General Relativity 2013

1. T tensor field (r,s), X vector field

(a)
$$(\mathcal{L}_X T)_p = \lim_{t\to 0} \frac{((\phi_{-t})_* T)_p - T_p}{t}$$

where ϕ_t is the diffeomorphism that maps a point p to the point parameter distance t along the integral curve of X

(b) In the coordinate chart (t, x^i) s.t. $X = \partial/\partial t$,

the diffeomorphism ϕ_t sends $x^\mu = (t_p, x^i_p)$ to $y^\mu = (t_p + t, x^i_p)$ and hence $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu{}_\nu$.

Given that for a diffeomorphism ϕ

$$[(\phi_*(T))^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}]_{\phi(p)} = \left(\frac{\partial y^{\mu_1}}{\partial x^{\rho_1}}\right)_p \cdots \left(\frac{\partial y^{\mu_r}}{\partial x^{\rho_r}}\right)_p \left(\frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}}\right)_p \cdots \left(\frac{\partial x^{\sigma_s}}{\partial y^{\nu_s}}\right)_p (T^{\rho_1\dots\rho_r}{}_{\sigma_1\dots\sigma_s})_p$$

then here

$$[((\phi_t)_* T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}]_{\phi_t(p)} = (T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s})_p$$

and

$$[((\phi_t)_* T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}]_p = (T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s})_{\phi_{-t}(p)}$$

From definition,

$$[(\mathcal{L}_X T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}]_p = \lim_{t\to 0}\frac{1}{t}\left[(((\phi_{-t})_* T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s})_p - (T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s})_p\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[(T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s})_{\phi_t(p)} - (T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_r})_p\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(t_p + t, x_p^i) - T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(t_p, x_p^i) \right]$$

$$= \left[\frac{\partial}{\partial t} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(t, x^i)\right]_{(t_p, x_p^i)}$$

In this chart, Lie derivative is simply the partial derivative w.r.t. t.

Hence,

it commutes with contraction

& satisfies the Leibniz rule

$$\mathcal{L}_X(S\otimes T) = (\mathcal{L}_X S)\otimes T + S\otimes(\mathcal{L}_X T)$$

$\therefore$ As these are relations between tensors, they hold in general.

(c) Similarly, in this chart

$$\mathcal{L}_X f = \frac{\partial}{\partial t} f = X(f) \qquad \text{as} \quad X = \frac{\partial}{\partial t} \text{ in this chart.}$$

$$(\mathcal{L}_X Y)^\mu = \frac{\partial}{\partial t} Y^\mu = [X, Y]^\mu$$

Both generalise as tensor equalities hold in all bases

$$\mathcal{L}_X f = X(f)$$

$$\mathcal{L}_X Y = [X, Y]$$

(d) Again, in the chart

$$(\mathcal{L}_X \omega)_\mu = \frac{\partial}{\partial t}\omega_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \underbrace{\partial_\mu X^\nu}_{=0 \text{ in this chart}}$$

$$= X^\nu \omega_{\mu,\nu} + \omega_\nu X^\nu{}_{,\mu} = X^\nu \omega_{\mu;\nu} + X^\nu \Gamma^\rho_{\mu\nu}\omega_\rho$$
$$+ \omega_\nu X^\nu{}_{;\mu} - \omega_\nu \Gamma^\nu_{\rho\mu} X^\rho$$

$$= X^\nu \omega_{\mu;\nu} + \omega_\nu X^\nu{}_{;\mu} \qquad \text{for a torsion-free connection}$$

Again, this is a tensor relation and must hold in any basis. In coordinate basis, can be written as before $(\mathcal{L}_X\omega)_\mu = X^\nu \omega_{\mu,\nu} + \omega_\nu X^\nu{}_{,\mu}$.

