Standard Model

Example sheet 1

1. $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ $\qquad \gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, $\gamma^{\mu\dagger} = (\gamma^0, -\boldsymbol{\gamma})$

$[X,\gamma^\mu] = 0 \quad \forall \mu \quad \Rightarrow \quad X \propto \mathbb{1}$ $\qquad \gamma'^\mu = S\gamma^\mu S^{-1}$ for some S

<u>Def</u> C : $C\gamma^{\mu T}C^{-1} = -\gamma^\mu \quad \Rightarrow$

$$[C^TC^{-1},\gamma^\mu] = C^TC^{-1}\gamma^\mu - \gamma^\mu C^TC^{-1} = C^TC^{-1}\gamma^\mu - (C\gamma^{\mu T})^TC^{-1}$$

$$= C^TC^{-1}\gamma^\mu + (\gamma^\mu C)^TC^{-1} = C^TC^{-1}\gamma^\mu + C^T\gamma^{\mu T}C^{-1}$$

$$= C^TC^{-1}\gamma^\mu - C^TC^{-1}\gamma^\mu = 0 \qquad \square$$

$$\Rightarrow \quad C^TC^{-1} \propto \mathbb{1} \quad \Rightarrow \quad C^T = cC$$

$$(C^T)^T = C = (cC)^T = cC^T = c^2C \quad \Rightarrow \quad c = \pm 1 \qquad \square$$

$$(\gamma^\mu C)^T = C^T\gamma^{\mu T} = cC\gamma^{\mu T} = -c\gamma^\mu C$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad (\gamma^5)^2 = 1 \qquad \{\gamma^5,\gamma^\mu\} = 0$$

$$(\gamma^5C)^T = C^T\gamma^{5T} = iC^T\gamma^{3T}\gamma^{2T}\gamma^{1T}\gamma^{0T} = -ic\gamma^3C\gamma^{2T}\gamma^{1T}\gamma^{0T}$$

$$= +i\gamma^3\gamma^2C^T\gamma^{1T}\gamma^{0T} = -i\gamma^3\gamma^2\gamma^1C^T\gamma^{0T} = +i\gamma^3\gamma^2\gamma^1\gamma^0C^T = c\gamma^5C$$

$$(\gamma^\mu\gamma^5C)^T = (\gamma^5C)^T\gamma^{\mu T} = c\gamma^5C\gamma^{\mu T} = -c\gamma^5\gamma^\mu C = c\gamma^\mu\gamma^5C$$

$$([\gamma^\mu,\gamma^\nu]C)^T = (\gamma^\nu C)^T\gamma^{\mu T} - (\gamma^\mu C)^T\gamma^{\nu T} = -c\gamma^\nu C\gamma^{\mu T} + c\gamma^\mu C\gamma^{\nu T}$$

$$= c\gamma^\nu\gamma^\mu C - c\gamma^\mu\gamma^\nu C = -c[\gamma^\mu,\gamma^\nu]C$$

$\gamma^\mu C$ and $[\gamma^\mu,\gamma^\nu]C$ are 10 (non-zero) matrices (indep) $\Rightarrow$ symmetric

$\therefore\ c = -1$

$$C(\gamma^{\mu_1\cdots\mu_r})^T C^{-1} = C(\gamma^{\mu_1}\cdots\gamma^{\mu_r})^T C^{-1} = C(\gamma^{\mu_r T}\cdots\gamma^{\mu_1 T})C^{-1}$$

$$= C\gamma^{\mu_r T}C^{-1}C\cdots\gamma^{\mu_1 T}C^{-1} = (-1)^r\gamma^{\mu_r}\cdots\gamma^{\mu_1}$$

$$= (-1)^r(-1)^{r-1}\cdots(-1)\gamma^{\mu_1}\cdots\gamma^{\mu_r} = (-1)^{\frac{1}{2}r(r+1)}\gamma^{\mu_1\cdots\mu_r}.$$

$$\Rightarrow (\gamma^{\mu_1\cdots\mu_r}C)^T = C(-1)^{\frac{1}{2}r(r+1)}\gamma^{\mu_1\cdots\mu_r}C$$

$$(-1)^{\frac{1}{2}r(r+1)} = \begin{cases} 1 & : r = 0,3 \mod 4 \\ -1 & : r = 1,2 \mod 4 \end{cases}$$

$$\sum_{r=0,3 \mod 4}\binom{2n}{r} \quad : \quad (\gamma^{\mu_1\cdots\mu_r}C)^T = C\gamma^{\mu_1\cdots\mu_r}C$$

