The simplest QFTs are free: $S(\phi)$ is (at most) quadratic

e.g. Let $\phi^a : \{pt\} \to \mathbb{R}^n$ $(a = 1, \dots, n)$

and $S(\phi) = \frac{1}{2} M(\phi, \phi) = \frac{1}{2} M_{ab} \phi^a \phi^b$ where $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive-definite and symmetric matrix. Then the partition fn $Z(M)$ is just a Gaussian:

$$\mathcal{Z}(M) = \int_{\mathbb{R}^n} d^n\phi \; e^{-\frac{1}{2\hbar} M(\phi,\phi)} = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}$$

Pf: Since M is symmetric $\exists$ orthogonal transformation $O : \mathbb{R}^n \to \mathbb{R}^n$ that diagonalizes it. The measure $d^n\phi$ is rotationally invariant, so in terms of the eigenvectors of M, this just reduces to product of n 1d Gaussian integrals

$$\int_{\mathbb{R}} dx \; e^{-\frac{mx^2}{2\hbar}} = \sqrt{\frac{2\pi\hbar}{m}}$$ where $m > 0$ is an eigenvalue of M. $\square$

A small generalization is useful: let $S(\phi) = \frac{1}{2} M(\phi, \phi) + J_a \phi^a$ for some constants J_a (J is a source in the classical case).

$$\mathcal{Z}(M, J) = \int_{\mathbb{R}^n} d^n\phi \; \exp\left[-\frac{1}{\hbar}\left(\frac{1}{2} M(\phi,\phi) + J \cdot \phi\right)\right]$$

Let $\tilde{\phi} = \phi + M^{-1}(J, \cdot)$ (i.e. $\tilde{\phi}^a = \phi^a + (M^{-1})^{ab} J_b$) and complete the square

$$\mathcal{Z}(M, J) = \int_{\mathbb{R}^n} d^n\tilde{\phi} \; \exp\left(-\frac{1}{2\hbar} M(\tilde{\phi}, \tilde{\phi}) + \frac{1}{2\hbar} M^{-1}(J, J)\right)$$

$$= \exp\left(\frac{1}{2\hbar} M^{-1}(J, J)\right) \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}$$

This is useful because it allows us to compute correlation fns. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and consider $\langle P(\phi) \rangle = \frac{1}{\mathcal{Z}(M)} \int_{\mathbb{R}^n} d^n\phi \; P(\phi) \exp\left(-\frac{1}{2\hbar} M(\phi,\phi)\right)$.

By linearity, it suffices to consider the case $P(\phi) = \prod_{i=1}^{m} l_i \cdot \phi$, $(l_i \cdot \phi = l_{ia} \phi^a)$

If m is odd, $\langle P(\phi) \rangle = 0$ since integral of an odd fn.

When $m = 2k$, we have

$$\langle (l_1 \cdot \phi) \dots (l_{2k} \cdot \phi) \rangle = \frac{1}{\mathcal{Z}(M)} \int d^n\phi \; (l_1 \cdot \phi) \dots (l_{2k} \cdot \phi) \exp\left(-\frac{1}{2\hbar} M(\phi,\phi) - \frac{J \cdot \phi}{\hbar}\right)\Bigg|_{J=0}$$

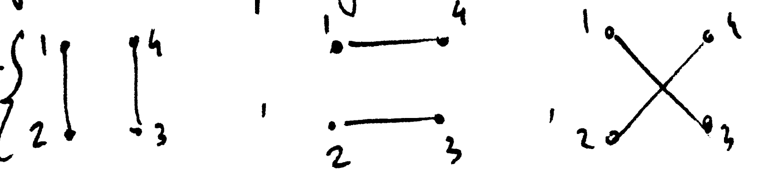
$$= \frac{(-\hbar)^{2k}}{\mathcal{Z}(M)} \int d^n\phi \prod_{i=1}^{2k} \left(l_i \cdot \frac{\partial}{\partial J}\right) \exp\left(-\frac{1}{2\hbar} M(\phi,\phi) - \frac{J \cdot \phi}{\hbar}\right)\Bigg|_{J=0}$$

$$= \frac{\hbar^{2k}}{\mathcal{Z}(M)} \prod_{i=1}^{2k} \left(l_i \cdot \frac{\partial}{\partial J}\right) \left[\int d^n\phi \; e^{-\frac{1}{2\hbar} M(\phi,\phi) - \frac{1}{\hbar} J \cdot \phi}\right]\Bigg|_{J=0}$$ (since $\int$ is absolutely convergent)

$$= \hbar^{2k} \prod_{i=1}^{2k} \left(l_i \cdot \frac{\partial}{\partial J} \right) \exp\left(\frac{M^{-1}(J,J)}{2\hbar} \right) \Bigg|_{J=0}$$

Each $l \cdot \frac{\partial}{\partial J}$ that acts on the exponential creates a factor $\frac{1}{\hbar} M^{-1}(J, l)$ in front. At the end, we'll set $J=0$, so get non-vanishing answer iff exactly k derivatives act on $\exp(\dots)$ and k act on factor in front.

