Symmetries, Fields and Particles
Example sheet 2

1. $g(t) = \mathrm{Exp}(tX) \in G \quad , t \in \mathbb{R} \quad , \quad X \in \mathcal{L}(G)$

$$\mathrm{Exp}(tX) = \mathbb{1} + tX + \frac{1}{2}(tX)^2 + \dots$$

$$g(t_1)g(t_2) = \mathrm{Exp}(t_1 X)\,\mathrm{Exp}(t_2 X) = \mathrm{Exp}(t_2 X)\,\mathrm{Exp}(t_1 X) = g(t_2)g(t_1)$$

as X commutes with itself (and by extension any power of itself).

$$g(t_1)g(t_2) = \mathrm{Exp}(t_1 X)\,\mathrm{Exp}(t_2 X) = \sum_{i=0}^{\infty} \frac{(t_1 X)^i}{i!} \sum_{j=0}^{\infty} \frac{(t_2 X)^j}{j!} = \sum_{i,j} \frac{t_1^i t_2^j X^{i+j}}{i!\,j!}$$

while

$$g(t_1+t_2) = \mathrm{Exp}((t_1+t_2)X) = \sum_i \frac{(t_1+t_2)^i X^i}{i!} = \sum_{i,j\le i} \frac{t_1^j t_2^{i-j} X^i}{i!}\binom{i}{j}$$

$$= \sum_{i,j\le i} \frac{t_1^j t_2^{i-j} X^i \, i!}{i!\,j!\,(i-j)!} = \sum_{i,j} \frac{t_1^i t_2^j X^{i+j}}{i!\,j!} \quad \text{relabelling } \begin{array}{l} j \to i \\ i-j \to j \end{array}$$

$\Rightarrow g(t_1)g(t_2) = g(t_1+t_2)$

Curve defines a subgroup by the above: $t=0$ is identity, $t \to -t$ inverse.

$g: t \longmapsto g(t)$ is injective by definition onto the subgroup

$U = \{g(t) : t \in \mathbb{R}\}$

Show injective:

assume $g(t_1) = g(t_2) \qquad | \cdot g(-t_1)$

$\Rightarrow g(t_1)g(-t_1) = g(t_1 - t_1) = g(0) = e = g(t_2)g(-t_1) = g(t_2 - t_1)$

but $\nexists\, t \neq 0 : g(t) = e \quad \Rightarrow \quad t_2 - t_1 = 0 \quad \Rightarrow \quad t_2 = t_1$

Hence, g is bijective $\Rightarrow (\mathbb{R}, +)$ is isomorphic to the subgroup U.

2. $$\exp X \cdot \exp Y = \left(\mathbb{1} + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 \dots\right)\left(\mathbb{1} + Y + \frac{1}{2}Y^2 + \frac{1}{6}Y^3 + \dots\right)$$

$$= \mathbb{1} + X + Y + XY + \frac{1}{2}X^2 + \frac{1}{2}Y^2 + \frac{1}{2}XY^2 + \frac{1}{2}X^2Y + \frac{1}{4}X^2Y^2 + \dots + \frac{1}{6}X^3 + \frac{1}{6}Y^3 + \dots$$

~~$\exp(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\dots)$~~

$$= \mathbb{1} + (X+Y) + \frac{1}{2}(X+Y)^2 + \frac{1}{2}XY - \frac{1}{2}YX + \frac{1}{2}XY^2 + \frac{1}{2}X^2Y + \frac{1}{4}X^2Y^2 + \dots + \frac{1}{6}X^3 + \frac{1}{6}Y^3 + \dots$$

$$= \mathbb{1} + (X+Y) + \frac{1}{2}(X+Y)^2 + \frac{1}{2}[X,Y] + \frac{1}{6}(X+Y)^3$$

$$+\tfrac{1}{3}X^2Y - \tfrac{1}{6}XYX - \tfrac{1}{6}YX^2 + \tfrac{1}{3}XY^2 - \tfrac{1}{6}YXY - \tfrac{1}{6}Y^2X + \dots$$

