One of main uses of symmetries is to constrain correlation f^ns. Suppose an operator changes under $\phi \mapsto \phi'$ as $\mathcal{O}(\phi) \mapsto \mathcal{O}(\phi')$ (i.e. the only change in the operator comes from its ϕ-dependence). Then if this transformation preserves $D\phi\, e^{-S[\phi]}$,

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{1}{\mathcal{Z}} \int D\phi\, e^{-S[\phi]}\, \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n))$$

$$(\text{trivially}) = \frac{1}{\mathcal{Z}} \int D\phi'\, e^{-S[\phi']}\, \mathcal{O}(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n))$$

$$= \frac{1}{\mathcal{Z}} \int D\phi\, e^{-S[\phi]}\, \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \quad \text{by invariance.}$$

$$\Rightarrow \langle \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}(\phi(x_n)) \rangle = \langle \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \rangle$$

so the correlation f^n of the original or transf^{d} operators agree in the same theory.

1) For example, suppose $\phi \mapsto \phi' := e^{i\alpha}\phi$, $\bar\phi \mapsto \bar\phi' := e^{-i\alpha}\bar\phi$, $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ preserves the scalar action + measure

(for example, the measure is preserved if we integrate over the same number of ϕ + $\bar\phi$ modes).

Consider the operators $\mathcal{O}_i(\phi, \bar\phi) := \bar\phi^{s_i}(x)\, \phi^{r_i}(x)$

We have $\mathcal{O}_i(\phi, \bar\phi) \mapsto \mathcal{O}_i(\phi', \bar\phi') = e^{i\alpha(r_i - s_i)}\mathcal{O}_i(\phi, \bar\phi)$ and so

$$\left\langle \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = \exp\left(i\alpha \sum_{i=1}^n (r_i - s_i)\right) \left\langle \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle.$$

Consequently, since this is true $\forall \alpha \in \mathbb{R}/2\pi\mathbb{Z}$, we have $\langle \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle = 0$ unless $\sum_i r_i = \sum_i s_i$ so we need an equal # of $\bar\phi, \phi$ insertions in total.

2) <u>Example</u>. Suppose $(M, g) = (\mathbb{R}^d, \delta)$ and consider a translation $x \mapsto x' = x - a$ for some const vector a. A scalar field ϕ transforms as $\phi(x) \mapsto \phi'(x) = \phi(x-a)$. If the action + measure are translationally invariant, then <u>scalar</u> operators [i.e. obeying $\mathcal{O}(\phi(x)) \mapsto \mathcal{O}(\phi(x-a))$] $(\mathcal{O}(x) \mapsto \mathcal{O}(x-a))$ have correlators

$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(x_1 - a) \cdots \mathcal{O}_n(x_n - a) \rangle$. Hence this correlation f^n depends only on the separations $x_{ij} = x_i - x_j$. Similarly, if the action + measure are also rotationally (Lorentz) invariant, then under $x \mapsto Lx$

$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(Lx_1) \cdots \mathcal{O}_n(Lx_n) \rangle$ (for scalar operators) so would depend only on x_{ij}^2.

Ward identities for correlators

For an infinitesimal transformation with $\phi \mapsto \phi + \varepsilon\delta\phi$ that is a symmetry when ε is a constant, for general $\varepsilon(x)$ we have

$$\int D\phi\, e^{-S[\phi]}\, \mathcal{O}_1(\phi(x_1))\dots\mathcal{O}_n(\phi(x_n)) = \int D\phi'\, e^{-S[\phi']}\, \mathcal{O}_1(\phi'(x_1))\dots\mathcal{O}_n(\phi'(x_n))$$

$$= \int D\phi\, e^{-S[\phi]}\left(1 - \int j^\mu(x)\,\partial_\mu\varepsilon(x)\, d^dx\right)\left(\mathcal{O}_1(\phi(x_1))\dots\mathcal{O}_n(\phi(x_n)) + \sum_{i=1}^n \varepsilon(x_i)\,\delta\mathcal{O}(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\right)$$

where $\delta\mathcal{O} = \frac{\partial\mathcal{O}}{\partial\phi}\delta\phi$ is the first order change.

The zeroth order terms cancel on the lhs/rhs. To lowest non-trivial order

$$0 = \int \partial^\mu\varepsilon(x)\, \langle j_\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle\, d^dx - \sum_{i=1}^n \varepsilon(x_i)\,\langle\delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle$$

$$\Rightarrow \int \varepsilon(x)\,\partial^\mu\langle j_\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle\, d^dx = -\sum_{i=1}^n\int\varepsilon(x)\,\delta^d(x-x_i)\,\langle\delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle\, d^dx$$

Since this holds for arbitrary $\varepsilon(x)$ (with compact support), we have

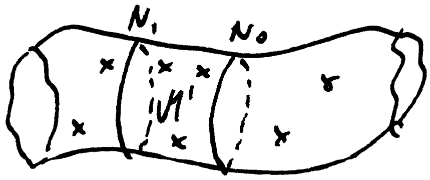
$$\partial^\mu\langle j_\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle = -\sum_{i=1}^n\delta^d(x-x_i)\,\langle\delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle$$

This is the Ward identity for correlation f^{n}s. It says the vector field $f_\mu(x,x_i) = \langle j_\mu(x)\prod_i\mathcal{O}_i(x_i)\rangle$ is divergence free except at the other insertions. To understand it, suppose M is compact without boundary and integrate the WI over all M.

$$0 = \int_M \partial^\mu(\langle j_\mu \prod\rangle)\, d^dx = -\sum_{i=1}^n\langle\delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle = \delta\langle\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle$$

i.e. the correlation f^n must be <u>invariant</u> under the transformation.

More generally, suppose $M' \subset M$ with $\partial M' = N_1 - N_0$ and let $x_i \in M'$ if $i \in I \subset \{1,\dots,n\}$.



Integrating over M' gives

$$\int_{M'}\partial^\mu(\langle j_\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle)\, d^dx = \int_{N_1} n^\mu\langle j_\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle\, d^{d-1}x - \int_{N_0} n^\mu\langle j_\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle\, d^{d-1}x$$

$$= \langle Q[N_1]\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle - \langle Q[N_0]\prod_{i=1}^n\mathcal{O}_i(x_i)\rangle = -\sum_{i\in I}\langle\delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle$$

In particular, if M' contains only one other insertion and we choose the region to be infinitesimally thin, then in the canonical picture we have

vacuum state

$$\langle\Omega|\mathcal{T}\{[\hat{Q},\hat{\mathcal{O}}_1(x_1)]\prod_{j=2}^n\hat{\mathcal{O}}_j(x_j)\}|\Omega\rangle = -\langle\Omega|\mathcal{T}\{\delta\hat{\mathcal{O}}_1(x_1)\prod_{j=2}^n\hat{\mathcal{O}}_j(x_j)\}|\Omega\rangle$$

Thus, in the canonical picture

$$\delta\hat{\mathcal{O}} = [\hat{Q},\hat{\mathcal{O}}]$$

change under some transformation — charge corresponding to it