Strong Cosmic Censorship (Penrose)

Let (Σ, h_{ab}, K_{ab}) be a geodesically complete asymptotically flat (with N ends) initial data for the vacuum Einstein equations. Then generically the maximal Cauchy development of the initial data is inextendible.

Null hypersurfaces

Def A null hypersurface is a hypersurface whose normal is everywhere null.

Example: Consider a const r surface in Schwarzschild in ingoing E.F. The normal $n = dr$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & & \\ 1 & 1-\frac{2M}{r} & & \\ & & 1/r^2 & \\ & & & 1/r^2 \sin^2\theta \end{pmatrix} \qquad n^2 = \left(1 - \frac{2M}{r}\right) \Rightarrow r = 2M, \; n^2 = 0$$

Let us look at $n^a|_{r=2M} = \left(\frac{\partial}{\partial v}\right)^a$. Let n_a be normal to a null hyp. $\mathcal{N}$. Then any (non-zero) vector X^a tangent to $\mathcal{N}$: $n_a X^a = 0$. which implies that either X^a is spacelike or X^a is parallel to n^a. In particular, note that n^a is tangent to the hypersurface. The integral curves ~~of~~ of n^a lie on $\mathcal{N}$.

Proposition: The integral curves of n^a are null geodesics. These are called the generators of $\mathcal{N}$.

Proof: Let $\mathcal{N}$ be given by an equation $f = $ const. for some fn f with $df \neq 0$ on $\mathcal{N}$. then we have $n = h\, df$ for some function h. Let $N = df$ (the integral curves of N are the same - up to reparametrisation - as n). Everywhere on $\mathcal{N}$: $N_a N^a = 0$

Since $N^a N_a$ is constant on $\mathcal{N}$:

$$\nabla_a (N_b N^b)\big|_{\mathcal{N}} = 2\alpha N_a \Rightarrow N^b \nabla_a N_b = \alpha N_a \qquad \text{but}$$

$$\nabla_a N_b = \nabla_a \nabla_b f = \nabla_b \nabla_a f = \nabla_b N_a \quad \text{which gives} \quad N^b \nabla_b N_a = \alpha N_a \qquad \square$$

Example: Take the Kruskal spacetime. Let $N = dU$ which is null everywhere ($g^{UU} = 0$) and normal to a family of null hypersurfaces ($U = $ const):

$$N^b \nabla_b N_a = N^b \nabla_b \nabla_a U = N^b \nabla_a \nabla_b U = N^b \nabla_a N_b = \frac{1}{2} \nabla_a (N^b N_b) = 0$$

and in this case N^a is tangent to ~~a~~ an affinely parametrised null geodesic.

Raising an index

$$N^a = -\frac{r}{16M^2} e^{\frac{r}{2M}} \left(\frac{\partial}{\partial V}\right)^a$$

Now let $\mathcal{N}$ be the surface $U = 0$ $(r = 2M)$ on $\mathcal{N}$. In that case $N^a \propto \left(\frac{\partial}{\partial V}\right)^a$

So V is an affinely parameter for these null geodesics

<u>Geodesic deviation</u>

<u>Def</u> A 1-parameter family of geodesics is a map:

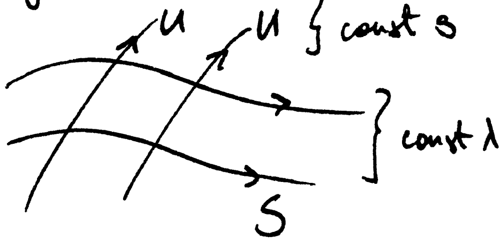
$$\gamma : I \times I' \longrightarrow \mathcal{M}$$

where I, I' are open intervals in $\mathbb{R}$.

(i) for fixed s, $\gamma(s,\lambda)$ is geodesic with affine parameter λ

(ii) the map $(s,\lambda) \longmapsto \gamma(s,\lambda)$ is smooth and one-to-one with a smooth inverse.

This implies that the family of geodesics defines a surface $\Sigma \subset \mathcal{M}$. Let U^a be the tangent vector to the geodesics and S^a be the vector tangent to the curves of constant λ.



In a chart x^μ, the geodesics are determined by

$$x^\mu(s,\lambda) \quad \text{with } S^\mu = \frac{dx^\mu}{ds}$$

$$x^\mu(s+\delta s, \lambda) = x^\mu(s,\lambda) + \delta s\, S^\mu + O(\delta s^2)$$

Therefore, $(\delta s, S^a)$ points from one geodesic to another. S^a is a deviation vector.

On the surface Σ we can use s, λ as coordinates

$$S = \frac{\partial}{\partial s}, \quad U = \frac{\partial}{\partial \lambda} \quad \text{on } \Sigma$$

On Σ, $0 = [S,U] \iff U^b \nabla_b S^a = S^b \nabla_b U^a$

This implies $U^c \nabla_c (U^b \nabla_b S^a) = R^a{}_{bcd} U^b U^c S^d \quad (*)$

Given an affinely parametrized geodesic γ with tangent U^a, a solution to $(*)$ is called a Jacobi field

<u>Geodesic congruence</u>

Let $U \subset \mathcal{M}$ be open. A geodesic congruence in U is a family of geodesics s.t. exactly <u>one</u> geodesic passes through $p \in U$.

Consider congruences for which all geodesics are of the same type. Then can choose $U^a U_a = \pm 1$ (in the spacelike or timelike case) and $U^a U_a = 0$ in the null case.

Consider a one parameter family of geodesics that belong to a congruence.

$$[S,U] = 0 \iff U^b \nabla_b S^a = B^a{}_b S^b \quad \text{where } B^a{}_b \equiv \nabla_b U^a .$$