Symmetries, Fields and Particles

Examples 1

1. $O(n) = \{ M \in GL(n;\mathbb{R}) : M^T M = I \}$

Closure under matrix multiplication:

$$(MM')^T (MM') = M'^T M^T M M' = M'^T M' = I \quad \forall\, M, M' \in O(n)$$

$$\Rightarrow MM' \in O(n)$$

Matrix multiplication is associative, identity is $I \equiv \mathbb{1}_n$ and unique inverse $M^{-1} = M^T$.

$U(n) = \{ U \in GL(n;\mathbb{C}) : U^\dagger U = I \}$

$$(UU')^\dagger (UU') = U'^\dagger U^\dagger U U' = U'^\dagger U' = I \quad \forall\, U, U' \in U(n)$$

$$\Rightarrow UU' \in U(n)$$

Other properties as before ($U^{-1} = U^\dagger$).

For $M \in GL(n;\mathbb{R})$, $M^T = M^\dagger$, hence $M^T M = I \Leftrightarrow M^\dagger M = I$,

$$O(n) = \{ M \in GL(n;\mathbb{R}) : M^T M = I \} = \{ M \in U(n) \cap GL(n;\mathbb{R}) \}$$

because $M^\dagger M = I$

$\Rightarrow O(n) \subset U(n)$ is a subgroup as $O(n)$ is a group $\square$

$SO(n) = \{ M \in O(n) : \det M = 1 \}$

$SU(n) = \{ U \in U(n) : \det U = 1 \}$

$\Rightarrow SO(n) = \{ M \in U(n) \cap GL(n;\mathbb{R}) : \det M = 1 \} \subset SU(n)$

a subgroup as $SO(n)$ is a group.

$$|U\underline{v}|^2 = (U\underline{v})^\dagger (U\underline{v}) = \underline{v}^\dagger U^\dagger U \underline{v} = \underline{v}^\dagger \underline{v} = |\underline{v}|^2 \quad \forall\, \underline{v} \in \mathbb{C}^n,\ U \in U(n)$$

$\underline{w} = U\underline{v}$ $\quad$ ~~$|w_i| = |U_{ij}||v_j|$~~ $\quad i,j = 1, \dots, n$

$v_i = p_i + i q_i$ $\quad$ $w_i = r_i + i s_i$ $\quad$ ~~$\arg(w_i) = \arg(U_{ij}) + \arg(v_j)$~~

$U_{ij} = A_{ij} + i B_{ij}$ $\quad$ $r_i = \Re w_i = \Re\{U_{ij} v_j\} = \Re\{(A_{ij} + iB_{ij})(p_j + i q_j)\}$

$$= A_{ij} p_j - B_{ij} q_j$$

$s_i = \Im w_i = \Im\{u_{ij} v_j\} = \Im\{(A_{ij} + iB_{ij})(p_j + iq_j)\} = A_{ij} q_j + B_{ij} p_j$

Map $\underline{w} \to \widetilde{\underline{w}} = (r_1, \dots, r_n, s_1, \dots, s_n)^T$ components $\widetilde{w}_\alpha$ $\alpha = 1, \dots, 2n$

$\underline{v} \to \widetilde{\underline{v}} = (p_1, \dots, p_n, q_1, \dots, q_n)^T$ $\widetilde{v}_\alpha$

$$\widetilde{\underline{w}} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \\ s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1n} & -B_{11} & \dots & -B_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \dots & A_{nn} & -B_{n1} & \dots & -B_{nn} \\ B_{11} & \dots & B_{1n} & A_{11} & \dots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ B_{n1} & \dots & B_{nn} & A_{n1} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

$\widetilde{\underline{w}} = \begin{pmatrix} r \\ s \end{pmatrix}, \widetilde{\underline{v}} = \begin{pmatrix} p \\ q \end{pmatrix}$

$$\Rightarrow U \to \widetilde{U} = \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right)$$

Now $\widetilde{\underline{w}}, \widetilde{\underline{v}} \in \mathbb{R}^{2n}$, $\widetilde{U} \in \mathrm{Mat}_{2n}(\mathbb{R})$

$U \in U(n) \Rightarrow U^\dagger U = U^*_{ji} U_{jk} = (A_{ji} - iB_{ji})(A_{jk} + iB_{jk})$

