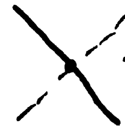
Effective QFT in $d=1$

Let's consider a theory of two real-valued fields $x, y: S^1 \to \mathbb{R}$ on a circle of circumference T.
As in $d=0$, we choose the action

$$S[x,y] = \int_{S^1} \left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}m^2x^2 + \frac{1}{2}M^2y^2 + \frac{\lambda}{4}x^2y^2 \right] dt$$

giving the momentum space Feynman rules

x ——— $1/(k^2+m^2)$ $\qquad$ y - - - - $1/(k^2+M^2)$



If we are only interested in correlators involving $x(t)$, we can integrate out the field $y(t)$ first,

$$\Rightarrow \int \mathcal{D}y \exp\left(-\frac{1}{2}\int_{S^1} y\left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda x^2(t)}{2}\right) y \, dt \right) \qquad (\text{p.c. } y\dot{y}|_0 = y\dot{y}|_T)$$

$$= \det\left[-\frac{d^2}{dt^2} + M^2 + \frac{\lambda x^2}{2} \right]^{-1/2}$$

$S_{eff}[x]$ thus contains a term

$$S_{eff}[x] = \int_{S^1} \frac{1}{2}(\dot{x}^2 + m^2x^2)\, dt + \frac{1}{2}\ln\det\left[-\frac{d^2}{dt^2} + M^2 + \frac{\lambda x^2}{2} \right]$$

$$= \quad \ldots \quad + \underbrace{\frac{1}{2}\operatorname{tr}\ln\left[-\frac{d^2}{dt^2} + M^2 \right]}_{\text{field independent}} + \frac{1}{2}\operatorname{tr}\ln\left[1 - \lambda\left(\frac{d^2}{dt^2} - M^2\right)^{-1}\frac{x^2}{2} \right]$$

We can expand the final term using the fact that the Green's function for $\left(\frac{d^2}{dt^2} - M^2\right)$ on S^1 is

$$\left(\frac{d^2}{dt^2} - M^2\right) G(t,t') = \delta(t-t') \qquad G(t,t') = \frac{1}{2M}\sum_{k\in\mathbb{Z}} \exp -M\left(|t-t'+\beta k|\right)$$

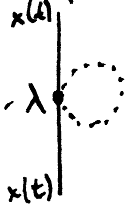
where $\beta = 1/T$

Hence we have

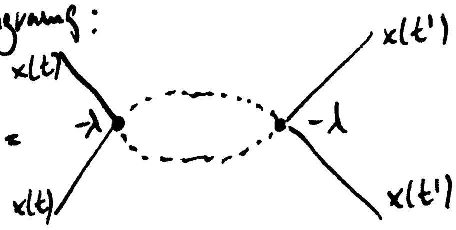
$$\operatorname{tr}\ln\left[1 - \lambda G(t,t')\frac{x^2}{2} \right] = \overset{①}{\frac{-\lambda}{2}\int_{S^1} G(t,t)\, x^2(t)\, dt} + \overset{②}{\frac{\lambda^2}{8}\int_{S^1\times S^1} dt\,dt'\, G(t,t')\, x^2(t)\, G(t,t')\, x^2(t')}$$

$+$ higher order

These terms are non-local! We can also see this non-locality from the corresponding Feynman diagrams:

① $=$ 

this is local

② $=$ 

this is non-local

Non-locality is bad news! To make progress, we note that if M is very large, then $G(t,t')$ could be suppressed for $t \neq t'$, so we can try to expand $x(t')$ around $t'-t$.

$$\int dt\,dt'\, G(t,t')^2 x^2(t)\, x^2(t') = \int dt\,dt'\, G(t,t')^2 x^2(t)\left[x^2(t) + 2x(t)\dot{x}(t)(t'-t) + \left(\dot{x}^2(t) + \frac{1}{2}x(t)\ddot{x}(t)\right)(t-t')^2 + \ldots \right]$$

Using the fact that $G(t,t')$ depends on t' only through $M(t'-t)$,

$$\frac{1}{M^2}\int dt\, \frac{\alpha}{M}x^4(t) + \frac{\beta}{M^3}\left[x^2\dot{x}^2 + \frac{1}{2}x\ddot{x} \right] + \frac{\gamma}{M^5}(\text{four derivative terms})$$

where α, β, γ are dimensionless

From dimensional analysis, every extra derivative is accompanied by a further power of $1/M$. Thus, provided $x(t)$ is slowly varying on scales of order $1/M$, we may hope to truncate this series.

$\Rightarrow$ At low energies $E \ll M$, our theory thus looks approximately local.

However, if we work with the truncated theory, we will get inconsistent / non-unitary results if we try to extrapolate our results to scales $E \sim M$.

In the theory of μ-decay, the effective action contains an interaction

$$\int d^4x \; \bar{\Psi}_e \, \mu \, \nu_e \, \rho \, G_{wk}$$

$\nwarrow$ coupling constant has mass dimension (-2)

At high energies, this 4-Fermi theory becomes non-unitary and G_{wk} is revealed as an approximation to $\frac{1}{k^2+M_W^2}$. W-boson propagator.

Quantum Gravity in d=1

In quantum gravity, we also include a (path) integral over all metrics on M, up to general co-ord invariance. We also sum over the possible topologies of M. In $d=1$ (and $d=2$ / strings) we can do this!

In $d=1$ a metric $g_{tt}(t) = e^2(t)$ is just a function of one variable. The only invariant of this metric is the total length $T = \int e(t)\, dt$, so the path integral over all possible metrics / diffeos is just an integral over $T \in (0,\infty)$

$$\int_0^\infty dT \int_{C_{[0,T]}[y,x]} \mathcal{D}x \; e^{-S[x]} \quad \text{with} \quad S[x] = \frac{1}{2}\int_0^T \dot{x}^2\, dt + \left(\frac{m^2}{2}\int_0^T dt\right)$$

— cosmological constant

$(p = \dot{x})$

$$= \int_0^\infty dT \, \langle y | e^{-HT} | x \rangle = \int_0^\infty dT \, \frac{d^n p \, d^n q}{(2\pi)^{2n}} \langle y | q \rangle \langle q | e^{-HT} | p \rangle \langle p | x \rangle$$

$$= \int_0^\infty dT \; e^{ip\cdot x - iq\cdot y} \, e^{-Tp^2/2} \, \delta^n(p-q) \, \frac{d^n p \, d^n q}{(2\pi)^n}$$

$$= \int_0^\infty \frac{dT}{(2\pi)^n} e^{ip\cdot(x-y)} \, e^{-Tp^2/2} \, d^n p = 2\int \frac{d^n p}{(2\pi)^n} e^{ip\cdot(x-y)} \frac{1}{p^2+(m^2)} = 2D(x-y)$$

for a scalar field on $\mathbb{R}^n$

$$S[\Phi] = \int d^n x \, \frac{1}{2}(\partial\Phi)^2 + \frac{m^2}{2}\Phi^2$$