R_c{}^{ef}{}_d = R_{cd}{}^{ef} \right) = 0 \quad \square$