$= A_{ji} A_{jk} + B_{ji} B_{jk} + i(A_{ji} B_{jk} - B_{ji} A_{jk}) = \delta_{ik}$

$\Rightarrow A_{ji} A_{jk} + B_{ji} B_{jk} = \delta_{ik}$, $A_{ji} B_{jk} - B_{ji} A_{jk} = 0$

or $A^T A + B^T B = I$, $A^T B - B^T A = 0$

$$\widetilde{U}^T \widetilde{U} = \left(\begin{array}{c|c} A^T & B^T \\ \hline -B^T & A^T \end{array}\right) \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right)$$

$$= \left(\begin{array}{c|c} A^T A + B^T B & -A^T B + B^T A \\ \hline -B^T A + A^T B & B^T B + A^T A \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) = I$$

$\Rightarrow \widetilde{U} \in O(2n)$

Consider UV, $(UV)_{ij} = U_{ik} V_{kj} = (A_{ik} + iB_{ik})(C_{kj} + iD_{kj})$

$= (A_{ik} C_{kj} - B_{ik} D_{kj}) + i(A_{ik} D_{kj} + B_{ik} C_{kj})$

$UV = (AC - BD) + i(AD + BC)$

$$\widetilde{(UV)} = \left(\begin{array}{c|c} AC-BD & -(AD+BC) \\ \hline (AD+BC) & AC-BD \end{array}\right)$$

$$\tilde{U}\tilde{V} = \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right)\left(\begin{array}{c|c} C & -D \\ \hline D & C \end{array}\right) = \left(\begin{array}{c|c} AC-BD & -(AD+BC) \\ \hline AD+BC & AC-BD \end{array}\right)$$

$\Rightarrow$ closure under matrix multiplication

$\therefore$ $U(n) \simeq \tilde{U}(n) \subseteq O(2n)$ and $\tilde{U}(n)$ is a subgroup of $O(2n)$.

Also, $U(n)$ is smooth, the map to $\tilde{U}(n)$ is smooth, and therefore $\tilde{U}(n)$ is a smooth subgroup of $O(2n)$. Thus, as the $\det M = -1$ ($M \in SO(2n)$) subset are disjoint from the $\det M = +1$ containing identity, then $\tilde{U}(n) \subseteq SO(2n)$ □

2. $M \in O(n)$, $M^T M = I$

$M_{ij} \equiv (\underline{v}_j)_i$ $\qquad M_{ji} M_{jk} = (v_i)_j (v_k)_j = \underline{v}_i^T \cdot \underline{v}_k = \delta_{ik}$

$\Rightarrow$ columns of M are orthonormal vectors

This gives $\frac{1}{2}n(n+1)$ constraints $\left(\binom{2}{n}\right)$, $\dim O(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$

$U \in U(n)$, $U^\dagger U = I$

$U_{ij} \equiv (\underline{u}_j)_i$ $\qquad U^*_{ji} U_{jk} = (\underline{u}^*_i)_j (\underline{u}_k)_j = \underline{u}_i^\dagger \underline{u}_k = \delta_{ik}$

$\Rightarrow$ columns of U are orthonormal complex vectors

This gives $\frac{1}{2}n(n\pm1)$ complex constraints, $n(n-1)$ real constraints, for $i \neq k$, and n real constraints for $i=k$.

$\dim U(n) = 2n^2 - n(n-1) - n = n^2$.

Suppose, for some i, $\underline{u}_i = \alpha_j \underline{u}_j$ $\; j \neq i$, $j = 1 \ldots n$, $\underline{u}_i = \sum_{j\neq i} \alpha_j \underline{u}_j$

Know that $\underline{u}_i^\dagger u_i = 1$ (no sum) $\Rightarrow \left(\sum_{j\neq i}\alpha_j \underline{u}_j\right)^\dagger \left(\sum_{k\neq i}\alpha_k \underline{u}_k\right)$

$$= \left(\sum_{j\neq i}\alpha_j^* \underline{u}_j^\dagger\right)\left(\sum_{k\neq i}\alpha_k \underline{u}_k\right) = \sum_{j,k\neq i}\alpha_j^*\alpha_k \underline{u}_j^\dagger \underline{u}_k = \sum_{j,k\neq i}\alpha_j^*\alpha_k \delta_{jk} = \sum_{j\neq i}|\alpha_j|^2 = 1$$

