Almost inertial coords x^μ $\quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ $\quad h = h^\mu{}_\mu = h_{\mu\nu}\eta^{\mu\nu}$

$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}$ $\quad \bar{h} = -h$

gauge transf^n $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

gauge condition $\partial^\nu \bar{h}_{\mu\nu} = 0 \quad \longrightarrow \quad \partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$

Velocities $\quad v^i = O(\varepsilon) \quad (\Rightarrow \Phi = O(\varepsilon^2)$ in Newtonian theory$)$

assume $h_{00} = O(\varepsilon^2)$, $h_{0i} = O(\varepsilon^3)$, $h_{ij} = O(\varepsilon^2)$

non-relativistic matter: $|\partial_i X| = O(X/L) \quad |\partial_0 X| = O(\varepsilon \frac{X}{L})$

$\hat{L} = (1 - h_{00}) \dot{t}^2 - 2h_{0i} \dot{t} \dot{x}^i + (\delta_{ij} + h_{ij}) \dot{x}^i \dot{x}^j \qquad \dot{x}^i = O(\varepsilon)$

$\hat{L} = 1 \Rightarrow \dot{t} = 1 + \frac{1}{2} h_{00} + \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j + O(\varepsilon^4)$

$\frac{d}{d\tau}\left[-2h_{0i}\dot{t} - 2(\delta_{ij} + h_{ij})\dot{x}^j\right] = -h_{00,i}\dot{t}^2 - 2h_{0j,i}\dot{t}\dot{x}^j - h_{jk,i}\dot{x}^j\dot{x}^k$

$-2\ddot{x}^i = -h_{00,i} \qquad O(\varepsilon^2/L)$

$\Phi = -\frac{1}{2} h_{00} \qquad \ddot{x}^i = -\Phi_{,i} \Rightarrow \frac{d^2 x^i}{dt^2} = -\partial_i \Phi + O(\varepsilon^4/L)$

$\nabla^2 \Phi = 4\pi\rho \qquad \Phi(t,x) = -\int d^3y \frac{\rho(t,y)}{|\underline{x} - \underline{y}|}$

u^a : 4-velocity of matter $\qquad u^i = O(\varepsilon) \xrightarrow{\text{normalisation}} u^0 = 1 + O(\varepsilon^2)$

$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$ $\qquad$ energy density in rest frame of matter $\quad \rho = T_{ab} u^a u^b$

momentum density $\quad -T_{0i} \sim \rho u_i = O(\rho\varepsilon)$

$T_{ij} \sim \rho u_i u_j = O(\rho \varepsilon^2)$

also : contribution from stresses in matter $\quad$ e.g. perfect fluid stress $\leftrightarrow$ pressure p

(e.g. $\frac{p}{\rho} \sim |\Phi| \sim 10^{-5}$ at centre of Sun) $\quad [p] = [\rho/c^2] \quad \therefore$ expect $p = O(\rho\varepsilon^2)$

$\therefore T_{ij} = O(\rho\varepsilon^2) \qquad T_{0i} = O(\rho\varepsilon) \qquad T_{00} = \rho(1 + O(\varepsilon^2))$

$$\bar{h}_{00} = \mathcal{O}(\varepsilon^2) \qquad \bar{h}_{ij} = \mathcal{O}(\varepsilon^2) \qquad \bar{h}_{0i} = h_{0i} = \mathcal{O}(\varepsilon^3)$$

$$\nabla^2 \bar{h}_{00} = -16\pi\rho(1+\mathcal{O}(\varepsilon^2)) \qquad \nabla^2 \bar{h}_{0i} = \mathcal{O}(\rho\varepsilon) \qquad \nabla^2 \bar{h}_{ij} = \mathcal{O}(\rho\varepsilon^2)$$

$\Vert\ \partial_i\partial^i$

$$\bar{h}_{0i} = \mathcal{O}(\bar{h}_{00}\,\varepsilon) = \mathcal{O}(\varepsilon^3)$$

$$\overset{\displaystyle -2\Phi}{\Vert} \qquad \overset{\displaystyle \Vert}{h_{0i}} \qquad \bar{h}_{ij} = \mathcal{O}(h_{00}\,\varepsilon^2) = \mathcal{O}(\varepsilon^4)$$

$$h_{00} = \frac{1}{2}\bar{h}_{00} + \mathcal{O}(\varepsilon^4) \quad \Rightarrow \quad \nabla^2\Phi = 4\pi\rho(1+\mathcal{O}(\varepsilon^2))$$

$$h_{ij} = -2\Phi\delta_{ij} + \mathcal{O}(\varepsilon^4)$$

Gravitational Waves

vacuum: $\partial^\rho\partial_\rho \bar{h}_{\mu\nu} = 0$ — $(*)$

$$\bar{h}_{\mu\nu}(x) = \mathrm{Re}\left[H_{\mu\nu}\, e^{ik_\rho x^\rho}\right] \qquad k_\mu = \text{wave vector}$$

