Definition

Real Lie algebra is of compact type if $\exists$ basis s.t.

$$K^{ab} = -\kappa \delta^{ab} \qquad \kappa \in \mathbb{R}^+$$

Theorem

Every complex semi-simple Lie algebra (of finite dim) has a real form of compact type.

5 Cartan Classification

$$\mathfrak{g} \begin{cases} \text{finite dim} \\ \text{simple} \\ \text{complex} \end{cases} \quad \text{Lie algebra}$$

classification (Cartan 1894)

$$\mathcal{L}_{\mathbb{C}}(SU(2)) = \text{span}_{\mathbb{C}}\{H, E_+, E_-\} \qquad [H, E_\pm] = \pm 2E_\pm \quad [E_+, E_-] = H$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E_\pm = \begin{pmatrix} 0 & 1/0 \\ 0/1 & 0 \end{pmatrix}$$

$$ad_H(E_\pm) = \pm 2 E_\pm \qquad ad_H(H) = 0$$

- We say that $X \in \mathfrak{g}$ is ad-diagonalisable (AD) if

$$ad_X : \mathfrak{g} \to \mathfrak{g} \quad \text{is diagonalisable}$$

A Cartan subalgebra. $\mathfrak{h}$ of $\mathfrak{g}$ is maximal abelian subalgebra containing only AD elements. Thus,

i) $H \in \mathfrak{h} \Rightarrow H$ is AD

ii) $H, H' \in \mathfrak{h} \Rightarrow [H, H'] = 0 \Rightarrow ad_H \circ ad_{H'} - ad_{H'} \circ ad_H = 0$

iii) If $X \in \mathfrak{g}$ is AD and $[X, H] = 0 \ \forall H \in \mathfrak{h}$ then $X \in \mathfrak{h}$.

In fact, all possible Cartan subalgebras have same dimension

$$r = \dim[\mathfrak{h}] \in \mathbb{N} \qquad \text{rank of } \mathfrak{g}$$

$\mathcal{L}_{\mathbb{C}}(SU(2))$ $\qquad H = \sigma_3$ is AD

$$E_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2) \quad \text{is not AD}$$

$\mathfrak{h} = \text{span}_{\mathbb{C}}\{H\}$ is a possible C.S.A. $\qquad \mathcal{L}_{\mathbb{C}}(SU(2))$ rank 1

Choose a basis $\{H^i, i=1,\dots,r\}$

$[H^i, H^j] = 0 \quad \forall i,j$

<u>Example</u> $\mathfrak{g} = \mathcal{L}_{\mathbb{C}}(SU(n)) = \left\{\begin{matrix}\text{traceless complex} \\ n\times n \text{ matrices}\end{matrix}\right\}$

$(H^i) = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha\, i+1}\,\delta_{\beta\, i+1} \qquad \alpha,\beta = 1,\dots,n \quad i = 1,\dots,n-1$

rank $[\mathfrak{g}] = n-1$

$[H^i, H^j] = 0 \quad \forall i,j$

$\Rightarrow (ad_{H^i} \circ ad_{H^j} - ad_{H^j} \circ ad_{H^i}) = 0$

Hence, r linear maps $ad_{H^i} : \mathfrak{g} \to \mathfrak{g}$ are <u>simultaneously diagonalisable</u>.

$\Rightarrow \mathfrak{g}$ spanned by <u>simultaneous eigenvectors</u> of ad_{H^i} $i = 1,\dots,r$

Eigenvector with,

- zero eigenval $\{H^j, j=1,\dots,r\}$

  $ad_{H^i}(H^j) = [H^i, H^j] = 0 \quad \forall i,j = 1,\dots,r$

- non-zero eigenval $\{E^\alpha ; \alpha \in \Phi\}$

  $ad_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha \qquad \alpha^i \in \mathbb{C} \quad i = 1,\dots,r \quad -(*)$

  <u>root</u> of $\mathfrak{g}$

  roots of $\mathcal{L}_{\mathbb{C}}(SU(2))$ ± 2

$H \in \mathfrak{h} \quad H = e_i H^i \qquad e_i \in \mathbb{C}$

$[H, E^\alpha] = \alpha(H) E^\alpha \quad -(\dagger) \qquad \alpha(H) = e_i \alpha^i \in \mathbb{C}$

each root defines a linear map $\alpha : \mathfrak{h} \to \mathbb{C}$

roots $\alpha \in \mathfrak{h}^*$ dual of CSA $\mathfrak{h}$

Roots are non-degenerate (proof omitted).

$\Rightarrow$ set of roots Φ consists of $d-r$ distinct elements of $\mathfrak{h}^*$

<u>Cartan-Weyl</u> basis for $\mathfrak{g}$, $B = \{H^i, i=1,\dots,r\} \cup \{E^\alpha ; \alpha \in \Phi\}$