$X \in \mathcal{L}(G)$ find a curve $C : \mathbb{R} \to G$

such that X is a tangent to C at e

$$g^{-1}(t)\frac{dg(t)}{dt} = X \qquad \forall t \in \mathbb{R} \quad -(*) \qquad g(0) = \mathbb{1}_n$$

$$\exp(M) = \sum_{l=0}^{\infty} \frac{1}{l!} M^l \in \mathrm{Mat}_n(F)$$

Solve $(*)$ by setting $g(t) = \exp(tX)$

check $g(0) = \exp(0) = \mathbb{1}_n$

$$\frac{dg(t)}{dt} = \sum_{l=1}^{\infty} \frac{1}{(l-1)!} t^{l-1} X^l = \exp(tX)X = g(t)X \quad \square \qquad \forall X \in \mathcal{L}(G) \ \exp(tX) \in G$$

Exercise

$$X \in \mathcal{L}(SU(N)) \Rightarrow \exp(tX) \in SU(N) \quad \forall t \in \mathbb{R}$$

with correct choice of range of t, I

$$S_{X,I} = \{ g(t) = \exp(tX), \ \forall t \in I \subset \mathbb{R} \}$$

is an abelian Lie subgroup of G

Reconstructing G from $\mathcal{L}(G)$

setting $t = 1$ we have a map

$$\exp : \mathcal{L}(G) \to G$$

1:1 in some neighbourhood of identity (proof omitted)

Given $X, Y \in \mathcal{L}(G)$ construct group elements

$$g_X = \exp(X), \quad g_Y = \exp(Y) \in G$$

$$g_X g_Y = \exp(Z) \qquad (g_X, g_Y \text{ near identity})$$

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) + \dots \in \mathcal{L}(G)$$

$\mathcal{L}(G)$ completely determines G in some neighbourhood of e

- $\exp$ is not globally 1:1

• not surjective when G is not connected

example $G = O(n)$, $SO(n)$

$\mathcal{L}(O(n)) = \{X \in \mathrm{Mat}_n(\mathbb{R}) : X + X^T = 0\}$ $\quad X \in \mathcal{L}(O(n)) \Rightarrow \mathrm{tr}\,X = 0$

$\det(\exp X) = \exp(\mathrm{tr}\,X) = +1 \quad \Rightarrow \exp X \in SO(n)$

more generally, the image of $\mathcal{L}(G)$ under exp is connected component of e.

• not injective when G has $U(1)$ subgroup

example $G = U(1)$

$\mathcal{L}(U(1)) = \{X = ix \;,\; x \in \mathbb{R}\}$

$g = \exp(X) = \exp(ix) \in U(1)$

elements ix $\quad ix + 2\pi i$ yield same group element

<u>$SU(2)$ vs $SO(3)$</u>

have seen that,

$\mathcal{L}(SU(2)) \simeq \mathcal{L}(SO(3))$ $\quad$ but $SU(2) \not\simeq SO(3)$

can construct a double cover $\quad$ a globally 2:1 map

$d : SU(2) \to SO(3)$ $\qquad A \in SU(2) \longmapsto d(A) \in SO(3)$

$d(A)_{ij} = \frac{1}{2}\mathrm{tr}_2(\sigma_i A \sigma_j A^\dagger)$ $\quad \overset{?\,Q4\,S2}{\in} SO(3)$

$d(A) = d(-A)$

this map provides an Homomorphism

$$SO(3) \simeq \frac{SU(2)}{\mathbb{Z}_2} \qquad \mathbb{Z}_2 = \{\mathbb{1}_2, -\mathbb{1}_2\}$$

$\nwarrow$ centre of $SU(2)$

$H \subset G$ is a normal subgroup

$ghg^{-1} \in H \quad \forall g \in G,\ h \in H$

define equivalence relation $\quad g, g' \in G$

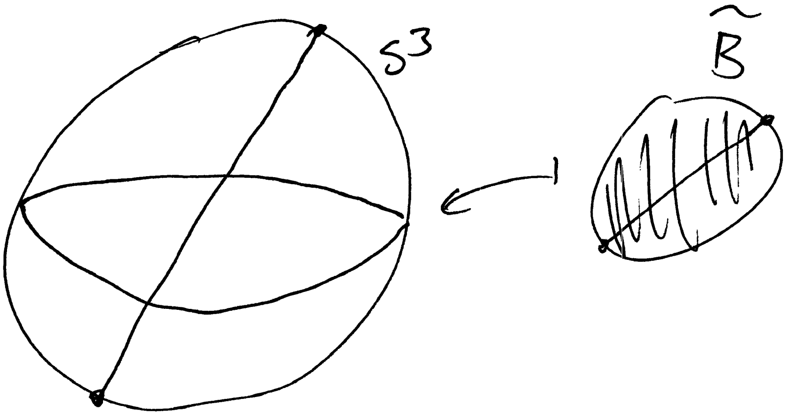
$g \sim g'$ if $g = hg' \quad h \in H$

equivalence classes $[g]$ under $\sim$ form a group $\frac{G}{H}$.

$\mathcal{M}(SU(2)) \simeq S^3$ $\qquad$ $\mathcal{M}(SO(3)) = \tilde{B}$

$\mathcal{M}(SO(3))$: $\tilde{B} \simeq \frac{S^3}{\mathbb{Z}_2}$

identify anti-podal points



Representations

For any group G, a <u>representation</u> is a set of non-singular matrices

$$g \mapsto D(g) \in Mat_n(F) \qquad n \in \mathbb{N}, F = \mathbb{R} \text{ or } \mathbb{C}$$

such that,

$$D(g_1)\,D(g_2) = D(g_1 g_2) \quad \forall g_1, g_2 \in G$$

$$D(e)\,D(g) = D(g) \quad \forall g \in G$$

$\underline{D(e) = \mathbb{1}_n}$

$$D(g^{-1})\,D(g) = D(g^{-1}g) = D(e) = \mathbb{1}_n \quad \forall g \in G$$

$\underline{D(g^{-1}) = (D(g))^{-1}}$

For any Lie algebra $\mathfrak{g}$ a representation is a set of matrices

$$\{ d(X) \in Mat_n(F),\ X \in \mathfrak{g} \}$$

such that

i) $[d(X_1), d(X_2)] = d([X_1, X_2]) \quad \forall X_1, X_2 \in \mathfrak{g}$

ii) $d(\alpha X_1 + \beta X_2) = \alpha d(X_1) + \beta d(X_2) \quad \forall X_1, X_2 \in \mathfrak{g} \quad \alpha, \beta \in F$