$$\sum_{r=1,2 \mod 4}\binom{2n}{r} \quad : \quad = -C\gamma^{\mu_1\cdots\mu_r}C$$

$$(1+i)^{2n} = \sum_{r=0}^{2n}\binom{2n}{r}i^{2n-r} = \sum_{r=0 \mod 4}\binom{2n}{r} - \sum_{r=2 \mod 4}\binom{2n}{r} + i\left(\sum_{r=1 \mod 4}\binom{2n}{r} - \sum_{r=3 \mod 4}\binom{2n}{r}\right)$$

$$\sum_{r=0 \mod 4}\binom{2n}{r} - \sum_{2}\binom{2n}{r} = \begin{cases} 0 & n = 1,3 \mod 4 \\ 2^n & n = 0 \mod 4 \\ -2^n & n = 2 \mod 4 \end{cases}$$

$$= (1+i)^{2n} = (2i)^n = 2^n i^n$$

$$\sum_{1}\binom{2n}{r} - \sum_{3}\binom{2n}{r} = \begin{cases} 0 & n = 0,2 \mod 4 \\ 2^n & n = 1 \mod 4 \\ -2^n & n = 3 \mod 4 \end{cases}$$

$$\sum_{r=0,3 \mod 4}\binom{2n}{r} - \sum_{r=1,2 \mod 4}\binom{2n}{r} = \begin{cases} 2^n & n = 0,3 \mod 4 \\ -2^n & n = 1,2 \mod 4 \end{cases}$$

$$\Rightarrow C = \begin{cases} 1 & : n = 0,3 \mod 4 \\ -1 & : n = 1,2 \mod 4 \end{cases} \Rightarrow C = (-1)^{\frac{1}{2}n(n+1)}$$

2. If $g'=0$, $\mathcal{L}_I = g\bar{\psi}\psi\phi + g'\bar{\psi}i\gamma^5\psi\phi$

$\hat{P}\psi(x)\hat{P}^{-1} = \gamma^0\psi(x_P)$

$\hat{P}\bar{\psi}(x)\hat{P}^{-1} = \bar{\psi}(x_P)\gamma^0$

$$\hat{P}\mathcal{L}_I(x)\hat{P}^{-1} = g\,\hat{P}\bar{\psi}(x)\psi(x)\phi(x)\hat{P}^{-1}$$

$$= \cancel{g\gamma^0}\, g\,\cancel{\gamma^0}\,\bar{\psi}(x_P)\gamma^0\hat{P}\psi(x)\phi(x)\hat{P}^{-1}$$

$$= g\,\cancel{\gamma^0}\,\bar{\psi}(x_P)\psi(x_P)\,\cancel{\gamma^0}\,\hat{P}\phi(x)\hat{P}^{-1} \qquad \text{as } (\gamma^0)^2 = 1$$

$$\Rightarrow \hat{P}\phi(x)\hat{P}^{-1} = \phi(x_P)$$

If $g=0$,

$$\hat{P}\mathcal{L}_I(x)\hat{P}^{-1} = g'\hat{P}\bar{\psi}(x)i\gamma^5\psi(x)\phi(x)\hat{P}^{-1}$$

$$= g'\bar{\psi}(x_P)\gamma^0 i\gamma^5\hat{P}\psi(x)\phi(x)\hat{P}^{-1}$$

$$= g'\bar{\psi}(x_P)\gamma^0 i\gamma^5\gamma^0\psi(x_P)\hat{P}\phi(x)\hat{P}^{-1}$$

$$= -g'\bar{\psi}(x_P)i\gamma^5\psi(x_P)\hat{P}\phi(x)\hat{P}^{-1}$$

$$\Rightarrow \hat{P}\phi(x)\hat{P}^{-1} = -\phi(x_P)$$

Parity won't be conserved if both g, g' are non-zero.

$$\hat{P}j_5^\mu(x)\hat{P}^{-1} = \hat{P}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\hat{P}^{-1} = \bar{\psi}(x_P)\gamma^0\gamma^\mu\gamma^5\hat{P}\psi(x)\hat{P}^{-1}$$

$$= \bar{\psi}(x_P)\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(x_P) = -\bar{\psi}(x_P)\gamma^0\gamma^\mu\gamma^0\gamma^5\psi(x_P)$$

$$= \begin{cases} -\bar{\psi}(x_P)\gamma^0\gamma^5\psi(x_P) & : \mu = 0 \\ \bar{\psi}(x_P)\gamma^i\gamma^5\psi(x_P) & : \mu = i \in \{1,2,3\} \end{cases}$$

