Definition (endpoint)

$p \in \mathcal{M}$ is a future endpoint of a future directed causal curve $\gamma : (a,b) \to \mathcal{M}$ if for any neighbourhood $\mathcal{O}$ of p, there exists t_0 s.t. $\gamma(t) \in \mathcal{O}$ $\forall t > t_0$. If γ has no future endpoint, we say γ is future inextendible.

E.g. $\gamma : (-\infty, \infty) \to \mathcal{M}$ with $(\mathcal{M}, g)$ Minkowski spacetime

$$\gamma(t) = (t, 0, 0, 0)$$

Def A geodesic is complete if an affine parameter for the geodesic extends $\pm\infty$. A spacetime is geodesically complete if all inextendible causal geodesics are complete.

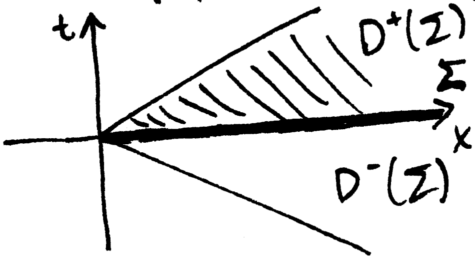
Def A singular spacetime is one that is inextendible and incomplete.

The initial value problem

Predictability

Def Let $(\mathcal{M}, g)$ be a time-orientable spacetime. A partial Cauchy surface Σ is a hypersurface for which no two points are connected by a causal curve.

Def (Domain of dependence) The future domain of dependence of Σ denoted by $D^+(\Sigma)$ is the set of $p \in \mathcal{M}$ s.t. every past-inextendible causal curve through p intersects Σ.



(*)

$$D(\Sigma) = \underbrace{D^+(\Sigma) \cup D^-(\Sigma)}_{\text{domain of dependence}}$$

Hyperbolic PDE $g^{ef}\nabla^c\nabla^d T_{abc\dots} = \dots$ ← only depends on $T_{ab\dots}$, $\nabla_c T_{ab\dots}$

$$0 = \nabla^a \nabla_a \psi = -\partial_t^2 \psi + \partial_x^2 \psi \quad \text{2D wave equation} \quad \psi(0,x),\ \partial_t(0,x)$$

Let's go back to (*) $\Rightarrow$ $\psi(0,x)$, $\partial_t \psi(0,x)$ on Σ, then I can't make predictions on $\mathcal{M} \setminus D(\Sigma)$.

Def A spacetime is globally hyperbolic if it admits a Cauchy surface: a partial Cauchy surface exists s.t. $D(\Sigma) = \mathcal{M}$.

Theorem (Wald) Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime,

i) there exists a global-time function $t : \mathcal{M} \to \mathbb{R}$ and is such that $-(dt)^a$ (normal to surfaces of constant t) is future-directed and timelike.

ii) surfaces of constant t are Cauchy surfaces and the spacetime is topologically $\mathbb{R} \times \Sigma$.

iii) $\mathcal{M}$ has the same topology.

What should we prescribe in GR to evolve Einstein equation?

Extrinsic curvature

Let Σ be a spacelike or timelike hypersurface with unit normal n^a, $n^a n_a = \pm 1$. (upper sign for timelike)

Lemma For any $p \in \Sigma$, let $h^a{}_b = \delta^a{}_b \mp n^a n_b$: $h^a{}_b n^b = 0$, then

1) $h^a{}_c h^c{}_b = h^a{}_b$

2) for any vector at p, X^a, we can decompose uniquely as $X^a_{\parallel} + X^a_{\perp}$ where

$X^a_{\parallel} = h^a{}_b X^b$ and $X^a_{\perp} = \pm n_b X^b n^a$

3) if X^a and Y^a are tangent to Σ $h^{ab} X_a Y_b = g^{ab} X_a Y_b$

Let N^a be a normal to Σ (not necessarily unit) at p, and consider parallel transport of N^a along a curve in Σ with tangent X^b. $X^b \nabla_b N^a = 0$

Take another vector Y^a tangent to Σ: $Y^a N_a = 0$ at p. Consider,

$X(Y^a N_a) = X^b \nabla_b (Y^a N_a) = N_a X^b \nabla_b Y^a = 0$

Def Up to now, n_a has been only defined on Σ, so extend n_a to a neighbourhood of Σ in an arbitrary way. The extrinsic curvature K_{ab} is defined

$K(X,Y) = -n_a (\nabla_{X_\parallel} Y_\parallel)^a$