$\forall \alpha, \beta \in \Phi$

$[h^\alpha, h^\beta] = 0$

$[h^\alpha, e^\beta] = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} e^\beta$

$$[e^\alpha, e^\beta] = \begin{cases} n_{\alpha,\beta}\, e^{\alpha+\beta} & : \alpha+\beta \in \Phi \\ h^\alpha & : \alpha+\beta = 0 \\ 0 & : \text{otherwise} \end{cases}$$

Simple root

$\Phi = \Phi_+ \cup \Phi_-$

$\delta \in \Phi_s \iff \begin{array}{l} \delta \in \Phi_+ \\ \delta \neq \alpha+\beta \quad \forall \alpha,\beta \in \Phi_+ \end{array}$

Properties of simple roots

i) if $\alpha, \beta \in \Phi$ are simple ($\alpha, \beta \in \Phi_s$) then $\alpha - \beta$ is not a root.

proof suppose $\alpha - \beta \in \Phi \Rightarrow$ i) $\alpha - \beta \in \Phi_+$
ii) $\alpha - \beta \in \Phi_- \Rightarrow \beta - \alpha \in \Phi_+$

i) $\alpha = (\alpha - \beta) + \beta \Rightarrow \alpha$ not simple #

ii) $\beta = (\beta - \alpha) + \alpha \Rightarrow \beta$ not simple # $\square$

ii) if $\alpha, \beta \in \Phi_s$ then "α-string thru β" $\{\beta + n\alpha \in \Phi\}$

has length $l_{\alpha,\beta} = 1 - \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{N} \cup \{0\}$

proof string consists of roots

$S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi,\ n \in \mathbb{Z} \quad n_- \leq n \leq n_+\}$

$(n_+ + n_-) = -\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Now $\alpha, \beta \in \Phi_s \Rightarrow \beta - \alpha \notin \Phi$

$\Rightarrow n_- = 0 \qquad \Rightarrow n_+ = -\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}^+ \cup \{0\}$ — $(*)$

$\Rightarrow$ length of string $l_{\alpha\beta} = n_+ + 1 = 1 - \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{N}$ $\square$

also from $(*)$

iii) $(\alpha,\beta) \leq 0 \quad \forall \alpha, \beta \in \Phi_s$

iv) Any positive root can be written as a linear combination of simple roots with positive integer coefficients.

$\beta \in \Phi_+ \Rightarrow \beta = \sum_i c_i \alpha_{(i)} \qquad \alpha_{(i)} \in \Phi_S \qquad c_j \in \mathbb{N} \cup \{0\}$

proof

- true if $\beta \in \Phi_S$
- if $\beta \notin \Phi_S \Rightarrow \beta = \beta_1 + \beta_2 \qquad \beta_1, \beta_2 \in \Phi_+$

if $\beta_1, \beta_2 \in \Phi_S$ then true

else if $\beta_1 \notin \Phi_S \Rightarrow \beta_1 = \beta_3 + \beta_4 \qquad \beta_3, \beta_4 \in \Phi_+$

and iterate ... □

v) simple roots linearly independent

proof consider all vectors $\lambda \in h^*_{\mathbb{R}}$ of form

$$\lambda = \sum_{i \in J} c_i \alpha_{(i)} \qquad c_i \in \mathbb{R} \setminus \{0\} \qquad \alpha_{(i)} \in \Phi_S \quad \forall i \in J$$

(↑ index set)

define $\lambda_+ = \sum_{i \in J_+} c_i \alpha_{(i)} \qquad J = J_+ \cup J_-$

$\lambda_- = -\sum_{i \in J_-} c_i \alpha_{(i)} \qquad J_+ = \{ i \in J,\ c_i > 0 \}$

$J_- = \{ i \in J,\ c_i < 0 \}$

$\Rightarrow \lambda = \lambda_+ - \lambda_- \qquad (\lambda_+, \lambda_-$ not both zero$)$

$\Rightarrow (\lambda, \lambda) = (\lambda_+, \lambda_+) + (\lambda_-, \lambda_-) - 2(\lambda_+, \lambda_-)$

$> -2(\lambda_+, \lambda_-) = +2 \sum_{i \in J_+} \sum_{j \in J_-} c_i c_j \alpha_{(i)} \alpha_{(j)} \geq 0$ by iii)

$\Rightarrow (\lambda, \lambda) > 0 \Rightarrow \lambda \neq 0$.

$\Rightarrow$ all vectors $\lambda = \sum_{i \in J} c_i \alpha_{(i)}$ are non-zero

$\Rightarrow \alpha_{(i)}$ are linearly independent □

vi) There are exactly $r = \text{Rank}[g]$ simple roots

$|\Phi_S| = r$

proof As simple roots are linearly independent

$\Rightarrow |\Phi_S| \leq r$

Suppose $|\Phi_S| < r$ (i.e. simple roots do not span $h^*_{\mathbb{R}}$)

Then $\exists$ $\lambda \in h^*_{\mathbb{R}}$ such that

$(\lambda, \alpha) = 0 \quad \forall \alpha \in \Phi_S$

$\Rightarrow$ (iv) $(\lambda, \alpha) = 0 \quad \forall \alpha \in \Phi$.

then $H_\lambda = \lambda_i H^i \in h$ is such that

$[H_\lambda, H] = 0 \quad \forall H \in h$

$[H_\lambda, E^\alpha] = (\lambda, \alpha) E^\alpha = 0 \quad \forall \alpha \in \Phi$

$\Rightarrow [H_\lambda, X] = 0 \quad \forall X \in g$ $\Rightarrow$ g has non-trivial ideal $j = \text{span}_{\mathbb{C}}\{H_\lambda\}$ #.

Now choose a basis for $h^*_{\mathbb{R}}$

$B = \{\alpha \in \Phi_S\} = \{\alpha_{(i)}, i = 1, \ldots, r\}$

Define <u>Cartan matrix</u>

$A^{ij} := \dfrac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}$ $r \times r$ matrix $\neq A^{ji}$ $\Rightarrow A^{ij} \in \mathbb{Z} \quad \forall i,j = 1, \ldots, r$

for each $\alpha_{(i)}$ we have an $sl(2)$ subalgebra

generators $\{h^i = h^{\alpha_{(i)}}, e^i_\pm = e^{\pm \alpha_{(i)}}\}$

$[h^i, e^i_\pm] = \pm 2 e^i_\pm$

$[e^i_+, e^i_-] = h^i$

algebra becomes $|\{h^i, e^i_\pm\}| = 3r$

$[h^i, h^j] = 0$

$[h^i, e^j_\pm] = \pm A^{ji} e^j_\pm$ no summation

$[e^i_+, e^j_-] = \delta_{ij} h_i$

if $i \neq j$ $[e^i_+, e^j_-] = [e^{\alpha_{(i)}}, e^{-\alpha_{(j)}}] = 0$ because $\alpha_{(i)} - \alpha_{(j)} \notin \Phi$

$[e^i_+, e^j_+] = \mathrm{ad}_{e^i_+}(e^j_+) \propto e^{\alpha_{(i)} + \alpha_{(j)}}$ if $\alpha_{(i)} + \alpha_{(j)} \in \Phi$

$\mathrm{ad}^n_{e^i_+}(e^j_+) = [e^i_+, [e^i_+, \ldots [e^i_+, e^j_+]] \propto e^{\alpha_{(j)} + n\alpha_{(i)}}$ if $\alpha_{(j)} + n\alpha_{(i)} \in \Phi$

$\hookrightarrow (\mathrm{ad}_{e^i_\pm})^{1 - A^{ji}} e^j_\pm = 0$