Suppose P is faithful to DAG $\mathcal{G}^0$. At each stage, the skeleton of $\mathcal{G}^0$ must be a subgraph of $\hat{\mathcal{G}}$.
Edges (j,k) remaining at termination will have

$$Z_j \not\perp\!\!\!\perp Z_k \mid Z_S \quad \forall\ S \subseteq (\mathrm{adj}(\hat{\mathcal{G}},k)\setminus\{j\}), (\mathrm{adj}(\hat{\mathcal{G}},j)\setminus\{k\})$$

In particular, $Z_j \not\perp\!\!\!\perp Z_k \mid Z_S \quad \forall\ S \subseteq (\mathrm{adj}(\mathcal{G}^0,k)\setminus\{j\}), (\mathrm{adj}(\mathcal{G}^0,j)\setminus\{k\})$

So such j,k must be adjacent in $\mathcal{G}^0$, so $\hat{\mathcal{G}}$ will be the skeleton of $\mathcal{G}^0$.

Prop 36 Suppose we have $j - \ell - k$ in the skeleton of a DAG (j,k not adjacent).

i) If $j \to \ell \leftarrow k$, then any S that d-separates j and k cannot contain ℓ.

ii) If $\exists S$ that d-separates j and k but $\ell \notin S$, then $j \to \ell \leftarrow k$.

To find v-structures, we perform

For all $j - \ell - k$ in $\hat{\mathcal{G}}$ do

If $\ell \notin S(j,k)$, then orient $j \to \ell \leftarrow k$

Other edges can be oriented using e.g. the acyclicity of DAGs.

A sample version of the PC algorithm replaces querying conditional independencies with conditional independence tests. If the data are multivariate normal, these tests are based on partial correlation.

$$\mathrm{Corr}(Z_j, Z_k \mid Z_S) = 0 \iff Z_j \perp\!\!\!\perp Z_k \mid Z_S$$

Given $x_1, \dots, x_n \overset{iid}{\sim} Z$, can estimate the partial correlation by regressing X_j and X_k on X_S and computing the correlation between the residuals.

4 Multiple testing

In many modern applications, often interested in testing many hypotheses $H_1, \dots, H_m$ simultaneously when m_0 of them are true nulls.

	Claim non-sig	Claim sig	Total
True nulls	N_{00}	N_{01}	m_0
False nulls	N_{10}	N_{11}	$m - m_0$
Total	$m - R$	R	m

Suppose we have p-values $p_1, \dots, p_m$ and $H_i : i \in I_0$ are the true nulls, so

$$\mathbb{P}(p_i \le \alpha) \le \alpha \quad \forall\, \alpha \in [0,1],\ i \in I_0 \qquad |I_0| = m_0$$

Traditionally, approaches have aimed to satisfy

Familywise error rate (FWER) $= \mathbb{P}(N_{01} \ge 1) \le \alpha$

Bonferroni correction rejects H_i if $p_i \leq \frac{\alpha}{m}$.

Thm 37 Using Bonferroni correction

$$\mathbb{P}(N_{01} \geq 1) \underset{\text{Markov}}{\leq} \mathbb{E}(N_{01}) \leq \frac{m_0 \alpha}{m} \leq \alpha$$

Proof

$$\mathbb{E}(N_{01}) = \mathbb{E}\Big(\sum_{i \in I_0} \mathbb{1}_{\{p_i \leq \frac{\alpha}{m}\}} \Big)$$

$$= \sum_{i \in I_0} \underbrace{\mathbb{P}\left(p_i \leq \frac{\alpha}{m}\right)}_{\leq \frac{\alpha}{m}}$$

$$\leq \frac{m_0 \alpha}{m} \leq \alpha \qquad \square$$

4.1 Closed testing procedure

Given family of hypotheses $\{H_i\}_{i=1}^m$ define its closure to be

$$\{H_I : I \subseteq \{1, \dots, m\}, I \neq \phi\}$$

where $H_I = \bigcap_{i \in I} H_i$ is the intersection hypothesis i.e. H_I is the hypothesis that all H_i, $i \in I$ are true.