4.2 SSB of a continuous symmetry

Consider an N-component real scalar field, $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ and

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\cdot(\partial_\mu \phi) - V(\phi) \quad , \quad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$$

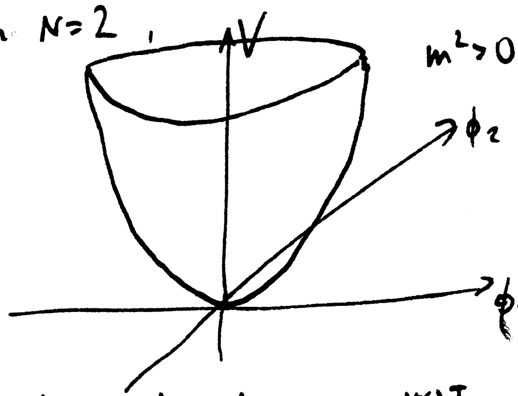
$\lambda > 0$, $\phi^2 = \phi\cdot\phi$, $\phi^4 = (\phi\cdot\phi)^2$

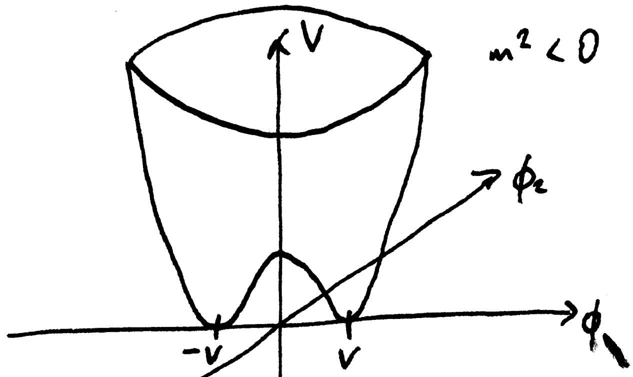
This is invariant under global $O(N)$ transforms of ϕ. Again, we're interested in $m^2 < 0$, in which case

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 + \text{const} \quad , \quad v^2 = \frac{-m^2}{\lambda} > 0 .$$

This is a Sombrero (or wine bottle) potential. Continuum of vacua (minima) with $\phi^2 = v^2$.

Sketch for $N=2$,





w.l.o.g. choose $\phi_0 = (0, 0, \dots, v)^T$ and study small fluctuations about this

$\phi(x) = (\pi_1(x), \pi_2(x), \dots, v + \sigma(x))^T$ where $\pi(x)$ has $N-1$ components and $\sigma(x)$ has 1 component. Then,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \pi)\cdot(\partial_\mu \pi) + \frac{1}{2}(\partial^\mu\sigma)(\partial_\mu\sigma) - V(\pi, \sigma)$$

(excitations in azimuthal dir)

$$V(\pi,\sigma) = \frac{1}{2}m_\sigma\sigma^2 + \lambda v(\sigma^2 + \pi^2)\sigma + \frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

(excitation in radial dir)

We see that the σ field has mass $m_\sigma = \sqrt{2\lambda v^2}$ but the $N-1$ π fields are massless.

Generalize this to a lets sym group G of $\mathcal{L}$ which is broken down to a subgroup $H \subset G$ by the vacuum. (We'll always be interested in normal subgroups.) Transformation is $\phi(x) \to g\phi(x)$, where $g \in G$ in some rep and $V(g\phi) = V(\phi)\ \forall g \in G$. Assume that G is ~~broken~~ spontaneously broken ~~by~~ and so vacuum is not unique but a manifold,

$$\Phi_0 = \{\phi_0 : V(\phi_0) = V_{min}\}$$

The invariant subgroup (or stability group) $H \subset G$ is

$$H = \{h \in G : h\phi_0 = \phi_0\}$$

Different vacua are related by

$$\phi_0' = g\phi_0 \quad \text{~~and~~ for some } g \in G \quad (\phi_0, \phi_0' \in \Phi_0)$$

The stability groups for different vacua are isomorphic. For ϕ_0', stability group is $H' = gHg^{-1}$. Group elements that map one vacuum to another are in coset space G/H and fall into equivalence classes

$g_1 \sim g_2$ if $\exists h \in H$ s.t. $g_1 = g_2 h$ (same left coset)

$\left(\phi_0' = g_1 \phi_0 \right.$ then $\phi_0' = g_2 h \phi_0 = g_2 \phi_0$: both g_1 and g_2 map $\left. \phi_0 \to \phi_0' \right)$

Correspondingly if $\phi_0' = g_1 \phi_0 = g_2 \phi_0 \Rightarrow g_2^{-1} g_1 \in H$ so there is one equivalence class (coset) for each $\phi_0' \in \Phi_0$.

$\Phi_0 \simeq G/H$ (if H is a normal subgroup then Φ_0 is a group).

Consider infinitesimal transform, $g\phi = \phi + \delta\phi$, $\delta\phi = i\alpha^a t^a \phi$ where $a = 1, \dots, \dim G$, t^a are generators of Lie algebra of G in rep of ϕ, and α^a are small params. G invariance means that $V(\phi + \delta\phi) = V(\phi)$ or expanding to first order,

$$V(\phi + \delta\phi) - V(\phi) = i\alpha^a (t^a \phi)_r \frac{\partial V}{\partial \phi_r} = 0 \qquad (*)$$

where $r = 1, \dots, N$ indexes components of ϕ in its rep. If ϕ_0 is a minimum of V,

$$V(\phi_0 + \delta\phi) - V(\phi_0) = \frac{1}{2} \delta\phi_r \underbrace{\frac{\partial^2 V}{\partial \phi_r \partial \phi_s}}_{\equiv m^2_{rs}} \delta\phi_s$$

Differentiate $(*)$,

$$\frac{\partial}{\partial \phi_s}\left[(t^a \phi)_r \frac{\partial V}{\partial \phi_r} \right] = \frac{\partial}{\partial \phi_s} (t^a \phi)_r \frac{\partial V}{\partial \phi_r}\bigg|_{\phi_0} + (t^a \phi)_r m^2_{sr} = 0$$