We'd written $\nabla = \nabla^{(0)} + a$ for some background $\nabla^{(0)}$, and imposed the gauge condition $\nabla^{(0)\mu} a_\mu$. Then, with n_f Dirac spinors in the R-rep^n of G the effective action for the background field $\nabla^{(0)} = \partial + A$ is

$$e^{-S_{eff}[A]} = e^{\left(-\frac{1}{4g_{YM}^2}\int F^a_{\mu\nu} F^{\mu\nu a}\, d^4x\right)} \frac{\det(\Delta_{0,adj})(\det \Delta_{1/2,R})^{n_f/2}}{\det(\Delta_{1,adj})^{1/2}}$$

where $\Delta_{j,R} = -\nabla^{(0)\mu}\nabla^{(0)}_\mu + \underbrace{2\left(\tfrac{1}{2}F^a_{\mu\nu} J^{\mu\nu}_{(j)}\right) t^a_{(R)}}_{\text{"magnetic moment"}}$

$$= -\partial^2 + \underbrace{\{\partial^\mu, A^a_\mu\}\, t^a_{(R)}}_{\Delta^{(1)}} + \underbrace{A^{\mu a} A^b_\mu\, t^a_{(R)} t^b_{(R)}}_{\Delta^{(2)}} + \underbrace{2\left(\tfrac{1}{2} F^a_{\mu\nu} J^{\mu\nu}_{(j)}\right) t^a_{(R)}}_{\Delta^{(J)}}$$

$$= -\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(J)}$$

The effective action involves $\log\det(\Delta_{j,R}) = \log\det\left(-\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(J)}\right)$

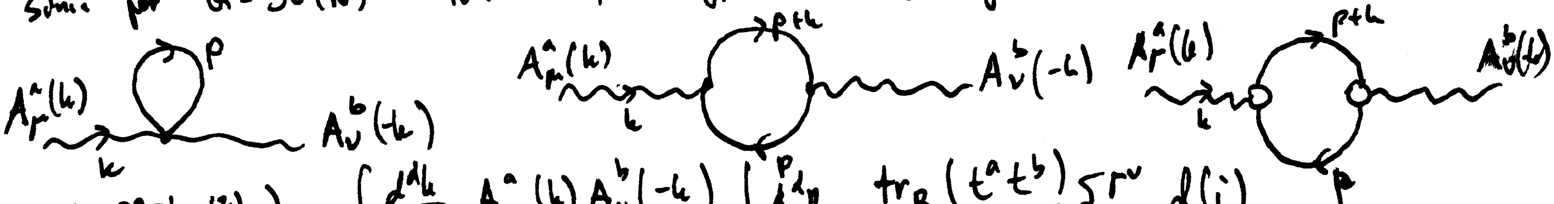
$$= \log\det(-\partial^2) + \operatorname{tr}\log\left(1 - (\partial^2)^{-1}(\Delta^1 + \Delta^2 + \Delta^J)\right)$$

We want to know the correction to the coupling $\frac{1}{4g_{YM}^2}$. Since everything is covariant w.r.t. the background $\nabla^{(0)}$, it's enough to just compute the quadratic terms in A_μ.

There are

$$\operatorname{tr}\left(-(\partial^2)^{-1}\Delta^{(2)}\right), \quad -\tfrac{1}{2}\operatorname{tr}\left((-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}\right), \quad -\tfrac{1}{2}\operatorname{tr}\left((-\partial^2)^{-1}\Delta^{(J)}(-\partial^2)^{-1}\Delta^{(J)}\right)$$

since for $G = SU(N)$ $\operatorname{tr} t^a = 0$, $\operatorname{tr}(J_{\mu\nu}) = 0$ for $J \in SO(d)$



$$\operatorname{tr}\left((-\partial^2)^{-1}\Delta^{(2)}\right) = \int \frac{d^dk}{(2\pi)^d} A^a_\mu(k) A^b_\nu(-k) \int \frac{d^dp}{(2\pi)^d} \frac{\operatorname{tr}_R(t^a t^b)}{p^2}\delta^{\mu\nu}\, d(j)$$

where $d(j) = \begin{cases} 1 & \text{scalar} \\ 4 & \text{Dirac} \\ 4 & \text{4-vector} \end{cases}$ denotes the # spin components running around the loop

Similarly, $\frac{1}{2}\operatorname{tr}\left((-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}\right) = \frac{d(j)}{2}\iint \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} \frac{(k+2p)_\mu (k+2p)_\nu}{p^2(p+k)^2} \operatorname{tr}_R(t^a t^b) A^a_\mu(k) A^b_\nu(-k)$

