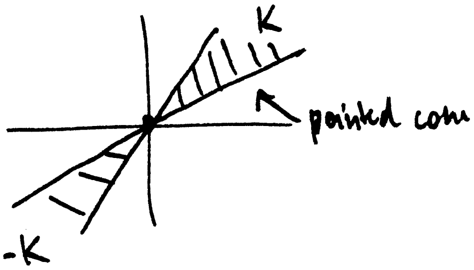
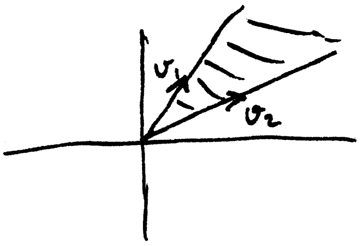
Def A cone $K \subseteq \mathbb{R}^n$ is pointed if $K \cap (-K) = \{0\}$



$\mathbb{R}^n$: <u>not</u> a pointed cone

Def A conic combination of $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is a linear combination $\lambda_1 v_1 + \dots + \lambda_k v_k$ where $\lambda_1, \dots, \lambda_k \geq 0$.

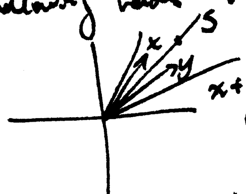


Def Let $S \subseteq \mathbb{R}^n$. The <u>conical hull</u> of S denoted $\text{cone}(S)$ is the smallest convex cone that contains S.

$$\text{cone}(S) = \bigcap_{\substack{K \text{ convex cone} \\ S \subseteq K}} K = \left\{ x \in \mathbb{R}^n \ \exists\, k \in \mathbb{N}_{\geq 1},\ s_1, \dots, s_k \in S,\ \lambda_1, \dots, \lambda_k \geq 0 : x = \sum_{i=1}^{k} \lambda_i s_i \right\}$$

Theorem (Minkowski theorem for cones)
Let K be a closed convex pointed cone in $\mathbb{R}^n$. Then K is the conical hull of its extreme rays.

Def (Extreme ray) Let $K \subseteq \mathbb{R}^n$ be a convex cone. A ray $S = \{\lambda x,\ \lambda \geq 0\}$ is <u>extreme</u> if the following holds $\forall x, y \in K$, $x + y \in S \Rightarrow x, y \in S$.



S not extreme
$x+y \in S$ but $x, y \notin S$ #

The positive semidefinite cone

S^n = space of $n \times n$ real symmetric matrices

Any $A \in S^n$ is diagonalizable in an orthonormal basis and eigenvalues are real.

S^n_+ = set of $n \times n$ real symmetric <u>positive semidefinite</u> matrices (i.e. eigenvalues non-negative)

S^n_{++} = " <u>positive definite</u> " (i.e. eigenvalues positive)

$A \succeq 0 \iff A$ positive semidefinite ; $A \succ 0 \iff A$ positive definite

Proposition The following are equivalent

(i) $A \in S^n_+$

(ii) $x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n$

(iii) There exists a lower triangular matrix L s.t. $A = LL^T$ (Cholesky decomposition)

(iv) All the principal minors of A are non-negative, i.e. $\det A[S,S] \geq 0 \quad \forall S \subseteq \{1, \dots, n\}$, $S \neq \emptyset$

Theorem S_+^n is a closed convex pointed cone with interior$(S_+^n) = S_{++}^n$.

Proof: $S_+^n = \{A \in S^n : x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n\}$

$$= \bigcap_{x \in \mathbb{R}^n} \underbrace{\{A \in S^n : x^T A x \geq 0\}}_{H_x}$$

• S_+^n is closed and convex as an intersection of closed half spaces. • S_+^n is a cone (trivial).

• S_+^n is pointed: need to show that $S_+^n \cap (-S_+^n) = \{0\}$.

If $A \in S_n^+ \cap (-S_n^+)$, then $\left.\begin{array}{l} A \in S_n^+ \Rightarrow \text{eigenvalues of } A \text{ are } \geq 0 \\ -A \in S_n^+ \Rightarrow \quad " \quad \leq 0 \end{array}\right\} \Rightarrow$ eigenvalues $= 0 \Rightarrow A = 0$

• interior$(S_+^n) = S_{++}^n$

Define the <u>spectral norm</u> of $A \in S^n$ as

$$\|A\| = \max_{\|x\|_2 = 1} \|Ax\|_2 = \max\{\lambda_{max}(A), -\lambda_{min}(A)\}.$$

We will show interior$(S_+^n) \subseteq S_{++}^n$:

Let $A \in$ interior(S_+^n). There exists $\varepsilon > 0$ s.t. $\{X \in S^n : \|A - X\| \leq \varepsilon\} \subseteq S_+^n$

Pick $X = A - \varepsilon I$ where I is the $n \times n$ identity matrix.

Since $\|A - X\| = \|\varepsilon I\| = \varepsilon$ we know that $X = A - \varepsilon I \in S_+^n$.

The eigenvalues of $A - \varepsilon I$ are the $(\lambda_i - \varepsilon)$ where λ_i are the eigenvalues of A.

Since $A - \varepsilon I \succeq 0$ we get $\lambda_i - \varepsilon \geq 0$, i.e. $\lambda_i \geq \varepsilon > 0 \Rightarrow A \succ 0$.

We now show the reverse interior$(S_+^n) \supseteq S_{++}^n$

Let $A \in S_{++}^n$. Let $\lambda_{min} > 0$ be the smallest eigenvalue of A.

$$B = \{X \in S^n : \|A - X\| \leq \lambda_{min}\}$$

Claim: $B \subseteq S_+^n$

Let $X \in B$. Since $\|A - X\| \leq \lambda_{min}$, we know that for any $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$,

$$-\lambda_{min} \leq u^T(A - X)u \leq \lambda_{min} \quad \Rightarrow \quad u^T X u \geq u^T A u - \lambda_{min}$$
$$\geq \lambda_{min} \text{ by definition of } \lambda_{min}$$

This shows that $X \succeq 0$. $\square$