Correlation functions

$$\langle y_1| e^{-HT}\, \mathcal{O}(t)\, |y_0\rangle = \int_{C_{[0,T]}[y_1,y_0]} Dx\; \mathcal{O}(x(t))\, e^{-S[x]}$$

For several operators, if $T > t_n > t_{n-1} > \dots > t_1 > 0$, then by the same argument we have

$$\langle y_1| e^{-HT}\, \hat{\mathcal{O}}(t_n) \dots \hat{\mathcal{O}}(t_1)\, |y_0\rangle = \int_{C_{[0,T]}[y_0,y_1]} Dx\; \mathcal{O}_n(x(t_n)) \dots \mathcal{O}_1(x(t_1))\, e^{-S[x]}$$

Note that while the $\hat{\mathcal{O}}$'s are operators in the canonical picture, the objects in the path integral (i.e. $\mathcal{O}_i(x(t_i))$) are just <u>functions</u>.

In particular, if $\{t_i\} \in (0,T)$ are a collection of times, then

$$\int Dx \prod_{i=1}^{n} \mathcal{O}_i(x(t_i))\, e^{-S[x]} = \langle y_1| e^{-HT}\, \mathcal{T}\Big\{\prod_{i=1}^{n} \hat{\mathcal{O}}(t_i)\Big\} |y_0\rangle$$

where $\mathcal{T}\{\hat{\mathcal{O}}_1(t_1)\, \hat{\mathcal{O}}_2(t_2)\} := \Theta(t_2 - t_1)\, \hat{\mathcal{O}}_2(t_2)\, \hat{\mathcal{O}}(t_1) + \Theta(t_1 - t_2)\, \hat{\mathcal{O}}_1(t_1)\, \hat{\mathcal{O}}(t_2)$ etc.

This is because in between t_i, t_j we evolve using $e^{-H(t_j - t_i)}$ which requires $t_j \geq t_i$.

<u>Remark</u> Note that we only get non-trivial correlation between operators at different t's because of the kinetic (or derivative) terms in $S[x]$. In the discretized version, we had

$$S_{kin}[x] = \sum_i \frac{1}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 \Delta t \qquad \text{(at least for target } \mathbb{R}^n\text{)}.$$

If the action had instead been purely $S_{pot}[x] = \sum_i V(x_i)$ (with no $x_{i+1} x_i$ terms) then the discretized integral could have factorized into a product of independent integrals over the x_i's.

$$\Rightarrow \langle \mathcal{O}_1(x(t_1))\, \mathcal{O}_2(x(t_2))\rangle = \langle \mathcal{O}_1(x(t_1))\rangle \langle \mathcal{O}_2(x(t_2))\rangle .$$

This is the same as we see perturbatively, where the kinetic term gives rise to <u>propagators</u>.

A more general class of local insertions would involve $\mathcal{O}(x(t), \dot{x}(t), \dots)$ and so we'd expect $\int Dx\; \mathcal{O}_1(x, \dot{x}, \dots)\big|_{t_1} \mathcal{O}_2(x, \dot{x}, \dots)\big|_{t_2} e^{-S[x]}$ to be related to the corresponding operator expression.

In basic QM, we'd have said (e.g. $S = \frac{1}{2}\int \dot{x}^2\, dt$) $p_a = \dot{x}_a$ and $[\hat{x}^a, \hat{p}_b] = \delta^a_b$.

How can we see this non-commutativity from the path integral?

The path integral measure

Last lecture we naively took $Dx \overset{?}{:=} \lim_{N\to\infty} \frac{1}{(2\pi \Delta t)^{nN/2}} \prod_{i=1}^{N} d^n x_i\; a(x_i)$ for curved target

and also $S_{free}[x] \overset{?}{=} \lim_{N\to\infty} \sum_{n=1}^{N-1} \frac{1}{2}\left(\frac{x_{n+1} - x_n}{\Delta t}\right)^2 \Delta t$ where $\Delta t = T/N$.

Before taking $N \to \infty$, the rhs here define a regularized path integral. (In higher dimensions, this is a lattice regularization.) Alternatively, we could decompose our field as

$$x^a(t) = \sum_{k\in\mathbb{Z}} x_k^a \, e^{2\pi i k t/T}$$ and then let

$$S_N[x] = \sum_{k=-N}^{N} \frac{1}{2} k^2 x_k^a x_{-k}^a$$ be a regularized action and

$D_N x := \prod_{|k|\le N} d^n x_k$ be the regularized path integral measure.