4. $\hat{C}\bar{\psi}(x) X \psi(x)\hat{C}^{-1} = \hat{C}\bar{\psi}(x)\hat{C}^{-1} X \hat{C}\psi(x)\hat{C}^{-1}$

$= -\psi^T C^{-1} X C \bar{\psi}^T = -\psi_\alpha (C^{-1}XC)_{\alpha\beta}\bar{\psi}_\beta$

$= +\bar{\psi}_\beta (C^{-1}XC)^T_{\beta\alpha}\psi_\alpha$ as $\{\psi_\alpha, \bar{\psi}_\beta\} = 0$

$= \bar{\psi}(x) X_c \psi(x)$ where $X_c := C X^T C^{-1}$

$\hat{T}\bar{\psi}(x) X \psi(x)\hat{T}^{-1} = \hat{T}\bar{\psi}(x)\hat{T}^{-1}\hat{T}(X\psi(x))\hat{T}^{-1}$

$= \hat{T}\bar{\psi}(x)\hat{T}^{-1} X^* \hat{T}\psi(x)\hat{T}^{-1} =$ as $\hat{T}$ antilinear

$= \bar{\psi}(x_T) B^{-1} X^* B \psi(x_T)$

$= \bar{\psi}(x_T) X_T \psi(x_T)$ where $X_T := B^{-1} X^* B$.

$\hat{C}\bar{\psi}(x)\psi(x)\hat{C}^{-1} = \bar{\psi}(x)\psi(x)$ $\quad$ $\hat{T}\bar{\psi}(x)\psi(x)\hat{T}^{-1} = \bar{\psi}(x_T)\psi(x_T)$

$\hat{C}\bar{\psi}(x) i\gamma^5 \psi(x)\hat{C}^{-1} = \bar{\psi}(x) i\gamma^5\psi(x)$ $\quad$ $\hat{T}\bar{\psi}(x) i\gamma^5\psi(x)\hat{T}^{-1} = -\bar{\psi}(x_T) i\gamma^5\psi(x_T)$

$\Gamma\, iC\gamma^{5T}C^{-1} = -i(\gamma^5 C)^T C^{-1} = i\gamma^5$ $\quad$ $\Gamma\, -B^{-1} i\gamma^{5*} B = -i\gamma^5$

$\hat{C}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\hat{C}^{-1} = +\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ $\quad$ $\hat{T}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\hat{T}^{-1} = \bar{\psi}(x_T)\gamma^{\mu\dagger}\gamma^5\psi(x)$

$\Gamma\, C(\gamma^\mu\gamma^5)^T C^{-1} = -(\gamma^5 C)^T \gamma^{\mu T} C^{-1} = \gamma^5\gamma^\mu$ $\quad$ $\Gamma\, B^{-1}\gamma^{\mu *}\gamma^{5*}B = \gamma^{\mu\dagger}\gamma^5$

$\langle 0|\bar{\psi}(0) i\gamma^5\psi(0)|\pi(p)\rangle \neq 0$, $\quad \hat{C}|\pi(p)\rangle = \eta_c|\pi(p)\rangle$

$\langle 0|\bar{\psi}(0) i\gamma^5\psi(0)|\pi(p)\rangle = \langle 0|\hat{C}^{-1}\hat{C}\bar{\psi}(0) i\gamma^5\psi(0)\hat{C}^{-1}\hat{C}|\pi(p)\rangle$

$= \eta_c\langle 0|\hat{C}\bar{\psi}(0) i\gamma^5\psi(0)\hat{C}^{-1}|\pi(p)\rangle = \eta_c\langle 0|\bar{\psi}(0) i\gamma^5\psi(0)|\pi(p)\rangle \neq 0$

$\Rightarrow \eta_c = 1$ $\quad$ $\hat{T}|\pi(p)\rangle = \eta_T|\pi(p_T)\rangle$

$\langle 0|\bar{\psi}(0) i\gamma^5\psi(0)|\pi(p)\rangle = \langle 0|\hat{T}^{-1}\hat{T}\bar{\psi}(0) i\gamma^5\psi(0)\hat{T}^{-1}\hat{T}|\pi(p)\rangle$

$= -\eta_T\langle 0|\bar{\psi}(0) i\gamma^5\psi(0)|\pi(p_T)\rangle \neq 0$

$\Rightarrow \eta_T = -1$

$\therefore \eta_c\eta_T\eta_P = 1 \Rightarrow \eta_P = -1$.

