Symmetries, Fields and Particles
Example sheet 3

1. (i) $\mathcal{L}_{\mathbb{C}}(SU(2)) \simeq \mathcal{L}(SL(2;\mathbb{C}))$

$L(SU(2)) = \{$ 2×2 traceless anti-Hermitian matrices $\}$

$L_{\mathbb{C}}(SU(2)) = \{$ 2×2 traceless $\mathbb{C}$ matrices $\} \simeq L(SL(2;\mathbb{C}))$

(ii) $L(SL(2;\mathbb{R})) = \{$ 2×2 traceless $\mathbb{R}$ matrices $\}$

$\Rightarrow L_{\mathbb{C}}(SL(2;\mathbb{R})) \simeq L(SL(2;\mathbb{C}))$

$\therefore$ $L_{\mathbb{C}}(SU(2))$ has (inequivalent) real forms ~~R~~: $L(SU(2))$ & $L(SL(2;\mathbb{R}))$

2. $L_{\mathbb{C}}(SO(2n)) = \{$ antisymmetric $2n \times 2n$ matrices $\}$

Cartan subalgebra

$$(H^i)^{\mu\nu} = \delta^{2i-1,\mu}\,\delta^{2i,\nu} - \delta^{2i,\mu}\,\delta^{2i-1,\nu} \qquad i=1,\dots,n \quad \mu,\nu=1,\dots,2n$$

$$\begin{pmatrix} \ddots & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}$$

i-th 2×2 block diagonal

Indeed $[H^i, H^j] = 0$ and are diagonalisable.

Suppose no additional X when $n=k-1$ s.t. $[X, H^i] = 0 \;\; \forall\, i \in \{1,\dots,k-1\}$.

Now consider $n=k$, if $[X, H^i] \neq 0$ for some $i \in \{1,\dots,k-1\}$

then this is still true when append zeros (a block diagonal).

Only need to consider

$$X = \left(\begin{array}{c|c} 0 & a\;b \\ \hline \begin{matrix}-a^T \\ -b^T\end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}\right)$$

Now $[X, H^k] = \left(\begin{array}{c|c} 0 & a\;b \\ \hline \begin{matrix}-a^T \\ -b^T\end{matrix} & 0 \end{array}\right)\left(\begin{array}{c|c} & \\ \hline & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) - \left(\begin{array}{c|c} & \\ \hline & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right)\left(\begin{array}{c|c} 0 & a\;b \\ \hline \begin{matrix}-a^T \\ -b^T\end{matrix} & 0 \end{array}\right)$

$$= \left(\begin{array}{c|c} 0 & -b\;a \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} 0 & 0 \\ \hline \begin{matrix}-b^T \\ a^T\end{matrix} & 0 \end{array}\right) \neq 0$$

And as $[X, H^k]$ has zeros in the upper block, it is also independent of matrices having these.

$\therefore$ No further X s.t. $[X, H^i] = 0 \ \forall \ i \in \{1, \dots, k\}$.

Hence, this is a Cartan subalgebra with basis.

Similarly for $\mathcal{L}_{\mathbb{C}}(SO(2n+1))$ as the odd extra row/column does not give a new basis element and for any matrix with non-zero value in the last row/column can select a basis element with which it does not commute.

Roots from non-zero eigenvalue in $\{E^\alpha ; \alpha \in \Phi\}$

$$ad_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha \qquad \alpha^i \in \mathbb{C}$$

