General Relativity
Example sheet 1

1a. $\mathbb{RP}^n = \mathbb{R}^{n+1}/\{0\}$ with $(x_1, \ldots, x_{n+1}) \sim \lambda(x_1, \ldots, x_{n+1})$
where $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Def^n subsets $\mathcal{O}_\alpha = \{(x) \in \mathbb{R}^{n+1} : x_\alpha \neq 0\}$

$\bigcup_\alpha \mathcal{O}_\alpha = \mathbb{RP}^n$ as $\{0\}$ is not in $\mathbb{RP}^n$.

Maps $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha \subset \mathbb{R}^n$

$$\phi_\alpha(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_{n+1}}{x_\alpha}\right)$$
$$\equiv (\tilde{x}_1, \ldots, \tilde{x}_n)$$

For $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = \{(x) \in \mathbb{R}^{n+1} : x_\alpha, x_\beta \neq 0\}$

$\phi_\alpha^{-1} : U_\alpha \to \mathcal{O}_\alpha$

$$\phi_\alpha^{-1}(\tilde{x}_1, \ldots, \tilde{x}_n) = \phi_\alpha^{-1}\left(\frac{x_1}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_{n+1}}{x_\alpha}\right)$$
$$= \left(\frac{x_1}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, 1, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_{n+1}}{x_\alpha}\right)$$

ϕ_α is 1-to-1 as $\phi_\alpha^{-1} \circ \phi_\alpha$ maps to scaled points which are identified in $\mathbb{RP}^{n+1}$.

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$$

$$\phi_\beta \phi_\alpha^{-1}(\tilde{x}_1, \ldots, \tilde{x}_n) = \phi_\beta\left(\frac{x_1}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, 1, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_{n+1}}{x_\alpha}\right)$$

or $= \phi_\beta(\tilde{x}_1, \ldots, \tilde{x}_{\alpha-1}, 1, \tilde{x}_\alpha, \ldots, \tilde{x}_n)$

$$= \left(\frac{x_1}{x_\alpha x_\beta}, \ldots, \frac{x_{\alpha-1}}{x_\alpha x_\beta}, \frac{1}{x_\beta}, \frac{x_{\alpha+1}}{x_\alpha x_\beta}, \ldots, \frac{x_{\beta-1}}{x_\alpha x_\beta}, \frac{x_{\beta+1}}{x_\alpha x_\beta}, \ldots, \frac{x_{n+1}}{x_\alpha x_\beta}\right)$$

is a smooth map.

$\therefore$ ϕ_α are charts for a <u>smooth manifold $\mathbb{RP}^n$</u>. $\square$

$\mathbb{RP}^n = \mathbb{R}^{n+1}/\{0\}$ with $X \sim \lambda X$, $X \in \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R}$, $X \neq \underline{0}$, $\lambda \neq 0$

$S^n = \{\underline{r} \in \mathbb{R}^{n+1} : |\underline{r}| = 1\}$ or equivalently

$= \mathbb{R}^{n+1}/\{0\}$ with $X \sim \lambda X$, $X \in \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R}$, $X \neq 0$, $\lambda > 0$

By identifying antipodal points on S^n, $\lambda > 0 \rightarrow \lambda \neq 0$ for the identification above. This gives exactly $\mathbb{RP}^n$.

$SO(3) \equiv$ rotations in $\mathbb{R}^3$ ~~$= \{R(\underline{n}, \theta) \in \mathbb{R}^3 \times \mathbb{R}^3 \; R(\underline{n}, \theta) \cos\theta\}$~~

$SO(3)$ is S^3 with antipodal points identified

as $SO(3)$ is parameterised as $(\underline{n}, \theta)$ where $\underline{n}$ is a unit vector in $\mathbb{R}^3$ and $\theta \in [0, 2\pi)$ $\Rightarrow$ defines ~~subset of~~ S^3.

Due to identification $(\underline{n}, \theta) \sim (-\underline{n}, 2\pi - \theta)$, antipodal points on S^3 are identified.

