Topics in Convex Optimization

3.1

1. $V \subseteq \mathbb{R}^n$ $\quad F_V = \{ Y \in S_+^n : \operatorname{im} Y \subseteq V \}$ is a face of the positive semidefinite cone

Proof:

F_V is convex : $\forall\, X, Y \in F_V$, $(\lambda X + \mu Y) x = \lambda X x + \mu Y x \in V \quad \forall x \in \mathbb{R}^n$

Assume $A, B \in S_+^n$ such that $\lambda A + \mu B \in F_V$ for some $\lambda, \mu \in \mathbb{R}_{\geq 0}$.

2. $X \in S_+^n$. Smallest closed face of S_+^n containing X is $F_{\operatorname{im} X}$.

Proof:

$F_{\operatorname{im} X}$ is a face as $\operatorname{im} X$ is a subspace of $\mathbb{R}^n$.

Any face F_V containing X must have $\operatorname{im} X \subseteq V$ by definition. Hence $F_{\operatorname{im} X}$ is smallest face.

$F_{\operatorname{im} X}$ is closed because S_+^n is closed.

3.2

- $A \in S^n_+$, $u \in \mathbb{R}^n$

$u^T A u = 0$. As $A \in S^n_+$, $\exists L \in \mathbb{R}^{n \times n}$: $A = LL^T$ (lower triangular)

$\Rightarrow u^T L L^T u = 0 \Rightarrow \|L^T u\|_2 = 0 \Rightarrow L^T u = 0$ by non-degeneracy of Euclidean inner product

$\therefore Au = LL^T u = L0 = 0$

$u^T A u = 0 \iff u \in \ker(A)$ (other direction obvious)

- $A \in S^n$, R invertible $n \times n$

$A \succeq 0$: $x^T R^T A R x = (x^T R^T) A (Rx) = (Rx)^T A (Rx) \geq 0 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow R^T A R \succeq 0$

$R^T A R \succeq 0$: $x^T A x = x^T (R R^{-1})^T A (R R^{-1}) x = (R^{-1}x)^T R^T A R (R^{-1}x) \geq 0 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow A \succeq 0$

$A \succ 0 \iff R^T A R \succ 0$ proceeds similarly

- $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \Rightarrow (x^T \; y^T) \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^T A x + x^T B y + y^T B^T x + y^T C y > 0 \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m$

$A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times m}$

By setting $y = 0$,

$\Rightarrow x^T A x > 0 \quad \forall x \in \mathbb{R}^n \quad \Rightarrow A \succ 0$

Now set $x = -A^{-1} B y$, (A is invertible as all eigenvalues positive)

$y^T B^T (A^{-1})^T A A^{-1} B y - y^T B^T (A^{-1})^T B y - y^T B^T A^{-1} B y + y^T C y > 0$

$\Rightarrow y^T C y - y^T B^T A^{-1} B y > 0 \quad \forall y \in \mathbb{R}^m$

$\Rightarrow C - B^T A^{-1} B \succ 0$

For the inverse consider $(x^T \; y^T) \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^T A x + x^T B y + y^T B^T x + y^T C y \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m$

Now $C - B^T A^{-1} B \succ 0$ implies as above.

Write $x = x' - A^{-1} B y$,

r.h.s: $(x' - A^{-1} B y)^T A (x' - A^{-1} B y) + x'^T B y + y^T B^T x' - y^T B^T (A^{-1})^T B y$
$- y^T B^T A^{-1} B y + y^T C y$

$= (x'^T A x' - y^T B^T (A^{-1})^T A x' - x'^T A A^{-1} B y + x'^T B y + y^T B^T x' \quad = S_1)$
$+ (y^T B^T (A^{-1})^T A A^{-1} B y - y^T B^T (A^{-1})^T B y - y^T B^T A^{-1} B y + y^T C y \quad = S_2)$

$S_2 > 0$ as $C - B^T A^{-1} B \succ 0$ (following steps above in inverse) $\forall y \in \mathbb{R}^m$

$S_1 = x'^T A x' > 0$ as $A \succ 0$ (as $(A^{-1})^T = A^{-1}$) $\forall x' \in \mathbb{R}^n \Rightarrow \forall x \in \mathbb{R}^n$

$\therefore \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0$.

- $A \succ B \succ 0$ then $A^{-1} \prec B^{-1}$.

Proof:
Start $B = I$,

$A \succ B = I \Rightarrow A - I \succ 0 \Rightarrow x^T A x > x^T x \quad \forall x \in \mathbb{R}^n$

Define $x = A^{-1/2} y$, ($A^{-1/2}$ is well defined as $A \succ 0 \Rightarrow A^{-1} \succ 0$)

$\Leftrightarrow \underbrace{y^T A^{-1/2} A A^{-1/2} y}_{\text{commute as simultaneously diagonalizable}} = y^T y > y A^{-1} y \quad \forall y \in \mathbb{R}^n$ (as $\ker(A) = \emptyset$)

$\therefore A^{-1} \prec I$

Now $A \succ B \Leftrightarrow B^{-1/2} A B^{-1/2} \succ I$ by a similar argument. Thus,

$\Leftrightarrow (B^{-1/2} A B^{-1/2})^{-1} \prec I \Leftrightarrow (B^{-1})^{-1/2} A^{-1} (B^{-1})^{-1/2} \prec I$

and therefore applying a similar argument yet again,

$\Leftrightarrow A^{-1} \prec B^{-1}$.

- $A, B \in S^n$, $A \succeq 0$, $B \succeq 0 \Rightarrow A \odot B \succeq 0$

Proof:
A, B have the eigenvalue decompositions ($i, j = 1, \dots, n$, $k, m = 1, \dots, n$)

$A_{ij} = \sum_k \lambda^{(k)} v_i^{(k)} v_j^{(k)}$ where $\lambda^{(k)} \geq 0 \ \forall k$ as $A \succeq 0$

$B_{ij} = \sum_m \mu^{(m)} u_i^{(m)} u_j^{(m)}$ where $\mu^{(m)} \geq 0 \ \forall m$ as $B \succeq 0$

$\Rightarrow (A \odot B)_{ij} = A_{ij} B_{ij} = \sum_{k,m} \lambda^{(k)} \mu^{(m)} v_i^{(k)} u_i^{(m)} v_j^{(k)} u_j^{(m)}$

Identify $w^{(km)} := v^{(k)} \odot u^{(m)}$ as a (not unit-normed) vector with $w_i^{(km)} = v_i^{(k)} u_i^{(m)}$.

$w^{(km)} \in \mathbb{R}^n$

Now

$x^T (A \odot B) x = \sum_{i,j} x_i (A \odot B)_{ij} x_j = \sum_{ijkm} \lambda^{(k)} \mu^{(m)} w_i^{(km)} x_i w_j^{(km)} x_j$

$= \sum_{k,m} \lambda^{(k)} \mu^{(m)} \underbrace{(x^T w^{(km)})^2}_{\geq 0} \geq 0 \quad \forall x \in \mathbb{R}^n \quad \therefore (A \odot B) \succeq 0$