(e) T is $(1,1)$

First, in the chart above

$$(\mathcal{L}_X T)^\mu{}_\nu = \frac{\partial}{\partial t} T^\mu{}_\nu = X^\rho T^\mu{}_{\nu,\rho} + T^\mu{}_\rho \underbrace{X^\rho{}_{,\nu}}_{0} - T^\rho{}_\nu \underbrace{X^\mu{}_{,\rho}}_{0}$$

$$= X^\rho T^\mu{}_{\nu;\rho} + X^\rho \Gamma^\sigma_{\nu\rho} T^\mu{}_\sigma - X^\rho \Gamma^\mu_{\sigma\rho} T^\sigma{}_\nu$$

$$+ T^\mu{}_\rho X^\rho{}_{;\nu} - T^\mu{}_\rho \Gamma^\rho_{\sigma\nu} X^\sigma$$

$\Gamma^\mu_{[\nu\sigma]} = 0$ torsion-free

$$- T^\rho{}_\nu X^\mu{}_{;\rho} + T^\rho{}_\nu \Gamma^\mu_{\sigma\rho} X^\sigma$$

$$= X^\rho T^\mu{}_{\nu;\rho} + T^\mu{}_\rho X^\rho{}_{;\nu} - T^\rho{}_\nu X^\mu{}_{;\rho}$$

Generalising to any bases.

Coordinate bases

$$(\mathcal{L}_X T)^\mu{}_\nu = X^\rho T^\mu{}_{\nu,\rho} + T^\mu{}_\rho X^\rho{}_{,\nu} - T^\rho{}_\nu X^\mu{}_{,\rho}$$

Then

$$(\mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T)^\mu{}_\nu$$

$$= X^\rho (\mathcal{L}_Y T)^\mu{}_{\nu,\rho} + (\mathcal{L}_Y T)^\mu{}_\rho X^\rho{}_{,\nu} - (\mathcal{L}_Y T)^\rho{}_\nu X^\mu{}_{,\rho}$$

$$- Y^\rho (\mathcal{L}_X T)^\mu{}_{\nu,\rho} + (\mathcal{L}_X T)^\mu{}_\rho Y^\rho{}_{,\nu} - (\mathcal{L}_X T)^\rho{}_\nu Y^\mu{}_{,\rho}$$

$$= \underbrace{X^\rho (Y^\sigma T^\mu{}_{\nu,\sigma})_{,\rho}}_{1} + \underbrace{X^\rho (T^\mu{}_\sigma Y^\sigma{}_{,\nu})_{,\rho} - X^\rho (T^\sigma{}_\nu Y^\mu{}_{,\sigma})_{,\rho}}_{2}$$

$$\underbrace{+ Y^\sigma T^\mu{}_{\rho,\sigma} X^\rho{}_{,\nu}}_{2} + \underbrace{T^\mu{}_\sigma Y^\sigma{}_{,\rho} X^\rho{}_{,\nu} - T^\sigma{}_\rho Y^\mu{}_{,\sigma} X^\rho{}_{,\nu}}_{3}$$

$$\underbrace{- Y^\sigma T^\rho{}_{\nu,\sigma} X^\mu{}_{,\rho}}_{2} - \underbrace{T^\rho{}_\sigma Y^\sigma{}_{,\nu} X^\mu{}_{,\rho} + T^\sigma{}_\nu Y^\rho{}_{,\sigma} X^\mu{}_{,\rho}}_{3}$$

$$- X \leftrightarrow Y$$

$$= \underbrace{[X,Y]^\sigma T^\mu{}_{\nu,\sigma}} + X^\rho T^\mu{}_\sigma Y^\sigma{}_{,\nu\rho} - X^\rho T^\sigma{}_\nu Y^\mu{}_{,\sigma\rho}$$

$$\underbrace{- Y^\rho T^\mu{}_\sigma X^\sigma{}_{,\nu\rho} + Y^\rho T^\sigma{}_\nu X^\mu{}_{,\sigma\rho}}_{2}$$

