Non-compact subgroups $GL(n;\mathbb{R})$

Orthogonal transformation $M \in O(n)$ $\quad M \mathbb{1}_n M^T = \mathbb{1}_n$

preserves metric ($\mathbb{1}_n$) on $\mathbb{R}^n$. $\quad n = p+q$

$O(p,q)$ transformations preserve metric of signature (p,q)

$$O(p,q) = \{ M \in GL(n;\mathbb{R}) : M^T \eta M = \eta \} \qquad \eta = \left(\begin{array}{c|c} \mathbb{1}_p & \\ \hline & -\mathbb{1}_q \end{array} \right)$$

groups are non-compact,

$$M = SO(1,1) \Rightarrow M = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \qquad \theta \in \mathbb{R}$$

Lorentz group : $O(3,1)$

Subgroups of $GL(n,\mathbb{C})$

- Unitary groups $U(n) = \{ U \in GL(n;\mathbb{C}) : U^\dagger U = \mathbb{1}_n \}$

  - unitary transformations $\underline{v} \in \mathbb{C}^n \to \underline{v}' = U \cdot \underline{v} \in \mathbb{C}^n$

    preserves length $|\underline{v}|^2 = \underline{v}^\dagger \cdot \underline{v}$

  - $U \in U(n) \Rightarrow U^\dagger U = \mathbb{1}_n \Rightarrow |\det U|^2 = 1 \Rightarrow \det U = e^{i\delta} \quad \delta \in \mathbb{R}$

- Special Unitary group $SU(n) = \{ U \in U(n) : \det U = 1 \}$

  $\dim[U(n)] = \underbrace{2n^2}_{\dim[GL(n;\mathbb{C})]} - \underbrace{n^2}_{\#\text{ real constraints}} = n^2 \qquad \dim[SU(n)] = n^2 - 1$

$G = U(1)$

$U(1) = \{ z \in \mathbb{C} ; |z| = 1 \} \qquad \mathcal{M}[U(1)] = S^1$

Two Lie groups G and G' are <u>isomorphic</u> if

$\exists$ 1:1 smooth map $J : G \to G'$ such that $\forall g_1, g_2 \in G$

$$J(g_1 g_2) = J(g_1) \cdot J(g_2)$$

General element $z = e^{i\theta}$ of $G = U(1)$ $\theta \in \mathbb{R}$, $\theta \sim \theta + 2\pi$
corresponds to a unique element, $M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$J : z(\theta) = e^{i\theta} \in U(1) \longmapsto M(\theta) \in SO(2)$ is 1:1 and

$J(z(\theta_1) z(\theta_2)) = M(\theta_1 + \theta_2) = M(\theta_1) M(\theta_2) = J(z(\theta_1)) J(z(\theta_2))$

$$\boxed{U(1) \simeq SO(2)}$$

$G = SU(2)$

Ex Show that $M(SU(2)) \simeq S^3$ $\quad x^2 + y^2 + z^2 + w^2 = 1$

Lie Algebra

A Lie algebra $\mathfrak{g}$ is a vector space ($\mathbb{R}$ or $\mathbb{C}$) with a bracket

$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$

i) Anti-symm

$[X, Y] = -[Y, X] \quad \forall X, Y \in \mathfrak{g}$

ii) Linearity

$[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z] \quad \forall X, Y, Z \in \mathfrak{g} \quad \alpha, \beta \in F$ ($\mathbb{R}$ or $\mathbb{C}$)

iii) Jacobi Identity

$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in \mathfrak{g}$

If a vector space V has an associative product

$* : V \times V \to V \qquad (V_1 * V_2) * V_3 = V_1 * (V_2 * V_3) \qquad \forall V_1, V_2, V_3 \in V$

then one can make a Lie algebra $\mathfrak{g}$ by setting $[X, Y] = X * Y - Y * X \quad \forall X, Y \in \mathfrak{g}$

"bracket" = "commutator" $\qquad \mathfrak{g} \subseteq V$

Lot of examples

V = vector space of matrices $\qquad *$ = matrix multiplication

- dimension of a Lie algebra is dimension of vector space

choose basis B for $\mathfrak{g}$ $\quad B = \{T^a, a = 1, \ldots, n, n = \dim(\mathfrak{g})\}$

- any $X \in \mathfrak{g}$ is written in components

$$X = X_a T^a \overset{\text{def}}{=} \sum_{a=1}^{n} X_a T^a \qquad X_a \in F$$

brackets of element $X, Y \in \mathfrak{g}$

$$[X, Y] = X_a Y_b [T^a, T^b]$$

structure constants

determined by the bracket of basis elements $[T^a, T^b] = f^{ab}{}_c T^c$

these obey,

$$f^{ba}{}_c = -f^{ab}{}_c \quad \in F \qquad \text{(antisym)}$$

<u>Ex</u> Jacobi $\Rightarrow f^{ab}{}_c f^{cd}{}_e + f^{da}{}_c f^{cb}{}_e + f^{bd}{}_c f^{ca}{}_e = 0 \quad \forall a,b,d,e = 1 \ldots \dim(\mathfrak{g})$

- Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic iff

$\exists$ a linear map (1:1) $f : \mathfrak{g} \to \mathfrak{g}'$ such that

$$[f(X), f(Y)] = f([X, Y]) \quad \forall X, Y \in \mathfrak{g}$$

classify (simple) Lie algebras up to isomorphism