In dim reg^n, these two diagrams combine to give

$$\int \frac{d^dk}{(2\pi)^d} A^a_\mu(k) A^b_\nu(-k) \left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)\pi(k^2)$$

where $\pi(k^2) = \dfrac{-1}{d-1} d(j) \operatorname{tr}_R(t^a t^b) \dfrac{\Gamma(2-d/2)\Gamma(d/2-1)^2}{\Gamma(d-2)} \dfrac{(k^2)^{\frac{d-4}{2}}}{2(4\pi)^{d/2}}$

We have $\mathrm{tr}_R(t^a t^b) = C(R)\,\delta^{ab}$ where for $G = SU(N)$, $C(\text{adj}) = N$
$C(\text{fund}) = 1/2$

Finally, $\frac{1}{2}\mathrm{tr}\left((-\partial^2)^{-1}\Delta^{(J)}(-\partial^2)^{-1}\Delta^{(J)}\right)$

$$= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} A^a_\mu(k)\, A^b_\nu(-k)\, (k^2\delta_{\mu\nu} - k_\mu k_\nu)\frac{1}{p^2(p+k)^2}\, C(j)\, \mathrm{tr}_R(t^a t^b)$$

where $C(j)$ is defined by $\mathrm{tr}_j(J_{\mu\nu}J_{\kappa\lambda}) = C(j)\left[\delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa}\right]$

explicitly, $C(j) = \begin{cases} 0 & \text{scalars} \\ 1 & \text{Dirac} \\ 2 & \text{4-vector} \end{cases}$

Again, in dim reg, this gives

$$\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} A^a_\mu(k)\, A_{\nu a}(-k)\,(k^2\delta_{\mu\nu} - k_\mu k_\nu)\,(k^2)^{\frac{d-4}{2}}\,\Gamma(2 - d/2)\frac{4}{(4\pi)^{d/2}}\,C(R)\,C(j)$$

Combining all the pieces, the effective action for A_μ is

$$S_{eff}[A] = \frac{1}{4g^2_{YM}}\int d^dx\, F^a_{\mu\nu}F^{\mu\nu}_a - \frac{\Gamma(2 - d/2)}{4}\int d^dx\,(-\partial^2)^{\frac{d-4}{2}} F^a_{\mu\nu}F^{\mu\nu}_a \times\left[\frac{1}{2}C_{adj,1} - C_{adj,0} - \frac{n_f}{2}C_{R,1/2}\right]$$

where $C_{R,j} = \frac{C(R)}{(4\pi)^2}\left[\frac{d(j)}{3} - 4C(j)\right]$

$$C_{R,j} = \frac{C(R)}{(4\pi)^2}\times\begin{cases} 1/3 & \text{scalars} \\ -8/3 & \text{Dirac} \\ -20/3 & \text{vectors}\end{cases}$$

The factor $\Gamma(2 - d/2)$ diverges as $d \to 4$, and we remove the divergence using counterterms. In $\overline{MS}$ scheme with scale μ ($g^2_{YM} = \mu^{4-d} g^2(\mu)$ ← dimensionless coupling)

We're left with logarithmic dependence on μ^2

$\Rightarrow$ independence of μ gives the condition

$$\mu\frac{d}{d\mu}\left[\frac{1}{g^2(\mu)} + \log\frac{\mu^2}{(\dots)}\left[\frac{1}{2}C_{adj,1} - C_{adj,0} - \frac{n_f}{2}C_{R,1/2}\right]\right] = 0$$

$$-\frac{2}{g^3(\mu)}\beta(g) + 2\left[\frac{1}{2}C_{adj,1} - C_{adj,0} - \frac{n_f}{2}C_{R,1}\right] = 0$$

$$\Rightarrow \beta(g) = g^3(\mu)\left[\frac{1}{2}C_{adj,1} - C_{adj,0} - \frac{n_f}{2}C_{R,1}\right]$$

$G = SU(N)$, $R = \text{fund}$

$$= -\frac{g^3(\mu)}{(4\pi)^2}\left[\frac{11}{3}C(\text{adj}) - \frac{4n_f}{3}C(R)\right] = \frac{-g^2}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2n_f}{3}\right]$$

<u>Conclusion</u>: For n_f sufficiently small, the β fn is <u>negative</u>. Hence, at least for small $g(\mu)$, the coupling <u>increases</u> as the scale is lowered and decreases as the scale is raised (i.e. $g^2(\mu') > g^2(\mu)$ if $\mu' < \mu$)

• This theory has a sensible continuum limit. • The coupling is large at low energies → confinement of quarks

In fact, in 1973, Gross-Coleman showed that only asymptotically free QFTs in $d=4$ are non-Abelian gauge theories.