If the energy is bounded below, there must be a <u>ground state</u> $|0\rangle$ satisfying $a|0\rangle = 0$. Excited states arise from repeated application of $a^\dagger$.

$|n\rangle = (a^\dagger)^n |0\rangle$ with $H|n\rangle = (n+\frac{1}{2})\omega|n\rangle$ (ignored normalization $\langle n|n\rangle \neq 1$).

This algebraic approach tells us the spectrum but not the explicit form of the wave function. In the Schrödinger rep, $\hat{p} = -i\frac{\partial}{\partial q}$.

$$a|0\rangle = 0 \Rightarrow \left(\frac{1}{\sqrt{2\omega}}\frac{\partial}{\partial q} + \sqrt{\frac{\omega}{2}}\,q\right)\Psi_0(q) = 0 \Rightarrow \left(\frac{\partial}{\partial q} + \omega q\right)\Psi_0(q) = 0$$

$$\Psi_0 \propto e^{-\omega q^2/2}$$

<u>Free Field Theory</u>

Let's apply SHO to free fields. Write $\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} + a_{\underline{p}}^\dagger\, e^{-i\underline{p}\cdot\underline{x}}\right)$ ops

$$\pi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left(a_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} - a_{\underline{p}}^\dagger\, e^{-i\underline{p}\cdot\underline{x}}\right)$$

[Can rewrite $\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\underline{p}\cdot\underline{x}}}{\sqrt{2\omega_p}}\left(a_{\underline{p}} + a_{-\underline{p}}^\dagger\right)$ relabelled $\underline{p} \to -\underline{p}$

now looks like $q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$]

<u>Claim</u> We can use comm relations to show

$$\left.\begin{aligned}
&[\phi(\underline{x}), \phi(\underline{y})] = 0 \\
&[\pi(\underline{x}), \pi(\underline{y})] = 0 \\
&[\phi(\underline{x}), \pi(\underline{y})] = i\delta^3(\underline{x}-\underline{y})
\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned}
&[a_{\underline{p}}, a_{\underline{q}}] = [a_{\underline{p}}^\dagger, a_{\underline{q}}^\dagger] = 0 \\
&[a_{\underline{p}}, a_{\underline{q}}^\dagger] = (2\pi)^3\delta^3(\underline{p}-\underline{q})
\end{aligned}\right.$$

check one way: assume this side, prove LHS

$$[\phi(\underline{x}), \pi(\underline{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_q}{\omega_p}}\left(-[a_{\underline{p}}, a_{\underline{q}}^\dagger]\, e^{i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}} + [a_{\underline{p}}^\dagger, a_{\underline{q}}]\, e^{-i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{-i}{2}\right)\left[-e^{-ip(\underline{x}-\underline{y})} - e^{ip(\underline{x}-\underline{y})}\right] = i\delta^3(\underline{x}-\underline{y})$$

Hamiltonian in terms of a_p and $a_p^\dagger$.

$$H = \frac{1}{2}\int d^3x\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

$$= \frac{1}{2}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\left[-\frac{\sqrt{\omega_p\omega_q}}{2}\left(a_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} - a_{\underline{p}}^\dagger\, e^{-i\underline{p}\cdot\underline{x}}\right)\left(a_{\underline{q}}\, e^{i\underline{q}\cdot\underline{x}} - a_{\underline{q}}^\dagger\, e^{-i\underline{q}\cdot\underline{x}}\right) + \right.$$

$$+\frac{1}{2\sqrt{\omega_p\omega_q}}\left(i\underline{p}\, a_p e^{i\underline{p}\cdot\underline{x}} - i\underline{p}\, a_p^\dagger e^{-i\underline{p}\cdot\underline{x}}\right)\left(i\underline{q}\, a_q e^{i\underline{q}\cdot\underline{x}} - i\underline{q}\, a_q^\dagger e^{-i\underline{q}\cdot\underline{x}}\right)$$

