Subgroups of $GL(n,\mathbb{R})$

- orthogonal groups

$$O(n) = \{ M \in GL(n,\mathbb{R}) : M^T M = \mathbb{1}_n \}$$

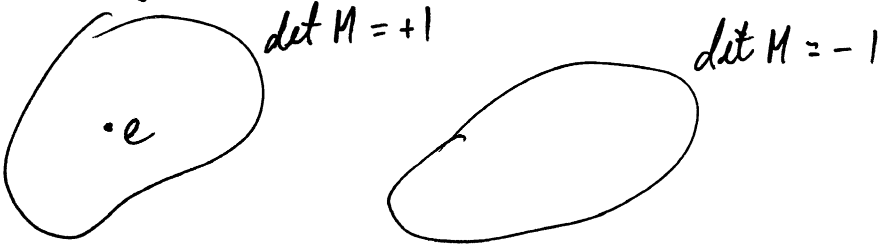
- orthogonal transformations, $\underline{v} \in \mathbb{R}^n \to \underline{v}' = M \cdot \underline{v} \in \mathbb{R}^n$

$M \in O(n)$ preserve length of vector $|\underline{v}'|^2 = \underline{v}'^T \cdot \underline{v}' = \underline{v}^T M^T M \underline{v} = \underline{v}^T \cdot \underline{v} = |\underline{v}|^2$

- $M \in O(n) \Rightarrow \det(M^T M) = \det(M^T)\det(M) = \det(M)^2 = 1$

$\Rightarrow \det M = \pm 1$

Continuity : $O(n)$ has two connected pieces



Special orthogonal group

$$SO(n) = \{ M \in O(n) : \det M = 1 \}$$

Given a frame $\{\underline{v}_1, \dots, \underline{v}_n\}$ in $\mathbb{R}^n$, an orthogonal transformation

$\underline{v}_a \in \mathbb{R}^n \to \underline{v}_a' = M \cdot \underline{v}_a \in \mathbb{R}^n$, $a = 1, \dots, n$

$M \in O(n)$ preserve sign of volume element,

$\Omega = \varepsilon^{i_1 i_2 \dots i_n} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}$ iff $\det M = +1 \Rightarrow M \in SO(n)$

$M \in O(n)$ $\det M = +1 \Rightarrow$ rotation

$M \in O(n)$ $\det M = -1 \Rightarrow$ reflection

Eigenvalues of $M \in O(n)$

$M \underline{v}_\lambda = \lambda \underline{v}_\lambda$

i) λ an eigenvalue $\Rightarrow \lambda^*$ an eigenvalue

ii) $|\lambda|^2 = 1$

i) $M \underline{v}_\lambda = \lambda \underline{v}_\lambda \Rightarrow M \underline{v}_\lambda^* = \lambda^* \underline{v}_\lambda^*$ $\quad \lambda^*$ is an eigenval of M $\square$

ii) for any complex $\underline{v} \in \mathbb{C}^n$

$$(M\underline{v}^*)^T M\underline{v} = \underline{v}^\dagger M^T M \underline{v} = \underline{v}^\dagger \cdot \underline{v}$$

if $\underline{v} = \underline{v}_\lambda$, $LHS = (M\underline{v}_\lambda^*)^T M\underline{v}_\lambda = |\lambda|^2 \underline{v}_\lambda^\dagger \underline{v}_\lambda = \underline{v}_\lambda^\dagger \cdot \underline{v}_\lambda \Rightarrow |\lambda|^2 = 1$ □

- **SO(2)** $\det M = +1$

$M \in SO(2) \Rightarrow M$ has eigenvals $\lambda = e^{i\theta}, e^{-i\theta}$ $\quad \theta \in \mathbb{R}$, $\theta \sim \theta + 2\pi$

$$M = M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ θ is rotation angle in $\mathbb{R}^2$

$$M(\theta_1)M(\theta_2) = M(\theta_2)M(\theta_1) = M(\theta_1+\theta_2)$$

$$\mathcal{M}(SO(2)) = S^1$$

- **SO(3)**

$M \in SO(3)$ $\quad M$ has eigenvals $\lambda = e^{i\theta}, e^{-i\theta}, 1$ $\quad \theta \in [0, 2\pi)$

normalised eigenvector for $\lambda = 1$

$\hat{\underline{n}} \in \mathbb{R}^3$ $\quad M\hat{\underline{n}} = \hat{\underline{n}}$, $\hat{\underline{n}}\cdot\hat{\underline{n}} = 1$ $\quad \theta$ = angle of rotation

specifies the axis of rotation (Q3)

