Advanced Quantum Field Theory
Example sheet 1

1. $\int d\theta\ \theta = 1$ and $\int d\theta\ 1 = 0$, $\{\theta^i, \theta^j\} = 0$

$$Z(\eta,\bar\eta) := \int d^n\theta\, d^n\bar\theta\, \exp(\bar\theta^i B_{ij}\theta^j + \bar\eta_i\theta^i + \bar\theta^i\eta_i)$$ B invertible $n\times n$ matrix

Define $\varsigma^i := \theta^i + (B^{-1})^{ij}\eta_j$ and $\bar\varsigma^i := \bar\theta^i + (B^{-1})^{ij}\bar\eta_j$, then

$d\varsigma^i = d\theta^i$ and $d\bar\varsigma^i = d\bar\theta^i$ as B and $\eta, \bar\eta$ are constant for the integration.

$$\Rightarrow Z(\eta,\bar\eta) = \int d^n\varsigma\, d^n\bar\varsigma\, \exp\Big[(\bar\varsigma^i - (B^{-1})^{ik}\bar\eta_k)B_{ij}(\varsigma^j - (B^{-1})^{jm}\eta_m)$$
$$+ \bar\eta_i\varsigma^i - \bar\eta_i(B^{-1})^{ij}\eta_j + \bar\varsigma^i\eta_i - (B^{-1})^{ij}\bar\eta_j\eta_i\Big]$$
$$= \int d^n\varsigma\, d^n\bar\varsigma\, \exp\Big[\bar\varsigma^i B_{ij}\varsigma^j + \bar\eta_k(B^{-1})^{km}\eta_m - \bar\eta_i(B^{-1})^{ij}\eta_j - (B^{-1})^{ij}\bar\eta_j\eta_i\Big]$$
$$= \exp(-\bar\eta_i(B^{-1})^{ij}\eta_j)\int d^n\varsigma\, d^n\bar\varsigma\, \exp(\bar\varsigma^i B_{ij}\varsigma^j)$$

Now $\{\varsigma^i, \varsigma^j\} = \{\theta^i + (B^{-1})^{ik}\eta_k, \theta^j + (B^{-1})^{jm}\eta_m\} = \{\theta^i, \theta^j\}$
$+ (B^{-1})^{ik}\{\eta_k, \theta^j\} + (B^{-1})^{jm}\{\theta^i, \eta_m\} + (B^{-1})^{ik}(B^{-1})^{jm}\{\eta_k, \eta_m\} = 0$.

$\exp(\bar\varsigma^i B_{ij}\varsigma^j) = \sum_k \frac{1}{k!}(\bar\varsigma^i B_{ij}\varsigma^j)^k$ and remembering that $\int d\theta\ 1 = 0$ and hence $\int d\varsigma\, 1 = 0$, further any term containing more than one ς^i for any given i is zero as $\varsigma\varsigma = -\varsigma\varsigma = 0$. Hence the only contributing term is $k=n$ as only $(\bar\varsigma^i B_{ij}\varsigma^j)^n$ has terms where each $\bar\varsigma^i$ and ς^j is represented exactly once.

$$\int d^n\varsigma\, d^n\bar\varsigma\, \exp(\bar\varsigma^i B_{ij}\varsigma^j) = \frac{1}{n!}\int d^n\varsigma\, d^n\bar\varsigma\, (\bar\varsigma^i B_{ij}\varsigma^j)^n$$
$$= \frac{1}{n!}\int d^n\varsigma\, d^n\bar\varsigma \sum_{\sigma,\bar\sigma\in S_n} \prod_{i,j} \bar\varsigma^{\bar\sigma_i} B_{\bar\sigma_i\sigma_j}\varsigma^{\sigma_j}$$

Ordering $\bar\varsigma^{\bar\sigma_i}$ (ς^{σ_j}) for integrating creates a factor of $\mathrm{sgn}(\bar\sigma)$ ($\mathrm{sgn}(\sigma)$) and all $\int = 1$

$$\Rightarrow = \frac{1}{n!}\sum_{\sigma,\bar\sigma\in S_n} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\bar\sigma)\prod_{i,j} B_{\bar\sigma_i\sigma_j} = \det B$$

$$\therefore \mathcal{Z}(\eta, \bar\eta) = \det B \; \exp(-\bar\eta_i (B^{-1})^{ij} \eta_j) \qquad \square$$

$$\langle \bar\theta^{i_1} \dots \bar\theta^{i_r} \theta^{j_1} \dots \theta^{j_s} \rangle := \frac{1}{\mathcal{Z}(0,0)} \int d^n\theta \, d^n\bar\theta \; \exp(-\bar\theta^i B_{ij} \theta^j) \, \bar\theta^{i_1} \dots \bar\theta^{i_r} \theta^{j_1} \dots \theta^{j_s}$$

$$= \frac{1}{\mathcal{Z}(0,0)} \int d^n\theta \, d^n\bar\theta \sum_k \frac{1}{k!} (-\bar\theta^i B_{ij} \theta^j)^k \, \bar\theta^{i_1} \dots \bar\theta^{i_r} \theta^{i_1} \dots \theta^{i_s}$$

