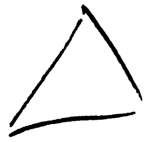
Interlude: what is $\mathbb{R}^n$?

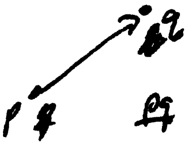
- metric topological space



$d(p,q) < r$ open ball
open sets $\equiv$ unions of open balls } need for manifolds

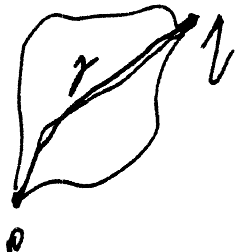
- vector space (linear space), $\underline{x}, \underline{y} \in \mathbb{R}^n \rightarrow \alpha \underline{x} + \beta \underline{y} \in \mathbb{R}^n$ } tangent space T_pM, $p \in M$

- affine space : set + translations



- inner product space, $\underline{x}, \underline{y} \rightarrow \underline{x} \cdot \underline{y} \in \mathbb{R}$, $\|\underline{x}\|^2 = \underline{x} \cdot \underline{x}$ } Riemann metrics / Isometric embeddings

2.2 Extremizing proper time



$p, q \in M$ connected by a time-like curve γ
Small deformations of γ also timelike, ∞ many timelike curves
Which curve is extremizing

$$\tau(\gamma) = \int_0^1 \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, du = \int_0^1 G(x, \dot{x}) \, du$$

$\gamma(0) = p$
$\gamma(1) = q$ $\quad \gamma : [0,1] \rightarrow M$, E-L eq (4 ODEs)

$$\frac{d}{du}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) = \frac{\partial G}{\partial x^\mu} \Rightarrow \frac{d}{du}\left(-\frac{1}{G} g_{\mu\nu} \dot{x}^\nu\right) - \left(-\frac{1}{2G} g_{\nu\rho,\mu} \dot{x}^\nu \dot{x}^\rho\right) = 0$$

Use proper time as a parameter along γ. where $g_{\nu\rho,\mu} \equiv \frac{\partial g_{\nu\rho}}{\partial x^\mu}$.

$$\frac{d\tau}{du} = G \quad , \quad \frac{d}{du} \rightarrow G \frac{d}{d\tau}$$

Exercise
$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0} \quad (*)$$
geodesic equations

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)$$
Christoffel symbols
[Not components of a tensor]

- $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$
- Minkowski space, inertial frame $g = -d(x^0)^2 + d\underline{x}^2 \to \Gamma^\mu_{\nu\rho} = 0$

  Change coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu) \to \tilde{\Gamma}_{\nu\rho}{}^\mu \neq 0$.

<u>Postulate of GR</u> Massive test bodies in GR follow curves of extremal proper time

<u>Exercise</u> The same geodesic equations arise from $L = G^2 = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$.

(Easier to obtain E-L eq, and to compute $\Gamma_{\nu\rho}{}^\mu$)

Note $\frac{\partial L}{\partial \tau} = 0 \to$ conservation law, $H = -\left(\dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} - L\right) = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

(constant, RHS $= -1$)

<u>Example</u> The Schwarzschild metric

$f = 1 - \frac{2m}{r}$, $m = \text{const}$, $x^\mu = (t, r, \theta, \phi)$

$$g = -f\,dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$t: \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{d}{d\tau}\left(2f\frac{dt}{d\tau}\right) = 0$$

$\ddot{t} + f^{-1}f'\dot{t}\dot{r} = 0$ compare with (*) $\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}f'/f$, otherwise $\Gamma^0_{\mu\nu} = 0$

2.3 Covariant derivative and Levi-Civita connection

How to differentiate tensors? E.g. V^a = vector field, $\frac{\partial V^\mu}{\partial x^\nu}$ <u>not</u> components of a tensor

<u>Def^n</u> A connection (covariant derivative) is a map $\nabla: (X,Y) \in T_pM \times T_pM \to$ $\to \nabla_X Y \in T_pM$ such that

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ (f, g any functions)

(ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

(iii) (Leibniz) $\nabla_X (fY) = X(f)\, Y + f\nabla_X Y$, $\nabla_X f \equiv X(f)$

$\forall X, Y, Z$ vector fields

<u>Def^n</u> A covariant derivative of a vector field Y is a $(1,1)$ tensor ∇Y such that

$$(\nabla Y)^a{}_b \equiv \nabla_b Y^a \equiv Y^a{}_{;b}$$