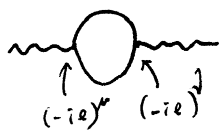
$$S[\psi, A] = \int d^dx \; \bar{\psi} (\not{D} + m) \psi$$



$$\not{D} = \gamma^\mu (\partial_\mu + ie A_\mu)$$

↑ canonically normalised

We are expanding $e^{-S[\psi, A]}$

$$\rightarrow -(-ie)^2 \int \mathrm{tr}\left(\gamma^\mu \frac{1}{i\not{D} + m} \gamma^\nu \frac{1}{i\not{D} + m}\right) \frac{d^dp}{(2\pi)^d}$$

↑ fermion around the loop

There are two (equivalent) ways to see where this extra minus sign comes from.

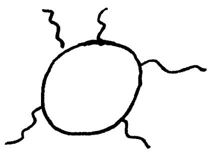
① Suppose $\langle \psi(x) \bar{\psi}(y) \rangle = S(x, y)$ is the Dirac propagator in position space

Then $\langle \psi(x) \bar{\psi}(y) \rangle = -\langle \bar{\psi}(y) \psi(x) \rangle$ $\cancel{= -S(y,x)}$ because $\bar{\psi}, \psi$ anticommuting.

Therefore in position space we'd get a term $(-ie)^2 \int d^dx \; \bar{\psi} \not{A} \psi(x) \int d^dy \; \bar{\psi} \not{A} \psi(y)$ from expanding the fermion interactions. Joining the electron fields with propagators and one of these terms is out of order $\Rightarrow$ we get a minus sign.

More generally

$$\bar{\psi} \not{A} \psi(x_1) \ldots \bar{\psi} \not{A} \psi(x_2) \ldots \bar{\psi} \not{A} \psi(x_n)$$



② Alternatively, if we perform the path integral over electrons, we get

$$\int DA\, D\bar{\psi}\, D\psi \; e^{-S[A, \bar{\psi}, \psi]} = \int DA \; \det(\not{D} + m) \; e^{-\frac{1}{4}\int F^2 dx}$$

$$= \int DA \; e^{-S_{eff}[A]}$$ where $S_{eff}[A] = \frac{1}{4}\int F^2 d^dx - \ln \det(\not{D} + m)$

$= \frac{1}{4}\int F^2 d^dx - \mathrm{tr} \ln(\not{D} + m)$

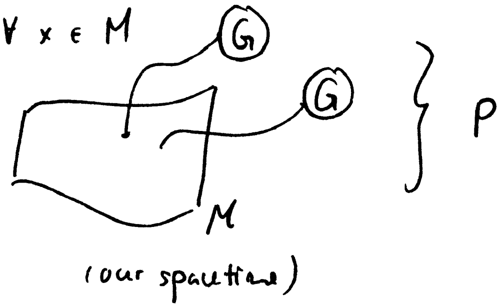
if bosons, would get $1/\det$ instead

and perturbatively, we get $\ln(\not{D} + m) = \ln(\not{\partial} + m + ie\not{A})$

$$= \ln(\not{\partial} + m) + \sum_{n=1}^{\infty} \frac{1}{n} \left(ie\not{A} \frac{1}{\not{\partial} + m} \, ie\not{A} \frac{1}{\not{\partial} + m} \cdots \frac{1}{\not{\partial} + m} \right)$$

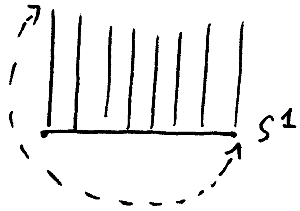
Non-Abelian Gauge Theories

Non-Abelian gauge theories are based on saying our world is a <u>principal G bundle</u>. This is a manifold P together with a projection $\pi : P \to M$ for some other manifold M, where $\pi^{-1}(x) \cong G$ (a Lie group) for $\forall x \in M$



G is often called the <u>structure group</u> or the <u>gauge group</u>.
(e.g. Maxwell theory has $G = U(1)$, while SM has $G = SU(3) \times SU(2) \times U(1)$)
There's a right action of $G : P \to P$ which preserves the fibres, i.e. given $g \in G$, we have $g : p \to pg$ where $\pi(pg) = \pi(p)$

