Scalar field action

$\Phi : M \to \mathbb{R}$ $\qquad$ action $S[\Phi] = \int_M d^4x \sqrt{-g}\, L$

Lagrangian $L = -\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi - V(\Phi)$ ← scalar potential

$\Phi \to \Phi + \delta\Phi \qquad \delta\Phi = 0$ on ∂M.

$$\delta S = S[\Phi + \delta\Phi] - S[\Phi] = \int_M d^4x \sqrt{-g}\left[-g^{ab} \nabla_a \Phi \nabla_b \delta\Phi - V'(\Phi)\delta\Phi\right]$$

$$= \int_M d^4x \sqrt{-g}\left[-\nabla_a(\delta\Phi \nabla^a \Phi) + \delta\Phi \nabla^a \nabla_a \Phi - V'(\Phi)\delta\Phi\right]$$

$$= -\underbrace{\int_{\partial M} d^3x \sqrt{|h|}\, \delta\Phi\, n_a \nabla^a \Phi}_{0} + \int_M d^4x \sqrt{-g}\left[\nabla^a \nabla_a \Phi - V'(\Phi)\right]\delta\Phi$$

$$\delta S = \int d^4x \frac{\delta S}{\delta \Phi}\delta\Phi \qquad \frac{\delta S}{\delta\Phi} = \sqrt{-g}\left(\nabla^a \nabla_a \Phi - V'(\Phi)\right)$$

$\delta S = 0$ for ~~action~~ arbitrary $\delta\Phi \iff \frac{\delta S}{\delta \Phi} = 0 \iff \nabla^a \nabla_a \Phi - V'(\Phi) = 0$

$V(\Phi) = \frac{1}{2} M^2 \Phi \quad \to$ KG eq

Einstein-Hilbert action

$S[g] = \int_M d^4x \sqrt{-g}\, L$ ← scalar built from g_{ab}

e.g. $L \propto R \quad \to \quad S_{EH}[g] = \frac{1}{16\pi}\int_M d^4x \sqrt{-g}\, R = \frac{1}{16\pi}\int_M R\, \epsilon$ ← vol. form

$g_{ab} \to g_{ab} + \delta g_{ab}$ ← tensor $\qquad$ demand $\delta S_{EH} = 0$

$g = \sum_\nu g_{\mu\nu} \Delta^{\mu\nu}$ (no sum μ)

$\Delta^{\mu\nu} = (-1)^{\mu+\nu} \times$ determinant of matrix obtained by deleting row μ and column ν from metric

indep't of $g_{\mu\nu}$

$$\frac{\partial g}{\partial g_{\mu\nu}} = \Delta^{\mu\nu} = g g^{\mu\nu}$$

$$\delta g = \frac{\delta g}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \quad \left(\sum_{\mu\nu}\right)$$

$$= g g^{\mu\nu} \delta g_{\mu\nu} = g g^{ab} \delta g_{ab}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{ab} \delta g_{ab}$$

$$\Rightarrow \delta\epsilon = \frac{1}{2} g^{ab} \delta g_{ab}\, \epsilon$$

$\delta\Gamma^a{}_{bc}$: tensor

normal coords @ p for unperturbed connection: $\left.\begin{array}{l} g_{\mu\nu,\rho}=0 \\ \Gamma^\mu{}_{\nu\rho}=0 \end{array}\right\}$ at p.

$$\therefore \delta\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma}\right) \quad \text{at } p$$

$$= \frac{1}{2} g^{\mu\sigma}\left(\delta_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma}\right) \qquad (\Gamma = 0 \text{ at } p)$$

$$\Rightarrow \delta\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}\left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d}\right) \quad p \text{ arbitrary}$$

$$\delta R^\mu{}_{\nu\rho\sigma} = \partial_\rho \delta\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \delta\Gamma^\mu{}_{\nu\rho} \quad \text{at } p \quad (\delta(\Gamma\Gamma) \sim \Gamma\delta\Gamma = 0 \text{ at } p)$$

$$= \nabla_\rho \delta\Gamma^\mu{}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^\mu{}_{\nu\rho} \quad \text{at } p$$

