Quantum Field Theory
Example sheet 1

1. $$L = \int_0^a dx \left[\frac{\sigma}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \right]$$

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{a}\right)$$

$$\frac{\partial y}{\partial t} = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \dot{q}_n \sin\left(\frac{n\pi x}{a}\right), \quad \frac{\partial y}{\partial x} = \frac{n\pi}{a}\sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} q_n \cos\left(\frac{n\pi x}{a}\right)$$

$$\Rightarrow L = \int_0^a dx \, \frac{2}{a}\left[\frac{\sigma}{2} \sum_{n=1}^{\infty} \dot{q}_n^2 \sin^2\left(\frac{n\pi x}{a}\right) - \left(\frac{n\pi}{a}\right)^2 \frac{T}{2} \sum_{n=1}^{\infty} q_n^2 \cos^2\left(\frac{n\pi x}{a}\right)\right]$$

$$\int_0^a dx \sin^2 \frac{n\pi x}{a} = \int_0^a dx \cos^2 \frac{n\pi x}{a} = \frac{a}{2}$$

$$\Rightarrow L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2}\left(\frac{n\pi}{a}\right)^2 q_n^2 \right]$$

$$\partial_t \frac{\partial L}{\partial \dot{q}_n} = \sigma \ddot{q}_n = \frac{\partial L}{\partial q_n} = -T\left(\frac{n\pi}{a}\right)^2 q_n$$

$$\Rightarrow \ddot{q}_n = -\frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n \qquad \text{EoM}$$

Hence for each q_n motion is independent.

$$\omega_n = \sqrt{\frac{T}{\sigma}}\left(\frac{n\pi}{a}\right).$$

2. $$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \qquad \left(\Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \right)$$

$$\phi \to \phi'(x) = \phi(\Lambda^{-1} x) = \phi(y)$$

$$\partial_\mu \phi \to \frac{\partial}{\partial x^\mu} \phi'(x) = \frac{\partial}{\partial x^\mu} \phi(\Lambda^{-1} x) = \frac{\partial}{\partial x^\mu}\phi(y) \qquad \text{where } y^\mu = \Lambda^{-1\mu}{}_\nu x^\nu$$

$$= \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \phi(y) = \Lambda^{-1\nu}{}_\mu \, \partial_\nu \phi(y)$$

~~Therefore,~~

$$\cancel{\partial_\mu \partial^\mu \phi + m^2 \phi \longrightarrow \partial^\mu \Lambda^{-1\nu}{}_\mu \partial_\nu \phi(y) + m^2 \phi(y)}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \longrightarrow \frac{1}{2}(\Lambda^{-1})^\nu{}_\mu \partial_\nu \phi(y)\, (\Lambda^{-1})^{\rho\mu}{}_{\rho}\, \partial^\rho \phi(y) - m^2 \phi(y)$$

$$= \frac{1}{2}(\Lambda^{-1})^\nu{}_\mu \partial_\nu \phi(y)\, (\Lambda^{-1})^\rho{}_\sigma \partial_\rho \phi(y)\, \eta^{\sigma\mu} - m^2 \phi^2(y)$$

Lorentz transform $(\Lambda^{-1})^\nu{}_\mu\, \eta^{\sigma\mu} (\Lambda^{-1})^\rho{}_\sigma = \eta^{\nu\rho}$

$$= \frac{1}{2}\partial_\nu \phi(y)\, \partial^\nu \phi(y) - m^2 \phi^2(y) = \mathcal{L}(y) \quad \square$$

3. $\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2$

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right] = \partial_\mu \partial^\mu \psi^* = \frac{\partial \mathcal{L}}{\partial \psi} = -m^2 \psi^* - \lambda \psi^* (\psi^* \psi)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\right) = \partial_\mu \partial^\mu \psi = \frac{\partial \mathcal{L}}{\partial \psi^*} = -m^2 \psi - \lambda \psi (\psi^* \psi)$$

$$\psi \to \psi + \delta\psi = \psi + i\alpha\psi, \quad \psi^* \to \psi^* + \delta\psi^* = \psi^* - i\alpha\psi^*$$

$$\partial_\mu \psi \to (1 + i\alpha)\partial_\mu \psi, \quad \partial_\mu \psi^* \to (1 - i\alpha)\partial_\mu \psi^*$$

$$\mathcal{L} \longrightarrow (1 - i\alpha)\partial_\mu \psi^* (1 + i\alpha)\partial^\mu \psi - m^2 (1 - i\alpha)(1 + i\alpha)\psi^* \psi$$

$$- \frac{\lambda}{2}\left[(1 - i\alpha)(1 + i\alpha)\right]^2 (\psi^* \psi)^2 = \mathcal{L} + \mathcal{O}(\alpha^2)$$