$\therefore SO(3) \equiv \mathbb{RP}^3$ by above

1b. Generating vector fields are given by $V|_p = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} G_\varepsilon(p)$ where G_ε is the 1-parameter (sub)group.

$\tilde{x}$ is the transformed coordinate

$$V_1 = \frac{d\tilde{x}}{d\varepsilon_1} \frac{\partial}{\partial \tilde{x}}\Big|_{\varepsilon_1 = 0} = \frac{\partial}{\partial x}$$

$$V_2 = \left(e^{\varepsilon_2} x \frac{\partial}{\partial \tilde{x}}\right)\Big|_{\varepsilon_2 = 0} = x\frac{\partial}{\partial x}$$

$$V_3 = \left(\frac{x^2}{(1-\varepsilon_3 x)^2} \frac{\partial}{\partial \tilde{x}}\right)\Big|_{\varepsilon_3 = 0} = x^2 \frac{\partial}{\partial x}$$

$x \rightarrow \frac{ax+b}{cx+d}$ are ~~generated by~~ $x = \frac{p}{q}$ $\qquad p \rightarrow a\frac{p}{q} + b \sim ap + bq$

$\qquad q \rightarrow cp/q + d \sim cp + dq$

$$\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M \in GL(2;\mathbb{R})} \begin{pmatrix} p \\ q \end{pmatrix}$$ and $\begin{pmatrix} p \\ q \end{pmatrix} \sim \lambda \begin{pmatrix} p \\ q \end{pmatrix}$ $\lambda \neq 0$ $\lambda \in \mathbb{R}$

$\Rightarrow M \sim \lambda M$ and w.l.o.g. $M \in SL(2;\mathbb{R})$

Near identity

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+\theta_1 & \theta_2 \\ \theta_3 & 1+\theta_4 \end{pmatrix} \quad \text{with } \theta_i \to 0 \text{ for } \#_2 .$$

$$\cancel{(1+\theta_1)(1+\theta_4) - \theta_2\theta_3 = 1}$$

$$\cancel{1 + \theta_1 + \theta_4 + \theta_1\theta_4 - \theta_2\theta_3 = 1}$$

$$\cancel{\theta_3 = \frac{\theta_1 + \theta_4 + \theta_1\theta_4}{\theta_2} \text{ if } \theta_2 \neq 0} \quad \cancel{\theta_1 + \theta_4 + \theta_1\theta_4 = 0}$$

Vector fields generating these are

$$\tilde{V}_1 = p\frac{\partial}{\partial p} \qquad \tilde{V}_3 = p\frac{\partial}{\partial q}$$

$$\tilde{V}_4 = q\frac{\partial}{\partial q} \qquad \tilde{V}_2 = q\frac{\partial}{\partial p}$$

In terms of x, $(x = p/q)$

$$V_1' = p\frac{\partial x}{\partial p}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x} = V_2$$

$$V_2' = q\frac{\partial x}{\partial p}\frac{\partial}{\partial x} = \frac{\partial}{\partial x} = V_1$$

$$V_3' = p\frac{\partial x}{\partial q}\frac{\partial}{\partial x} = -x^2\frac{\partial}{\partial x} = -V_3$$

$$V_4' = q\frac{\partial x}{\partial q}\frac{\partial}{\partial x} = -x\frac{\partial}{\partial x} = -V_2$$

the three vector fields generating the previous transformations

As the matrix transformation generated by these vector fields, $M \in SL(2;\mathbb{R})$, these vector fields V_α are the Lie algebra $L(SL(2,\mathbb{R})) = sl(2;\mathbb{R})$.

