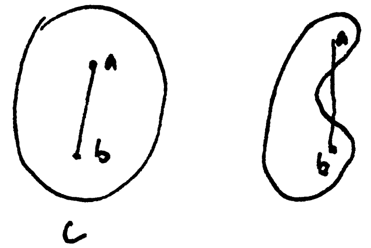
Topics in Convex Optimization

Def A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall a, b \in C$ $\forall \lambda \in [0,1]$ $\lambda a + (1-\lambda) b \in C$



- <u>Halfspace</u>: $\{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$ where $a \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$
- <u>Unit balls</u>: $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ where $\|\cdot\|$ norm on $\mathbb{R}^n$

Prop (operations that preserve convexity)

- Let $C \subseteq \mathbb{R}^n$ convex and $A : \mathbb{R}^n \to \mathbb{R}^n$ linear, then $A(C)$ is convex.
- Let $C_1, C_2 \subseteq \mathbb{R}^n$ convex. Then $C_1 \cap C_2$ is convex.

$$C = \{a \in \mathbb{R}^{n+1} : a_0 + a_1 x + \dots + a_n x^n \geq 0 \quad \forall x \in \mathbb{R}\}$$
$$= \bigcap_{x \in \mathbb{R}} \underbrace{\{a \in \mathbb{R}^{n+1} : a_0 + a_1 x + \dots + a_n x^n \geq 0\}}_{H_x \text{ halfspace (definition}}$$

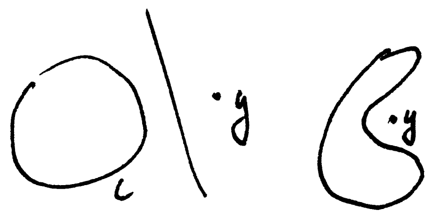
Def (Dimension of a convex set)
Let $C \subseteq \mathbb{R}^n$ be convex. The <u>dimension</u> of C is the dim of the smallest affine space containing C.
C is called <u>full dimensional</u> in $\mathbb{R}^n$ if $\dim C = n$.

Def (Convex hull)
Let $S \subseteq \mathbb{R}^n$. The convex hull of S, denoted $\mathrm{conv}(S)$ is the smallest convex set that contains S.

$$\mathrm{conv}(S) = \bigcap_{\substack{C \text{ convex} \\ S \subseteq C}} C = \{x \in \mathbb{R}^n : \exists k \in \mathbb{N}, s_1, \dots, s_k \in S \quad x = \sum \lambda_i s_i \quad \lambda_1, \dots, \lambda_k \in [0,1] \ \sum \lambda_i = 1\}$$

Exercise (Caratheodory theorem)
Let $S \subseteq \mathbb{R}^n$. Show that any element of $\mathrm{conv}(S)$ can be expressed as a convex combination of at most $n+1$ points in S.

Theorem (Separating hyperplane theorem)
Let $C \subseteq \mathbb{R}^n$, $y \notin C$. C convex.
Then there exist $a \in \mathbb{R}^n \setminus \{0\}$ $b \in \mathbb{R}$
s.t. $\begin{cases} \langle a, x\rangle \leq b & \forall x \in C \\ \langle a, y\rangle \geq b \end{cases}$



Proof: We'll assume C is closed. Let $p_C(y) = \arg\min_x \{\|y - x\|, x \in C\}$

$$\langle y - p_C(y), x - p_C(y)\rangle \leq 0 \quad \forall x \in C$$

$a = y - p_C(y) \neq 0$ since $y \in C$ $\qquad b = \langle a, p_C(y)\rangle$

This choice of a, b satisfies..

$\langle a, x\rangle \leq b \qquad \langle a, y\rangle \geq b \qquad \forall x \in C$

$\left(\langle a, x\rangle - b = \langle y - p_C(y), x\rangle - \langle y - p_C(y), p_C(y)\rangle = \langle y - p_C(y), x - p_C(y)\rangle \leq 0 \right)$

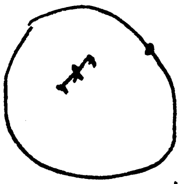
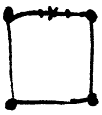
Def (Faces and extreme points)

Let $C \subseteq \mathbb{R}^n$ be convex. A set $F \subseteq C$ is called a face if:

(i) F is convex

(ii) $\forall x \in F \ \forall a, b \in C, \lambda \in (0,1)$ s.t. $x = \lambda a + (1-\lambda) b \Rightarrow a, b \in F$

If F is a singleton $F = \{x_0\}$ x_0 is called an extreme point

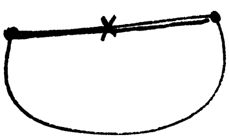
Theorem (Minkowski)

Let C be closed and bounded convex set. Let $\mathrm{ext}(C)$ be the extreme points of C. Then $C = \mathrm{conv}(\mathrm{ext}(C))$.

Lemma Let $C \subseteq \mathbb{R}^n$ be full-dimensional closed convex set

(i) Let $F \subseteq G \subseteq C$ s.t. G is a face of C and F a face of G, then F is a face of C

(ii) Let $x_0 \in C \setminus \underbrace{\mathrm{int}(C)}_{\text{interior of } C}$. Then there exists face F of C with $\dim F < \dim C$ s.t. $x_0 \in F$



Proof of (ii): $x_0 \notin \mathrm{int}(C)$ Separating hyperplane theorem: $\exists\, a \neq 0 \ b \in \mathbb{R}$ s.t. $\langle a, x_0\rangle = b \quad \langle a, x\rangle \leq b \quad \forall x \in \mathrm{int}(C)$

$F = C \cap \{x : \langle a, x\rangle = b\}$ a face that contains x_0, $\dim \leq n-1 < n = \dim(C)$ ∎

Proof of theorem

We want to show $C = \mathrm{conv}(\mathrm{ext}(C))$. The inclusion $C \supseteq \mathrm{conv}(\mathrm{ext}(C))$ is trivial.

We need to show $C \subseteq \mathrm{conv}(\mathrm{ext}(C))$.