Invariance under Weyl transformations

$$h_{\mu\nu} \mapsto e^{\lambda} h_{\mu\nu}$$

$$h^{\mu\nu} \mapsto e^{-\lambda} h^{\mu\nu}$$

$$\sqrt{-h} \mapsto e^{\lambda} \sqrt{-h}$$

$$X^a \mapsto X^a$$

$$\eta_{ab} \mapsto \eta_{ab}$$

Invariant under conformal transformations

Local symmetries: 2d diffeomorphisms, Weyl invariance } world-sheet

Global spacetime symmetries:

Translations: $X^a \mapsto X^a + T^a$ ← constant vector in spacetime

$$\partial_\mu X^a \mapsto \partial_\mu X^a$$

Lorentz invariance:

$X^a \to L^a{}_b X^b$ $\quad$ $L^a{}_b$ is a Lorentz transformation $L^a{}_b L^c{}_d \eta_{ac} = \eta_{bd}$

$L^a{}_b \in O(d-1, 1)$ in d spacetime dims

$$\partial_\mu X^a \partial_\nu X^b \eta_{ab} \to \partial_\mu (L^a{}_c X^c) \partial_\nu (L^b{}_d X^d) \eta_{ab} = \partial_\mu X^c \partial_\nu X^d \eta_{cd}$$

Poincaré is the semi-direct product of translations and Lorentz transforms.

This Polyakov action is invariant under the Poincaré group.

Can formulate the string in a curved spacetime by replacing η_{ab} by $g_{ab}(X)$

$$I = -\frac{T}{2}\int d^2\xi \sqrt{-h}\, h^{\mu\nu} \partial_\mu X^a \partial_\nu X^b g_{ab}(X)$$

Equations of motion are generalisations of the geodesic eq. (non-linear).

Still has local Weyl and Diffeomorphism invariance.

World-sheet metric $h_{\mu\nu}$ - in general has 3 components

Weyl invariance : -1

Diffeos : -2

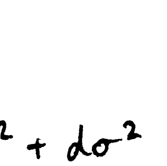
Can fix the metric to be $-d\tau^2 + d\sigma^2$ { This works globally if the world-sheet is a cylinder, a plane or a torus.

For non-trivial topology, can replace this by a metric of constant curvature (2d versions of de Sitter or anti-de Sitter)

$$I = \frac{1}{2}T\int d\sigma\, d\tau \left(\dot{X}^2 - X'^2\right)$$

$\dot{} \equiv d/d\tau$, $\quad {}' \equiv d/d\sigma$ $\quad$ $\dot{X}^2 = \dot{X}^a \dot{X}^b \eta_{ab}$

Motion in the plane is a change of coordinates and does nothing.



If the point moves ⊥ plane, you stretch the world-sheet.

Thus two directions of change of X are not physical string transformations. One of these directions is timelike and the other spacelike.

EOM for X^a $\quad -\frac{\partial^2 X^a}{\partial \tau^2} + \frac{\partial^2 X^a}{\partial \sigma^2} = 0$ (2-d wave equation)

$$\sigma^\pm = \tau \pm \sigma$$

$$\frac{\partial^2 X^a}{\partial \sigma^+ \partial \sigma^-} = 0 \qquad X^a_L(\sigma^+) + X^b_R(\sigma^-) = X^a(\sigma)$$

left moving $\qquad$ right moving

Metric: $-d\tau^2 + d\sigma^2$ $\qquad \tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ $\quad \sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$

$$\longrightarrow \frac{1}{4}(-\cancel{d\sigma^{+2}} - 2d\sigma^+ d\sigma^- - \cancel{d\sigma^{-2}} + \cancel{d\sigma^{+2}} - 2d\sigma^+ d\sigma^- + \cancel{d\sigma^{-2}})$$

