$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta_{\mu\nu}\right) = T_{\nu\mu}$$

$T_{00} = \mathcal{E}$ $\quad T_{0i} = -S_i$ $\quad T_{ij} = t_{ij}$

$\partial^\mu T_{\mu\nu} = 0$

__Def__ The energy-momentum tensor of a Maxwell field in a general spacetime is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}g_{ab}\right)$$

__Ex__ Show Maxwell eqs $\Rightarrow \nabla^a T_{ab} = 0$

__Postulate__ energy, momentum & stresses of matter are described by an __energy-momentum tensor__ a symmetric $(0,2)$ tensor that is conserved: $\nabla^a T_{ab} = 0$

O: observer at p, 4-velocity $u^a$ $\qquad$ choose LIF at p s.t. $u^\mu = (1,0,0,0)$

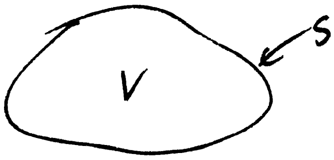
orthonormal basis $e_0^a = u^a$ $\quad e_1^\mu = (0,1,0,0)$ $\quad e_2^\mu = (0,0,1,0)$ $\quad e_3^\mu = (0,0,0,1)$

EP: $\mathcal{E} \equiv T_{00} = T_{ab}e_0^a e_0^b = T_{ab}u^a u^b$ is energy density at p as measured by O

$S_i = -T_{0i}$ momentum density
$t_{ij} = T_{ij}$ stress tensor
} measured by O

energy-momentum current measured by O $\quad j^a = -T^a{}_b u^b$ $\qquad$ cpts $j^\mu = (\mathcal{E}, S_i)$

surface of const $t = x^0$



$$\frac{d}{dt}\int_V \mathcal{E}\,dV = -\int_S \underline{S}\cdot\underline{n}\,dA$$

Newtonian theory: energy of grav field $-\frac{1}{8\pi}(\nabla\Phi)^2$

$g_{\mu\nu,\rho}$ & $g^{\mu\nu}{}_{,\rho}$ ? normal coords @ p $\Rightarrow$ vanishes @ p

Perfect fluid: 4-velocity vector field $u^a$

scalar fields $\rho, p$ $\qquad T_{ab} = (\rho + p)u_a u_b + p\,g_{ab}$ $\qquad$ $p=0$ geodesic flow

$T_{ab}u^a u^b = (\rho + p) - p = \rho$ : energy density measured by observer O comoving with fluid

p : pressure measured by O

__Ex__ Show $\nabla^a T_{ab} = 0$ implies eqs of motion of fluid $\quad u^a\nabla_a\rho + (\rho+p)\nabla_a u^a = 0$

$$(\rho+p)u^b\nabla_b u^a = -(g^{ab} + u^a u^b)\nabla_b p$$

$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \qquad \Lambda^{-1/2} \sim 10^{9}$ ly

## 7 Maps between manifolds

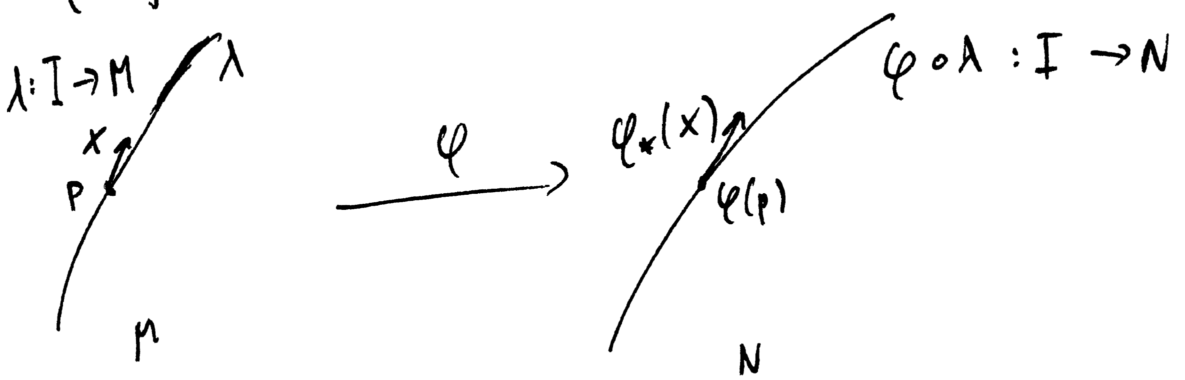
**Def** $M, N$ manifolds of dim $m, n$ $\quad \varphi : M \to N$ is smooth iff