$$= \mathbb{1} + (X+Y) + \tfrac{1}{2}(X+Y)^2 + \tfrac{1}{6}(X+Y)^3 + \tfrac{1}{2}[X,Y] + \cancel{\tfrac{1}{6}X[X,Y] + \tfrac{2}{6}X^2Y - \tfrac{2}{6}YX^2}$$

$$\cancel{+\tfrac{1}{6}[X,Y]Y + \tfrac{2}{6}XY^2 - \tfrac{2}{6}Y^2X} + \tfrac{1}{12}X[X,Y] - \tfrac{1}{12}[X,Y]X + \tfrac{1}{4}X^2Y - \tfrac{1}{4}YX^2$$

$$+ \tfrac{1}{12}[X,Y]Y - \tfrac{1}{12}Y[X,Y] + \tfrac{1}{4}XY^2 - \tfrac{1}{4}Y^2X + \dots$$

$$= \mathbb{1} + (X+Y) + \tfrac{1}{2}(X+Y)^2 + \tfrac{1}{6}(X+Y)^3 + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]]$$

$$+\tfrac{1}{4}X^2Y - \tfrac{1}{4}XYX + \tfrac{1}{4}[X,Y]X + \tfrac{1}{4}XY^2 - \tfrac{1}{4}YXY + \tfrac{1}{4}Y[X,Y] + \dots$$

$$= \mathbb{1} + (X+Y) + \tfrac{1}{2}(X+Y)^2 + \tfrac{1}{6}(X+Y)^3 + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]]$$

$$+ \tfrac{1}{4}(X+Y)[X,Y] + \tfrac{1}{4}[X,Y](X+Y) + \dots$$

$$= \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \dots\right)$$

3. $g(t) = \mathrm{Exp}(it\sigma_1)$

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as σ_1 is diagonal

$$g(t) = \mathrm{Exp}(it\sigma_1) = \begin{pmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{pmatrix} \in SU(2)$$

Group manifold of $SU(2)$ is S^3, this subgroup is isomorphic to $U(1)$, hence describes a S^1 "slice" of S^3.

4. $A \in SU(2)$

$$R(A)_{ij} = \tfrac{1}{2}\mathrm{tr}_2(\sigma_i A \sigma_j A^\dagger) \quad , \quad \sum_{j=1}^{3} (\sigma_j)_{\alpha\beta}(\sigma_j)_{\delta\gamma} = 2\delta_{\alpha\gamma}\delta_{\delta\beta} - \delta_{\alpha\beta}\delta_{\delta\gamma}$$

$$\left(R(A)R(A)^T\right)_{ij} = R(A)_{ik}R(A)_{jk} = \tfrac{1}{4}\mathrm{tr}_2(\sigma_i A \sigma_k A^\dagger)\,\mathrm{tr}_2(\sigma_j A \sigma_k A^\dagger)$$

$$= \tfrac{1}{4}\left\{(\sigma_i)_{\alpha\beta}A_{\beta\gamma}(\sigma_k)_{\gamma\delta}A^\dagger_{\delta\alpha}\right\}\left\{(\sigma_j)_{\alpha\beta}A_{\beta\gamma}(\sigma_k)_{\gamma\delta}A^\dagger_{\delta\alpha}\right\}$$

$$= \tfrac{1}{4}(\sigma_i)_{\alpha\beta}A_{\beta\gamma}A^\dagger_{\delta\alpha}(\sigma_j)_{\varepsilon\varsigma}A_{\varsigma\eta}A^\dagger_{\kappa\varepsilon}(\sigma_k)_{\gamma\delta}(\sigma_k)_{\eta\kappa}$$

$$= \tfrac{1}{2}(\sigma_i)_{\alpha\beta}A_{\beta\gamma}A^\dagger_{\eta\alpha}(\sigma_j)_{\varepsilon\varsigma}A_{\varsigma\eta}A^\dagger_{\gamma\varepsilon} - \tfrac{1}{4}(\sigma_i)_{\alpha\beta}A_{\beta\gamma}A^\dagger_{\gamma\alpha}(\sigma_j)_{\varepsilon\varsigma}A_{\varsigma\eta}A^\dagger_{\eta\varepsilon}$$