$\therefore \mathcal{L}$ inv't under this transformation

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} i\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} i\psi^*$$

$$= i\psi\, \partial^\mu \psi^* - i\psi^* \partial^\mu \psi$$

$$\partial_\mu j^\mu = \partial_\mu (i\psi\, \partial^\mu \psi^*) - \partial_\mu (i\psi^* \partial^\mu \psi)$$

$$= i\left[\partial_\mu \psi\, \partial^\mu \psi^* - \partial_\mu \psi^* \partial^\mu \psi + \psi\, \partial_\mu \partial^\mu \psi^* - \psi^* \partial_\mu \partial^\mu \psi\right]$$

$$= i\left[m^2 \psi^* \psi + \lambda (\psi^* \psi)^2 - m^2 \psi^* \psi \pm \lambda (\psi^* \psi)^2\right]$$

$$= 0 \quad \square$$

4. $\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2 \phi_a\phi_a \quad a \in \{1,2,3\},\ \phi_a \in \mathbb{R}$

$\eta^b \eta_b = 1$

$\phi_a \to \phi_a + \theta \epsilon_{abc}\eta_b \phi_c$

$\partial^\mu \phi_a \to \partial^\mu \phi_a + \theta \epsilon_{abc}\eta_b \partial^\mu \phi_c$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2\phi_a\phi_a \to \frac{1}{2}(\partial^\mu\phi_a + \theta\epsilon_{abc}\eta_b\partial^\mu\phi_c)(\partial_\mu\phi_a + \theta\epsilon_{abc}\eta_b\partial_\mu\phi_c)$$

$$- \frac{1}{2}m^2(\phi_a + \theta\epsilon_{abc}\eta_b\phi_c)(\phi_a + \theta\epsilon_{abc}\eta_b\phi_c)$$

$$= \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \underbrace{\theta\epsilon_{abc}\eta_b\partial_\mu\phi_a\partial^\mu\phi_c}_{\to 0} - \frac{1}{2}m^2\phi_a\phi_a$$

$$- \underbrace{\cancel{\tfrac{1}{2}} m^2 \theta\epsilon_{abc}\eta_b\phi_a\phi_c}_{\to 0} + \mathcal{O}(\theta^2) = \mathcal{L}$$

because $\epsilon_{abc}\partial_\mu\phi_a\partial^\mu\phi_c = 0$

$\epsilon_{abc}\phi_a\phi_c = 0$.

~~$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\epsilon_{abc}\eta_b\phi_c = \epsilon_{abc}\eta_b\phi_c\partial^\mu\phi_a$~~

Get three currents, for each η_b, $b = 1, 2, 3$,

$$j_b^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\epsilon_{abc}\phi_c = \epsilon_{abc}\phi_c\partial^\mu\phi_a$$

As $\partial_\mu j_b^\mu = 0$,

$$Q_a \equiv \int d^3x\, j_a^0 = \int d^3x\, \epsilon_{abc}\dot\phi_b\phi_c$$

are conserved charges.

ϕ_a satisfy field eq.

$$\partial_\mu\partial^\mu\phi_a + m^2\phi_a = 0$$

$$\Rightarrow \ddot\phi_a - \nabla^2\phi_a + m^2\phi_a = 0$$

$$\Rightarrow \dot Q_a = \int d^3x\, \epsilon_{abc}(\ddot\phi_b\phi_c + \underbrace{\dot\phi_b\dot\phi_c}_{\to 0})$$

$$= \int d^3x\, \epsilon_{abc}\phi_c(\nabla^2\phi_a - \underbrace{m^2\phi_a}_{\to 0})$$

$$= \int d^3x\, \epsilon_{abc}\phi_c\nabla^2\phi_a = \int d^3x\, \epsilon_{abc}\partial_i[\phi_c\partial^i\phi_a] \to 0$$