Indeed, $[V_1, V_2] = V_1$

$[V_2, V_3] = V_3$ hence $sl(2,\mathbb{R})$ is a subalgebra of $vect(\mathbb{R})$

$[V_3, V_1] = -2V_2$

~~Subalgebra of $vect(\mathbb{R})$, $g = \{V_i = f_i(x)\frac{\partial}{\partial x} : [V_i, V_j] \in g \ \forall V_i, V_j \in g\}$~~

$\Rightarrow f_i f_j' - f_j f_i'$ ~~$= f_k$ for some i, j, k where $V_i, V_j, V_k \in g$~~

Finite dim subalgebras of $vect(\mathbb{R})$ $\to$ poly subalgebras

$$f(x)\frac{\partial}{\partial x} \longrightarrow \text{poly}(x)\frac{\partial}{\partial x}$$

$x^k \frac{\partial}{\partial x}$, $N > l$ subalgebra with N largest exp

$\left[x^N \frac{\partial}{\partial x}, x^l \frac{\partial}{\partial x}\right] = (l-N)\, x^{N+l-1} \frac{\partial}{\partial x}$ $N+l-1 \leq N$ by assumption

$l \leq 1$

$l = 0 : \left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}\right\}$ || $l=1 : \left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\}$

$\hookrightarrow \simeq \left\{x\frac{\partial}{\partial x}, x^N\frac{\partial}{\partial x}\right\}$

2. $\hat{T} : T_p(M) \longrightarrow T_p^*(M)$

$X \in T_p(M) \longmapsto w \in T_p^*(M)$, $w = \hat{T}(X)$

$T : (X, Y) \longrightarrow (\hat{T}(Y))(X)$

$= w(X)$ where $w = \hat{T}(Y)$

$= w_\mu X^\mu$

$T = g_{\mu\nu}$: $g_{\mu\nu} X^\mu Y^\nu = w_\mu X^\mu$ with $w_\mu = g_{\mu\nu} Y^\nu$

T is a tensor $(0,2)$, as the map goes from vectors to scalars.

$V : T_p(M) \longrightarrow T_p(M)$

$X^\mu \longrightarrow Y^\mu = V^\mu{}_\nu X^\nu$ $\quad$ V is a tensor $(1,1)$ as both sides contracted with covectors give scalars.

Identity $\delta^\mu{}_\nu$ is tensor $(1,1)$.

3. V^{ab}

$S_{ab} = S_{ba}$

$A_{ab} = -A_{ba}$

$$V^{ab} S_{ab} = V^{ba} S_{ba} = V^{ba} S_{ab}$$

$$= \frac{1}{2}(V^{ab} S_{ab} + V^{ba} S_{ab}) = V^{(ab)} S_{ab}$$

$$V^{ab} A_{ab} = V^{ba} A_{ba} = -V^{ba} A_{ab}$$

$$= \frac{1}{2}(V^{ab} A_{ab} - V^{ba} A_{ab}) = V^{[ab]} A_{ab}$$

4. In some basis, consider matrix ~~K~~ with elements $[K^{\mu\nu}]$ over $\mathbb{C}$.

For an inner product $\langle \cdot , \cdot \rangle$, (matrix $K_{ij} = K^{\mu\nu}$ with $i=\mu, j=\nu$)

$$|\langle K_i | K_j \rangle|^2 \le \langle K_i | K_i \rangle \langle K_j | K_j \rangle \qquad \text{(Cauchy-Schwarz)}$$

where consider K_i, K_j ~~column~~ vectors

This equality holds when $K_i \sim K_j$ linearly dependent. If equality holds for each pair i,j, then all rows are linearly dependent, i.e.

$$K_{ij} = A_i B_j$$ ~~or Hoz~~ (rows B_j with factors A_i multiplying each).

$$\cancel{\langle K_i | K_j \rangle = \left(\sum_m K_{im} K_{jm} \right)^2 \neq}$$

This approach is not tensorial, i.e. not basis-invariant. However, if this holds in one basis, it must hold in all basis by tensor transformation law.

Could express in a basis invariant manner, if possible to construct a suitable inner product using tensors on the manifold (possible in Riemannian geometry using metric tensor), but pseudo-Riemannian metric unsuitable).

Trying to use det would have similar problems as ε is not a tensor.

In general, it is obvious from above that cannot be written as a direct product.