$$= \frac{1}{2}(\sigma_i)_{\alpha\beta}\,\delta_{\beta\varepsilon}\,\delta_{\alpha\delta}\,(\sigma_j)_{\varepsilon\delta} - \frac{1}{4}(\sigma_i)_{\alpha\beta}\,\delta_{\alpha\beta}\,(\sigma_j)_{\varepsilon\delta}\,\delta_{\delta\varepsilon}$$

$$= \frac{1}{2}(\sigma_i)_{\alpha\beta}(\sigma_j)_{\beta\alpha} - \frac{1}{4}(\sigma_i)_{\alpha\alpha}(\sigma_j)_{\delta\delta}$$

$$= \frac{1}{2}\,\mathrm{tr}_2(\sigma_i\sigma_j) = \cancel{\frac{1}{2}\mathrm{tr}_2(i\epsilon_{ijk}\sigma_k)}\ \frac{1}{2}\,\mathrm{tr}_2(i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}_2)$$

$$= \delta_{ij}$$

$\therefore\ R(A) \in O(3)$

As $R(\mathbb{1}_2) = \frac{1}{2}\mathrm{tr}_2(\sigma_i\sigma_j) = \mathbb{1}_2$, $\det(R(\mathbb{1}_2)) = 1$ and $SU(2)$ is connected, $R: SU(2) \longrightarrow O(3)$ is smooth, then all $R(A)$ must be connected to $\mathbb{1}$.

$\therefore\ R(A) \in SO(3)$

$$A = a_0 I_2 + i\sigma_i a_i \quad \}\ a_0^2 + a_i a_i = 1$$

$$\cancel{R(A)_{ij} \overset{?}{=} \frac{1}{2}\mathrm{tr}_2(\sigma_i A \sigma_j A^\dagger) = \frac{1}{2}\mathrm{tr}_2\Big(\sigma_i \frac{(\mathbb{1}_2 + \sigma_k R_{kl}\sigma_l)}{2\sqrt{1+\mathrm{Tr}_3 R}}\,\sigma_j \frac{(\mathbb{1}_2 + \sigma_m^\dagger R_{mn}^\dagger \sigma_n^\dagger)}{2\sqrt{1+\mathrm{Tr}_3 R}}\Big)}$$

$$\cancel{= \frac{1}{2}\mathrm{tr}_2\left(\frac{\sigma_i(\mathbb{1}_2 + \sigma_k R_{kl}\sigma_l)\sigma_j(\mathbb{1}_2 + \sigma_m^\dagger R_{mn}^\dagger \sigma_n^\dagger)}{4(1+\mathrm{Tr}_3 R)}\right)}$$

Write $R_{ij}(A)$ and solve for a_0, a_i ...

$$\cancel{= \frac{1}{8(1+\mathrm{Tr}_3 R)}\,\mathrm{tr}_2\big(\sigma_i(\mathbb{1}_2 + \sigma_k R_{kl}\sigma_l)\sigma_j(\mathbb{1}_2 + \sigma_m R_{mn}\sigma_n)\big)}$$

$$\cancel{(\sigma_i + i\epsilon_{ikp}\sigma_p R_{kl}\sigma_l)(\sigma_j + i\epsilon_{jmq} R_{mn}\sigma_n)}$$

$$\cancel{(\sigma_i - \epsilon_{ikp}\epsilon_{plr} R_{kl}\sigma_r)(\sigma_j - \epsilon_{jmq}\epsilon_{qns}\sigma_s R_{mn})}$$

$$\cancel{(\sigma_i - (\delta_{il}\delta_{kr} - \delta_{ir}\delta_{kl})\sigma_r R_{kl})(\sigma_j - (\delta_{jn}\delta_{ms} - \delta_{js}\delta_{mn})\sigma_s R_{mn})}$$

