# Conic programming

**Def** A cone $K \subseteq \mathbb{R}^n$ is called <u>proper</u> if it is closed, convex, pointed and has nonempty interior.

We saw last time that $S^n_+$ = positive semidefinite cone is proper.

Let $K$ be a proper cone in $\mathbb{R}^n$.

A conic program on $K$ is an optimization problem of the form:

$$\text{minimize } \langle c, x\rangle \leftarrow \text{linear cost function specified by } c \in \mathbb{R}^n$$

$$\text{subject to } Ax = b \leftarrow \text{linear equality constraint}$$

$$x \in K \leftarrow \text{conic constraint}$$

1. Linear programming

$K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0\}$ (nonnegative orthant)

$$\text{minimize } c^T x$$
$$\text{subject to } Ax = b, \quad x \geq 0$$

2. Semidefinite programming

$K = S^+_n = \{X \in S^n : X \succcurlyeq 0\}$ = positive semidefinite matrices

$$\text{minimize } \langle C, X\rangle$$
$$\text{subject to } \mathcal{A}(X) = b, \quad X \succcurlyeq 0$$

Linear programming

Eg: minimize $2x + y$ s.t. $x + y = 1$, $x \geq 0$, $y \geq 0$

linear program with $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A = [1 \ \ 1]$, $b = 1$

Example from signal processing

Let $M \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^n$ $(m < n)$

<u>Goal</u>: Find vector $x \in \mathbb{R}^n$ that satisfies $Mx = d$ with the smallest $\ell_1$ norm $(\|x\|_1 = \sum |x_i|)$

① $\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \|x\|_1$ subject to $Mx = d$

② $\underset{x, y \in \mathbb{R}^n}{\text{minimize}} \ \sum_{i=1}^n y_i$ subject to $-y \leq x \leq y$, $Mx = d$

**Prop** Assume $x$ is feasible for ①. Then there exists $y \in \mathbb{R}^n$ s.t. $(x, y)$ feasible for ② and $\sum_{i=1}^n y_i \leq \|x\|_1$.

Conversely, assume $(x, y)$ is feasible for ②. Then $x$ is feasible for ① and $\|x\|_1 \leq \sum_{i=1}^n y_i$.

Proof:
- For the first direction take $y_i = |x_i|$.
- For the second direction: if $(x, y)$ feasible for ②, then

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n \underbrace{\max(x_i, -x_i)}_{\leq y_i} \leq \sum_{i=1}^n y_i$$

③ $\underset{x, y, u, v \in \mathbb{R}^n}{\text{minimize}} \ \sum_{i=1}^n y_i$ subject to $u = y - x$, $v = y + x$, $Mx = d$, $u, v \geq 0$

④ $\underset{x^1, x^2, y^1, y^2, u, v \in \mathbb{R}^n}{\text{minimize}} \ \sum_{i=1}^n (y^1)_i - (y^2)_i$ subject to $u = (y^1 - y^2) - (x^1 - x^2)$, $v = (y^1 - y^2) + (x^1 - x^2)$, $M(x^1 - x^2) = d$, $u, v, x^1, x^2, y^1, y^2 \geq 0$

6n variables

m + 2n equality constraints

$$A = \begin{array}{c} \\ m \\ n \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} x^1 & x^2 & y^1 & y^2 & u & v \end{array} \\ \begin{bmatrix} M & -M & 0 & 0 & 0 & 0 \\ I & -I & -I & I & I & 0 \\ -I & I & -I & I & 0 & I \end{bmatrix} \end{array}, \quad b = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{array}{c} x^1 \\ x^2 \\ y^1 \\ y^2 \\ u \\ v \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

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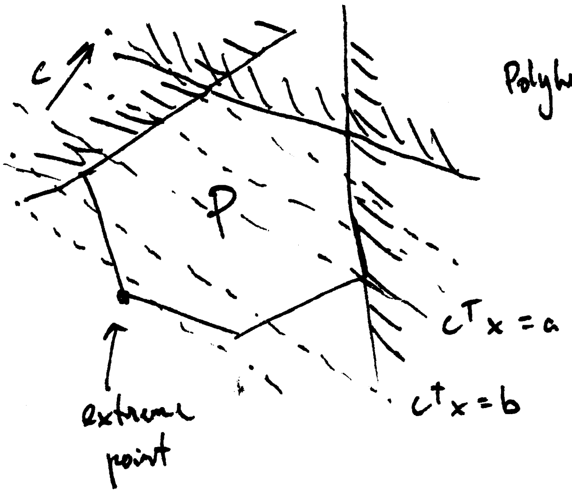
minimize $c^T x$

subj to $Ax \leq b$ $\quad A: m \times n$

$(a_i^T x \leq b \quad i = 1, \ldots, m)$




Polyhedron (intersection of finite number of halfspaces)