5. $\eta_{ab} x'^a x'^b = \eta_{ab} \Lambda^a{}_c x^c \Lambda^b{}_d x^d = \eta_{ab} x^a x^b \quad \forall x$

$$\Rightarrow \eta_{ab} = \eta_{cd} \Lambda^c{}_a \Lambda^d{}_b$$

$$\eta_{ab} = \eta_{ba}$$

$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b$ where $\omega^{ab} = -\omega^{ba}$

$$\Rightarrow \eta_{cd} \Lambda^c{}_a \Lambda^d{}_b = \eta_{cd} (\delta^c{}_a + \omega^c{}_a)(\delta^d{}_b + \omega^d{}_b)$$

$$= \eta_{cd} \delta^c{}_a \delta^d{}_b + \eta_{cd} \delta^c{}_a \omega^d{}_b + \eta_{cd} \delta^d{}_b \omega^c{}_a + \eta_{cd} \omega^c{}_a \omega^d{}_b$$

$$= \eta_{ab} + \eta_{ad} \omega^d{}_b + \eta_{cb} \omega^c{}_a + \eta_{cd} \omega^c{}_a \omega^d{}_b$$

$$= \eta_{ab} + \cancel{\eta_{ad}\omega^d{}_b} \; \omega_{ab} + \omega_{ba} + \omega_{da} \omega^d{}_b$$

$$= \eta_{ab} \pm \omega_{ad} \omega^d{}_b = \cancel{\eta_{ab} - \omega_{ad}\,\eta^{de}\,\omega_{eb}}$$

$$= \eta_{ab} + O(\omega^2) \quad \square$$

$$\omega^a{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{rotation } \theta \text{ about } x^3 .$$

$$\omega^a{}_b = \begin{pmatrix} 0 & -v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{boost } v \text{ along } x^1 .$$

6. $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x) = \phi(x^\mu - \omega^\mu{}_\nu x^\nu)$$

$$= \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

$$\cancel{\partial_\sigma \phi(x) \to \partial_\sigma \phi(x) - \partial_\sigma (\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x))}$$

$$\cancel{= \partial_\sigma \phi(x) - \omega^\mu{}_\nu \partial_\sigma (x^\nu \partial_\mu \phi(x))}$$

$$\delta \mathcal{L} = (\partial\mathcal{L}/\partial\phi)\,\delta\phi + (\partial\mathcal{L}/\partial(\partial_\mu\phi))\,\delta(\partial_\mu\phi)$$

$$= \frac{\partial \mathcal{L}}{\partial\phi}\left(-\omega^\mu_{\ \nu} x^\nu \partial_\mu\phi\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\left(-\omega^\mu_{\ \nu}\,\partial_\sigma(x^\nu\partial_\mu\phi)\right)$$

$$= \frac{\partial \mathcal{L}}{\partial\phi}\left(-\omega^\mu_{\ \nu} x^\nu \partial_\mu\phi\right) \pm \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\right)\left(-\omega^\mu_{\ \nu} x^\nu\partial_\mu\phi\right)$$

$$\pm\, \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\,\omega^\mu_{\ \nu} x^\nu \partial_\mu\phi\right) \qquad \text{Use EoM}$$

$$= -\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\,\omega^\mu_{\ \nu}\, x^\nu\partial_\mu\phi\right)$$

Similar to $T^{\mu\nu}$ deriv $\mathcal{L} \to \mathcal{L} - \omega^\mu_{\ \nu} x^\nu \partial_\mu \mathcal{L}$

~~$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$~~ $\quad$ ~~$\delta\mathcal{L} \stackrel{?}{=} -\partial_\mu(\omega^\mu_{\ \nu}x^\nu\mathcal{L})$~~

$$\Rightarrow \delta\mathcal{L} = -\omega^\mu_{\ \nu} x^\nu\partial_\mu\mathcal{L} = -\partial_\mu(\omega^\mu_{\ \nu}x^\nu\mathcal{L}) \quad \cancel{= -\omega^\mu_{\ \mu}\mathcal{L} - \omega^\mu_{\ \nu}x^\nu\partial_\mu\mathcal{L}}$$

as $-\omega^\mu_{\ \mu}\mathcal{L} = 0$ because ω ~~antisym~~ traceless

Noether's theorem

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\omega^\sigma_{\ \nu}x^\nu\partial_\sigma\phi + \omega^\mu_{\ \nu}x^\nu\mathcal{L}$$

