For each $h \in G$ we have smooth maps.

$L_h \quad G \to G \quad g \in G \to hg \in G$

$R_h \quad G \to G \quad g \in G \to gh \in G$

Left and right translations.

Maps are surjective

$\forall g' \in G, \ \exists g \in G$ s.t. $L_h(g) = g'$

set $g = h^{-1} g'$

and injective

$\forall g, g'. \quad L_h(g) = L_h(g') \Rightarrow g' = g$

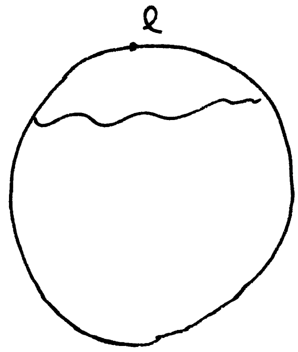
If $L_h(g) = L_h(g')$, $\quad hg = hg'$

$h^{-1}hg = h^{-1}hg' \Rightarrow g = g'$

This means map

$(L_h)^{-1} = L_{h^{-1}}$ exists and is smooth

L_h and R_h are diffeomorphisms of G



Introduce coordinate $\{\theta^i\}$, $i = 1, \dots, D$ in some region containing e.

$g = g(\theta) \in G \qquad g(0) = e$

$g' = g(\theta') = L_h(g) = h g(\theta)$

In coordinates, L_h, specified by D real function $\theta'^i = \theta'^i(\theta)$ smooth.

L_h is a diffeomorphism, means Jacobi matrix $J^i_j(\theta) = \dfrac{\partial \theta'^i}{\partial \theta^j}$ $i = 1 \dots D$ exists and is invertible ($\det J \neq 0$)

Map $L_h: G \to G$ induces a linear map L_h^* tangent vectors at g to $L_h(g)$ tangent vectors

$$L_h^*: T_g(G) \to T_{hg}(G)$$

$L_h^* \quad v = v^i \dfrac{\partial}{\partial \theta^i} \in T_g(G)$

$v' = v'^i \dfrac{\partial}{\partial \theta'^i} \in T_{hg}(G)$

$$\boxed{v'^i = J^i_j v^j}$$

Definition

A vector field V of G specifies a tangent vector

$$V(g) \in \mathcal{T}_g(G) \quad \text{at each } g \in G$$

In coordinates

$$V(\theta) = V^i(\theta) \frac{\partial}{\partial \theta^i} \quad \in \quad \mathcal{T}_{g(\theta)}(G)$$

smooth if $V^i(\theta)$ are continuous and differentiable.

Start from a tangent vector at e

$$w \in \mathcal{T}_e(G)$$

can define a vector field

$$\boxed{V(g) = L_g^*(w)} \quad \forall g \in G$$

J^i_j invertible. $\det(J) \neq 0$

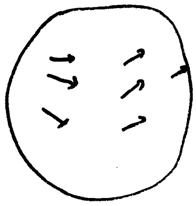
$V(g)$ smooth and non-vanishing

Starting from a basis $\{w_a\}$ $a = 1, \dots, D$ for $\mathcal{T}_e(G)$

get D independent nowhere vanishing vector fields on G.

$$V_a(g) = L_g^*(w_a)$$

Poincare - Hopf Harry - Ball theorem



Any smooth vector field on S^2 has at least two zeros. (or one double zero).

Have $M(G) \neq S^2$

In fact

dim $G = 2$, compact $\Rightarrow M(G) = T^2 \Rightarrow G = U(1) \times U(1)$

Matrix Lie group

$$G \subset \mathrm{Mat}_n(F)$$

$\forall h \in G$, $X \in \mathcal{L}(G)$, we have

$$L_h^*(X) = hX \quad \in \mathcal{T}_h(G)$$

Proof

curve $C: t \in \mathbb{R} \longrightarrow g(t) \in G$ $\quad g(0) = e$, $\dot{g}(0) = X$

near $t = 0$

$$g(t) = \mathbb{1}_n + tX + O(t^2)$$

Define a new curve $C' : t \in \mathbb{R} \longmapsto h(t) = h \cdot g(t) \in G$

$$h(t) = h + t\,h \cdot X + O(t^2) \text{ , i.e. } h'(0) = hX \text{ , } hX \in T_h(G)$$