The integral is again non-vanishing for terms that have exactly one θ^i and $\bar\theta^i$. If $r, s > n$, $\exists\, i$: term contains $\dots \theta^i \dots \theta^i \dots$ (or $\bar\theta$) and hence all terms vanish. If $r \neq s$, $\nexists\, k$: get each θ^i and $\bar\theta^i$ exactly once. Thus,

$$= \frac{\delta^{rs}}{\mathcal{Z}(0,0)} \int d^n\theta \, d^n\bar\theta \; \frac{1}{(n-r)!} (-\bar\theta^i B_{ij} \theta^j)^{n-r} \, \bar\theta^{i_1} \dots \bar\theta^{i_r} \theta^{j_1} \dots \theta^{j_s}$$

$$= \frac{\delta^{rs} (-1)^{n-r}}{\mathcal{Z}(0,0)} \sum_{\substack{\sigma \in S_n \setminus \{j_1, \dots, j_s\} \\ \bar\sigma \in S_n \setminus \{i_1, \dots, i_r\}}} \frac{\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\bar\sigma)}{(n-r)!} \prod_{i,j} B_{\bar\sigma_i \sigma_j} = \frac{\delta^{rs} (-1)^{n-r}}{\mathcal{Z}(0,0)} \det B[\bar S, S]$$

where $B[\bar S, S]$ means the minor of B obtained by removing rows $i \in \bar S$ and columns $j \in S$, and $\bar S = \{i_1, \dots, i_r\}$, $S = \{j_1, \dots, j_s\}$.

$$\therefore \langle \bar\theta^{i_1} \dots \bar\theta^{i_r} \theta^{j_1} \dots \theta^{j_s} \rangle = \frac{\delta^{rs} (-1)^{n-r}}{\det B} \det B[\bar S, S] \quad \text{up to -sign}$$

$$\langle \bar\theta^{i_1} \dots \bar\theta^{i_r} \theta^{j_1} \dots \theta^{j_s} \rangle = \frac{\partial}{\partial \eta_{i_1}} \cdots \frac{\partial}{\partial \eta_{i_r}} \frac{\partial}{\partial \bar\eta_{j_1}} \cdots \frac{\partial}{\partial \bar\eta_{j_s}} \exp(-\bar\eta_i (B^{-1})^{ij} \eta_j) \Big|_{\eta = \bar\eta = 0}$$

$$= \begin{cases} 0 & : r \neq s \\ 0 & : r, s > n \\ \sum_{\sigma \in S_r} \varepsilon(\sigma) (B^{-1})^{i_1 j_{\sigma(1)}} \cdots (B^{-1})^{i_r j_{\sigma(r)}} & : r = s \leq n \end{cases}$$

This is a better idea.

2. $\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \; e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4}$, $\lambda > 0$

(a) Expand in λ,

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \left[e^{-\frac{1}{2}x^2} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{-\lambda}{4!}\right)^{\ell} x^{4\ell} \right]$$ Taylor expansion

Truncating this series,

$$\mathcal{Z}_n(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{-\lambda}{4!}\right)^{\ell} \int_{\mathbb{R}} dx \; e^{-\frac{1}{2}x^2} x^{4\ell}$$

$u = \frac{1}{2}x^2$

$du = x\,dx$

$dx = \frac{du}{\sqrt{2u}}$

$$= \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{-\lambda}{4!}\right)^{\ell} \frac{2^{2\ell}}{\sqrt{2}} \, 2 \int_0^{\infty} du \; e^{-u} \, u^{2\ell + \frac{1}{2} - 1}$$

$\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x} \, dx$

$$= \frac{1}{\sqrt{\pi}} \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{-\lambda}{3!}\right)^{\ell} \Gamma\left(2\ell + \frac{1}{2}\right)$$

$\Gamma(z) = (z-1)\,\Gamma(z-1)$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$= \sum_{\ell=0}^{n} \left(-\frac{\lambda}{4!}\right)^{\ell} \frac{(4\ell)!}{4^{\ell}(2\ell)!\,\ell!}$$

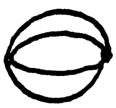
$\Rightarrow \Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})\,\Gamma((n-1)+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\Gamma(\frac{1}{2})$

