Standard Model 10

$$\chi^\mu = -\partial_y^\mu \int d\sigma \, [\rho(\sigma) D(y-x,\sigma) + \hat{\rho}(\sigma) D(x-y,\sigma)]$$

For spacelike $(x-y)^2 \Rightarrow D(x-y,\sigma) = D(y-x,\sigma)$

Also require commutation ~~between~~ spacelike interval to vanish, $\chi^\mu = 0$

$\Rightarrow$ must have $\rho(\sigma) = -\hat{\rho}(\sigma)$

$$\therefore \langle 0| [j^\mu(y), \phi_n(x)] |0\rangle = -\partial_y^\mu \int d\sigma \, \rho(\sigma) \, i\Delta(y-x;\sigma)$$

where $i\Delta(x,\sigma) = D(x,\sigma) - D(-x,\sigma)$

$$= \int \frac{d^4k}{(2\pi)^3} \delta(k^2-\sigma) \, \varepsilon(k^0) \, e^{-ik\cdot x}$$

change $k^\mu \to -k^\mu$ get $D(-x,\sigma)$

where $\varepsilon(k^0) = \begin{cases} +1 & k^0 > 0 \\ -1 & k^0 < 0 \end{cases}$

Current conservation $\partial_\mu^y j^\mu(y) = 0 \Rightarrow$

$$-\partial_y^2 \int d\sigma \, \rho(\sigma) \, i\Delta(y-x,\sigma) = 0$$

KG: $(\partial^2 + \sigma)\Delta = 0$

$$-\int d\sigma (-\sigma) \, \rho(\sigma) \, i\Delta(y-x,\sigma) = 0$$

$$\therefore \int d\sigma \, \sigma \rho(\sigma) \, i\Delta(y-x,\sigma) = 0$$

This is true $\forall \, y-x$, especially timelike $y-x \Rightarrow \sigma\rho(\sigma) = 0$

Two cases:

1. $\rho(\sigma) = 0 \Rightarrow t_{nm}\phi_m(x) = 0$, t is not a broken symmetry generator
2. $\rho(\sigma) \propto \delta(\sigma)$. This is the case of interest

Let $\rho(\sigma) = N\delta(\sigma)$ — N dimensionful const

$$\langle 0| [j^\mu(y), \phi_n(x)] |0\rangle = -\partial_y^\mu \int d\sigma \, N\delta(\sigma) \, i\Delta(y-x,\sigma) = -iN\partial_y^\mu \Delta(y-x,0)$$

First, prove a useful identity $\int d^3y \, \Delta(y-x,0) = -(y_0 - x_0)$

Since $\int d^3x \exp(-ik\cdot x) = (2\pi)^3 \delta^3(k)$

$$\int d^3x \, i\Delta(x,0) = \lim_{\sigma\to 0} \int dk^0 \, \delta((k^0)^2 - \sigma) \, \varepsilon(k^0) \, e^{-ik^0 x_0}$$

$$= \lim_{\sigma\to 0} \int dk^0 \left[\frac{\delta(k^0 - \sqrt{\sigma})}{|2\sqrt{\sigma}|} - \frac{\delta(k^0+\sqrt{\sigma})}{|-2\sqrt{\sigma}|} \right] \varepsilon(k^0) \, e^{-ik^0 x_0}$$

$$= \lim_{\sigma\to 0} \frac{1}{2\sqrt{\sigma}} \left(e^{-i\sqrt{\sigma}x_0} - e^{i\sqrt{\sigma}x_0} \right) = -ix_0$$

Therefore

$$i\langle 0|[Q,\phi_n(x)]|0\rangle = N^a\int d^3y\ \partial^0\Delta(y-x,\sigma) = -N$$

$$\Rightarrow \langle t_{nm}\phi_m(x)\rangle_0 = i\langle 0|[Q,\phi_n(x)]|0\rangle = -N$$

$$\Rightarrow \rho(\sigma) = -t_{nm}\langle 0|\phi_m(x)|0\rangle\ \delta(\sigma)$$

Going back to the expression for $\rho^\mu(p)$

$$\rho^\mu(p) = ip^\mu\,\theta(p^0)N\delta(p^2) = \sum_n \delta(p-p_n)\langle 0|j^\mu(0)|n\rangle\langle n|\phi_n(0)|0\rangle$$

- Massless due to $\delta(p^2)$
- A delta-f^n can only arise from single particle; multiparticle states would contribute to a continuum extending down to $\sigma=0$

Now put back to label for symmetry generator. Each broken symmetry generator corresponds to one Goldstone boson.

Dimensional analysis & L.I $\rightarrow$ can parameterise the matrix elements as

$$\langle 0|j^{a\mu}(0)|B_b(p)\rangle = i\underbrace{F^a{}_b}_{[M]^{-1}} p^\mu$$

$$\langle B_b(p)|\phi_n(0)|0\rangle = \underbrace{Z_{bn}}_{\text{dimensionless}}$$

$|B_b(p)\rangle$ are spin zero states as $\phi(0)|0\rangle$ is rotationally inv. and massless.

Full expression:

$$ip^\mu\theta(p^0)N^a\delta(p^2) = \sum_b\int\frac{d^3k}{2|\vec{k}|}\delta^4(k-p)\langle 0|j^{a\mu}(0)|B_b(k)\rangle\langle B_b(k)|\phi_n(0)|0\rangle$$

as integral

$$\int\frac{d^3k}{2|\vec{k}|}\delta^4(k-p)\,ip^\mu N^a = \int\frac{d^3k}{2|\vec{k}|}\delta^4(k-p)\,ik^\mu\sum_b F^a{}_b Z_{bn} \Rightarrow N^a = \sum_b F^a{}_b Z_{bn}$$

$$\Rightarrow \sum_b F^a{}_b Z_{bn} = -t^a{}_{nm}\langle 0|\phi_m(x)|0\rangle$$

As there are dim H generators of H which are unbroken, there are $n = \dim G - \dim H$ broken generators, and the same # of $\rho^a(\sigma)$ which has non-zero contribution at $\sigma=0$ (case 2)

Each broken generator $\leftrightarrow$ one Goldstone boson $\Rightarrow$ n Goldstone bosons

N.B. We have assumed L.I. theory with dim > 2. Counting more subtle in non-relativistic theories. Also the proof requires the space of states to have +ve norm (gauge theories exempt)