$$\cancel{(\sigma_i - \sigma_k R_{ki} + \sigma_i \mathrm{Tr}_3 R)(\sigma_j - \sigma_m R_{jm} + \sigma_j \mathrm{Tr}_3 R)}$$

$$\cancel{\sigma_i\sigma_j(1+\mathrm{Tr}_3 R)^2 - \sigma_k R_{ki}\sigma_j(1+\mathrm{Tr}_3 R) - \sigma_m R_{jm}\sigma_i(1+\mathrm{Tr}_3 R) + \sigma_k\sigma_m R_{ki} R_{jm}}$$

$$\cancel{\mathrm{tr}_2(\overset{\delta_{ij}}{\sigma_i\sigma_j} + \sigma_i\sigma_j\sigma_k\sigma_l R_{kl} + \sigma_i\sigma_j\sigma_m\sigma_n R_{mn} + \sigma_i\sigma_j\sigma_k\sigma_l\sigma_m\sigma_n R_{kl}R_{mn})}$$

$$\cancel{= \mathrm{tr}_2(\delta_{ij}\mathbb{1}_2 + \sigma_i\sigma_j \mathrm{Tr}_3 R + i\epsilon_{klm}\sigma_i\sigma_j\sigma_m R_{kl} + \ldots}$$

$$\cancel{\mathrm{tr}_2(\delta_{ij}(1+\mathrm{Tr}_3 R)\mathbb{1}_2 - \sigma_i\sigma_n R_{jn} + \sigma_i\sigma_n R_{nj} + \ldots + (\sigma_n R_{in})(\sigma_m R_{jm})}$$

$$+(\sigma_n R_{ni} - \sigma_n R_{in})(\sigma_m R_{mj} - \sigma_m R_{jm})$$

$$\cancel{\mathrm{tr}_2(\delta_{ij}(1+\mathrm{Tr}_3 R)\mathbb{1}_2}$$

5. $D(g) = \mathrm{Exp}(d(X))$ $\qquad g = \mathrm{Exp}(X)$

$$D(g_1)\,D(g_2) = \mathrm{Exp}(d(X_1))\,\mathrm{Exp}(d(X_2)) \qquad X_1, X_2 \in L(G)$$

$$= \mathrm{Exp}\left(d(X_1) + d(X_2) + \tfrac{1}{2}[d(X_1), d(X_2)] + \tfrac{1}{12}[d(X_1),[d(X_1), d(X_2)]] - \tfrac{1}{12}\ldots + \ldots\right)$$

$$= \mathrm{Exp}\left(d(X_1) + d(X_2) + \tfrac{1}{2}d([X_1, X_2]) + \tfrac{1}{12}d([X_1,[X_1, X_2]]) + \ldots\right)$$

because d is a repn of $L(G)$

while

$$D(g_1 g_2) = D(\mathrm{Exp}(X_1)\,\mathrm{Exp}(X_2)) = D(\exp(X_1 + X_2 + \tfrac{1}{2}[X_1, X_2] + \ldots))$$

$$D(g_1)D(g_2) = \mathrm{Exp}(d(X_1 + X_2 + \tfrac{1}{2}[X_1, X_2] + \ldots)) \text{ by linearity of } d$$

$\therefore\ D(g_1 g_2) = D(g_1)\,D(g_2)$ $\quad$ considering $g = \mathrm{Exp}(X_1 + X_2 + \tfrac{1}{2}[X_1, X_2] + \ldots) = \mathrm{Exp}(X)$

D is not a repn of G as $\exp : X \longmapsto$ is not 1:1,
i.e. may exist $g \in G$ $\nexists X : g = \exp X$ $X \in L(G)$.

