$(\alpha, \beta) \leq 0 \qquad \alpha, \beta \in \Phi_s \quad \alpha \neq \beta$

$\{ h^i = h^{\alpha(i)} \,,\; e^i_\pm = e^{\pm \alpha(i)} \,,\; i = 1, \dots, r \}$

$[h^i, h^j] = 0$

$[h^i, e^j_\pm] = \pm A^{ji} e^j_\pm$

$[e^i_+, e^j_-] = \delta^{ij} h^i$

Cartan matrix

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}$$

Serre relation

$$(\mathrm{ad}_{e^i_+})^n e^j_+ = \underbrace{[e^i_+, [e^i_+, \dots [e^i_+}_{n \text{ times}}, e^j_+]]]$$

$\propto e^{\alpha_{(j)} + n\alpha_{(i)}}$ if $\alpha_{(j)} + n\alpha_{(i)} \in \Phi$

$= 0$ otherwise

depends on $\alpha_{(i)}$ string thru $\alpha_{(j)}$

length

$$\ell_{ij} = 1 - A^{ji} = 1 - \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(i)}, \alpha_{(i)})} \in \mathbb{N}$$

$$(\mathrm{ad}_{e^i_\pm})^{1-A^{ji}} e^j_\pm = 0$$

Cartan: A finite dim, simple, complex Lie algebra is uniquely determined by Cartan matrix.

i) classify Cartan matrices.
ii) reconstruct $\mathfrak{g}$ from A^{ij}

Constraints on the Cartan matrix

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})} \in \mathbb{Z}$$

a) $A^{ii} = 2 \qquad i = 1, \dots, r$

symmetry of inner product

b) $A^{ij} = 0 \Leftrightarrow A^{ji} = 0$

c) $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$

non-degenerate euclidean inner product $(\, , \,)$ on $h_{\mathbb{R}}^*$

d) $\det A > 0$

$\lambda = \sum_{i=1}^{r} \lambda^i \alpha_{(i)}$ $\quad \mu = \sum_{i=1}^{r} \mu^i \alpha_{(i)} \in h_{\mathbb{R}}^*$ $\quad$ inner product $(\lambda, \mu) = (K^{-1})_{ij} \lambda^i \mu^j$

$(K^{-1})_{ij} = (\alpha_{(i)}, \alpha_{(j)})$ $\quad i, j = 1, \dots, r$

K^{-1} is a real symmetric matrix

$O(K^{-1})O^T = \mathrm{diag}(l_1, \dots, l_r)$ $\quad l_i \in \mathbb{R}$

eigenvector

$v_l = \sum_{i=1}^{r} v_l^i \alpha_{(i)}$ $\quad v_l^i \in \mathbb{R}$

$\sum_{j=1}^{r} (K^{-1})_{ij} v^j = l \sum_{j=1}^{r} \delta_{ij} v^j_l$

$(v_l, v_l) = \sum_{i,j} (K^{-1})_{ij} v_l^i v_l^j = l \sum_{i=1}^{r} (v_l^i)^2 > 0$

$\Rightarrow$ eigenvalue $l > 0$ $\quad \Rightarrow \det(K^{-1}) > 0$ ///

Cartan matrix

$A^{ij} = S^{ik} D_k{}^j$ $\quad S^{ik} = (\alpha_{(i)}, \alpha_{(k)}) = (K^{-1})_{ik}$ $\quad \det S > 0$

$D_k{}^j = \frac{2}{(\alpha_{(j)}, \alpha_{(j)})} \delta_k^j$ $\quad \det D > 0$

$\Rightarrow \det A = \det S \det D > 0$ $\square$

---

reducible Cartan matrix

$$A = \left(\begin{array}{c|c} A^{(1)} & 0 \\ \hline 0 & A^{(2)} \end{array} \right)$$

$\Rightarrow$ not simple. (semi-simple)

e) A irreducible

rank $=1$ $\Rightarrow$ $A = 2$

rank $=2$ $\quad A = \begin{pmatrix} 2 & m \\ l & 2 \end{pmatrix}$ $\quad \det A > 0 \Rightarrow ml < 4 \quad m, l \in \mathbb{Z}_{\leq 0}$

$(m,l) = (-1,-1), (-1,-2), (-1,-3)$

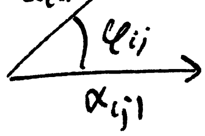
Diagrams

Information in Cartan matrix $\rightsquigarrow$ diagram

- Draw a node O for each simple root $\alpha_{(i)} \in \Phi_S \quad i = 1, \dots, r$
- Join the nodes corr roots $\alpha_{(i)}$ and $\alpha_{(j)}$

$\max\{|A^{ij}|, |A^{ji}|\} \in \{0,1,2,3\}$ — if different lengths draw arrow from longer to shorter

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})} = \frac{2|\alpha_{(i)}|}{|\alpha_{(j)}|} \cos\varphi_{ij}$$



$$A^{ji} = \frac{2|\alpha_{(j)}|}{|\alpha_{(i)}|} \cos\varphi_{ij}$$

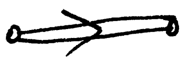
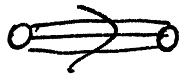
$$\cos^2\varphi_{ij} = \frac{1}{4} A^{ij} A^{ji}$$

$$\frac{|\alpha_{(i)}|}{|\alpha_{(j)}|} = \sqrt{\frac{A^{ij}}{A^{ji}}}$$

Exercise 1) $A^{ij}A^{ji} = \{0,1,2,3\}$ find all vals for A^{ij}

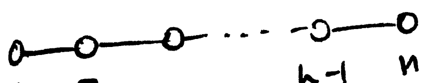
2) simple Lie algebra has roots of at most 2 distinct length

Rank 2

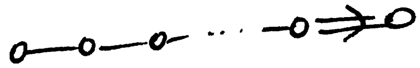
	A
	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$
	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

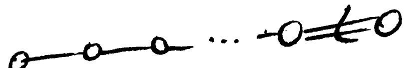
Solution (Cartan)

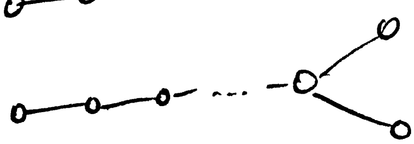
4 infinite families labelled $n \in \mathbb{N}$ $n \geq 1$
$n = \text{Rank } g$

A_n : 

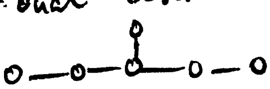
$\mathcal{L}_{\mathbb{C}}(SU(n+1))$

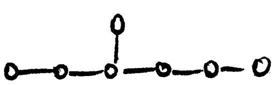
B_n :  $\mathcal{L}_{\mathbb{C}}(SO(2n+1))$

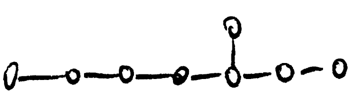
C_n :  $\mathcal{L}_{\mathbb{C}}(Sp(2n))$

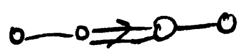
D_n :  $\mathcal{L}_{\mathbb{C}}(SO(2n))$

5 exceptional cases

E_6 : 

E_7 : 

E_8 : 

F_4 : 

G_2 :