$H_{\mu\nu}$: const symmetric ~~matrix~~ complex matrix. polarisation

$(*) \Rightarrow k^\rho k_\rho = 0 \qquad \partial^\nu \bar{h}_{\mu\nu} = 0 \Rightarrow k^\nu H_{\mu\nu} = 0$ transverse

e.g. wave moving in z-direction

$k^\mu = \omega(1,0,0,1) \qquad \exp(ik_\mu x^\mu) = \exp(-i\omega(t-z))$

transverse: $H_{\mu0} + H_{\mu3} = 0$

$$\partial^\nu \bar{h}_{\mu\nu} \to \partial^\nu \bar{h}_{\mu\nu} + \partial^\nu\partial_\nu \xi_\mu \Rightarrow \partial^\nu \bar{h}_{\mu\nu} = 0 \text{ preserved if } \partial^\rho\partial_\rho \xi_\mu = 0$$

$$\xi_\mu = X_\mu e^{ik_\rho x^\rho} \qquad \bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial^\rho\xi_\rho$$

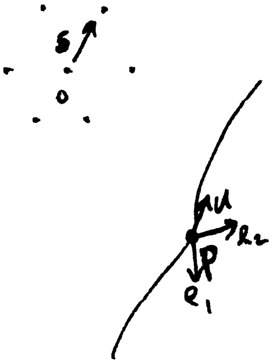
$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu X_\nu + k_\nu H_\mu - \eta_{\mu\nu} k^\rho X_\rho)$$

Exercise: Show that can choose X_μ s.t. $H_{0\mu} = 0$ and $H^\mu{}_\mu = 0$

$$\Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$$

e.g. wave in z-direction $H_{3\mu} = -H_{0\mu} = 0 \Rightarrow H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$H_+, H_\times$ const $\Rightarrow$ 2 independent polarisations



S^a : (infinitesimal) deviation vector

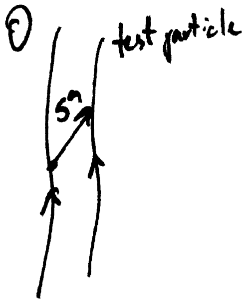
geodesic observer   basis $\{e_\alpha\}$ for $T_p(M)$

$$e_0^{\ a} = u^a \quad (\text{4-velocity})$$

$$e_i^{\ a} \quad \text{spacelike}$$

parallelly transported frame: $u^b \nabla_b u^a = 0$ (geodesic)

$$u^b \nabla_b e_i^{\ a} = 0$$



$$u^b \nabla_b (u^c \nabla_c S_a) = R_{abcd}\, u^b u^c S^d$$

$$\times\, e_\alpha^{\ a} \Rightarrow u^b \nabla_b (u^c \nabla_c (e_\alpha^{\ a} S_a)) = R_{abcd}\, e_\alpha^{\ a} u^b u^c S^d$$

$$S_\alpha = e_\alpha^{\ a} S_a$$

$$\frac{d^2 S_\alpha}{d\tau^2} = R_{abcd}\, e_\alpha^{\ a} u^b u^c \underbrace{e_\beta^{\ d} S^\beta}_{S^d}$$

on RHS $u^\mu \approx (1,0,0,0)$

$$\frac{d^2 S_\alpha}{d\tau^2} = \underbrace{R_{\mu 0 0 \nu}}_{\frac{1}{2}\frac{\partial^2 h_{\mu\nu}}{\partial t^2}} e_\alpha^{\ \mu} e_\beta^{\ \nu} S^\beta \quad (h_{0\mu} = 0)$$

$e_1^{\ \mu} \approx (0,1,0,0) \qquad e_2^{\ \mu} \approx (0,0,1,0) \qquad e_3^{\ \mu} \approx (0,0,0,1)$

$\alpha = 0$: $\dfrac{d^2 S_0}{d\tau^2} = 0 \qquad \alpha = 3$: $\dfrac{d^2 S_3}{d\tau^2} = 0$