$$\Rightarrow \delta R^a{}_{bcd} = \nabla_c \delta\Gamma^a{}_{bd} - \nabla_d \delta\Gamma^a{}_{bc} \quad p \text{ arbitrary}$$

$$\delta R_{ab} = \nabla_c \delta\Gamma^c{}_{ab} - \nabla_b \delta\Gamma^c{}_{ac}$$

variation of g^{ab}

$$\delta R = \delta(g^{ab} R_{ab}) = g^{ab}\delta R_{ab} + R_{ab}\delta g^{ab}$$

$$\delta(g_{\mu\rho} g^{\rho\nu}) = \delta(\delta_\mu{}^\nu) = 0 \quad \Rightarrow \quad \delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd}$$

$$\delta R = -g^{ac} g^{bd} R_{ab} \delta g_{cd} + g^{ab}\left(\nabla_c \delta\Gamma^c{}_{ab} - \nabla_b \delta\Gamma^c{}_{ac}\right)$$

$$\delta R = -R^{ab}\delta g_{ab} + \nabla_c\left(g^{ab}\delta\Gamma^c{}_{ab}\right) - \nabla_b\left(g^{ab}\delta\Gamma^c{}_{ac}\right)$$

$$= -R^{ab}\delta g_{ab} + \nabla_a X^a \qquad X^a = g^{bc}\delta\Gamma^a{}_{bc} - g^{ab}\delta\Gamma^c{}_{bc}$$

$$\delta S_{EH} = \frac{1}{16\pi}\int_M \delta(\epsilon R) = \frac{1}{16\pi}\int_M \epsilon\left(\frac{1}{2}R^{ab}\delta g_{ab} - R^{ab}\delta g_{ab} + \nabla_a X^a\right)$$

$$= \frac{1}{16\pi}\int_M d^4x\,\sqrt{-g}\,\left(\frac{1}{2}R g^{ab}\delta g_{ab} - R^{ab}\delta g_{ab} + \nabla_a X^a\right)$$

surface term

$X^a = 0$ on ∂M if δg_{ab} has support in compact region that doesn't intersect ∂M

$\Rightarrow$ surface term 0

$$\therefore \delta S_{EH} = \frac{1}{16\pi}\int_M d^4x\,\sqrt{-g}\,\left(-G^{ab}\right)\delta g_{ab}$$

i.e. $\dfrac{\delta S_{EH}}{\delta g_{ab}} = -\dfrac{1}{16\pi}\sqrt{-g}\,G^{ab}$ $\qquad \delta S_{EH} = 0 \;\forall\, \delta g_{ab} \Rightarrow G^{ab} = 0$ Vac. Einstein eq.

Ex Show that vac Einstein eq with Λ obtained by extremising

$$S_{EH\Lambda} = \frac{1}{16\pi}\int_M d^4x\,\sqrt{-g}\,(R - 2\Lambda)$$

Energy-momentum tensor

Assume $S_{matter} = \int d^4x \sqrt{-g}\, L_{matter}$ ← scalar function of matter fields, metric & its derivatives

$$T^{ab} := \frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g_{ab}}$$

i.e. if $g_{ab} \to g_{ab} + \delta g_{ab}$ then $\delta S_{matter} = \frac{1}{2} \int_M d^4x \sqrt{-g}\, T^{ab} \delta g_{ab}$

$$T^{ab} = T^{ba}$$

e.g. $S = \int_M d^4x\, \varepsilon \left[-\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi - V(\Phi) \right]$

$$\delta S = \int_M d^4x\, \varepsilon \left[\frac{1}{2} \nabla^a \Phi \nabla^b \Phi + \frac{1}{2} \left(-\frac{1}{2} g^{cd} \nabla_c \Phi \nabla_d \Phi - V \right) g^{ab} \right] \delta g_{ab}$$

$$= \int_M d^4x \sqrt{-g} \left[\cdots \right] \delta g_{ab}$$

$$\Rightarrow T^{ab} = \nabla^a \Phi \nabla^b \Phi + \left(-\frac{1}{2} g^{cd} \nabla_c \Phi \nabla_d \Phi - V(\Phi) \right) g^{ab}$$