On the other hand, $\underline{u}_k^\dagger \underline{u}_i = \sum_{j\neq i}\alpha_j \underline{u}_k^\dagger \underline{u}_j = \alpha_k = 0 \;\; \forall k \neq i$ $\therefore$ No $\underline{u}_i$ in the linear span of other $\underline{u}_j$s

3. $\left[R(\underline{n},\theta)\,\underline{n}\right]_i = R(\underline{n},\theta)_{ij}\, n_j = \cos\theta\,\delta_{ij} n_j + (1-\cos\theta) n_i n_j n_j$

$\quad - \sin\theta\,\epsilon_{ijk} n_k n_j = \cos\theta\, n_i + (1-\cos\theta)\, n_i = n_i$

$\Rightarrow$ $\underline{n}$ is an eigenvector of $R(\underline{n},\theta)$ with eigenvalue 1.

$\{\underline{n}, \underline{m}, \underline{\tilde{m}}\}$, $n_i n_i = 1$, $m_i m_i = 1$, $n_i m_i = 0$, $\tilde{m}_i = \epsilon_{ijk} n_j m_k$

$\Rightarrow \tilde{m}_i \tilde{m}_i = 1$, $\tilde{m}_i n_i = 0$

$\left[R(\underline{n},\theta)\,\underline{m}\right]_i = R(\underline{n},\theta)_{ij}\, m_j = \cos\theta\,\delta_{ij} m_j + (1-\cos\theta)\, n_i n_j m_j$

$\quad - \sin\theta\,\epsilon_{ijk} n_k m_j = \cos\theta\, m_i + \sin\theta\,\tilde{m}_i$

$\left[R(\underline{n},\theta)\,\underline{\tilde{m}}\right]_i = R(\underline{n},\theta)_{ij}\,\tilde{m}_j = \cos\theta\,\delta_{ij}\tilde{m}_j + (1-\cos\theta)\, n_i n_j \tilde{m}_j$

$\quad - \sin\theta\,\epsilon_{ijk} n_k \tilde{m}_j = \cos\theta\,\tilde{m}_{\cancel{j}i} - \sin\theta\,(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, n_k n_l m_m$

$\quad = \cos\theta\,\tilde{m}_i - \sin\theta\, m_i$

In basis $\{\underline{n}, \underline{m}, \underline{\tilde{m}}\}$,

$$R(\underline{n},\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$ rotation by angle θ about $\underline{n}$

$\det R(\underline{n},\theta) = 1$ & $R(\underline{n},\theta)^T R(\underline{n},\theta) = I$ $\Rightarrow R(\underline{n},\theta) \in SO(3)$.

4. $G = \left\{ U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \alpha,\beta \in \mathbb{C} : |\alpha|^2 - |\beta|^2 = 1 \right\}$

$\alpha = 1, \beta = 0 \to I$

$UU' = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}\begin{pmatrix} \alpha' & \beta' \\ \beta'^* & \alpha'^* \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \beta\beta'^* & \alpha\beta' + \alpha'^*\beta \\ \alpha'\beta^* + \alpha^*\beta'^* & \alpha^*\alpha'^* + \beta^*\beta' \end{pmatrix} \in G$

$\Big[\ \alpha^*\alpha'^*\alpha\alpha' + \cancel{\alpha^*\alpha'^*\beta\beta'^*} + \cancel{\alpha\alpha'\beta^*\beta'} + \beta^*\beta'\beta\beta'^*$

$\quad -(\cancel{\alpha\beta'\alpha'\beta^*} + \alpha'^*\beta\alpha'\beta^* + \alpha\beta'\alpha^*\beta'^* + \cancel{\alpha'^*\alpha^*\beta\beta'^*})$

$\quad = |\alpha||\alpha'|^2 + |\beta||\beta'|^2 - |\alpha'|^2|\beta|^2 - |\alpha|^2|\beta'|^2 = (|\alpha|^2 - |\beta|^2)(|\alpha'|^2 - |\beta'|^2) = 1\ \Big]$

$u^{-1} = \begin{pmatrix} \alpha^* & -\beta \\ -\beta^* & \alpha \end{pmatrix}$.

$\therefore$ G is a group.

Lie group if G is a smooth manifold.

$\dim G = 4 - 1 = 3$.

