Killing form $K : \mathfrak{g} \times \mathfrak{g} \to F$

$$K(X,Y) = \mathrm{Tr}[\mathrm{ad}_X \circ \mathrm{ad}_Y] = K^{ab} X_a Y_b \qquad K^{ab} = f^{ad}{}_c f^{bc}{}_d$$

Q: Why natural?

A: Obey invariance condition $\forall X, Y, Z \in \mathfrak{g}$

$$K([Z,X],Y) + K(X,[Z,Y]) = 0 \qquad -(*)$$

aside For each $Z \in \mathfrak{g}$ define infinitesimal symmetry transformation δ_Z

$\forall X \in \mathfrak{g} \quad X \longrightarrow (X + \delta_Z X) \in \mathfrak{g} \qquad \delta_Z X = [Z,X]$

in components $X = X_a T^a \qquad Z = Z_a T^a$

$\delta_Z X = [\cancel{X}, X]$

$(\delta_Z X)_a = f^{cd}{}_a Z_c X_d$

invariance condition

$\delta_Z (K[X,Y]) = 0$

$\delta_Z (K^{ab} X_a Y_b)$

$= K^{ab} (\delta_Z X)_a Y_b + K^{ab} X_a (\delta_Z Y)_b$

$\Leftrightarrow K([Z,X],Y) + K(X,[Z,Y]) = 0$

Check $(*)$

$K([Z,X],Y) = \mathrm{Tr}(\mathrm{ad}_{[Z,X]} \circ \mathrm{ad}_Y)$

by adj repn $\mathrm{ad}_{[Z,X]} = \mathrm{ad}_Z \circ \mathrm{ad}_X - \mathrm{ad}_X \circ \mathrm{ad}_Z$

$\Rightarrow K([Z,X],Y) = \mathrm{Tr}(\mathrm{ad}_Z \circ \mathrm{ad}_X \circ \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Z \circ \mathrm{ad}_Y)$

$K(X,[Z,Y]) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Z \circ \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z)$

LHS of $(*)$ $= \mathrm{Tr}(\mathrm{ad}_Z \circ \mathrm{ad}_X \circ \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z) = 0$ □

Definition

A Lie algebra is semi-simple if it has no abelian ideals.

$\mathfrak{g}$ semi-simple $\Leftrightarrow \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_n$, $\mathfrak{g}_i$ simple

Theorem (Cartan)

K non-degenerate $\Longleftrightarrow \mathfrak{g}$ semi-simple

$\left(\begin{array}{l} \forall X \neq 0 \; X \in \mathfrak{g} \\ \exists Y : K(X,Y) \neq 0 \end{array}\right)$

Proof (forward direction)

Assume K non-degenerate.

Suppose $\mathfrak{g}$ not semi-simple $\Rightarrow$ $\mathfrak{g}$ has an abelian ideal $\mathfrak{j} \subset \mathfrak{g}$

$\dim(\mathfrak{g}) = D$, $\dim(\mathfrak{j}) = d$

choose a basis $B = \{T^a\}$, $a = 1, \dots, D$

$= \underbrace{\{T^i , i = 1, \dots, d\}}_{\text{span } \mathfrak{j}} \cup \{T^\alpha , \alpha = 1, \dots, D-d\}$

$\mathfrak{j}$ abelian , $[T^i, T^j] = 0$ $\quad i,j = 1, \dots, d$ $\quad \Rightarrow f^{ij}{}_a = 0$

$\mathfrak{j}$ ideal , $[T^\alpha, T^j] = f^{\alpha j}{}_k T^k$ $\quad \Rightarrow f^{\alpha j}{}_\beta = 0$

$X = X_a T^a \in \mathfrak{g}$

$Y = Y_i T^i \in \mathfrak{j}$

$K(X,Y) = K^{ai} X_a Y_i$ $\quad$ where $K^{ai} = f^{ad}{}_c f^{ic}{}_d$ $\quad$ by ~~that~~ conditions

$$= f^{aj}{}_\alpha f^{i\alpha}{}_j = 0$$

$\Rightarrow K(X,Y) = 0 \quad \forall X \in \mathfrak{g} \quad \Rightarrow K$ degenerate #

K non-degenerate $\Rightarrow$ $\mathfrak{g}$ semi-simple $\square$

K non-degenerate $\Rightarrow K^{ab}$ is invertible

Define inverse $K_{ab} K^{bc} = \delta_a{}^c$

Complexification

Given a real Lie algebra, real a basis

$\{T^a , a = 1, \dots, \dim \mathfrak{g}\}$

$[T^a, T^b] = f^{ab}{}_c T^c \qquad f^{ab}{}_c \in \mathbb{R}$

$\mathfrak{g} = \text{span}_{\mathbb{R}}\{T^a ; a = 1, \dots, \dim \mathfrak{g}\}$

Define

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{T^a \, ; \, a = 1, \dots, \dim \mathfrak{g}\}$$ same bracket,

$\mathfrak{g}_{\mathbb{C}}$ is complexification of $\mathfrak{g}$ | $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$

<u>Example</u>

$$\mathcal{L}(SU(2)) = \mathrm{span}_{\mathbb{R}}\{T^a = -\tfrac{i}{2}\sigma_a \, , \, a = 1, 2, 3\}$$

$$= \{2\times 2 \text{ traceless } \overset{\text{anti-}}{\text{Hermitian}} \text{ matrices}\}$$

Complexification

$$\mathcal{L}_{\mathbb{C}}(SU(2)) = \mathrm{span}_{\mathbb{C}}\{T^a = -\tfrac{i}{2}\sigma_a \, , \, a = 1, 2, 3\}$$

$$= \{2\times 2 \text{ traceless } \mathbb{C} \text{ matrices}\}$$

Complex Lie algebras can have more than one real form

$$\mathcal{L}_{\mathbb{C}}(SU(2)) \longleftarrow \mathcal{L}(SU(2))$$

$$\mathcal{L}_{\mathbb{C}}(SU(2)) \longleftarrow \mathcal{L}(SL(2;\mathbb{R}))$$

All finite-dim complex Lie algebras simple.