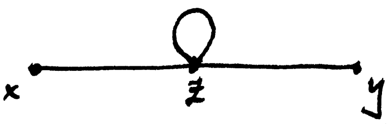
Feynman's original approach to QFT was from the worldline perspective. To compute a correlation f^n such as $\langle \Phi(x_1) \dots \Phi(x_n) \rangle$ in Φ^4 theory on $\mathbb{R}^n$ say, we consider all 4-valent graphs with n external legs with one endpoint at each of the x_i, $i=1,\dots,n$.
e.g. we get a contribution to $\langle \Phi(x) \Phi(y) \rangle$ from the graph



We associate to this the 1d QG expression

$$\int d^n z \int_0^\infty dT_1 dT_2 dT_3 \int_{C_{T_1}[x,z]} Dx\, e^{-S_{T_1}[x]} \int_{C_{T_2}[z,z]} Dx\, e^{-S_{T_2}[x]} \int_{C_{T_3}[z,y]} Dx\, e^{-S_{T_3}[x]}$$

where $S_T[x] = \frac{1}{2}\int_0^T \dot{x}^2 dt + \frac{m^2}{2}\int_0^T dt$ and the integrals over T's are the 1d quantum gravity part of the path integral (aka Schwinger params for the graph).

$$\int d^n z \int dT_1 dT_2 dT_3 \, \langle z | e^{-HT_1} | x \rangle \langle z | e^{-HT_2} | z \rangle \langle y | e^{-HT_3} | z \rangle$$

$$= \int \frac{d^n p}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{e^{ip\cdot(x-z)}}{p^2+m^2} \frac{e^{iq\cdot(y-z)}}{q^2+m^2} \frac{e^{il\cdot(z-z)}}{l^2+m^2}$$

$$= \int \frac{d^n p}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{e^{ip\cdot(x-y)}}{(p^2+m^2)^2} \frac{1}{l^2+m^2}$$ as in the $\mathbb{R}^n$ Feynman expansion

4 Symmetries of the Path Integral

In classical field theory, Nöther's thm relates symmetries to conservation laws. Suppose $\delta\phi = \varepsilon f(\phi, \partial\phi)$ is a transformation of the fields. The most common case is when $f(\phi,\partial\phi)$ depends on ϕ only locally in which case we can think of the transformation as being generated by the vector

$$V_f = \int_M d^n x \sqrt{g}\, f(\phi, \partial\phi) \frac{\delta}{\delta\phi(x)}$$ on the space of fields.

If the action $S[\phi]$ is invariant under this transformation when ε is constant, then for general $\varepsilon(x)$ we must have ($(M,g) = (\mathbb{R}^n, \delta)$)

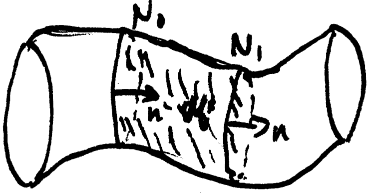
$$\delta S[\phi] = \int_M d^n x \, j^\mu(x) \partial_\mu \varepsilon$$ for some <u>field-dependent current</u> $j_\mu(x)$.

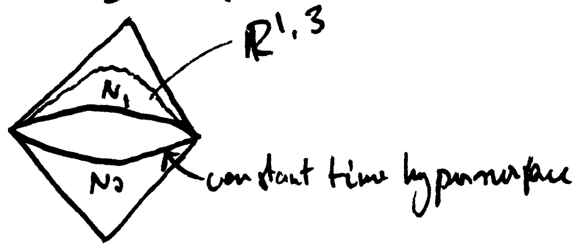
If we choose $\varepsilon(x)$ to have compact support, then $\delta S = -\int_M d^n x \, (\partial^\mu j_\mu)\, \varepsilon(x)$.
<u>On sol^{n}s</u> of the field eq^{n}s, we know the action is stationary under arbitrary variation, so $\delta S = 0 \Rightarrow \partial^\mu j_\mu = 0$ since this holds $\forall\, \varepsilon(x)$.

We define the charge $Q[N]$ associated to a codimension-one hypersurface $N \subset M$ by

$$Q[N] = \int_N (n^\mu j_\mu) \, d^{d-1}x \quad \text{where } n^\mu \text{ is the normal to } N.$$

If N_0, N_1 are two such hypersurfaces bounding a region $M' \subset M$

e.g. 

e.g. 

Then integrating over M', $\delta S = 0 \Rightarrow$

$$0 = \int_{M'} \partial^\mu j_\mu \, d^d x = \int_{N_1} n^\mu j_\mu \, d^{d-1}x - \int_{N_0} n^\mu j_\mu \, d^{d-1}x = Q[N_1] - Q[N_0]$$

e.g. choose $\varepsilon(x) = 1$ inside M', 0 else.

Ward identities

Nöther's th^m used the classical eom, so we must re-examine this in the quantum theory.

Suppose a trans$^{f^m}$ $\phi(x) \mapsto \phi'(x)$ of the fields has the property

$$\mathcal{D}\phi \, e^{-S[\phi]} = \mathcal{D}\phi' \, e^{-S[\phi']} \qquad \text{for constant } \varepsilon$$

(in practice, usually just look for symmetries that preserve $S[\phi]$ and then try to find a def^n of $\mathcal{D}\phi$ that also preserves this symmetry).

The path integral measure, when regularised, may break some symmetries of the classical action. If so, there are two possibilities:

① The symmetry could be restored in the classical limit (typically, this means $\exists$ a regularized p.i. measure manifestly inv^t under $\phi \mapsto \phi'$ but we just didn't use it)
e.g. rotational invariance in a lattice theory

② The symmetry is **anomalous** (i.e. broken in the quantum theory). In this case, no inv^t p.i. measure exists. e.g. scale invariance in QED with $m_{\text{electron}} = 0$

Suppose $\mathcal{D}\phi = \mathcal{D}\phi'$ for $\varepsilon = \text{const}$, then for any $\varepsilon(x)$

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} \underset{\uparrow \text{trivial relabelling}}{=} \int \mathcal{D}\phi' \, e^{-S[\phi']} = \int \mathcal{D}\phi \, e^{-S[\phi]} \left(1 - \int j_\mu \partial^\mu \varepsilon \, d^d x\right)$$

to first order in ε.

Note that in general j_μ can receive contributions both from $S[\phi]$ and $\mathcal{D}\phi$. The term $O(\varepsilon^0)$ cancels, so

$$0 = \int_M \mathcal{D}\phi \, e^{-S[\phi]} \int j^\mu \partial_\mu \varepsilon \, d^d x = -\int_M \varepsilon \, \partial^\mu \langle j_\mu(x) \rangle \, d^d x \quad \text{for } \varepsilon(x) \text{ of compact support.}$$

Thus $\partial^\mu \langle j_\mu(x) \rangle = 0$ so the expectation value of the current $\hat{j}_\mu(x)$ is conserved, as in the classical theory.