(this is analogous to a "high-energy cut-off".)

Q: Does the $\lim_{N\to\infty} S_N[x]$ or $\lim_{N\to\infty} D_N x$ <u>exist</u>? A: <u>No!</u>

Proof (sketch):

A Lebesgue measure $d\mu$ on a vector space $\mathbb{R}^D$ obeys:

- For all open subsets (non-empty) $U \subset \mathbb{R}^D$, $\mathrm{vol}(U) = \int_U d\mu > 0$
- If U' is obtained by translation of U, $\mathrm{vol}(U') = \mathrm{vol}(U)$
- Every $x \in \mathbb{R}^D$ is contained in at least one open neighborhood $U_x \ni x$ with $\mathrm{vol}(U_x) < \infty$.

Claim: $\nexists$ no non-trivial Lebesgue measure on an infinite dim vector space

Let $C(L)$ be an open hypercube of side length L. Then on $\mathbb{R}^D$ $C(L)$ contains 2^D hypercubes of length $L/2$

$$\mathrm{vol}(C(L)) \geq \sum_{i=1}^{2^D} \mathrm{vol}(C_i(L/2)) = 2^D \mathrm{vol}(C(L/2))$$

so as $D\to\infty$, we must have $\mathrm{vol}\, C(L/2) \to 0$ for any finite L. $\square$

<u>Non-commutativity in QM</u>

Discretization of the path integral plays an important role in $[\hat{x},\hat{p}] \neq 0$.

Let $T > t_+ > t > t_- > 0$ and consider (with $S = \frac{1}{2}\int \dot{x}^2 dt$)

$$\int Dx \; x(t)\dot{x}(t_-) e^{-S} = \langle y_1 | e^{-H(T-t)} \hat{x} \, e^{-H(t-t_-)} \hat{p} \, e^{-Ht_-} | y_0 \rangle$$

$$\int Dx \; x(t)\dot{x}(t_+) e^{-S} = \langle y_1 | e^{-H(T-t_+)} \hat{p} \, e^{-H(t_+-t)} \hat{x} \, e^{-Ht} | y_0 \rangle$$

As we take the limit $\lim t_\pm \to t$ difference of rhs becomes

$$\langle y_1 | e^{-H(T-t)} [\hat{x},\hat{p}] e^{-Ht} | y_0 \rangle = \langle y_1 | e^{-HT} | y_0 \rangle \quad (\neq 0)$$

On the other hand, in the continuum path integral, limit seems to give same expression, so diff $=0$.

We <u>need</u> to regularize. Let's regularize by replacing

$$\lim_{\substack{t_-\to t \\ t_+\to t}} [x(t)\dot{x}(t_-) - x(t)\dot{x}(t_+)] = x_t\left(\frac{x_t - x_{t-\Delta t}}{\Delta t}\right) - x_t\left(\frac{x_{t+\Delta t} - x_t}{\Delta t}\right)$$

so we've stopped taking limit as soon as any part of the discretized derivative touches x_t.

The only x_t dependence is $\int d^n x_t \, K_{\Delta t}(x_{t+\Delta t}, x_t)\, x_t \left[\frac{x_t - x_{t-\Delta t}}{\Delta t} - \frac{x_{t+\Delta t} - x_t}{\Delta t}\right] K_{\Delta t}(x_t, x_{t-\Delta t})$

$= -\int d^n x_t \; x_t \frac{\partial}{\partial t}\left[K_{\Delta t}(x_{t+\Delta t}, x_t)\, K_{\Delta t}(x_t, x_{t-\Delta t})\right]$ since $K_{\Delta t}(x_t, x_{t-\Delta t}) \sim \exp\left(-\frac{(x_t - x_{t-\Delta t})^2}{2\Delta t}\right)$

$= \int d^n x_t \, K_{\Delta t}(x_{t+\Delta t}, x_t)\, K_{\Delta t}(x_t, x_{t-\Delta t}) = K_{2\Delta t}(x_{t+\Delta t}, x_{t-\Delta t})$ so we get the same as in the operator approach.