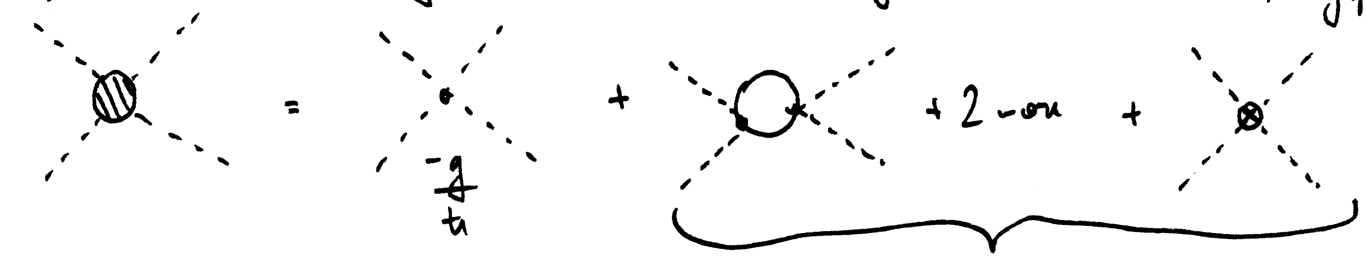
$\lambda\phi^4$: to subleading order, with a dim reg scale μ, we have $\lambda = g(\mu)\,\mu^{4-d}$



$$+ \quad \frac{3g^2}{32\pi^2}\log\left(\frac{\mu^2}{m^2}\right) \quad + \mathcal{O}(\hbar)$$

Nothing physical (such as this 4-pt f^n) can depend on our arbitrary scale μ. Consequently, the coupling $g(\mu)$ must run so that

$$\mu\frac{\partial}{\partial\mu}\left(-\frac{g}{\hbar} + \frac{3g^2}{32\pi^2}\log\frac{\mu^2}{m^2} + \mathcal{O}(\hbar)\right) = 0$$

$$\Rightarrow \beta(g) = \frac{3g^2\hbar}{32\pi^2} > 0 \qquad \text{for } g^2 > 0$$

Solving for $g(\mu)$ we find

$$\frac{1}{g(\mu)} = \frac{1}{g(\mu')} + \frac{3}{16\pi^2}\log\frac{\mu'}{\mu}$$ which relates the couplings at different scales.

In particular, for any $g(\mu) > 0$, there is a scale

$$\mu' = \mu\, e^{16\pi^2/3g(\mu)} \gg \mu \text{ at which } g(\mu') = \infty$$

so pert^n theory breaks down in the UV.

Renormalization of QED

In 4 dimensions, the classical action for QED in Euclidean signature is

$$S[A,\Psi] = \int d^4x\, \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \bar\Psi \not{D}\Psi + m\bar\Psi\Psi$$ where $\not{D}\Psi = \gamma^\mu(\partial_\mu + iA_\mu)\Psi$ and $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$.

To do pert^n theory, we'd like the photon kinetic term to be canonically normalized, so we introduce

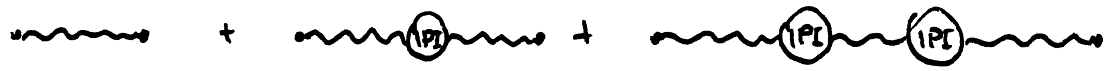
$$A^{\text{new}}_\mu = \frac{1}{e}A^{\text{old}}_\mu \quad \text{whereupon} \quad S[A,\Psi] = \int d^4x\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi(\not\partial + ie\not{A})\Psi + m\bar\Psi\Psi\right]$$

The original photon field necessarily has $[A^{\text{old}}] = 1$, so in d-dim, $[e] = \frac{4-d}{2}$.

Therefore, $[A^{\text{new}}] = [A^{\text{old}}] - [e] = \frac{d-2}{2}$. We can also introduce a dimensionless coupling by $e^2 = \mu^{4-d}g^2(\mu)$ in terms of some arbitrary scale μ.

Let's consider the exact photon propagator in momentum space, i.e.

$\Delta_{\mu\nu}(q) := \int d^dx\; e^{iq\cdot x} \langle A_\mu(x) A_\nu(0)\rangle$ in Lorentz gauge $\partial^\mu A_\mu = 0$



$$= \Delta^0_{\mu\nu}(q) + \Delta^0_\mu{}^\rho(q)\, \Pi_\rho{}^\sigma(q)\, \Delta^0_{\sigma\nu}(q) + \Delta^0_\mu{}^\rho\, \Pi_\rho{}^\sigma\, \Delta^0_\sigma{}^\lambda\, \Pi_\lambda{}^\kappa\, \Delta^0_{\kappa\nu} + \dots$$

$\frac{1}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$ (the Classical Lorentz gauge propagator)

where $\Pi_{\rho\sigma}(q) =$  is the photon self-energy.

