③ Non-abelian internal symmetries

A theory of the form

$$\mathcal{L} = \sum_{a=1}^{N} \left(\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} \sum_{a=1}^{N} \phi_a^2 - g \left(\sum_{a=1}^{N} \phi_a^2 \right)^2 \right.$$

is inv't under $G = SO(N)$ or $U(N/2)$ if the field suitably complexified.
$SU(3)$ is 8-fold way.

$\exists$ a trick to determine the current. Know that $\delta \mathcal{L} = 0$. Re-do the transformation with $\alpha = \alpha(x)$. We'll find $\delta \mathcal{L} = (\partial_\mu \alpha(x)) h^\mu(\phi)$.

Since $\delta \mathcal{L} = 0$ when $\alpha = \text{const}$, we must have $\delta S = \int \delta \mathcal{L} = -\int \alpha(x) \partial_\mu h^\mu$ which means that EoM are satisfied (so $\delta S = 0$ $\forall$ variations including $\delta \alpha = \alpha(x) \phi$).

We have $\partial_\mu h^\mu = 0$, so identify $h^\mu = j^\mu$ is the conserved current.

The Hamiltonian Formulation can also accommodate field theories. Define the conjugate momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$, not to be confused with total momentum P_i. The Hamiltonian density $\mathcal{H} = \pi(x) \dot{\phi}(x) - \mathcal{L}(x)$ where, as in classical mechanics, we eliminate $\dot{\phi}$ in favour of $\pi(x)$ everywhere in $\mathcal{H}$.

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi). \quad \Rightarrow \quad \pi = \dot{\phi}$$

$$\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi)$$

The Hamiltonian is ~~the spatial integral~~ then $H = \int d^3x \, \mathcal{H}$ - agrees with the total energy E from Noether's theorem. Hamilton's eq are $\dot{\phi} = \partial \mathcal{H} / \partial \pi$, $\dot{\pi} = -\partial \mathcal{H} / \partial \phi$ give us EoM of $\phi(x)$. The Hamiltonian formulation is not manifestly Lorentz inv't. Of course, the physics remain unchanged and so it must really be Lorentz inv't.

Canonical Quantization

Recall in QM, canonical quantization says to take generalized coords q_a and momenta p_a and promote them to ops. We replace Poisson brackets with commutators $[q_a, p^b] = i \delta_a{}^b$. We will do the same for $\phi_a(\underline{x})$ and ~~$\phi_b(\underline{x})$~~ $\pi_b(\underline{x})$.

A quantum field is an op valued f^n of space obeying the commutator relations

$$[\phi_a(\underline{x}), \phi_b(\underline{y})] = 0 \qquad [\pi^a_a(\underline{x}), \pi^b(\underline{y})] = 0 \qquad \text{and}$$

$$[\phi_a(\underline{x}), \pi^b(\underline{y})] = i\delta^3(\underline{x}-\underline{y})\,\delta_a^b$$

We're in the Schrödinger picture, so ops $\phi_a(\underline{x})$ and $\pi^a_a(\underline{x})$ don't depend on t — all the t dependence is in the states which evolve by the usual Schrödinger eq.

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

We want to know the spectrum of H. Typically, this is very hard: we have an ∞ # of d.o.f., one for each point $\underline{x}$ in space. However, for certain theories (known as free theories) can find new dof in which all states evolve independently.

Free field theories have ~~LLL~~ $\mathcal{L}$ which are quadratic in the fields $\Rightarrow$ EoM linear. the simplest free theory is the classical Klein-Gordon theory for a real scalar field $\phi(x,t)$: $\partial_\mu\partial^\mu\phi + m^2\phi = 0$. To see why it is free, take the Fourier transform.

$$\phi(\underline{x},t) = \int\frac{d^3p}{(2\pi)^3}\, e^{i\underline{p}\cdot\underline{x}}\,\tilde{\phi}(\underline{p},t) \quad \xrightarrow{KG}$$

$$\left[\frac{\partial^2}{\partial t^2} + (|p|^2+m^2)\right]\tilde{\phi}(\underline{p},t) = 0 \quad \text{which describes a harmonic oscillator at}$$

$\omega_p = \sqrt{|p|^2+m^2}$ So the sols to the classical KG eq is a superposition of simple harmonic oscillators, each vibrating at a different frequency (and arbitrary amplitude).

To quantise $\phi(\underline{x},t)$, need to quantise this ∞ # harmonic oscillators

Review of simple harmonic oscillator

Schrödinger picture with $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$ where $[p,q] = -i$. To find the spectrum we define creation & annihilation operators

$$a = \frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}\,q, \quad a^\dagger = -\frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}\,q \qquad \Leftrightarrow$$

$$q = \frac{1}{\sqrt{2\omega}}(a+a^\dagger), \quad p = -i\sqrt{\frac{\omega}{2}}\,(a-a^\dagger)$$

Substituting $[a, a^\dagger] = 1$.

$$H = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$[H, a^\dagger] = \omega a^\dagger \qquad [H, a] = -\omega a$$

these ensure that a, $a^\dagger$ take us between energy eigenstates.

Let $H|E\rangle = E|E\rangle$ then $Ha^\dagger|E\rangle = (E+\omega)\, a^\dagger|E\rangle$ and

$$Ha\,|E\rangle = (E-\omega)\, a\,|E\rangle .$$

Get a ladder of energy eigenstates with energy

$$\ldots,\ E-2\omega,\ E-\omega,\ E,\ E+\omega,\ E+2\omega,\ \ldots$$