R reducible

$$R(X) = \left(\begin{array}{c|c} A(X) & B(X) \\ \hline 0 & C(X) \end{array}\right) \qquad \forall X \in g$$

R fully reducible

$$R(X) = \left(\begin{array}{c|c|c} R_1(X) & & \\ \hline & \ddots & \\ \hline & & R_n(X) \end{array}\right) \qquad \forall X \in g$$

R_i irreducible $i=1,\dots,n$

Important Fact

If R_i $i=1,\dots,m$ are finite-dim irreps of a simple g then

$$R_1 \otimes R_2 \otimes \dots \otimes R_m = \tilde{R}_1 \oplus \tilde{R}_2 \oplus \dots \oplus \tilde{R}_{\tilde{m}} \quad \text{some irreps } \tilde{R}_j \; j=1,\dots,\tilde{m}.$$

Tensor product of $\mathcal{L}(SU(2))$ reps

Let R_Λ and $R_{\Lambda'}$ be irreps of $\mathcal{L}(SU(2))$ highest weights $\Lambda, \Lambda' \in \mathbb{N} \cup \{0\}$

reps space V_Λ and $V_{\Lambda'}$. $\dim(R_\Lambda) = \Lambda + 1$ $\quad \dim(R_{\Lambda'}) = \Lambda' + 1$

- form tensor product $R_\Lambda \otimes R_{\Lambda'}$ with reps space

$$V_\Lambda \otimes V_{\Lambda'} = \mathrm{span}_{\mathbb{R}}\{ v \otimes v' \; ; \; v \in V_\Lambda, \; v' \in V_{\Lambda'} \}$$

$\forall X \in \mathcal{L}(SU(2))$

$$(R_\Lambda \otimes R_{\Lambda'})(X)(v \otimes v') = (R_\Lambda(X) v) \otimes v' + v \otimes (R_{\Lambda'}(X) v')$$

fully reducible reps of $\mathcal{L}(SU(2))$ dimension

$$\dim(R_\Lambda \otimes R_{\Lambda'}) = (\Lambda+1)(\Lambda'+1)$$

$$R_\Lambda \otimes R_{\Lambda'} = \bigoplus_{\substack{\Lambda'' \in \mathbb{Z} \\ \Lambda'' \geq 0}} \mathcal{L}^{\Lambda''}_{\Lambda,\Lambda'} R_{\Lambda''}$$

$\mathcal{L}^{\Lambda''}_{\Lambda,\Lambda'}$ some non-negative weights, Littlewood-Richardson coeffs

- V_Λ has a basis $\{v_\lambda\}$ $\qquad R(H) v_\lambda = \lambda v_\lambda \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\lambda \in S_\Lambda = \{-\Lambda, -\Lambda+2, \dots, \Lambda\}$$

- $V_{\Lambda'}$ has a basis $\{v'_{\lambda'}\}$

$$\lambda' \in S'_{\Lambda'} = \{-\Lambda', -\Lambda'+2, \dots, \Lambda'\}$$

$B = \{ v_\lambda \otimes v'_{\lambda'} : \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'} \}$ basis for $V_\Lambda \otimes V_{\Lambda'}$

$$(R_\Lambda \otimes R_{\Lambda'})(H)(v_\lambda \otimes v'_{\lambda'}) = (R_\Lambda(H) v_\lambda) \otimes v'_{\lambda'} + v_\lambda \otimes (R_{\Lambda'}(H) v'_{\lambda'})$$
$$= (\lambda + \lambda')(v_\lambda \otimes v'_{\lambda'})$$

Weight set of $R_\Lambda \otimes R_{\Lambda'}$

$$S_{\Lambda,\Lambda'} = \{ \lambda + \lambda' : \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'} \}$$

Highest weight is $\Lambda + \Lambda'$ with multiplicity one.

$\Rightarrow \mathcal{L}^{\Lambda+\Lambda'}_{\Lambda,\Lambda'} = 1$ $\qquad R_{\Lambda+\Lambda'}$

$$R_\Lambda \otimes R_{\Lambda'} = R_{\Lambda+\Lambda'} \oplus \tilde{R}_{\Lambda,\Lambda'}$$

$\tilde{R}_{\Lambda,\Lambda'}$ has weight set $\tilde{S}_{\Lambda,\Lambda'}$: $S_{\Lambda,\Lambda'} = S_{\Lambda+\Lambda'} \cup \tilde{S}_{\Lambda,\Lambda'}$

$S_{\Lambda+\Lambda'} = \{ -\Lambda-\Lambda', -\Lambda-\Lambda'+2, \ldots, +\Lambda+\Lambda' \}$ then iterate

<u>Example</u> $\Lambda = \Lambda' = 1$

$S_1 = \{-1, +1\}$ fundamental

$$S_{1,1} = \{-1, +1\} + \{-1, +1\} = \{-2, 0, 0, +2\}$$
$$= \underbrace{\{-2, 0, +2\}}_{S_2} \cup \underbrace{\{0\}}_{S_0}$$

$$R_1 \otimes R_1 = R_2 \oplus R_0$$

$$\mathcal{L}^{\Lambda''}_{1,1} = \delta_{\Lambda'',2} + \delta_{\Lambda'',0}$$

familiar from QM : spin $\frac{1}{2} \otimes$ spin $\frac{1}{2}$ = spin $1 \oplus$ spin 0

$$R_N \otimes R_M = R_{|N-M|} \oplus R_{|N-M|+2} \oplus \cdots \oplus R_{N+M}$$

The Killing Form

Given a vector space V over F ($\mathbb{R}$ or $\mathbb{C}$). An inner product is a symmetric bilinear map

$$i : V \times V \longrightarrow F$$

i is non-degenerate if, for every $v \in V$, $\exists\, w \in V$ such that,

$$i(v, w) \neq 0 \qquad (v \neq 0)$$

Q: Is there a "natural" inner product on $\mathfrak{g}$?

A: The Killing Form $\quad \forall\, X, Y \in \mathfrak{g}$

trace of linear map

$$K : \mathfrak{g} \times \mathfrak{g} \longrightarrow F \qquad K(X,Y) = \mathrm{Tr}(ad_X \circ ad_Y)$$

$$(ad_X \circ ad_Y) : \mathfrak{g} \rightarrow \mathfrak{g}$$

$$Z \in \mathfrak{g} \longmapsto [X, [Y, Z]] \in \mathfrak{g}$$

Choose a basis $\{T^a\}$, $a = 1, \dots, n$ for $\mathfrak{g}$

$$X = X_a T^a$$
$$Y = Y_a T^a$$
$$Z = Z_a T^a$$

$$[T^a, T^b] = f^{ab}{}_c T^c$$

$$[X, [Y, Z]] = X_a Y_b Z_c [T^a, [T^b, T^c]]$$

$$= X_a Y_b Z_c f^{ad}{}_e f^{bc}{}_d T^e$$

$$= M(X,Y)^c_e Z_c T^e \qquad M(X,Y)^c_e = X_a Y_b f^{ad}{}_e f^{bc}{}_d$$

$$K(X,Y) = \mathrm{Tr}_0 [M(X,Y)] = K^{ab} X_a Y_b$$

$$K^{ab} = f^{ad}{}_e f^{be}{}_d$$