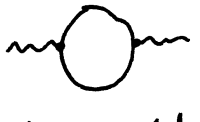
Last time, we found

 $= \Pi^{1\text{-loop}}_{\mu\nu}(q) = q^2\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\pi^{1\text{-loop}}(q^2)$

where $\pi^{1\text{-loop}}(q^2) = \dfrac{-8\,g^2(\mu)\,\Gamma(2-d/2)}{(4\pi)^{d/2}} \displaystyle\int_0^1 dx\; x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}$

where $\Delta = m^2 + x(1-x)q^2$

This result is finite in $d=2,3$ but diverges at $d=4$. We use counterterms to remove the divergence as we analytically continue, setting $d = 4-\varepsilon$ and $\varepsilon \to 0^+$.

$$\not{D} = \not{\partial} + ie\not{A}$$

$$S^{CT}[A,\psi] = \int \tfrac{1}{4}\,\delta Z_3\, F_{\mu\nu}F^{\mu\nu} + \delta Z_2\,\bar\psi\not{D}\psi + \delta m\,\bar\psi\psi \; d^dx$$

Notice that the counterterms multiply gauge invariant contributions, i.e. do not have separate counterterms for $\bar\psi\not\partial\psi$ and $e\bar\psi\not{A}\psi$. So it's important that $q^\mu \Pi_{\mu\nu} = 0$ for this procedure to work.

For $\Pi^{1\text{-loop}}_{\mu\nu}$ the appropriate correction is δZ_3. As $\varepsilon \to 0^+$ we have

$$\pi^{1\text{-loop}}(q^2) \sim \frac{-g^2(\mu)}{2\pi^2}\int_0^1 dx\; x(1-x)\left[\frac{2}{\varepsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{\Delta}\right) + O(\varepsilon)\right]$$

The counterterm  $= -\dfrac{\delta Z_3}{4}\displaystyle\int F_{\mu\nu}F^{\mu\nu}\,d^4x$ and must to chosen to remove the $1/\varepsilon$ singularity, and in $\overline{MS}$ scheme we also remove $(-\gamma + \ln 4\pi)$

$$\Rightarrow \pi^{\overline{MS}}(q^2) \sim \frac{+g^2(\mu)}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{m^2 + q^2 x(1-x)}{\mu^2}\right)$$

which is finite in $d=4$

Rmk/ The log Δ has a branch point when $m^2 + q^2x(1-x) = 0$. For $x \in [0,1]$ ~~this~~ $x(1-x) \in [0, \tfrac{1}{4}]$ so the branch cut is inaccessible with real Euclidean momenta. However in Lorentzian signature, $q^2 = \underline{q}^2 - E^2$, so the branch-cut occurs when $(E^2 - \underline{q}^2)x(1-x) \geq m^2$, which can be reached whenever $E^2 \geq (2m)^2$

This is exactly the threshold energy for creating a real e^+, e^- pair.

The QED β-f^n

To relate this "1-loop exact" photon propagator to the β-f^n for the electromagnetic coupling we rescale back to our original $A_\mu^{old} = e A_\mu^{new}$, where we have

$$S_{eff}[A^{old}] = \frac{1}{4g^2}\left[1 - \Pi(0)\right]\int F_{\mu\nu}F^{\mu\nu}\, d^4 z \qquad \Delta_{\mu\nu}^{exact} = \frac{\Delta^0_{\mu\nu}(q)}{1-\Pi(q^2)}$$

$$= \frac{1}{4}\left[\frac{1}{g^2(\mu)} - \underbrace{\frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2}{\mu^2}\right)}_{\uparrow\ \Pi^{\overline{ms}}(0)}\right]\int F_{\mu\nu}F^{\mu\nu}\, d^4 z$$

Since nothing physical can depend on arbitrary scale μ, we must have

$$0 = \mu\frac{\partial}{\partial\mu}\left[\frac{1}{g^2(\mu)} - \frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{m^2}{\mu^2}\right)\right]$$

$$\Rightarrow \beta(g) = \frac{g^3}{12\pi^2} + \text{higher order} \qquad \int_0^1 x(1-x) = \frac{1}{6}$$

$$\Rightarrow \frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu')} + \frac{1}{6\pi^2}\ln\left(\frac{\mu'}{\mu}\right) \quad \text{is the running coupling}$$

Suppose $\mu \sim m_e$ where we measure the $\frac{g^2(m_e)}{4\pi} \approx \frac{1}{137}$. Then $\exists$ a scale μ' given by $\mu' = m_e e^{6\pi^2/g^2(m_e)} \sim 10^{286}$ GeV where $g^2(\mu')$ diverges.