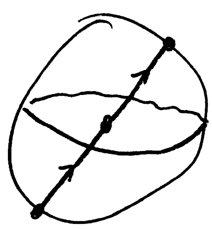
$$M(\hat{\underline{n}}, \theta)_{ij} = \cos\theta\,\delta_{ij} + (1-\cos\theta)\,n_i n_j - \sin\theta\,\varepsilon_{ijk} n_k$$

note identification

$$M(\hat{\underline{n}}, 2\pi - \theta) = M(-\hat{\underline{n}}, \theta)$$

To uniquely specify a group element, restrict θ, $0 \leq \theta \leq \pi$

$(\hat{\underline{n}}, +\pi) \sim (-\hat{\underline{n}}, \pi)$



also note that $M(\hat{\underline{n}}, 0) = \mathbb{1}_3$ for any $\hat{\underline{n}}$.

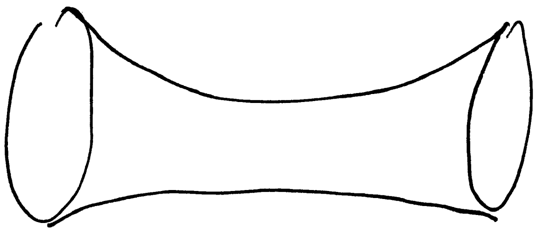
Consider vector $\underline{w} = \theta\hat{\underline{n}}$, lie in region, $B_3 = \{\underline{w} \in \mathbb{R}^3 ; |\underline{w}| \leq \pi\} \subset \mathbb{R}^3$

with boundary $\partial B_3 = \{\underline{w} \in \mathbb{R}^3 : |\underline{w}| = \pi\} \cong S^2$. identify antipodal points

$\theta \in [0, 2\pi)$ $\vdash\!\!\!-\!\!\!-\!\!\!\dashv$ I $\xrightarrow{\theta = 0 \sim \theta = 2\pi}$ $\bigcirc$ S^1

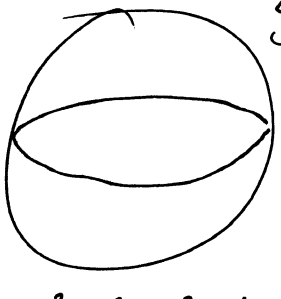
Resulting manifold is,

- <u>compact</u>: closed and bounded subset of $\mathbb{R}^n$



H^2 ✗

$x^2 - y^2 - z^2 = 1$



S^2 ✓

$x^2 + y^2 + z^2 = 1$

- <u>connected</u>: continuous path between any two group elements

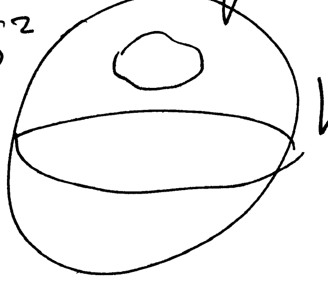
$O(n)$ ✗

$SO(n)$ ✓

- <u>not simply connected</u>

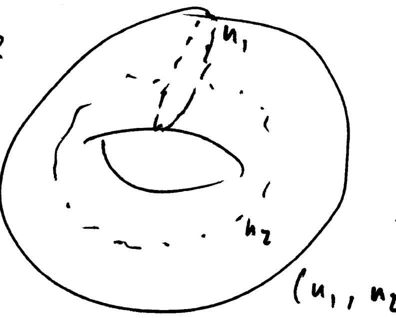
$\mathcal{M}$ is simply connected if all loops $\ell(\theta): S^1 \to \mathcal{M}$ are <u>contractible</u>

S^2



✓

T^2



✗

$(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$

e.g. $SO(3)$ $\quad \ell(\theta) \quad \mathcal{M}(\underline{w} = \theta \hat{\underline{n}}) \quad \theta \in [0, \pi)$

non-contractible loop $\quad \mathcal{M}(\underline{w} = -(2\pi - \theta)\hat{\underline{n}}) \quad \theta \in [\pi, 2\pi)$

homotopy group $\quad \pi_1(\mathcal{M})$

equivalence classes of loops $\quad \ell: S^1 \to \mathcal{M}$

$\pi_1(S^2) = \phi$

$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$

$\pi_1(SO(3)) = \mathbb{Z}_2 = \{\pm 1\}$