Integration

Def^n M oriented n-dimensional : $\varphi: \mathcal{O} \to U$ RH coord chart, coords x^μ.
X : n-form, the <u>integral of X over $\mathcal{O}$</u> is

$$\int_{\mathcal{O}} X = \int_U dx^1 \dots dx^n \, X_{12\dots n}$$

Ex: show this is indept of choice of RH coords on $\mathcal{O}$ i.e. get the same result for any other chart $\varphi' : \mathcal{O} \to U'$.

Def^n RH charts $\varphi_\alpha : \mathcal{O}_\alpha \to U_\alpha$ atlas for M. "Partition of unity" $h_\alpha : M \to [0,1]$.

$h_\alpha(p) = 0$ if $p \in \mathcal{O}_\alpha$, $\sum_\alpha h_\alpha(p) = 1 \quad \forall p$

Define $\int_M X \equiv \sum_\alpha \int_{\mathcal{O}_\alpha} h_\alpha X$

Diffeo: $\phi : M \to M$ is <u>orientation-preserving</u> iff $\varphi^*(\varepsilon)$ equivalent to ε. $\forall$ orientations ε, then

$$\int_M \varphi^*(X) = \int_M X$$

Def^n ε: volume form. The <u>volume ~~form~~ of M</u> is $\int_M \varepsilon$. If $f : M \to \mathbb{R}$, define $\int_M f = \int_M f\varepsilon$

Notation: $\int_M f \equiv \int_M d^n x \sqrt{|g|}\, f$

Submanifolds, Stokes thm

Def^n S, M oriented manifolds, dimensions m, n respectively $m < n$
$\varphi : S \to M$ is an <u>embedding</u> if it is 1-1 ($\varphi[S]$ doesn't intersect itself) and $\forall p \in S \ \exists \ \mathcal{O}$ of p s.t. $\varphi^{-1} : \varphi[S] \to S$ is smooth. If φ is an embedding, then $\varphi[S]$ is an <u>embedded submanifold</u> of M. A <u>hypersurface</u> is an embedded submfd of dimension $n-1$.

Def^n The <u>integral of m-form X over $\phi[S]$</u> is $\int_{\varphi[S]} X \equiv \int_S \varphi^*(X)$

$$X = dY \qquad \int_{\varphi[S]} dY = \int_S d(\varphi^*(Y))$$

Def^n A <u>manifold with ~~M~~ boundary M</u> same as manifold except
$\varphi_\alpha : \mathcal{O}_\alpha \to U_\alpha \qquad U_\alpha$ = open subset of $\frac{1}{2}\mathbb{R}^n = \{(x^1, \dots, x^n) : x^1 \le 0\}$
The <u>boundary of M</u>, denoted ∂M, is set of points with $x^1 = 0$. ∂M is a manifold of dimension $n-1$, coord charts $(x^2, \dots, x^n)$
M oriented $\Rightarrow$ ∂M oriented, $(x^2, \dots, x^n)$ is RH on ∂M if $(x^1, \dots, x^n)$ is RH on M

<u>Stokes theorem</u> N oriented n-dim mfd with boundary, X : $(n-1)$-form.

$$\int_N dX = \int_{\partial N} X$$

regard ∂N as hypersurface in N $\quad \varphi : \partial N \to N \quad p \mapsto p$.

Often N = region of M $\Rightarrow$ ∂N hypersurface in M.

<u>Example</u> Σ: hypersurface in M with boundary $\partial\Sigma$. F Maxwell field in M

$$\frac{1}{4\pi}\int_{\partial\Sigma} {*F} \underset{\text{Stokes}}{=} \frac{1}{4\pi}\int_{\Sigma} d{*F} \underset{\text{Maxwell}}{=} -\int_{\Sigma} {*j} \equiv Q[\Sigma]$$

defines charge on Σ.

<u>Gauss' Law</u>

<u>Defn</u> $X \in T_p(M)$ is <u>tangent to $\varphi[S]$ at p</u> if X is tangent at p to a curve in $\varphi[S]$.

$n \in T_p^*(M)$ is <u>normal</u> to $\varphi[S]$ if $n(X) = 0$ $\forall$ X tangent to $\varphi[S]$ at p.

Remark: vectors tangent to $\varphi[S]$ at p: vector space of dim $m = \dim \varphi[S]$

normals .... $n-m$, ~~any normals~~

Any 2 normals to a <u>hypersurface</u> are proportional.

<u>Defn</u> A hypersurface is $\left\{\begin{matrix}\text{timelike}\\ \text{spacelike}\\ \text{null}\end{matrix}\right\}$ if any normal is $\cancel{\left\{\begin{matrix}\text{timelike}\\ \text{spacelike}\\ \text{null}\end{matrix}\right\}}$ $\left\{\begin{matrix}\text{spacelike}\\ \text{timelike}\\ \text{null}\end{matrix}\right\}$.

Remark: M mfd with boundary. Consider curve in ∂M, tangent X^a

$$dx^1(X) = X(x^1) = \frac{dx^1}{dt} = 0$$

t parameter along curve

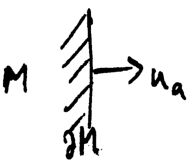
$\therefore$ $dx^1(X) = 0$ $\forall$ X tangent to ∂M

$\therefore$ dx^1 is normal to ∂M.

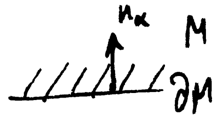
∂M timelike / spacelike $\Rightarrow$ <u>unit normal</u>

$$n_a = \frac{(dx^1)_a}{\sqrt{\pm g^{bc}(dx^1)_b(dx^1)_c}} \Rightarrow \cancel{g_{ab}}\, g^{ab} n_a n_b = \pm 1$$

∂M timelike (or g Riemannian)



∂M spacelike



<u>Divergence theorem</u> ∂M timelike/spacelike:

$$\int_M d^n x \sqrt{|g|}\, \nabla_a X^a = \int_{\partial M} d^{n-1} x \sqrt{|h|}\, n_a X^a$$

∇: Levi-Civita connection on M; X^a: vec field on M

h_{ab}: pull-back of g_{ab} to ∂M w/ det h

$n_a X^a$ scalar on M, pull-back to ∂M on RHS.