$$h_{\mu\nu} = \bar{h}_{\mu\nu} = \mathrm{Re}\left(H_{\mu\nu}\, e^{i k_\rho x^\rho} \right)$$

$$\frac{d^2 S_\alpha}{d\tau^2} = \frac{1}{2} \frac{\partial^2 h_{\mu\nu}}{\partial \tau^2} e_\alpha^{\ \mu} e_\rho^{\ \nu} S^\rho$$

$$\Rightarrow \frac{d^2 S_0}{d\tau^2} = \frac{d^2 S_3}{d\tau^2} = 0$$

$$S_0 = \frac{dS_0}{d\tau} = 0 \Rightarrow S_0 = 0$$

$$\frac{dS_3}{d\tau} = 0 \Rightarrow S_3 = \text{const}$$

Observer: $x^\mu = (\tau, 0, 0, 0)$ $\quad \therefore t \approx \tau$

+ polarization $(H_\times = 0)$

$$\frac{d^2 S_1}{d\tau^2} = -\frac{1}{2} \omega^2 |H_+| \cos(\omega\tau - \alpha)\, S_1$$

avg H_+

$$\frac{d^2 S_2}{d\tau^2} = +\frac{1}{2} \omega^2 |H_+| \cos(\omega t - \alpha)\, S_2$$

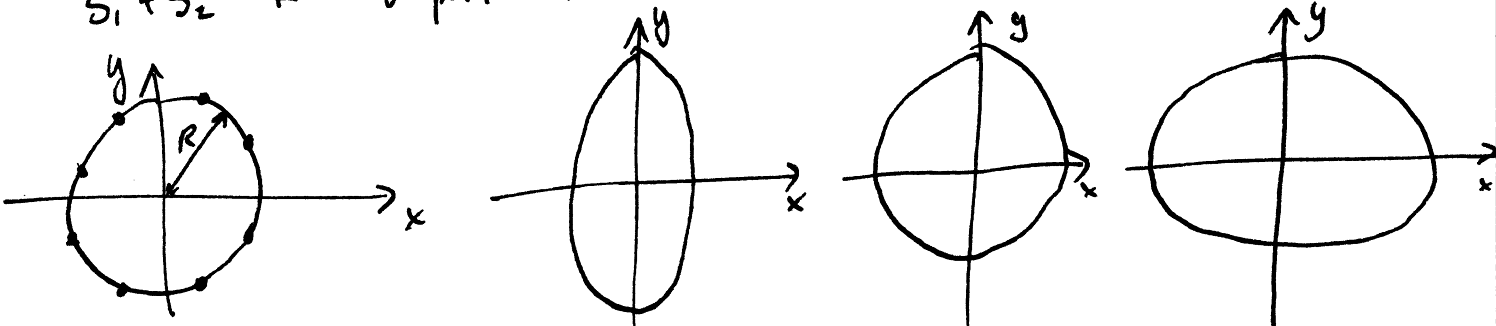
zeroth order : $S_1 = \bar{S}_1 \qquad S_2 = \bar{S}_2$

consts

1st order: $S_1 = \bar{S}_1 \left(1 + \frac{1}{2} |H_+| \cos(\omega t - \alpha) \right), \quad S_2 = \bar{S}_2 \left(1 - \frac{1}{2} |H_+| \cos(\omega t - \alpha) \right)$

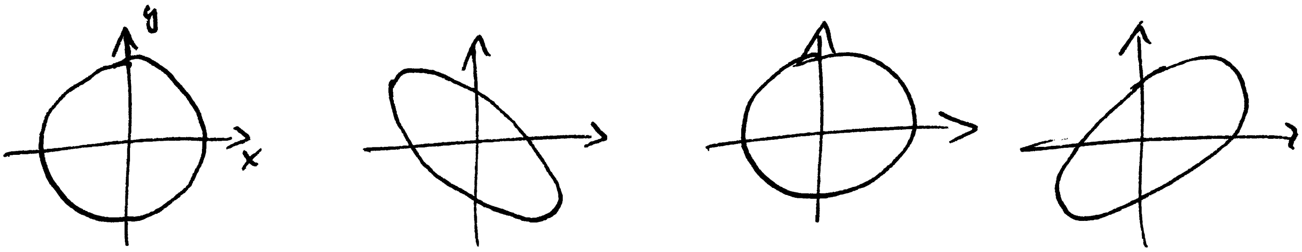
$\bar{S}_1, \bar{S}_2$: average displacement

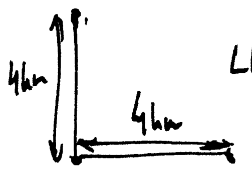
$\bar{S}_1^2 + \bar{S}_2^2 = R^2 \quad \forall$ particles



$\omega\tau = \frac{\pi}{2} + \alpha \qquad \omega\tau = \pi + \alpha \qquad \omega\tau = \frac{3\pi}{2} + \alpha \qquad \omega\tau = 2\pi + \alpha$

