

General Relativity
Example about 1)

1a. $\mathbb{R}\mathbb{P}^n = \mathbb{R}^{n+1}/\{0\}$ with $(x_1, \dots, x_{n+1}) \sim \lambda(x_1, \dots, x_{n+1})$
where $\lambda \in \mathbb{R}, \lambda \neq 0$.

Def'ns charts $O_\alpha = \{(x) \in \mathbb{R}^{n+1} : x_\alpha \neq 0\}$

then $\bigcup_\alpha O_\alpha = \mathbb{R}\mathbb{P}^n$ as $\{0\}$ is not in $\mathbb{R}\mathbb{P}^n$.

Maps $\phi_\alpha: O_\alpha \rightarrow U_\alpha \subset \mathbb{R}^n$

$$\begin{aligned}\phi_\alpha(x_1, \dots, x_{n+1}) &= \left(\frac{x_1}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha} \right) \\ &\equiv (\tilde{x}_1, \dots, \tilde{x}_n)\end{aligned}$$

For O_α and O_β , $O_\alpha \cap O_\beta = \{(x) \in \mathbb{R}^{n+1} : x_\alpha, x_\beta \neq 0\}$

$\phi_\alpha^{-1}: U_\alpha \rightarrow O_\alpha$

$$\begin{aligned}\phi_\alpha^{-1}(\tilde{x}_1, \dots, \tilde{x}_n) &= \phi_\alpha^{-1}\left(\frac{x_1}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha}\right) \\ &= \left(\frac{x_1}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, 1, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha}\right)\end{aligned}$$

ϕ_α is 1-to-1 as $\phi_\alpha^{-1} \circ \phi_\alpha$ maps to related points which are identified in $\mathbb{R}\mathbb{P}^{n+1}$.

$\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(O_\alpha \cap O_\beta) \rightarrow \phi_\beta(O_\alpha \cap O_\beta)$

$$\phi_\beta \circ \phi_\alpha^{-1}(\tilde{x}_1, \dots, \tilde{x}_n) = \phi_\beta\left(\frac{x_1}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, 1, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha}\right)$$

$$\text{or } = \phi_\beta(\tilde{x}_1, \dots, \tilde{x}_{\alpha-1}, 1, \tilde{x}_\alpha, \dots, \tilde{x}_n)$$

$$= \left(\frac{x_1}{x_\alpha x_\beta}, \dots, \frac{x_{\alpha-1}}{x_\alpha x_\beta}, \frac{1}{x_\beta}, \frac{x_{\alpha+1}}{x_\alpha x_\beta}, \dots, \frac{x_{\beta-1}}{x_\alpha x_\beta}, \frac{x_{\beta+1}}{x_\alpha x_\beta}, \dots, \frac{x_{n+1}}{x_\alpha x_\beta}\right)$$

is a smooth map.

$\therefore \phi_\alpha$ are charts for a smooth manifold $\mathbb{R}\mathbb{P}^n$. \square

$\mathbb{R}\mathbb{P}^n = \mathbb{R}^{n+1}/\{\mathbf{0}\}$ with $X \sim \lambda X$, $X \in \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R}$, $X \neq \mathbf{0}$, $\lambda \neq 0$

$$S^n = \left\{ \underline{x} \in \mathbb{R}^{n+1} : |\underline{x}| = 1 \right\} \text{ or equivalently}$$

$$= \mathbb{R}^{n+1}/\{\mathbf{0}\} \text{ with } X \sim \lambda X, X \in \mathbb{R}^{n+1}, \lambda \in \mathbb{R}, X \neq \mathbf{0}, \lambda > 0$$

By identifying anti-podal points on S^n , $\lambda > 0 \rightarrow \lambda \neq 0$ for the identification above. This gives exactly $\mathbb{R}\mathbb{P}^n$.

$$SO(3) = \text{rotations in } \mathbb{R}^3 = \cancel{\{ R(\underline{n}, \theta) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid R(\underline{n}, \theta) \text{ is } \cos \theta \text{ diag} \}}$$

$SO(3)$ is S^3 with anti-podal points identified

as $SO(3)$ is parameterised as (\underline{n}, θ) where \underline{n} is a unit vector in \mathbb{R}^3 and $\theta \in [0, 2\pi) \Rightarrow$ defines surface of S^3 .

