Def^n The extrinsic curvature tensor K_{ab} is defined @ p as

$$K(X,Y) = -n_a (\nabla_{X_{\parallel}} Y_{\parallel})^a \quad (*)$$ where X and Y are vector fields on M.

Lemma K_{ab} is independent of how n_a is extended and $K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d$.

Proof: r.h.s $(*)$

$$-n_d X_{\parallel}^c \nabla_c Y_{\parallel}^d \underset{Y^d_{\parallel} n_d = 0}{\longrightarrow} Y^d_{\parallel} X^c_{\parallel} \nabla_c n_d = X^a h_a{}^c Y^b h_b{}^d \nabla_c n_d \quad \square$$

Pick a different extension n'_a, then define $m \equiv n' - n$. On Σ, $m = 0$. Then,

$$X^a Y^b (K'_{ab} - K_{ab}) = X^c_{\parallel} Y^d_{\parallel} \nabla_c m_d$$
$$= \nabla_{X_{\parallel}} (Y^d_{\parallel} m_d) = X_{\parallel}\underbrace{(Y^d_{\parallel} m_d)}_{0}$$

$\Big(\nabla_c (Y^d_{\parallel} m_d) =$ on Σ: $= (\nabla_c Y^d_{\parallel}) \cancelto{0}{m_d} + Y^d_{\parallel} \nabla_c m_d \Big)$

Recall that $n^a n_a = \pm 1 \Rightarrow n^a \nabla_c n_a = 0$.

$$K_{ab} = h_a{}^c (g_b{}^d \mp n_b n^d) \nabla_c n_d = h_a{}^c \nabla_c n_b$$

Lemma $K_{ab} = K_{ba}$ $-K_{ab}$ is symmetric 2-tensor .

Proof: Let $f: M \to \mathbb{R}$ be constant on Σ with $df \neq 0$ on Σ.

Let X^a be tangent to Σ. Then $X(f) = 0$.

Then $n_a = \alpha (df)_a$ where choose α s.t. n_a is a unit normal covector.

So now we can extend n_a to ~~the~~ a neighbourhood of Σ. Consider

$$\nabla_c n_d = \alpha_c \nabla_d f + (\nabla_c \log \alpha) n_d$$

when you project with h

$$K_{ab} = \alpha h_a{}^c h_b{}^d \nabla_c \nabla_d f = K_{ba} \quad \square$$

Exercise $K = \frac{1}{2} \mathcal{L}_n h$.

The Gauss - Codacci equations

A tensor at $p \in \Sigma$ is invariant under a projection $h_a{}^b$ if

$$\underbrace{T^{a_1 \dots a_r}{}_{b_1 \dots b_s}}_{(r,s)\text{ tensor on } M} = h_{c_1}{}^{a_1} \cdots h_{c_r}{}^{a_r} h_{b_1}{}^{d_1} \cdots h_{b_s}{}^{d_s} T^{c_1 \dots c_r}{}_{d_1 \dots d_s}$$

Proposition: A covariant deriv D on Σ can be identified by the projection of the covariant derivative ~~on~~ M (∇) .

$$D_a T^{b_1 \dots b_r}{}_{c_1 \dots c_s} = h_a{}^d h_{e_1}{}^{b_1} \cdots h_{e_r}{}^{b_r} h_{c_1}{}^{d_1} \cdots h_{c_s}{}^{d_s} \nabla_d T^{e_1 \dots e_r}{}_{d_1 \dots d_s}$$

Lemma D is the Levi-Civita connection associated with h and D is torsion free.

Proof: $\nabla_a h_{bc} = \mp n_c \nabla_a n_b \mp n_b \nabla_a n_c$, $h_{bc} = g_{bc} \mp n_b n_c$

because $n^a h_{ac} = 0$. Then $\nabla_a h_{bc} = 0$ □

To show torsion-free, let $f : \Sigma \to \mathbb{R}$ and extend it to a function $f : M \to \mathbb{R}$.

Then $\nabla_a \nabla_b f = h_a{}^c h_b{}^d \nabla_c (h_d{}^e \nabla_e f) = h_a{}^c h_b{}^e \nabla_c \nabla_e f + h_a{}^c h_b{}^d \nabla_c h^e{}_d \nabla_e f$

The second term involves ↑ symmetric in (a,b)

$$h_a{}^c h_b{}^d \nabla_c h_d{}^e = g^{ef} h_a{}^c h_b{}^d \nabla_c h_{df} = \mp g^{ef} h_a{}^c h_b{}^d n_f \nabla_c n_d = \mp n^e K_{ab}$$

which is symmetric □

__Proposition__ Denote the Riemann tensor associated with D_a on Σ as $R'^a{}_{bcd}$. This is given by

$$R'^a{}_{bcd} = h^a{}_e h_b{}^f h_c{}^g h_d{}^h R^e{}_{fgh} \pm 2 K_{[c}{}^a K_{d]b}$$

Proof: Look at Ricci identity for Σ,

$X^b R'^a{}_{bcd} = 2 D_{[c} D_{d]} X^a$ where X^a is tangent to Σ.

The r.h.s.

$$D_c D_d X^a = h_c{}^e h_d{}^f h_g{}^a \nabla_e (D_f X^g) = h_c{}^e h_d{}^f h_g{}^a \nabla_e (h_f{}^h h_i{}^g \nabla_h X^i)$$
$$= h_c{}^e h_d{}^h h^a{}_i \nabla_e \nabla_h X^i + h^e{}_c h^f{}_d h^a{}_i (\nabla_e h^h{}_f) \nabla_h X^i +$$
$$+ h^e{}_c h^h{}_d h^a{}_g (\nabla_e h^g{}_i) \nabla_h X^i \quad (*)$$

But we have seen that

$$h^c{}_a h_b{}^d \nabla_c h^e{}_d = \mp n^e K_{ab} \quad (**)$$

so we can use this on $(*)$

$$D_c D_d X^a = h^e{}_c h_d{}^h h^a{}_i \nabla_e \nabla_h X^i \mp K_{cd} h^a{}_i n^h \nabla_h X^i \mp K_c{}^a n_i h^h{}_d \nabla_h X^i$$

The final term

$$\mp K_c{}^a h_d{}^h \nabla_h (\cancel{n_i X^i})^{0} \pm K^a{}_c X^i h^h{}_d \nabla_h n_i = \pm K^a{}_c K_{bd} X^b$$

Note that if we antisymmetrize in $[cd]$

$$R'^a{}_{bcd} X^b = 2 h_{[c}{}^e h_{d]}{}^f h_g{}^a \nabla_e \nabla_f X^g \pm 2 K_{[c}{}^a K_{b]}{}^d X^b$$

$$2 h_c{}^e h_d{}^f h_g{}^a \nabla_{[e} \nabla_{f]} X^g = h_c{}^e h_d{}^f h_g{}^a h^h{}_b R^g{}_{hef} X^b$$ □

__Lemma__ The Ricci scalar of Σ is

$$R' = R \mp 2 R_{ab} n^a n^b \pm K^2 \mp K^{ab} K_{ab}$$

__Exercise__ Codacci's eq^n

$$D_a K_{bc} - D_b K_{ac} = h_a{}^d h_b{}^e h_c{}^f n^g R_{defg}.$$