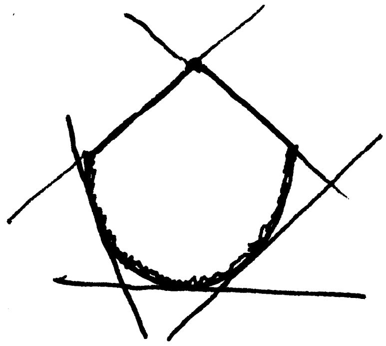
Recap

- Convex sets have two "dual" descriptions
  1) Convex hull of points (internal desc)
  2) Intersection of halfspaces

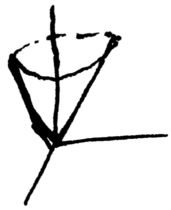


- Conic programs

$$\begin{array}{lll} \text{minimize} & \langle c, x\rangle & (p^*) \\ \text{subject to} & \mathcal{A}(x) = b & \\ & x \in K \text{ proper cone} & \end{array}$$

(LP): Linear programming: $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$

(SOCP): Second-order cone programming: $K = \{x \in \mathbb{R}^n \times \mathbb{R}_+ : \|x\|_2 \leq t\}$



(SDP): Semidefinite programming: $K = S^n_+ = \{X \in S^n : X \succeq 0\}$

"$LP \subset SOCP \subset SDP$"

Duality

$$\begin{array}{lll} \text{maximize} & \langle b, y\rangle & (d^*) \\ \text{subject to} & c = z + \mathcal{A}^*(y) & \\ & z \in K^* & \end{array}$$

Weak duality: $p^* \geq d^*$

Strong duality: $p^* = d^*$ assuming Slater condition holds (either primal or dual)

Slater condition (Primal): $\exists\, x \in \operatorname{int}(K)$ s.t. $\mathcal{A}(x) = b$ (strict feasibility)

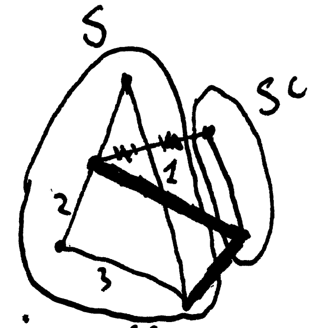
" (Dual): $\exists\, z \in \operatorname{int}(K^*)$ and y s.t. $c = z + \mathcal{A}^*(y)$

Binary quadratic optimization

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximize}} & x^T Q x \\ \text{s.t.} & x_i \in \{-1, +1\}, \quad i = 1, \dots, n \end{array}$$

$Q \in S^n \qquad x^T Q x = \sum_{i,j} Q_{ij} x_i x_j$

Maximum Cut Problem



Let $G = (V, E)$ be a undirected graph with vertex set V and edge set E.

Weight function $w : E \to \mathbb{R}_+$

A cut in G is a partition of V into two parts (S, S^c) where $S^c = V \setminus S$.

The value of this is $\sum_{\substack{i \in S \\ j \in S^c}} w_{ij}$.

Maximum cut

Find a cut in $G=(V,E)$ with maximum value.

Let's formulate the maximum cut problem as a binary quadratic optimization problem.

$i \in S \rightarrow x_i = +1$

$i \in S^c \rightarrow x_i = -1$

Value of cut given by $x \in \{-1,+1\}^V$:

$$\left(\tfrac{1}{2}\ \tfrac{1}{4}\right)\sum_{i,j\in V} w_{ij}(x_i - x_j)^2$$

Maximum cut : maximize $\frac{1}{2}\sum_{i,j\in V} w_{ij}(x_i - x_j)^2 = x^T L_G x$ ← Laplacian of graph $L_G \in S^n$

(V^*) s.t. $x_i \in \{-1,+1\}$, $i=1,\dots,n$

$$(L_G)_{ij} = \begin{cases} \sum_{k\neq i} w_{ik} & \text{if } i=j \\ -w_{ij} & \text{if } i\neq j \end{cases}$$

Semidefinite relaxation: (p^*)

$$\begin{aligned} \underset{X\in S^n}{\text{maximize}} \quad & \operatorname{trace}(L_G X) \\ \text{s.t.} \quad & X \succeq 0 \\ & X_{ii} = 1 \quad i=1,\dots,n \end{aligned}$$

__Claim:__ $p^* \geq v^*$ Proof : ⟶

$x \in \{-1,+1\}^n$

$X = xx^T \succeq 0$

$X_{ii} = \cancel{4}x_i^2 = \cancel{4}1$

$x^T L_G x = tr(L_G x x^T) = tr(L_G X)$

__Claim:__ If the solution X of (SDP) is rank-one then $p^* = v^*$.

Proof: If $\operatorname{rk} X = 1$, then $X = xx^T$

Since $X_{ii} = 1\ \forall i=1,\dots,n$ we have $x_i^2 = 1$

Since $x^T L_G x = \operatorname{trace}(L_G X)$ we have $V^* \geq p^*$

But since we have $p^* \geq v^*$, it holds $p^* = V^*$.