Due to identification $(\underline{n}, \theta) \sim (-\underline{n}, 2\pi - \theta)$, anti-podal points on S^3 are identified.

$$\therefore SO(3) \cong \mathbb{R}\mathbb{P}^3 \text{ by above}$$

1b. Generating vector fields are given by $V|_p = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} G_\varepsilon(p)$ where G_ε is the 1-parameter (sub)group.

\tilde{x} is the transformed coordinate

$$V_1 = \frac{d\tilde{x}}{d\varepsilon} \frac{\partial}{\partial \tilde{x}} \Big|_{\varepsilon=0} = \frac{\partial}{\partial x}$$

$$V_2 = \left(e^{\varepsilon_2} x \frac{\partial}{\partial \tilde{x}} \right) \Big|_{\varepsilon_2=0} = x \frac{\partial}{\partial x}$$

$$V_3 = \left(\frac{x^2}{(1-\varepsilon_3 x)^2} \frac{\partial}{\partial \tilde{x}} \right) \Big|_{\varepsilon_3=0} = x^2 \frac{\partial}{\partial x}$$

$x \rightarrow \frac{ax+b}{cx+d}$ are generated by $x \mapsto \frac{ax}{cx+d}$ $p \rightarrow \frac{ap+b}{cp+d}$ $\sim ap + bq$
 $q \rightarrow cpq + d \sim cq + dq$

$$\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M \in GL(2; \mathbb{R})} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\text{and } \begin{pmatrix} p \\ q \end{pmatrix} \sim \lambda \begin{pmatrix} p \\ q \end{pmatrix} \quad \lambda \neq 0 \quad \lambda \in \mathbb{R}$$

$$M \in GL(2; \mathbb{R})$$

$$\Rightarrow M \sim \lambda M \text{ and w.l.o.g.}$$

$$M \in SL(2; \mathbb{R})$$

New identity

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+\theta_1 & \theta_2 \\ \theta_3 & 1+\theta_4 \end{pmatrix} \text{ with } \theta_i \rightarrow 0 \text{ for } i \geq 2.$$

$$\begin{aligned} (1+\theta_1)(1+\theta_4) - \theta_2\theta_3 &= 1 \\ 1 + \theta_1 + \theta_4 + \theta_1\theta_4 - \theta_2\theta_3 &= 1 \\ \theta_3 &= \frac{\theta_1 + \theta_4 + \theta_1\theta_4}{\theta_2} \quad \text{if } \theta_2 \neq 0 \quad \cancel{\text{else } \theta_1 + \theta_4 + \theta_1\theta_4 = 0} \end{aligned}$$

Vector fields generating these are

$$\begin{aligned} \tilde{V}_1 &= p \frac{\partial}{\partial p} & \tilde{V}_3 &= p \frac{\partial}{\partial q} \\ \tilde{V}_4 &= q \frac{\partial}{\partial q} & \tilde{V}_2 &= q \frac{\partial}{\partial p} \end{aligned}$$

In terms of x , ($x = p/q$)

$$\begin{aligned} V_1' &= p \frac{\partial x}{\partial p} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x} = V_2 \\ V_2' &= q \frac{\partial x}{\partial q} \frac{\partial}{\partial x} = -x \frac{\partial}{\partial x} = V_1 \\ V_3' &= p \frac{\partial x}{\partial q} \frac{\partial}{\partial x} = -x^2 \frac{\partial}{\partial x} = -V_3 \\ V_4' &= q \frac{\partial x}{\partial q} \frac{\partial}{\partial x} = -x \frac{\partial}{\partial x} = -V_2 \end{aligned}$$

