A generic QFT has an asymptotic expansion

$$\mathcal{Z}(\hbar) \sim (2\pi\hbar)^{n/2} \frac{e^{-S(\phi_0)/\hbar}}{\sqrt{\det \partial_a \partial_b S|_{\phi_0}}} \left[1 + a_1 \hbar + a_2 \hbar^2 + \dots\right]$$

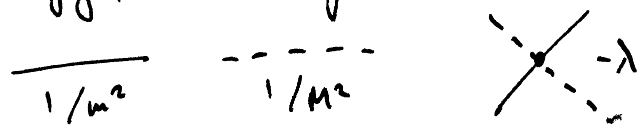
In some special ~~theories~~, the leading term is the <u>exact</u> answer! Typically this happens for SUSY theories.

<u>An effective theory</u>

Suppose we have two scalar fields $\phi, \chi \in \mathbb{R}$ and consider the action

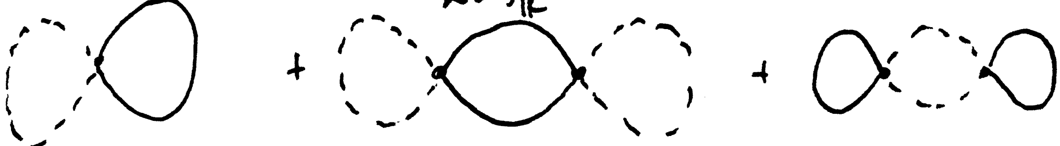
$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2$$

Correspondingly, we have Feynman rules ($\hbar = 1$)



We can use these to compute $\langle f(\phi, \chi)\rangle = \frac{1}{Z_0}\int_{\mathbb{R}^2} d\phi\, d\chi\, e^{-S(\phi,\chi)/\hbar}$

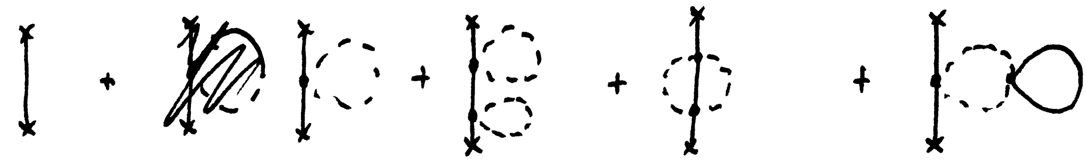
e.g. $\ln\left(\frac{Z}{Z_0}\right) =$



$$\frac{-\lambda}{4m^2M^2} + \frac{\lambda^2}{16m^4M^4} + \frac{\lambda^2}{16m^4M^4}$$

$+$  $\frac{\lambda^2}{8m^4M^4}$

e.g. $\langle\phi^2\rangle =$



$$\frac{1}{m^2} + \frac{-\lambda}{2m^4M^2} + \frac{\lambda^2}{4m^6M^4} + \frac{\lambda^2}{2m^6M^4} + \frac{\lambda^2}{4m^6M^4}$$

Let's arrive at this result in a different way. Suppose we think of χ as "heavy", so we cannot access it directly using our experimental equipment. In particular, if we're only interested in correl^n f^{n}s that depend only on ϕ, i.e. $f(\phi,\chi) = f(\phi)$, then we could try to 'integrate out' χ first. We define the <u>effective action</u> for ϕ, $S_{eff}(\phi)$, by

$$S_{eff}(\phi) = -\hbar \ln\left[\int_{\mathbb{R}} d\chi\, e^{-S(\phi,\chi)/\hbar}\right]$$

Generically, we can only find an asymp series for $S_{eff}(\phi)$ but in this example we have

$$\int_{\mathbb{R}} d\chi\, e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2 + \lambda\phi^2/2}}$$

$$\Rightarrow S_{eff}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\hbar}{2}\ln\left[1 + \frac{\lambda\phi^2}{2M^2}\right] + \frac{\hbar}{2}\ln\left(\frac{M^2}{2\pi\hbar}\right)$$

$$S_{eff}(\phi) \sim \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots + \frac{\hbar}{2}\ln\left(\frac{M^2}{2\pi\hbar}\right)$$

$$=: \frac{m_{eff}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \dots + \frac{\hbar}{2}\ln\left(\frac{M^2}{2\pi\hbar}\right)$$

where $\lambda_{2k} := (-1)^{k+1}\frac{\hbar(2k)!}{2^{k+1}k!}\frac{\lambda^k}{M^{2k}}$.

