QED cont. $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

EoM $\partial_\mu F^{\mu\nu} = 0$

We must be careful about the defn of 3-vectors - $\underline{E}$ and $\underline{B}$ were defined with no minus from $\eta^{\mu\nu}$.

We have $\underline{E} = -\nabla\phi - \dot{A}$ $\qquad \underline{B} = \nabla \times \underline{A}$

$$\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \partial_i \qquad A^\mu = (\phi, \underline{A}) \quad \text{i.e. } \underline{A} = (A^1, A^2, A^3)$$

$$E = (F_{01}, F_{02}, F_{03}) = (-F^{01}, -F^{02}, -F^{03})$$

Look at F_{0i}, look at defn of $\underline{B} = \nabla \times \underline{A}$ if $\underline{B} = (B_x, B_y, B_z)$

$$\Rightarrow B_z = \partial_1 A^2 - \partial_2 A^1 = -\partial_1 A_2 + \partial_2 A_1 = -F_{12}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Bianchi identity: $(\lambda=3, \mu=1, \nu=2)$ $\quad \dot{\underline{B}} = -\nabla \times \underline{E}$

EoM: $\quad \nabla \cdot \underline{E} = 0$, $\quad \dot{\underline{E}} = \nabla \times \underline{B}$

The massless vector field A_μ has 4 dof but the photon (γ) only has 2 polarisations:

(i) A_0 field is not dynamical: it has no kinetic term in $\mathcal{L}$; A_0 is fully determined by $\nabla \cdot \underline{E} = 0$
Thus, if given A_i and $\dot{A}_0$ at some initial time t_0,
$\Rightarrow \nabla^2 A_0 + \nabla \cdot \dot{\underline{A}} = 0$ which has solution $A_0(\underline{x}) = \int d^3x' \frac{\nabla \cdot \dot{A}(\underline{x}')}{4\pi |\underline{x} - \underline{x}'|}$, so A_0 is not independent;

(ii) The theory has a large symmetry group:

$$A_\mu(\underline{x}) \longrightarrow A_\mu(x) + \partial_\mu \lambda(x) \qquad \left(\lim_{x\to\infty} \lambda(x) \to 0\right)$$

$$F_{\mu\nu} \longrightarrow \partial_\mu(A_\nu + \partial_\nu \lambda) - \partial_\nu(A_\mu + \partial_\mu \lambda) = F_{\mu\nu} \quad \text{inv't}$$

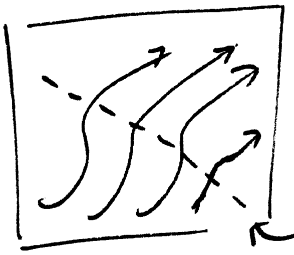
Local symmetry that rather than taking a physical state to another physical state, is a redundancy of the description (no conservation laws). Known as local / gauge symmetries.

Maxwell's eq do not specify the evolution of A_μ:

$$[\eta_{\mu\nu}(\partial_\rho \partial^\rho) - \partial_\mu \partial_\nu] A^\nu = 0$$ $\Rightarrow$ not invertible: annihilates any f^n of the form $\partial_\mu \lambda(x)$.

Thus, given A_μ: and $\dot{A}_i$ at t_0, no way to uniquely determine evolution of A_μ: can't distinguish between A_μ and $A_\mu + \partial_\mu \lambda(x)$. OK if A_μ and $A_\mu + \partial_\mu \lambda$ correspond to the same physical state. However, formulating the theory only in terms of gauge inv't objects like $\underline{E}, \underline{B}$ does not work. Have to introduce quantities we cannot measure.

Configuration space of the system foliated by gauge orbits:



All states on a given line can be reached by a gauge transformation and are identified as physically equivalent.

We pick a representative from each gauge orbit (all physically equivalent, but a good gauge cuts all the orbits). Different choices of representation are called gauges (some of which will be more useful in different situations).

Will consider 2 different gauges:

① Lorentz gauge : $\partial_\mu A^\mu = 0$.

Can always pick this: Suppose start with A'_μ s.t. $\partial_\mu A'^\mu = f$. Then, choose $A_\mu = A'_\mu + \partial_\mu \lambda$ where $\partial_\mu \partial^\mu \lambda = -f$ which can always solve.

In fact, $\partial_\mu A^\mu = 0$ doesn't pick a unique representation from the gauge orbit. We are always free to make a further gauge transformation with $\partial_\mu \partial^\mu \lambda = 0$ which has non-trivial solns.

② Coulomb gauge (aka radiation gauge) $\nabla \cdot \underline{A} = 0$

We can make use of the residual gauge transformation to pick $\nabla \cdot \underline{A} = 0$. Since A_0 is fixed by $\int d^3x' \frac{\nabla \cdot \dot{A}(\underline{x}')}{4\pi |\underline{x} - \underline{x}'|}$ we find that $A_0 = 0$. This gauge breaks Lorentz invariance.

But easier to see the physical d.o.f.: 3 comps in $\underline{A}$ satisfy a single constraint $\nabla \cdot \underline{A} = 0 \Rightarrow 2$ d.o.f. (2 polarization states)

Quantization of the EM field

The first subtlety comes in computing the momentum π^μ conjugate to A^μ:

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \qquad \pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\dot{A}^i + \partial^i A^0 = -F^{0i} = +E^i$$

$$H = \int d^3x \, \pi^i \dot{A}_i - \mathcal{L} = \int d^3x \, \frac{1}{2}(\underline{E}^2 + \underline{B}^2) - A_0 (\nabla \cdot \underline{E})$$

Hamilton's eq for A_0 acts as a Lagrange multiplier imposing $\nabla \cdot E = 0$, a constraint on the system treating $\underline{A}$ as a physical d.o.f.