$:H: = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p$

In gravity, the sum of zero point fluctuations should appear on RHS of Einstein's eq as $\Lambda = E_0/V$ ← cosmological const

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

observations suggest $\Lambda \sim (10^{-3}\,eV)^4$ which provides 75% of the Universe's energy budget. Why don't the zero point energies of the SM fields contribute (10^{15} times larger)?

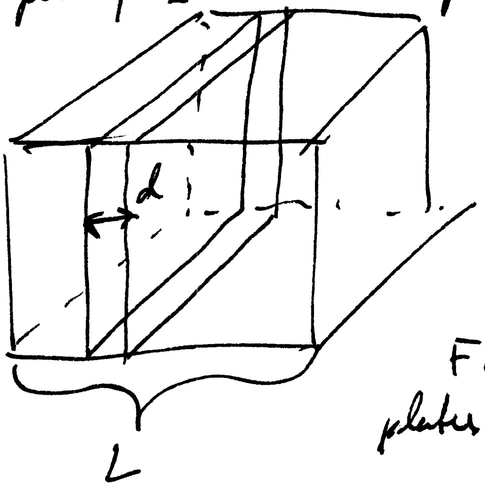
Applications

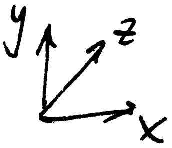
1) The Casimir effect

We happily set $E_0 = 0$ claiming that only energy differences are measured.
But $\exists$ a situation where differences in the vacuum fluctuations themselves can be measured.
To regulate the IR divergences, we make x^1 direction periodic, a box with size L and impose periodic boundary conditions.

$\phi(\underline{x}) = \phi(\underline{x} + L\underline{\hat{n}})$ where $\underline{\hat{n}} = (1,0,0)$ i.e. $\phi(x,y,z) = \phi(x+L,y,z)$.

We put 2 reflecting plates in the box, some distance $d \ll L$ apart. The plates impose $\phi(\underline{x}) = 0$ on the plates.





The presence of the plates means that the momentum of the fields inside them is quantised $\quad E = |p|$

$$\underline{p} = \left(\frac{\pi n}{d}, p_y, p_z\right) \quad n \in \mathbb{Z}$$

For a massless scalar field, the energy between the plates $E(d) = L^2 \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{\pi n}{d}\right)^2 + p_y^2 + p_z^2}$

$E(L-d)$ is the energy outside plates. The total energy

$E = E(d) + E(L-d)$ depends on d

$\Rightarrow \exists$ a force on the plates 'Casimir effect' - predicted 1945, observed 1958

In the lab, the effect is due to the electric field, the plates impose the B.C. We've modeled the effect with scalar field. $E \to \infty$ - comes from high $|p|$ modes.
Try to neglect modes where $|p| \gg a^{-1}$ for some distance scale $a \ll d$: the UV cut off. beyond which the high momentum modes would break through the plates. Cutoff by

$E(d) = L^2 \sum_n \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\ldots}\; e^{-a\sqrt{p_y^2 + p_z^2 + \left(\frac{n\pi}{d}\right)^2}}$ (as $a \to 0$, get prev expr).

Do the simpler problem in 1+1 dimensions.

$$E_{1+1}(d) = \frac{\pi}{2d}\sum_{n=1}^{\infty} n e^{-an\pi/d} = -\frac{1}{2}\frac{\partial}{\partial a}\sum_n e^{-an\pi/d} = -\frac{1}{2}\partial_a \frac{1}{1-e^{-a\pi/d}}$$

$$= \frac{\pi}{2d}\frac{e^{a\pi/d}}{(e^{a\pi/d}-1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}(a^2) \quad (a \ll d)$$

$$E = E(d) + E(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L-d}\right) + \mathcal{O}(a^2)$$

still ∞ as $a \to 0$!

Force is $\frac{\partial E}{\partial d} = \frac{\pi}{24d^2} + \mathcal{O}(d^2/L^2) + \mathcal{O}(a^2)$

is finite as $L \to \infty$ and $a \to 0$.

In 3+1 dimensions $\frac{1}{L^2}\frac{\partial E}{\partial d} = \frac{\pi^2}{480 d^4}$ (true Casimir force is 2x this due to the 2 pol states of the photon).

Recovering Particles

It's easy to verify $[H, a_p^\dagger] = \omega_p a_p^\dagger$ and $[H, a_p] = -\omega_p a_p$ which means (like SHO) we can construct energy eigenstates by acting with $a_p^\dagger$.

Let $|p\rangle = a_p^\dagger|0\rangle$. Then $H|p\rangle = \omega_p|p\rangle$ with $\omega_p^2 = p^2 + m^2$ but $\vec{p}^2 = p^2 + m^2$.

We interpret $|p\rangle$ as the momentum eigenstate of a particle of mass m and momentum p.

Note that in KG $\mathcal{L} = \ldots + \frac{1}{2}m^2\phi^2 + \ldots$ m is the mass of the quantised particle

From now on, will write E_p rather than ω_p.

Let's check this interpretation. After normal ordering,

$$\underline{P} = -\int \pi(\underline{x})\,\nabla\phi(\underline{x})\,d^3\underline{x} = \int \frac{d^3p}{(2\pi)^3}\, p\, a_p^\dagger a_p \quad \text{so}$$

$\underline{P}|p\rangle = p|p\rangle$ i.e. the state has total momentum p.

Can also act with Ang momentum operator $J^i = \epsilon^{ijk}\int d^3x$

to discover $J^i|p\rangle\big|_{p} = 0$ i.e spin 0).

We can create multi-part^c states by acting with more $a_p^\dagger$s. We have the n-part^c state

$|p_1, p_2, \ldots, p_n\rangle = a_{p_1}^\dagger a_{p_2}^\dagger \ldots a_{p_n}^\dagger|0\rangle$. $|p, q\rangle = |q, p\rangle \Rightarrow$ 2-part^c are symmetric under interchange $\Rightarrow$ bosons. Full Hilbert space is spanned by $|0\rangle, a_p^\dagger|0\rangle, a_p^\dagger a_q^\dagger|0\rangle$, etc. is known as Fock space. There is also an operator which counts the number of particles.

$N = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p$. so $N|p_1, \ldots, p_n\rangle = n|p_1, \ldots, p_n\rangle$. $[N, H] = 0 \Rightarrow$ part^c # conserved in the free theory.