$\mathcal{L}_{\mathbb{C}}(SO(3))$

$$H^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Any } E^\alpha \text{ must have } \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix}$$

$$[H^1, E^\alpha] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -a \\ -b & +a & 0 \end{pmatrix} = \alpha^1 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} b = \alpha^1 a \\ -a = \alpha^1 b \end{matrix} \quad \Rightarrow -a = (\alpha^1)^2 a \quad \text{and as } a \neq 0 \Rightarrow (\alpha^1)^2 = -1$$

$$\therefore \alpha^1 = \pm i \qquad E^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}, \quad E^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}$$

$$\Phi = \{\pm i\}$$

$\mathcal{L}_{\mathbb{C}}(SO(4))$

$$H^1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

$$E^\alpha = \begin{pmatrix} 0 & & a & b \\ & 0 & c & d \\ -a & -c & 0 & \\ -b & -d & & 0 \end{pmatrix} \qquad [H^1, E^\alpha] = \begin{pmatrix} 0 & 0 & c & d \\ 0 & 0 & -a & -b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & -a & 0 & 0 \\ d & -b & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c & d \\ 0 & 0 & -a & -b \\ -c & +a & 0 & 0 \\ -d & +b & 0 & 0 \end{pmatrix} = \alpha^1 \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c = \alpha^1 a & a = -\alpha^1 c \\ d = \alpha^1 b & b = -\alpha^1 d \end{cases} \qquad [H^2, E^\alpha] = \begin{pmatrix} 0 & 0 & b & -a \\ 0 & 0 & d & -c \\ -b & -d & 0 & 0 \\ a & c & 0 & 0 \end{pmatrix} = \alpha^2 E^\alpha$$

$$\Rightarrow \begin{cases} b = \alpha^2 a & a = -\alpha^2 b \\ d = \alpha^2 c & c = -\alpha^2 d \end{cases}$$

$(\alpha^1)^2 = (\alpha^2)^2 = -1$

$$\alpha^1, \alpha^2 = +i, +i \quad , \quad E^{++} = \begin{pmatrix} 0 & & 1 & i \\ & 0 & i & -1 \\ -1 & -i & 0 & \\ -i & 1 & & 0 \end{pmatrix}$$

$$\alpha^1, \alpha^2 = +i, -i \quad , \quad E^{+-} = \begin{pmatrix} 0 & & 1 & -i \\ & 0 & i & 1 \\ -1 & -i & 0 & \\ i & -1 & & 0 \end{pmatrix}$$

$$\alpha^1, \alpha^2 = -i, +i \quad , \quad E^{-+} = \begin{pmatrix} 0 & & 1 & i \\ & 0 & -i & 1 \\ -1 & i & 0 & \\ -i & -1 & & 0 \end{pmatrix}$$

$$\alpha^1, \alpha^2 = -i, -i \quad , \quad E^{--} = \begin{pmatrix} 0 & & 1 & -i \\ & 0 & -i & -1 \\ -1 & i & 0 & \\ i & 1 & & 0 \end{pmatrix}$$

3. $\alpha \in \Phi$ for a simple complex Lie algebra $\mathfrak{g}$ of finite dimension

$$k\alpha \in \Phi, \ k \in \mathbb{C} \overset{?}{\Rightarrow} k = \pm 1$$

$[H, E^\alpha] = \alpha(H) E^\alpha$

$\kappa(H, E^\alpha) = 0 \qquad \forall H \in \mathfrak{h}$

$\kappa(E^\alpha, E^\beta) = 0$

$\underline{\alpha \in \Phi} \Rightarrow -\alpha \in \Phi$ and $\kappa(E^\alpha, E^{-\alpha}) \neq 0$

As $(\alpha, \beta) \in \mathbb{R} \ \forall \alpha, \beta \in \Phi \Rightarrow (\alpha, k\alpha) \in \mathbb{R} \Rightarrow k(\alpha,\alpha) \in \mathbb{R}$

and as $(\alpha,\alpha) \in \mathbb{R} \Rightarrow k \in \mathbb{R}$

Considering root strings

$$\frac{2(\alpha, k\alpha)}{(\alpha,\alpha)} = 2k \in \mathbb{Z} \quad , \quad \frac{2(k\alpha, \alpha)}{(k\alpha, k\alpha)} = \frac{2}{k} \in \mathbb{Z}$$

