$$\frac{dO_H}{dt} = i[H, O_H]$$

We can check that in terms of $O_H = \phi$, i.e. $\frac{\partial\phi}{\partial t} = i[H,\phi]$ means that the Heisenberg op ϕ satisfies the KG eq $\partial_\mu \partial^\mu \phi + m^2\phi = 0$.

We write the Fourier Transform of $\phi(x)$ by using

$$e^{iHt} a_p e^{-iHt} = e^{-iE_p t} a_p \qquad [H, a_p] - E_p a_p$$

$$e^{iHt} a_p^\dagger e^{-iHt} = e^{+iE_p t} a_p^\dagger$$

i.e. $\phi(\underline{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$

Causality

ϕ and π satisfy equal-time comm relations

e.g. $[\phi(\underline{x}, t), \pi(\underline{y}, t)] = i\delta^3(\underline{x}-\underline{y})$. What about arbitrary space-time separations?

In particular, causality requires that all space-like separated ops commute, i.e.

$[O(x), O(y)] = 0 \quad \forall\, (x-y)^2 < 0$.

this ensures that a measurement at x can't affect a measurement at y. Do we have this?

Define $\Delta(x-y) \equiv [\phi(x), \phi(y)]$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \quad \text{Lorentz invariant as } \int\frac{d^3p}{2E_p} \text{ is.}$$

- Doesn't vanish for t-like separation:

$[\phi(\underline{x}, 0), \phi(\underline{x}, t)] \sim e^{-imt} - e^{imt}$

- Vanishes for space-like separations:

Note that $\Delta(x,y) = 0$ at equal times

L.I. $\Rightarrow$ It can only depend on $(x-y)^2$, so it must vanish $\forall (x-y)^2 < 0$

$[\phi(\underline{x}, t), \phi(\underline{y}, t)] = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\underline{p}^2+m^2}} \left(e^{i\underline{p}\cdot(\underline{x}-\underline{y})} - e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}\right) = 0$ (change sign of $\underline{p}$).

Also holds in an interacting theory. But $\Delta(x-y)$ is a c-f^n only in the free theory.

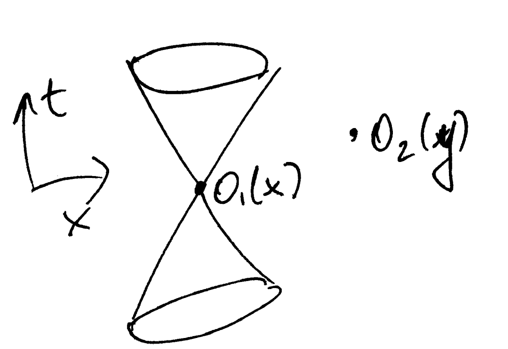
Propagator

Prepare a part^c at y, what is the amplitude at a point x?

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_{p'}}}\underbrace{\overbrace{\langle 0|a_p a_{p'}^\dagger|0\rangle}^{=[a_p,\, a_{p'}^\dagger]}}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')} e^{-ip\cdot x + ip'\cdot y}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{ip(x-y)} \equiv D(x-y),$$ the propagator

For space-like separations $(x-y)^2 < 0$, one can show that it decays as $D(x-y) \sim e^{-m|\mathbf{x}-\mathbf{y}|}$ $\neq 0$!



The quantum field leaks out of the light cone. But we've seen that space-like measurements commute!

$$\Delta(x-y) = [\phi(\mathbf{x}), \phi(\mathbf{y})] = D(x-y) - D(y-x)$$
$$= 0 \text{ if } (x-y)^2 < 0.$$

When $(x-y)^2 < 0$, $\exists$ no LI way to order the events. If a part^c can travel in a spacelike direction $\mathbf{x} \to \mathbf{y}$, it can just as easily go $\mathbf{y} \to \mathbf{x}$. In a measurement, these two amplitudes cancel.

With a $\mathbb{C}$ scalar: $[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})] = 0$ outside the light cone. The interpretation now is that the amplitude for the part^c to propagate $x \to y$ cancels the amplitude for the anti-part^c going $y \to x$. ($\mathbb{R}$ scalar ϕ part^c = anti-part^c)

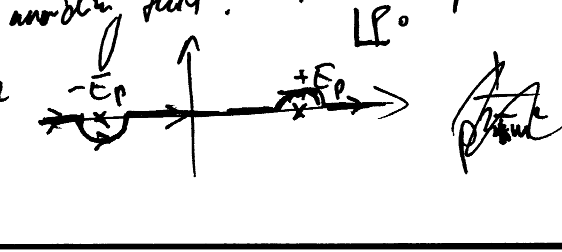
Feynman Propagator

$$\Delta_F(x-y) = \langle 0|\underset{\uparrow \text{ time ordering op}}{T}\,\phi(x)\phi(y)|0\rangle = \begin{cases} \langle 0|\phi(x)\phi(y)|0\rangle & \text{if } x^0 > y^0 \\ \langle 0|\phi(y)\phi(x)|0\rangle & \text{if } y^0 > x^0 \end{cases}$$

Claim $$\Delta_F = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2}e^{-ip\cdot(x-y)}$$ Ill defined since for each $\mathbf{p}$ the integrand over p^0 has a pole at $(p^0)^2 = \mathbf{p}^2 + m^2$.

Need a prescription for avoiding these.

We def^n the contour to be



$$\frac{1}{p^2-m^2} = \frac{1}{p^{02}-E_p^2} = \frac{1}{(p^0-E_p)(p^0+E_p)}$$

So the residue of the pole at $p^0 = \pm E_p$ is $\pm \frac{1}{2E_p}$.

When $x^0 > y^0$, we close the contour in the lower half plane $p^0 \to -i\infty$, so

$e^{-p^0(x^0-y^0)} \to e^{-\infty} \to 0$. Then $\int dp^0$ picks up residue at $p^0 = E_p$.

$$\text{So } \Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4} \frac{(-2\pi i)}{2E_p}\, i\, e^{-iE_p(x^0-y^0) + i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

When $y^0 > x^0$, we close the contour in the upper half plane $p^0 \to +i\infty$, so

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4} \frac{2\pi i}{(-2E_p)}\, i\, e^{iE_p(x^0-y^0) + i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

flip sign

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(y-x)}$$

Instead of specifying the contour,

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon}$$

$\varepsilon > 0$ infinitesimal