$$S_{\Lambda_0}[\phi] = \int d^dx \; \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \sum_i \Lambda_0^{d-d_i}\, g_i(\Lambda_0)\, \mathcal{O}_i(\phi, \partial\phi)$$

where $[\mathcal{O}_i] = d_i$. We now define a regularised path integral by

$$\mathcal{Z}(\Lambda_0, g_i(\Lambda_0)) := \int_{C^\infty(M)_{\leq \Lambda_0}} D\phi \; e^{-S_{\Lambda_0}[\phi]/\hbar}$$

where $C^\infty(M)_{\leq\Lambda_0}$ is the space of functions on M that can be written as a sum of eigenvalues of the Laplacian w/ eigenvalue $\leq \Lambda_0^2$ (i.e. modes of "energy" $\leq \Lambda_0$).

$\Big\lceil$ If M is compact, then $C^\infty(M)_{\leq\Lambda_0}$ is finite dimensional and the measure $D\phi$ really exists,

e.g. $M = T^d$ of period L $\Rightarrow$ $\phi(x) = \sum_{\underline{n}\in\mathbb{Z}^d} \hat{\phi}_{\underline{n}}\, e^{\frac{2\pi i}{L}\underline{n}\cdot x}$

and we integrate over modes where $\underline{n}\cdot\underline{n} \leq \Lambda_0^2 \left(\frac{L}{2\pi}\right)^2$.

If M is non-compact (e.g. $M = \mathbb{R}^d$) there are further subtleties here due to infra-red divergences: we'll mostly ignore these and pretend we're working on $\mathbb{R}^d$. $\Big\rfloor$

<u>Integrating out modes</u>

We split $\varphi(x)$ into "low" and "high" energy modes as

$$\varphi(x) = \int_{|p|\leq\Lambda_0} \frac{d^dp}{(2\pi)^d}\, \tilde{\varphi}(p)\, e^{ip\cdot x} = \int_{0\leq|p|<\Lambda} \frac{d^dp}{(2\pi)^d}\, \tilde{\varphi}(p)\, e^{ip\cdot x} + \int_{\Lambda\leq|p|\leq\Lambda_0} \frac{d^dp}{(2\pi)^d}\, \tilde{\varphi}(p)\, e^{ip\cdot x}$$

for some $\Lambda < \Lambda_0$

$=: \phi(x) + \chi(x)$ respectively.

Let's consider the effective theory we obtain by integrating out χ. We define

$$S_\Lambda[\phi] := -\hbar \log\left[\int_{C^\infty(M)_{\Lambda<|p|\leq\Lambda_0}} D\chi \; e^{-S_{\Lambda_0}[\phi,\chi]/\hbar}\right] \qquad (*)$$

to be the scale Λ effective action. Note: $D\varphi = D\phi\, D\chi$.

We can iterate this procedure, picking a new scale $\Lambda' < \Lambda$ and similarly defining $S_{\Lambda'}[\phi']$ for the modes with $|p| \leq \Lambda'$. For this reason,

$(*)$ is known as the (Wilsonian) <u>renormalization group equation</u> for the effective action.

We also write

$$S_\Lambda[\phi] = \frac{1}{2}\int_M d^dx\ \frac{Z_\Lambda}{2}(\partial\phi)^2 + (m^2+\delta m^2)\frac{Z_\Lambda\phi^2}{2} + \sum_i \Lambda^{d-d_i} g_i(\Lambda)\, \mathcal{O}_i(\phi,\partial\phi)$$

for our generic scale Λ effective action.

Here, $g_i(\Lambda)$ are the shifted couplings, with contributions both from the original action, and from the quantum corrections obtained from χ.

N.B. if (by some miracle) a certain coupling receives <u>no</u> new corrections then

$$g_i(\Lambda) = \left(\frac{\Lambda_0}{\Lambda}\right)^{d-d_i} g_i(\Lambda_0) .$$

The factors Z_Λ, δm^2 account for the fact that there could be new contributions to the kinetic term for ϕ. Z_Λ is called the <u>wavefunction renormalization</u>.