$$+ \underbrace{T^\mu{}_\sigma Y^\sigma{}_{,\rho} X^\rho{}_{,\nu} + T^\sigma{}_\nu Y^\rho{}_{,\sigma} X^\mu{}_{,\rho} - T^\mu{}_\sigma X^\sigma{}_{,\rho} Y^\rho{}_{,\nu} - T^\sigma{}_\nu X^\rho{}_{,\sigma} Y^\mu{}_{,\rho}}_{3}$$

$$= [X,Y]^\sigma\, T^\mu{}_{\nu,\sigma} + T^\mu{}_\sigma\, [X, Y_{,\nu}]^\sigma - T^\sigma{}_\nu\, [X, Y_{,\sigma}]^\mu$$
$$- T^\mu{}_\sigma\, [Y, X_{,\nu}]^\sigma + T^\sigma{}_\nu\, [Y, X_{,\sigma}]^\mu$$

$$= [X,Y]^\sigma\, T^\mu{}_{\nu,\sigma} + T^\mu{}_\sigma\, [X,Y]^\sigma{}_{,\nu} - T^\sigma{}_\nu\, [X,Y]^\mu{}_{,\sigma}$$

$$= \left(\mathcal{L}_{[X,Y]}\, T\right)^\mu{}_\nu$$ Holds in general. $\square$

2. (a) bookwork $\quad T^{ij} = \partial_k (T^{ik} x^j) - (\partial_k T^{jk}) x^j \quad \ldots$

(b) Stars at $\pm(R\cos\Omega t,\ R\sin\Omega t,\ 0)$

Non-rel Newtonian orbit $\quad T_{00} = M\delta^{(3)}(\underline{x} - \underline{x}(t))$

$$I_{ij} = I^+_{ij} + I^-_{ij} \qquad I^\pm_{ij} = M x^\pm_i x^\pm_j \;//$$

Quadrupole formula: $\langle P \rangle = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle \qquad Q_{ij} = I_{ij} - \frac{1}{3} I_{kk} \delta_{ij}$

$$Q^\pm_{ij} = M x^\pm_i x^\pm_j - \frac{1}{3} M |x^\pm|^2 \delta_{ij}$$

$$\dddot{Q}^\pm_{ij} = M(\dddot{x}^\pm_i x^\pm_j + 3\ddot{x}^\pm_i \dot{x}^\pm_j + 3\dot{x}^\pm_i \ddot{x}^\pm_j + x^\pm_i \dddot{x}^\pm_j)$$

$$= -4M\Omega^2 (\dot{x}^\pm_i x^\pm_j + x^\pm_i \dot{x}^\pm_j) \qquad \text{Now } x^\pm_i = -x^\mp_i$$

$$= -4M\Omega^2 (\dot{x}^+_i x^+_j + x^+_i \dot{x}^+_j)$$

$$\Rightarrow \frac{1}{5} \dddot{Q}_{ij} \dddot{Q}_{ij} = \frac{1}{5} 8M\Omega^2 (\dot{x}^+_i x^+_j + x^+_i \dot{x}^+_j)\, 8M\Omega^2 (\dot{x}^+_i x^+_j + x^+_i \dot{x}^+_j)$$

$$\Rightarrow \frac{1}{5} \dddot{Q}_{ij} \dddot{Q}_{ij} = \frac{64\Omega^4 M^2}{5} \left(2|\dot{x}^+|^2 |x^+|^2 + 2\underbrace{(\dot{x}^+ \cdot x)^2}_{\to 0} \right)$$

$$= \frac{128\Omega^6 M^2 R^4}{5}$$

Kepler's law $\quad \Omega^2 = \dfrac{M}{4R^3} \qquad \langle P \rangle = \dfrac{2}{5} \dfrac{M^5}{R^5} \;//$

For $\langle P \rangle$ large, need $\dfrac{M}{R}$. $\quad R >$ size of objects $> 2M$

$R \sim$ size $\sim 2M$

tightly bound $\quad$ BH or NS

3. (a) $g_{ab} \to g_{ab} + \delta g_{ab}$

(i) $\delta \Gamma^a_{bc}$ is difference of 2 connections $\Rightarrow$ tensor

Normal coords @ p (w.r.t unperturbed metric) ~~g_{ab}~~ $g_{\mu\nu,\rho} = 0$
$\Gamma^\mu_{\nu\rho} = 0$

$$\delta \Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} \right) \overset{\partial = \nabla \text{ in normal coords}}{=} \dots \text{[result]}$$

$$= \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma} \right)$$

Tensorial $\Rightarrow$ holds in all frames $\Rightarrow$ latin indices $\Rightarrow$ result.

