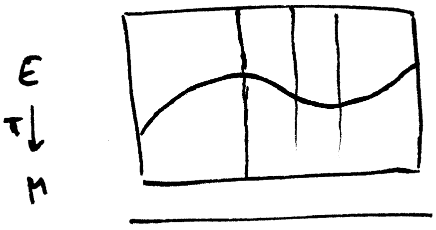
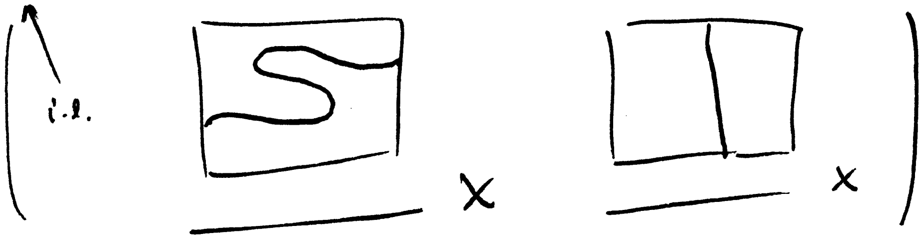
Charged matter is a <u>section</u> of a v.b. $E \xrightarrow{\pi} M$. A section is a map $s: M \to E$ that obeys $\pi \cdot s = id$.





1) For example, in electromagnetism, a $\mathbb{C}$-scalar field ϕ of charge q isn't really a function on M, but rather a section of a v.b. of rank 1 over M. This is because under a gauge transform (i.e. change of local trivialization) we have

$$\phi(x) \longmapsto \underbrace{e^{iq\lambda(x)}}_{\rho(U(1))} \phi(x)$$

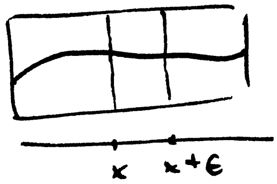
which is the transformation behaviour of an element of $E|_x$.

$\uparrow \rho$ is $\rho: U(1) \to$ Unitary 1×1 matrices $\subset GL(1,\mathbb{C})$

2) QCD has gauge group $G = SU(3)$ and quarks / antiquarks lie in the fundamental / antifundamental rep^{n}s. i.e. $q(x) \in E|_x = \mathbb{C}^3$. Under a change of local trivialization the 'value' of the quark field changes by $q(x) \longmapsto \rho(t(x))\, q(x)$ where $\rho: SU(3) \to Mat(3,\mathbb{C})$.

<u>Connections + Covariant derivatives</u>

We need a new notion of how to differentiate a section $s: M \to E$ because for any finite ϵ, the expression $\dfrac{s(x+\epsilon) - s(x)}{\epsilon}$ doesn't make sense because $s(x+\epsilon) \in E|_{x+\epsilon}$ while $s(x) \in E|_x$.



Let $\Omega^0_M(E)$ be the space of smooth sections $s: M \to E$, and let $\Omega^1_M(E)$ be the space of all smooth sections of E with values in 1-forms (covectors) on M.

A <u>connection</u> ∇ is defined to be a linear map $\nabla: \Omega^0_M(E) \to \Omega^1_M(E)$ s.t.

i) linearity $\nabla(\alpha_1 s_1 + \alpha_2 s_2) = \alpha_1 \nabla s_1 + \alpha_2 \nabla s_2 \quad \forall\, s_1, s_2 \in \Omega^0_M(E),\ \alpha_1, \alpha_2$ const

ii) Leibnitz $\nabla(fs) = (df)s + f(\nabla s) \quad \forall\, s \in \Omega^0_M(E)$ and $f \in C^\infty(M)$

Given a vector field $V(x)$ on M, the covariant derivative of s in the direction V is the map $\nabla_V: \Omega^0_M(E) \to \Omega^0_M(E)$ defined by $\nabla_V: s \longmapsto V \lrcorner (\nabla s)$ $(= V^\mu \nabla_\mu s)$

Given any two connections ∇, ∇', the difference obeys

$(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)s$ so $\nabla' - \nabla: \Omega^0_M(E) \to \Omega^1_M(E)$ that is linear over $C^\infty(M)$. Hence it must be some element of $\Omega^1_M(\mathrm{End}\, E)$ (i.e. matrix valued 1-form $(A_\mu(x))^a{}_b$). In particular, in $U \subset M$ with a trivialisation Φ,

we have $\Phi \circ s : U \longrightarrow U \times \mathbb{C}^n$ Then we can think of $s_\Phi(x)$ as just
$$x \longmapsto (x, s_\Phi(x))$$
a collection of n functions and we could define a 'trivial' connection just as $\nabla = d$.