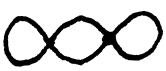
$= \frac{(2n)!}{4^n n!}\sqrt{\pi}$

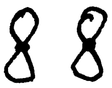
$\ell=0$ $\quad$ $\ell=1$ $\quad$ $\ell=2$ $\quad$ $\ell=3$

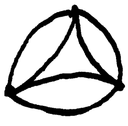
$$\mathcal{Z} = 1 + (-\lambda) \times \frac{1}{8} + \lambda^2 \times \frac{35}{384} + (-\lambda^3) \times \frac{385}{3072} + \dots$$

$\ell = 1$: $\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 8$ $\;//\;$ $\sum_{\Gamma \in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} = \frac{1}{8}$

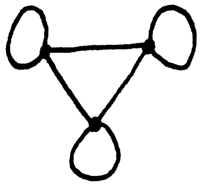
$\ell = 2$: $\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 4! \times 2 = 48$

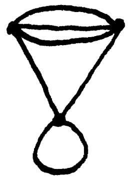
$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 2^3 \times 2 = 16$

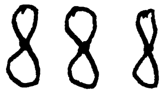
$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 8 \times 8 \times 2 = 128$ $\;//\;$ $\sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} = \frac{1}{48} + \frac{1}{16} + \frac{1}{128} = \frac{35}{384}$

$\ell = 3$: $\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 2^3 \times 3! = 48$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 2^4 \times 2 = 32$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 2^3 \times 3! = 48$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 3! \times 2 \times 2 = 24$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 8^2 \times 3! = 3072$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 16 \times 8 = 128$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 48 \times 8 = 384$

$$\sum_\Gamma \frac{1}{|\mathrm{Aut}\,\Gamma|} = \frac{1}{48} + \frac{1}{32} + \frac{1}{48} + \frac{1}{24} + \frac{1}{3072} + \frac{1}{128} + \frac{1}{384} = \frac{395}{3072}.$$

(b) [Z_n{1_lambda}.pdf] Diverges at $\lambda = \frac{1}{10}$ after $n \gtrsim 24$.

(c)
$$|a_\ell \lambda^\ell| = \frac{(4\ell)!}{4^\ell (2\ell)!\, \ell!\, (4!)^\ell} \lambda^\ell \approx \frac{e^{4\ell \ln 4\ell - 4\ell}}{4^\ell e^{2\ell \ln 2\ell - 2\ell} e^{\ell \ln \ell - \ell} 24^\ell} \lambda^\ell = \exp\left[4\ell \ln 4\ell - 2\ell \ln 2\ell - \ell - \ell \ln \ell - \ell \ln 4 - \ell \ln 24\right] \lambda^\ell = \cancel{\exp[4\ell \ln 4\ell - 2\ell \ln 2\ell - \ell \ln 4\ell - \ell \ln 24]}$$

$$= \exp\left[4\ell \ln 4 + 4\ell \ln \ell - 2\ell \ln 2 - 2\ell \ln \ell - \cancel{\ell} - \ell \ln \ell - \ell \ln 4 - \ell \ln 24\right] \lambda^\ell$$

$$= \exp\left[2\ell \ln 4 + \ell \ln \ell - \ell - \ell \ln 24\right] = \exp\left[\ell \ln \ell - \ell - \ell \ln 3/2\right] \lambda^\ell$$

$$= \exp\left[\ell \ln \ell - \ell - \ell \ln 3/2 + \ell \ln \lambda\right] = e^{\ell \ln(2\ell\lambda/3e)}$$

$$\Rightarrow \frac{d a_\ell \lambda^\ell}{d\lambda} \approx e^{\ell \ln(2\ell\lambda/3e)} \left[\ln(2\ell\lambda/3e) + \frac{2\ell\lambda/3e}{2\ell\lambda/3e}\right] = 0 \quad \text{at minimum}$$

$$\Rightarrow \ln(2\ell\lambda/3e) = -1 \qquad \Rightarrow \ell \approx \frac{3}{2\lambda} \quad \text{at minimum w.r.t. } \ell.$$

Provided $|\lambda| < \frac{3}{2}$, $\ell_{min} \approx \frac{3}{2\lambda}$. $|\ell_{min}| \approx \frac{3}{2|\lambda|} > 1$, then the Borel transform

$$\mathcal{B}Z(\lambda) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a_\ell \lambda^\ell \approx \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-\lambda)^\ell e^{\ell \ln(2\ell/3) - \ell}$$

$$\approx \sum_{\ell=0}^{\infty} (-\lambda)^\ell e^{\ell \ln(2\ell/3) - \ell - \ell \ln \ell + \ell} = \sum_{\ell=0}^{\infty} \left(-\frac{2\lambda}{3}\right)^\ell$$

This converges for $|\lambda| < \frac{3}{2}$ by the ratio test

$$\frac{|b_{\ell+1}|}{|b_\ell|} = \frac{\left|-\frac{2\lambda}{3}\right|^{\ell+1}}{\left|-\frac{2\lambda}{3}\right|^{\ell}} = \left|\frac{2\lambda}{3}\right| < 1 \qquad \text{when } |\lambda| < \frac{3}{2}.$$

In fact, $\mathcal{B}\mathcal{Z}(\lambda) \approx \frac{1}{1+\frac{2\lambda}{3}}$ when $|\lambda| < \frac{3}{2}$.

