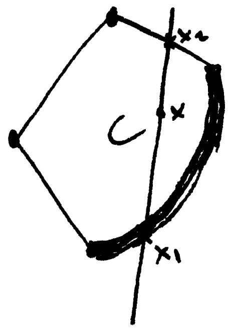
Minkowski's theorem

Let C be a closed and ~~convex~~ bounded convex set in $\mathbb{R}^n$. Let $ext(C)$ be the set of extreme points of C.
Then $C = conv(ext(C))$



Proof By induction on dimension of convex set:

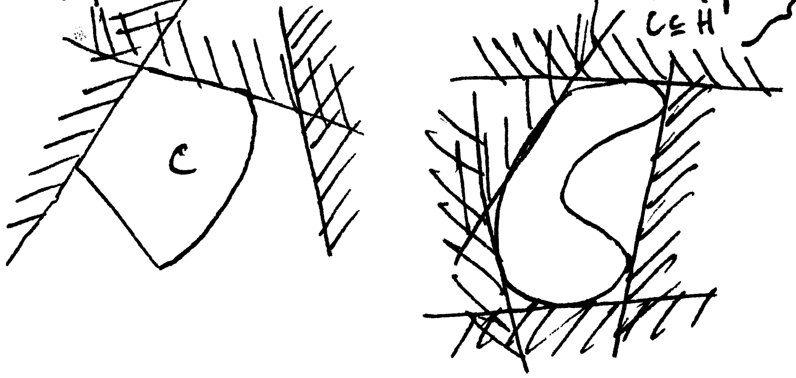
dim = 0 : C is a point (trivial)

dim = 1 : C is a segment (trivial)

$x \in conv(\{x_1, x_2\})$

Since x_1, x_2 lie on the boundary of C, there is a low dim face F_i s.t. $x_i \in F_i$.

Theorem Let C be a closed convex set in $\mathbb{R}^n$. Then C is equal to the intersection of all halfspaces that contain C : $C = \underbrace{\bigcap_{\substack{H \text{ halfspace} \\ C \subseteq H}} H}_{D}$.



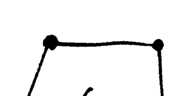
Proof

$C \subseteq D$: trivial

$D \subseteq C$: Let $x \in D$ assume for contradiction that $x \notin C$.

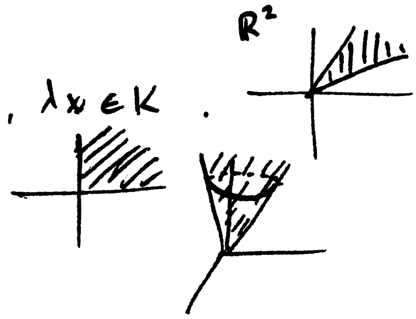
(Strict) Separating hyperplane theorem: $\exists\, a \in \mathbb{R}^n \setminus \{0\}, b \in \mathbb{R}$ s.t. $\begin{cases} \langle a, x \rangle < b \\ \langle a, y \rangle > b \quad \forall y \in C \end{cases}$

$H = \{y : \langle a, y \rangle \geq b\}$ halfspace that contains C #



Duality: "Internal" representation of C : C = convex hull of points

"External" representation of C : C = intersection of halfspaces



Def (Cone): A set $K \subseteq \mathbb{R}^n$ is called a cone if $\forall x \in K$ $\forall \lambda \in \mathbb{R}_{\geq 0}$, $\lambda x \in K$.

Examples:

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \quad i = 1, \dots, n\}$ (non-negative orthant)

$Q^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_2 \leq t\} \subset \mathbb{R}^{n+1}$ (icecream cone)

$S^n_+ = \{X : n \times n$ real symmetric positive semidefinite matrices$\}$ (positive semidefinite cone)

Def (Dual cone) Let K be a cone. The dual cone of K is defined as

$$K^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \quad \forall x \in K\}$$

Theorem Let K be cone. Then K^* is a closed convex cone. Furthermore, if K is closed and convex, then $(K^*)^* = K$.

Proof: • By definition $K^* = \bigcap_{x \in K} \underbrace{\{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0\}}_{H_x \text{ closed halfspace}}$

$\Rightarrow$ K^* is closed convex as an intersection of closed convex sets.

• The proof that $(K^*)^* = K$ when K is closed and convex is left as an exercise (similar to the proof that any convex set is the intersection of halfspaces that contain it).

Def (Extreme ray of a cone) Let K be a cone in $\mathbb{R}^n$. An extreme ray of K is a subset S of K of the form $S = \mathbb{R}_+ v = \{\lambda v : \lambda \geq 0\}$ where $v \neq 0$ s.t. for any $x, y \in K$ $x + y \in S \Rightarrow x, y \in S$.