6. $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$

$$\bar{\psi}_\pm \gamma^5 = \psi_\pm^\dagger \gamma^0 \gamma^5 = \frac{1}{2}\psi^\dagger(1 \pm \gamma^{5\dagger})\gamma^0\gamma^5 = -\frac{1}{2}\psi^\dagger(\gamma^5 \pm (\gamma^5)^2)\gamma^0$$

$$= \cancel{\tfrac{1}{2}\psi^\dagger(\pm} \frac{1}{2}\psi^\dagger(-\gamma^5 \mp 1)\gamma^0 = \mp \frac{1}{2}\psi^\dagger(1 \pm \gamma^5)\gamma^0$$

$$= \mp \bar{\psi}_\pm \qquad \square$$

$$\Psi_\pm := \begin{pmatrix} \psi_\pm \\ C\bar{\psi}_\mp^T \end{pmatrix}$$

$$\bar{\Psi}_\pm = \Psi_\pm^\dagger \gamma^0 = (\psi_\pm^\dagger, \bar{\psi}_\mp^{T\dagger} C^\dagger)\gamma^0 = (\bar{\psi}_\pm, (\gamma^{0\dagger}\psi_\mp)^T C^\dagger \gamma^0)$$

$$= (\bar{\psi}_\pm, \psi_\mp^T \gamma^{0*} C^\dagger \gamma^0) = (\bar{\psi}_\pm, -\psi_\mp^T (\gamma^0 C)^* \gamma^0)$$

$$= (\bar{\psi}_\pm, -\psi_\mp^T (\gamma^0 C)^\dagger \gamma^0) = (\bar{\psi}_\pm, -\psi_\mp^T C^\dagger)$$

$$= (\bar{\psi}_\pm, -\psi_\mp^T C^{-1}) \qquad \square$$

$$\mathcal{L}_m = \frac{1}{2}\Psi_+^T C^{-1} M \Psi_+ - \frac{1}{2}\bar{\Psi}_+ M^* C \bar{\Psi}_+^T$$

$M^T = M$

$[M, C] = 0$

$[M, \gamma^\mu] = 0$

$$\mathcal{L}_m^\dagger = \frac{1}{2}\Psi_+^\dagger M^\dagger (C^{-1})^\dagger \Psi_+^{T\dagger} - \frac{1}{2}\bar{\Psi}_+^{T\dagger} C^\dagger M^{*\dagger} \bar{\Psi}_+^\dagger$$

$$= \frac{1}{2}\bar{\Psi}_+ \gamma^0 M^* C (\bar{\Psi}_+ \gamma^0)^T - \frac{1}{2}(\gamma^{0\dagger}\Psi_+)^T C^{-1} M^T \gamma^{0\dagger} \Psi_+$$

$$= \frac{1}{2}\bar{\Psi}_+ M^* \gamma^0 C \gamma^{0T} \bar{\Psi}_+^T - \frac{1}{2}\Psi_+^T \gamma^{0*} C^{-1} \gamma^{0\dagger} M \Psi_+$$

$$= -\frac{1}{2}\bar{\Psi}_+ M^* \gamma^0 (\gamma^0 C)^T \bar{\Psi}_+^T - \frac{1}{2}\Psi_+^T (\gamma^0 (\gamma^0 C)^T)^* M \Psi_+$$

$$= -\frac{1}{2}\bar{\Psi}_+ M^* C \bar{\Psi}_+^T + \frac{1}{2}\Psi_+^T C^{-1} M \Psi_+ = \mathcal{L}_m \qquad \square$$

(i) $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} e^{i\theta}$ $\qquad m \in \mathbb{R}_{\geq 0} \qquad \theta \in \mathbb{R}/2\pi\mathbb{Z}$

$$\mathcal{L}_m = \frac{1}{2} \Psi_+^T C^{-1} \mathcal{M} \Psi_+ - \frac{1}{2} \bar{\Psi}_+ \mathcal{M}^* C \bar{\Psi}_+^T$$

$$\psi_\pm \longrightarrow \psi_\pm e^{\mp i\theta/2} \qquad \bar{\psi}_\pm \longrightarrow e^{\pm i\theta/2} \bar{\psi}_\pm$$

$$\Rightarrow \Psi_\pm \longrightarrow \begin{pmatrix} \psi_\pm e^{\mp i\theta/2} \\ C \bar{\psi}_\mp^T e^{\mp i\theta/2} \end{pmatrix} = \Psi_\pm e^{\mp i\theta/2}$$

$$\bar{\Psi}_\pm \longrightarrow \left(\bar{\psi}_\pm e^{\pm i\theta/2} , -\psi_\mp^T C^{-1} e^{\pm i\theta/2} \right) = \bar{\Psi}_\pm e^{\pm i\theta/2}$$