while $T^\mu_{\ \nu} = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \mathcal{L}\,\delta^\mu_\nu$

$$\Rightarrow j^\mu = -\omega^\rho_{\ \nu}\left[T^\mu_{\ \rho}\, x^\nu\right].$$

$\rho, \nu = 0 \Rightarrow \epsilon_{i\rho\nu} = 0$

Rotation about axis x^i, $(\omega_i)^\rho_{\ \nu} = \epsilon_{i\ \nu}^{\ \rho}\,\theta$ by angle θ

$$\Rightarrow 3 \text{ currents } (j_i)^\mu = -\epsilon_{i\ \nu}^{\ \rho}\left[T^\mu_{\ \rho}\,x^\nu\right] = -\epsilon_{i\rho\nu}\,T^{\mu\rho}x^\nu$$

$$= \epsilon_{i\nu\rho}\,T^{\mu\rho}x^\nu$$

$$Q_i \equiv \int d^3x\, 2\,(j_i)^0 = \epsilon_{ijk}\int d^3x\,(x^j T^{0k} - x^k T^{0j}).$$

Lorentz boost along x^i, $(\omega_i)^\rho{}_\nu = -v\left(\delta^{\rho 0}\delta^{i}_{\nu} + \delta^{\rho i}\delta^{0}_{\nu}\right)$ for velocity v

$\Rightarrow$ currents $(j^i)^\mu = \left(\delta^{\rho 0}\delta^{i}_{\nu} + \delta^{\rho i}\delta^{0}_{\nu}\right) T^{\mu}{}_{\rho}\, x^\nu$

$$= T^{\mu 0} x^i + T^{\mu i} x^0$$

$$Q^i = \int d^3x \left(T^{00} x^i + T^{0i} x^0\right) \qquad \text{conserved}$$

$$\underbrace{\frac{d}{dt} Q^i}_{=0} = \frac{d}{dt}\int d^3x \left(x^i T^{00}\right) + \frac{d}{dt}\int d^3x \left(x^0 T^{0i}\right)$$

$$= \frac{d}{dt}\int d^3x \left(x^i T^{00}\right) + \underbrace{\int d^3x\, T^{0i}}_{p^i = \text{const}} + \underbrace{x^0 \frac{d}{dt}\int d^3x\, T^{0i}}_{\frac{d}{dt}p^i = 0}$$

$$= \frac{d}{dt}\int d^3x \left(x^i T^{00}\right) = \text{const}$$

Centre of energy moves with constant velocity.

7. $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$A_\mu \rightarrow A_\mu + \partial_\mu \xi \qquad \xi = \xi(x)$$

$$\partial_\nu A_\mu \rightarrow \partial_\nu A_\mu + \partial_\nu \partial_\mu \xi$$

$$F_{\mu\nu} \longrightarrow \partial_\mu A_\nu + \partial_\mu \partial_\nu \xi - \partial_\nu A_\mu - \partial_\nu \partial_\mu \xi = F_{\mu\nu}$$

$$\therefore \mathcal{L} \rightarrow \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\rho} F_{\mu\nu} F_{\sigma\rho} = -\frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\rho}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\sigma A_\rho - \partial_\rho A_\sigma\right)$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)}\partial^\nu A_\sigma - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}\eta^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}$$

$$T^{\mu\nu} \neq T^{\nu\mu} = -F^{\nu\sigma}\partial^\mu A_\sigma + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

as $T^{\mu\nu} - T^{\nu\mu} = -F^{\mu\sigma}\partial^\nu A_\sigma + F^{\nu\sigma}\partial^\mu A_\sigma \neq 0$

$$T^{\mu\nu} \longrightarrow -F^{\mu\sigma}(\partial^\nu A_\sigma + \partial^\nu\partial_\sigma\xi) + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

$$= T^{\mu\nu} - \underbrace{F^{\mu\sigma}\partial^\nu\partial_\sigma\xi}_{\neq 0 \text{ in general}}$$

from EoM

$= \partial_\rho[F^{\rho\mu}A^\nu]$ as $\partial_\rho F^{\rho\mu} = 0$

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu$$

Currents

$$= -F^{\mu\rho}\partial^\nu A_\rho - F^{\rho\mu}\partial_\rho A^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

