- A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace which is also a Lie algebra
- An ideal of $\mathfrak{g}$ is a subalgebra st,

$$[X, Y] \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}$$

Examples
- Every Lie algebra $\mathfrak{g}$ has two "trivial" ideals

$$\mathfrak{h} = \{0\} \quad \text{and} \quad \mathfrak{h} = \mathfrak{g}$$

- the derived algebra of $\mathfrak{g}$

$$i = [\mathfrak{g}, \mathfrak{g}] \stackrel{\text{def}^n}{=} \mathrm{span}_F \{ [X, Y] : X, Y \in \mathfrak{g} \} \qquad F = \mathbb{R} \text{ or } \mathbb{C}$$

- The centre $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$

$$\mathfrak{z}(\mathfrak{g}) = \{ X \in \mathfrak{g} : [X, Y] = 0 \quad \forall Y \in \mathfrak{g} \}$$

An abelian Lie algebra $\mathfrak{g}$ $\quad [X, Y] = 0 \quad \forall X, Y \in \mathfrak{g} \iff \mathfrak{z}(\mathfrak{g}) = \mathfrak{g}$

Def^n $\mathfrak{g}$ is simple if it is non-abelian and possesses no non-trivial ideals

$$\Rightarrow \mathfrak{z}(\mathfrak{g}) = \phi, \quad i(\mathfrak{g}) = \mathfrak{g}$$

(Then have non-degenerate invariant inner product.)

Lie algebras from Lie groups

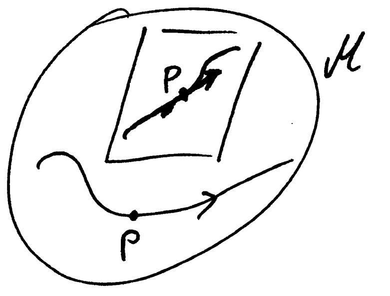
- Preliminaries

M is a smooth manifold of dimension D and $p \in M$ is a point. Introduce coordinates $\{x^i\}$ $i = 1 \dots D$ in some region $P \subset M$ with point p origin $x^i = 0$.

The tangent space $T_p(M)$ to M at p is a D-dimensional vector space spanned by $\left\{ \frac{\partial}{\partial x^j} \right\}$ $j = 1 \dots D$ act on functions $f : M \to \mathbb{R}$.

A tangent vector

$$V = v^i \frac{\partial}{\partial x^i} \in T_p(\mathcal{M})$$



acts of functions $f = f(x)$ as

$$V \cdot f = v^i \frac{\partial}{\partial x^i} f(x) \Big|_{x=0}$$

- Consider a smooth curve

$$C : \mathbb{R} \longrightarrow \mathcal{M}$$

passing through point $p \in \mathcal{M}$

Refer to coordinates — parameter of C

$$C : t \in \mathbb{R} \longmapsto \{ x^i(t) \in \mathbb{R} \quad i = 1, \dots, D \}$$

$x^i(t)$ cts and differentiable (once) with $x^i(0) = 0$

- Tangent vector of a curve C

$$V_c = \dot{x}^i(0) \frac{\partial}{\partial x^i} \in T_p(\mathcal{M}) \qquad \dot{x}^i(t) = \frac{d x^i(t)}{dt}$$

– Acting on functions $f = f(x)$, V_c corresponds to derivative of f along C

$$V_c \cdot f = \dot{x}^i(0) \frac{\partial}{\partial x^i} f(x) \Big|_{x=0} = \frac{df}{dt} \qquad f = f(x(t)) \qquad \text{(chain rule)}$$

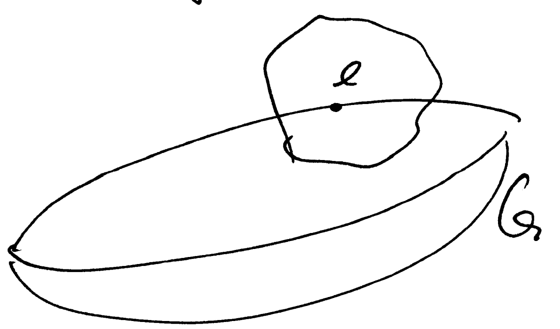
curve $C \longrightarrow V_c \in T_p(\mathcal{M})$

$V \in T_p(\mathcal{M})$ can find curve C such that $V = V_c$ (∞ many curves C)

- the Lie algebra $\mathcal{L}(G)$ of Lie group G

G is a Lie group of dimension D. Introduce coords $\{\theta^i\}$ $i = 1 \dots D$ near $e \in G$

$$g = g(\theta) \in G, \quad g(0) = e$$



Tangent space at identity $T_e(G)$ admits a Lie bracket

$[\cdot,\cdot] : T_e(G) \times T_e(G) \to T_e(G)$ such that $\mathcal{L}(G) = (T_e(G), [\cdot,\cdot])$

is a Lie algebra.

- ~~Use~~ matrix Lie groups

$G \subset Mat_n(F) \qquad n \in \mathbb{N} \quad , F = \mathbb{R}$ or $\mathbb{C}$

Can map tangent vectors to matrices

$\rho : T_e(G) \longrightarrow Mat_n(F)$

$$v^i \frac{\partial}{\partial \theta^i} \in T_e(G) \longmapsto v^i \frac{\partial}{\partial \theta^i} g(\theta) \Big|_{\theta=0} \in Mat_n(F)$$

identify $T_e(G)$ with the ~~space of matrices~~ subspace of $Mat_n(F)$ spanned by

$$\left\{ \frac{\partial g(\theta)}{\partial \theta^i} \Big|_{\theta=0} \right\} \quad i = 1 \dots D$$

Now there is an obvious candidate for bracket,

$[X,Y] = XY - YX \quad \forall X,Y \in T_e(G)$ bracket is the matrix commutator

Easy to check i) - iii) i) antisym ii) linear iii) Jacobi

Remains to show closure $[X,Y] \in \mathcal{L}(G) \quad \forall X,Y \in \mathcal{L}(G)$

Let C be a smooth curve on G passing through identity

$C : t \longmapsto g(t) \in G \quad , \; g(0) = \mathbb{1}_n$

$$\frac{dg(t)}{dt} = \frac{d\theta^i(t)}{dt} \frac{\partial g(\theta)}{\partial \theta^i} \qquad \dot{g}(0) = \frac{dg(t)}{dt}\Big|_{t=0} = \dot{\theta}^i(0) \frac{\partial g(\theta)}{\partial \theta^i}\Big|_{\theta=0} \in T_e(G)$$