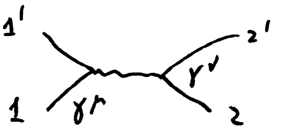
This is known as Landau pole $\Rightarrow$ pure QED does not exist as a continuum QFT.

Physics of Vacuum Polarisation

To see the physics of the shifted kinetic term, consider scattering two charged particles (distinguishable) of charge e_1, e_2

$\uparrow$ Dirac spinors

$$S(12 \longrightarrow 1'2') = -\frac{e_1 e_2}{4\pi}\,\delta^4(p_1 + p_2 - p_{1'} - p_{2'})\,\bar{u}_{1'}\gamma^\mu u_1\,\Delta_{\mu\nu}(\underset{\uparrow\, q = p_1 + p_2}{q})\,\bar{u}_{2'}\gamma^\nu u_2$$



$$\Delta_{\mu\nu}(q) = \frac{\Delta^0_{\mu\nu}(q)}{1-\pi(q^2)}$$

To lead order

$$\approx -\frac{e_1 e_2}{4\pi}\delta^4(p_1+p_2-p_{1'}-p_{2'})\,\bar{u}_{1'}\gamma^\mu u_1\,\Delta^0_{\mu\nu}\,\bar{u}_{2'}\gamma^\nu u_2\,\left[1+\pi(q^2)+\dots\right]$$

On-shell $\bar{u}_{1'}\gamma^\mu u_1\,\Delta^0_{\mu\nu}\,\bar{u}_{2'}\gamma^\nu u_2 = \bar{u}_{1'}\gamma^\mu u_1\,\bar{u}_{2'}\gamma_\mu u_2\,\frac{1}{q^2}$

So the quantum correction modifies the classical by $[1+\pi(q^2)]$. In the non-relativistic limit $|q^0| \ll |q|$ and $\bar{u}_{1'}\gamma^\mu u_1 \approx \begin{pmatrix} -i\delta_{m_1 m_{1'}} \\ 0 \end{pmatrix}$ $m_1 = \sigma_3$ (spin) angular momentum quantum #'s of 1 and 1'

Consequently

$$S(12 \to 1'2') \approx \frac{e_1 e_2}{4\pi |q|^2}\delta^4(p_1+p_2-p_{1'}-p_{2'})\left[1+\pi(q^2)\right]\delta_{m_1 m_{1'}}\delta_{m_2 m_{2'}}$$

This is just what we get in non-relativistic QM using Born approximation to scatter off a scalar potential (scalar ⇒ spin not modified)

$$V(r) = e_1 e_2 \int \frac{d^3 q}{(2\pi)^3}\left[\frac{1+\pi(|q|^2)}{|q|^2}\right]e^{iq\cdot r}$$

In a regime where $|q|^2 \ll m_e^2$ we have

$$\pi(|q|^2) = \pi(0) + \frac{g^2(\mu)}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(1+\frac{x(1-x)|q|^2}{m^2}\right)$$

$$\approx \pi(0) + \frac{g^2(\mu)}{60\pi^2}\frac{|q|^2}{m^2}$$

$$\Rightarrow V(r) \approx e_1 e_2 \int \frac{d^3 q}{(2\pi)^3}\left[\frac{1+\pi(0)}{|q|^2} + \frac{g^2}{60\pi^2 m^2}\frac{|q|^2}{|q|^2}\right]e^{iq\cdot r}$$

$$\approx \frac{e_1 e_2\left[1+\pi(0)\right]}{4\pi r} + \frac{g^2}{60\pi^2 m^2}\delta^3(r)$$

↑ short range modification

This modification of the Coulomb force is attributed to "screening".
This leads to a measured shift in the energy levels of $l=0$ bound state of H.