Proof

$$\text{LHS} \underset{(4.1)}{=} \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma]$$

$$= \tfrac{1}{2}\{[S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \gamma^\rho[S^{\mu\nu}, \gamma^\sigma]\}$$

$$= \tfrac{1}{2}\{\gamma^\mu\eta^{\nu\rho}\gamma^\sigma - \gamma^\nu\eta^{\mu\rho}\gamma^\sigma + \gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \gamma^\rho\gamma^\nu\eta^{\mu\sigma}\}$$

$\Rightarrow$ Claim 4.2 Thus $S^{\mu\nu}$ is a repn of Lorentz group

We introduce a Dirac spinor $\psi_\alpha(x)$ $\alpha \in \{1,2,3\}$

L.T. $\psi^\alpha(x) \longrightarrow S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x)$

where Λ is a finite Lorentz transformation $\Lambda = \exp[\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}]$

and $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$ $\longleftarrow$ 4×4 matrix

Q: is the Spinor repn the usual vector repn.?

A: No. Look at specific L.T.s to show this.

Rotation: $S^{ij} = \frac{1}{4}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0\end{pmatrix}\right] = -\frac{i}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$

Write $\Omega_{ij} = -\varepsilon_{ijk}\phi^k$

$$\Rightarrow S[\Lambda] = \exp(\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}) = \begin{bmatrix} e^{i\underline{\phi}\cdot\underline{\sigma}/2} & 0 \\ 0 & e^{i\underline{\phi}\cdot\underline{\sigma}/2}\end{bmatrix}$$

Consider a rotation by 2π around x^3 axis

$\Omega_{12} = -\Omega_{21} = -\phi_3$ $\quad \underline{\phi} = (0, 0, 2\pi)$

$$\Rightarrow S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3}\end{pmatrix} = -\mathbb{1}$$

under a 2π rotation $\psi^\alpha(x) \longrightarrow -\psi^\alpha(x)$ different to vector repn.

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\mu\nu}M^{\mu\nu}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ & & 0 & \phi_3 \\ & & -\phi_3 & 0 \end{pmatrix}$$

$\phi_3 = 2\pi$, $\Lambda = \mathbb{1}_4$

Boosts as spinors

$$S^{0i} \overset{(\ast\ast)}{=} \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i\end{pmatrix}$$

Write boost param $\Omega_{0i} = -\Omega_{i0} = \chi_i$

So $S[\Lambda] = \begin{bmatrix} e^{\underline{\chi}\cdot\underline{\sigma}/2} & 0 \\ 0 & e^{-\underline{\chi}\cdot\underline{\sigma}/2} \end{bmatrix}$

For rotation, $S[\Lambda]$ is unitary since $S^{+}[\Lambda]S[\Lambda] = \mathbb{1}$, but for boosts, $S[\Lambda]$ isn't unitary.

$\nexists$ finite dim ~~rep~~ unitary repn. of the Lorentz group (algebra)

Chiral repn $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$

only unitary if $S^{\rho\sigma}$ are anti-Hermitian, i.e. $(S^{\mu\nu})^{\dagger} = -S^{\mu\nu}$

But $(S^{\mu\nu})^{\dagger} = -\frac{1}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}]$

- $S^{\mu\nu}$ can be anti-Hermitian if all γ^{μ} are either Hermitian or anti-Hermitian. This can't be arranged, since $[\gamma^0]^2 = \mathbb{1} \Rightarrow$ real eigenvalues $[\gamma^i]^2 = -\mathbb{1} \Rightarrow$ im. eigenvalues. γ^0 ~~and~~ can be Hermitian and γ^i can then be anti-Hermitian

Chiral repn has $(\gamma^0)^{\dagger} = \gamma^0$, $(\gamma^i)^{\dagger} = -\gamma^i$

<u>Constructing an action</u>

What Lorentz covariant E.o.m. by constructing a L.I. action.

- Naively, define $\psi^{\dagger}(x) = (\psi^{*})^{T}(x)$

Q: is $\psi^{\dagger}(x)\psi(x)$ a Lorentz scalar?

$\psi(x) \longmapsto S[\Lambda]\psi(\Lambda^{-1}x)$

so $\psi^{\dagger}(x) \longmapsto \psi^{\dagger}(\Lambda^{-1}x)(S[\Lambda])^{\dagger}$ but $(S[\Lambda])^{\dagger}S[\Lambda] \neq \mathbb{1}$

Clue: pick a repn where γ^0 Hermitian γ^i anti-Hermitian

Then $\gamma^0\gamma^{\mu}\gamma^0 = (\gamma^{\mu})^{\dagger}$

$(S^{\mu\nu})^{\dagger} = \frac{1}{4}[(\gamma^{\mu})^{\dagger}, (\gamma^{\mu})^{\dagger}] = -\gamma^0 S^{\mu\nu}\gamma^0$

$S[\Lambda]^{\dagger} = \exp(\underset{\substack{\uparrow \\ \text{real}}}{\tfrac{1}{2}\Omega_{\rho\sigma}} S^{\rho\sigma\dagger}) \underset{\substack{\uparrow \\ \text{use } (\gamma^0)^2 = \mathbb{1}}}{=} \gamma^0 S[\Lambda]^{-1}\gamma^0$

With this in mind, we define the Dirac adjoint of ψ

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$$

Claim 4.3

$\bar{\psi}(x)\psi(x)$ is a lorentz scalar

$$\bar{\psi}(x)\psi(x) = \psi^\dagger(x)\gamma^0\psi(x) : \xrightarrow{L.T} \psi^\dagger(\Lambda^{-1}(x))\, S[\Lambda]^\dagger \gamma^0 S[\Lambda]\psi(x)$$

$$= \psi^\dagger(\Lambda^{-1}(x))\gamma^0\psi(\Lambda^{-1}x)$$

$$= \bar{\psi}(\Lambda^{-1}x)\,\psi(\Lambda^{-1}x)$$

Claim 4.4

$\bar{\psi}\gamma^\mu\psi$ is a lorentz 4-vector

$$\bar{\psi}\gamma^\mu\psi \xrightarrow{L.T.} \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi$$

we need $S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$ — ①

Working infinitesimally

$$\Lambda^\mu{}_\nu = \exp\left[\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right] \qquad S[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

To first order in Ω

$$[S^{\rho\sigma}, \gamma^\mu] = \underbrace{(M^{\rho\sigma})^\mu{}_\nu}\gamma^\nu$$

$$(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu)\gamma^\nu$$

$$= \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho \leftarrow \text{claim } 4.1$$

Claim 4.5 $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a lorentz tensor

The sym. part is a lorentz scalar $\propto \eta^{\mu\nu}\bar{\psi}\psi$
whilst the anti-sym part is a lorentz tensor $\propto \bar{\psi}S^{\mu\nu}\psi$
(proof similar to above)

Hi

Armed with these objects, we can construct a lorentz inv. action.

$$S = \int d^4x\; \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x)$$

The Dirac Eq.

The e.o.m. is E.L. of $\psi, \bar{\psi}$ independently.

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0} \quad \text{Dirac Equation}$$

Note: 1^{st} order in derivative, unlike K.G.

Hi

~ Slash notation ~

We often need to contract with γ^μ

$$A^\mu \gamma_\mu = \not{A}$$

$$(i\not{\partial} - m)\psi = 0$$