} the three vector fields generating
the previous transformations

As the matrix transformation generated by these vector fields, $M \in SL(2; \mathbb{R})$,
these vector fields V_α are the Lie algebra $L(SL(2, \mathbb{R})) = \mathfrak{sl}(2, \mathbb{R})$.

$$\text{Indeed, } [V_1, V_2] = V_1$$

$$[V_2, V_3] = V_2 \quad \text{hence } \mathfrak{sl}(2, \mathbb{R}) \text{ is a subalgebra of } \text{vect}(\mathbb{R})$$

$$[V_3, V_1] = -2V_2$$

Subalgebra of $\text{vect}(\mathbb{R})$, $\mathfrak{g} = \left\{ V_i = f_i(x) \frac{\partial}{\partial x} : [V_i, V_j] \in \mathfrak{g} \quad \forall V_i, V_j \in \mathfrak{g} \right\}$

$$\Rightarrow f_i f_j - f_j f_i = f_k \quad \text{for some } i, j, k \text{ where } V_i, V_j, V_k \in \mathfrak{g}$$

$$2. \hat{f}: T_p(M) \rightarrow T_p^*(M)$$

$$x \in T_p(M) \mapsto w \in T_p^*(M), w = \hat{f}(x)$$

$$T: (x, y) \rightarrow (\hat{f}(y))(x)$$

$$= w(x) \quad \text{where } w = \hat{f}(y)$$

$$= w_p x^r$$

$$T = g_{\mu\nu} : g_{\mu\nu} X^r Y^\nu = w_p x^r \quad \text{with } w_p = g_{\mu\nu} Y^\nu$$

T as a tensor $\overset{(0,2)}{\sim}$ as the map goes from vectors to scalars.

$$V: T_p(M) \rightarrow T_p(M)$$

$$X^r \rightarrow Y^r = V^m_{\nu} X^\nu \quad V \text{ is a tensor } (1,1) \text{ as both roles contracted with covectors give scalars}$$

Identity δ^m_ν is tensor $(1,1)$.

$$3. V^{ab}$$

$$V^{ab} S_{ab} = V^{ba} S_{ba} = V^{ba} S_{ab}$$

$$S_{ab} = S_{ba}$$

$$A_{ab} = -A_{ba}$$

$$= \frac{1}{2} (V^{ab} S_{ab} + V^{ba} S_{ab}) = V^{(ab)} S_{ab}$$

$$V^{ab} A_{ab} = V^{ba} A_{ba} = -V^{ba} A_{ab}$$

$$= \frac{1}{2} (V^{ab} A_{ab} - V^{ba} A_{ab}) = V^{[ab]} A_{ab}$$

4. In some basis, consider matrix with elements $[K^{\mu\nu}]$ over \mathcal{C} .

For an inner product $\langle \cdot, \cdot \rangle$, (matrix $K_{ij} = K^{\mu\nu}$ with $i=p, j=v$)

$$|\langle K_i | K_j \rangle|^2 \leq \langle K_i | K_i \rangle \langle K_j | K_j \rangle \quad (\text{Cauchy-Schwarz})$$

where consider K_i, K_j ~~column~~ vectors

This equality holds when $K_i \sim K_j$ linearly dependent. If equality holds for each pair i, j , then all rows are linearly dependent, i.e.

$$K_{ij} = A_i B_j \quad \text{as } A_i \text{ rows } B_j \text{ with factors } A_i \text{ multiplying each.}$$

$$\langle K_i | K_j \rangle = \left(\sum_m K_{im} K_{jm} \right)^2$$

This approach is not tensorial, i.e. not basis-invariant. However, if they holds in one basis, it must hold in all basis by tensor transformation law.

Could express in a basis invariant manner, if possible to construct a suitable inner product using tensors on the manifold (possible in Riemannian geometry using metric tensor), but ~~not~~ pseudo-Riemannian metric non-invertible).

