A Lorentz inv't theory should have eq of motion s.t. if $\phi(x)$ is a solution, then so is $\phi(\Lambda^{-1}x)$. We can ensure this by requiring that S is inv't under Lorentz transformations.

The KG theory : $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

L.T: $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$

$$\partial_\mu\phi(x) \to \frac{\partial}{\partial x^\mu}\left(\phi(\Lambda^{-1}x)\right) = \frac{\partial}{\partial x^\mu}\left(\phi(y)\right) \quad \text{where } y^\mu = \Lambda^{-1\mu}{}_\nu x^\nu$$

$$= \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\left(\phi(y)\right) = (\Lambda^{-1})^\nu{}_\mu(\partial_\nu\phi)(y)$$

Symmetries

The role of symmetries in QFT is more important than in part mechanics

$\exists$ Lorentz symmetry, gauge symmetries, global symmetries, supersymmetries

Noether's Theorem for Field theories

Every cts symmetry of Lagrangian gives rise to a conserved current $j^\mu(x)$ s.t. EoM $\Rightarrow \partial_\mu j^\mu = 0$, i.e. $\partial_0 j^0 + \nabla\cdot\underline{j} = 0$

conserved current $\Rightarrow$ conserved charge $Q = \int_{\mathbb{R}^3} j^0\, d^3\underline{x}$

this follows because $\frac{dQ}{dt} = \int_{\mathbb{R}^3}\frac{dj^0}{dt}\,d^3\underline{x} = -\int_{\mathbb{R}^3}\nabla\cdot\underline{j}\,d^3\underline{x} = 0$

assuming $j^i \to 0$ as $\underline{x}\to\infty$.

Consider making an arbitrary transformation $\phi_a \to \phi_a + \delta\phi_a$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) = \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]}_{\text{" } 0 \text{ when EoM satisfied}}\delta\phi_a$$
$$+\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)$$

If the specific transformation $\delta\phi_a = X_a(\phi_b)$ is a symmetry, then $\delta\mathcal{L} = 0$
$\Rightarrow \partial_\mu j^\mu = 0$ where $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} X_a$. A slight generalisation: relax conditions for a symmetry & still get a conserved current. Say that X_a is a symmetry if $\delta\mathcal{L} = \partial_\mu F^\mu(\phi)$ for some $F^\mu(\phi)$. Repeating calculation:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} X_a(\phi) - F^\mu(\phi).$$

① space + time → energy + momentum
Recall that in classical mechanics, spatial invariance ⇒ conservation of momentum, inv'ce w.r.t. time translation ⇒ conservation of energy. Something similar in FT.

Consider $x^\mu \to x^\mu + \varepsilon^\mu$: $\phi_a(x) \to \phi_a(x) + \varepsilon^\nu\partial_\nu\phi_a(x)$

A Lagrangian that has no explicit x^μ dependence transforms as

$\mathcal{L}(x) \to \mathcal{L}(x) + \varepsilon^\nu\partial_\nu\mathcal{L}(x)$ giving rise to 4 currents, $\nu = 0,1,2,3$

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L} \equiv (T^\mu)_\nu$$

this satisfies $\partial_\mu T^\mu{}_\nu = 0$ where $T^\mu{}_\nu$ is the energy-momentum tensor.

The 4 conserved quantities are

$E = \int d^3x\, T^{00}$, the total energy of the field configuration

$\underline{P}^i = \int d^3x\, T^{0i}$, the total momentum of the field configuration

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e.g. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

$\Rightarrow T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$, so $E = \int d^3x\left(\frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right)$

In this example $T^{\mu\nu}$ symmetric $\mu \leftrightarrow \nu$. $\underline{P}^i = \int d^3x\, \dot\phi\,\partial^i\phi$

Others won't be, but we can always massage it into a symmetric form by adding $\sigma^{\mu\nu}$,

$\sigma^{\mu\nu} = T^{\mu\nu} + \partial_\rho\Gamma^{\rho\mu\nu}$ with $\Gamma^{\rho\mu\nu}$ antisymmetric in $\rho \leftrightarrow \mu$,

$\Rightarrow \partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = 0$. A symmetric energy-momentum tensor of this form is on the RHS of Einstein's field eqs.

② Internal Symmetries

Consider a $\mathbb{C}$ scalar $\Psi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \partial_\mu \Psi^* \partial^\mu \Psi - V(\Psi^* \Psi)$$

where $V(\Psi^*\Psi) = m^2 \Psi^*\Psi + \frac{\lambda}{2}(\Psi^*\Psi)^2 + \ldots$

To find EoM, could expand in ϕ_1, ϕ_2 + continue as before but ~~...~~ its easier (and equivalent) to treat Ψ, Ψ^* as independent variables

e.g. $\left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)}\right]$:

$$\partial_\mu \partial^\mu \Psi + m^2 \Psi + \lambda(\Psi^*\Psi)\Psi + \ldots = 0$$

the $\mathcal{L}$ has a symmetry $\Psi \to e^{i\alpha}\Psi$ or $\delta\Psi = i\alpha\Psi$

and $\delta\Psi^* = -i\alpha\Psi^*$ $(\Psi^* \to e^{-i\alpha}\Psi^*)$. $\delta\mathcal{L} = 0$

gives a current $j^\mu = i(\partial^\mu\Psi^*)\Psi - i(\partial^\mu\Psi)\Psi^*$

Associated conserved charges of this type have the interpretation of electric charge, or part # (e.g. baryon # or lepton #).