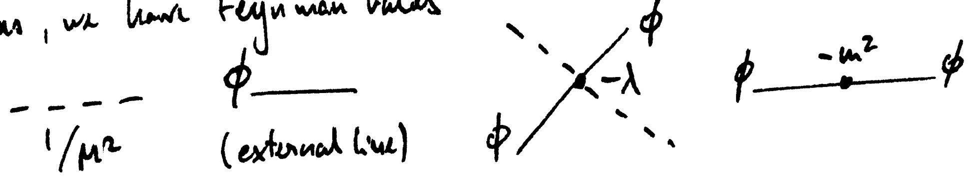
Remarks:

1) Integrating out χ has generated an infinite series of new interactions for ϕ in $S_{eff}(\phi)$.
We have $m^2 \longrightarrow m_{eff}^2 = m^2 + \frac{\hbar\lambda}{2M^2}$ so the effective mass of ϕ is also shifted.
Notice the new vertices are <u>quantum</u> effects: they vanish as $\hbar \to 0$. They are also suppressed by powers of $1/M^2$.

2) The original action had a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $(\phi, \chi) \longmapsto (\pm\phi, \pm\chi)$.
This symmetry is preserved and we do not generate any vertices $\sim \frac{\lambda_{2k+1}}{(2k+1)!}\phi^{2k+1}$.

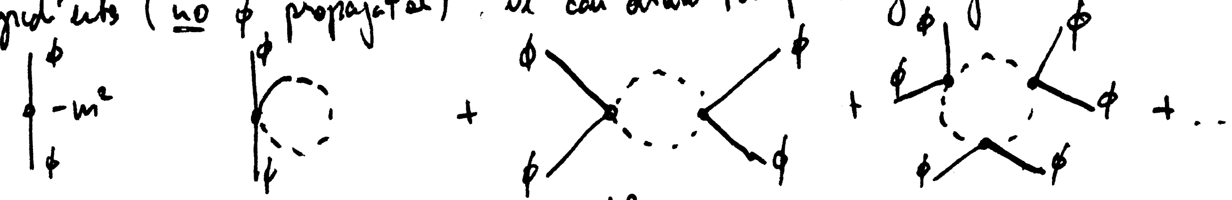
3) $S_{eff}(\phi)$ also contains a field independent term $\frac{\hbar}{2}\ln\left(\frac{M^2}{2\pi\hbar}\right)$. This plays no role in correlation fns $\langle f(\phi)\rangle$. However, this is the biggest problem in physics — this term contributes to Λ.

Let's think about where these (new vertices) terms have come from.
We now return to $S(\phi, \chi)$ and treat ϕ as non-dynamical (since just doing χ path integral).
Thus, we have Feynman rules



With these ingredients (<u>no</u> ϕ propagator), we can draw the following diagrams:

$-S_{eff}(\phi) =$



$$= \underbrace{\frac{-m^2}{2}\phi^2 - \frac{\lambda}{4M^2}\phi^2}_{} + \underbrace{\frac{\lambda^2}{16M^4}\phi^4}_{} - \underbrace{\frac{\lambda^3}{48M^6}\phi^6}_{} + \dots$$

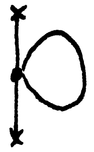
$$= -\frac{m_{eff}^2}{2}\phi^2 + \frac{\lambda_4\phi^4}{4!} + \frac{\lambda_6\phi^6}{6!} + \dots$$

We see that the new / shifted couplings in $S_{eff}(\phi)$ are generated by <u>loops</u> of χ fields.

- Generically, we can't hope to compute $S_{eff}(\phi)$ analytically, but we can use the Feynman diagrams to construct an asymptotic series for $S(\phi, \hbar)$.
- We should generically imagine starting from a complicated theory, i.e. $S(\phi, \chi)$ is ~~actually~~ "really" $S_{eff}(\phi, \chi)$ obtained by integrating out even higher mass fields.

Let's now compute $\langle \phi^2 \rangle$ using the effective theory:

$$S_{eff}(\phi) = \frac{m_{eff}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \dots$$

$\langle \phi^2 \rangle =$  $+$  $+ \dots$

$$\frac{1}{m_{eff}^2} - \frac{\lambda_4}{2 m_{eff}^6} + \dots$$

this agrees w/ earlier calculations correct to order λ^2.