Trying to understand how similar problems as \mathcal{E} is not a tensor.

In general, it is obvious from above that cannot be written as a direct product.

Can always write as (consider ∂_i for $i=1 \dots n$, and $(A_i)^M = \partial_i^r$)

$$K^{\mu\nu} = \sum_i (A_i)^M (B_i)^N \quad (\text{sum of direct products})$$

$$5. F_{\mu\nu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \rightarrow \tilde{F}_{\mu\nu} = \tilde{g}_{\mu\rho} \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} F_{\sigma\rho} = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} \frac{\partial^2 f}{\partial x^\sigma \partial x^\rho} \\ = \frac{\partial^2 f}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} \quad (\text{chain rule})$$

$\therefore F_{\mu\nu}$ is a ~~2D~~ (0,2) tensor

$$6. g_{\mu\nu} \rightarrow \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} g_{\sigma\rho} = \tilde{g}_{\mu\nu}$$

$$\tilde{g} = \det(\tilde{g}_{\mu\nu}) = \det\left(\frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\right) \det\left(\frac{\partial x^\rho}{\partial \tilde{x}^\nu}\right) \det(g_{\sigma\rho})$$

$$\tilde{g} = \det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) g \quad \text{scaled by } \eta \text{ of Jacobian det.}$$

$$7. [e_\mu, e_\nu] = e_\mu^\alpha \frac{\partial e_\nu}{\partial x^\alpha} - e_\nu^\alpha \frac{\partial e_\mu}{\partial x^\alpha} = \gamma^\rho_{\mu\nu} e_\rho$$

for components β gives ...

Contract with ω_β^σ ,

$$e_\mu^\alpha \frac{\partial e_\nu^\sigma}{\partial x^\alpha} \omega_\beta^\sigma - e_\nu^\alpha \frac{\partial e_\mu^\sigma}{\partial x^\alpha} \omega_\beta^\sigma = \gamma^\rho_{\mu\nu} e_\rho^\sigma \omega_\beta^\sigma$$

$$\Rightarrow \frac{\partial}{\partial x^\alpha} \left(e_\mu^\alpha e_\nu^\beta \omega_\beta^\sigma \right) - e_\nu^\beta \omega_\beta^\sigma \frac{\partial e_\mu^\alpha}{\partial x^\alpha} - e_\mu^\alpha e_\nu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha}$$

$$- \frac{\partial}{\partial x^\alpha} \left(e_\nu^\alpha e_\mu^\beta \omega_\beta^\sigma \right) + e_\mu^\beta \omega_\beta^\sigma \frac{\partial e_\nu^\alpha}{\partial x^\alpha} + e_\nu^\alpha e_\mu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha}$$

$$= \gamma^\sigma_{\mu\nu}$$

$$\frac{\partial}{\partial x^\alpha} \left(e_\mu^\alpha e_\nu^\sigma \right) - \delta_\nu^\sigma \frac{\partial e_\mu^\alpha}{\partial x^\alpha} - e_\mu^\alpha e_\nu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} \\ - \frac{\partial}{\partial x^\alpha} \left(e_\nu^\alpha e_\mu^\sigma \right) + \delta_\mu^\sigma \frac{\partial e_\nu^\alpha}{\partial x^\alpha} + e_\nu^\alpha e_\mu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} = \gamma^\sigma_{\mu\nu}$$

$$\Rightarrow e_\mu^\alpha e_\nu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} - e_\nu^\alpha e_\mu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} = -\gamma^\sigma_{\mu\nu}$$

