$\bar{h}_{0i} = -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$ $\quad r \gg \tau \quad \tau \gg d \quad$ + time-indept

$\bar{h}_{00} = \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$ $\quad$ (+ linear int) + time indept

$\bar{h}_{00} \sim \frac{4E}{r} \qquad E = \int d^3x\, T_{00} \qquad \partial_0 E = 0$

$\bar{h}_{0i} = -\frac{4P_i}{r} \qquad P_i = \int d^3x(-T_{0i})$ $\quad$ total momentum of matter $\partial_0 P_i = 0$

$\partial_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu T^{\mu\nu} = 0$

Centre of momentum frame $P_i = 0$ $\qquad M = E$

total mass

$$\therefore \bar{h}_{00}(t,\underline{x}) \approx \frac{4M}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

$$\bar{h}_{0i}(t,\underline{x}) \approx -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

Energy in gravitational waves

$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{O(\varepsilon)} + \underbrace{h^{(2)}_{\mu\nu}}_{O(\varepsilon^2)}$

1st order $G^{(1)}_{\mu\nu}[h] = 0$

2nd order: linear in $h^{(2)}$ or quadratic in h

calculate by setting $h = 0$ $\quad G^{(2)}_{\mu\nu}[h]$

$\Rightarrow G^{(1)}_{\mu\nu}[h^{(2)}]$

$$\therefore G_{\mu\nu}[g] = G^{(1)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h]$$

$$G^{(2)}_{\mu\nu}[h] = R^{(2)}_{\mu\nu}[h] - \frac{1}{2}R^{(1)}[h]h_{\mu\nu} - \frac{1}{2}R^{(2)}[h]\eta_{\mu\nu}$$

$$R^2[h] = \eta^{\mu\nu}R^{(2)}_{\mu\nu}[h] - h^{\mu\nu}R^{(1)}_{\mu\nu}[h]$$

Ex sheet 3:

Show $R^{(2)}_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}\partial_\rho\partial_\sigma h_{\mu\nu} + \dots$

Assume vacuum $(T_{\mu\nu} = 0)$

$O(\varepsilon)$: $G^{(1)}_{\mu\nu}[h] = 0 \quad -(*)$

$O(\varepsilon^2)$: $G^{(1)}_{\mu\nu}[h^{(2)}] = 8\pi\, t_{\mu\nu}[h]$

$$t_{\mu\nu}[h] = -\frac{1}{8\pi}G^{(2)}_{\mu\nu}[h]$$
$$= -\frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\eta_{\mu\nu}\right)$$

assuming h obeys $(*)$

$g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0$

$O(\varepsilon)$: $\partial^\mu G^{(1)}_{\mu\nu}[h] = 0 \quad -(\dagger)$ $\qquad$ <u>h arbitrary</u>

$O(\varepsilon^2)$: $\partial^\mu \left(G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h] \right) + h\, \underbrace{G^{(1)}[h]}_{=0 \text{ by } (*)} = 0$

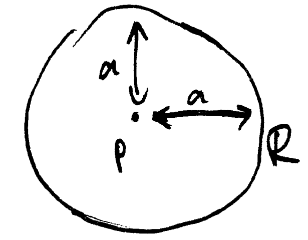
$(\dagger) \Rightarrow \partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] = 0$

$\therefore\ \partial^\mu G^{(2)}_{\mu\nu}[h] = 0$ if h obeys $(*)$ ...

$\Rightarrow \partial^\mu t_{\mu\nu}[h] = 0$

$t_{\mu\nu}$ is not gauge invariant under $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Can show $\int_{t=\text{const}} d^3x\, t_{00}$ **is** gauge invariant.



p arbitrary $\qquad p \in R \subset \mathbb{R}^4$

$$\langle X_{\mu\nu} \rangle_p = \int_R X_{\mu\nu}(x)\, W(x)\, d^4x$$

$W(x) > 0 \qquad \int_R W\, d^4x = 1 \qquad W = 0$ on ∂R.

Far from source assume grav radiation wavelength $\lambda \sim \tau$

cpts $X_{\mu\nu} \sim x \qquad \partial_\rho X_{\nu\rho} \sim x/\lambda$

$$\langle \partial_\rho X_{\nu\rho} \rangle \underset{\int \text{parts}}{=} -\int_R X_{\nu\rho}\, \underbrace{\partial_\rho W}_{\sim W/a}\, d^4x \sim \frac{x}{a} \int W\, d^4x = \frac{x}{a}$$

Choose $a \gg \lambda$, averaging reduces cpts of $\partial_\rho X_{\nu\rho}$ by $\lambda/a \ll 1$.

$\Rightarrow$ can neglect total derivatives inside $\langle \dots \rangle$

$$\langle A\, \partial B \rangle = \langle \partial(AB) \rangle - \langle (\partial A) B \rangle \approx -\langle (\partial A) B \rangle$$

Ex sheet 3. Show $\langle \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h] \rangle = 0$ in vacuum

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi} \langle \partial_\mu \bar{h}_{\rho\sigma}\, \partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_\mu \bar{h}\, \partial_\nu \bar{h} - 2\, \partial_\sigma \bar{h}^{\rho\sigma} \partial_{(\mu} \bar{h}_{\nu)\rho} \rangle$$

$\langle t_{\mu\nu} \rangle$ is gauge invariant

LIGO: $100\ \text{Hz} \Rightarrow \lambda = 3000$ km $\qquad a \gg \lambda$ (not relevant)

Ex. For plane grav wave, show

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi} \left(|H_+|^2 + |H_\times|^2 \right) k_\mu k_\nu = \frac{\omega^2}{32\pi} \left(|H_+|^2 + |H_\times|^2 \right) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Quadrupole Formula

average energy flux $-\langle t_{0i} \rangle$

time averaged power radiated across sphere of $r = \text{const}$

$$\langle P \rangle = -\int r^2 \langle t_{0i} \rangle\, \hat{x}_i\, d\Omega$$

unit normal: $\hat{x}_i$ $\qquad$ S^2 vol element: $d\Omega$

$$\langle t_{0i} \rangle = \frac{1}{32\pi} \langle \partial_0 \bar{h}_{\rho\sigma}\, \partial_i \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_0 \bar{h}\, \partial_i \bar{h} \rangle$$

$$= \langle \partial_0 \bar{h}_{jk}\, \partial_i \bar{h}_{jk} - 2\, \partial_0 \bar{h}_{0j}\, \partial_i \bar{h}_{0j} + \partial_0 \bar{h}_{00}\, \partial_i \bar{h}_{00} - \frac{1}{2} \partial_0 \bar{h}\, \partial_i \bar{h} \rangle$$

$\bar{h}_{jk}(t, \underline{x}) \propto \frac{2}{r} \ddot{I}_{jk}(t-r) \qquad \partial_0 \bar{h}_{jk} \propto \frac{2}{r} \dddot{I}_{jk}(t-r)$

$$\partial_i \bar{h}_{jk} = \left(-\frac{2}{r} \dddot{I}_{jk} - \frac{2}{r^2} \ddot{I}_{jk} \right) \hat{x}_i \approx -\frac{2}{r} \hat{x}_i \dddot{I}_{jk} \quad \text{for } r \gg \tau$$

$$\left(\partial_i r = \frac{x_i}{r} = \hat{x}_i \right) \qquad -\frac{1}{32\pi} \int r^2 d\Omega \langle \partial_0 \bar{h}_{jk} \partial_i \bar{h}_{jk} \rangle \hat{x}_i = \frac{1}{2} \langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle_{tr}$$