$$+\frac{m^2}{2\sqrt{\omega_p\omega_q}}\left(a_p e^{i\underline{p}\cdot\underline{x}}+a_p^\dagger e^{-i\underline{p}\cdot\underline{x}}\right)\left(a_q e^{i\underline{q}\cdot\underline{x}}+a_q^\dagger e^{-i\underline{q}\cdot\underline{x}}\right)\Big]$$

$$=\frac14\int\frac{d^3p}{(2\pi)^3}\left[\left(-\omega_p+\frac{\underline{p}^2}{\omega_p}+\frac{m^2}{\omega_p}\right)\left(a_p a_{-p}+a_p^\dagger a_{-p}^\dagger\right)+\left(\omega_p+\frac{\underline{p}^2}{\omega_p}+\frac{m^2}{\omega_p}\right)\left(a_p a_p^\dagger + a_p^\dagger a_p\right)\right]$$

$$=\frac14\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}\left(\omega_p^2+\underline{p}^2+m^2\right)\left(a_p a_p^\dagger+a_p^\dagger a_p\right) \qquad \text{as } \omega_p^2=\underline{p}^2+m^2$$

$$\Rightarrow H=\int\frac{d^3p}{(2\pi)^3}\,\omega_p\left(a_p^\dagger a_p+\frac12\underbrace{[a_p,a_p^\dagger]}_{(2\pi)^3\delta^3(0)}\right)$$

which is simply the Hamiltonian for an ∞ # uncoupled SHOs each with frequency $\omega_p=\sqrt{\underline{p}^2+m^2}$.

The vacuum

Following the SHO, we define the vacuum $|0\rangle$ s.t. $a_p|0\rangle=0$. So the energy of the ground state comes from the 2nd term in H. Since $[a_p,a_q^\dagger]=(2\pi)^3\delta^3(\underline{p}-\underline{q})$.

We have $H|0\rangle=\frac12\int\frac{d^3p}{(2\pi)^3}\,\omega_p\,(2\pi)^3\delta^3(\underline{0})|0\rangle=\infty|0\rangle$ QFTs are full of ∞s!

But they tell us something important. Often that we are asking a stupid question.

Let's explore this ∞. In fact $\exists$ 2 ∞s in this. The first is due to space being big ("infrared divergence"). Put universe in a box  and impose periodic boundary conditions on the field. $(2\pi)^3\delta^3(\underline{0})=\lim_{L\to\infty}\int_{-L/2}^{L/2}e^{i\underline{p}\cdot\underline{x}}\Big|_{p=0}=V$

the volume of ∞ universe. So $\delta^3(\underline{0})$ divergence arises because we are computing the total energy, not the energy density $\mathcal{E}_0$.

$$\mathcal{E}_0=\frac{E}{V}=\int\frac{d^3p}{(2\pi)^3}\frac12\omega_p \quad \text{which is still } \infty\ (\omega_p^2=\underline{p}^2+m^2).$$

It's the sum of ground state energies for each SHO but $\mathcal{E}_0\to\infty$ from $|\underline{p}|\to\infty$.

this is a high frequency divergence ("ultraviolet divergence") at short distances.

this arises because we assumed that our theory is valid to arbitrarily short distances / high momenta. So the $\int$ could be cut off at high momentum in some way (e.g. lattice).

In non-gravitational physics, we only care about energy differences. So we can simply redefine H by subtracting the ∞ $\Rightarrow H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p$ s.t. $H|0\rangle = 0$.

The difference between this H and the previous one is an ordering ambiguity in going from the classical to the quantum theory.

e.g. if we defined $H = \frac{1}{2}(\omega q - ip)(\omega q + ip)$ (classically $H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}$).

Classically the same as our original choice. Upon quantization $H = \omega a^\dagger a$

We write a <u>normal ordered</u> string of ops

$$\phi_1(\underline{x}_1) \cdots \phi_n(\underline{x}_n) \quad \text{as} \quad :\phi_1(\underline{x}_1) \cdots \phi_n(\underline{x}_n):$$

as the usual product but with all annihilation ops on the RHS, so we could write

$$:H: = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p$$