In this case,

$$\mathcal{Z}(\lambda) \overset{?}{=} \int_0^\infty dz\, e^{-z}\, \mathcal{B}\mathcal{Z}(z\lambda) = \int_0^\infty dz\; e^{-z} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a_\ell \lambda^\ell z^\ell$$

and as the sum converges

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a_\ell \lambda^\ell \int_0^\infty e^{-z} z^\ell\, dz = \sum_{\ell=0}^{\infty} a_\ell \lambda^\ell = \mathcal{Z}(\lambda) \quad \square$$

(But $|z\lambda| < \frac{3}{2}$ not true really.)

(d)

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx\; e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4}$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \left(\frac{\lambda}{6}\right)^{-1/4} \int_0^\infty du\; e^{-u} (4u)^{-3/4} e^{-\sqrt{6u/\lambda}}$$

$x \geq 0$

$u = \frac{\lambda}{4!}x^4$, $x^2 = \sqrt{\frac{4!\,u}{\lambda}} = \frac{2\sqrt{6}}{\sqrt{\lambda}}\sqrt{u}$

$du = \frac{\lambda}{6} x^3 dx$, $dx = \left(\frac{\lambda}{6}\right)^{-1}\left(\frac{24u}{\lambda}\right)^{-3/4} du$

$= \left(\frac{\lambda}{6}\right)^{-1/4} (4u)^{-3/4} du$

Expanding in $1/\sqrt{\lambda}$,

$$= \frac{1}{2\sqrt{\pi}} \left(\frac{\lambda}{6}\right)^{-1/4} \int_0^\infty du\; e^{-u}\, u^{-3/4} \sum_{L=0}^{\infty} \frac{1}{L!}\left(\frac{6}{\lambda}\right)^{L/2} u^{L/2} (-1)^L$$

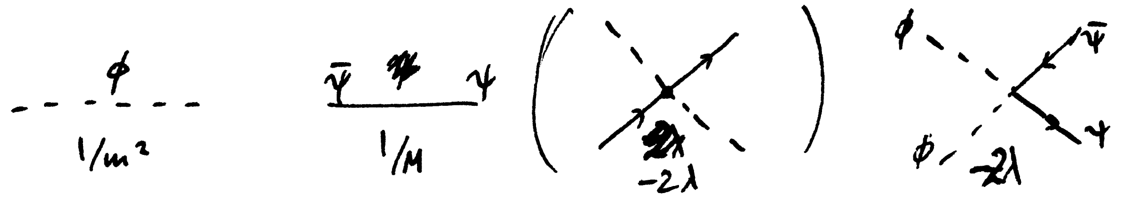
$$= \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \left(\frac{6}{\lambda}\right)^{\frac{L}{2}+\frac{1}{4}} \int_0^\infty du\; e^{-u}\, u^{L/2 + \frac{1}{4} - 1}$$

$$= \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \left(\frac{6}{\lambda}\right)^{\frac{L}{2}+\frac{1}{4}} \Gamma\left(\frac{L}{2} + \frac{1}{4}\right)$$

For $\lambda = \frac{1}{10}$, need to take ~ 95 terms, and soon afterwards, gnuplot breaks.

3. $S(\phi, \psi, \bar\psi) = \frac{m^2}{2}\phi^2 + \frac{M}{2}\bar\psi\psi + \lambda\phi^2\bar\psi\psi$

(a)



(b) $S_{eff}(\phi) := -\hbar \log\left[\int d\psi\, d\bar\psi\; e^{-S(\phi,\psi,\bar\psi)/\hbar}\right]$

$$\int d\psi\, d\bar\psi\, \exp\left(-\frac{m^2}{2\hbar}\phi^2 - \frac{M}{2\hbar}\bar\psi\psi - \lambda\phi^2\bar\psi\psi/\hbar\right)$$

$$= \exp\left(-\frac{m^2}{2\hbar}\phi^2\right)\int d\psi\, d\bar\psi\, \exp\left[-\left(\frac{M}{2\hbar} + \frac{\lambda\phi^2}{\hbar}\right)\bar\psi\psi\right]$$

$$= -\exp\left(-\frac{m^2}{2\hbar}\phi^2\right)\left(\frac{M}{2\hbar} + \frac{\lambda\phi^2}{\hbar}\right)$$ as in question 1.