$(j^\mu)^\nu \equiv \Theta^{\mu\nu}$ require $\partial_\nu(j^\mu)^\nu = 0$

$$= F_\rho{}^\mu(\partial^\nu A^\rho - \partial^\rho A^\nu) + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

$$= F_\rho{}^\mu F^{\nu\rho} + \dots$$

$$= -F^\mu{}_\rho F^{\nu\rho} + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

$$\partial_\nu\Theta^{\mu\nu} = \partial_\nu T^{\mu\nu} - \partial_\nu(F^{\rho\mu}\partial_\rho A^\nu)$$

$$= -\partial_\nu(F^\mu{}_\rho F^{\nu\rho}) + \frac{1}{4}\eta^{\mu\nu}\partial_\nu(F_{\sigma\rho}F^{\sigma\rho})$$

$$= -F^\nu{}_\rho\,\partial_\nu F^{\mu\rho} - \cancel{F^\mu{}_\rho\,\partial_\nu F^{\nu\rho}}^{\,\to 0} + \frac{1}{2}\eta^{\mu\nu}F_{\sigma\rho}\partial_\nu F^{\sigma\rho}$$

$$= -F_{\nu\rho}\partial^\nu F^{\mu\rho} + \frac{1}{2}\eta^\mu{}_\nu F_{\sigma\rho}\partial^\nu F^{\sigma\rho}$$

Bianchi identity

$$= -F_{\nu\rho}\partial^\nu F^{\mu\rho} - \frac{1}{2}\eta^\mu{}_\nu F_{\sigma\rho}(\partial^\sigma F^{\rho\nu} + \partial^\rho F^{\nu\sigma})$$

$$= -F_{\nu\rho}\partial^\nu F^{\mu\rho} - \frac{1}{2}F_{\sigma\rho}\partial^\sigma F^{\rho\mu} - \frac{1}{2}F_{\sigma\rho}\partial^\rho F^{\mu\sigma}$$

$$= \frac{1}{2}F_{\nu\rho}\partial^\nu F^{\rho\mu} - \frac{1}{2}F_{\sigma\rho}\partial^\rho F^{\mu\sigma} = 0$$

$$\Theta^{\mu\nu} = -F^{\mu}{}_{\rho}F^{\nu\rho} + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

$\Theta^{\mu\nu} = \Theta^{\nu\mu}$ trivially in this form

$\Theta^{\mu\nu} \longrightarrow \Theta^{\mu\nu}$ as $A^{\mu} \longrightarrow A^{\mu} + \partial^{\mu}\xi$ as $F^{\mu\nu} \longrightarrow F^{\mu\nu}$

$$tr(\Theta) = \Theta^{\mu}{}_{\mu} = -F^{\mu}{}_{\rho}F_{\mu}{}^{\rho} + \frac{1}{4}\eta^{\mu}{}_{\mu}F_{\sigma\rho}F^{\sigma\rho}$$

$$= -F_{\mu\rho}F^{\mu\rho} + F_{\sigma\rho}F^{\sigma\rho} = 0 \quad \square$$

8. $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_{\mu}C^{\mu}$ where $F_{\mu\nu} = \partial_{\mu}C_{\nu} - \partial_{\nu}C_{\mu}$

EoM $\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}C_{\nu})}\right) = \partial_{\mu}\left(-F^{\mu\nu}\right) = -\partial_{\mu}F^{\mu\nu}$

$$= \frac{\partial\mathcal{L}}{\partial C_{\nu}} = m^2 C^{\nu}$$

or $\partial_{\mu}\partial^{\mu}C^{\nu} - \partial_{\mu}\partial^{\nu}C^{\mu} + m^2 C^{\nu} = 0$

Take

$m^2\partial_{\nu}C^{\nu} = \partial_{\nu}\partial_{\mu}\partial^{\nu}C^{\mu} - \partial_{\nu}\partial_{\mu}\partial^{\mu}C^{\nu} = 0$

$\Rightarrow \partial_{\nu}C^{\nu} = 0$ when $m \neq 0$.