Contract with $\omega^\mu_\gamma, \omega^\nu_\delta$

~~$$e_\mu^\alpha e_\nu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} - e_\nu^\alpha e_\mu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} - e_\nu^\alpha \omega_\delta^\nu e_\mu^\beta \omega_\gamma^\mu \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha}$$~~

$$\delta_\mu^\alpha \delta_\nu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} - \delta_\nu^\alpha \delta_\mu^\beta \frac{\partial \omega_\beta^\sigma}{\partial x^\alpha} = -\gamma^\sigma_{\mu\nu} \omega^\mu_\gamma \omega^\nu_\delta$$

$$\Rightarrow \frac{\partial \omega^\sigma_\delta}{\partial x^\alpha} - \frac{\partial \omega^\sigma_\delta}{\partial x^\alpha} = -\gamma^\sigma_{\mu\nu} \omega^\mu_\gamma \omega^\nu_\delta$$

Coordinate induced basis

$$e_\mu = \frac{\partial}{\partial y^\mu} \Rightarrow e_\mu^\alpha = \delta_\mu^\alpha$$

$$\Rightarrow \frac{\partial e_\mu^\alpha}{\partial y^\beta} = \frac{\partial \delta_\mu^\alpha}{\partial y^\beta} = 0 \quad \therefore [e_\mu, e_\nu] = 0 \quad \forall \mu, \nu$$

$$\text{If } [e_\mu, e_\nu] = 0 \quad \forall \mu, \nu \Rightarrow \gamma^\rho_{\mu\nu} = 0 \quad \forall \rho, \mu, \nu$$

$$\Rightarrow (+) \quad \frac{\partial \omega_\delta^\sigma}{\partial x^\alpha} = \frac{\partial \omega_\delta^\sigma}{\partial x^\alpha} \quad \forall \delta, \alpha$$

Contract with arbitrary e_μ^δ ,

$$e_\mu^\delta \frac{\partial \omega_\delta^\sigma}{\partial x^\alpha} = (e_\mu \omega^\sigma)_\gamma = (\delta_\mu^\sigma)_\gamma \quad \cancel{\Rightarrow \delta_\mu^\delta \frac{\partial \omega_\delta^\sigma}{\partial x^\alpha}}$$

$$\cancel{\frac{\partial}{\partial x^\alpha} (e_\mu^\delta \omega_\delta^\sigma)} - \omega_\delta^\sigma \frac{\partial e_\mu^\delta}{\partial x^\alpha} = (\delta_\mu^\sigma)_\gamma \quad \text{with } \cancel{\omega_\delta^\sigma}$$

$$\cancel{\frac{\partial \omega_\delta^\sigma}{\partial x^\alpha} e_\mu^\delta} = (\delta_\mu^\sigma)_\gamma e_\nu^\gamma$$

$$8. ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$(a) r = \sqrt{x^2 + y^2 + z^2} \quad \cos\theta = \frac{z}{r} \quad \tan\phi = \frac{y}{x}$$

$$\Rightarrow x = r \cos\phi \sin\theta \quad y = r \sin\phi \sin\theta \quad z = r \cos\theta$$

$$dx = \cos\phi \sin\theta dr - r \sin\phi \sin\theta d\phi + r \cos\phi \cos\theta d\theta$$

$$dy = \sin\phi \sin\theta dr + r \cos\phi \sin\theta d\phi + r \sin\phi \cos\theta d\theta$$

$$dz = \cos\theta dr + r \sin\theta d\theta$$

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$(b) \tilde{t} = t \quad \tilde{x} = \sqrt{x^2 + y^2} \cos(\phi - \omega t) \quad \tilde{y} = \sqrt{x^2 + y^2} \sin(\phi - \omega t) \quad \tilde{z} = z$$

$$d\tilde{t} = dt \quad d\tilde{z} = dz \quad \rho^2 = x^2 + y^2 \quad x = \rho \cos\phi, \\ y = \rho \sin\phi$$

$$d\tilde{x} = d\rho \cos(\phi - \omega t) - \rho \sin(\phi - \omega t)(d\phi - \omega dt)$$

$$d\tilde{y} = d\rho \sin(\phi - \omega t) + \rho \cos(\phi - \omega t)(d\phi - \omega dt)$$

