The Word-Takahashi identity in QED

The QED action $S[A,\Psi] = \int \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\not{D}\Psi + m\bar{\Psi}\Psi$ is invariant under the global transformations

$$\Psi(x) \mapsto e^{i\alpha}\Psi(x) \qquad \bar{\Psi}(x) \mapsto e^{-i\alpha}\bar{\Psi}(x) \qquad A_\mu(x) \mapsto A_\mu(x)$$

for $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. The path integral measure is also invariant provided we integrate over equal #'s of ψ and $\bar{\psi}$ modes.

Now consider the local transformations

$$\psi \mapsto e^{i\alpha(x)}\psi \qquad \bar{\psi} \mapsto e^{-i\alpha(x)}\bar{\psi} \qquad A_\mu \mapsto A_\mu$$

(N.B. this is <u>not</u> a gauge transformation, because we don't change A_μ itself). The classical action is no longer invariant, and $\delta S = \int j^\mu \partial_\mu \alpha \, d^4x$ with $j^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x)$. This leads to a Ward identity

$$\partial^\mu\langle j_\mu(x)\Psi(x_1)\bar{\Psi}(x_2)\rangle = -\delta^4(x-x_1)\langle\Psi(x_1)\bar{\Psi}(x_2)\rangle + \delta^4(x-x_2)\langle\Psi(x_1)\bar{\Psi}(x_2)\rangle$$

(since $\delta\Psi \sim \Psi$, $\delta\bar{\Psi} \sim \bar{\Psi}$). Let's look at this in momentum space. We define

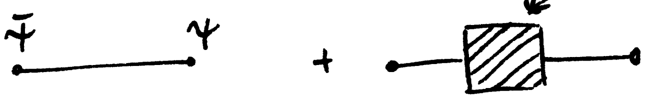
$$\int d^4x_1\, d^4x_2\, e^{ik_1\cdot x_1} e^{-ik_2\cdot x_2}\langle\Psi(x_1)\bar{\Psi}(x_2)\rangle$$

$$= \int d^4y\, d^4x_2\, e^{i(k_1-k_2)\cdot x_2} e^{ik_1\cdot y}\langle\Psi(y)\bar{\Psi}(0)\rangle \qquad \text{where } y = x_1 - x_2$$

$$=: \delta^4(k_1-k_2)\, S(k_1)$$

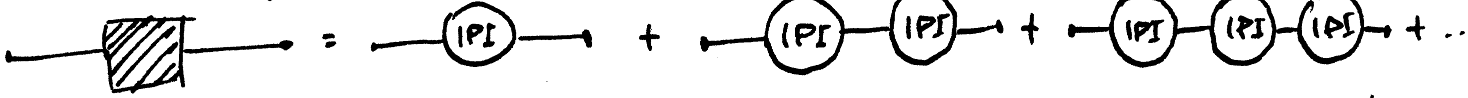
where $S(k) = \int d^4y\, e^{ik\cdot y}\langle\Psi(y)\bar{\Psi}(0)\rangle$ is the 2-pt f^n of the electron field in momentum space.

all possible non-trivial Feynman graphs

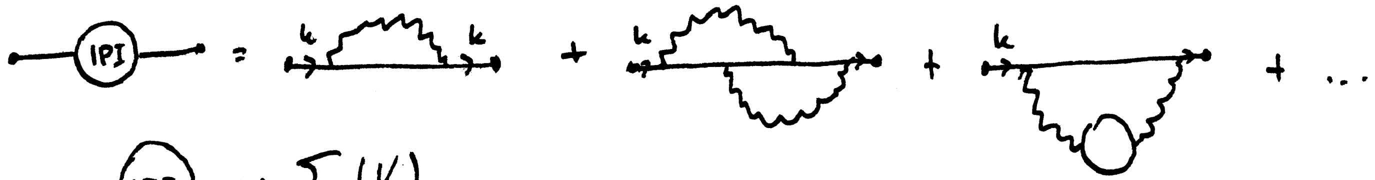
$S(k) =$ 

$$= \frac{1}{i\not{k} + m} + \text{quantum corrections}$$

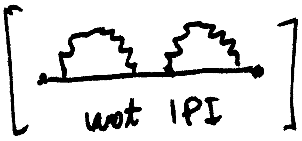
We can usefully write



where (1PI) is the sum of all <u>One-particle irreducible</u> graphs (i.e. connected graph that cannot be made disconnected by cutting any single internal line)



(1PI) $=: \Sigma(\not k)$



. $\Sigma(\not k)$ is called the <u>electron self-energy</u>. In terms of $\Sigma(\not k)$ the exact propagator is

$$S(k) = \frac{1}{i\not k + m} + \frac{1}{i\not k + m}\Sigma(\not k)\frac{1}{i\not k + m} + \frac{1}{i\not k + m}\Sigma(\not k)\frac{1}{i\not k + m}\Sigma(\not k)\frac{1}{i\not k + m} + \dots$$

$$= \frac{1}{i\not k + m - \Sigma(\not k)}$$ by summing the geometric series.

