Prop 34 Two DAGs are Markov equivalent iff they have the same skeleton and the same v-structures.

A Markov equivalence can be conveniently represented by a CPDAG which contains an edge (j,k) iff at least one member of the equivalence class has edge (j,k).

Faithfulness

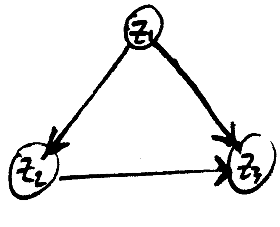
Consider the SEM

$$\begin{aligned} Z_1 &= \varepsilon_1 \\ Z_2 &= \alpha Z_1 + \varepsilon_2 \\ Z_3 &= \beta Z_1 + \gamma Z_2 + \varepsilon_3 \end{aligned} \qquad \varepsilon \sim N_3(0, I)$$

Then $(Z_1, Z_2, Z_3)^T \sim N_3(0, \Sigma) = P$

$$\Sigma = \begin{pmatrix} 1 & \alpha & \beta + \alpha\gamma \\ \alpha & 1+\alpha^2 & \alpha\beta + \gamma(\alpha^2+1) \\ \beta+\alpha\gamma & \alpha\beta+\gamma(\alpha^2+1) & \beta^2+\gamma^2(1+\alpha^2)+2\alpha\beta\gamma+1 \end{pmatrix}$$

P is Markov w.r.t. $\mathcal{G}$



Now if $\beta + \alpha\gamma = 0$ (e.g. if $\beta = -1$, $\alpha, \gamma = 1$) then $Z_1 \perp\!\!\!\perp Z_3$.

We will have $\Sigma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

Note P satisfies causal minimality w.r.t. $\mathcal{G}$.

Removing $1 \to 2 \Rightarrow Z_1 \perp\!\!\!\perp Z_2$ #

$2 \to 3 \Rightarrow Z_2 \perp\!\!\!\perp Z_3 \mid Z_1$

$\mathrm{Var}\left(\begin{pmatrix} Z_2 \\ Z_3 \end{pmatrix} \middle| Z_1\right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$ #

$1 \to 3 \Rightarrow Z_1 \perp\!\!\!\perp Z_3 \mid Z_2$

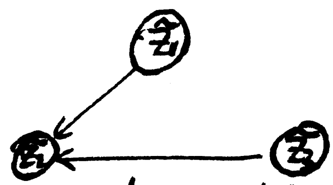
$\mathrm{Var}\left(\begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} \middle| Z_2\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$ #

P can also be generated by SEM

$\tilde{Z}_1 = \tilde{\varepsilon}_1 \sim N(0, 1)$

$\tilde{Z}_3 = \tilde{\varepsilon}_3 \sim N(0, 2)$

$\tilde{Z}_2 = \tilde{Z}_1 + \frac{1}{2}\tilde{Z}_3 + \tilde{\varepsilon}_2$ where $\tilde{\varepsilon}_2 \sim N(0, \frac{1}{2})$



Corresponding DAG is $\tilde{\mathcal{G}}$. Can check P satisfies causal minimality w.r.t. $\tilde{\mathcal{G}}$.

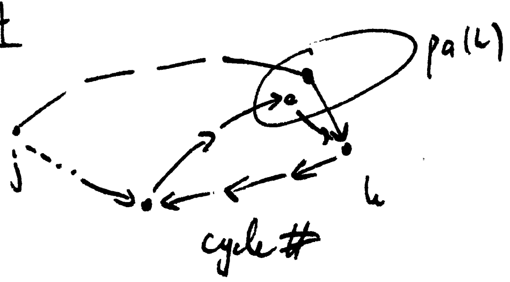
Def^n Say P is faithful to a DAG $\mathcal{G}$ if it is Markov w.r.t. $\mathcal{G}$ and $\forall\ A, B, S \subseteq \{1, \dots, p\}$ disjoint $(A, B \neq \emptyset)$

$$A, B \text{ d-separated by } S \Longleftarrow Z_A \perp\!\!\!\perp Z_B \mid Z_S$$

3.7.2 The PC algorithm

Prop 35 If nodes j and k are adjacent in DAG $\mathcal{G}$, then no set can d-separate them. If they are not adjacent and π is a topological order with $\pi(j) < \pi(k)$ then they can be d-separated by $pa(k)$.

Proof



Consider path $j = j_1, \dots, j_m = k$. If $j_{m-1} \to k$ then $j_{m-1} \in pa(k)$ so the path would be blocked at j_{m-1}. Suppose $j_{m-1} \leftarrow k$ and let l be the largest l' s.t. $j_{l'-1} \to j_{l'} \leftarrow j_{l'+1}$ (this must exist since otherwise we would have $j \in de(k)$). For the path not be blocked, j_l must have a descendant in $pa(k)$, but this cannot be the case as it would imply the existence of a cycle. $\square$

Denote the set of nodes that are adjacent to a node j in graph $\mathcal{G}$ by $adj(j, \mathcal{G})$.

PC algorithm part I: finding the skeleton (population version)

Set $\hat{\mathcal{G}}$ to be the complete undirected graph. Set $l = -1$.

- Repeat
  - $l \to l+1$
  - Repeat
    - Select a new ordered pair of nodes j, k that are adjacent in $\hat{\mathcal{G}}$, and such that $|adj(\hat{\mathcal{G}}, j) \setminus \{k\}| \geq l$.
    - Repeat
      - Choose a new $S \subseteq adj(\hat{\mathcal{G}}, j) \setminus \{k\}$ with $|S| = l$.
      - If $Z_j \perp\!\!\!\perp Z_k \mid Z_S$ then delete $j - k$ in $\hat{\mathcal{G}}$. Set $S(j,k) = S(k,j) = S$.
    - Until edge $j-k$ deleted or all S have been chosen
  - Until all relevant pairs j, k have been chosen
- Until $l = p-2$