$\Rightarrow k \in \{\pm\tfrac{1}{2}, \pm 1, \pm 2\}$ $\qquad S_{\alpha,\beta} \leftarrow$ α-string through β

However, considering

$\alpha \to 2\alpha$ excludes $\tfrac{1}{2}\alpha$.

w.l.o.g $k \in \{\pm 1, \pm 2\}$

for $S_{\alpha,\alpha}$, $\frac{2(\alpha,\alpha)}{(\alpha,\alpha)} = 2$ but

$S_{\alpha,\alpha} = \{-2\alpha, -\alpha, 0, \alpha, 2\alpha\}$ #

$\therefore k = \pm 1$ $\square$

5. (i)

$$A_{ij}A_{ji} = 4\,\frac{(\alpha_{(i)},\alpha_{(j)})}{(\alpha_{(j)},\alpha_{(j)})}\,\frac{(\alpha_{(j)},\alpha_{(i)})}{(\alpha_{(i)},\alpha_{(i)})} < 4 \quad \text{by Cauchy-Schwartz}$$

as $(\,,\,)$ shown to be inner product (positive definite) and strict inequality as no colinear roots by question 3 (and -1 not included in Cartan matrix).

(ii) $l, m, n = 0 \Rightarrow$ Cartan matrix block diagonal $\Rightarrow$ algebra not simple #

$m = 0 \Rightarrow m' = 0$ by symmetry of inner product

$$A = \begin{pmatrix} 2 & l & 0 \\ l' & 2 & n \\ 0 & n' & 2 \end{pmatrix} \qquad \begin{matrix} 0 \le ll' < 4 \\ 0 \le nn' < 4 \end{matrix}$$

$$\det A = 2(4 - nn') - 2ll' = 8 - 2nn' - 2ll' > 0$$

$$nn' + ll' < 4$$

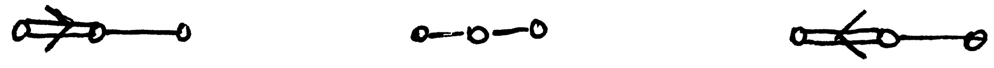
$l, n \neq 0$ as otherwise get a reducible matrix.

nn'	ll'
2	1
1	2
1	1

relabelling (as 1 and 3 both orthogonal to 2)

$\therefore$ Using $(\alpha,\beta) \le 0 \;\forall\, \alpha,\beta \in \Phi_S$

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$



$C_3 \qquad A_3\,(=D_3) \qquad B_3$

These are all rank 3 simple Lie algebras. No additional solutions with $m \neq 0$ as no cyclic Dynkin diagrams, or from $\det A > 0$

$$\underbrace{8 + \underset{\le 0}{lm'n} + \underset{\le 0}{ml'n'}}_{<6} > \underbrace{2(mm' + nn' + ll')}_{>6} \quad \# \text{ when none of } l,m,n \neq 0 \;\square.$$

6. $\mathcal{L}_{\mathbb{C}}(SU(3)) = \{ 3\times 3$ $\mathbb{C}$ traceless matrices $\}$

$$(H^i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha\, i+1}\delta_{\beta\, i+1} \qquad \alpha,\beta = 1,\dots,3 \qquad i = 1,\dots,2$$

$$H^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{CSA}$$

$$[H^1, E^\alpha] = \alpha^1 E^\alpha, \quad [H^2, E^\alpha] = \alpha^2 E^\alpha$$

For $E^\alpha = \begin{pmatrix} 0 & a & b \\ a' & 0 & c \\ b' & c' & 0 \end{pmatrix}$ get $\begin{pmatrix} 0 & 2a & b \\ -2a' & 0 & -c \\ -b' & c' & 0 \end{pmatrix} = \alpha^1 \begin{pmatrix} 0 & a & b \\ a' & 0 & c \\ b' & c' & 0 \end{pmatrix}$

and $\begin{pmatrix} 0 & -a & b \\ a' & 0 & 2c \\ -b' & -2c' & 0 \end{pmatrix} = \alpha^2 \begin{pmatrix} 0 & a & b \\ a' & 0 & c \\ b' & c' & 0 \end{pmatrix}$

