$(i\partial\!\!\!/ - m)\psi = 0$

The Dirac eq. mixes the different components of ψ. However, each individual component satisfies the Klein-Gordon eq:

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = -(\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\psi = 0$$

but $\gamma^\nu \gamma^\mu \partial_\mu \partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu \mathbb{1}$

$$\Rightarrow -(\partial_\mu \partial^\mu + m^2)\psi^\alpha = 0$$

Chiral Spinors

For choice of γ^μ, $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$.

The rep of the Lorentz group is block diagonal

$$S[\Lambda] = \begin{pmatrix} e^{\frac{1}{2}\underline{\chi}\cdot\underline{\sigma}} & 0 \\ 0 & e^{-\frac{1}{2}\underline{\chi}\cdot\underline{\sigma}} \end{pmatrix} \quad \text{for boosts}$$

$$\begin{pmatrix} e^{i/2\, \underline{\phi}\cdot\underline{\sigma}} & 0 \\ 0 & e^{i/2\, \underline{\phi}\cdot\underline{\sigma}} \end{pmatrix} \quad \text{for rotations}$$

Hence the Dirac rep is reducible. It decomposes into 2 irreps

u_+, u_- $\quad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$

$\to$ 2 $\mathbb{C}$ component objects : Weyl / chiral spinors

They transform in the same way under rotations:

$$u_\pm \to e^{i\underline{\phi}\cdot\underline{\sigma}/2}\, u_\pm \quad \text{and oppositely under boosts:}$$

$$u_\pm \longrightarrow e^{\pm \underline{\chi}\cdot\underline{\sigma}/2}\, u_\pm .$$

u_+ is the $(0, \frac{1}{2})$ rep of the Lorentz group, u_- is the $(\frac{1}{2}, 0)$ rep.

$SU(2)_L$ $\quad$ $SU(2)_R$

and ψ is in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

The Weyl Equation

Decompose $\mathcal{L}$ in terms of Weyl spinors:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = (u_+ \;\; u_-)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[i \begin{pmatrix} 0 & \partial_t + \sigma^i \partial_i \\ \partial_t - \sigma^i \partial_i & 0 \end{pmatrix}\right.$$

$- m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big] \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = i u_-^\dagger \sigma^\mu \partial_\mu u_- + i u_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ - m(u_+^\dagger u_- + u_-^\dagger u_+)$

where $\sigma^\mu = (\mathbb{1}, \underline{\sigma})$, $\bar{\sigma}^\mu = (\mathbb{1}, -\underline{\sigma})$.

Massive fermion requires both u_+ and u_-; a massless fermion can be described by only u_+ (or u_-) along with $i \bar{\sigma}^\mu \partial_\mu u_+ = 0$ <u>Weyl eq</u>.

<u>Degrees of freedom (d.o.f.)</u>

ψ^α : 4 $\mathbb{C}$ rows = 8 $\mathbb{R}$ d.o.f.

EoM is 1st order, not 2nd order (related to $\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger$ not $\propto \dot{\psi}$)

$\therefore$ In the phase space for the Dirac spinor has 4 $\mathbb{R}$ d.o.f.

2 for part$^{\text{l}}$ (spin ↑ + spin ↓) | 2 d.o.f. for a Weyl fermion
2 for anti-part$^{\text{l}}$ (spin ↑ + spin ↓)

<u>γ_5</u> $S[\Lambda]$ comes block diagonal because we choose a specific rep for γ^μ. It's called the chiral rep precisely because the decomposition of ψ is $\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$.
What happens for other reps related by $\gamma^\mu \to U \gamma^\mu U^{-1}$, $\psi \to U\psi$?
$S[\Lambda]$ is not always block diagonal. What is an inv^t way to define Weyl spinors?

Introduce a 5th γ matrix
$$\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$
Can check that
$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{and} \quad (\gamma^5)^2 = \mathbb{1}$$
Also $\{S^{\mu\nu}, \gamma^5\} = 0$ i.e. Lorentz scalar

Since $(\gamma 5)^2 = \mathbb{1}$, can define proj op^s i.e.
$$P_\pm = \frac{1}{2}(1 \pm \gamma^5) \qquad P_+^2 = P_+ \,, \; P_-^2 = P_- \quad \text{and } P_+ P_- = P_- P_+ = 0$$
Define chiral spinors as $\underset{(-)}{\psi_\pm} = \underset{(+)}{P_\pm} \psi$ form irreps of the Lorentz group.
$\psi_\pm$ are often called left/right handed spinors.

<u>Parity</u>

$\psi_\pm$ are related by parity. The Lorentz group is def^n as $x^\mu \longmapsto \Lambda^\mu{}_\nu x^\nu$ i.e.
$\eta^{\nu\sigma} \Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma = \eta^{\mu\rho}$. So far we have considered ~~only~~ cts transforms (continuously connected to $\mathbb{1}$).

But $\exists$ 2 discrete symmetries:

Time-reversal $T: x^0 \to -x^0, \; x^i \to x^i$ (anti-unitary)

Parity $P: x^0 \to x^0, \; x^i \to -x^i$ (important for spinors)

Since γ^μ transforms like a vector, we should have

$$P: \gamma^0 \to \gamma^0, \quad \gamma^i \to -\gamma^i$$

$$P: \gamma^\mu \to \gamma^0 \gamma^\mu \gamma^0$$

$$P: \gamma^5 \to -\gamma^5$$

so P exchanges LH/RH spinors.

$$P \psi_\pm(\underline{x}, t) \to \psi_\mp(-\underline{x}, t)$$

For a Dirac spinor, can implement the change $u_\pm \to u_\mp$ by

$$P \psi(\underline{x}, t) \to \gamma^0 \psi(-\underline{x}, t)$$

For interactions,

$P: \bar{\psi}\psi(\underline{x}, t) \to \bar{\psi}\psi(-\underline{x}, t)$ transforms as a scalar

$\underline{\bar{\psi}\gamma^\mu\psi}$: $P: \bar{\psi}\gamma^0\psi(\underline{x}, t) \to \bar{\psi}\gamma^0\psi(-\underline{x}, t)$

$\bar{\psi}\gamma^i\psi(\underline{x}, t) \to \bar{\psi}\gamma^0\gamma^i\gamma^0\psi(-\underline{x}, t)$ as a 4-vector

$= -\bar{\psi}\gamma^i\psi(-\underline{x}, t)$

$\bar{\psi} S^{\mu\nu} \psi$ transforms as a tensor