The Wilson-Fisher Critical Pt

Wilson-Fisher critical point found another, non-trivial f.p. by expanding in $\epsilon = 4-d > 0$. They proved $\exists$ a new fixed point with

$$g_2^* = -\frac{1}{6}\epsilon + O(\epsilon^2) \qquad g_4^* = \frac{\epsilon}{3a} + O(\epsilon^2) \quad , \quad g_{2k}^* \sim O(\epsilon^k) \quad \text{for } k \geq 3$$

To study the behaviour near this critical pt, we again expand $g_i = g_i^*\big|_{WF} + \delta g_i$ in the β-f^{n}s we found earlier (LPA) to lowest non-trivial order. In the (g_2, g_4) subspace, we have

$$\Lambda \frac{\partial}{\partial \Lambda}\begin{pmatrix}\delta g_2 \\ \delta g_4\end{pmatrix} = \begin{pmatrix} \epsilon/3 - 2 & -a(1+\frac{\epsilon}{6}) \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix}\delta g_2 \\ \delta g_4\end{pmatrix} + O(\epsilon^2, \delta g^2)$$

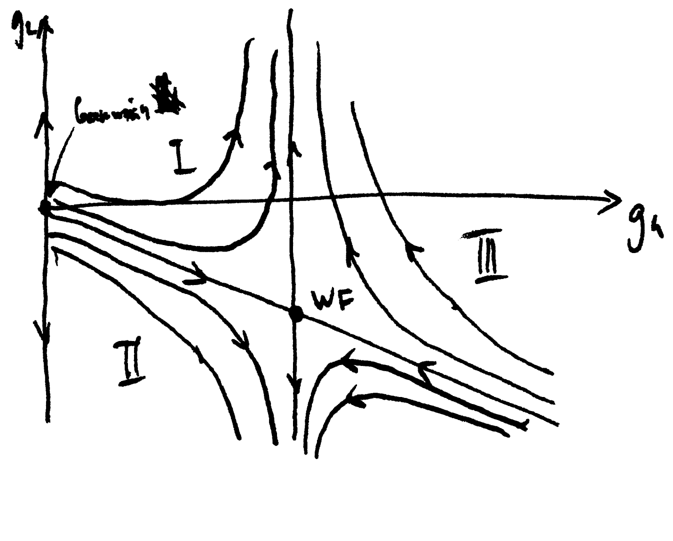
$\Rightarrow$ the eigenvalues / eigenvectors are

$$\frac{\epsilon}{3} - 2, \quad v_1 = \begin{pmatrix}1 \\ 0\end{pmatrix} \qquad \text{and} \quad \epsilon, \quad v_2 = \begin{pmatrix}-a(3+\epsilon/2) \\ 2(3+\epsilon)\end{pmatrix}$$

In $d = 4-\epsilon$, $\quad a = \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\Big|_{d=4-\epsilon} \sim \frac{1}{16\pi^2} + \frac{\epsilon}{32\pi^2}\left(1-\gamma + \ln 4\pi\right) + O(\epsilon^2)$

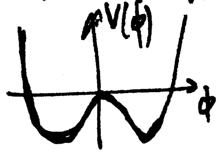
where $\gamma \approx 0.577\ldots$ is the Euler-Mascheroni constant, using the asymptotic expansion for $\Gamma(-\epsilon/2) \sim -\frac{2}{\epsilon} - \gamma + O(\epsilon)$

We see that $\sigma_2 \sim \phi^2$ is relevant, while $\sigma_4 \sim a(3+\epsilon/2)\phi^2 + 2(3+\epsilon)\phi^4$ is irrelevant. We thus have the following picture of RG evolution:



Theories in region I are massless free in the deep UV, but flow to become massive + interacting in the IR.

Theories in region II behave similarly, but have $m^2 < 0$. Consequently $\phi = 0$ is a local maximum of the effective potential, so the vacuum will



have $\langle \phi \rangle \neq 0$ and will spontaneously break the $\mathbb{Z}_2$-symmetry $\phi \mapsto -\phi$.

Theories in region III do not have a sensible continuum limit, at least within perturbation theory.

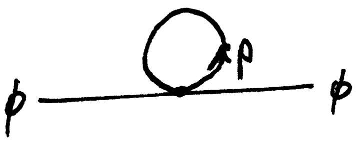
Perturbative Renormalization

Consider the scalar theory

$$S_{\Lambda_0}[\phi] = \int \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right] d^4x \qquad (\text{in } d=4)$$

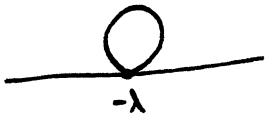
We expect that in perturbation theory (i.e. near the Gaussian crit pt) m^2 is relevant and λ is marginally irrelevant.