$\alpha = x + iy$
$\beta = z + iw$ $\qquad |\alpha|^2 - |\beta^2| = 1 \Rightarrow x^2 + y^2 - z^2 - w^2 = 0$

$\mathcal{M} = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 - z^2 - w^2 = 0\}$

As $|\underline{x}|^2 = x^2 + y^2 + z^2 + w^2 = 2(x^2 + y^2)$ is not bounded for $\underline{x} = (x, y, z, w)$, then $\mathcal{M}$ is a non-compact subset of $\mathbb{R}^4$.

5. $SU(2)$. $U^\dagger U = \mathbb{1}_2$

$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad U^\dagger U = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} =$

$= \begin{pmatrix} \alpha^*\alpha + \gamma^*\gamma & \alpha^*\beta + \gamma^*\delta \\ \alpha\beta^* + \gamma\delta^* & \beta^*\beta + \delta^*\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\alpha^*\alpha + \gamma^*\gamma = 1$
$\beta^*\beta + \delta^*\delta = 1$ $\quad \Rightarrow \quad \delta = -\frac{\alpha^*\beta}{\gamma^*}$ if $\gamma \neq 0$
$\alpha^*\beta + \gamma^*\delta = 0$

$\beta^*\beta + \frac{\alpha\beta^*}{\gamma}\frac{\alpha^*\beta}{\gamma^*} = 1 \Rightarrow \beta^*\beta\left(1 + \frac{\alpha^*\alpha}{\gamma^*\gamma}\right) = 1$

while $\gamma = 0 \Rightarrow \beta = 0$. $\qquad \Rightarrow \beta^*\beta = \gamma^*\gamma$

Identify $|\gamma| = |\beta|$ and $\Rightarrow |\delta| = |\alpha|$

if $\alpha = |\alpha| e^{i\theta}$, $\beta = |\beta| e^{i\phi}$ and $|\alpha|^2 + |\beta|^2 = 1$

$\delta = |\alpha| e^{i\theta'} \quad \gamma = |\beta| e^{i\phi'}$

$\Rightarrow \alpha^*\beta + \gamma^*\delta = |\alpha| e^{-i\theta} |\beta| e^{i\phi} + |\beta| e^{-i\phi'} |\alpha| e^{i\theta'} = 0$

$\Rightarrow e^{i(\phi - \theta)} + e^{i(\theta' - \phi')} = 0$ in general

$\phi - \theta = \theta' - \phi' + \pi \Rightarrow \phi' = \pi - \phi + \theta' + \theta \Rightarrow \gamma = -\beta^* e^{i(\theta' + \theta)}$

~~$\alpha\beta^* + \gamma\delta^* = |\alpha| e^{i\theta} |\beta| e^{-i\phi} + |\beta| e^{i(\pi - \phi + \theta' + \theta)} |\alpha| e^{-i\theta'} = 0$~~

Also $\det U = +1 \qquad \alpha\delta - \beta\gamma = |\alpha|^2 e^{i(\theta + \theta')} + |\beta|^2 e^{i(\theta + \theta')}$
$= e^{i(\theta' + \theta)} = 1 \Rightarrow \theta' = -\theta$.

$$\therefore \quad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$

$$\alpha = a_0 + i a_3$$

$$\beta = a_2 + i a_1$$

then $U = a_0 I + i \underline{a} \cdot \underline{\sigma}$

and $a_0^2 + a_1^2 + a_2^2 + a_3^2 = a_0^2 + \underline{a} \cdot \underline{a} = 1$

$$\begin{aligned}
UU' &= (a_0 I + i\underline{a}\cdot\underline{\sigma})(a_0' I + i\underline{a}'\cdot\underline{\sigma}) \\
&= a_0 a_0' + i a_0' \underline{a}\cdot\underline{\sigma} + i a_0 \underline{a}'\cdot\underline{\sigma} - (\underline{a}\cdot\underline{\sigma})(\underline{a}'\cdot\underline{\sigma}) \\
&= a_0 a_0' + i(a_0'\underline{a} + a_0\underline{a}')\cdot\sigma - a_i a_j' \sigma_i \sigma_j \\
&= a_0 a_0' + i(a_0'\underline{a} + a_0\underline{a}')\cdot\sigma - i a_i a_j' \varepsilon_{ijk}\sigma_k - a_i a_j' \delta_{ij} \\
&= (a_0 a_0' - \underline{a}\cdot\underline{a}') + i(a_0'\underline{a} + a_0\underline{a}' - \underline{a}\times\underline{a}')\cdot\sigma \quad .
\end{aligned}$$

6. V becomes a Lie algebra by definition, its bracket

$[\cdot,\cdot] : V \times V \longrightarrow V \qquad [X,Y] = X * Y - Y * X$

i) linear by linearity of $*$

ii) antisymmetric trivially

iii) Jacobi identity shown by expanding.

