$X = X^\mu e_\mu$ vector, $w = w_\mu f^\mu$ co-vector (1-form) $\quad w(X) = \ldots = w_\mu X^\mu = X \lrcorner w$

If $T_pM = \text{span}\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$, dual basis $T_p^*M = \text{span}\{dx^1, \ldots, dx^n\}$

wedge product : $dx^i \wedge dx^j = -dx^j \wedge dx^i$ [skew part of tensor product]

2-forms

$f: M \to \mathbb{R}$ 0-form

gradient $df|_p$ 1-form at p

$df|_p(X) = \{X \lrcorner df|_p\} = X(f)|_p$ for any p

so $df = \frac{\partial f}{\partial x^\mu} dx^\mu$ (in a coordinate basis), so that $X(f) = X^\mu \frac{\partial f}{\partial x^\mu}$

1-form $\Omega = \Omega_\mu dx^\mu$

k-form $\Omega = \frac{1}{k!}\Omega_{\mu\nu\ldots\rho}\, dx^\mu \wedge dx^\nu \wedge \ldots \wedge dx^\rho \in \Lambda^k(M)$

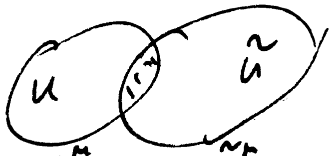
Exterior differentiation $\quad d: \Lambda^k(M) \to \Lambda^{k+1}(M)$; $d^2 = 0$

$$d\Omega = \sum_{\mu,\nu} \frac{\partial X_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu$$

$$d\Omega(X,Y) = X(\Omega(Y)) - Y(\Omega(X)) - \Omega([X,Y])$$

any 1-form Ω
vector fields X, Y
(exercise).

More than 1 open set



$U \cap \tilde{U}$ $\quad \tilde{x}^\mu(x^\nu)$

1-form Ω

On U, $\Omega = \Omega_\mu dx^\mu$; on $\tilde{U}$, $\Omega = \tilde{\Omega}_\mu d\tilde{x}^\mu$. On $U \cap \tilde{U}$

$$\Omega = \tilde{\Omega}_\mu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu = \Omega_\nu dx^\nu \quad \Rightarrow \boxed{\Omega_\nu = \tilde{\Omega}_\mu \frac{\partial \tilde{x}^\mu}{\partial x^\nu}}$$

transformation of co-vectors

1.3 Tensors

Abstract index notation (Penrose)

[Greek letters] $\mu, \nu, \ldots$ basis components, e.g. X^μ, η_μ [equations only valid in a basis]

[Latin letters] $a, b, \ldots$ "abstract index"

X^a is a vector $\qquad X^a, X^b, X^c$ same vector

η_a is a 1-form

<u>Def</u> A tensor of type (r, s) at $p \in M$ is a multilinear map.

$$T : \underbrace{T_p^*(M) \times T_p^*(M) \times \dots \times T_p^*(M)}_{r} \times \underbrace{T_p(M) \times T_p(M) \times \dots \times T_p(M)}_{s} \longrightarrow \mathbb{R}.$$

e.g. • $r=0,\ s=1$ 1-form.

• $r=1,\ s=0$ vector

linear map $X \in T_p(M)$, maps $\eta \in T_p^*(M)$ to $X^a \eta_a = X(\eta) = X \lrcorner \eta$.

$\{e_\mu\}$ basis of $T_p(M)$, $\{f^\mu\}$ basis of $T_p^*(M)$

Components of T w.r.t. $\{e_\mu\}, \{f^\nu\}$.

$$T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \equiv T(f^{\mu_1}, \dots, f^{\mu_r}, e_{\nu_1}, \dots, e_{\nu_s}).$$

Abstract index

$$T \longleftrightarrow T^{a_1 \dots a_r}{}_{b_1 \dots b_s}.$$

Tensors at $p \in M$ form a vector space of dimension n^{r+s}.

$(2,1)$ tensor T, $\underset{\in T_p^*(M)}{(\eta, \omega)}$ 1-forms, $X \in T_p(M)$

$$T(\eta, \omega, X) = T(\eta_\mu f^\mu, \omega_\nu f^\nu, X^\rho e_\rho)$$
$$= \eta_\mu \omega_\nu X^\rho T(f^\mu, f^\nu, e_\rho) = \eta_\mu \omega_\nu X^\rho \overbrace{T^{\mu\nu}{}_\rho}^{\text{components}}$$

Contraction (r,s) tensor $\longrightarrow (r-1, s-1)$ tensor. Trace

e.g. $T^{ab}{}_c \longrightarrow T^a{}_a{}^b = S^b$

$(2,1)$ tensor $\longrightarrow$ vector

Tensor product of a (p,q) tensor S and (r,s) tensor T is a $(p+r, q+s)$ tensor $S \otimes T$ defined by

$$S \otimes T(\underbrace{\omega_1, \dots, \omega_p, \eta_1, \dots, \eta_r}_{p+r \text{ 1-forms}}, \underbrace{X_1, \dots, X_q, Y_1, \dots, Y_s}_{q+s \text{ vectors}})$$

$$= \underbrace{S(\omega_1, \dots, \omega_p, X_1, \dots, X_q)}_{\in \mathbb{R}} \cdot \underbrace{T(\eta_1, \dots, \eta_r, Y_1, \dots, Y_s)}_{\in \mathbb{R}}$$

$$(S \otimes T)^{a_1 \dots a_p b_1 \dots b_r}{}_{c_1 \dots c_q d_1 \dots d_s} = S^{a_1 \dots a_p}{}_{c_1 \dots c_q} T^{b_1 \dots b_r}{}_{d_1 \dots d_s}$$

(anti) symmetrization of a $(0,2)$ tensor T

$$S(X,Y) = \frac{1}{2}\left[T(X,Y) + T(Y,X)\right] \qquad S_{ab} = \frac{1}{2}\left(T_{ab} + T_{ba}\right) = T_{(ab)}$$

$$A(X,Y) = \frac{1}{2}\left[T(X,Y) - T(Y,X)\right] \qquad A_{ab} = \frac{1}{2}\left(T_{ab} - T_{ba}\right) = T_{[ab]}$$

Wedge product: $\omega, \eta \in T^*_p(M)$

$$\omega \wedge \eta \equiv \frac{1}{2}(\omega \otimes \eta - \eta \otimes \omega) \in \Lambda^2(M)$$

Symmetrization in general:

$$(3.1) \quad T^{(ab)c}{}_d = \frac{1}{2}\left(T^{abc}{}_d + T^{bac}{}_d\right)$$

$$T_{[abcd]} = \frac{1}{3!}\left(T_{abcd} + T_{dbc} + T_{cdb} - T_{cbd} - T_{dcb} - T_{bdc}\right)$$