6. (a) $[T^a, T^b] = f^{ab}{}_c T^c$

$$[R(T^a), R(T^b)] = R([T^a, T^b]) = R(f^{ab}{}_c T^c) = f^{ab}{}_c R(T^c)$$

$$[\bar{R}(T^a), \bar{R}(T^b)] = [R(T^a)^*, R(T^b)^*] = [R(T^a), R(T^b)]^*$$

$$= f^{ab}{}_c R(T^c)^* = f^{ab}{}_c \bar{R}(T^c) \quad \text{as } f^{ab}{}_c \in \mathbb{R}.$$

(b) $R(T^a) = -\frac{1}{2}i\sigma_a$, $\quad \overset{\sigma_3}{\frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}$, $\quad \overset{\sigma_2}{\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$, $\quad \overset{\sigma_1}{\frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}}$

$\bar{R}(T^a) = \frac{1}{2}i\sigma_a^*$, $\quad \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\quad \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\quad \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Both provide a fundamental repn of $L(SU(2))$. $\bar{R} = SRS^{-1}$, $S = \sigma_2 \Rightarrow$ equiv / isomorphic

Weights are the same as $Rv_\lambda = \lambda v_\lambda \Rightarrow (Rv_\lambda)^* = \bar{R}v_\lambda^* = \lambda^* v_\lambda^* = -\lambda v_\lambda^*$ as anti-H must have im e'val.
and weight set $\{-\Lambda, \ldots, \Lambda-2, \Lambda\} \Rightarrow -\lambda \in \{\ldots\}\ \forall \lambda \in \{\ldots\}$

7. $R_1 : \underset{\sim}{L} \to V_1$

$R_2 : \underset{\sim}{L} \to V_2$

$$R_1 \otimes R_2(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X) \qquad X \in L$$

$$[R_1 \otimes R_2(X), R_1 \otimes R_2(Y)] = [R_1(X) \otimes I_2, R_1(Y) \otimes I_2]$$
$$+ [R_1(X) \otimes I_2, I_1 \otimes R_2(Y)] + [I_1 \otimes R_2(X), R_1(Y) \otimes I_2]$$
$$+ [I_1 \otimes R_2(X), I_1 \otimes R_2(Y)]$$
$$= [R_1(X), R_1(Y)] \otimes I_2 + I_1 \otimes [R_2(X), R_2(Y)]$$
$$= R_1([X,Y]) \otimes I_2 + I_1 \otimes R([X,Y]) = R_1 \otimes R_2([X,Y])$$

$$R_1 \otimes R_2(\alpha X + \beta Y) = R_1(\alpha X + \beta Y) \otimes I_2 + I_1 \otimes R_2(\alpha X + \beta Y)$$
$$= \alpha R_1(X) \otimes I_2 + \beta R_1(Y) \otimes I_2 + \alpha I_1 \otimes R_2(X) + \beta I_1 \otimes R_2(Y)$$
$$= \alpha R_1 \otimes R_2(X) + \beta R_1 \otimes R_2(Y)$$

$\therefore R_1 \otimes R_2$ is a repn of L.

8. $R_N \otimes R_M \quad N, M \in \mathbb{Z}_+ \cup \{0\}$, R_Λ irrepn of $\mathcal{L}(SU(2))$ w/ highest weight Λ

$R_N : \underset{\sim}{g} \to V_N$, $u \in V_N$

$R_M : \underset{\sim}{g} \to V_M$, $v \in V_M$

$R_N(H) u_\lambda = \lambda u_\lambda$, $R_M(H) v_{\lambda'} = \lambda' v_{\lambda'}$

$R_N \otimes R_M$ has eigenvectors $u_\lambda \otimes v_{\lambda'}$:

$$R_N \otimes R_M(H)(u_\lambda \otimes v_{\lambda'}) = (R_N(H) \otimes I_2)(u_\lambda \otimes v_{\lambda'}) + (I_1 \otimes R_M(H))(u_\lambda \otimes v_{\lambda'})$$
$$= R_N(H) u_\lambda \otimes v_{\lambda'} + u_\lambda \otimes R_M(H) v_{\lambda'} = \lambda u_\lambda \otimes v_{\lambda'} + \lambda' u_\lambda \otimes v_{\lambda'}$$
$$= (\lambda + \lambda') u_\lambda \otimes v_{\lambda'}$$

Hence weight set

$S_{N \otimes M} \{ \lambda + \lambda' : \lambda \in S_N, \lambda' \in S_M \}$ where S denotes the corresponding weight sets

$S_N = \{-N, \dots, N-2, N\}$

$S_M = \{-M, \dots, M-2, M\}$

How many ways to put n balls into 2 buckets with capacities N and M?

