$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho \quad -(\dagger)$$

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$$

φ_t sym transform of T $\forall t \Rightarrow \mathcal{L}_X T = 0$

φ_t 1-param family of isometries $\Rightarrow \mathcal{L}_X g = 0 \Rightarrow \nabla_a X_b + \nabla_b X_a = 0$ (Killing's eq)

Solutions are called <u>Killing vector fields</u> (KVFs).

If $g_{\mu\nu}$ independent of coordinate z then $(\dagger) \Rightarrow X = \frac{\partial}{\partial z}$ is a KVF.

KVF $X^a \Rightarrow \exists$ coords (t, x^i) s.t. $X = \frac{\partial}{\partial t}$ and $(\dagger)$ $\Rightarrow$ metric independent of t in this chart

<u>Lemma</u> X^a KVF, V^a tangent to affinely parameterised geo. Then $X_a V^a$ is const along geo.

Proof: $\underset{\substack{\uparrow \\ \text{affine parameter}}}{\frac{d}{d\tau}}(X_a V^a) = V(X_a V^a) = \nabla_V (X_a V^a) = V^b \nabla_b (X_a V^a)$

$$= \underbrace{\cancel{V^a V^b \nabla_b X_a}}_{\text{Killing eq}} \to 0 + \underbrace{\cancel{X_a V^b \nabla_b V^a}}_{\text{geo eq}} \to 0 = 0$$

Ex $J^a = \underset{\substack{\uparrow \\ \text{en tensor}}}{T^a{}_b} X^b \leftarrow$ KVF. Show $\nabla_a J^a = 0$ (J^a is a conserved current.)

<u>Linearised Theory</u> $M = \mathbb{R}^4$ globally defined coords x^μ – "almost inertial"

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad |h_{\mu\nu}| \ll 1$

$\eta_{\mu\nu} \leftarrow \operatorname{diag}(-1,1,1,1)$

$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$

$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho + O(h^2)$

$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma})$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\tau}\left(\partial_\rho \Gamma^\tau_{\nu\sigma} - \partial_\sigma \Gamma^\tau_{\nu\rho}\right)$$

$$= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right)$$

$$R_{\mu\nu} = \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \qquad \text{where } h \equiv h^\mu{}_\mu$$

$$G_{\mu\nu} = \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h\right)$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \quad \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\,\eta_{\mu\nu} \qquad \bar{h} = \bar{h}^\mu{}_\mu = -h$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$-\frac{1}{2}\partial^\rho\partial_\rho \bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}$$

— linearised Einstein equation

(g, T) equivalent to $(\varphi_*(g), \varphi_*(T))$

1-param family of diffeos φ_t $\qquad \varphi_0 = \text{identity}$

$(\varphi_{t*}(\eta))_{\mu\nu}$ close to $\text{diag}(-1,1,1,1)$ for small t.

$$(\varphi_{-t})_*(T) = T + t\mathcal{L}_X T + O(t^2) = T + \mathcal{L}_\xi T + O(t^2)$$

ξ^μ small $\qquad \xi^a = tX^a$

T = em tensor: $(\mathcal{L}_\xi T)_{\mu\nu}$ 2nd order : neglect

$\Rightarrow T_{\mu\nu}$ gauge invariant to 1st order $\qquad$ Sim for $R_{\mu\nu\rho\sigma}$

$$(\varphi_{-t})_*(g) = g + \mathcal{L}_\xi g + \dots = \eta + h + \mathcal{L}_\xi \eta + \dots$$

($\mathcal{L}_\xi h$ 2nd order)

gauge symmetry $\quad h_{\mu\nu} \longrightarrow h_{\mu\nu} + \underbrace{(\mathcal{L}_\xi \eta)_{\mu\nu}}_{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu} \quad \xi$ small $\quad (*)$

cf electromagnetism $\quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$

can choose Λ to impose gauge condition $\partial^\mu A_\mu = 0$

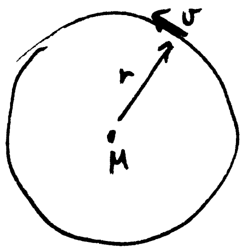
Ex: Show $\partial^\nu \bar{h}_{\mu\nu} \longrightarrow \partial^\nu \bar{h}_{\mu\nu} + \partial^\nu\partial_\nu \xi_\mu$ under $(*)$

Choose ξ_μ s.t. $\partial^\nu \partial_\nu \xi^*_\mu = -\partial^\nu \bar{h}_{\mu\nu}$ $\Rightarrow$ can impose gauge condition
$$\partial^\nu \bar{h}_{\mu\nu} = 0$$

Linearized Einstein eq: $-\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 8\pi T_{\mu\nu}$

<u>Newtonian Limit</u>

$x^\mu = (t, x^i)$ 3-velocity of particle $v^i = \frac{dx^i}{d\tau}$ assume $v^i = O(\varepsilon)$.



$$\frac{v^2}{r} = +\frac{M}{r^2} \qquad v^2 = +\frac{M}{r} = -\Phi \qquad v^2 \sim |\Phi|$$
$\Rightarrow$ expect $\Phi = O(\varepsilon^2)$

Assume $h_{00} = O(\varepsilon^2)$, $h_{0i} = O(\varepsilon^3)$, $h_{ij} = O(\varepsilon^2)$

L: length scale over which $h_{\mu\nu}$ varies $|\partial_i X| = O(X/L)$ (X: cpt of $h_{\mu\nu}$)

<u>Assume</u> $\partial_0 X = O\left(\frac{\varepsilon X}{L}\right)$

(e.g. $\Phi = -\dfrac{M}{|\underline{x} - \underline{x}(t)|}$ $L \sim |\underline{x} - \underline{x}(t)|$ $|\dot{\underline{x}}| \sim \varepsilon$)

$$\hat{L} = (1 - h_{00})\dot{t}^2 - 2h_{0i}\dot{t}\dot{x}^i - (\delta_{ij} + h_{ij})\dot{x}^i \dot{x}^j \qquad \left(\dot{} = \frac{d}{d\tau}\right)$$

$$\hat{L} = 1 \Rightarrow \dot{t} = 1 + \frac{1}{2}h_{00} + \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + O(\varepsilon^4) \quad \text{--- } (*)$$

$$\frac{d}{d\tau}\left[-2h_{0i}\dot{t} - 2(\delta_{ij} + h_{ij})\dot{x}^j\right] = -h_{00,i}\dot{t}^2 - 2h_{0j,i}\dot{t}\dot{x}^j - h_{jk,i}\dot{x}^j\dot{x}^k$$

$$-2\ddot{x}^i + \ldots = -h_{00,i} + \ldots$$

$$\ddot{x}^i = \frac{1}{2}h_{00,i} \quad \text{(leading order)}$$

$$\Phi \equiv -\frac{1}{2}h_{00} \qquad \ddot{x}^i = -\partial_i \Phi$$

$$(*) \Rightarrow \frac{d^2x^i}{dt^2} = -\partial_i \Phi \quad \text{leading order} \qquad (\dagger)$$

Ex: show corrections to $(\dagger)$ are $O(\varepsilon^4/L)$