We also define the <u>exact electromagnetic vertex</u> $\Gamma_\mu(k_1,k_2)$ by

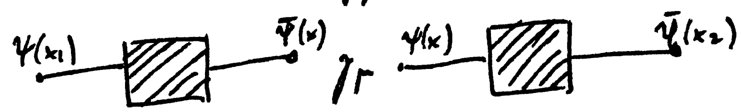
$$\int d^4x\, d^4x_1\, d^4x_2 \langle j_\mu(x)\Psi(x_1)\bar\Psi(x_2)\rangle e^{ip\cdot x} e^{ik_1\cdot x_1} e^{-ik_2\cdot x_2}$$

$$= \int d^4x\, d^4x_1\, d^4x_2 \langle j_\mu(x-x_2)\Psi(x_1-x_2)\bar\Psi(0)\rangle e^{ip\cdot(x-x_2)} e^{ik\cdot(x_1-x_2)} e^{i(p+k_1-k_2)\cdot x_2}$$

$$=: \delta^4(p+k_1-k_2)\, S(k_1)\Gamma_\mu(k_1,k_2)S(k_2)$$ ← exact electron propagator

To understand this, note that since $j_\mu(x) = \bar\Psi(x)\gamma_\mu\Psi(x)$, we have

$$\langle \Psi(x_1) j_\mu(x) \bar\Psi(x_2)\rangle \supset \langle\Psi(x_1)\bar\Psi(x)\rangle\gamma_\mu\langle\Psi(x)\bar\Psi(x_2)\rangle$$



↳ $S(k_1) \quad \gamma_\mu \quad S(k_2)$

these diagrams don't tell us about quantum corrections to the electron-photon <u>vertex</u>.
The remaining diagrams join the two exact electron propagators together

e.g. 

+ ... at leading non-trivial order.

<u>Thus</u> $\Gamma_\mu(k_1,k_2) = \gamma_\mu$ + quantum corrections.

Returning to the Ward identity, we have in momentum space

$$(k_1-k_2)_\mu S(k_1)\Gamma^\mu(k_1,k_2)S(k_2) = iS(k_1) - iS(k_2)$$

$$(k_1-k_2)_\mu \Gamma^\mu(k_1,k_2) = iS^{-1}(k_2) - iS^{-1}(k_1)$$

$$= i\left(i\not k_2 + m - \Sigma(\not k_2) - i\not k_1 - m + \Sigma(\not k_1)\right)$$

$$= (k_1 - k_2)_\mu \gamma^\mu + i\left(\Sigma(\not{k}_1) - \Sigma(\not{k}_2)\right)$$

The <u>point</u> is that quantum corrections to the electron term $i\bar{\Psi}\not{\partial}\Psi$ must be related to quantum corrections to the vertex $\bar{\Psi}A\Psi$. This identity (order-by-order in perturbation theory) is an important check of gauge invariance — the covariant term $i\bar{\Psi}\not{D}\Psi$ should be treated as a whole.

The subtlety is that we <u>assumed</u> $j_\mu(x) = \bar{\Psi}\gamma_\mu\Psi$, i.e. we didn't allow for a change in $\mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}$. If we've regularized by imposing a cut-off $k^2 \leq \Lambda_0$, this is not compatible with $\Psi(x) \to e^{i\alpha(x)}\Psi(x)$.

<u>Wilsonian Renormalization</u>

Suppose we regularize by integrating only over Fourier components of fields with $k^2 < \Lambda_0$. We start from an action

$$S_{\Lambda_0}[\phi] = \int d^dx\ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \sum_i g_i \Lambda_0^{d-d_i}\, \mathcal{O}_i(\phi, \partial\phi)$$

where $\mathcal{O}_i(\phi,\partial\phi)$ are monomials in fields + derivatives and $[\mathcal{O}_i] = d_i$

$$[(\partial\phi)^2] = d \qquad [\partial_\mu] = 1 \qquad \Rightarrow \qquad [\phi] = \frac{d-2}{2}$$

e.g. $[\mathcal{O} = \phi^4] = 2(d-2) = d_{\phi^4}$

$[\mathcal{O} = \phi^2(\partial\phi)^2] = 2(d-2) + 2 = d_{\phi^2(\partial\phi)^2}$.

The power of ~~to~~ the cut-off Λ_0 are a convention chosen so that the couplings g_i are dimensionless.