Duality

(P) $\underset{x,y\in\mathbb{R}}{\text{minimize}}$ $2x+y$

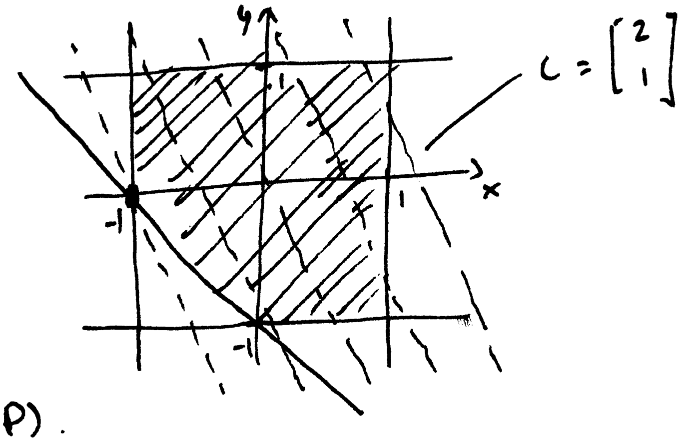
s.t. $x+y+1\geq 0$

$x+1\geq 0$

$y+1\geq 0$

$-x+1\geq 0$

$-y+1\geq 0$



Call p^* the optimal value of (P).

<u>Upper bound on p^*</u>: If (x,y) is any feasible point for (P) then $p^*\leq 2x+y$

e.g. $(x,y)=(0,0)$ is feasible so $p^*\leq 0$

$(x,y)=(-1,0)$ is feasible so $p^*\leq -2$

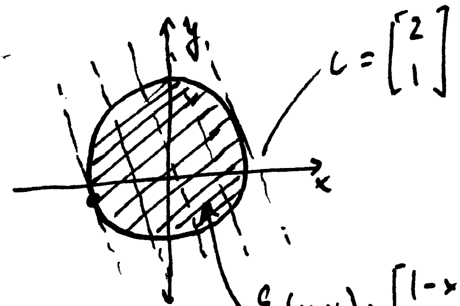
<u>Lower bound on p^*</u>

If we sum up the first two inequalities, we get that any feasible (x,y) must satisfy $2x+y+2\geq 0$. This shows $p^*\geq -2$.

<u>An example in semidefinite programming</u>

(P') $\underset{x,y}{\text{minimize}}$ $2x+y$

s.t. $\begin{bmatrix}1-x & y\\ y & 1+x\end{bmatrix}\succeq 0$



$\left\{(x,y):\begin{bmatrix}1-x & y\\ y & 1+x\end{bmatrix}\succeq 0\right\}$

Call p^* the optimal value on (P').

If $A=\begin{bmatrix}a & b\\ b & c\end{bmatrix}\succeq 0$, then $\underbrace{\mathrm{Tr}\left(\begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}1-x & y\\ y & 1+x\end{bmatrix}\right)}_{\text{trace inner product}}\geq 0$ for any feasible (x,y) of (P')

$\left(x^TMx=\mathrm{Tr}(\underbrace{xx^T}_{\succeq 0}M)\right)$

$$\mathrm{Tr}\left(\begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}1-x & y\\ y & 1+x\end{bmatrix}\right)=a(1-x)+by+by+c(1+x)=(c-a)x+2by+c+a\geq 0$$

if $\begin{bmatrix}a & b\\ b & c\end{bmatrix}\succeq 0$

Let $\alpha=\frac{\sqrt{5}}{2}$

Take $a=\alpha-1$

$c=\alpha+1$

$b=\frac{1}{2}$

Claim: $\begin{bmatrix}a & b\\ b & c\end{bmatrix}\succeq 0$

Proof: $a+c=2\alpha\geq 0$ $\quad$ $\det=ac-b^2=\alpha^2-1-\frac{1}{4}\geq 0$

$\mathrm{Tr}=2x+y+\sqrt{5}$ $\quad\Rightarrow\quad p^*\geq -\sqrt{5}$

Conic programming

Let $K \subseteq \mathbb{R}^n$ be a proper cone

(P) minimize $\langle c, x \rangle$

s.t. $\mathcal{A}(x) = b$ $\qquad \mathcal{A}: \mathbb{R}^n \to \mathbb{R}^m$ linear map

$x \in K$

Call p^* the optimal value of (P).

Lower bounds on p^*:

Assume I can find $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ s.t.

$$c = z + \mathcal{A}^*(y), \quad z \in K^*.$$

Then any feasible point x of (P) satisfies $\langle c, x \rangle \geq \langle b, y \rangle$

Proof:
$$\begin{aligned} \langle c, x \rangle &= \langle z + \mathcal{A}^*(y), x \rangle \\ &= \langle z, x \rangle + \langle y, \mathcal{A}(x) \rangle \\ &= \langle z, x \rangle + \langle y, b \rangle \\ &\geq \langle y, b \rangle \quad \text{using } z \in K^*, x \in K. \end{aligned}$$

<u>Dual problem to (P)</u>

(D) maximize $\langle b, y \rangle$

s.t. $c = z + \mathcal{A}^*(y)$

$z \in K^*$

<u>Theorem</u> Let p^* (respectively d^*) be the optimal values of (P) (resp. (D))

(i) Weak duality: $p^* \geq d^*$

(ii) Strong duality: Assume (P) is strictly feasible, i.e. there exists $x \in \operatorname{int}(K)$ s.t. $\mathcal{A}(x) = b$. Then $p^* = d^*$.