Vacuum to Vacuum Amplitudes

Consider the theory with a source $J(x)$ added:

$$\mathcal{H} = H_0 + H_{int} - J(x)\,\phi(x)$$

EoM:

$$\partial_\mu \partial^\mu \phi + \dots = J(x) \quad \Rightarrow \quad \phi = 0 \text{ is no longer a sol}^{n}$$

Consider the interaction picture with $(H_0 + H_{int})$ as "free", $-J\phi$ as the interaction.

Start with vacuum (despite the source) $|\Omega\rangle$

$$\langle \Omega | U_I(-\infty, \infty) | \Omega \rangle = \langle \Omega | T \exp[i \int d^4x\, J(x)\, \phi_H(x) | \Omega \rangle$$

$$= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n\, J(x_1) \cdots J(x_n)\, G^{(n)}(x_1, \dots, x_n)$$

Correlation f^{n}s contain the vacuum to vacuum information.

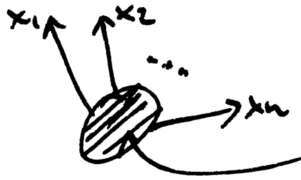
$W[J] = \langle \Omega | U_I(-\infty, \infty) | \Omega \rangle$ where $W[J]$ is a functional, known as a generating f^{n} for $G^{(n)}(x_1, \dots, x_n)$ since

$$G^{(n)}(x_1, \dots, x_n) = (-i)^n \left. \frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0} .$$

Can compute $W[J]$ by the rules earlier, amended to include the vertex $iJ(x)$.

Often, first define a related generating f^{n}

$Z[J] = \langle 0 | S | 0 \rangle_J$ which sums over all Feynman diagrams, and order n, is given by



all diagrams

$$Z[J] = e^{W[J]}$$

sums only connected diagrams

The Dirac Equation

Scalar fields: LT $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x)$

But most particles have spin. This arises naturally in field theory by considering fields with a non-trivial L.T. A familiar e.g. vector field $A_\mu(x)$

LT: $A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1} x)$

In general,

$$\phi^a(x) \longrightarrow D^a{}_b(\Lambda)\, \phi^b(\Lambda^{-1} x)$$

where $D^a{}_b(\Lambda)$ is a repr of the Lorentz group, i.e.

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2) \quad , \quad D(\Lambda^{-1}) = D(\Lambda)^{-1} \quad , \quad D(\mathbb{1}) = \mathbb{1}$$

Find by considering infinitesimal L.T.s.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon\,\omega^\mu{}_\nu + \mathcal{O}(\varepsilon^2)$$

Then $\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho\, \eta^{\sigma\rho} = \eta^{\mu\nu}$

$$\Rightarrow (\delta^\mu{}_\sigma + \varepsilon\,\omega^\mu{}_\sigma)(\delta^\nu{}_\rho + \varepsilon\,\omega^\nu{}_\rho)\,\eta^{\sigma\rho} = \eta^{\mu\nu} + \mathcal{O}(\varepsilon^2)$$

$$\Rightarrow \omega^{\mu\nu} + \omega^{\nu\mu} = 0$$ i.e. ω is anti-symmetric

(6 $\mathbb{R}$ dof : $\frac{4\times 3}{2}$) : 3 rotations, 3 boosts

Introduce a basis of these six 4x4 antisymmetric matrices:

$$(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$$

e.g. $(M^{01})^\mu{}_\nu = \eta \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ boost in x^1 dir

$(M^{12})^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}$ rotation in x^1x^2 plane

Can write all $\omega^\mu{}_\nu$ as linear combinations of these:

$$\omega^\mu{}_\nu = \frac{1}{2}(\Omega_{\rho\sigma} M^{\rho\sigma})^\mu{}_\nu$$ ($M^{\rho\sigma}$ generators of the Lie algebra)

They obey the Lorentz Lie algebra relations:

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}$$

A finite LT can be expressed as the exponential

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right)$$

The Spinor Representation

We search for other matrices which satisfy the Lie algebra relations – helping to construct spinor repn.

Define the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} \qquad \mu,\nu = 0,1,2,3$$

4 matrices s.t. $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ $(\mu \neq \nu)$ and $(\gamma^0) = \mathbb{1}$. $(\gamma^i)^2 = -\mathbb{1}$

The simplest repn is in terms of 4×4 matrices

e.g. $\gamma^0 = \begin{pmatrix} 0_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0_2 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0_2 & \sigma^i \\ -\sigma^i & 0_2 \end{pmatrix}$ (*)

where σ^i are the Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbb{1}_2$

Can construct many other repns of the Clifford algebra

e.g. $U\gamma^\mu U^{-1}$ for any invertible constant matrix U

<u>Claim</u> Up to the equiv ~~th~~ $\gamma^\mu \to U\gamma^\mu U^{-1}$,

$\exists$ a unique irrep of the Clifford algebra.

(*) provide an example, known as the <u>chiral</u> rep.

Proof omitted.

To relate this to the Lorentz group, consider

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & : \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & : \rho \neq \sigma \end{cases} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \eta^{\rho\sigma} \quad (**)$$

by Clifford algebra

<u>Claim</u> $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$

<u>Proof</u>: $\text{LHS} = \frac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] = \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu$

$$= \frac{1}{2}\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu$$

$$- \frac{1}{2}\{\gamma^\rho, \gamma^\mu\}\gamma^\nu + \frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu$$

$$= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \quad \text{by Clifford algebra}$$