$$S_{eff}(\phi) = \frac{m^2}{2}\phi^2 - \hbar\log\left(\frac{M}{2\hbar} + \frac{\lambda\phi^2}{\hbar}\right)$$

$$= \frac{m^2}{2}\phi^2 - \hbar\log\left(1 + \frac{2\lambda}{M}\phi^2\right) - \hbar\log\left(\frac{M}{2\hbar}\right)$$

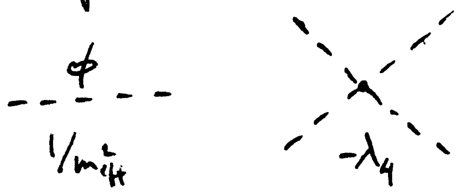
$$= \frac{m^2}{2}\phi^2 - \hbar\frac{2\lambda}{M}\phi^2 + \hbar\,\frac{2\lambda^2}{M^2}\phi^4 + \dots - \hbar\log\left(\frac{M}{2\hbar}\right)$$

$$=: \sum_{n=0}^{\infty}\frac{\lambda_n}{n!}\phi^n$$

where $\lambda_0 = -\hbar\log\left(\frac{M}{2\hbar}\right)$, $\lambda_2 = m_{eff}^2 = m^2 - \frac{4\hbar\lambda}{M}$

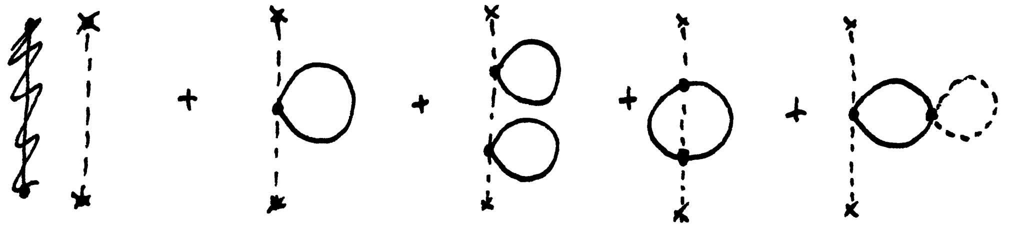
$$\lambda_{2k} = (-1)^k \frac{(2k)!}{k}\hbar\frac{2^k\lambda^k}{M^k} \qquad \left(\lambda_4 = \frac{48\lambda^2}{M^2}\right.$$

Can compute correlation functions only depending on ϕ. Now Feynman rules



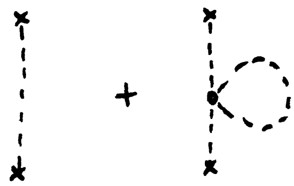
... (all $2k$-point vertices)

(c) Original:

$\frac{1}{2}\langle\phi^2\rangle =$ 

$$= \frac{1}{m^2} + \frac{2\lambda}{m^4 M} + \frac{4\lambda^2}{m^6 M^2} + \frac{4\lambda^2}{m^6 M^2} + \frac{4\lambda^2}{2m^6 M^2} + \dots$$

Effective:

$\frac{1}{2}\langle\phi^2\rangle =$ 

$$\frac{1}{m_{\text{eff}}^2} + \frac{-\lambda_4}{2m_{\text{eff}}^6} + \dots$$

$$\frac{1}{2}\langle\phi^2\rangle = \frac{\int d\phi\, d\psi\, d\bar\psi\, \frac{1}{2}\phi^2 \exp\left[-\frac{m^2}{2}\phi^2 - \frac{M}{2}\bar\psi\psi - \lambda\phi^2\bar\psi\psi\right]}{\int d\phi\, d\psi\, d\bar\psi \exp\left[-\frac{m^2}{2}\phi^2 - \frac{M}{2}\bar\psi\psi\right]}$$

$$= -\frac{2}{M}\frac{m}{\sqrt{2\pi}}\int d\psi\, d\bar\psi \frac{\sqrt{2\pi}}{(m^2+2\bar\psi\psi\lambda)^{3/2}}\frac{1}{2}e^{-\frac{M}{2}\bar\psi\psi}$$

$$= \frac{1}{2}\frac{1}{m^2} + \frac{3\lambda}{m^2 M}$$

$$= \frac{1}{m^2 - \frac{4\lambda}{M}} - \frac{24\lambda^2/M^2}{\left(m^2 - \frac{4\lambda}{M}\right)^3} + O(\lambda^3)$$

$$= \frac{1}{m^2} + \frac{4\lambda}{m^4 M} + \frac{16\lambda^2}{m^6 M^2} - \frac{24\lambda^2}{m^6 M^2} + O(\lambda^3)$$

$$= \frac{1}{m^2} + \frac{4\lambda}{m^4 M} - \frac{8\lambda^2}{m^6 M^2} + O(\lambda^3)$$

Something with factors of 2 ...