Can always write as (consider $i = 1 \dots n$, and $(A_i)^\mu = \delta_i^{\ \mu}$).

$$K^{\mu\nu} = \sum_i (A_i)^\mu (B_i)^\nu \qquad \text{(sum of direct products)}$$

The dets of <u>all</u> minors of rank > 1 vanish $\iff K^{\mu\nu} = A^\mu B^\nu$

$\tilde{K} : T_p^* \longrightarrow T_p$ $\qquad \eta, \theta \in T_p^*$

$\theta(\tilde{K}(\eta)) = K(\theta, \eta)$

$\longrightarrow \mathrm{rank}(\tilde{K}) = 1$

5. $F_{\mu\nu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \longrightarrow \tilde{F}_{\mu\nu} = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu} F_{\sigma\rho} = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu}\frac{\partial^2 f}{\partial x^\sigma \partial x^\rho}$

$$= \frac{\partial^2 f}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} \quad \text{(chain rule)}$$

$\therefore F_{\mu\nu}$ is a $\cancel{(1,1)}$ $(0,2)$ tensor

$F'_{\mu\nu} = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial}{\partial x'^\nu}\frac{\partial f}{\partial x^\alpha}\right)$ $\quad df = 0$

$$= \frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}F_{\beta\alpha} + \frac{\partial x^\beta}{\partial x'^\mu}\underbrace{\left(\frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x'^\nu}\right)}_{\neq 0}\underbrace{\frac{\partial f}{\partial x^\alpha}}_{0}$$

$F(X,Y) = X(Y(f))|_p \quad T_p \times T_p \to \mathbb{R}$

$F(X, \alpha Y_1 + \beta Y_2) = X(\alpha Y_1(f) + \beta Y_2(f))$

$= \cancel{X(\alpha)Y_1(f)}^{0} + \alpha X(Y_1(f)) + \cancel{X(\beta)Y_2(f)}^{0} + \beta X(Y_2(f))$

$X_1(f) = df\, Y_1 = 0$

6. $g_{\mu\nu} \longrightarrow \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu} g_{\sigma\rho} = \tilde{g}_{\mu\nu}$

$\tilde{g} = \det(\tilde{g}_{\mu\nu}) = \det\left(\frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\right)\det\left(\frac{\partial x^\rho}{\partial \tilde{x}^\nu}\right)\det(g_{\sigma\rho})$

$\tilde{g} = \det^2\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) g$ scaled by sq of Jacobian det.

7. $[e_\mu, e_\nu] = e_\mu^\alpha \frac{\partial e_\nu}{\partial x^\alpha} - e_\nu^\alpha \frac{\partial e_\mu}{\partial x^\alpha} = \gamma^\rho{}_{\mu\nu} e_\rho$

for components β gives ...

Contract with $\omega^\sigma{}_\beta$,

$$e_\mu^\alpha \frac{\partial e_\nu^\beta}{\partial x^\alpha}\omega^\sigma{}_\beta - e_\nu^\alpha \frac{\partial e_\mu^\beta}{\partial x^\alpha}\omega^\sigma{}_\beta = \gamma^\rho{}_{\mu\nu} e_\rho^\beta \omega^\sigma{}_\beta$$

$$\Rightarrow \frac{\partial}{\partial x^\alpha}\left(e_\mu^\alpha e_\nu^\beta \omega^\sigma{}_\beta\right) - e_\nu^\beta \omega^\sigma{}_\beta \frac{\partial e_\mu^\alpha}{\partial x^\alpha} - e_\mu^\alpha e_\nu^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha}$$

$$- \frac{\partial}{\partial x^\alpha}\left(e_\nu^\alpha e_\mu^\beta \omega^\sigma{}_\beta\right) + e_\mu^\beta \omega^\sigma{}_\beta \frac{\partial e_\nu^\alpha}{\partial x^\alpha} + e_\nu^\alpha e_\mu^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha}$$

$$= \gamma^\sigma{}_{\mu\nu}$$

$$\frac{\partial}{\partial x^\alpha}\left(e_\mu{}^\alpha \delta_\nu{}^\sigma\right) - \delta_\nu{}^\sigma \frac{\partial e_\mu{}^\alpha}{\partial x^\alpha} - e_\mu{}^\alpha e_\nu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha}$$

$$-\frac{\partial}{\partial x^\alpha}\left(e_\nu{}^\alpha \delta_\mu{}^\sigma\right) + \delta_\mu{}^\sigma \frac{\partial e_\nu{}^\alpha}{\partial x^\alpha} + e_\nu{}^\alpha e_\mu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} = \gamma^\sigma{}_{\mu\nu}$$

$$\Rightarrow e_\mu{}^\alpha e_\nu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} - e_\nu{}^\alpha e_\mu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} = -\gamma^\sigma{}_{\mu\nu}$$