Then any other $\nabla : U \to E|_U$ can be expressed as
$$\nabla s = ds + As \quad \text{for some } A \in \Omega^1_U(\mathrm{End}\, E) \quad \begin{cases} \text{connection 1-form} \\ \text{gauge field} \end{cases}$$
where the particular A depends on our trivialisation Φ.

Suppose $U_\alpha \cap U_\beta \neq \emptyset$ with trivialisations Φ_α, Φ_β and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{End}\, E$ denote the transition f^ns. We have $s_\alpha = g_{\alpha\beta}\, s_\beta$ and <u>by def^n</u> $(\nabla s)_\alpha = g_{\alpha\beta} (\nabla s)_\beta$.

It follows that $A_\alpha = -g_{\alpha\beta}\, d(g_{\alpha\beta})^{-1} + g_{\alpha\beta} A_\beta (g_{\alpha\beta})^{-1}$

e.g. in the ~~Abelian~~ $r=1$ case, $g_{\alpha\beta}$ are just complex functions, so
$$A_\beta = g\, dg^{-1} + g A_\alpha g^{-1}$$
$$= g\, dg^{-1} + A = i(d\lambda - iA) \quad \text{if } g = e^{-i\lambda}$$

<u>Curvature</u>

Given a connection ∇, we can extend its action to elements $s_{[\mu_1 \dots \mu_p]}(x) \in \Omega^p_M(E)$.

as $\nabla : \Omega^p_M(E) \longrightarrow \Omega^{p+1}_M(E)$ with again $\nabla(\alpha_1 s_1 + \alpha_2 s_2) = \alpha_1 \nabla s_1 + \alpha_2 \nabla s_2$ for $s_{1,2} \in \Omega^p_M(E)$ and $\alpha_{1,2}$ const, and now
$$\nabla(\omega \wedge s) = (d\omega)s + (-1)^{|q|} \omega \wedge (\nabla s) \quad \text{where } \omega \in \Omega^q_M,\ s \in \Omega^{p-q}_M(E)$$
∇ is thus a 'covariant generalisation of the de Rham operator d'. However, whereas $d^2 \equiv 0$, we have instead
$$\nabla^2(\omega \wedge s) = \nabla(d\omega \wedge s + (-1)^q \omega \wedge (\nabla s))$$
$$= d^2\omega \wedge s + (-1)^{q+1} d\omega \wedge \nabla s + (-1)^q d\omega \wedge \nabla s + (-1)^{2q} \omega \wedge (\nabla^2 s)$$
$$= \omega \wedge (\nabla^2 s).$$
So ∇^2 is linear over Ω^q_M &, so it must ~~correspond~~ be a multiplicative operator:
$$\nabla^2(s) = F_\nabla s \quad \text{for some } F_\nabla \in \Omega^2_M(\mathrm{End}\, E)$$
$$\text{in comp}^n: \quad F_\nabla = \tfrac{1}{2}(F_{\mu\nu}(x))^a{}_b\, dx^\mu \wedge dx^\nu$$
In a local trivialisation Φ, we had $\nabla s|_U = ds + As|_U$, so
$$\nabla^2 s|_U = \nabla(ds + As)|_U$$
$$= d^2 s|_U + d(As)|_U + A(ds + As)|_U$$
$$= (dA + A \wedge A)|_U$$

Thus locally $F = dA + A \wedge A$

$$= (\partial_\mu A_\nu + A_\mu A_\nu)\, dx^\mu \wedge dx^\nu$$

$$= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu)\, dx^\mu \wedge dx^\nu$$

$$= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{[A_\mu, A_\nu]}_{\text{matrix commutator}})\, dx^\mu \wedge dx^\nu$$

Let's also compute $\nabla^3 s \begin{cases} = \nabla(\nabla^2 s) = \nabla(Fs) = (\nabla F)_\wedge s + F \wedge (\nabla s) \\ = \nabla^2(\nabla s) = F \wedge \nabla s \end{cases}$

These are compatible iff $\nabla(F_\nabla) \equiv 0$ known as the Bianchi identity. On U,

$$\Big(\nabla(F)|_U = (d+A)(dA + A \wedge A)$$

$$= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A$$

$$= d^2 A + dA \wedge A - A \wedge dA + A\, dA + A^3 - dA \wedge A - A^3$$

$$= 0 \quad \checkmark \Big)$$