$N+M: \quad N+M$

$N+M-2: \quad (N-2)+M, \quad N+(M-2)$

$N+M-4: \quad (N-4)+M, \quad (N-2)+(M-2), \quad N+(M-4)$

$\vdots$

$N+M-2\min(N,M) = |N-M| \quad \dots$

$N+M-2(\min(N,M)+1): \quad \dots$

$\vdots$

$-(N+M)+2\min(N,M) = -|N-M|$

$\vdots$

$-(N+M) \quad : -N-M$

} $N+M-2n$ has multiplicity $n+1$

} multiplicity $\min(N,M)+1$

$\therefore R_N \otimes R_M = R_{|N-M|} \oplus R_{|N-M|+2} \oplus \dots \oplus R_{N+M}$

as step ~~operator~~ ops allow to construct e'vector set starting at highest weight and e'v can be made orthogonal.

$\dim(R_N \otimes R_M) = \dim(R_N)\dim(R_M) = (N+1)(M+1)$

$\dim(R_{|N-M|} \oplus \dots \oplus R_{N+M}) = \dim(R_{|N-M|}) + \dots + \dim(R_{N+M})$

$= (|N-M|+1) + \dots + (N+M+1)$

$$= 2\,\frac{\frac{N+M+1}{2}\left(\frac{N+M+1}{2}+1\right)}{2} - 2\,\frac{\frac{|N-M|}{2}\left(\frac{|N-M|}{2}+1\right)}{2} = \frac{(N+M)^2+4(N+M)+3-(N-M)^2+\dots}{4}$$

$= NM + (N+M) + 1 = (N+1)(M+1)$ ✓

9. (a) $X = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ $\quad [Q, P] = Z$

$[Z, Q] = [Z, P] = 0$

$\mathrm{Span}_{\mathbb{R}}\{Z\}$ is an abelian ideal $\Rightarrow$ Lie algebra not semi-simple

$[T^a, T^b] = f^{ab}{}_c T^c$

$f^{QP}{}_Z = -f^{PQ}{}_Z = 1$

$\kappa^{ab} = f^{ad}{}_c f^{bc}{}_d \Rightarrow \kappa^{ab} = 0$ degenerate $\Leftrightarrow$ ~~not~~ not semi-simple

(b) L real, compact : basis $\{T^a\}$: $\kappa^{ab} = -\kappa\delta^{ab}$ $\quad \kappa > 0$

$I \subseteq L \quad [L, I] \subseteq I$

$I_\perp := \{X \in L : \kappa(X, Y) = 0 \quad \forall Y \in I\}$

$\kappa(X, [Y, Z]) = -\kappa([Y, X], Z)$ by invariance of Killing form

$= 0 \quad \forall\ Y \in L,\ X \in I \Rightarrow [Y, X] \in I \quad , \quad Z \in I_\perp$

$\therefore [Y, Z] \in I_\perp \quad \forall\ Y \in L,\ Z \in I_\perp$

$\Rightarrow I_\perp \subseteq L \quad [L, I_\perp] \subseteq I_\perp$ $\square$

$I, I_\perp$ ideals $\Rightarrow [I, I_\perp] \subseteq I \cap I_\perp$

$I \cap I_\perp = \{0\}$ since κ is euclidean

$L = I \oplus I_\perp$ as I and $I_\perp$ span L.

Any semi-simple real Lie algebra ~~is~~ $M = \bigoplus_i S_i$ w/ S_i simple by iterating the above recursively and using that M real compact $\Rightarrow$ I real compact.

Need to show that a complexified simple Lie algebra is simple (as complexification of a sum is the sum of complexifications). This is not trivial.