The quadratic term in the action receives corrections from the diagram with 2 external ϕ fields. At 1-loop, the only such diagram is



$$= \frac{-\lambda}{2(2\pi)^4} \int_{|p|\le\Lambda_0} \frac{d^4p}{p^2+m^2}$$ if we integrate out all modes with $|p|\le\Lambda_0$.

Using the fact that $\mathrm{Vol}(S^3) = 2\pi^2$, we have



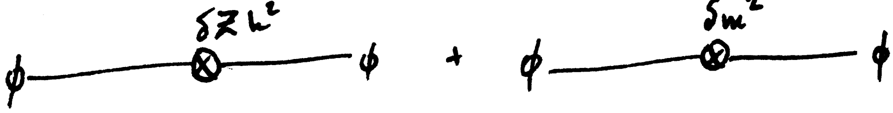
$$= \frac{-\lambda\,\mathrm{Vol}(S^3)}{2(2\pi)^4} \int_0^{\Lambda_0} \frac{p^3\,dp}{p^2+m^2} = \frac{-\lambda m^2}{32\pi^2} \int_0^{\Lambda_0^2/m^2} \frac{x\,dx}{1+x}$$

$$= \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2 \ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right]$$

As expected, this diverges as $\Lambda_0 \to \infty$. To obtain a finite limit in the continuum, we should tune (m^2, λ) as we take the lim $\Lambda_0 \to \infty$. In practice, we allow for counterterms

$$S^{CT}[\phi] = \hbar \int \left[\frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4 \right] d^4x$$

The factor of $\hbar$ means we have further contributions to the quadratic term



Thus, including these ~~contributions~~ counterterms the 1-loop contribution is

$$\frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2 \ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right] + \delta Z\, k^2 + \delta m^2$$

We now tune $(\delta Z, \delta m^2)$ by hand, so that this expression is finite as $\Lambda_0 \to \infty$.

The On-Shell Renormalisation Scheme

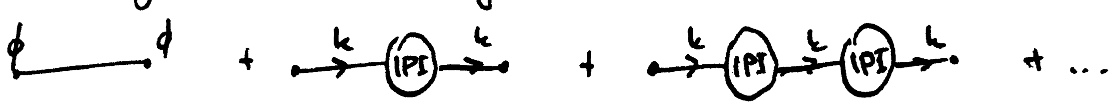
There is a lot of freedom in our choice of $(\delta Z, \delta m^2)$ because we only need a finite limit. Any choice of how to fix this freedom is called a renormalisation scheme.

In the on-shell ren. scheme, we ask that the exact momentum space propagator $\int d^4x\, e^{ik\cdot x} \langle \phi(x)\phi(0)\rangle$ has

- a simple pole when $k^2 = -m^2_{phys}$
- unit residue at this pole

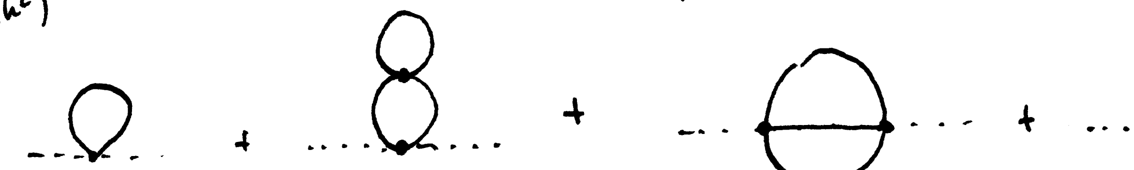
[Classically, the propagator is $\frac{1}{k^2+m^2}$ where m^2 is the original mass term in $S[\phi]$.

In the quantum theory, the exact propagator is

$\Delta(k^2) =$ 

$$= \frac{1}{k^2+m^2} + \frac{-1}{k^2+m^2}\Pi(k^2)\frac{1}{k^2+m^2} + \frac{1}{k^2+m^2}\Pi(k^2)\frac{1}{k^2+m^2}\Pi(k^2)\frac{1}{k^2+m^2} + \dots$$

$$= \frac{1}{k^2+m^2+\Pi(k^2)} \quad \text{where } -\Pi(k^2) \text{ is } \sum(\text{1PI graphs})$$

e.g. $-\Pi(k^2) =$ 

+ counterterms

Thus in on-shell scheme, if we choose our original parameter $m^2 = m^2_{phys}$

$$\Pi(-m^2_{phys}) = 0 \quad , \quad \frac{\partial}{\partial k^2}\Pi(k^2)\Big|_{k^2=-m^2_{phys}} = 0$$

To leading order,

$$-\Pi(k^2) = \delta m^2 + k^2\delta Z + \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right]$$

and the on-shell conditions force

$$\delta Z = 0 + O(\lambda^2),\ \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right] + O(\lambda^2)$$