Contract with $\omega^\mu{}_\gamma, \omega^\nu{}_\delta$

$$e_\mu{}^\alpha \omega^\mu{}_\gamma e_\nu{}^\beta \omega^\nu{}_\delta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} - e_\nu{}^\alpha \omega^\nu{}_\delta e_\mu{}^\beta \omega^\mu{}_\gamma \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha}$$

$$\delta^\alpha{}_\gamma \delta^\beta{}_\delta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} - \delta^\alpha{}_\delta \delta^\beta{}_\gamma \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} = -\gamma^\sigma{}_{\mu\nu}\omega^\mu{}_\gamma \omega^\nu{}_\delta$$

$$\Rightarrow \frac{\partial \omega^\sigma{}_\delta}{\partial x^\gamma} - \frac{\partial \omega^\sigma{}_\gamma}{\partial x^\delta} = -\gamma^\sigma{}_{\mu\nu}\omega^\mu{}_\gamma \omega^\nu{}_\delta$$

Coordinate induced bases

$$e_\mu = \frac{\partial}{\partial y^\mu} \Rightarrow e_\mu{}^\alpha = \delta_\mu{}^\alpha$$

$$\Rightarrow \frac{\partial e_\mu{}^\alpha}{\partial y^\beta} = \frac{\partial \delta_\mu{}^\alpha}{\partial y^\beta} = 0 \qquad \therefore [e_\mu, e_\nu] = 0 \quad \forall \mu,\nu$$

If $[e_\mu, e_\nu] = 0 \quad \forall \mu,\nu \Rightarrow \gamma^\rho{}_{\mu\nu} = 0 \quad \forall \rho,\mu,\nu$

$$\Rightarrow (\dagger) \quad \frac{\partial \omega^\sigma{}_\delta}{\partial x^\gamma} = \frac{\partial \omega^\sigma{}_\gamma}{\partial x^\delta} \quad \forall \gamma,\delta \quad \forall \sigma \qquad d(\omega^\sigma) = 0 \text{ or } \mathrm{curl}(\omega^\sigma) = 0$$

~~Contract with arbitrary~~ $\cancel{e_\mu{}^\delta}$, Poincaré lemma $\Rightarrow \omega^\sigma = dy^\sigma$ (local)

$$\cancel{e_\mu{}^\delta \frac{\partial \omega^\sigma{}_\delta}{\partial x^\gamma} = (e_\mu \omega^\sigma)_\gamma = (\delta_\mu{}^\sigma)_\gamma \Rightarrow e_\mu{}^\delta \frac{\partial \omega^\sigma{}_\delta}{\partial}}$$

$$\cancel{\frac{\partial}{\partial x^\gamma}\left(e_\mu{}^\delta \omega^\sigma{}_\delta\right)^{\to 0} - \omega^\sigma{}_\delta \frac{\partial e_\mu{}^\delta}{\partial x^\gamma} = (\delta_\mu{}^\sigma)_\gamma} \qquad \text{~~with~~ } \cancel{e_\nu{}^\gamma}$$

$$\cancel{\omega^\sigma{}_\delta e_\nu{}^\gamma \frac{\partial e_\mu{}^\delta}{\partial x^\gamma} = (\delta_\mu{}^\sigma)_\gamma e_\nu{}^\gamma}$$

8. $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

(a) $r = \sqrt{x^2+y^2+z^2}$ $\cos\theta = \frac{z}{r}$ $\tan\phi = \frac{y}{x}$

$\Rightarrow x = r\cos\phi\sin\theta$ $y = r\sin\phi\sin\theta$ $z = r\cos\theta$