The simplest examples are to take $M = S^1$ and $G = \mathbb{R}$. Then we have two possible $\mathbb{R}$-bundles: the cylinder and the moibius strip

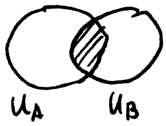


For an open set $U \subset M$, we have a local trivialisation Φ

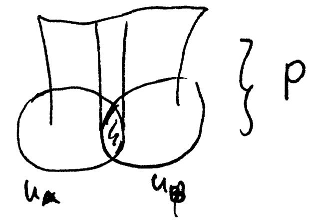
$$\Phi : P|_{\pi^{-1}(U)} \cong U \times G$$

Suppose $\{U_\alpha\}$ are a collection of open sets and we are given trivialisations Φ_α on each U_α. We have that $\Phi_\alpha : p \longrightarrow (x, \phi_\alpha(p))$ and similarly
$\uparrow$ group element

$\Phi_\beta : p' \longrightarrow (x', \phi_\beta(p'))$ Now suppose U_α and U_β overlap



then on the intersection we can compare the two trivialisations.



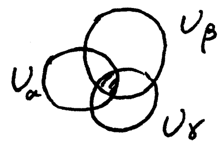
Since $\Phi_\alpha(p), \Phi_\beta(p) \in G$, we must have

$$\Phi_\beta(p) = \Phi_\alpha(p)\, t_{\alpha\beta} \quad \text{for some } t_{\alpha\beta} \in G.$$

If we want to compare these trivialisations throughout the overlapping set, $t_{\alpha\beta}$ may need to vary with $x \in U_\alpha \cap U_\beta$. We define a <u>transition f^n</u>

$$T_{\alpha\beta} : U_\alpha \cap U_\beta \to G \qquad \text{that obeys} \quad t_{\alpha\beta}^{-1}(x) = t_{\beta\alpha}(x) \quad \forall\, x \in U_\alpha \cap U_\beta$$
$$x \to t_{\alpha\beta}(x)$$

and also $t_{\alpha\beta}(x)\, t_{\beta\gamma}(x) = t_{\alpha\gamma}(x) \quad \forall\, x \in U_\alpha \cap U_\beta \cap U_\gamma$



In physics, the most application is when $M \cong \mathbb{R}^n$ and all the U_α are the whole space, so we are just comparing different trivialisations of $\pi^{-1}(\mathbb{R}^n) \cong \mathbb{R}^n \times G$

- For example, in EM, $M \cong \mathbb{R}^{3,1}$, $G = U(1)$, so our local trivialisations are a choice of <u>gauge</u> and transition f^{n}s are gauge transformations
$$t_{\alpha\beta}(x) = e^{i\lambda_{\alpha\beta}(x)} \qquad \left(\psi(x) \to e^{i\lambda_{\alpha\beta}(x)} \psi(x) \right)$$
- In GR, we take $G = GL(n, \mathbb{R})$ if $\dim M = n$. Then transition f^{n}s allow us to change coordinates $t_{\alpha\beta}(x) = \dfrac{\partial y^\nu(x)}{\partial x^\mu(x)}$

Vector bundles

To a Lie group G we often want to study its representations. A <u>representation</u> is a map $\rho : G \to \mathrm{Mat}(r, \mathbb{C})$ where $\rho(gh) = \rho(g)\rho(h)$

Picking a repn for a principle G bundle gives a vector bundle of rank r. This is $E \xrightarrow{\pi} M$ s.t. we have local trivialisation $\Phi : E|_{\pi^{-1}(U)} \to U \times \mathbb{C}^r$ and transition f^n $T_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Mat}(r, \mathbb{C})$

If the original P has structure group $G \subset GL(r, \mathbb{C})$ then the transition f^{n}s preserve some extra structure.

If $U(r) = G$, then the transition f^{n} preserve the inner product

$$\langle z_1, z_2 \rangle = \sum_i \bar{z}_1^i z_2^i$$

If $G = SU(r)$, then the transition f^{n} also have unit det.