$$\Gamma^{(n)}_\Lambda(x_1,\dots,x_n; g_i(\Lambda)) = \left[\frac{Z_\Lambda}{Z_{s\Lambda}}\right]^{n/2} \Gamma^{(n)}_{s\Lambda}(x_1,\dots,x_n; g_i(s\Lambda)) \qquad 0<s<1$$

Now let's consider <u>rescaling</u> $x^\mu \to s x^\mu$. The action is invariant provided

$$\int d^dx\, (\partial\phi)^2 \to s^{d-2}\int d^dx\, (\partial\phi)^2$$ so choose $\phi(sx) = s^{\frac{2-d}{2}}\phi(x)$ and similarly

$\Lambda \to \Lambda/s$ for the couplings. With this scaling

$$\Gamma^{(n)}_\Lambda(x_1,\dots,x_n; g_i(\Lambda)) = \left[\frac{Z_\Lambda}{Z_{s\Lambda}}\right]^{n/2} \Gamma^{(n)}_{s\Lambda}(x_1,\dots,x_n; g_i(s\Lambda))$$

$$= \left[\frac{Z_\Lambda}{Z_{s\Lambda}} s^{2-d}\right]^{n/2} \Gamma^{(n)}_\Lambda(sx_1,\dots,sx_n; g_i(s\Lambda))$$ ← n.B. we don't change the values of the g_i.

Equivalently, if $y_i = s x_i$, then

$$\Gamma^{(n)}_\Lambda\left(\frac{y_1}{s},\dots,\frac{y_n}{s}; g_i(\Lambda)\right) = \left[\frac{Z_\Lambda}{Z_{s\Lambda}} s^{2-d}\right]^{n/2} \Gamma^{(n)}_\Lambda(y_1,\dots,y_n; g_i(s\Lambda))$$

In a Lorentz invariant theory, $\Gamma(\{y_i\})$ depends only on $|y_i - y_j|$, so on the left as probing the theory at long distances $|y_i - y_j|/s$. The ~~coupling~~ equality says we can equally compute holding the insertions fixed, but flowing down to the couplings $g_i(s\Lambda)$ appropriate for the low-energy theory.

Infinitesimally, let $s = 1-\delta s$ with $0<\delta s \ll 1$. Then

$$\left[\frac{Z_\Lambda (1-\delta s)^{2-d}}{Z_{(1-\delta s)\Lambda}}\right]^{n/2} \approx 1 - \left[\frac{d-2}{2} + \gamma_\phi\right]\delta s$$ where γ_ϕ is the anomalous dimension of ϕ

Classically, we'd expect $\langle\phi(sx_1),\dots,\phi(sx_n)\rangle$ to scale with s as $\left(\frac{d-2}{2}\right)n$.

The true scaling dimension is $\Delta_\phi = \frac{d-2}{2} + \gamma_\phi$

RG Flow

To begin, suppose we start from a theory where the couplings $g_i = g_i^*$ s.t. $\beta_i(g_j^*) = 0$ e.g. the Gaussian theory with $m^2 = 0$ and $g_i^* = 0$.

More generally, we could have a theory where either 1) classical dimension of all couplings are zero + quantum corrections vanish, or 2) the quantum corrections compensate for non-zero classical dimension.

In either case, the couplings g_i^* are independent of scale

$\gamma_\phi(g_i^*) = \gamma_\phi^*$ too is independent of scale.

The RG eq^n for the 2-point function then becomes

$$0 = \left(\Lambda\frac{\partial}{\partial\Lambda} + \cancel{\beta_i(g_j^*)}\frac{\partial}{\partial g_i} + 2\gamma_\phi(g_j^*)\right)\Gamma_\Lambda^{(2)}(x,y)$$

$$\Rightarrow \Lambda\frac{\partial}{\partial\Lambda}\Gamma_\Lambda(x,y) = -2\gamma_\phi^*\,\Gamma_\Lambda^{(2)}(x,y)$$

On dimensional grounds, $\Gamma_\Lambda^{(2)}(x,y;g_i^*) = f(\Lambda|x-y|, g_i^*)\,\Lambda^{d-2}$.

Then the RG eq^n tells us

$$\Gamma_\Lambda^{(2)}(x,y;g_i^*) = \frac{\Lambda^{d-2}\,c(g_i^*)}{\Lambda^{2\Delta_\phi}\,|x-y|^{2\Delta_\phi}} \propto \frac{c(g_i^*)}{|x-y|^{2\Delta_\phi}}$$

so the anomalous dimensions "correct" the dependence on $|x-y|$.

Now suppose we start close to, but not at, a scale-invariant theory. Set ~~the~~ $g_i = g_i^* + \delta g_i^*$.

We have
$$\Lambda\frac{\partial g_i}{\partial\Lambda}\bigg|_{g_i^*+\delta g_i} = B_{ij}(\{g_i^*\})\,\delta g_j + O(\delta g^2)$$

since $\beta_i(g_j^*) = 0$ by def^n. Let σ_j be an eigenvector of B_{ij} w/ eigenvalue $\Delta_i - d$.

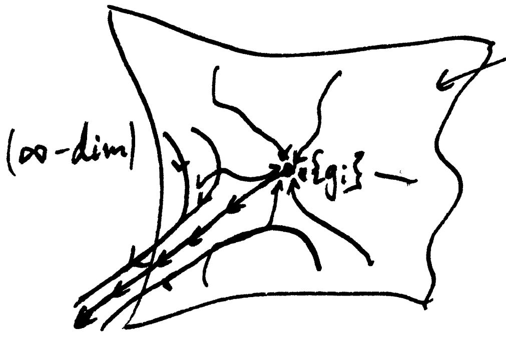
This "eigencoupling" σ_i corresponds to turning on some combinations of interactions in S.

Classically, such an operator would have mass dim d_i, and we define

$\gamma_i = \Delta_i - d_i$ to be the <u>anomalous dimension of the operator</u>.

$$\Rightarrow \Lambda\frac{\partial\sigma_i}{\partial\Lambda} = (\Lambda_i - d)\,\sigma_i \quad\Rightarrow\quad \sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d}\sigma_i(\Lambda_0)$$ to this order

<u>Case 1)</u> $\Delta_i > d$. Then as we lower the cut-off from Λ_0 to $\Lambda \to 0$, we have $\sigma_i(\Lambda) \to 0$. So we flow back to the theory at g_i^* as we move to low energies.



C = critical surface (spanned by eigenvectors of B_{ij} with $\Delta_i > d$)

These operators are called <u>irrelevant</u>.

There are always infinitely many such operators, because we can increase Δ_i by adding more derivatives / fields ($d > 2$).

Case 2) If $\Delta_i < d$ then $\sigma(\Lambda)$ increases as we go to the IR. These operators hence become more significant: they're called relevant. There are only finitely many such relevant operators ($d > 2$). Any RG trajectory emanating from g_i^* is called a critical trajectory.

A more generic QFT will start at scale Λ_0 with both relevant + irrelevant operators turned on. The RG flows of these theories focus on the critical trajectory in the IR. This focusing is called universality: it's the reason physics works!