Def^n A covariant derivative of a v. field Y is a $(1,1)$ tensor ∇Y s.t.

$$(\nabla Y)(X) = \nabla_X Y \in T_p M$$

Write $(\nabla Y)^a{}_b = \nabla_b Y^a = Y^a{}_{;b}$

$$(\nabla_X Y)^a = X^b \nabla_b Y^a = X^b Y^a{}_{;b}$$

Pick a basis $\{e_\mu\}$ of T_pM and consider $\nabla_{e_\rho} e_\nu = \Gamma^\mu{}_{\nu\rho} e_\mu$

Notation $\nabla_{e_\rho} e_\nu = \nabla_\rho e_\nu$

Calculate $\nabla_X Y = \nabla_X (Y^\mu e_\mu) = X(Y^\mu) e_\mu + Y^\mu \nabla_X(e_\mu)$

$$= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu X^\nu \underbrace{\nabla_\nu(e_\mu)}_{\Gamma^\rho{}_{\mu\nu} e_\rho}$$

$$= X^\nu \left(e_\nu (Y^\mu) + Y^\rho \Gamma^\mu{}_{\rho\nu}\right) e_\mu$$

$$\Rightarrow (\nabla_X Y)^\mu = X^\nu \left(e_\nu(Y^\mu) + Y^\rho \Gamma^\mu{}_{\rho\nu}\right)$$

In coord basis: $Y^\mu{}_{;\nu} = \partial_\nu Y^\mu + \Gamma^\mu{}_{\rho\nu} Y^\rho$

∇ for general tensors given by Leibniz rule;

$$T(r,s) \longrightarrow \nabla T\ (r, s+1)$$

e.g. for 1-form η: $(\nabla_X \eta)(Y) = \nabla_X(\eta(Y)) - \eta(\nabla_X Y)$

$= X(\eta_\mu Y^\mu) - \eta_\mu (\nabla_X Y)^\mu$ Exercise: $\eta_{\mu;\nu} = e_\nu(\eta_\mu) - \Gamma^\rho{}_{\mu\nu}\eta_\rho$

Extend to general bases (w/ $e_\nu = \partial_{x^\nu}$)

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s;\sigma} = \partial_\sigma T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \sum_{j=1}^{r} \Gamma^{\mu_j}{}_{\sigma\rho} T^{\mu_1\cdots\mu_{j-1}\rho\mu_{j+1}\cdots\mu_r}{}_{\nu_1\cdots\nu_s}$$

$$- \sum_{j=1}^{s} \Gamma^\rho{}_{\nu_j\sigma} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_{j-1}\rho\nu_{j+1}\cdots\nu_s}$$

Notation: $f_{,\mu} = \partial_\mu f$ $\quad f_{,[\mu\nu]} = 0$

$$X^a{}_{;b} = \nabla_b X^a \quad f_{;[\mu\nu]} = -\Gamma^\rho{}_{[\mu\nu]} f_{,\rho}$$

Def^n ∇ is torsion free if $\nabla_{[a b]} f = 0 \quad \forall f: \mathcal{M} \to \mathbb{R}$ equiv. if $\Gamma^\rho{}_{[\mu\nu]} = 0$ in any coord basis.

Lemma X, Y v. fields; ∇ torsion-free $\Rightarrow \nabla_X Y - \nabla_Y X = [X, Y]$

Proof Tensor eq.: prove it in a coord basis

$$X^\nu Y^\mu{}_{;\nu} - Y^\nu X^\mu{}_{;\nu} = [X,Y]^\mu + 2\Gamma^\mu{}_{[\rho\nu]} X^\nu Y^\rho = [X,Y]^\mu.$$

<u>Thm</u> (Fundamental thm of Riemannian Geometry) : Given a (pseudo) Riemannian mfd (M, g) $\exists$ unique torsion-free connection ∇ s.t. $\nabla g = 0$ (called the Levi-Civita connection).

<u>Proof</u> X, Y, Z v.fields

$X(g(Y,Z)) = \nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
defof X(func) , $\nabla_X g = 0$ and Leibniz rule .

$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$. $Z(\dots) = \dots$

(1) + (2) − (3) : $X(\dots) + Y(\dots) - Z(\dots) = g(\nabla_X Y + \nabla_Y X, Z) - g(\nabla_Z X - \nabla_X Z, Y)$
$+ g(\nabla_Y Z - \nabla_Z Y, X) = 2g(\nabla_X Y, Z) - g([X,Y], Z) - g([Z,X], Y) + g([Y,Z], X)$
by Lemma above

This uniquely determines $\nabla_X Y$ (as g invertible)

$g(\nabla_X Y, Z) = \frac{1}{2}(\text{LHS} + g(\dots) + g(\dots) - g(\dots))$ □

In coord basis $e_\mu = \partial_{x^\mu}$, $[e_\mu, e_\nu] = 0$

$g(\nabla_\rho e_\nu, e_\sigma) = \frac{1}{2}(e_\rho(g_{\nu\sigma}) + e_\nu(g_{\sigma\rho}) - e_\sigma(g_{\rho\nu}))$

LHS $= g(\Gamma^\tau{}_{\nu\rho} e_\tau, e_\sigma) = \Gamma^\tau{}_{\nu\rho}\, g(e_\tau, e_\sigma) = \Gamma^\tau{}_{\nu\rho}\, g_{\tau\sigma}$

Contract with $g^{\mu\sigma}$: $g_{\tau\sigma} g^{\mu\sigma} = \delta_\tau{}^\mu$

$\Rightarrow \Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma})$

Same Christoffel symbols as geodesic eqns.

$d^2_{\tau^2} X^\mu + \Gamma^\mu{}_{\rho\nu} \dot{X}^\nu \dot{X}^\rho = 0$

$d^2_{\tau^2} X^\mu = d_\tau X^\mu(X(\tau)) = d_\tau X^\nu \frac{\partial X^\mu}{\partial X^\nu} = X^\nu X^\mu{}_{,\nu}$

so $X^\nu(X^\mu{}_{,\nu} + \Gamma^\mu{}_{\nu\rho} X^\rho) = 0 \Rightarrow \nabla_X X = 0$