For C^0,

$\partial_{\mu}\partial^{\mu}C_0 - \partial_{\mu}\partial_0 C^{\mu} + m^2 C_0 = 0$

$\ddot{C}_0 + \partial_i\partial^i C_0 - \ddot{C}_0 - \partial^i\dot{C}_i + m^2 C_0 = 0$

$\Rightarrow \partial_i\partial^i C_0 + m^2 C_0 = \partial^i\dot{C}_i$

$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{C}_{\mu}}$. $\mathcal{L} = -\frac{1}{4}(\partial_{\mu}C_{\nu} - \partial_{\nu}C_{\mu})(\partial^{\mu}C^{\nu} - \partial^{\nu}C^{\mu}) + \frac{1}{2}m^2 C_{\mu}C^{\mu}$

$\Pi^{\mu} = -F^{0\mu} = \partial^{\mu}C^0 - \partial^0 C^{\mu}$, $\Pi_i = \partial_i C^0 - \dot{C}_i$

$\Pi_0 = 0$

$$\mathcal{H} = \Pi^\mu \dot{C}_\mu - \mathcal{L} = \dot{C}_i \partial^i C^0 \pm \dot{C}^i \dot{C}_i + \tfrac{1}{4}(\partial_0 C_i - \partial_i C_0)(\partial^0 C^i - \partial^i C^0)$$

$$+ \tfrac{1}{4}(\partial_i C_0 - \partial_0 C_i)(\partial^i C^0 - \partial^0 C^i) + \tfrac{1}{4}(\partial_i C_j - \partial_j C_i)(\partial^i C^j - \partial^j C^i)$$

$$\pm \tfrac{1}{2} m^2 C_0 C^0 \pm \tfrac{1}{2} m^2 C_i C^i$$

$$= \partial^i C^0 (\partial_i C^0 - \Pi_i) - (\partial_i C^0 - \Pi_i)(\partial^i C^0 - \Pi^i)$$

$$+ \tfrac{1}{2}(-\Pi_i)(-\Pi^i) + \tfrac{1}{4}(\partial_i C_j - \partial_j C_i)(\partial^i C^j - \partial^j C^i)$$

$$\pm \tfrac{1}{2} m^2 C_0 C^0 \pm \tfrac{1}{2} m^2 C_i C^i$$

$$= (\partial^i C^0)\Pi_i - \tfrac{1}{2}\Pi_i \Pi^i + \tfrac{1}{4}(\partial_i C_j - \partial_j C_i)(\partial^i C^j - \partial^j C^i)$$

$$\pm \tfrac{1}{2} m^2 C_0 C^0 \pm \tfrac{1}{2} m^2 C_i C^i$$

9. $x^\mu \to x'^\mu = \lambda x^\mu \qquad \phi(x) \to \phi'(x) = \lambda^{-D}\phi(\lambda^{-1}x)$

$$\partial_\mu \phi \to \frac{\partial}{\partial x^\mu}\phi'(x) = \frac{\partial}{\partial x^\mu}\lambda^{-D}\phi(\lambda^{-1}x) = \lambda^{-D}\frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\phi(y)$$

with $y^\mu = \lambda^{-1}x^\mu$

$$= \lambda^{-D-1}\partial_\mu \phi$$

EoM $\partial_\mu \partial^\mu \phi + m^2\phi + gp\,\phi^{p-1} = 0$

invariant ($D=1$, $m=0$, $p=4$)

$\partial_\mu \partial^\mu \phi + 4g\phi^3 = 0$

~~$\Rightarrow \partial_\mu\phi$ won't if $D \neq 1$.~~

$$S = \int d^4x \ \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2}m^2\phi^2 - g\phi^p$$

$$S \to S' = \int d^4x' \ \tfrac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - \tfrac{1}{2}m^2\phi'^2 - g\phi'^p$$

$$= \int d^4x \ \lambda^4 \left(\tfrac{1}{2}\lambda^{-2(D+1)}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\lambda^{-2D}\phi^2 - g\lambda^{-pD}\phi^p\right)$$

$D=1$ so that derivative terms ~~coeff~~ $\lambda^{-2(D+1)}$ cancels with λ^4

Require $m=0$ and $p=4$ (as $D=1$), to get $S' = S$

In $n+1$ dimensions $d^{n+1}x \to d^{n+1}x' = \lambda^{n+1} d^{n+1}x \Rightarrow n+1 = 2(D+1)$ or $D = \frac{n+1}{2} - 1$

for m: $n+1-2D = (n+1)-(n+1)+2 \neq 0$ so $m=0\ \forall n$, $n+1-pD = 0 \Rightarrow p = \frac{n+1}{D} = 2\frac{n+1}{n-1}$.