We have

$$\mathcal{Z}(\Lambda_0, g_i(\Lambda_0)) = \int_{C^\infty(M)_{\leq\Lambda_0}} \mathcal{D}\varphi\, e^{-S_{\Lambda_0}[\varphi]/\hbar} = \int_{C^\infty(M)_{\leq\Lambda}} \mathcal{D}\phi\, e^{-S_\Lambda[\phi]/\hbar} = \mathcal{Z}(\Lambda, g_i(\Lambda))$$

"all" modes "low-energy" modes

provided $S_\Lambda[\phi]$ is obtained as in $(*)$. Hence partition f^n is independent of Λ:

$$\Lambda\frac{d\mathcal{Z}(\Lambda, g_i(\Lambda))}{d\Lambda} = \Lambda\left.\frac{\partial\mathcal{Z}}{\partial\Lambda}\right|_{g_i} + \left.\frac{\partial\mathcal{Z}}{\partial g_i}\right|_\Lambda \Lambda\frac{\partial g_i}{\partial\Lambda} = 0 \qquad (**)$$

which says that any explicit dependence of $\int_{\leq\Lambda} \mathcal{D}\phi\, e^{-S[\phi]/\hbar}$ on Λ is compensated by the Λ dependence of the couplings (+ wave f^n ren. + mass ren.)

$(**)$ is called the <u>Callan-Symanzik eq^n</u> for the partition f^n.

We let $\beta_i(g_j) := \Lambda\frac{\partial g_i}{\partial\Lambda}$ be the <u>beta-function</u> of the coupling g_i.

Generically, $\beta_i(g_j) = (d_i - d)\, g_i + \beta_i^{\text{quantum}}(g_j)$

where β_i^{quantum} accounts for the shift in couplings from the $\mathcal{D}\chi$ integral.

We also define the <u>anomalous dimension</u> of ϕ by

$$\gamma_\phi := -\frac{1}{2}\Lambda\, \frac{\partial \ln Z_\Lambda}{\partial\Lambda}$$

wave f^n renormalization factor

Of course, at any given scale we can absorb Z_Λ (say) by defining a new field

$\varphi(x) = \sqrt{Z_\Lambda}\,\phi$ so as to give $\varphi(x)$ canonically normalised terms.

Correlation functions & Anomalous dimension

Suppose we wish to compute

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int_{C^\infty(M)_{s\Lambda}} D\phi\, e^{-S_\Lambda[Z_\Lambda^{1/2}\phi\,,\,g_i(\Lambda)]}\,\phi(x_1)\cdots\phi(x_n)$$

In terms of $\varphi(x) = Z_\Lambda^{1/2}\phi(x)$, this is

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = Z_\Lambda^{-n/2}\langle \varphi(x_1)\cdots\varphi(x_n)\rangle$$

which gives some function $\Gamma_\Lambda^{(n)}(\{x_i\}, g_i, Z_\Lambda)$.

Now suppose $s<1$, and that we have chosen to insert fields only with energies below $s\Lambda < \Lambda$. Then we should equally be able to compute the correlator using $S_{s\Lambda}$. We find

$$Z_{s\Lambda}^{-n/2}\,\Gamma_{s\Lambda}^{(n)}(x_1,\dots,x_n;\, g_i(s\Lambda)) = Z_\Lambda^{-n/2}\,\Gamma_\Lambda^{(n)}(x_1,\dots,x_n;\, g_i(\Lambda))$$

Infinitesimally, this is

$$\Lambda\frac{d}{d\Lambda}\Gamma_\Lambda^{(n)}(\{x_i\};g_i(\Lambda)) = \left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma_\phi\right)\Gamma_\Lambda^{(n)}(\{x_i\};g_i(\Lambda)) = 0$$

This is the Callan-Symanzik eq^n for the correlation function.