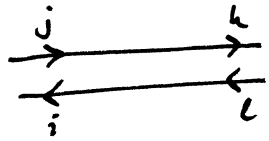
4.7 $\mathcal{Z}(a;N) = \int d^{2N}M \exp\left(-\frac{1}{2}\mathrm{tr}(M^2) - \frac{a}{N}\mathrm{tr}(M^4)\right)$

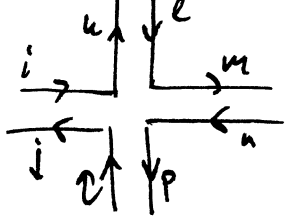
(a) $\int d^{2N}M \exp\left(-\frac{1}{2}\mathrm{tr}\, M^2 + \mathrm{tr}\, M\cdot J - \frac{a}{N}\mathrm{tr}(M^4)\right) = \mathcal{Z}(J,a)$

$$-\frac{1}{2}M^i{}_j M^j{}_i + M^i{}_j J^j{}_i = -\frac{1}{2}\mathrm{tr}\,(M+J)(M+J) + \frac{1}{2}\mathrm{tr}\, JJ$$

$$\Rightarrow \langle M_i{}^j M_k{}^l\rangle_{a=0} = \frac{\partial}{\partial J^i{}_j}\frac{\partial}{\partial J^k{}_l}\frac{\mathcal{Z}(J,0)}{\mathcal{Z}(0,0)}\Big|_{J=0} = \frac{\partial}{\partial J}\frac{\partial}{\partial J}\exp\left(-\frac{1}{2}\mathrm{tr}\, J\cdot J\right)\Big|_{J=0}$$

$$= \delta^l{}_i\, \delta^j{}_k$$





$$\frac{a}{N}\delta^k{}_i\,\delta^m{}_l\,\delta^p{}_n\,\delta^j{}_q$$

$$\ln \frac{Z(a;N)}{Z(0;N)} = N \ln \sum_{l=0}^{\infty} (-a)^l N^{-l} \frac{(4l)!}{4^l (2l)!\, l!}$$

$$= N \sum_{m=0}^{\infty} (-1)^m \frac{1}{m} \left(\sum_{l=1}^{\infty} (-a)^l N^{-l} \frac{(4l)!}{4^l (2l)!\, l!} \right)^m$$

$$= N \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-a)^n N^{-n} F_{n,m}$$ for some combinatoric constant $F_{n,m}$.

5. $\mathcal{H} = L^2(\mathbb{R}^n, d^n x)$

$P : \hat{x}^a \to -\hat{x}^a$

(a) $$\mathrm{Tr}_{\mathcal{H}}(P e^{-TH}) = \int d^n y \langle y | P e^{-HT} | y \rangle$$

$$= \int d^n y \langle -y | e^{-HT} | y \rangle = \int_{\mathbb{R}^n} d^n y \int_{C_T[-y,y]} \mathcal{D}x \; e^{-S}$$

(b) $$\langle \psi_f | e^{-TH} | \psi_i \rangle = \int d^n y_0 \, d^n y_1 \langle \psi_f | y_1 \rangle \langle y_1 | e^{-HT} | y_0 \rangle \langle y_0 | \psi_i \rangle$$

$$= \int d^n y_0 \, d^n y_1 \, \bar{\psi}_f(y_1) \psi_i(y_0) \langle y_1 | e^{-HT} | y_0 \rangle$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} d^n y_0 \, d^n y_1 \, \bar{\psi}_f(y_1) \psi_i(y_0) \int_{C_T[y_1, y_0]} \mathcal{D}x \; e^{-S}$$

Suppose $n=1$, $V = \frac{1}{2} m \omega^2 x^2$,

$$\langle x | e^{-TH} | y \rangle = \sqrt{\frac{m\omega}{2\pi \sin \omega t}} \exp\left(-m\omega \frac{(x^2+y^2)\cos \omega t - 2xy}{2 \sin \omega t} \right)$$

(a) $\Rightarrow$ $$\mathrm{Tr}_{\mathcal{H}}(P e^{-TH}) = \int dy \sqrt{\frac{m\omega}{2\pi \sin \omega T}} \exp\left(-m\omega \frac{2y^2 \cos \omega T - 2y^2}{2 \sin \omega T} \right)$$

$$= \sqrt{\frac{m\omega}{2\pi \sin \omega T}} \sqrt{\frac{\sin \omega T}{m\omega(\cos \omega T - 1)}} \sqrt{\pi} = \frac{1}{\sqrt{2(\cos \omega T - 1)}} \quad \times$$