<u>Ex</u> Show that for x-polarized wave





LIGO expect $H_+, H_\times \sim 10^{-21}$, $\frac{\delta L}{L} \sim 10^{-21}$

Field far from source $\quad \partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$

$$\bar{h}_{\mu\nu}(t, \underline{x}) = 4 \int d^3x \, \frac{T_{\mu\nu}(t - |\underline{x} - \underline{x}'|, \underline{x}')}{|\underline{x} - \underline{x}'|}$$

Euclidean $|\cdot\cdot|$

Assume matter confined to region near origin of size d

For far from source $\quad r = |\underline{x}| \gg |\underline{x}'| \approx d$

$$|\underline{x} - \underline{x}'|^2 = r^2 - 2\underline{x}\cdot\underline{x}' + \underline{x}'^2 = r^2\left(1 - \frac{2}{r}\hat{\underline{x}}\cdot\underline{x}' + \mathcal{O}\left(\frac{d^2}{r^2}\right)\right) \qquad \hat{\underline{x}} = \underline{x}/r$$

$$|\underline{x} - \underline{x}'| \approx r - \hat{\underline{x}}\cdot\underline{x}' + \mathcal{O}\left(\frac{d^2}{r}\right)$$

$$T_{\mu\nu}(t - |\underline{x} - \underline{x}'|, \underline{x}') = T_{\mu\nu}(\underset{t-r}{t'}, \underline{x}') + \underbrace{\hat{\underline{x}}\cdot\underline{x}'(\partial_0 T_{\mu\nu})(t', \underline{x}')}_{\mathcal{O}\left(d\frac{T_{\mu\nu}}{\tau}\right)} + \dots$$

τ timescale for $T_{\mu\nu}$ variation $\quad \partial_0 T_{\mu\nu} \sim T_{\mu\nu}/\tau$

assume $\frac{d}{\tau} \ll 1$: non-relativistic source (assume ~~[illegible]~~)

$$\bar{h}_{ij}(t, x) \approx \frac{4}{r}\int d^3\underline{x}' \, T_{ij}(t', \underline{x}') , \quad t' = t - r$$

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \qquad \partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji} \qquad \partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i}$$

e-m conservation: $\quad \partial_\mu T^{\mu\nu} = 0$

$$\int d^3x \, T^{ij} = \int d^3x \left[\partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j\right]$$

$$= -\int d^3x \, (\partial_k T^{ik}) x^j = \int d^3x \, (\partial_0 T^{i0}) x^j$$

$$= \partial_0 \int d^3x \, T^{i0} x^j$$

symmetrize: $\displaystyle \int d^3x \, T^{ij} = \partial_0 \int d^3x \, T^{0(i} x^{j)}$

$$= \partial_0 \int d^3x \left[\frac{1}{2}\partial_k (T^{0k} x^i x^j) - \frac{1}{2}\partial_k T^{0k} x^i x^j\right]$$

$$= -\frac{1}{2}\partial_0 \int d^3x \, \partial_k T^{0k} x^i x^j$$

$$= \frac{1}{2}\partial_0 \int d^3x \, \partial_0 T^{00} x^i x^j = \frac{1}{2}\partial_0\partial_0 \int d^3x \, T^{00} x^i x^j = \frac{1}{2}\ddot{I}_{ij}(t)$$

$$I_{ij}(t) \equiv \int d^3x \; T_{00}(t,\underline{x})\, x^i x^j$$

$$\bar{h}_{ij}(t,\underline{x}) \approx \frac{2}{r}\, \ddot{I}_{ij}(t-r) \qquad r \gg d \qquad \tau \gg d$$

2nd moment of energy density

$$\partial_0 \bar{h}_{0i} \approx \partial_j \left(\frac{2}{r} \ddot{I}_{ij}(t-r) \right) \Rightarrow \bar{h}_{0i} \approx \partial_j \left(\frac{2}{r} \dot{I}_{ij}(t-r) \right) \qquad \partial_i r = \frac{x_i}{r} = \hat{x}_i$$

$$= \underbrace{-2\frac{\hat{x}_j}{r^2} \dot{I}_{ij}(t-r)}_{\frac{I_{ij}}{r^2\tau}} \underbrace{- 2\frac{\hat{x}_j}{r} \ddot{I}_{ij}(t-r)}_{\frac{I_{ij}}{r\tau^2}}$$

assume $r \gg \tau$ "radiation zone"

$$\bar{h}_{0i} \approx -\frac{2\hat{x}_j}{r} \ddot{I}_{ij}(t-r)$$

$$\partial_0 \bar{h}_{00} = \partial_i \left(-\frac{2\hat{x}_j}{r} \ddot{I}_{ij}(t-r) \right)$$

$$\bar{h}_{00} = \partial_i \left(-\frac{2\hat{x}_j}{r} \dot{I}_{ij}(t-r) \right) \approx \frac{2\,\hat{x}_i \hat{x}_j}{r} \ddot{I}_{ij}(t-r)$$

$$\bar{h}_{00} \approx \frac{4}{r} \int d^3\underline{x}' \; T_{00}(t',\underline{x}') = \frac{4E}{r}$$

$$I_{ij} \sim E d^2$$

$$\bar{h}_{00} \sim \frac{E d^2}{r \tau^2}$$