Standard Model 9

$\Phi_0 \simeq G/H$ ← sym grp of $\mathcal{L}$; H ← stability group of vacuum

↑ manifold of vacuum

$$V(\phi_0) - V(\phi_0 + \delta\phi) = \frac{1}{2}\delta\phi_r M_{rs}^2 \delta\phi_s$$

Differentiate $0 = \frac{\partial}{\partial\phi_s}\left[(t^a\phi)_r \frac{\partial V}{\partial\phi_r}\right] = \frac{\partial}{\partial\phi_s}(t^a\phi)_r \underbrace{\frac{\partial V}{\partial\phi_r}\Big|_{\phi_0}}_{=0} + (t^a\phi)_r M_{sr}^2$

Hence $(t^a\phi)_r M_{sr}^2 = 0$ at $\phi = \phi_0$

Two ways to satisfy this:

① Unbroken symmetry: $\delta\phi = 0$ $(g\phi_0 = \phi_0) \Rightarrow t^a\phi_0 = 0$

② If $\exists$ some $g \in G$ s.t. $\exists a$ with $t^a\phi_0 \neq 0$, then $(t^a\phi_0)$ is an eigenvector of M_{rs}^2 with eigenvalue $= 0$

Generators of $H \subset G$ are $\hat{t}^i$ with $i = 1, 2, \dots, \dim H$ and $\hat{t}^i\phi_0 = 0$ (unbroken sym)

For compact, semi-simple G, define a group invariant scalar product and orthogonality. Choose a basis of Lie algebra

$t^a = \{\hat{t}^i, \hat{\theta}^{\hat{a}}\}$ where $\hat{\theta}^{\hat{a}}$ are orthogonal to $\hat{t}^i$ $(\mathrm{Tr}\, \hat{t}^i\hat{\theta}^{\hat{a}} = 0)$

↑ $\dim G$; ↑ $\dim H$; ↑ $\dim G - \dim H$

and $\hat{\theta}^{\hat{a}}$ is a unique zero eigenvector of M_{rs}^2 for $\hat{a} = 1, \dots, \dim G - \dim H$

∴ We have $\dim G - \dim H$ massless modes (Goldstone boson or Nambu-Goldstone bosons) and in general $N - (\dim G - \dim H)$ massive modes

↑ # eigenvalues of M^2

This is the classical proof for Goldstone's theorem.

E.g. for $O(N)$ model $G = O(N)$, $H = O(N-1)$, $\Phi_0 = S^{N-1}$

dim: $\frac{N(N-1)}{2}$; dim: $\frac{(N-1)(N-2)}{2}$

Expect $\frac{(N-1)}{2}(N - (N-2)) = N - 1$ massless modes. This is what we expect: $N-1$ massless π fields. Also $N - (N-1) = 1$ massive σ field.

↑ corresponds to "broken symmetry", i.e. group actions that take one vacuum to another.

4.3 Goldstone Theorem

Consider SSB in a fully quantum way. H, G same as before.

A scalar field gets a non-zero vacuum expection value (VEV)

$$\langle 0 | \phi(x) | 0 \rangle = \phi_0 \neq 0$$

The VEV is invariant under $h \in H$, $\langle 0 | h \phi(x) | 0 \rangle = \phi_0$ but not invariant under $g' \in G$ and $g' \notin H$. Lie algebra of G is $t^a, a = 1, \ldots, \dim G$.

Lie algebra of H is $\hat{t}^i$, $i = 1, \ldots, \dim H$.

G sym of $\mathcal{L}$ $\Rightarrow$ conserved current and charges

$$Q_a = \int d^3x\, j^0{}_a(x) = \int d^3x\, \pi(x)\, t_a \phi(x)$$

Drop the a index. and to be explicit, index each component of the field

$$i\alpha^a t_{nm} \phi_m(x) = \underbrace{-[Q^a, \phi_n(x)]}_{\text{a repn of } G \text{ on } \phi_n(x)} \alpha \qquad \text{from the expression for } Q$$

To investigate excitations from SSB, consider

$$\chi^\mu = \langle 0 | [j^\mu(y), \phi_n(x)] | 0 \rangle$$
$$= \sum_n \Big[\langle 0 | j^\mu(y) \underbrace{|n\rangle\langle n|}_{\text{complete set of eigenstates of } P \text{ 4-momentum}} \phi_n(x) | 0 \rangle - \langle 0 | \phi_n(x) | n \rangle \langle n | j^\mu(y) | 0 \rangle \Big]$$

Assume translational invariance of vacuum, and write $j^\mu(y) = e^{i\hat{P}\cdot y} j^\mu(0) e^{-i\hat{P}\cdot y}$

$$\chi^\mu = i \int \frac{d^4p}{(2\pi)^3} \left[\rho^\mu(p) e^{-ip\cdot(y-x)} - \tilde{\rho}^\mu(p) e^{ip\cdot(y-x)} \right]$$

where

$$i\rho^\mu(p) = (2\pi)^3 \sum_n \delta^4(p - \underset{\substack{\uparrow \\ \text{4-momentum of } |n\rangle}}{p_n}) \langle 0 | j^\mu(0) | n \rangle \langle n | \phi_n(0) | 0 \rangle$$

$$i\tilde{\rho}^\mu(p) = (2\pi)^3 \sum_n \delta^4(p - p_n) \langle 0 | \phi_n(0) | n \rangle \langle n | j^\mu(0) | 0 \rangle$$

Källen-Lehmann spectral rep

From L.I. of ρ^μ and $\tilde{\rho}^\mu$ they must be $\propto p^\mu$. Physical states have $p_0 > 0$

$$\rho^\mu(p) = p^\mu \theta(p^0) \rho(p^2), \quad \tilde{\rho}^\mu(p) = p^\mu \theta(p^0) \tilde{\rho}(p^2)$$

Now recall $D(y-x, \sigma) = \langle 0 | \phi(y) \phi(x) | 0 \rangle = \int \frac{d^4p}{(2\pi)^3} \theta(p^0) \delta(p^2 - \sigma) e^{-ip\cdot(x-y)}$

$$\chi^\mu = i \int \frac{d^4p}{(2\pi)^3} p^\mu \theta(p^0) \left[\rho(p^2) e^{-ip\cdot(y-x)} - \tilde{\rho}(p^2) e^{ip\cdot(y-x)} \right]$$

$$= -\frac{\partial}{\partial y_\mu} \int \frac{d^4p}{(2\pi)^3} \theta(p^0) \left[\rho(p^2) e^{-ip\cdot(y-x)} + \tilde{\rho}(p^2) e^{ip\cdot(y-x)} \right]$$

$$= -\frac{\partial}{\partial y_\mu} \int d\sigma \left[\rho(\sigma) D(y-x, \sigma) + \tilde{\rho}(\sigma) D(-y+x, \sigma) \right]$$