Although we are calling these states 'particle', they aren't localised - they are momentum eigenstates.

We can create a localised state via a Fourier transform

$$|\underline{x}\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\underline{p}\cdot\underline{x}}\, |\underline{p}\rangle$$

More generally we create a wavepacket and insert $\psi(p)$

$$|\psi\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\underline{p}\cdot\underline{x}}\, \psi(\underline{p})\, |\underline{p}\rangle$$ which is partially localised in space and momentum

(e.g. $(\psi(p)) \propto e^{-p^2/2m}$

Neither $|\underline{x}\rangle$ nor $|\psi\rangle$ are e'states of H like non-relativistic QM

Relativistic normalisation

We define vacuum s.t. $\langle 0|0\rangle = 1$ and 1 particle state $|\underline{p}\rangle = a_p^\dagger |0\rangle$ satisfies $\langle \underline{p}|\underline{q}\rangle = (2\pi)^3 \delta^3(\underline{p}-\underline{q})$

Is this Lorentz invariant? It's not obvious because we've only written 3-vectors. What could go wrong with Lorentz invariance?

Suppose we have a Lorentz transformation $p^\mu \rightarrow \Lambda^\mu{}_\nu p^\nu = p'^\mu$

We would want the 2 states to be related by a unitary transformation on $|\underline{p}\rangle \longrightarrow U(\Lambda)|\underline{p}'\rangle = |\underline{p}\rangle$ but we haven't been careful about normalisation, so there's no reason why we wouldn't get

$$|p\rangle \rightarrow \underbrace{f(\underline{p},\underline{p}')}_{\text{some unknown } f^n} |\underline{p}'\rangle$$

The trick is to look at an object we know to be Lorentz invariant e.g. the identity operator.

$$\mathbb{1} = \underbrace{\int \frac{d^3p}{(2\pi)^3}}\ \underbrace{|\underline{p}\rangle\langle\underline{p}|}_{\text{individually not L.I.}}$$

Claim: $\int \frac{d^3p}{2E_p}$ is L.I.

Proof: $\int d^4p$ is L.I. relativistic dispersion relation $p_0^2 = E_p^2 = \underset{\text{3-momentum}}{p^2} + m^2$ is L.I

The Sol for p_0 has two branches, $\pm E_p$, but the choice of branch is L.I.
(can't change the sign by L. boosts)

So the following comb is L.I.

$$\int d^4p\, \delta(p_0^2 - \underline{p}^2 - m^2) = \int \frac{d^3p}{2p_0}\Big|_{p_0 = E_p}$$
take $p_0 > 0$ branch

· Claim 2:

$2E_p\, \delta^3(\underline{p} - \underline{q})$ is L.I.

proof: $\underbrace{\int \frac{d^3p}{E_p}}_{\text{L.I.}} 2E_p \delta^3(\underline{p} - \underline{q}) = 1$

From this, we learn that the correctly normalized states are

$$\underset{\text{4-vector}}{|p\rangle} = \sqrt{2E_p}\, |\underline{p}\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

These new states satisfy $\langle p | q \rangle = (2\pi)^3\, 2E_p\, \delta^3(\underline{p} - \underline{q})$ and we can re-write the identity as $\mathbb{1} = \int \frac{d^3p}{(2\pi)^3\, 2E_p} |p\rangle\langle q|$
relativistic states

We can define relativistically normalised creation / annihilation operators

$$a^\dagger(p) = \sqrt{2E_p}\, a^\dagger_{\underline{p}}$$

$$\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} a^\dagger_p\, e^{i\underline{p}\cdot\underline{x}} + \dots$$

$\mathbb{C}$ Scalar field

$$\mathcal{L} = (\partial_\mu \psi^*)(\partial^\mu \psi) - \overset{\text{real}}{\mu^2} \psi^* \psi$$

E-L $\quad \partial_\mu \partial^\mu \psi + \mu^2 \psi = 0$

$\quad \partial_\mu \partial^\mu \psi^* + \mu^2 \psi^* = 0$

· check!

