$D(g_1)\, D(g_2) = D(g_1 g_2) \quad \forall g_1, g_2 \in G$

$D(g)\, D(e) = D(e)\, D(g) = D(g) \; \forall g \in G \qquad D(g) \in Mat_n(F)$

$\Rightarrow D(e) = \mathbb{1}_n$ $\qquad$ D non-singular

$D(g)\, D(g^{-1}) = D(e) = \mathbb{1}_n \quad \forall g \in G$

$D(g^{-1}) = (D(g))^{-1}$

The dimension of a representation is dimension of matrices

- Matrices act linearly on a vector space $V = F^n$ <u>representation space</u>

Show reps D of $G \rightsquigarrow$ rep d of $\mathcal{L}(G)$.

Let D be a repr of dim n of matrix Lie group $G \subset Mat_n(F)$ (dim $D = n \neq$ dim G for dimension of matrices M).

For each $X \in \mathcal{L}(G)$ construct a curve

$C: t \in \mathbb{R} \longmapsto g(t) \in G \qquad$ with $g(0) = \mathbb{1}_n \;,\; \dot{g}(0) = X$

curve $D(g(t))$ in $Mat_n(F)$

$$d(X) = \frac{d}{dt} D(g(t)) \Big|_{t=0} \in Mat_n(F)$$

<u>Claim</u> d is a repr of $\mathcal{L}(G)$

<u>Proof</u> Show $[d(X), d(Y)] = d([X, Y]) \quad \forall X, Y \in \mathcal{L}(G)$

For any $X_1, X_2 \in \mathcal{L}(G)$

$C_1: t \longmapsto g_1(t) \in G \qquad g_1(0) = g_2(0) = \mathbb{1}_n$

$C_2: t \longmapsto g_2(t) \in G \qquad \dot{g}_1(0) = X_1$

$\qquad\qquad\qquad\qquad\qquad\quad \dot{g}_2(0) = X_2$

As before,

$h(t) = g_1^{-1}(t)\, g_2^{-1}(t)\, g_1(t)\, g_2(t) \in G$

Taylor expansion

$h(t) = \mathbb{1}_n + t^2 [X_1, X_2] + \mathcal{O}(t^3)$

D is a repr of G, $D(h) = D(g_1)^{-1} D(g_2)^{-1} D(g_1) D(g_2)$

LHS: $D(h) = D(\mathbb{1}_m + t^2[X_1, X_2] + \dots)$

$$= D(\mathbb{1}_m) + t^2 \frac{d^2}{dt^2} D(h(t))\Big|_{t=0} + \dots$$

$$= \mathbb{1}_n + t^2 d([X_1, X_2]) + \mathcal{O}(t^3)$$

$$D(g_1) = \mathbb{1}_n + t\,d(X_1) + \mathcal{O}(t^2)$$

RHS: $D(g_1)^{-1} D(g_2)^{-1} D(g_1) D(g_2) \approx \mathbb{1}_n + t^2 [d(X_1), d(X_2)] + \dots$

RHS = LHS $\Rightarrow$ $d([X_1, X_2]) = [d(X_1), d(X_2)]$ □

Representations of Lie Algebras

g be a Lie algebra of dim D,

- trivial repr, d_0 $\quad d_0(X) = 0$ (1x1 matrix) $\quad \forall X \in g \quad \dim(d_0) = 1$
- if $g = \mathcal{L}(G)$ for $G \subset \mathrm{Mat}_n(F)$, have fundamental repr $\quad d_f(X) = X \quad \forall X \in g \quad \dim(d_f) = n$
- All Lie algebras have an adjoint repr, d_{Adj}

$$\dim(d_{Adj}) = \dim(g) = D$$

For all $X \in g$ define a linear map,

$$ad_X : g \to g$$

$$Y \in g \longmapsto ad_X(Y) = [X, Y] \in g$$

ad_X is a linear map between vector spaces of dim D $\leadsto$ $D \times D$ matrix

basis $B = \{T^a, a = 1 \dots D\}$ for g

$$X = X_a T^a \quad , \quad Y = Y_a T^a$$

$$\Rightarrow [X, Y] = X_a Y_b [T^a, T^b] = X_a Y_b f^{ab}{}_c T^c$$

in their basis,

$$[ad_X(Y)]_c = (R_X)_c^{\ b}\, Y_b$$

$$\boxed{(R_X)_c^{\ b} = X_a f^{ab}{}_c}$$

adjoint repr, $d_{Adj}(X) = ad_X \quad \forall X \in \mathfrak{g}$

$$[d_{Adj}(X)]^b{}_c = (R_X)^b{}_c$$

check that d_{Adj} is a repr.

i) $\forall X, Y \in \mathfrak{g}$

$$[d_{Adj}(X), d_{Adj}(Y)] = d_{Adj}([X,Y]) \qquad (*)$$

Proof $d_{Adj}(X) = ad_X \quad , d_{Adj}(Y) = ad_Y$

have $\forall Z \in \mathfrak{g}$ $\quad (d_{Adj}(X) \circ d_{Adj}(Y))(Z) = [X,[Y,Z]]$

$$(d_{Adj}(Y) \circ d_{Adj}(X))(Z) = [Y,[X,Z]]$$

$[d_{Adj}(X), d_{Adj}(Y)](Z) = [X,[Y,Z]] - [Y,[X,Z]]$ LHS of $(*)$

while

$d_{Adj}([X,Y])(Z) = ad_{[X,Y]}(Z) = [[X,Y],Z]$ RHS of $(*)$

$$(LHS - RHS)(Z) = [X,[Y,Z]] - [Y,[X,Z]] - [[X,Y],Z]$$

$$= [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \quad \text{(Jacobi)} \ \square$$

ii) $\forall X, Y \in \mathfrak{g} \quad \alpha, \beta \in F$

$d_{Adj}(\alpha X + \beta Y) = \alpha\, d_{Adj}(X) + \beta\, d_{Adj}(Y)$ because ad_X, ad_Y linearly II.