Classical Field theory

A field is a physical quantity defined at every point of space-time $(\underline{x}, t)$.
Classical particle mechanics: finite number of generalised coords $q_a(t)$.
In field theory, interested in the dynamics of $\phi_a(\underline{x}, t)$ both labels
∞ d.o.f. (at least 1 for each $\underline{x}$)
$\underline{x}$ now relegated from a dynamical variable to a label.
eg $E_i(\underline{x}, t)$, $B_i(\underline{x}, t)$ $\quad i \in \{1, 2, 3\}$ labels direction
these 6 can be derived from 4 fields $A_\mu(\underline{x}, t)$, $\mu \in \{0, 1, 2, 3\}$
where $E_i = \partial A_i / \partial t - \partial A_0 / \partial x^i$ and $B_i = \frac{1}{2} \varepsilon_{ijk} \partial A_k / \partial x^j$
often, we write $A^\mu = (\phi, \underline{A})$
The dynamics of the field is governed by a Lagrangian which is a f^n of $\phi(\underline{x}, t)$, $\dot{\phi}(\underline{x}, t)$ and $\nabla \phi(\underline{x}, t)$.

$$L = \int d^3x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

Lagrangian density

Action $S = \int_{t_1}^{t_2} dt \, L(t) = \int d^4x \, \mathcal{L}$

NB. In particle mechanics, L depends on $\dot{q}$ and not $\ddot{q}$. In field theory, $\mathcal{L}$ should depend on $\dot{\phi}$ but could still depend on $\nabla\phi$, $\nabla^2\phi$, etc.
With an eye to Lorentz invariance, only consider $\mathcal{L}$ depending on $\nabla\phi$.

$[S] = 0$ since $[d^4x] = -4 \Rightarrow [\mathcal{L}] = 4$
Principle of least action gives us the eq of motion
Vary the path of $\int$, keeping end-points fixed and require $\delta S = 0$.

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\}$$

$$= \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\right] \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right)\right\}$$

the last term vanishes for any term that decays at spatial ∞ and obeys $\delta\phi_a(\underline{x}, t_1) = 0 = \delta\phi_a(\underline{x}, t_2)$.

Requiring $\delta S = 0$ $\Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0$

these are the Euler–Lagrange eq's for a field

e.g. 1) Klein-Gordon eq for a real scalar field $\phi(\underline{x}, t)$

$$\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \frac{1}{2} \dot\phi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2$$

$(L = T - V$: $T = \int d^3x \, \frac{1}{2}\dot\phi^2$, $V = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2\right])$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi = (\dot\phi, -\nabla\phi) \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\Rightarrow \ddot\phi - \nabla^2\phi + m^2\phi = 0$$

which we write

$$\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$ Klein-Gordon equation

An obvious generalization of $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$

$$\Rightarrow \partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0$$

2) Maxwell's equations in vacuo

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu + \frac{1}{2} (\partial_\mu A^\mu)^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial_\rho A^\rho \eta^{\mu\nu}$$

$$\Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu\partial^\mu A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$\equiv -\partial_\mu F^{\mu\nu}$$

exercise: check EL eqs reproduce $\partial_i E_i = 0$, $\dot{E}_i = \varepsilon_{ijk}\partial_j B_k$

We can rewrite $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

In these eqns, the $\mathcal{L}$ is local, i.e. the terms do not couple $\phi(\underline{x},t)$ to $\phi(\underline{y},t)$ with $\underline{x}\neq\underline{y}$ (e.g. $\int d^3\underline{x}\, d^3\underline{y}\, \phi(\underline{x})\,\phi(\underline{x}-\underline{y})$).

A priori, there is no reason for this, $\underline{x}$ is merely a label and we do couple other indices together e.g. $\partial_3 A_0\, \partial_0 A_3$.
The closest we get to non-locality for $\underline{x}$ is $\phi(\underline{x})$ coupled to $\phi(\underline{x}+\delta\underline{x})$ in $(\nabla\phi)^2$. Nature seems to be local, we shall consider local $\mathcal{L}$'s.

Lorentz Invariance

We wish to construct relativistic field theories s.t. $\underline{x}$ and t are on equal footing; the Lagrangian should be invariant under a Lorentz transformation.

$$x^\mu \Rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{with} \quad \Lambda^\mu{}_\sigma\, \eta^{\sigma\tau}\, \Lambda^\nu{}_\tau = \eta^{\mu\nu}$$

e.g. $\Lambda^\mu{}_\sigma = \begin{pmatrix} 1 &&& \\ & 1 && \\ && \cos\theta & -\sin\theta \\ && \sin\theta & \cos\theta \end{pmatrix}$ describes a rotation by angle θ around z-axis,

while $\Lambda^\mu{}_\sigma = \begin{pmatrix} \gamma & -\gamma v && \\ -\gamma v & \gamma && \\ && 1 & \\ &&& 1 \end{pmatrix}$, $\gamma = \frac{1}{\sqrt{1-v^2}}$ is a boost by v along x-axis.

The Lorentz transformations form a Lie group under matrix multiplication, have a representation on the fields.
For a scalar field, this $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$
This is an active transformation:

$\cdot\, x_0 \longrightarrow \cdot\, x_0' = \Lambda x_0$ is the position of the new max

place where field is max

We could have used passive transforms where we simply label points. then: $\phi(x) \longrightarrow \phi(\Lambda x)$. Does not matter since dealing with Lorentz invariant theories.