Let $x^\mu(\lambda)$ be any tangent future-directed causal curve with tangent $V^\mu = \frac{dx^\mu}{d\lambda}$. Assume that $r(\lambda_0) \leq 2M$, then $r(\lambda) \leq 2M \quad \forall \lambda > \lambda_0$. $\square$

$$0 \geq \left(-\frac{\partial}{\partial r}\right)\cdot V = -\frac{dv}{d\lambda} \Rightarrow \frac{dv}{d\lambda} \geq 0 \quad (*)$$ (Both $-\frac{\partial}{\partial r}$ and V are causal and future-directed.)

and

$$-2\left(\frac{dv}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) = -V^2 + \left(\frac{2M}{r}-1\right)\left(\frac{dv}{d\lambda}\right)^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2 \quad (**)$$ (Because V is causal)

We conclude that

$$\frac{dr}{d\lambda} \leq 0 \quad \text{if } r \leq 2M.$$

If $r < 2M$, then $\frac{dr}{d\lambda} \leq 0$ has to be strict. From $(**)$ $\frac{d\Omega}{d\lambda} = \frac{dv}{d\lambda} = 0$ but then $V^\mu = 0$ $\#$

Hence if $r(\lambda_0) < 2M$, then $r(\lambda)$ is monotonically decreasing $\forall \lambda \geq \lambda_0$.

Finally $r(\lambda_0) = 2M$, if $\frac{dr}{d\lambda} < 0$ for $\lambda > \lambda_0$ ✓

$\frac{dr}{d\lambda} = 0$ for $\lambda > \lambda_0$ ✓

$\frac{dr}{d\lambda} > 0$ for $\lambda \geq \lambda_0$?

Exactly @ $\lambda = \lambda_0$, $r(\lambda_0) = 2M$. So from $(**)$, $\left.\frac{dv}{d\lambda}\right|_{\lambda=\lambda_0} \neq 0$, or else $V^\mu = 0$ $\#$. So from $(*)$ $\frac{dv}{d\lambda} > 0$.

Very close to $v \simeq v_0 = v(\lambda_0)$, we can interchange v and λ. We can divide $(**)$ by $(dv/d\lambda)^2$.

$$2\frac{dr}{dv} \leq 1 - \frac{2M}{r}.$$

Take $v_0 < v_1 < v_2$, then $$2\int_{r(v_1)}^{r(v_2)} \frac{dr}{1-\frac{2M}{r}} \leq v_2 - v_1, \quad v_1 \to v_0,$$

$\nwarrow$ diverges if $v_1 \to v_0$

$\square$

Take $r(\lambda) > 2M$: note that outgoing geodesics can reach infinity. The surface $r = 2M$ is an event horizon.

Detecting black holes (impressive)

1) No upper bound on the mass. Note for cold stars $M \lesssim 3M_\odot$

2) Have to be small $(R = 2M)$ $\begin{cases} M = M_\odot, & R = 3\text{ km} \\ M = M_\oplus, & R = 0.9\text{ cm}\end{cases}$

Orbits around a black hole

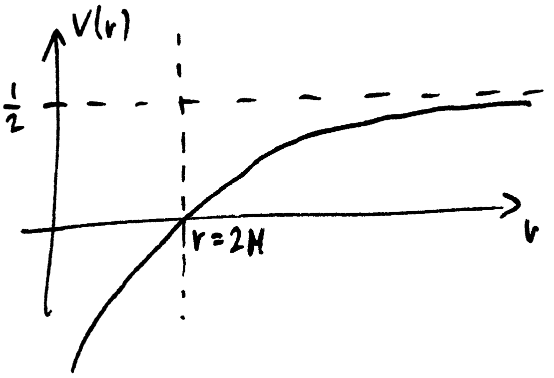
Consider timelike particles

$$\frac{\dot r^2}{2} + V(r) = \frac{E^2}{2}, \quad V(r) = \frac{1}{2}\left(1+\frac{h^2}{r^2}\right)\left(1-\frac{2M}{r}\right)$$

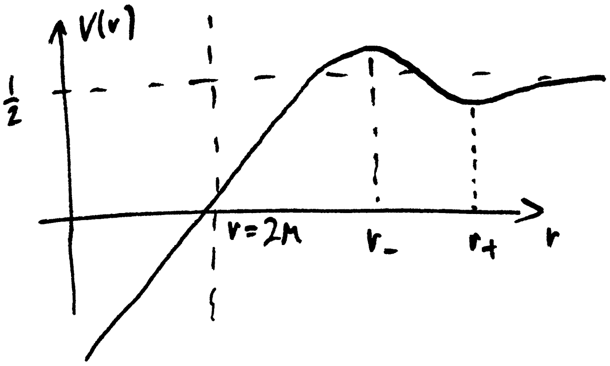
Turning points

$$V'(r_\pm) = 0 \Rightarrow r_\pm = \frac{h^2 \pm \sqrt{h^4 - 12h^2M^2}}{2M}$$

If $h^2 < 12M^2$,



$h^2 > 12M^2$



It is

$$3M < r_- < 6M < r_+$$

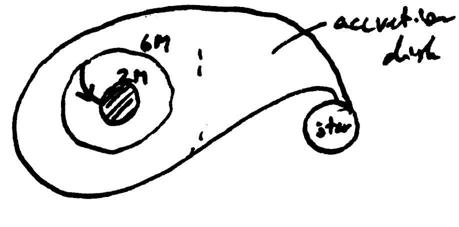
the orbit with $r = 6M$ is called the innermost stable circular orbit (ISCO).

The energy for these orbits

$$E_\pm = \frac{r_\pm - 2M}{r^{1/2}(r_\pm - 3M)^{1/2}}, \quad E \simeq \underbrace{1}_{\text{rest mass}} - \frac{M}{2r}$$

Second for binaries ($M \sim 100 M_\odot$)

$E = \sqrt{\frac{8}{9}}$



$1 - E \sim 0.06$

<u>White holes</u>

Take an outgoing radial null geodesic

$$u = t - r_* = \text{const} \qquad (v = t + r_* = \text{const for ingoing})$$

The Schwarzschild metric now becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2$$

can be extended to any $r > 0$; $\frac{dr}{d\tau} = 1$ for lines of const u.

No signal can reach $r < 2M$ if sent from $r > 2M$.