$$d\tilde{x}^2 + d\tilde{y}^2 = d\rho^2 + \rho^2(d\phi^2 - 2\omega d\phi dt + \omega^2 dt^2)$$

$$dx = \cos\phi d\rho - \rho \sin\phi d\phi$$

$$dy = \sin\phi d\rho + \rho \cos\phi d\phi$$

$$\Rightarrow d\tilde{x}^2 + d\tilde{y}^2 = dx^2 + dy^2 + \rho^2(-2\omega d\phi dt + \omega^2 dt^2)$$

$$\tan(\phi - \omega t) = \frac{\tilde{y}}{\tilde{x}} \quad \Rightarrow \quad \frac{d\phi - \omega dt}{\cos^2(\phi - \omega t)} = \frac{\tilde{x} d\tilde{y} - \tilde{y} d\tilde{x}}{\tilde{x}^2}$$

$$\Rightarrow d\tilde{x}^2 + d\tilde{y}^2 = d\rho^2 + \frac{1}{\rho^2} (\tilde{x} d\tilde{y} - \tilde{y} d\tilde{x})^2$$

$$\rho = r \sin\theta \quad d\rho = \sin\theta dr + r \cos\theta d\theta$$

$$d\tilde{z} = dz = \cos\theta dr - r \sin\theta d\theta$$

~~$$d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 = dr^2 + r^2 d\theta^2 + \frac{1}{r^2 \sin^2\theta} (\tilde{x} d\tilde{y} - \tilde{y} d\tilde{x})^2$$~~

$$= dr^2 + r^2 d\theta^2 + \frac{(\tilde{x} d\tilde{y} - \tilde{y} d\tilde{x})^2}{\tilde{x}^2 + \tilde{y}^2}$$

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 \stackrel{+d\tilde{z}^2}{=} \frac{(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2}{\tilde{x}^2 + \tilde{y}^2} + \rho^2 d\phi^2$$

$$= -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 \stackrel{+d\tilde{z}^2}{=} \frac{(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2}{\tilde{x}^2 + \tilde{y}^2} + (\tilde{x}^2 + \tilde{y}^2) \left[\frac{\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x}}{\tilde{x}^2 + \tilde{y}^2} + \omega d\tilde{t} \right]^2$$

$$= -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + 2\omega(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})d\tilde{t} + (\tilde{x}^2 + \tilde{y}^2)\omega^2 d\tilde{t}^2$$

If $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$,

$$g_{\mu\nu} = \begin{pmatrix} -1 + (\tilde{x}^2 + \tilde{y}^2)\omega^2 & -\omega\tilde{y} & \omega\tilde{x} & 0 \\ -\omega\tilde{y} & 1 & 0 & 0 \\ \omega\tilde{x} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow g^{\mu\nu} = \begin{pmatrix} -1 & -\omega\tilde{y} & \omega\tilde{x} & 0 \\ -\omega\tilde{y} & 1 - \omega^2\tilde{y}^2 & \omega\tilde{y}\tilde{x} & 0 \\ \omega\tilde{x} & \omega^2\tilde{y}\tilde{x} & 1 - \omega^2\tilde{x}^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$1. \quad \mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$t: \frac{\partial}{\partial t} \left(-2 \left(1 - \frac{2M}{r} \right) \dot{t} \right) = 0$$

$$r: \frac{\partial}{\partial r} \left(\frac{2r}{1 - \frac{2M}{r}} \right) = -\frac{2M}{r^2} \dot{t}^2 \not\equiv \frac{r^2 \frac{2M}{r^2}}{\left(1 - \frac{2M}{r} \right)^2} + 2r\dot{\theta}^2 + 2r\sin^2\theta \dot{\phi}^2$$

$$\theta: \frac{\partial}{\partial \theta} (2r^2\dot{\theta}) = 2r^2\sin\theta\cos\theta\dot{\phi}^2$$