7. $M = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad a,b,c \in \mathbb{R}$

$G = \{ M \quad a,b,c \in \mathbb{R} \}$

closure:

$$MM' = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b'+ac'+b \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix} \in G \quad \forall M, M' \in G$$

identity:

$a,b,c = 0$

inverse:

$$M^{-1} = \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ G is a group

G is smooth in a, b, c $\Rightarrow$ G is a Lie group

$\mathcal{M}(G) = \mathbb{R}^3$, non-abelian (as seen above from MM').

$$M(t) = \begin{pmatrix} 1 & t\theta_1 & t\theta_2 \\ 0 & 1 & t\theta_3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{dM}{dt} = \begin{pmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & \theta_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L(G) = \left\{ \begin{pmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & \theta_3 \\ 0 & 0 & 0 \end{pmatrix} \ \theta_1, \theta_2, \theta_3 \in \mathbb{R} \right\}$$

$$TT' = \begin{pmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & \theta_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \theta_1' & \theta_2' \\ 0 & 0 & \theta_3' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \theta_1\theta_3' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \forall T, T' \in L(G)$$

$$[T, T'] = \begin{pmatrix} 0 & 0 & \theta_1\theta_3' - \theta_1'\theta_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\in L(G))$$

<u>Not</u> simple $\left([L(G), L(G)] \neq L(G) \right)$, i.e. non-trivial ideal $[L(G), L(G)]$).

8. $(T^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$

$$[T^{ij}, T^{kl}] = \delta_{i\alpha}\delta_{j\beta}\delta_{k\beta}\delta_{l\gamma} - \delta_{k\alpha}\delta_{l\beta}\delta_{i\beta}\delta_{j\gamma}$$
$$= \delta_{i\alpha}\delta_{l\gamma}\delta_{jk} - \delta_{k\alpha}\delta_{j\gamma}\delta_{il} = T^{il}\delta_{jk} - T^{kj}\delta_{il}$$

If $[T^{ij}, T^{kl}] = f^{ijkl}{}_{mn} T^{mn}$,

$$f^{ijkl}{}_{mn} = \delta_{jk}\delta_{im}\delta_{ln} - \delta_{il}\delta_{km}\delta_{jn}$$

9. $U = \exp(iH) = \sum_n \frac{1}{n!}(iH)^n$

$H^\dagger = H \quad \Rightarrow \quad (iH)^\dagger = -(iH)$

$$U^\dagger U = \left(\sum_m \frac{1}{m!}(iH)^{\dagger m}\right)\left(\sum_n \frac{1}{n!}(iH)^n\right) = \exp(-iH)\exp(iH) = I$$

as $\exp A \exp B = \exp(A+B)$ if $[A,B]=0$.

$\operatorname{tr} H = 0 \qquad \det U = \exp(\operatorname{tr}(iH))$

$H = V^{-1} D V$ where D is a diagonal matrix (eigenvalues d_i)

$$\operatorname{tr} H = \operatorname{tr}(V^{-1}DV) = \operatorname{tr}(VV^{-1}D) = \operatorname{tr} D = \sum_i d_i = 0$$

$$\det(\exp iH) = \det\left[\exp(iV^{-1}DV)\right] = \det \sum_m \frac{i^m}{m!}(V^{-1}DV)^m$$

$$= \det \sum_m \frac{i^m}{m!} V^{-1} D^m V = \det(V^{-1}\exp(iD)V) =$$

$$= \det \exp(iD) = \prod_i \exp(id_i) = \exp\left(i\sum_i d_i\right) = 1$$

These results ~~show that~~ agree with the Lie algebra of $U(n)$ being the anti Hermitian matrices and that of $SU(n)$ also being traceless, under the map.