CW basis

$$B_{CW} = \{H^1, H^2, E^{\pm\alpha}, E^{\pm\beta}, E^{\pm(\alpha+\beta)}\}$$

$$\dim[A^2] = 2+6 = 8 = \dim(\mathcal{L}_{\mathbb{C}}(SU(2)))$$

Representations of g

Let R be a repn of g of dim $N \in \mathbb{N}$

$$\begin{matrix} H^i \\ E^\alpha \end{matrix} \xrightarrow{R} \begin{matrix} R(H^i) \\ R(E^\alpha) \end{matrix} \in Mat_N(\mathbb{C})$$

- Assume $R(H^i)$ are diagonalisable

$$[R(H^i), R(H^j)] = R([H^i, H^j]) = 0 \quad \forall i,j$$

$\Rightarrow \{H^i\}$ are simultaneously diagonalisable

$\Rightarrow V \equiv \mathbb{C}^N$ spanned by the simultaneous eigenvectors of $\{R(H^i)\}$

$$V = \bigoplus_{\lambda \in S_R} V_\lambda \quad \text{where } \forall v \in V_\lambda \quad R(H^i) v = \lambda^i v$$

- Eigenvalue $\lambda \in h^*$ is a weight of R

S_R is weight set. The weights have multiplicity $m_\lambda = \dim V_\lambda \geq 1$

- $[H^i, E^\alpha] = \alpha^i E^\alpha \Rightarrow ad_{H^i} E_\alpha = \alpha^i E_\alpha$

$$R_{adj}(H^i) E_\alpha = \alpha^i E_\alpha$$

Roots $\alpha \in \Phi$ are weights of the adjoint repn.

- Consider action of the step operators $R(E^\alpha)$ $\alpha \in \Phi$ on V_λ

$$R(H^i) R(E^\alpha) v = R(E^\alpha) R(H^i) v + [R(H^i), R(E^\alpha)] v$$

$$= R(E^\alpha)(\lambda^i + \alpha^i) v$$

$$R(H^i)\left(R(E^\alpha) v\right) = (\lambda^i + \alpha^i)\left(R(E^\alpha) v\right)$$

Thus $\forall v \in V_\lambda$ we find $R(E^\alpha) v$ has weight $(\lambda^i + \alpha^i)$ if $\lambda \in S_R$, 0 otherwise.

Consider the action of $sl(2)_\alpha$ on V

$$\{R(h^\alpha), R(e^\alpha), R(e^{-\alpha})\}$$

Each generator defn a linear map on V $\Rightarrow$ V is repn space for some repn R_α of $sl(2)_\alpha$

Reconstruction

Cartan-Weyl basis $\{H^i, E^\alpha\}$

- Cartan matrices determine simple roots

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})} = \frac{2|\alpha_{(i)}|}{|\alpha_{(j)}|} \cos\varphi_{ij}$$

- Root strings

$$l_{ij} = 1 - A_{ji} \in \mathbb{N}$$

Example: $g = A_2$ o—o

Cartan matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

given two simple roots α and $\beta \in \Phi$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\beta,\alpha)}{(\beta,\beta)} = -1 \Rightarrow \quad |\alpha| = |\beta|, \quad \cos\varphi_{\alpha\beta} = \frac{2\pi}{3}$$

$\alpha, \beta \in \Phi_S \qquad \pm(\alpha - \beta) \notin \Phi$

length of root string α thru β $\quad l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2$

$l_{\beta,\alpha} = 2$

$\begin{matrix}\beta + n\alpha \\ \alpha + \tilde{n}\beta\end{matrix} \in \Phi \qquad n, \tilde{n} \in \{0, 1\}$

$\Rightarrow \alpha, \beta, \alpha + \beta \in \Phi$

$-\alpha, -\beta, -\alpha-\beta \in \Phi$ terminates

Root system of A^2

$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$

$(\alpha+\beta, \alpha+\beta) = (\alpha,\alpha) + (\beta,\beta) + 2(\alpha,\beta) = (\alpha,\alpha)(2-1) = (\alpha,\alpha)$

All roots have the same length

$|\alpha| = |\beta| = |\alpha+\beta|$