$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \qquad X^\mu = \frac{dx^\mu}{d\tau} \qquad \nabla_X X = 0$$

—

<u>Def</u> Let $(M, g)$ be a (pseudo) Riemannian mfd with a connection $\nabla$.
An affinely parametrised geodesic is an integral curve of a vector field $X$ such that $\nabla_X X = 0$.

~

freedom $t \to at + b$ ($t$ - affine parameter)

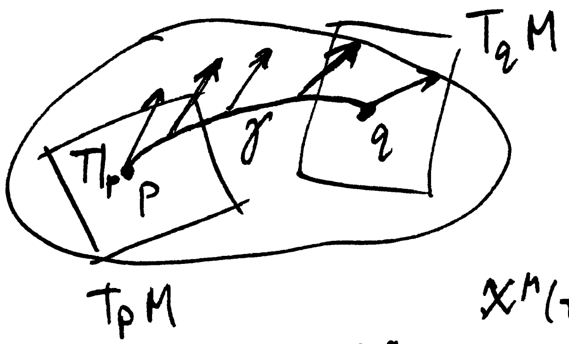
* A different parametrisation, $t = t(u)$, $\underbrace{\frac{dx^\mu}{du}}_{Y^\mu} = \underbrace{\frac{dx^\mu}{dt}}_{X^\mu}\underbrace{\frac{dt}{du}}_{\neq 0 \text{ function}}$

$\nabla_Y Y = fY$ . for some $f$ [same geodesic, different parameter]

* Let $p \in M$, $\exists$ normal coordinates in a neighbourhood of $p$ s.t. $\Gamma^\mu_{(\nu\rho)}|_p = 0$.
Components of $g$ at $p$ are $(-1, 1, 1, 1)$.
(Normal coordinates $\equiv$ inertial frame at $p$).

* Parallel transport

| how to compare tensors at $p, q$ given a geodesic?




A tensor $T$ is parallely transported along a curve $\gamma$ if $\nabla_X T = 0$, when $X = \dot{\gamma}$.

$x^\mu(\tau)$ $X^\mu = \frac{dx^\mu}{d\tau}$

e.g. $T^a{}_b$ $\cancel{0 = \nabla_X T =}$

$$0 = \nabla_X T = \underbrace{X^\rho T^\mu{}_{\nu,\rho}}_{\frac{d}{d\tau}T^\mu{}_\nu} + \Gamma^\mu_{\rho\sigma}T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma}T^\mu{}_\rho X^\sigma$$

A system of (1st order) ODEs for $T^\mu{}_\nu(x(\tau))$. $\exists$ unique solution given $T^\mu{}_\nu|_p$.
Isomorphism between tensor spaces ~~at~~ at $p, q$.

<u>Postulate</u> In GR $\left\{\begin{matrix}\text{massive}\\ \text{massless}\end{matrix}\right\}$ free particles move on $\left\{\begin{matrix}\text{timelike}\\ \text{null}\end{matrix}\right\}$ geodesics of the Levi-Civita connection of the metric.

* timelike: $\tau$ proper time $\to g(X, X) = -1$, $\tau' = \tau + b$

* null , $g(X,X)=0$ , $\tau' = a\tau + b$ (affine transformations)

## 2.4 The Riemann tensor

Def The Riemann curvature tensor of a connection $\nabla$ is a map

$$R : T_pM \times T_pM \times T_pM \longrightarrow T_pM$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad (*)$$

(1,3) tensor $R^a{}_{bcd}$ s.t. $R^a{}_{bcd} Z^b X^c Y^d = (R(X,Y)Z)^a$

Need to check the linearity.

Antisymmetric in $(X,Y)$ so need to check $X \to fX$ and $Z \to fZ$, $f: M \to \mathbb{R}$

$$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z$$

$$= f\nabla_X \nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]} Z + Y(f)\nabla_X Z$$

$$= f R(X,Y)Z$$

ex: show that $R(X,Y)fZ = fR(X,Y)Z$.

Coordinate basis $\{e_\mu = \frac{\partial}{\partial x^\mu}\}$, $[e_\mu, e_\nu] = 0$

$$R(e_\rho, e_\sigma)e_\nu \overset{(*)}{=} \underbrace{\nabla_\rho(\Gamma^\tau_{\nu\sigma} e_\tau)}_{(\partial_\rho \Gamma^\tau_{\nu\sigma}) e_\tau + \Gamma^\tau_{\nu\sigma} \underbrace{(\nabla_\rho e_\tau)}_{\Gamma^\kappa_{\rho\tau} e_\kappa}} - \nabla_\sigma(\Gamma^\tau_{\nu\rho} e_\tau)$$

$$= e_\mu \underbrace{(\partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma})}_{R^\mu{}_{\nu\rho\sigma}}$$

$R^\mu{}_{\nu\rho\sigma}$ = components of the Riemann tensor in a coord basis

So $R^a{}_{bcd} = 0$ in Minkowski (Euclidean) space (tensor! and can take $g = \mathrm{diag}(-1,1,1,1)$).

Say that $g$ is flat iff $R^a{}_{bcd} = 0$. (or $\nabla$ is flat).

Def The Ricci curvature is a (0,2) tensor $R_{ab} \equiv R^c{}_{acb}$

Exercise (alternative def of Riemann): Ricci identity

$$[\nabla_c, \nabla_d] Z^a = R^a{}_{bcd} Z^b$$