Expand $\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} + c_{\underline{p}}^\dagger\, e^{-i\underline{p}\cdot\underline{x}} \right)$

$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_{\underline{p}}^\dagger\, e^{-i\underline{p}\cdot\underline{x}} + c_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} \right)$

and $\quad \pi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = \dot\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{E_p}}{\sqrt{2E_p}} \left(b_p^\dagger\, e^{-i\underline{p}\cdot\underline{x}} - c_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} \right)$

$\pi^\dagger = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_p}{2}} \left(b_{\underline{p}}\, e^{i\underline{p}\cdot\underline{x}} - c_{\underline{p}}^\dagger\, e^{-i\underline{p}\cdot\underline{x}} \right.$

Recall that the theory has a conserved charge classically.

$$Q = i\int d^3x\ \dot{\psi}^*\psi - \psi^*\dot{\psi} = i\int d^3x\ (\pi\psi - \pi^*\psi^*)$$

Commutator relations are $[\psi(x), \pi(y)] = i\delta^3(x-y)$, $[\psi(x), \pi^\dagger(y)] = 0$ $\Leftrightarrow$ $[b_p, b_q^\dagger] = (2\pi)^3\delta^3(p-q)$, $[c_p, c_q^\dagger] = (2\pi)^3\delta(p-q)$

· Claim, after normal ordering

$$Q = \int \frac{d^3p}{(2\pi)^3}\left(c_p^\dagger c_p - b_p^\dagger b_p\right) = \cancel{\int\frac{d^3p}{(2\pi)^3}}\,(\underbrace{N_c}_{\#\ c\ \text{particles}} - \underbrace{N_b}_{\#\ b\ \text{part}^c})$$

They are interpreted as part^c & anti-part^c (both spin 0, mass μ)

For a real scalar field ϕ, part^s = anti-part^c

$[Q, H] = 0 \Rightarrow Q$ is conserved (we are in a free theory, no big deal as N_c and N_b are separately conserved. But in an interacting theory, N_c and N_b are not conserved but Q is)

The Heisenberg Picture

Although our theory is L.I., it's not so obvious.

$\phi(x)$ depends on x not t, and the states evolve in t by the S. equation.

$$i\frac{d|p\rangle}{dt} = H|p\rangle = E_p|p\rangle \Rightarrow |p(t)\rangle = e^{iE_pt}|p(0)\rangle$$

Things look better in the H. picture, with t-dep assigned to the op.s.

$$O_H(t) = e^{iHt}O_s e^{-iHt} \qquad \frac{dO_H(t)}{dt} = i[H, O_H]$$

$O_H = O_S$ at $t=0$. In this language, op. satisify equal t comm relations.

$$[\phi(x,t), \phi(y,t)] = 0 = [\pi(x,t), \pi(y,t)]$$

$$[\phi(x,t), \pi(y,t)] = i\delta^3(x-y)$$

We can check H. eq for ϕ, i.e. $\frac{d\psi}{dt} = i[H,\phi]$ means that the H op ϕ satisfies the K.G. e.g. $\partial_\mu\partial^\mu\phi + m^2\phi = 0$

We write the Fourier transform of $\phi(x)$ by using

$$e^{iHt} a_p e^{-iHt} = e^{-iE_p t} a_p \qquad [H, a_p] = -E_p a_p$$

$$e^{iHt} a_p^\dagger e^{-iHt} = e^{iE_p t} a_p^\dagger$$

absorbs the energy factor

gives $\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip_\mu x^\mu} + a_p^\dagger e^{ip_\mu x^\mu} \right)$

Causality

ϕ, π satisfy equal time comm. relations

$[\phi(x,t), \pi(y,t)] = i\delta^3(x-y)$ What if we chose $\pi(y,t')$?

In particular, causality requires that all space-like operators commute.

i.e. $[O(x), O(y)] = 0 \quad \forall (x-y)^2 < 0$

$\uparrow$ 4-vector

This ensures that measurements at x can't affect measurements at y.

Do we have this in our set-up.

Define $\Delta(x-y) = [\phi(x), \phi(y)]$ RHS ops, LHS $\mathbb{C}$-f^n $\leftarrow$ what do we know about this? We know it is L.I.

$$\Delta(x-y) = \int \underbrace{\frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}}_{\text{L.I.}} \underbrace{\left(e^{-ip(x-y)} - e^{ip(x-y)} \right)}_{\text{L.I}}$$

- It doesn't vanish for time like separation

$$[\phi(x,0), \phi(x,t)] \sim e^{-imt} - e^{imt}$$

- It vanishes for space-like separation

Note that $\Delta(x-y) = 0$ at equal times. But L.I $\Rightarrow$ can only depend on $(x-y)^2$ so it must vanish $\forall (x-y)^2 < 0$.

N.B.

$$[\phi(x,t), \phi(y,t)] = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{p^2+m^2}} \left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)} \right) = 0$$

$\uparrow$ change sign of p integral to flip the sign here

- Also holds in interacting theory. But comm. is only a $\mathbb{C}$ f^n in free theory.