$a=1$: $E^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$c=1$: $E^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ } simple roots

$b=1$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha+\beta$

$a'=1$: $-\alpha$, $b'=1$: $-\alpha-\beta$, $c'=1$: $-\beta$

Killing form in CSA

$$K^{11} = \frac{1}{W}\mathrm{Tr}\left[ad_{H^1} \circ ad_{H^1}\right] = \frac{2}{W}\left(\alpha^{1\,2} + \beta^{1\,2} + (\alpha^1+\beta^1)^2\right) = \frac{12}{W}$$

$$K^{22} = \frac{1}{W}\mathrm{Tr}\left[ad_{H^2} \circ ad_{H^2}\right] = \frac{2}{W}\left(\alpha^{2\,2} + \beta^{2\,2} + (\alpha^2+\beta^2)^2\right) = \frac{12}{W}$$

$$K^{12} = \frac{1}{W}\mathrm{Tr}\left[ad_{H^1} \circ ad_{H^2}\right] = \frac{2}{W}\left(\alpha^1\alpha^2 + \beta^1\beta^2 + (\alpha^1+\beta^1)(\alpha^2+\beta^2)\right) = \frac{-6}{W}$$

$$\Rightarrow K \propto \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad K^{-1} \propto \begin{pmatrix} 2 & +1 \\ +1 & 2 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$ This is the Cartan matrix of A_2. $\square$

7. B_2 $\quad$ o⇛o $\quad$ Cartan matrix $\quad A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

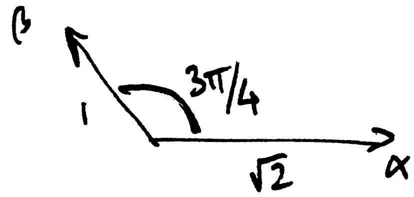
$(=C_2)$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -1 \qquad \frac{2(\beta,\alpha)}{(\beta,\beta)} = -2$$

$|\alpha| = |\beta|\sqrt{2}$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2\frac{|\beta|}{|\alpha|}\cos\varphi_{\alpha\beta} = -1$$

$\cos\varphi_{\alpha\beta} = \frac{-1}{\sqrt{2}} \quad \Rightarrow \quad \varphi_{\alpha\beta} = \frac{3\pi}{4}$



$$l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2$$

$\Rightarrow \beta\ ,\ \beta+\alpha$

$$l_{\beta\alpha} = 1 - \frac{2(\beta,\alpha)}{(\beta,\beta)} = 3$$

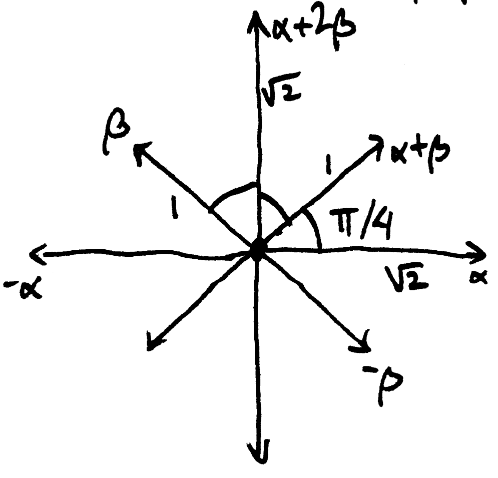
$\Rightarrow \alpha\ ,\ \alpha+\beta\ ,\ \alpha+2\beta$

Check that no further strings.