We'll find $\Pi_\rho{}^\sigma(q) = q^2\left(\delta_\rho^\sigma - \frac{q_\rho q^\sigma}{q^2}\right)\pi(q^2)$ for some scalar function $\pi(q^2)$

The operator $P_\rho{}^\sigma := \left(\delta_\rho^\sigma - \frac{q_\rho q^\sigma}{q^2}\right)$ is a projection operator onto transverse polarizations

Since $P_\rho{}^\sigma P_\sigma{}^\lambda = P_\rho{}^\lambda$, we have

$$\Delta_{\mu\nu}(q) = \Delta^0_{\mu\nu}\left(1 + \pi(q^2) + \pi^2(q^2) + \dots\right) = \frac{\Delta^0_{\mu\nu}}{1-\pi(q^2)}$$

Just as the classical propagator came from the kinetic term

$$S_{kin} = \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^dx = \frac{1}{2}\int q^2\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\tilde{A}^\mu(-q)\,\tilde{A}^\nu(q)\, d^dq$$

so too our exact propagator is what we'd get from an action whose quadratic term is

$$S_{quad} = \frac{1}{2}\int (1-\pi(q^2))\, q^2\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\tilde{A}^\mu(-q)\,\tilde{A}^\nu(q)\, d^dq$$

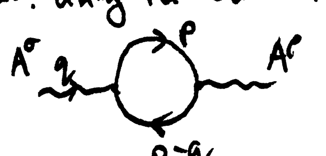
Expanding $\pi(q^2)$ around $q^2 = 0$, the leading term $\pi(0)$ just corrects the kinetic term of the photon.

$$S_{quad} \sim \frac{1}{4}(1-\pi(0))\int F_{\mu\nu}F^{\mu\nu}\, d^dx \;+\; \text{higher derivative terms}$$

irrelevant in $d=4$

<u>Computing the self-energy</u>

To leading order, using the classical action, we have

$\Pi^{\rho\sigma}_{1\text{-loop}}(q^2) =$  $= g^2\mu^{4-d}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{tr}\left[\frac{(-i\not{p}+m)\gamma^\rho}{\underbrace{p^2+m^2}_{\equiv A}}\;\frac{(-i(\not{p}-\not{q})+m)\gamma^\sigma}{\underbrace{(p-q)^2+m^2}_{\equiv B}}\right]$

To compute, first note $\frac{1}{AB} = \frac{1}{B-A}\left[\frac{1}{A} - \frac{1}{B}\right] = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$

$$\Rightarrow \int_0^1 \frac{dx}{[(p^2+m^2)(1-x)+((p-q)^2+m^2)x]^2} = \int_0^1 \frac{dx}{[p^2+m^2-2x\,p\cdot q+q^2x]^2}$$

$$= \int_0^1 \frac{dx}{[(p-xq)^2+m^2+q^2x(1-x)]^2}$$

Letting $p' := p-qx$, our loop integral becomes (dropping the prime)

$$\frac{g^2\mu^{4-d}}{(2\pi)^d}\int d^dp \int_0^1 dx \; \frac{\mathrm{tr}\left[(-i(\not p+\not q x)+m)\gamma^\rho(-i(\not p-\not q(1-x))+m)\gamma^\sigma\right]}{[p^2+\Delta]^2}$$

where $\Delta := m^2+q^2x(1-x)$. We next do the trace over Dirac spinor indices, treating these as in $d=4$.

$$\mathrm{tr}(\gamma^\rho\gamma^\sigma) = 4\delta^{\rho\sigma}, \quad \mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) = 4(\delta^{\mu\rho}\delta^{\nu\sigma}-\delta^{\mu\nu}\delta^{\rho\sigma}+\delta^{\mu\sigma}\delta^{\rho\nu})$$

$$\Rightarrow \mathrm{tr}(\ldots) = 4\Big[-(p+qx)^\rho(p-q(1-x))^\sigma + (p+qx)\cdot(p-q(1-x))\delta^{\rho\sigma}$$
$$- (p+qx)^\sigma(p-q(1-x))^\rho + m^2\delta^{\rho\sigma}\Big]$$

Whenever $d\in\mathbb{N}$, we'd certainly get zero if any component of p^σ appears an odd # of times. Consequently, in the numerator, can replace

$$p^\rho p^\sigma \to \frac{p^2\delta^{\rho\sigma}}{d}, \quad p^\mu p^\rho p^\nu p^\sigma \to \frac{(p^2)^2}{d(d+2)}[\delta^{\mu\nu}\delta^{\rho\sigma}+\delta^{\mu\rho}\delta^{\nu\sigma}+\delta^{\mu\sigma}\delta^{\rho\nu}]$$

The integrals are given in terms of Γ-functions. We obtain

$$\Pi^{\rho\sigma}_{1\text{-loop}}(q) = \frac{-4g^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\,\frac{\delta^{\rho\sigma}(-m^2+x(1-x)q^2)+\delta^{\rho\sigma}(m^2+x(1-x)q^2)-2x(1-x)q^\rho q^\sigma}{\Delta^{2-d/2}}$$

$$= q^2\left(\delta^{\rho\sigma}-\frac{q^\rho q^\sigma}{q^2}\right)\pi_{1\text{-loop}}(q^2)$$

where $\pi_{1\text{-loop}}(q^2) := \dfrac{-8g^2(\mu)\,\Gamma(2-d/2)}{(4\pi)^{d/2}}\displaystyle\int_0^1 dx\; x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}$.