(b) $\Rightarrow \langle \psi_f | e^{-TH} | \psi_i \rangle =$ $\qquad \psi_{i,f}(x) = \sqrt[4]{\frac{m\omega}{\pi}} \exp\left(-\frac{1}{2} m\omega x^2\right) = \psi_0(x)$

$$= \int dy_0 dy_1 \sqrt{\frac{m\omega}{\pi}} \exp\left(-m\omega \frac{y_0^2+y_1^2}{2}\right) \sqrt{\frac{m\omega}{2\pi \sin\omega T}} \exp\left(-m\omega \frac{(y_0^2+y_1^2)\cos\omega T - 2y_0y_1}{2\sin\omega T}\right)$$

$$= \int dy_0 dy_1 \frac{m\omega}{\pi} \frac{1}{\sqrt{2\sin\omega T}} \exp\left(-m\omega \frac{(y_0^2+y_1^2)(\cos\omega T + \sin\omega T) - 2y_0y_1}{2\sin\omega T}\right)$$

$$= \int dy_0 dy_1 \frac{m\omega}{\pi} \frac{1}{\sqrt{2\sin\omega T}} \exp\left(-m\omega \frac{[y_1 - y_0/(\cos\omega T + \sin\omega T)]^2(\cos\omega T + \sin\omega T)}{2\sin\omega T}\right)$$

$$\times \exp\left(-m\omega \frac{y_0^2(\cos\omega T + \sin\omega T - 1/(\cos\omega T + \sin\omega T))}{2\sin\omega T}\right)$$

$$= \int dy_0 \sqrt{\frac{m\omega}{\pi}} \frac{1}{\sqrt{\cos\omega T + \sin\omega T}} \exp\left(-m\omega \frac{y_0^2 \cos\omega T}{\cos\omega T + \sin\omega T}\right)$$

$$= \frac{1}{\sqrt{\cos\omega T}}$$

$$\langle x | e^{-HT} | y \rangle = \sqrt{\frac{m\omega}{2\pi \sinh\omega T}} \exp\left(-m\omega \frac{(x^2+y^2)\cosh\omega T - 2xy}{2\sinh\omega T \hbar}\right)$$

$$\mathrm{tr}_x P e^{-HT} = \frac{1/2}{\cosh\frac{\omega T}{2}}$$

$$\langle \psi_0 | e^{-HT} | \psi_0 \rangle = e^{-\omega T/2}$$

In QM,

$$\mathrm{tr}_x P e^{-HT} = \sum_{n=0}^{\infty} \langle n | P e^{-HT} | n \rangle = \sum_{n=0}^{\infty} (-1)^n e^{-T \frac{1}{2}\omega(2n+1)}$$

$$\langle 0 | e^{-HT} | 0 \rangle = e^{-\omega T/2}$$

6. $$S=\int d\tau\left[\frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2+\bar\psi\frac{\partial\psi}{\partial\tau}+\frac{1}{2}\lambda^2 h'(x)^2+\lambda h''(x)\bar\psi\psi\right]$$

λ const
$h(x)$ smooth f^n
x real bosonic
$\psi,\bar\psi$ fermionic

(a) $$\delta x = \varepsilon\bar\psi-\bar\varepsilon\psi$$
$$\delta\psi=\varepsilon(-\dot x+\lambda h'(x))$$
$$\delta\bar\psi=\bar\varepsilon(\dot x+\lambda h'(x))$$

$\varepsilon,\bar\varepsilon$ const fermionic

$\dot x=\partial x/\partial\tau$

$$\delta S=\int d\tau\Big[\dot x\,\delta\dot x+\delta\bar\psi\frac{\partial\psi}{\partial\tau}+\bar\psi\,\delta\dot\psi+\lambda^2h'(x)\,\delta h'(x)$$
$$+\lambda\,\delta h''(x)\bar\psi\psi+\lambda h''(x)\,\delta\bar\psi\,\psi+\lambda h''(x)\bar\psi\,\delta\psi\Big]$$

$$\delta\dot x=\partial_\tau\delta x=\varepsilon\dot{\bar\psi}-\bar\varepsilon\dot\psi$$
$$\delta\dot\psi=\partial_\tau\delta\psi=\varepsilon(-\ddot x+\lambda\,\partial_\tau h'(x))$$
$$\delta h'(x)=\frac{\partial}{\partial x}\delta h=\frac{\partial}{\partial x}\frac{\partial h}{\partial x}\delta x=h''(x)\,\delta x$$

