Recap: Root string $S_{\alpha,\beta} = \{\beta + \rho\alpha \in \Phi, \rho \in \mathbb{Z}\}$ for β not proportional to α, and $S_{\alpha,\alpha} = \{n\alpha, n \in \mathbb{Z}\}$

$$(\alpha, \beta) = \frac{1}{N} \sum_{\delta \in \Phi} (\alpha, \delta)(\beta, \delta)$$

$$R_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \in \mathbb{R}$$

$$\frac{2R_{\alpha,\beta}}{(\beta,\beta)} = \frac{1}{N} \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta} \in \mathbb{R}$$

$$\Rightarrow (\alpha, \beta) \in \mathbb{R}$$

The real geometry of roots

● Roots $\alpha \in \Phi$ are elements of $\mathfrak{h}^*$. In general not lin. ind.

(# roots $\underset{\dim(\mathfrak{g})\nearrow}{d} - \underset{\nwarrow \dim(\mathfrak{h})}{r} \geq r$ where $r = \dim(\mathfrak{h}^*)$) But:

Prop: $\boxed{\text{Roots span } \mathfrak{h}^*}$

Proof:

Suppose not, $\exists \lambda \in \mathfrak{h}^*$ s.t.

$$(\lambda, \alpha) = (K^{-1})_{ij} \lambda^i \alpha^j = K^{ij} \lambda_i \alpha_j = 0 \quad \forall \alpha$$

Def $H_\lambda = \lambda_i H^i \in \mathfrak{h}$

● $[H_\lambda, H] = 0 \quad \forall H \in \mathfrak{h}$

$[H_\lambda, E^\alpha] = (\lambda, \alpha) E^\alpha = 0 \quad \forall \alpha \in \Phi$

i.e. $[H_\lambda, X] = 0 \quad \forall X \in \mathfrak{g}$

But then $\text{span}_{\mathbb{C}}\{H_\lambda\}$ is a non-trivial ideal, contradiction to simplicity

So far $\alpha \in \mathfrak{h}^*$ $\nwarrow$ complex vector space

Now define a real subspace $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$

$$\mathfrak{h}^*_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}} \{\alpha_{(i)} : i = 1, \dots r\}$$

$$\mathfrak{h}^* = \mathrm{Span}_{\mathbb{C}} \{\alpha_{(i)} : i = 1, \dots r\}$$ So can write any root $\beta \in \Phi$

$\nwarrow$ some subset of roots

$$\beta = \sum_{i=1}^{r} \beta^i \alpha_{(i)} \qquad \beta^i \text{ complex}$$

$$\underset{\substack{\uparrow \\ \text{real} \\ ⑱}}{(\beta, \alpha_{(j)})} = \sum_{i=1}^{r} \beta^i \underset{\substack{\uparrow \\ \text{real} \\ ⑱}}{(\alpha_{(i)}, \alpha_{(j)})}$$

since $(\alpha, \beta) \in \mathbb{R}$ $\forall \alpha, \beta \in \Phi$ any subset of them must also have real

$\Rightarrow \beta^i$ real as they solve real lin. eq^n

$$\Rightarrow \beta \in \mathfrak{h}^*_{\mathbb{R}} \quad \forall \phi \in \Phi$$

Inner product of any two general elements of $\mathfrak{h}^*_{\mathbb{R}}$ (no necessarily $\in \Phi$)

$$\lambda = \sum_{i=1}^{r} \lambda^i \alpha_{(i)} \in \mathfrak{h}^*_{\mathbb{R}}$$

$$\mu = \sum_{i=1}^{r} \mu^i \alpha_{(i)} \in \mathfrak{h}^*_{\mathbb{R}} \qquad \lambda^i, \mu^i \in \mathbb{R}$$

$$(\lambda, \mu) = \sum_{i,j} \lambda^i \mu^i \underset{\text{real } ⑱}{(\alpha_{(i)}, \alpha_{(j)})} \in \mathbb{R}$$

$$(\lambda, \lambda) \overset{⑰}{=} \frac{1}{N} \sum_{\delta \in \Phi} (\lambda, \delta)^2 \geq 0$$ because $(\lambda, \delta)^2$ are squares of real

