$$Z_{YM} = \int_{\mathcal{A}/\mathcal{G}} \underset{?}{D\mu}\; e^{-S_{YM}[\nabla]/\hbar}$$

$\uparrow$ not affine

Consider the finite dimensional case:

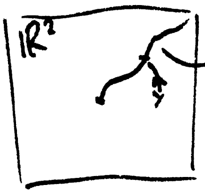
$$\int_{\mathbb{R}^2} dx\,dy\; e^{-S[x,y]} = \underset{\text{vol } SO(2)}{(2\pi)} \int_{\mathbb{R}_>} dr\, r\, e^{-S(r)}$$

Suppose $f(\underline{x}) : \mathbb{R}^2 \to \mathbb{R}$ obeys

(i) for each $\underline{x} \in \mathbb{R}^2$, $\exists$ some rotation $R \in SO(2)$ s.t. $f(R\underline{x}) = 0$

(ii) f is non-degenerate: $f(R\underline{x}) = f(\underline{x})$ iff $R = \mathrm{id}$

The curve $C = \{\underline{x} \in \mathbb{R}^2 : f(\underline{x}) = 0\}$ is then a choice of embedding $\mathbb{R}_> \hookrightarrow \mathbb{R}^2$



$C = \{f(\underline{x}) = 0\}$

Now consider the integral $\int_{\mathbb{R}^2} dx\,dy\; \delta(f(\underline{x}))\, e^{-S(x,y)}$.

The δ-fn restricts us to C (known as the gauge slice) but it depends not just on C but on the choice of f. E.g. if we replace $f \longmapsto cf$ for some $c \in \mathbb{R}$,

$$\delta(f(\underline{x})) \longmapsto \delta(cf(\underline{x})) = \frac{1}{|c|}\delta(f(\underline{x}))$$ so our integral changes.

To remove this dependence, let

$$\Delta_f(\underline{x}) := \frac{\partial}{\partial\theta} f(R_\theta \underline{x})\Big|_{\theta=0}$$ where $R_\theta \in SO(2)$ is a rotation.

then the new integral

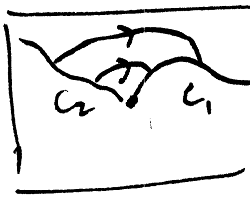
$$\int_{\mathbb{R}^2} dx\,dy\; \delta(f(\underline{x}))\,|\Delta_f(\underline{x})|\, e^{-S(x,y)}$$ has the following property:

i) It's independent of the choice of gauge-fixing function f

(e.g. if $f(\underline{x}) \mapsto c(r) f(\underline{x})$, then for $c > 0$, $\delta(cf) = \frac{1}{|c|}\delta(f)$

and $\Delta_{cf}(\underline{x}) = c(r)\Delta_f$)

ii) It's even independent of the choice of C itself. For suppose C_1, C_2 are two such curves defined by f_1, f_2. Then at each value of r, $\exists$ some rotation $R_{\theta(r)} \in SO(2)$ s.t. $f_2(\underline{x}) \propto f_1(R_{\theta(r)}\underline{x})$



Let $\underline{x}' = R_\theta(r)\,\underline{x}$ (note $|\underline{x}'| = |\underline{x}|$). Then since the action and measure obey $S(x',y') = S(x,y)$, $dx'dy' = dx\,dy$, the 'path' integrals using f_1, f_2 agree.

<u>Ex</u> Choose C to be the x-axis, with $f(\underline{x}) := y$. Then under a rotation, $y \longmapsto y\cos\theta - x\sin\theta$, so $\Delta_f(\underline{x}) = \frac{\partial}{\partial\theta}\left(y\cos\theta - x\sin\theta\right)\Big|_{\theta=0} = -x$

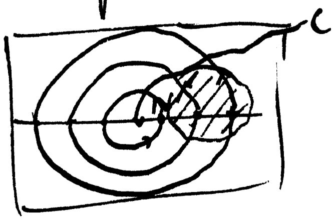
and we have

$$\int_{\mathbb{R}^2} dx\,dy\;\delta(f)\,\Delta_f(\underline{x})\,e^{-S(x,y)} = \int_{\mathbb{R}^2} dx\,dy\;\delta(y)\,|x|\,e^{-S(x,y)} = \int_{-\infty}^{\infty} dx\,|x|\,e^{-S(x,0)}.$$

Since the action was rotationally invariant, $S(x,0)$ depends only on $|x|$, so we have $2\int_0^\infty d|x|\,|x|\,e^{-S(|x|)} = 2\int_0^\infty r\,dr\;e^{-S(r)}$ ($r = |x|$ on $y=0$)

$$\Rightarrow 2\int_0^\infty dr\,r\,e^{-S(r)} = \int_{\mathbb{R}^2} dx\,dy\;\delta(f)\,|\Delta_f(\underline{x})|\,e^{-S(x,y)}$$

The factor of 2 is a "Gribov ambiguity": our choice of f wasn't quite non-degenerate. In pert theory we won't be sensitive to this global factor.



