AQFT

What is QFT?

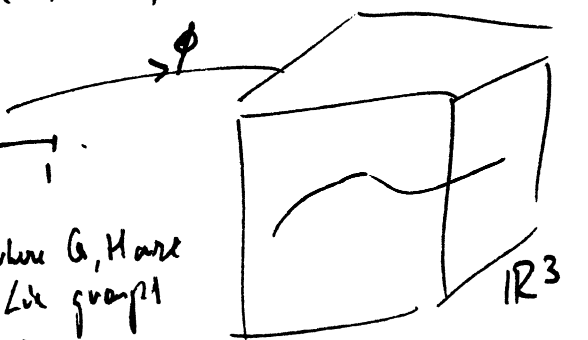
- It's the quantum version of a field theory (duh!)
  - e.g. QED as QFT of photons + electrons
  - Condensed matter applications, e.g. vibrations of a crystal (phonons)
  - Phase transitions (e.g. boiling water)
  - Topology (e.g. Jones polynomial - knot invariants, Donaldson theory - inequiv differentiable structures on $\mathbb{R}^4$)

Building a QFT

1) Pick a space. Usually, this is some (pseudo-) Riemannian mfd (M, g)
   - e.g. in particle physics $(M,g) = (\mathbb{R}^4, \eta)$ ← Minkowski
     in CM physics $(M,g) = (\mathbb{R}^3, \delta)$ ← Euclidean
   - In string theory, have fields living on a Riemann surface Σ and $[g]$ specified.
   - QFT for knots, M to be some oriented 3-mfd (e.g. S^3) with no metric

In this course, we'll usually take $(M,g) = (\mathbb{R}^d, \delta)$ ← flat, Euclidean

2) Pick some fields. Simplest choice $\phi : M \to \mathbb{R}, \mathbb{C}, \ldots$ (scalar field)
   More generally $\phi : M \to N$ (N also a Riemannian manifold, "target space")
   e.g. QM is a QFT! $M = I = [0,1]$ (so $\dim(M) = 1$)
   and fields $\phi : I \to \mathbb{R}^3$



In string theory, often have fields
$\phi : \Sigma \to N$ where N is a Calabi-Yau mfd

In pion physics, $\pi(x)$ describes a map $(\mathbb{R}^4, \eta) \to G/H$ where G, H are Lie groups

Can also have fields with non-trivial spin, or fermions, or gauge fields (i.e. a connection on a principal bundle).

3) Whatever fields we choose, let $\mathcal{C}$ be the space of all field configurations i.e. a point $\phi \in \mathcal{C}$ represents a picture of our field across all of M. This $\mathcal{C}$ is some sort of function space and will typically be ∞-dim. This ∞-dimensionality makes QFT hard + also interesting.

3) Choose an action $S : \mathcal{C} \to \mathbb{R}$. We often choose our action to be local, in the sense we assume $\exists\, \mathcal{L}(\phi, \partial\phi, \ldots)$ s.t.

$$S[\phi] = \int_M d^d x \sqrt{g}\, \mathcal{L}(\phi, \partial\phi, \ldots)$$

↑ integral over a single copy of M (We'll think more about this later.)

e.g. $S[\phi] = \int d^4x \ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$

$S[A,\Psi] = \frac{1}{4}\int d^4x \ F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}(\not{D}+m)\Psi$

Why can't we also choose e.g. $S[A,\dots] = \int F^2 + F^4 + F^2 \Box F^2 + \dots$ etc. $\cosh(FF)$

$\to$ actually we can, and must!

What do we want to compute?

In this course, the main object we'll study is the partition function

$$\boxed{\mathcal{Z} = \int_{\mathcal{C}} D\phi \ e^{-S[\phi]/\hbar}}$$

Remarks: i) The $-$ sign in the exponential is for Euclidean signature (M,g)

ii) The feature of $\exp(-S[\phi]/\hbar)$ means that as $\hbar \to 0$, the dominant contribution to $\mathcal{Z}$ comes from stationary pts of $S[\phi]$ over $\mathcal{C}$. (pf steepest descent)

iii) The effect of $e^{-S/\hbar}$ is to try to suppress "wild" contributions to $\mathcal{Z}$, i.e. where ϕ is varying rapidly or ϕ takes a very large value.

However, ~~xxxxx~~

iv) The measure $D\phi$ on $\mathcal{C}$ doesn't exist! Heuristically, as in stat phys we have a competition $\underset{\text{"entropy"}}{D\phi} \quad e^{-S[\phi]/\hbar}$ $\nwarrow$ suppress wild configs

Understanding how to make sense of $\mathcal{Z}$ + actually use it is our first main goal.

2 QFT in $d=0$

In $d=0$, if M is connected, only choice is $M = \{pt\}$. There's no possibility for a field to have "spin", so our fields are scalars and simplest choice is

$\phi : \{pt\} \to \mathbb{R}$ i.e. just a real variable

Similarly, $\mathcal{C} \cong \mathbb{R}$ and the action $S(\phi)$ is just a function of our real variable ϕ. The path integral measure $D\phi$ is just the standard measure $d\phi$ on $\mathbb{R}$.

$\mathcal{Z} = \int_{\mathbb{R}} d\phi \ e^{-S[\phi]/\hbar}$ where we assume $S(\phi)$ is chosen so this converges

More generally, we may wish to compute correlation functions, i.e. we pick some $f(\phi)$ and compute $\langle f(\phi)\rangle = \frac{1}{\mathcal{Z}}\int_{\mathbb{R}} d\phi \, f(\phi)\, e^{-S(\phi)/\hbar}$. In practice, we usually pick $f(\phi)$ to be a polynomial.

In this case, $\frac{1}{\mathcal{Z}} e^{-S(\phi)/\hbar}$ is a probability distribution on $\mathbb{R}$ and $\langle f(\phi)\rangle$ is the expectation value of f in this prob distⁿ.

Our action is also taken to have a series expansion in ϕ, so e.g. $S(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{n=3}^{N} \frac{g_n\phi^n}{n!}$ where N is even.

The partition f^n $\mathcal{Z} = \mathcal{Z}(m^2, g_n)$ and similarly $\langle f(\phi)\rangle = \langle f\rangle(m^2, g_n, \dots)$ but nothing depends on ϕ since we integrated over all $\phi \in \mathcal{C}$.