Equality can only hold if $(\lambda, \delta) = 0$ $\forall \delta \in \Phi$
$\Rightarrow \lambda = 0$ using non-degeneracy of inner product on $\mathfrak{h}^*$

Summary

Roots $\alpha \in \Phi$ live in a real vector space $\mathfrak{h}^*_{\mathbb{R}} \cong \mathbb{R}^r$ $r = \mathrm{Rank}(\mathfrak{g})$

Euclidean inner product: $\forall \lambda, \mu \in \mathfrak{h}^*_{\mathbb{R}}$

(i) $(\lambda, \mu) \in \mathbb{R}$

(ii) $(\lambda, \lambda) \geq 0$

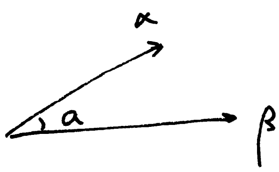
(iii) $(\lambda, \lambda) = 0 \Leftrightarrow \lambda = 0$

As $(\alpha, \alpha) > 0$ $\forall \alpha \in \Phi$ define length

$$|\alpha| = +(\alpha, \alpha)^{1/2} > 0$$

- the inner product takes the standard form

$$(\alpha, \beta) = |\alpha| |\beta| \cos\theta$$



⑮ : $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \qquad \forall \alpha, \beta \in \Phi$

$\Rightarrow \frac{2|\beta|}{|\alpha|} \cos\varphi \in \mathbb{Z}$ – (19a)

$\frac{2(\beta,\alpha)}{(\beta,\beta)} = \frac{2|\alpha|}{|\beta|} \cos\varphi \in \mathbb{Z}$ – (19a)

(19a) × (19b) $\quad 4\cos^2\varphi \in \mathbb{Z}$

$\Rightarrow \cos\varphi = \frac{\pm\sqrt{n}}{2} \qquad n \in \mathbb{N} \cup \{0\}$

$n \in \{0, 1, 2, 3, 4\}$

Solutions : $\varphi = 0 \quad \alpha = \beta$ (show in examples that $\alpha = 1 \times \beta$)

$\varphi = \pi/2 \quad (\alpha, \beta) = 0$

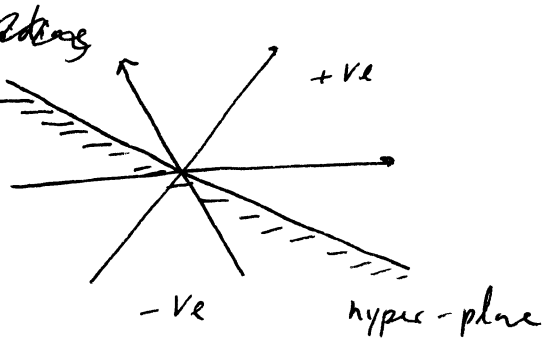
$\varphi = \pi \quad \alpha = -\beta$ (show in example that $\alpha = 1 \times (-\beta)$)

$\varphi = \pi/6, \pi/4, \pi/3 \qquad (\alpha, \beta) > 0$

$\varphi = 2\pi/3, 3\pi/4, 5\pi/6 \qquad (\alpha, \beta) < 0$

Simple Roots

Divide roots $\alpha \in \Phi$ into +ve and -ve by ~~adding~~ adding a hyper plane in $h^*_{\mathbb{R}} \cong \mathbb{R}^r$



$\Phi = \Phi_+ \cup \Phi_-$

$\forall \alpha, \beta \in \Phi$

i) $\alpha \in \Phi_+ \Leftrightarrow -\alpha \in \Phi_-$

ii) $\alpha, \beta \in \Phi_\pm \Rightarrow \alpha + \beta \in \Phi_\pm$

A simple root is a positive root which cannot be written as a sum of two positive roots.

$$\underset{\substack{\uparrow \\ \text{simple roots}}}{\delta \in \Phi_s} \Leftrightarrow \begin{array}{l} \delta \in \Phi_+ \\ \delta \neq \alpha + \beta \ \forall \alpha, \beta \in \Phi_+ \end{array}$$