$$\delta h=\frac{\partial h}{\partial x}\delta x=h'(x)\,\delta x$$
$$\partial_\tau h'=\frac{\partial}{\partial x}\frac{\partial}{\partial\tau}h=\frac{\partial}{\partial x}\frac{\partial x}{\partial\tau}\frac{\partial}{\partial x}h=\dot x\,h''(x)$$

$$\delta h''(x)=h'''(x)\,\delta x$$

$$\Rightarrow\delta S=\int d\tau\Big[\dot x\varepsilon\dot{\bar\psi}-\dot x\bar\varepsilon\dot\psi+\bar\varepsilon(\dot x+\lambda h'(x))\dot\psi+\bar\psi\varepsilon(-\ddot x+\lambda\dot x h''(x))$$
$$+\lambda^2h'(x)h''(x)\cancel{\delta x}\,(\varepsilon\bar\psi-\bar\varepsilon\psi)+\lambda h'''(x)\underbrace{(\varepsilon\bar\psi-\bar\varepsilon\psi)\bar\psi\psi}_{=0\text{ as }\psi,\bar\psi\text{ fermionic}}$$
$$+\lambda h''(x)\bar\varepsilon(\dot x+\lambda h'(x))\psi+\lambda h''(x)\bar\psi\varepsilon(-\dot x+\lambda h'(x))\Big]$$

$$=\int d\tau\left[\dot x\varepsilon\dot{\bar\psi}+\bar\varepsilon\lambda h'(x)\dot\psi-\bar\psi\varepsilon\ddot x+\lambda h''(x)\bar\varepsilon\dot x\psi\right]$$

$$=\int d\tau\left[\partial_\tau(\dot x\varepsilon\bar\psi)+\lambda\,\partial_\tau(\bar\varepsilon h'(x)\psi)\right]=0$$ as ~~complete total der~~ boundary terms vanish

$$\bar Q=\frac{\partial\mathcal L}{\partial\dot x}\bar\psi+\frac{\partial\mathcal L}{\partial\dot\psi}(-\dot x+\lambda h'(x))-\delta\mathcal L=2\dot x\bar\psi-\bar\psi\dot x+\bar\psi\lambda h'(x)-\dot x\psi=\dot x\bar\psi+\lambda h'(x)\bar\psi$$

$$Q=\frac{\partial\mathcal L}{\partial\dot x}\psi+\frac{\partial\mathcal L}{\partial\dot{\bar\psi}}(\dot x+\lambda h'(x))-\delta\mathcal L=\dot x\psi-\lambda h'(x)\psi$$

(b) $H = \frac{\partial \mathcal{L}}{\partial \dot\psi}\dot\psi + \frac{\partial \mathcal{L}}{\partial \dot{\bar\psi}}\dot{\bar\psi} + \frac{\partial \mathcal{L}}{\partial \dot x}\dot x - \mathcal{L}$

$$= \tfrac{1}{2}p^2 - \tfrac{1}{2}\lambda^2 h'(x)^2 - \tfrac{1}{2}\lambda h''(x)(\bar\psi\psi - \psi\bar\psi)$$

$$\{Q,\bar Q\} = \{p\psi - \lambda h'(x)\psi,\ p\bar\psi + \lambda h'(x)\bar\psi\} \qquad \{\psi,\bar\psi\} = 1$$

$$= p^2 - \lambda^2 h'(x)^2 - \lambda h'(x)\psi p\bar\psi - p\bar\psi\lambda h'(x)\psi$$
$$+ p\psi\lambda h'(x)\bar\psi + \lambda h'(x)\bar\psi p\psi$$

$$= p^2 - \lambda^2 h'(x)^2 - \lambda(p h'(x) - h'(x)p)(\bar\psi\psi - \psi\bar\psi)$$

$$= p^2 - \lambda^2 h'(x) - \lambda[p, h'(x)](\bar\psi\psi - \psi\bar\psi) = 2H$$

as $[p, h'(x)] = [p, x]h''(x) = h''(x)$

Suppose $|\Psi\rangle$ is normalised energy eigenstate with $H|\Psi\rangle = E|\Psi\rangle$,

$$2E = \langle\Psi|2H|\Psi\rangle = \langle\Psi|\{Q,\bar Q\}|\Psi\rangle = \langle\Psi|Q\bar Q + \bar Q Q|\Psi\rangle$$

$$= \langle\Psi|Q\bar Q|\Psi\rangle + \langle\Psi|\bar Q Q|\Psi\rangle = \|\bar Q|\Psi\rangle\|^2 + \|Q|\Psi\rangle\|^2 \geq 0$$

$\Rightarrow E \geq 0$ and in the limiting groundstate with $E=0$ $\Rightarrow Q|\Psi\rangle = \bar Q|\Psi\rangle = 0$.

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