(ii) In normal coords,

$$\delta R^\mu{}_{\nu\rho\sigma} = \partial_\rho \delta \Gamma^\mu_{\nu\sigma} - \partial_\sigma \delta \Gamma^\mu_{\nu\rho}$$

Contract $\mu\rho$ ~~$\delta R_{\nu\sigma}$~~ $\delta R_{\nu\sigma} \overset{\partial = \nabla}{=} \nabla_\mu \delta \Gamma^\mu{}_{\nu\sigma} - \nabla_\sigma \delta \Gamma^\mu_{\mu\nu}$

Tensorial $\Rightarrow$ holds $\forall$ frames $\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ac}$

(iii) $$\delta R = \delta(g^{ab} R_{ab}) = g^{ab} \delta R_{ab} + \underbrace{(\delta g^{ab})}_{-g^{ac} g^{bd} \delta g_{cd}} R_{ab}$$

$$= g^{ab} \left(\nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ac} \right) - R^{ab} \delta g_{ab}$$

$$= -R^{ab} \delta g_{ab} + \nabla_c \left(g^{ab} \delta \Gamma^c_{ab} \right) - \nabla_b \left(g^{ab} \delta \Gamma^c_{ac} \right)$$

$$= -R^{ab} \delta g_{ab} + \nabla_a \left[g^{ad} \nabla^b \delta g_{bd} - \frac{1}{2} g^{ad} \nabla_d \left(g^{bc} \delta g_{bc} \right) \right]$$

$$- \nabla_b \left(\frac{1}{2} g^{ab} g^{cd} \left[\nabla_a \delta g_{dc} + \nabla_c \delta g_{da} - \nabla_d \delta g_{ac} \right] \right)$$

$$= -R^{ab} \delta g_{ab} + \nabla^a \nabla^b \delta g_{ab} - \nabla^c \nabla_c \left(g^{ab} \delta g_{ab} \right) \quad //$$

(b) ~~δ~~ $S = \int d^4x \sqrt{-g}\, f(R) + S_{\text{matter}}$

(i) $$\delta S = \underbrace{\int d^4x \left[(\delta\sqrt{-g})\, f(R) + \sqrt{-g}\, f'(R)\, \delta R \right]}_{①} + \frac{\delta S_{matter}}{\delta g_{ab}}\, \delta g_{ab}$$

$$① = \int d^4x \left[\frac{1}{2}\sqrt{-g}\, g^{ab} f\, \delta g_{ab} + \sqrt{-g}\, f' \left(-R^{ab}\delta g_{ab} + \nabla^a\nabla^b \delta g_{ab} - \nabla^c\nabla^c (g^{ab}\delta g_{ab}) \right) \right]$$

use div thm (dropping surface terms) twice

$$= \int d^4x \left[\frac{1}{2} g^{ab} f - f' R^{ab} + \nabla^b\nabla^a f' - g^{ab}\nabla^c\nabla_c f' \right] \sqrt{-g}\, \delta g_{ab}$$

$$\Rightarrow E_{ab} = f' R_{ab} - \frac{1}{2} g_{ab} f - \nabla_b (f''\nabla_a R) + g_{ab}\nabla^c (f''\nabla_c R)$$

= result //

(ii) S diffeomorphism invariant. Consider diffeo $\delta g_{ab} = 2\nabla_{(a}\xi_{b)}$

$$0 = \delta S_{grav} = -2\int \sqrt{-g}\, E^{ab}\, \nabla_{(a}\xi_{b)}$$

div thm, dropping boundary

$$= -2\int \sqrt{-g}\, \cancel{\nabla_a(E^{ab}\xi_b)} - \xi_b \nabla_a E^{ab} = 0 \quad \text{for any } \xi$$

$$\therefore \nabla_a E^{ab} = 0$$

(iii) Higher order derivs.