$\psi_A \circ \varphi \circ \psi_\alpha^{-1}$ is smooth $\forall$ charts $\psi_\alpha$ of $M$, $\psi_A$ of $N$

$\mathbb{R}^m \to \mathbb{R}^n$

**Def** $\varphi : M \to N \qquad f : N \to \mathbb{R}$ smooth. The pull-back of $f$ by $\varphi$ is

$\varphi^*(f) : M \to \mathbb{R} \qquad \varphi^*(f) = f \circ \varphi \quad \left( \varphi^*(f)(p) = f(\varphi(p)) \right)$



**Def** $\varphi : M \to N$ smooth, $p \in M$ $\quad X \in T_p(M)$, $\lambda$ curve in $M$ with tangent $X$ @ $p$

The push-forward of $X$ by $\varphi$ is $\varphi_*(X) \in T_{\varphi(p)}(N)$ : tangent to curve $\varphi \circ \lambda$ at $\varphi(p)$.

**Lemma** $f : N \to \mathbb{R} \quad (\varphi_*(X))(f) = X(\varphi^*(f))$

Proof: $LHS = \left[\frac{d}{dt}\left(f \circ (\varphi \circ \lambda)\right)(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\underbrace{(f \circ \varphi)}_{\varphi^*(f)} \circ \lambda\right)(t)\right]_{t=0} = RHS$

$\lambda(0) = p$

**Ex** $x^\mu$ coords on $M \qquad \varphi \leftrightarrow$ map $y^\alpha(x)$

$y^\alpha$ on $N \qquad$ show $(\varphi_*(X))^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p X^\mu$

**Def** $\varphi : M \to N$ smooth $p \in M \quad \eta \in T^*_{\varphi(p)} N$. The pull-back of $\eta$ by $\varphi$ is

$\varphi^*(\eta) \in T^*_p M \qquad \varphi^*(\eta)(X) = \eta(\varphi_*(X)) \quad \forall X \in T_p M$

**Lemma** $f : N \to \mathbb{R}$ $\qquad \varphi^*(df) = d(\varphi^*(f))$

Proof: $(\varphi^*(df))(\underset{\substack{\Uparrow \\ T_pM}}{X}) = (df)(\varphi_*(X)) \underset{\text{def of } d}{=} \varphi_*(X)(f) \overset{\text{lemma}}{=} X(\varphi^*(f))$

$= d(\varphi^*(f))(X)$

def of $d$

**Ex** Show $(\varphi^*(\eta))_\mu = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \eta_\alpha$

Pull-back $(0,s)$ tensor $S$ : $(\varphi^*(S))(\underbrace{X_1, \dots, X_s}_{\in T_pM}) = S(\varphi_*(X_1), \dots, \varphi_*(X_s))$

Push-forward $(r,0)$ tensor $T$ : $(\varphi_*(T))(\underbrace{\eta_1, \dots, \eta_r}_{\in T^*_{\varphi(p)}M}) = T(\varphi^*(\eta_1), \dots, \varphi^*(\eta_r))$

$$\Rightarrow (\varphi^*(S))_{\mu_1 \dots \mu_s} = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right)_p \cdots \left(\frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}}\right)_p S_{\alpha_1 \dots \alpha_s}$$

$$(\varphi_*(T))^{\alpha_1 \dots \alpha_r} = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right)_p \cdots \left(\frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}\right)_p T^{\mu_1 \dots \mu_r}$$

**Example**: $M = S^2$, $N = \mathbb{R}^3$ $\qquad \varphi : M \to N \qquad x^\mu = (\theta, \varphi) \longmapsto y^\alpha = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$

$g$ : Euclidean metric on $\mathbb{R}^3$

$\varphi^*(g)$ = unit round metric on $S^2$ $\qquad (*) \Rightarrow (\varphi^*(g))_{\mu\nu} = \mathrm{diag}(1, \sin^2\theta)$