$$= -d\sigma^+ d\sigma^-$$

$h_{++} = 0 \qquad h_{--} = 0 \qquad h_{+-} = h_{-+} = -1/2$

$h^{++} = 0 \qquad h^{--} = 0 \qquad h^{+-} = h^{-+} = -2$

$\sqrt{-h} = 1/2 \qquad \partial_\pm = \frac{1}{2}\left(\frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial\sigma}\right)$

transforming like a vector when acting on scalars

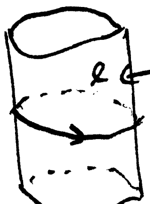
$I = 2T \int d\sigma^+ d\sigma^- \, \partial_+ X^a \partial_- X^b \eta_{ab}$ $\quad \Rightarrow$ EOM for X^a is $\partial_+\partial_- X^a = 0$

Sol^n $X^a = X^a_L(\sigma^+) + X^a_R(\sigma^-)$

Boundary conditions

Open strings: N or D for each end of the string in each direction

Closed strings:



ℓ ← length of closed path wrapping the cylinder at constant τ

$$X^a(\sigma, \tau) = X^a(\sigma + l, \tau)$$

Energy-momentum tensor must vanish:

$$T_{\mu\nu} \sim \left(\partial_\mu X^a \partial_\nu X^b - \frac{1}{2} h_{\mu\nu} \partial_\rho X^a \partial_\sigma X^b h^{\rho\sigma}\right)\eta_{ab}$$

$T_{+-} = 0$

$T_{++} = 0 \qquad \partial_+ X^a \partial_+ X^b \eta_{ab} = 0$ $\qquad$ Indefiniteness of η_{ab} $\Rightarrow$ possible to solve these equations

$T_{--} = 0 \qquad \partial_- X^a \partial_- X^b \eta_{ab} = 0$

Conservation of energy-momentum tensor:

$D_\mu T^{\mu\nu} = 0 \qquad \partial_+ T^{++} + \cancel{\partial_- T^{++}} = 0 \qquad \checkmark$

$\cancel{\partial_+ T^{--}} + \partial_- T^{--} = 0$

$V_\mu = h_{\mu\nu} V^\nu \qquad V_+ = h_{+-} V^-$

Thus $\partial_+ T_{--} = 0 \qquad \partial_- T_{++} = 0$ $\quad T_{++} = T_{++}(\sigma^+)$

$T_{--} = T_{--}(\sigma^-)$

This tells you that there are an infinite number of conservation laws

$\partial_- T_{++} = 0$ $\qquad$ Integrate over a surface of constant τ on the world sheet

$\partial_-(f(\sigma^+) T_{++}) = 0 \qquad \int d\sigma \, f(\sigma^+) T_{++} = \text{constant}$ (from Gauss theorem)

$= L_f \Leftarrow$ Virasoro charge corresponding to f

$I = +\frac{1}{2} T \int d\sigma d\tau \left(\dot{X}^2 - X'^2\right)$

Momentum conjugate to X^a: $\quad \Pi_a = \frac{\delta I}{\delta \dot{X}^a} = T \dot{X}_a$

Hamiltonian $H = \int d\sigma\, \Pi_a \dot{X}^a - L = \frac{1}{2}T \int d\sigma \left(\dot{X}^2 - X'^2\right)$ [Basically the total energy]

$$= T \int d\sigma \left(\partial_+ X\right)^2 + \left(\partial_- X\right)^2$$

Poisson Brackets $\{\Pi_a(\sigma,\tau), \Pi_b(\sigma',\tau)\} = 0$

$$\{X^a(\sigma,\tau), X^b(\sigma',\tau)\} = 0$$

$$\{X^a(\sigma,\tau), \Pi_b(\sigma',\tau)\} = \delta^a_b\, \delta(\sigma-\sigma')$$

$\Pi_b = T\dot{X}_b \Rightarrow \{X^a(\sigma,\tau), \dot{X}^b(\sigma',\tau)\} = \frac{1}{T}\eta^{ab}\,\delta(\sigma-\sigma')$

Note that one of the X's has the "wrong" sign.