Two repn R_1, R_2 of $\underset{\sim}{g}$ are <u>isomorphic</u> if $\exists$ non-singular matrix S s.t.

$$R_2(X) = S\, R_1(X)\, S^{-1} \quad \forall X \in \underset{\sim}{g}$$

Repn. R with repn space V has an <u>invariant subspace</u> $U \subset V$

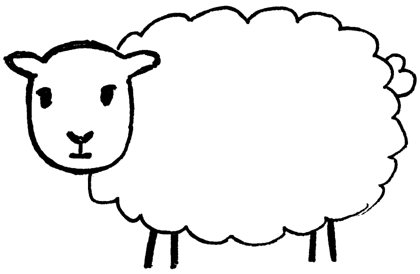
$$R(X)\, u \in U \quad \forall X, u$$

trivial subspace

$$U = \{0\}, \quad U = V$$

An <u>irreducible repn</u> R of $\underset{\sim}{g}$ has no non-trivial subspaces.

Finite dimensional irreps of $\mathcal{L}(SU(2))$



<u>Real basis</u>

$$\{T^a = -\tfrac{1}{2} i \sigma_a,\ a = 1,2,3\}$$

$$\Rightarrow [T^a, T^b] = f^{ab}{}_c\, T^c$$

$$f^{ab}{}_c = \varepsilon_{abc} \qquad a,b,c = 1,2,3$$

- $\mathcal{L}(SU(2)) = \mathrm{Span}_{\mathbb{R}} \{T^a;\ a = 1,2,3\}$

New basis (complex basis)

$$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

↑ not in $\mathcal{L}(SU(2))$, this a complex linear combination

Actually the basis for $\mathcal{L}_{\mathbb{C}}(SU(2)) = \mathrm{Span}_{\mathbb{C}} \{T^a;\ a = 1,2,3\}$

$$[H, E_\pm] = \pm 2 E_\pm$$

$$[E_+, E_-] = H$$

$$ad_H(E_\pm) = \pm 2 E_\pm \qquad ad_H(X) = [H, X] \quad \forall X \in SU(2)$$

$$ad_H(H) = 0$$

$\{H, E_\pm, E_-\}$ are eigenvectors of

$$ad_H \; \mathcal{L}(su(2)) \longrightarrow \mathcal{L}(su(2))$$

with eigenvalues $\{0, +2, -2\}$ known as "roots" of $\mathcal{L}(su(2))$

★ repn R of $\mathcal{L}(su(2))$ ~~with repn space V~~ in repn space V (finite dimn, irreps) ★

- H is diagonalisable.

  Assume $R(H)$ is also diagonalisable.

  $\Rightarrow V$ is spanned by eigenvectors of $R(H)$

  $$\boxed{R(H)\, v_\lambda = \lambda\, v_\lambda} \qquad \lambda \in \mathbb{C}$$

  ↑ weights of R

- $E_\pm$ are known step operators $\qquad // \; R([H, E_\pm]) = \pm 2 R(E_\pm)$

$$R(H)\big[R(E_\pm)\, v_\lambda\big] = \Big(R(E_\pm)\, R(H) + [R(H), R(E_\pm)]\Big)\, v_\lambda$$

$$= (\lambda \pm 2)\big[R(E_\pm)\, v_\lambda\big]$$

$R(E_\pm)\, v_\lambda$ are also eigen vectors of $R(H)$ but with eigenvalues $(\lambda \pm 2)$

- A finite dimensional rep must have <u>heighest weight</u> $\Lambda \in \mathbb{C}$

  s.t. $R(H)\, V_\Lambda = \Lambda\, V_\Lambda$

  $R(E_+)\, V_\Lambda = 0$

- If R is irrep, we expect to get all remaining basis vectors by acting with $R(H)$, $R(E_\pm)$.

$$\underline{V_{\Lambda-2n}} = \underline{(R(E_-))^n\, V_{\Lambda-2n}} \qquad n \in \mathbb{N}$$

$$R(E_+)\, V_{\Lambda-2n} = R(E_+)\, R(E_-)\, V_{\Lambda-2n+2}$$

$$= R(E_-)\, R(E_+)\, V_{\Lambda-2n+2} + (\Lambda - 2n + 2)\, V_{\Lambda-2n+2} \quad (*)$$

$V_\Lambda \xrightarrow{R(E_-)} V_{\Lambda-2} \to V_{\Lambda-4}$, $\quad V_{\Lambda-2} \xrightarrow{R(E_+)} V_\Lambda$

Setting $n = 1$,

$$R(E_+)\, V_{\Lambda-2} = \Lambda\, V_\Lambda$$

Take $n = 2$

$$R(E_+)\, V_{\Lambda-4} = R(E_-)\, R(E_+)\, V_{\Lambda-2} + (\Lambda - 2)\, V_{\Lambda-2}$$

$$= \Lambda\, R(E_-)\, V_\Lambda + (\Lambda - 2)\, V_{\Lambda-2} = (2\Lambda - 2)\, V_{\Lambda-2}$$

$R(E_+)\, V_{\Lambda-2n} \propto V_{\Lambda-2n+2}$ (not any other eigenvector with the same eigenvalues)

- Set const of proportionality

$$R(E_+)\, V_{\Lambda-2n} = r_n\, V_{\Lambda-2n+2} \longrightarrow \text{sub into } \circledast$$

$$\left.\begin{array}{l} \Rightarrow r_n = r_{n-1} + \Lambda - 2n + 2 \quad - (\alpha) \\ R(E_+)\, V_\Lambda = 0 \quad - (\beta) \quad \Rightarrow r_0 = 0 \end{array}\right\} \quad \underline{r_n = (\Lambda + 1 - n)\, n}$$

- For R to be finite dimn, we must have lowest weight $\Lambda - 2N$

$$R(E_-)\, V_{\Lambda-2N} = 0$$

$$\Rightarrow r_{N+1} = 0 \quad \Rightarrow (\Lambda - N)(N+1) = 0, \quad \boxed{N = \Lambda}$$

- **Conclusion**

Finite dimensional irreps R_Λ of $\mathcal{L}(SU(2))$ are labelled by a positive integer $\Lambda \geq 0$

- weight-set:

$$\{-\Lambda, -\Lambda+2, \ldots \Lambda\}$$

non-degenerate, $\dim(R_\Lambda) = \Lambda + 1$

	dim	
R_0	1	trivial
R_1	2	fundamental
R_3	3	adjoint

SU(2) repns from $\mathcal{L}(SU(2))$ repns

- Smooth map $D : G \longrightarrow GL(n, F)$

$$D(g_1)\, D(g_2) = D(g_1 g_2) \quad \forall g_1, g_2 \in G$$

Locally, parameterise group elements $A \in SU(2)$

$$A = \mathrm{Exp}(X) \qquad X \in \mathcal{L}(SU(2))$$

$*\ \mathrm{Exp}(\vec{n}\cdot\vec{\sigma}) = \cos|\vec{n}|\, \mathbb{1} + i(\hat{n}\cdot\vec{\sigma}) \sin|\vec{n}|$

Starting from irreps of $\mathcal{L}(SU(2))$

$$\underline{D_\Lambda(A) = \mathrm{Exp}(R_\Lambda(X))} \qquad \Lambda \in \mathbb{Z},\ \Lambda \geq 0$$

$\rightsquigarrow$ repn of $SU(2)$