2 Riemannian geometry

2.1 Metric

Notion of a distance for curves in $\mathbb{R}^3$

$\underline{x}(t)$, $a < t < b$, $L = \int_a^b \sqrt{\frac{d\underline{x}}{dt} \cdot \frac{d\underline{x}}{dt}}\, dt$

Generalise the scalar product of tangent vector with itself

$g : T_p(M) \times T_p(M) \to \mathbb{R}$, $(0,2)$ tensor

<u>Def</u> A metric tensor at p is a $(0,2)$ tensor g such that

(i) symmetric : $g(X,Y) = g(Y,X) \quad \forall X, Y$

(ii) non-degenerate : $g(X,Y) = 0 \quad \forall Y \Leftrightarrow X = 0$

<u>Notation</u> $g(X,Y) = \langle X, Y \rangle_g = X \cdot Y = g_{ab} X^a Y^b$

diagonalise g at point p $\to$ signature (# positive eigenvals, # negative eigenvals)

Riemannian : $+ + \dots +$, Lorentzian : $- + + \dots +$.

<u>Def</u> A Riemannian (or a Lorentzian = pseudo-Riemannian) manifold is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a smooth manifold, and g is a Riemannian (Lorentzian) metric tensor field.

A space-time in GR is <u>assumed</u> to be a Lorentzian manifold.

Length of a smooth curve $\gamma : (a,b) \to M$ with tangent vector field X.

$L = \int_a^b \sqrt{\left| g(X,X)\big|_{\lambda(t)} \right|}\, dt$ (independent of parametrisation)

Coordinate basis $g = ds^2 = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$ skip, and assume $g_{\mu\nu} = g_{\nu\mu}$

<u>Examples</u> 1) Euclidean metric on $\mathbb{R}^n$, $g = d\underline{x} \cdot d\underline{x}$

2) Minkowski metric on $\mathbb{R}^{1,n}$, $g = -d(x^0)^2 + d\underline{x} \cdot d\underline{x}$, $\underline{x} \in \mathbb{R}^n$

3) Round two sphere , $g = d\theta^2 + \sin^2\theta\, d\phi^2$, $\theta \in (0, \pi)$

This metric is induced from $d\underline{x} \cdot d\underline{x}$ in $\mathbb{R}^3$ (isometric embedding)

(check another chart, and overlap relations)

Def An inverse metric is a symmetric $(2,0)$ tensor field g^{ab}, s.t.

$$g^{ab} g_{bc} = \delta^a{}_c \quad , \quad g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \text{ coordinate basis}$$

Metric gives a natural isomorphism $T_p(M) \simeq T_p^*(M)$

$$X^a \to X_a = X^b g_{ab} \quad , \quad \eta_a \to \eta^a = g^{ab} \eta_b \qquad \text{(Raising / lowering indices)}$$

$$T^a{}_b{}^c = T^{adc} g_{bd}$$

Lorentzian signature n-dimensions $0 \le \mu, \nu, \ldots \le n-1$

Orthonormal basis $\{e_\mu\}$ for $T_p(M)$. $g(e_\mu, e_\nu) = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$

Not unique: $e'_\mu = (\Lambda^{-1})^\nu{}_\mu e_\nu$: $\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ (Lorentz transform at p).

A vector $X \in T_p(M)$ is timelike / null / spacelike if
lightlike

$$g(X,X) < 0 \,/\, = 0 \,/\, > 0$$

On a Riemannian mfd. $\cos(\angle(X,Y)) = \dfrac{g(X,Y)}{\sqrt{g(X,X)}\sqrt{g(Y,Y)}}$

This can still be defined but for spacelike vectors.

Def $\gamma(u)$ timelike curve with $\gamma(u) = p \in M$, if tangent vector X^a is time-like.

Proper time from $\gamma(0) = p$ to $\gamma(1) = q$ is

$$\tau = \int_0^1 du \sqrt{-g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}} \qquad \text{where } X^\mu = \frac{dx^\mu}{du}$$

Equivalently $d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$ along the curve.

Could use τ as a parameter along γ instead of u.

4-velocity $u^\mu = \dfrac{dx^\mu}{d\tau}$ (unit, timelike)