③ Final attempt: ask what states are defined by

$$\partial^\mu A^+_\mu(x)|\psi\rangle = 0 \quad - (\phi) \quad \text{Gupta - Bleuler}$$

ensuring that $\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0$ so $\partial_\mu A^\mu$ has vanishing matrix elements between physical states.

The linearity of (ϕ) means that the physical states span a Hilbert space $\mathcal{H}_{phys}$

Decompose a basis state of $\mathcal{H}_{phys}$ in Fock space

$$|\psi\rangle = |\psi_T\rangle|\phi\rangle$$

transverse $(a_p^{1,2})^+$ — time-like, longitudinal $(a_p^{0,3})^+$

$$(\phi) \Rightarrow (a_k^3 - a_k^0)|\phi\rangle = 0 \quad -(**)$$

In general, $|\phi\rangle$ will be a lin. comb. of states, containing comb. of timelike + longitudinal phonons.

$$|\phi\rangle = \sum_{n=0}^{\infty}|\phi_n\rangle \qquad |\phi_0\rangle = |0\rangle$$

$|\phi_n\rangle$ contains n t-like or long. γ, each obeying (**)

$$\langle\phi_n|\phi_m\rangle = \delta_{n0}\delta_{m0}$$

All -ve norm states are removed by (**)

Treat zero-norm states as gauge equivalent to $|0\rangle$

2 states which differ only in the longitudinal or timelike photons are said to be physically equivalent.

This stipulation only makes sense if no physical observables depend on $|\phi_n\rangle$ for $n = 1, 2, \ldots$

Check $H = \int \frac{d^3p}{(2\pi)^3}|\vec{p}|\left(\sum_{i=1}^{3} a_p^{i+} a_p^i - a_p^{0+}a_p\right)$

But $(a_k^3 - a_k^0)|\psi\rangle = 0$

$$\Rightarrow \langle\psi|a_p^{3+}a_p^3|\psi\rangle = \langle\psi|a_p^{0+}a_p^0|\psi\rangle$$

So t-like and longitudinal pieces cancel in H.

So t-like and long. pieces cancel in H – you just get contributions from the transverse states.

In general, matrix elements involving any gauge invariant operator evaluated on physical states are independent of the C_n.

The Feynman propagator is

$$\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2+i\varepsilon} \left(\eta_{\mu\nu} + (\alpha-1) \frac{p_\mu p_\nu}{p^2} \right) e^{-ip\cdot(x-y)}$$

Coupling to Matter

Interacting theory of light + matter:

Simplest possibility:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu$$

e.o.m.

$$\partial_\mu F^{\mu\nu} = j^\nu$$

So we require

$$0 = \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0 \longrightarrow \text{conserved current}$$

Coupling to fermions

$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ had internal symmetry

$\psi \to e^{-i\alpha}\psi$, $\bar{\psi} \to e^{i\alpha}\bar{\psi}$, $\alpha \in \mathbb{R}$, which leads to

$j^\mu = \bar{\psi}\gamma^\mu\psi$, so let's try

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \cancel{\tfrac{1}{2}}\, \bar{\psi}(i\partial\!\!\!/ - m)\psi - \overset{\text{coupling constant}}{e}\, \bar{\psi}\gamma^\mu A_\mu \psi$$

Q: have we lost gauge invariance?

A: No

Rewrite $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!/ - m)\psi$

where $D_\mu = \partial_\mu + ieA_\mu$ is called the covariant derivative.

Claim: $\mathcal{L}$ is g.i. under

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$
$$\psi \to e^{-ie\lambda(x)} \psi$$

Proof: only worry about $i\bar{\psi}\not{D}\psi$

$$D_\mu \psi \Rightarrow \partial_\mu \psi + ieA_\mu \psi \longrightarrow \partial_\mu (e^{-ie\lambda}\psi) + ieA_\mu (e^{-ie\lambda}\psi)$$
$$+ ie(\partial_\mu \lambda)(e^{-ie\lambda}\psi)$$
$$= e^{-ie\lambda} D_\mu \psi$$

Hence $\bar{\psi}\not{D}\psi \to \bar{\psi}\not{D}\psi$

The coupling const e has the interpretation of electric charges. This follows from $\partial_\mu F^{\mu\nu} = ej^\nu$

In EM, j^0 is the charge density, but as a quantum operator

$$Q = e\int d^3x\, \bar{\psi}\gamma^0\psi = e\int \frac{d^3p}{(2\pi)^3}\sum_s (b_p^{s\dagger} b_p^s - c_p^{s\dagger} c_p^s)$$
$$= e \times (\#\text{particles} - \#\text{antiparticles})$$

In QED, usually write e in terms of the fine structure constant

$$\alpha \approx \frac{1}{137} = \frac{e^2}{4\pi} \Rightarrow e = \sqrt{4\pi\alpha} = 0.3$$

Coupling to complex scalars

$\mathbb{R}$ scalar field – no suitable conserved current. Can't couple to A_μ. For $\mathbb{C}$ scalar field φ, we can use current from complex phase rotations $\varphi \to e^{-i\alpha}\varphi$.

$$\mathcal{L}_{int} = -i\,[(\partial_\mu\varphi)^*\varphi - \varphi^*\partial_\mu\varphi]\,A^\mu$$

but this doesn't work $\rightarrow$ lost gauge invariance.
as current depends on the derivative of φ, so adding $j^\mu A_\mu$ changes the conserved current we got from $\varphi \to e^{-i\alpha}\varphi$

We solve both simultaneously by noting that

$D_\mu \varphi = \partial_\mu \varphi + ieA_\mu \varphi$ transforms as $D_\mu \varphi \longrightarrow e^{-i\alpha} D_\mu \varphi$

So we can construct a g.i. $\mathcal{L}$ by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu \varphi)^\dagger D^\mu \varphi$$

Current is $j^\mu = i\left[(D_\mu\varphi)^\dagger \varphi - \varphi^\dagger (D_\mu \varphi)\right]$

<u>Minimal Coupling</u>

ϕ^a (bosonic / fermionic)

U(1) gauge symmetry $\phi^a \longrightarrow e^{i\lambda^a(x)} \phi^a$ (no s.c.)

Coupling to γ by $\partial_\mu \phi^a \longrightarrow D_\mu \phi^a = \partial_\mu \phi^a + ie\lambda^a A_\mu \phi^a$