(iv) Einstein in vacuo with Λ: $R_{ab} = \Lambda g_{ab}$

$$E_{ab} = 0 \iff f'(R = 4\Lambda)\,(R_{ab} = \Lambda g_{ab}) - \frac{1}{2} f(4\Lambda)\, g_{ab} = 0$$

$$\iff \frac{f'(4\Lambda)}{f(4\Lambda)} = \frac{1}{2\Lambda}$$

Have $\iff$ thus necessary and sufficient.

4. $ds^2 = \frac{1}{z^2}\left(-f(z)^2 dt^2 + dx^2 + dy^2 + f(z)^{-2} dz^2\right)$

$e^0 = \frac{f}{z} dt \qquad e^1 = \frac{1}{z} dx \qquad e^2 = \frac{1}{z} dy \qquad e^3 = \frac{1}{zf} dz$

Connection 1-forms: $\quad de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$

$de^0 = \left(\frac{f}{z}\right)' dz \wedge dt = -\left(\frac{f}{z}\right)' z^2 e^0 \wedge e^3$

$\Rightarrow \omega_{03} = -z^2 \left(\frac{f}{z}\right)' e^0$ //

$de^1 = -\frac{1}{z^2} dz \wedge dx = f e^1 \wedge e^3 \quad \Rightarrow \omega_{13} = -f e^1$ //

simil $\qquad \omega_{23} = -f e^2$ //

Check: $0 = de^3 = \omega_{\nu 3} \wedge e^\nu = 0$ // Rest 0 by symmetry

Curvature 2-forms: $\Theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$

$\Rightarrow \Theta^0{}_1 = d\omega^0{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 = z^2\left(\frac{f}{z}\right)' e^0 \wedge f e^1 = f z^2 \left(\frac{f}{z}\right)' e^0 \wedge e^1$

$\Rightarrow \Theta_{01} = -f z^2 \left(\frac{f}{z}\right)' e^0 \wedge e^1$

$\Theta_{01} = (-f f' z + f^2) e^0 \wedge e^1$

$= -\frac{1}{2}(f^2)' z + f^2 = \left(1 + \frac{1}{2}\alpha z^3\right) e^0 \wedge e^1$

$\Theta_{02} = \ldots = \left(1 + \frac{1}{2}\alpha z^3\right) e^0 \wedge e^2 \qquad \Theta_{12} = (-1 + \alpha z^3) e^1 \wedge e^2$

$\Theta_{03} = \ldots = \left(1 - \alpha z^3\right) e^0 \wedge e^3 \qquad \Theta_{13} = \left(-1 - \frac{1}{2}\alpha z^3\right) e^1 \wedge e^3$

$\Theta_{23} = \left(-1 - \frac{1}{2}\alpha z^3\right) e^2 \wedge e^3$

Rest 0 by symm

b) $\Theta^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} e^\rho \wedge e^\sigma \qquad$ Einstein: $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \Rightarrow R = 4\Lambda$

$R_{ab} = \Lambda g_{ab}$

$R_{00} = R_0{}^\alpha{}_{0\alpha} = R_{0101} + R_{0202} + R_{0303} \qquad \Lambda = -3$

$= +\left(1 + \frac{1}{2}\alpha z^3\right) + \left(1 + \frac{1}{2}\alpha z^3\right) + \left(1 - \alpha z^3\right) = 3$

$R_{01} = R_{0212} + R_{0313} = 0 \qquad R_{02} = 0 \qquad R_{11} = -3 \quad R_{22} = -3 \quad R_{33} = -3$

etc.