gauge field

gauge transformation

$$\delta_X A_\mu = -\varepsilon \partial_\mu X + \varepsilon [X, A_\mu]$$

gauge symmetry

$$X : \mathbb{R}^{3,1} \to \mathcal{L}(G)$$

$$\delta_X \phi = \varepsilon R(X(x)) \phi \quad \in V$$

covariant derivative

$$D_\mu \phi = \partial_\mu \phi + R(A_\mu) \phi$$

want:

$$\boxed{\delta_X (D_\mu \phi) = \varepsilon R(X) D_\mu \phi} \quad — \text{(27)}$$

proof:

$$\delta_X (D_\mu \phi) = \delta_X (\partial_\mu \phi + R(A_\mu) \phi)$$

$$= \partial_\mu (\delta_X \phi) + R(A_\mu) \delta_X \phi + R(\delta_X A_\mu) \phi$$

$$= \partial_\mu (\varepsilon R(X) \phi) + \varepsilon R(A_\mu) R(X) \phi - \varepsilon R(\partial_\mu X) \phi + \varepsilon R([X, A_\mu])$$

$$= \cancel{\varepsilon R(\partial_\mu X) \phi} + \varepsilon R(X) \partial_\mu \phi$$

$$+ \varepsilon R(X) R(A_\mu) \phi + \cancel{\varepsilon [R(A_\mu), R(X)] \phi}$$

$$\cancel{- \varepsilon R(\partial_\mu X) \phi} + \cancel{\varepsilon [R(X), R(A_\mu)]}$$

$$= \varepsilon R(X) \partial_\mu \phi + \varepsilon R(X) R(A_\mu) \phi$$

$$= \varepsilon R(X) D_\mu \phi$$

(27) ensures that

$$\delta_X [(D_\mu \phi, D^\mu \phi)] = \varepsilon (R(X) D_\mu \phi, D^\mu \phi) + \varepsilon (D_\mu \phi, R(X) D^\mu \phi) = 0$$

when $R(X)^\dagger = - R(X)$

a unitary representation

$$\hat{\mathcal{L}}_\phi = (D_\mu\phi \, . \, D^\mu\phi) - W[(\phi, \phi)]$$

$$\delta_X \hat{\mathcal{L}}_\phi = 0 \quad \forall X \in \mathcal{L}(G)$$

We also need a gauge inv. kinetic term for our gauge field

$A_\mu : \mathbb{R}^{3,1} \longrightarrow \mathcal{L}(G)$

field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in \mathcal{L}(G)$$

$$\delta_X A_\mu = -\varepsilon \partial_\mu X + \varepsilon [X, A_\mu]$$

$$\boxed{\delta_X F_{\mu\nu} = \varepsilon [X, F_{\mu\nu}] \in \mathcal{L}(G)}$$

proof:

$$\delta_X F_{\mu\nu} = \partial_\mu (\delta_X A_\nu) - \partial_\nu (\delta_X A_\mu) + [\delta_X A_\mu, A_\nu] + [A_\mu, \delta_X A_\nu]$$

$$= \cancel{-\varepsilon \partial_\mu \partial_\nu X} + \varepsilon \partial_\mu (\cancel{[X, A_\mu]}) + \cancel{\varepsilon \partial_\nu \partial_\mu X} - \varepsilon \partial_\nu (\cancel{[X, A_\mu]})$$

$$- \varepsilon [\partial_\mu X, A_\nu] - \varepsilon [A_\mu, \partial_\nu X]$$

$$+ \varepsilon ([[X, A_\mu], A_\nu]) + \varepsilon [A_\mu, [X, A_\nu]]$$

$$= \varepsilon [X, \partial_\mu A_\nu] - \varepsilon [X, \partial_\nu A_\mu] - \varepsilon ([A_\nu, [X, A_\mu]] + [A_\mu, [A_\nu, X]])$$

Jacobi

$$= \varepsilon [X, \partial_\mu A_\nu - \partial_\nu A_\mu] + \varepsilon [X, [A_\mu, A_\nu]]$$

$$= \varepsilon [X, F_{\mu\nu}]$$

Use killing form on $\mathcal{L}(G)$

$$\mathcal{L}_A = \frac{1}{g^2} K(F_{\mu\nu}, F^{\mu\nu})$$ ← Yang Mills

This is gauge invariant due to the invariance of the killing form

- $$\delta_X \mathcal{L}_A = \frac{1}{g^2} k(\delta_X F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{g^2} k(F_{\mu\nu}, \delta_X F^{\mu\nu})$$
  $$= \frac{1}{g^2}\left[k([X, F_{\mu\nu}], F^{\mu\nu}) + k(F_{\mu\nu}, [X, F^{\mu\nu}])\right]$$
  $$= 0$$

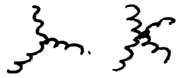
Simple $\Rightarrow$ real form of compact type.

$\Rightarrow B = \{T^a, a = 1, \dots \dim G\}$

$K^{ab} = k(T^a, T^b) = -K\delta^{ab}$

- $$\mathcal{L}_A = -\frac{K}{g^2}\sum_{a=1}^{d} F_{\mu\nu a} F^{\mu\nu a}$$

↳ $(\partial A)^2$, $A^2 \partial_\mu A$, A^4

gauge fields interact with themselves 

$\mathfrak{g}$ semi-simple

R_{Λ_i} irreps

$G = U(1) \times SU(2) \times SU(3)$