In finite dimensions, if $S(\underline{x})$ is invariant under the same transformation $\underline{x} \mapsto R_\theta\,\underline{x}$ with parameters θ^a, $a = 1,\dots,\dim G$ ($R_\theta \in G$) then we need one gauge-fixing condition $f^a(\underline{x})$ per parameter. Then

$$\Delta_f := \det\left(\frac{\partial f^a(R_\theta\,\underline{x})}{\partial\theta^b}\Big|_{\theta^a=0}\right)$$

is known as the <u>Fade'ev-Popov determinant.</u>

We take our path integral over the affine space, but include a factor of

$$|\Delta_f|\prod_{a=1}^{\dim G}\delta(f^a(\underline{x})).$$

In Y-M, this means we have

$$\mathcal{Z} = \int_{\mathcal{A}/\mathcal{G}} D\mu\;e^{-S_{YM}} \;\text{"="}\; \int_{\mathcal{A}} DA\;\delta[f]\,|\Delta_f(A)|\,e^{-S_{YM}[A]}$$

where if the gauge (structure) grp is G, then $\delta[f] \text{ "=" } \prod_{x\in M}\prod_{a=1}^{\dim G}\delta(f^a(A(x)))$ i.e. our gauge conditions $f^a(A(x)) = 0$ must hold at each $x \in M$.

To make this tractable, write

$$\delta[f] = \int Dh\;e^{i\int h_a(x)\,f^a(A(x))}$$

where $h_a(x)$ ($a = 1,\dots,\dim G$)

is a Lagrange multiplier. Similarly, recalling $\det(M) = \int d^n c \, d^n \bar{c} \, e^{-\bar{c}Mc}$ for fermionic variables $(c, \bar{c})$, we write the F.P. det as

$$\Delta_f = \int D\bar{c} \, Dc \, \exp\left(- \int_{M \times M} d^n x \, d^n y \, \bar{c}_a(x) \frac{\delta f^a(A^\lambda(x))}{\delta \lambda^b(y)}\Big|_{\lambda=0} c^b(y) \right)$$

where $\lambda^b(y)$ are our gauge parameters, and A^λ is the gauge transformed field.

The fermionic fields $c / \bar{c}$ are known as <u>ghosts</u> / <u>antighosts</u>. They are scalars on M, valued in $\mathfrak{g}$.

E.g. We often pick Lorenz gauge $f^a(A) := \partial^\mu A^a_\mu$

Under a gauge transformation $A \longmapsto A^\lambda = A + \nabla\lambda$

A in adjoint

$$(A^\lambda)^a_\mu = A^a_\mu + \partial_\mu \lambda^a + f^a{}_{bc} A^b_\mu \lambda^c$$

$$\Rightarrow \det \frac{\partial f^a(A^\lambda(x))}{\partial \lambda^b(y)} = \left[\delta^a{}_c \partial^{(x)}_\mu + f^a{}_{bc} A^b{}_\mu(x) \right] \delta^{(d)}(x-y)$$

$$\Rightarrow S_{gh} = \int d^d x \, d^d y \, \bar{c}(x) \left(\delta^a{}_c \partial^{(x)}_\mu + f^a{}_{bc} A^b{}_\mu(x) \right) \delta^{(d)}(x-y) \, c^b(y)$$

$$= \int_M d^d x \, \bar{c}_a (\nabla_\mu c)^a \qquad (\nabla_\mu c)^a = \partial_\mu c^a + f^a{}_{bc} A^b_\mu c^c$$

Thus the whole YM integral is

$$\mathcal{Z} = \int DA \, D\bar{c} \, Dc \, Dh \, \exp\left[-\frac{1}{4 g^2_{YM}} \int F^a_{\mu\nu} F^{\mu\nu a} d^d x + i \int h_a \partial^\mu A^a_\mu \, d^d x + \int \bar{c}_a (\nabla c)^a \, d^d x \right]$$