$dx = \cos\phi\sin\theta\, dr - r\sin\phi\sin\theta\, d\phi + r\cos\phi\cos\theta\, d\theta$
$dy = \sin\phi\sin\theta\, dr + r\cos\phi\sin\theta\, d\phi + r\sin\phi\cos\theta\, d\theta$
$dz = \cos\theta\, dr - r\sin\theta\, d\theta$

$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

(b) $\tilde{t} = t$ $\tilde{x} = \sqrt{x^2+y^2}\cos(\phi - \omega t)$ $\tilde{y} = \sqrt{x^2+y^2}\sin(\phi - \omega t)$ $\tilde{z} = z$

$d\tilde{t} = dt$ $d\tilde{z} = dz$ $\rho^2 = x^2+y^2$ $x = \rho\cos\phi$
$y = \rho\sin\phi$

$d\tilde{x} = d\rho\cos(\phi-\omega t) - \rho\sin(\phi-\omega t)(d\phi - \omega dt)$
$d\tilde{y} = d\rho\sin(\phi-\omega t) + \rho\cos(\phi-\omega t)(d\phi - \omega dt)$

$d\tilde{x}^2 + d\tilde{y}^2 = d\rho^2 + \rho^2(d\phi^2 - 2\omega d\phi\, dt + \omega^2 dt^2)$

$dx = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$
$dy = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$
$dx^2 + dy^2 = d\rho^2 + \rho^2 d\phi^2$

$\Rightarrow d\tilde{x}^2 + d\tilde{y}^2 = dx^2 + dy^2 + \rho^2(-2\omega d\phi\, dt + \omega^2 dt^2)$

$\tan(\phi - \omega t) = \frac{\tilde{y}}{\tilde{x}}$ $\Rightarrow$ $\frac{d\phi - \omega dt}{\cos^2(\phi-\omega t)} = \frac{\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x}}{\tilde{x}^2}$

$\Rightarrow d\tilde{x}^2 + d\tilde{y}^2 = d\rho^2 + \frac{1}{\rho^2}(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2$

$\rho = r\sin\theta$ $d\rho = \sin\theta\, dr + r\cos\theta\, d\theta$
$d\tilde{z} = dz = \cos\theta\, dr - r\sin\theta\, d\theta$

$d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 = dr^2 + r^2 d\theta^2 + \frac{1}{r^2\sin^2\theta}(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2$

$= dr^2 + r^2 d\theta^2 + \frac{(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2}{\tilde{x}^2 + \tilde{y}^2}$

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 - \frac{(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2}{\tilde{x}^2 + \tilde{y}^2} + \rho^2 d\phi^2$$

$$= -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 - \frac{(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})^2}{\tilde{x}^2 + \tilde{y}^2} + (\tilde{x}^2 + \tilde{y}^2)\left[\frac{\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x}}{\tilde{x}^2 + \tilde{y}^2} + \omega d\tilde{t}\right]^2$$

$$= -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 + 2\omega(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x})d\tilde{t} + (\tilde{x}^2 + \tilde{y}^2)\omega^2 d\tilde{t}^2$$

If $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$,

$$g_{\mu\nu} = \begin{pmatrix} -1 + (\tilde{x}^2 + \tilde{y}^2)\omega^2 & -\omega\tilde{y} & \omega\tilde{x} & 0 \\ -\omega\tilde{y} & 1 & 0 & 0 \\ \omega\tilde{x} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow g^{\mu\nu} = \begin{pmatrix} -1 & -\omega\tilde{y} & \omega\tilde{x} & 0 \\ -\omega\tilde{y} & 1 - \omega^2\tilde{y}^2 & \omega^2\tilde{y}\tilde{x} & 0 \\ \omega\tilde{x} & \omega^2\tilde{y}\tilde{x} & 1 - \omega^2\tilde{x}^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