$$\Phi = \{\pm\alpha\ ,\ \pm\beta\ ,\ \pm(\alpha+\beta)\ ,\ \pm(\alpha+2\beta)\}$$

$$(\alpha+\beta,\alpha+\beta) = (\alpha,\alpha) + (\beta,\beta) + 2(\alpha,\beta) =$$
$$= (\alpha,\alpha)\left(1 + \frac{1}{2} - 1\right) = \cancel{\frac{1}{\sqrt{2}}}\ \frac{1}{2}(\alpha,\alpha) = (\beta,\beta)$$

$$(\alpha+2\beta,\alpha+2\beta) = (\alpha,\alpha) + 4(\beta,\beta) + 4(\alpha,\beta)$$
$$= (\beta,\beta)\big(2 + 4 + (-4)\big) = (\alpha,\alpha)$$



8. A_2

$\{h^\alpha, h^\beta, e^{\pm\alpha}, e^{\pm\beta}, e^{\pm\theta}\}$ $\qquad \theta = \alpha+\beta$

$[h^\alpha, h^\beta] = 0$

$[h^\alpha, e^{\pm\alpha}] = \pm 2\, e^{\pm\alpha}$

$[h^\beta, e^{\pm\beta}] = \pm 2\, e^{\pm\beta}$

$[h^\alpha, e^{\pm\beta}] = \pm \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e^{\pm\beta} = \mp e^{\pm\beta}$

$[h^\beta, e^{\pm\alpha}] = \pm \frac{2(\beta,\alpha)}{(\beta,\beta)} e^{\pm\alpha} = \mp e^{\pm\alpha}$

$[h^\alpha, e^{\pm\theta}] = \pm \frac{2(\alpha,\alpha+\beta)}{(\alpha,\alpha)} e^{\pm\theta} = \pm e^{\pm\theta}$

$[h^\beta, e^{\pm\theta}] = \pm \frac{2(\beta,\alpha+\beta)}{(\beta,\beta)} e^{\pm\theta} = \pm e^{\pm\theta}$

$[e^{+\alpha}, e^{-\alpha}] = h^\alpha$

$[e^{+\beta}, e^{-\beta}] = h^\beta$

$[e^{+\theta}, e^{-\theta}] = h^\theta = h^\alpha + h^\beta$ completes the $sl(2)_\theta$ subalgebra

$[e^\alpha, e^\beta] = n_{\alpha\beta}\, e^\theta$

$[e^{-\alpha}, e^{-\beta}] = n_{-\alpha,-\beta}\, e^{-\theta}$

$[e^\alpha, e^{-\beta}] = [e^{-\alpha}, e^\beta] = 0$

$[e^\alpha, e^\theta] = 0 \qquad [e^{-\alpha}, e^{-\theta}] = 0 \qquad [e^{-\alpha}, e^\theta] = n_{-\alpha,\theta}\, e^\beta$

$[e^\beta, e^\theta] = 0 \qquad [e^{-\beta}, e^{-\theta}] = 0 \qquad [e^{-\beta}, e^\theta] = n_{-\beta,\theta}\, e^\alpha$

$[e^\alpha, e^{-\theta}] = n_{\alpha,-\theta}\, e^{-\beta}$

$[e^\beta, e^{-\theta}] = n_{\beta,-\theta}\, e^{-\alpha}$

$[e^{-\alpha},[e^\alpha, e^\beta]] = n_{\alpha\beta}\, n_{-\alpha,\theta}\, e^\beta$

$= -[e^\beta,[e^{-\alpha}, e^\alpha]] - [e^\alpha,[e^\beta, e^{-\alpha}]]$

$= +[e^\beta, h^\alpha] = e^\beta \quad \Rightarrow \quad n_{-\alpha,\theta}\, n_{\alpha,\beta} = 1 \quad \Rightarrow$

$$\left|\begin{array}{l}
[e^\alpha, e^\beta] = n_+ e^\theta \\
[e^{-\alpha}, e^{-\beta}] = n_- e^{-\theta} \\
[e^{-\alpha}, e^\theta] = \frac{1}{n_+} e^\beta \\
[e^{-\beta}, e^\theta] = -\frac{1}{n_+} e^\alpha
\end{array}\right.$$