And $\mathcal{L}_m \to \mathcal{L}_m$ with $\mathcal{M}$ real and positive

Then,

$$\mathcal{L}_m = \frac{1}{2} \Psi_+^T C^{-1} \mathcal{M} \Psi_+ - \frac{1}{2} \bar{\Psi}_+ \mathcal{M}^* C \bar{\Psi}_+^T$$

$$= \frac{1}{2} \left(\psi_+ , C \bar{\psi}_-^T \right) C^{-1} \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ C \bar{\psi}_-^T \end{pmatrix}$$

$$- \frac{1}{2} \left(\bar{\psi}_+ , -\psi_-^T C^{-1} \right) \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} C \begin{pmatrix} \bar{\psi}_+ \\ -\psi_-^T C^{-1} \end{pmatrix}$$

$$= \frac{1}{2} \left(\psi_+ , C \bar{\psi}_-^T \right) C^{-1} \begin{pmatrix} C \bar{\psi}_-^T \\ \psi_+ \end{pmatrix} m - \frac{1}{2} \left(-\psi_-^T C^{-1} , \bar{\psi}_+ \right) m C \begin{pmatrix} \bar{\psi}_+ \\ -\psi_-^T C^{-1} \end{pmatrix}$$

$$= \frac{m}{2} \left(\psi_+ \bar{\psi}_-^T + C \bar{\psi}_-^T C^{-1} \psi_+ \right) - \frac{m}{2} \left(-\psi_-^T \bar{\psi}_+ + \bar{\psi}_+ C \psi_-^T C^{-1} \right)$$

$$\left[\bar{\psi}_\pm = \frac{1}{2} \psi^\dagger (1 \pm \gamma^{5\dagger}) \gamma^0 = \frac{1}{2} \psi^\dagger \gamma^0 (1 \mp \gamma^5) \right.$$
$$= \frac{1}{2} \bar{\psi} (1 \mp \gamma^5)$$

$$= \frac{m}{2} \Big(\frac{1}{2} (1 + \gamma^5) \psi \, \frac{1}{2} (1 \pm \gamma^{5T}) \bar{\psi}^T + C \frac{1}{2} (1 \pm \gamma^{5T}) \bar{\psi}^T C^{-1} \frac{1}{2} (1 \pm \gamma^5) \psi$$

$$+ \frac{1}{2} \psi^T (1 - \gamma^{5T}) \frac{1}{2} \bar{\psi} (1 - \gamma^5) + \frac{1}{2} \bar{\psi} (1 \pm \gamma^5) C \frac{1}{2} \psi^T (1 - \gamma^{5T}) C^{-1} \Big)$$

$$\mathcal{L}_K = \bar\psi\, i\partial\!\!\!/\, \psi$$

$$\bar\Psi_+ i\partial\!\!\!/\, \Psi_+ = (\bar\psi_+ , -\psi_-^T C^{-1})\, i\partial\!\!\!/ \begin{pmatrix} \psi_+ \\ C\bar\psi_-^T \end{pmatrix}$$

$$= \bar\psi_+ i\partial\!\!\!/\, \psi_+ - \psi_-^T C^{-1} i\partial\!\!\!/\, C \bar\psi_-^T$$

$$= \bar\psi_+ i\partial\!\!\!/\, \psi_+ - \psi_-^T C^{-1}\gamma^\mu C\, i\partial_\mu \bar\psi_-^T$$

$$= \bar\psi_+ i\partial\!\!\!/\, \psi_+ + \psi_-^T \gamma^{\mu T} i\partial_\mu \bar\psi_-^T$$

$$= \bar\psi_+ i\partial\!\!\!/\, \psi_+ + (\bar\psi_- i\overleftarrow{\partial\!\!\!/}\, \psi_-)^T = \bar\psi_+ i\partial\!\!\!/\, \psi_+ + \bar\psi_- i\partial\!\!\!/\, \psi_-$$

$$\mathcal{L}_K = \bar\psi\, i\partial\!\!\!/\, \psi = (\bar\psi_+ + \bar\psi_-)\, i\partial\!\!\!/\, (\psi_+ + \psi_-)$$