Given any smooth curve

$$C : t \in \mathbb{R} \longmapsto g(t) \in G$$

$$\dot{g}(t) \in T_{g(t)}(G)$$

tangent space near identity

$$\Rightarrow g^{-1}(t)\,\dot{g}(t) = L^{*}_{g^{-1}(t)}(\dot{g}(t)) \in \mathcal{L}(G) \quad \forall\, t \in R$$

- Conversely, given $X \in \mathcal{L}(G)$, we can reconstruct a curve

$C: \mathbb{R} \longrightarrow G$ by solving ODE

$$g^{-1}(t)\,\dot{g}(t) = X \quad \forall t$$

with b.c. $g(0) = e$

* Define exponential of a matrix $M \in Mat_n(F)$ by Taylor Series

$$\mathrm{Exp}(M) \overset{def}{=} \mathbb{1}_n + M + \frac{M^2}{2!} + \dots = \sum_{l=0}^{\infty} \frac{M^l}{l!}$$

- $g^{-1}(t)\dfrac{dg(t)}{dt} = X \quad \forall t \in \mathbb{R}$, $g(0) = \mathbb{1}_n$

has solution $g(t) = \mathrm{Exp}(tX)$

* Check:

$$g(0) = \mathrm{Exp}(0) = \mathbb{1}_n$$

$$\frac{dg(t)}{dt} = \mathrm{Exp}(tX)X = g(t)X \text{ , } \forall\, X \in \mathcal{L}(G) \text{ , } \mathrm{Exp}(tX) \in G$$

* Exercise sheet 1 Q9

· $X \in \mathcal{L}(SU(n)) \Rightarrow \mathrm{Exp}(tX) \in SU(n) \quad \forall t \in \mathbb{R}$

* With correct choice of range of t (sheet 2 Q1)

$S_{X,J} = \{ g(t) = \mathrm{Exp}(tX), \ \forall t \in J \subset \mathbb{R} \}$

is an abelian Lie subgroup of G of $\dim = 1$ } one parameter subgroup

Reconstructing G from $\mathcal{L}(G)$

Setting $t = 1$ we have a map

$\mathrm{Exp} : \mathcal{L}(G) \to G$

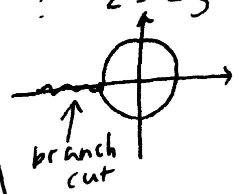
1:1 in some neighbourhood of the identity

(not proven here)

$w = \exp(z)$

$w = 1 + y$ ← small

define $z = \mathrm{Log}(1+y)$ ← principle value



single valued for $|y| < 1$

Given $X, Y \in \mathcal{L}(G)$, construct group elements

$g_X = \mathrm{Exp}(X)$, $g_Y = \mathrm{Exp}(Y) \in G$

$g_X g_Y = \mathrm{Exp}(Z)$ (provided $g_X g_Y$ is near the identity)

where

$$Z = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) \cdots$$

(Baker – Campbell – Haussdrorff)

{ is an element of $\mathcal{L}(G)$ due to closure of Lie algebra under bracket operation

$\mathcal{L}(G)$ completely determines the Lie group in some neighbourhood of the identity. But exp is not globally 1:1.

- Not surjective when G is not connected

Example : $G = O(n)$, $SO(n)$

$\mathcal{L}(O(n)) = \mathcal{L}(SO(n)) = \{ X \in \mathrm{Mat}_n(\mathbb{R}), \ X + X^T = 0 \}$ → $\mathrm{Tr}(X) = 0$

$\det(\mathrm{Exp}\,X) = \exp(\mathrm{Tr}\,X) = 1 \Rightarrow \mathrm{Exp}\,X \in SO(n)$

More generally, the image of Lie algebra under Exp is connected component of the identity in G.