$$\phi: \frac{\partial}{\partial \phi} (2r^2\sin^2\theta\dot{\phi}) = 0$$

$$\frac{2M}{r^2} \ddot{r} + \left(1 - \frac{2M}{r}\right) \ddot{t} = 0$$

$$\Gamma_{\mu\nu}^0 = \begin{pmatrix} 0 & \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\ddot{r}}{1 - \frac{2M}{r}} - \frac{\frac{2M}{r^2} \dot{r}^2}{\left(1 - \frac{2M}{r}\right)^2} = -\frac{M}{r^2} \dot{t}^2 - \frac{\frac{M}{r^2} \dot{r}^2}{\left(1 - \frac{2M}{r}\right)^2} + r \dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2$$

$$2r\ddot{r}\dot{\theta} + r^2 \ddot{\theta} = r^2 \sin \theta \cos \theta \dot{\phi}^2$$

$$\Gamma_{\mu\nu}^1 = \begin{pmatrix} \frac{M}{r^2} \left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ -\frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & -r \left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & r \sin^2 \theta \left(1 - \frac{2M}{r}\right) & 0 \end{pmatrix}$$

$$\Gamma_{\mu\nu}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \sin \theta \cos \theta \\ 0 & 0 & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2r\ddot{r}\sin^2 \theta \dot{\phi} + 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0$$

$$\Gamma_{\mu\nu}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & 0 & 0 & \cot \theta \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Independent of t and ϕ :

$\left(1 - \frac{2M}{r}\right) \dot{t} = \epsilon$ constant (conserved quantity - "energy")

$r^2 \sin^2 \theta \dot{\phi} = l_c$ constant (conserved quantity - "angular momentum")

to. to.

10. Timelike geodesic $ds^2 < 0$,

geodesic equations

$$t: \frac{\partial}{\partial \tau} \left(-2t^{-2}\dot{t} \right) = 2t^{-3}(\dot{t}^2 - \dot{x}^2)$$

$$x: \frac{\partial}{\partial \tau} (2t^{-2}\dot{x}) = 0 \Rightarrow t^{-2}\dot{x} = \cancel{p} \text{ constant}$$

~~$$= \cancel{-2t^{-2}\ddot{t}} + \cancel{2t^{-3}\dot{t}^2} - \cancel{2t^{-3}(\dot{t}^2 - \dot{x}^2)}$$~~

~~$$\ddot{t} - t^{-1}\dot{t}^2 + t^{-1}\dot{x}^2 - t^{-2}\dot{x}\cancel{2t^{-3}(\dot{t}^2 - \dot{x}^2)} = 0$$~~

~~$$\Rightarrow \ddot{t} = p^2 t^2 + \dot{x}^2 \quad \text{constant}$$~~

~~$$\ddot{t} = p^2 t^2 + \dot{x}^2$$~~

Choose $0 > ds^2 = -dt^2 = t^{-2}(-dt^2 + dx^2)$ parameter τ .

$$\Rightarrow \cancel{dt^2} - \dot{x}^2 = t^2$$

From above,

$$t^2 - p^2 t^4 = t^2 \Rightarrow \left(\frac{dt}{d\tau} \right)^2 = p^2 t^4 + t^2$$

$$\tau = \int \frac{dt}{\sqrt{t^2 + p^2 t^4}} = \frac{1}{2} \log \frac{\sqrt{1+p^2 t^2} - 1}{\sqrt{1+p^2 t^2} + 1}$$

$$\Rightarrow \sqrt{1+p^2 t^2} = -\cancel{t} \coth \tau$$

$$p^2 t^2 = \frac{1}{\sinh^2 \tau} \quad \dot{x}^2 = p t^2 = \frac{1}{p \sinh^2 \tau} \Rightarrow x = \frac{1}{p} \int \frac{d\tau'}{\sinh^2 \tau'} = -\frac{\coth \tau}{p}$$

$$t = \frac{1}{p \sinh \tau}$$

$$\therefore x^2 - t^2 = \frac{1}{p^2}$$