$[e^\alpha, e^{-\theta}] = \frac{1}{n_-} e^{-\beta}$

$[e^\beta, e^{-\theta}] = -\frac{1}{n_-} e^{-\alpha}$

$[e^{-\beta},[e^\alpha, e^\beta]] = n_{\alpha\beta}\, n_{-\beta,\theta}\, e^\alpha$

$= -[e^\alpha,[e^\beta, e^{-\beta}]] = -[e^\alpha, h^\beta] = -e^\alpha$

$[e^\alpha,[e^{-\alpha}, e^{-\beta}]] = n_{\alpha,-\beta}\, n_{\alpha,-\theta}\, e^{-\beta} \Rightarrow n_{-\beta,\theta}\, n_{\alpha,\beta} = -1$

$= -[e^{-\beta},[e^\alpha, e^{-\alpha}]] = [h^\alpha, e^{-\beta}] = e^{-\beta}$

$[e^\beta,[e^{-\alpha}, e^{-\beta}]] = n_{-\alpha,-\beta}\, n_{\beta,-\theta}\, e^{-\alpha} \qquad \Rightarrow n_{\alpha,-\theta}\, n_{-\alpha,-\beta} = 1$

$= -[e^{-\alpha},[e^{-\beta}, e^\beta]] = [e^{-\alpha}, h^\beta] = -e^{-\alpha} \qquad \Rightarrow n_{\beta,\theta}\, n_{-\alpha,-\beta} = -1$

Finally, from

$$[[e^\alpha, e^\beta], [e^{-\alpha}, e^{-\beta}]] = [n_+ e^\theta, n_- e^{-\theta}] = (h^\alpha + h^\beta) n_+ n_-$$

$$= -[e^{-\beta}, [[e^\alpha, e^\beta], e^{-\alpha}]] - [e^{-\alpha}, [e^{-\beta}, [e^\alpha, e^\beta]]]$$

$$= +[e^{-\beta}, e^\beta] + [e^{-\alpha}, e^\alpha] = -(h_\alpha + h_\beta)$$

$\Rightarrow n_+ n_- = -1$

n_+ can finally be absorbed into e^θ.

9. A_2 representation R with Dynkin labels $(\Lambda^1, \Lambda^2) = (2, 0)$

$\Rightarrow \Lambda = 2w_{(1)}$

- $\Lambda = 2w_{(1)} \in S_R$

  $\Rightarrow \Lambda - \alpha_{(1)} = 2w_{(1)} - (2w_{(1)} - w_{(2)}) = +w_{(2)} \in S_R$

  $\Lambda - 2\alpha_{(1)} = 2w_{(1)} - 2(2w_{(1)} - w_{(2)}) = -2w_{(1)} + 2w_{(2)} \in S_R$

- $\lambda = w_{(2)} \in S_R$

  $\Rightarrow \lambda - \alpha_{(2)} = w_{(2)} - (2w_{(2)} - w_{(1)}) = -w_{(2)} + w_{(1)} \in S_R$

- $\lambda = -2w_{(1)} + 2w_{(2)} \in S_R$

  $\Rightarrow \lambda - \alpha_{(2)} = -2w_{(1)} + 2w_{(2)} - (2w_{(2)} - w_{(1)}) = -w_{(1)} \in S_R$

  $\lambda - 2\alpha_{(2)} = -2w_{(1)} + 2w_{(2)} - 2(2w_{(2)} - w_{(1)}) = -2w_{(2)} \in S_R$

- $\lambda = -w_{(2)} + w_{(1)} \in S_R$

  $\Rightarrow \lambda - \alpha_{(1)} = -w_{(2)} + w_{(1)} - (2w_{(1)} - w_{(2)}) = -w_{(1)} \in S_R$

Terminates

$$S_R = \{2w_{(1)}, -2w_{(1)} + 2w_{(2)}, -2w_{(2)}, w_{(2)}, -w_{(2)} + w_{(1)}, -w_{(1)}\}$$