$$= \bar\psi_+ i\partial\!\!\!/\, \psi_+ + \bar\psi_- i\partial\!\!\!/\, \psi_- + \bar\psi_+ i\partial\!\!\!/\, \psi_- + \bar\psi_- i\partial\!\!\!/\, \psi_+$$

$$\ulcorner\ \bar\psi_- i\partial\!\!\!/\, \psi_+ = \bar\psi_- i\partial\!\!\!/\, \tfrac{1}{2}(1+\gamma^5)\psi = \tfrac{1}{2}\bar\psi_-(1-\gamma^5)\, i\partial\!\!\!/\, \psi$$

$$= \tfrac{1}{4}\bar\psi(1+\gamma^5)(1-\gamma^5)\, i\partial\!\!\!/\, \psi = 0$$

$$\bar\psi_+ i\partial\!\!\!/\, \psi_- = 0 \quad \lrcorner$$

$$\mathcal{L}_K = \bar\psi_+ i\partial\!\!\!/\, \psi_+ + \bar\psi_- i\partial\!\!\!/\, \psi_- = \bar\Psi_+ i\partial\!\!\!/\, \Psi_+ \quad \square$$

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_m = \bar\Psi_+ i\partial\!\!\!/\, \Psi_+ + \tfrac{1}{2}\Psi_+^T C^{-1} M \Psi_+ - \tfrac{1}{2}\bar\Psi_+ M^* C \bar\Psi_+^T$$

$$\Psi_+ : \quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Psi_+]}\right) = \partial_\mu\left(\bar\Psi_+ i\gamma^\mu\right)$$

$$= \frac{\partial\mathcal{L}}{\partial\Psi_+} = \Psi_+^T C^{-1} M$$

$$\Rightarrow i\gamma^{\mu T}\partial_\mu \bar\Psi_+^T + M C^{-1}\Psi_+ = 0$$

$$\Rightarrow i C\gamma^{\mu T}\partial_\mu \bar\Psi_+^T + M\Psi_+ = 0$$

$$\Rightarrow i\partial\!\!\!/\, C\bar\Psi_+^T - M\Psi_+ = 0$$

$$\bar{\Psi}_+ : \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}_+)}\right) = 0$$

$$= \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_+} = i\not{\partial}\,\Psi_+ - \mathcal{M}^* C\,\bar{\Psi}_+^T$$

$$\Rightarrow i\not{\partial}\Psi_+ - \mathcal{M}^* C\,\bar{\Psi}_+^T = 0 \qquad \square$$

EoM:

$$\begin{pmatrix} i\not{\partial} & -\mathcal{M}^* C \\ -\mathcal{M} & i\not{\partial}\, C \end{pmatrix}\begin{pmatrix} \Psi_+ \\ \bar{\Psi}_+^T \end{pmatrix} = 0$$

Hence require

$$\det\begin{pmatrix} i\not{\partial} & -\mathcal{M}^* C \\ -\mathcal{M} & i\not{\partial}\, C \end{pmatrix} \det\left(-\not{\partial}^2 C - \mathcal{M}\mathcal{M}^* C\right) = 0$$

$$\Rightarrow \det\left(-\not{\partial}^2 - \mathcal{M}\mathcal{M}^*\right) = 0$$

For plane wave solutions,

$$\det\left(-\not{p}^2 - \mathcal{M}\mathcal{M}^*\right) = 0 \qquad \Rightarrow \qquad \det\left(p^2 - \mathcal{M}^*\mathcal{M}\right) = 0 \qquad \square$$

(ii) $\hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T) \qquad \hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B^{-1}$

$B = \pm\gamma^5 C$

$$\hat{T}\Psi_+(x)\hat{T}^{-1} = \hat{T}\begin{pmatrix} \psi_+(x) \\ C\bar{\psi}_-^T(x) \end{pmatrix}\hat{T}^{-1} = \begin{pmatrix} B\psi_+(x_T) \\ \hat{T}C\bar{\psi}_-^T(x)\hat{T}^{-1} \end{pmatrix}$$

$$\hat{T}\mathcal{L}_m\hat{T}^{-1} = \frac{1}{2}\hat{T}\Psi_+^T C^{-1}\mathcal{M}\Psi_+\hat{T}^{-1} - \frac{1}{2}\hat{T}\bar{\Psi}_+\mathcal{M}^* C\bar{\Psi}_+^T\hat{T}^{-1}$$