9. $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \qquad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$$t: \frac{\partial}{\partial\tau}\left(-2\left(1 - \frac{2M}{r}\right)\dot{t}\right) = 0$$

$$r: \frac{\partial}{\partial\tau}\left(\frac{2\dot{r}}{1 - \frac{2M}{r}}\right) = -\frac{2M}{r^2}\dot{t}^2 \pm \frac{\dot{r}^2 \frac{2M}{r^2}}{\left(1 - \frac{2M}{r}\right)^2} + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2$$

$$\theta: \frac{\partial}{\partial\tau}\left(2r^2\dot{\theta}\right) = 2r^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$\phi: \frac{\partial}{\partial\tau}\left(2r^2\sin^2\theta\,\dot{\phi}\right) = 0$$

$$\frac{2M}{r^2}\dot{r}\dot{t} + \left(1-\frac{2M}{r}\right)\ddot{t} = 0 \qquad \Gamma^0{}_{\mu\nu} = \begin{pmatrix} 0 & \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

$$\frac{\ddot{r}}{1-\frac{2M}{r}} - \frac{\frac{2M}{r^2}\dot{r}^2}{\left(1-\frac{2M}{r}\right)^2} = -\frac{M}{r^2}\dot{t}^2 - \frac{\frac{M}{r^2}\dot{r}^2}{\left(1-\frac{2M}{r}\right)^2} + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2$$

$$\Gamma^1{}_{\mu\nu} = \begin{pmatrix} \frac{M}{r^2}\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ & -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ & & -r\left(1-\frac{2M}{r}\right) & 0 \\ & & & r\sin^2\theta\left(1-\frac{2M}{r}\right) \end{pmatrix}$$

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r^2\sin\theta\cos\theta\,\dot{\phi}^2 \qquad \Gamma^2{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & r^{-1} & 0 \\ & & 0 & 0 \\ & & & \sin\theta\cos\theta \end{pmatrix}$$

$$2r\dot{r}\sin^2\theta\,\dot{\phi} + 2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} + r^2\sin^2\theta\,\ddot{\phi} = 0$$

$$\Gamma^3{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & r^{-1} \\ & & 0 & \cot\theta \\ & & & 0 \end{pmatrix}$$

Independent of t and ϕ:

$\left(1-\frac{2M}{r}\right)\dot{t} = \epsilon$ constant (conserved quantity – "energy")

$r^2\sin^2\theta\,\dot{\phi} = k$ constant (conserved quantity – "angular momentum")

[illegible]

10. Timelike geodesic $ds^2 < 0$,

AdS_2 metric

geodesic equations

$$t: \frac{\partial}{\partial \tau}\left(-2t^{-2}\dot{t}\right) = 2t^{-3}\left(\dot{t}^2 - \dot{x}^2\right)$$

$$x: \frac{\partial}{\partial \tau}\left(2t^{-2}\dot{x}\right) = 0 \qquad \Rightarrow \quad t^{-2}\dot{x} = p \quad \text{constant}$$

Choose $0 > ds^2 = -d\tau^2 = t^{-2}(-dt^2 + dx^2)$ parameter τ.

$$\Rightarrow \dot{t}^2 - \dot{x}^2 = t^2$$

From above,

$$\dot{t}^2 - p^2 t^4 = t^2 \quad \Rightarrow \quad \left(\frac{dt}{d\tau}\right)^2 = p^2 t^4 + t^2$$

$$\tau = \int \frac{dt}{\sqrt{t^2 + p^2 t^4}} = \frac{1}{2}\log\frac{\sqrt{1+p^2t^2} - 1}{\sqrt{1+p^2t^2} + 1}$$

$$\Rightarrow \sqrt{1+p^2t^2} = -\coth\tau$$

$$p^2 t^2 = \frac{1}{\sinh^2\tau}$$

$$\dot{x} = pt^2 = \frac{1}{p\sinh^2\tau} \Rightarrow x = \frac{1}{p}\int^\tau \frac{d\tau'}{\sinh^2\tau'} = -\frac{\coth\tau}{p} + \text{const.}$$

$$t = \frac{\pm 1}{p\sinh\tau}$$

$$\therefore x^2 - t^2 = \frac{1}{p^2}$$