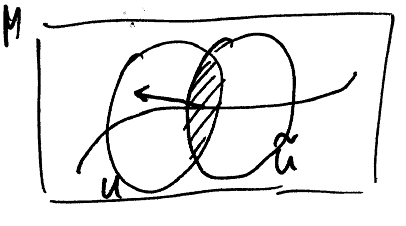

Vector field X

(U,ϕ), x^μ, $\mu=1\dots n$ $(\tilde U,\tilde\phi)$, $\tilde x^\mu$, $\mu = 1\dots n$

On $U\cap\tilde U$ $\tilde x^\mu = \tilde x^\mu(x^\nu)$

On U, $X = X^\mu(x)\dfrac{\partial}{\partial x^\mu}$, on $\tilde U$ $X = \tilde X^\mu(\tilde x)\dfrac{\partial}{\partial \tilde x^\mu}$

On $U\cap\tilde U$ (chain rule) $\dfrac{\partial}{\partial x^\mu} = \dfrac{\partial \tilde x^\nu}{\partial x^\mu}\dfrac{\partial}{\partial \tilde x^\nu}$

$$X = X^\mu \frac{\partial \tilde x^\nu}{\partial x^\mu}\frac{\partial}{\partial \tilde x^\nu} = \tilde X^\nu \frac{\partial}{\partial x^\nu} \longrightarrow \tilde X^\nu = A^\nu{}_\mu X^\mu, \quad A^\nu{}_\mu = \frac{\partial \tilde x^\nu}{\partial x^\mu}$$

May use a general (non-coordinate) basis $\{e_\mu\}$ $\mu=1\dots n$ $X = X^\mu e_\mu$

<u>Def</u> A Lie bracket of two vector fields X, Y is a vector field $[X,Y]$ defined by its action on functions

$$[X,Y]f = X(Y(f)) - Y(X(f)) \quad \forall f \text{ on } M$$

properties: (i) antisymmetry $[X,Y] = -[Y,X]$

(ii) Jacobi $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall X,Y,Z$

<u>Example</u> $M=\mathbb{R}^2$, $X = x\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$, $Y = \dfrac{\partial}{\partial x}$

$$[X,Y]f = x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) = -\frac{\partial f}{\partial x} = -Y(f)$$

$\Rightarrow [X,Y] = -Y$

Components (in coordinate basis)

$$[X,Y] = [X,Y]^\mu \frac{\partial}{\partial x^\mu}, \quad [X,Y]^\mu = X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}$$

Note that $\left[\dfrac{\partial}{\partial x^\mu}, \dfrac{\partial}{\partial x^\nu}\right] = 0$ but $[e_\mu, e_\nu] \neq 0$ in general.

<u>Converse</u>

If $X_1, \dots, X_m$ where $m \leq n$ are linearly independent v. fields on M and s.t.

$[X_i, X_j] = 0$ $\forall i,j$ then, near $p\in M$, $\exists$ coordinate chart

(U, ϕ) s.t. $p \in U$ and $X_i = \frac{\partial}{\partial x^i}$ $(x^1, \dots, x^n)$

Def A Lie algebra is a vector space g with an antisymmetric, bilinear operator $[\cdot,\cdot] : g \times g \to g$ which satisfies the Jacobi identity.

If g is finite dimensional and X_α $\alpha = 1, \dots, \dim(g)$ is a basis

$\exists f_{\alpha\beta}{}^{\gamma}$ s.t. $[X_\alpha, X_\beta] = f_{\alpha\beta}{}^{\gamma} X_\gamma$ (structure constants)

Example 1 $\exists$ two 2D Lie algebras up to isomorphism.

a) $[X, Y] = -Y$ b) $[X, Y] = 0$

Example 2 (infinite dimensional) Diff (S^1) [or diff$(\mathbb{R})$]

$x \in \mathbb{R}$ $X_\alpha = -x^{\alpha+1}\frac{\partial}{\partial x}$ $\alpha \in \mathbb{Z}$ $[X_\alpha, X_\beta] = (\alpha - \beta) X_{\alpha+\beta}$

[Later: Killing vector fields span a Lie algebra].

Group action on a manifold G = group, M = manifold

$\rho : G \times M \to M$ s.t. $\rho(g, p) \equiv g(p)$ $g \in G, p \in M$

(i) $e(p) = p$ $\forall p$

(ii) $g_1(g_2(p)) = (g_1 \circ g_2) p$ $\forall g_1, g_2 \in G, p \in M$

Example (Euclidean Lie algebra $E(2)$). $M = \mathbb{R}^2$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$G_\varepsilon \subset G$ 1-parameter subgroups

$G_\theta \Rightarrow \tilde{x} = \cos\theta\, x - \sin\theta\, y$, $\tilde{y} = \sin\theta\, x + \cos\theta\, y$

$G_a \Rightarrow \tilde{x} = x + a$, $\tilde{y} = y$

$G_b \Rightarrow \tilde{x} = x$, $\tilde{y} = y + b$.

each is generated by a vector field

$$X|_p = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} G_\varepsilon(p)$$

$$X_\theta = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad , \quad X_a = \left(\frac{d\tilde{x}}{da}\frac{\partial}{\partial \tilde{x}} + \frac{d\tilde{y}}{da}\frac{\partial}{\partial \tilde{y}}\right)\Big|_{a=0} = \frac{\partial}{\partial x} \quad , \quad X_b = \frac{\partial}{\partial y}$$

$$[X_a, X_\theta] = X_b \quad , \quad [X_a, X_b] = 0 \quad , \quad [X_b, X_\theta] = -X_a$$

<u>Covectors (forms)</u>

Def E = real vector space, E^* = dual vector space i.e. space of all linear maps on E (i.e. $E \to \mathbb{R}$)

If E finite-dimensional then $\dim(E) = \dim(E^*)$.

Say $\{e_\mu\}$ $\mu = 1 \dots n$ span E. Dual basis $\{f^\mu\}$ of E^* is defined by

$$f^\mu(e_\nu) = \delta^\mu{}_\nu$$

by linearity $f^\mu(X) = f^\mu(X^\nu e_\nu) = X^\nu \underbrace{f^\mu(e_\nu)}_{\delta^\mu{}_\nu} = X^\mu$

$\exists$ isomorphism $\Phi : E \simeq (E^*)^*$

$$X \in E \quad \Phi(X)(w) = w(X) \quad \forall w \in E^*$$

Def A covector (1-form) at p is an element of $T^*_p(M)$ = cotangent space

If $w \in T^*_p(M)$, $w = w_\mu f^\mu$ (in a basis f^μ)

$$w(X) = w_\mu f^\mu(X^\nu e_\nu) = w_\mu X^\nu f^\mu(e_\nu) = w_\mu X^\mu$$

(no raising/lowering of indices)