~~$\mathcal{L} \to \mathcal{L}' = \lambda^{-4}\mathcal{L} = \mathcal{L} - 4\eta\mathcal{L}$, $\lambda = 1+\eta$, $D^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}$~~

$$\delta \mathcal{L} = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right]$$

On the other hand,

$$\delta\mathcal{L} = \frac{\partial\mathcal L}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal L}{\partial(\partial_\mu\phi)}\,\delta(\partial_\mu\phi)$$

Consider infinitesimal scaling $\lambda = 1+\eta$,

$$x^\mu \longrightarrow x^\mu + \eta x^\mu$$

$$\phi \longrightarrow \lambda^{-D}\phi(\lambda^{-1}x) = \lambda^{-D}\phi[(1-\eta)x] = (1-D\eta)\left[\phi(x) - \eta x^\mu\frac{\partial\phi}{\partial x^\mu}\right] + O(\eta^2)$$

$$= (1-D\eta)(\phi - \eta x^\mu\partial_\mu\phi)$$

$$= \phi - (D\phi + x^\mu\partial_\mu\phi)\eta$$

$$\partial_\mu\phi \longrightarrow \partial_\mu\phi - \eta D\partial_\mu\phi - \eta\,\partial_\mu(x^\nu\partial_\nu\phi)$$

$$= \partial_\mu\phi - \eta(D\partial_\mu\phi + \partial_\mu\phi + x^\nu\partial_\nu\partial_\mu\phi)$$

$$\frac{\partial\mathcal L}{\partial\phi} = -m^2\phi - gp\phi^{p-1}, \quad \frac{\partial\mathcal L}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$$

$$\cancel{\delta\mathcal L = \frac{\partial\mathcal L}{\partial\phi}\delta\phi + \frac{\partial\mathcal L}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = -(-m^2\phi - gp\phi^{p-1})(D\phi + x^\mu\partial_\mu\phi)\eta}$$

$$\cancel{-\partial^\mu\phi((D+1)\partial_\mu\phi + x^\nu\partial_\nu\partial_\mu\phi)\eta = \eta\Big[m^2D\phi^2 + gpD\phi^p}$$

$$\cancel{+ m^2\phi x^\mu\partial_\mu\phi + gp\phi^{p-1}x^\mu\partial_\mu\phi - (D+1)\partial^\mu\phi\partial_\mu\phi - x^\nu(\partial^\mu\phi)\partial_\nu\partial_\mu\phi\Big]}$$

$D=1,\ m=0,\ p=4$, $\quad \mathcal L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - g\phi^4$

$$\frac{\partial\mathcal L}{\partial\phi}\delta\phi = 4g\phi^3(\phi + x^\mu\partial_\mu\phi)\eta = \eta(4 + x^\mu\partial_\mu)g\phi^4$$

$$\frac{\partial\mathcal L}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = -(\partial^\mu\phi)\eta(2 + x^\nu\partial_\nu)\partial_\mu\phi = -\eta(4 + x^\mu\partial_\mu)\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi$$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \eta\left[(4 + x^\mu\partial_\mu) g\phi^4 - (4 + x^\mu\partial_\mu)\tfrac{1}{2}\partial_\nu\phi\,\partial^\nu\phi\right]$$

$$= \eta\left[4g\phi^4 - 2\partial_\mu\phi\,\partial^\mu\phi - x^\mu\partial_\mu\mathcal{L}\right] = -\eta(4 + x^\mu\partial_\mu)\mathcal{L}$$

$$= -\eta\,\partial_\mu(x^\mu\mathcal{L})$$

Also

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] = -\partial_\mu\left[\partial^\mu\phi\,\eta(\phi + x^\nu\partial_\nu\phi)\right]$$

Hence,

$$D^\mu = (\partial^\mu\phi)(\phi + x^\nu\partial_\nu\phi) - x^\mu\mathcal{L}$$

Also $\quad T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$,

so $\quad D^\mu = x_\nu T^{\mu\nu} + \phi\,\partial^\mu\phi$.