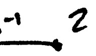
Let σ denote a (complete) pairing of the set $\{1, 2, \dots, 2k\}$ and Π_{2k} be the set of all such pairings.

e.g. $\Bigg\{$  $\Bigg\} = \Pi_4$

In general $|\Pi_{2k}| = \dfrac{(2k)!}{2^k\, k!}$. We have

$$\langle (l_1 \cdot \phi) \dots (l_{2k} \cdot \phi) \rangle = \hbar^k \sum_{\sigma \in \Pi_{2k}} \prod_{i \in \{1,\dots,2k\}/\sigma} M^{-1}(l_i, l_{\sigma(i)}) \qquad (*)$$

i.e. sum over all inequivalent ways of joining the l_i's into pairs via M^{-1}.

e.g. $\langle (l_1 \cdot \phi)(l_2 \cdot \phi) \rangle = \hbar\, M^{-1}(l_1, l_2)$ 

$$\langle (l_1 \cdot \phi) \dots (l_4 \cdot \phi) \rangle = \hbar^2 \Big[M^{-1}(l_1, l_2) M^{-1}(l_3, l_4) + M^{-1}(l_1, l_3) M^{-1}(l_2, l_4) + M^{-1}(l_1, l_4) M^{-1}(l_2, l_3) \Big]$$

(*) This is <u>Wick's theorem</u> for this 0d QFT. M^{-1} plays the role of the "propagator".

<u>Interacting theories</u>

Physically interesting theories contain interactions, i.e. $S(\phi)$ is non-quadratic. Typically, $\int d^n\phi\, P(\phi) \exp(-S(\phi)/\hbar)$ involve transcendental functions, and we usually cannot perform these integrals analytically.

In perturbation theory, we use the fact that

$$\mathcal{Z}(\hbar, \dots) = \int_{\mathbb{R}^n} d^n\phi \exp(-S(\phi)/\hbar)$$ has an <u>asymptotic ~~series~~ expansion</u> as $\hbar \to 0^+$.

(It can't have a Taylor series expansion around $\hbar = 0$, because clearly $\mathcal{Z}(\hbar, \dots)$ is non-analytic at $\hbar = 0$; it diverges whenever $\mathrm{Re}(\hbar) < 0$.)

Suppose $S(\phi)$ is smooth with a global minimum at $\phi = \phi_0 \in \mathbb{R}^n$ where $\left.\dfrac{\partial^2 S}{\partial\phi^a \partial\phi^b}\right|_{\phi_0}$ is positive definite. Then (by Laplace's method / Watson's lemma)

$$\mathcal{Z}(\lambda, \dots) \sim (2\pi\hbar)^{n/2} \frac{\exp(-S(\phi_0)/\hbar)}{\sqrt{\det \partial_a \partial_b S(\phi_0)}} \left(1 + A\hbar + B\hbar^2 + \dots \right)$$

Remark:
The leading term involves the action evaluated on the classical sol^n ϕ_0 and is called the semi-classical term. The remaining Taylor series are called quantum corrections.

Remark:
Let $\mathcal{Z}_N(\hbar)$ be the first N terms on rhs. Then to say $\mathcal{Z}_N(\hbar)$ is an asymptotic series for $\mathcal{Z}(\hbar)$ implies that for any fixed $N \in \mathbb{N}$

$\lim_{\hbar \to 0^+} \frac{|\mathcal{Z}(\hbar) - \mathcal{Z}_N(\hbar)|}{\hbar^N} = 0$. Thus, as $\hbar \to 0^+$ we get an arbitrarily good approximation to $\mathcal{Z}(\hbar)$ from any finite N. But the series will diverge if we try to fix $\hbar \in \mathbb{R}_{>0}$ and include more + more terms.

e.g. Let's consider a single scalar field ϕ with action $S(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$ where $m^2, \lambda > 0$. The action has unique global minimum at $\phi_0 = 0$, where $S(\phi_0) = 0$, $\partial^2 S|_{\phi_0 = 0} = m^2$, so leading term in asym expansion of $\mathcal{Z}(\hbar, m; \lambda)$ is $\frac{(2\pi\hbar)^{1/2}}{m}$.

Further, $\mathcal{Z}(\hbar, m, \lambda) = \int_{\mathbb{R}} d\phi \; e^{-\left(\frac{m^2\phi^2}{2\hbar} + \frac{\lambda\phi^4}{4!\hbar}\right)} = \frac{\sqrt{2\hbar}}{m} \int d\tilde{\phi} \; e^{-\tilde{\phi}^2} e^{-\left(\frac{4\lambda\hbar}{4! m^4}\tilde{\phi}^4\right)}$

$$\sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \left(\frac{-\lambda\hbar 4}{m^4 4!}\right)^n \frac{1}{n!} \int_{\mathbb{R}} d\tilde{\phi} \; e^{-\tilde{\phi}^2} \tilde{\phi}^{4n}$$

$$= \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \left(\frac{-\lambda\hbar 4}{m^4 4!}\right)^n \frac{1}{n!} \Gamma(2n + 1/2)$$