The interaction picture

Let's return to simple QM. In the Schrödinger picture, the states evolve with t

$i\frac{\partial|\psi\rangle_S}{\partial t} = H|\psi\rangle_S$ while ops O_S are t independent. In the Heisenberg picture, the states are fixed but ops evolve in t, $O_H(t) = e^{iHt}O_S e^{-iHt}$, $|\psi\rangle_H = e^{iHt}|\psi\rangle_S$.

The interaction picture is a hybrid of the two. We split the Hamiltonian up $H = H_0 + H_{int}$. The t-dependence of ops is governed by H_0, whereas the t-dependence of states is controlled by H_{int}. Although the split into H_0, H_{int} is a priori arbitrary, it's useful when H_0 is simple (e.g. a free field theory).

$$|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S, \quad O_I(t) = e^{iH_0 t}O_S e^{-iH_0 t}$$

$$H_I \equiv (H_{int})_I = e^{iH_0 t}H_{int}e^{-iH_0 t}$$ interaction Hamiltonian, in the interaction picture

The S.E. is $i\frac{d|\psi\rangle_S}{dt} = H_S|\psi\rangle_S$

$$i\frac{d}{dt}\left(e^{-iH_0 t}|\psi\rangle_I\right) = (H_0 + H_{int})_S\, e^{-iH_0 t}|\psi\rangle_I$$

$$H_0 e^{-iH_0 t}|\psi\rangle_I + i e^{-iH_0 t}\frac{d|\psi\rangle_I}{dt} = H_{0_S}e^{-iH_0 t}|\psi\rangle_I + H_{int_S}e^{-iH_0 t}|\psi\rangle_I$$

$$\Rightarrow i\frac{d|\psi\rangle_I}{dt} = \underbrace{e^{iH_0 t}H_{int_S}e^{-iH_0 t}}_{\to H_I(t)}|\psi\rangle_I \qquad -(*)$$

Dyson's Formula

Try a solution to $(*)$

$$|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle_I$$

unitary time evolution operator

s.t.

$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$ and $U(t,t) = 1$ and $i\frac{dU}{dt} = H_I(t)U$

If H_I were an ordinary function, we could solve this by $U(t,t_0) = \exp\left[-i\int_{t_0}^{t} H_I(t')\,dt'\right]$ but the trouble is that H_I is an op and we have ordering ambiguities since $[H_I(t'), H_I(t'')] \neq 0$ for $t' \neq t''$.

Claim The solⁿ is given by Dyson's formula

$$U(t,t_0) = T\exp\left[-i\int_{t_0}^{t} H_I(t')\,dt'\right]$$ where T stands for time ordering:

ops evaluated at earlier times appear on the RHS

$$T\{O_1(t_1)\,O_2(t_2)\} = \begin{cases} O_1(t_1)\,O_2(t_2) & : t_1 > t_2 \\ O_2(t_2)\,O_1(t_1) & : t_2 > t_1 \end{cases}$$

The exponential is understood in terms of a power series expansion.

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')\,dt' + \frac{(-i)^2}{2}\left\{\int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, H_I(t'')\,H_I(t') + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')\,H_I(t'')\right\} + \dots$$

the last two terms double up since

$$\underbrace{\int_{t_0}^{t} dt' \int_{t'}^{t} dt''}_{t''>t'}\, H_I(t'')\,H_I(t') = \underbrace{\int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt'}_{t'<t''}\, H_I(t'')\,H_I(t')$$

$$= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')\,H_I(t'')$$ by relabelling

$$\Rightarrow U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, H_I(t') + (-i)^2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')\,H_I(t'') + \dots$$

Under T, all ops commute (since their order is already fixed) so

$$i\frac{\partial}{\partial t}\left\{T\exp\left[-i\int_{t_0}^{t} dt'\, H_I(t')\right]\right\} = T\left[H_I(t)\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)\right]$$

$$= H_I(t)\,T\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)$$

since t on the upper limit of $\int$ is the latest time, so we pull $H_I(t)$ to the LHS.

<u>A first look at scattering</u>

We now apply the interaction picture to QFT, starting with $H_{int} = g\psi^*\psi\phi$.

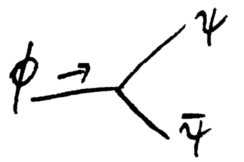
Expand $\phi \sim a + a^\dagger$ ~~…~~ create/destroy a ϕ: meson

$\psi \sim b + c^\dagger$ destroys a ψ or creates an anti partᶜ ~~$\psi^* \sim b^\dagger$ & c~~ $\psi \sim$ nucleon

$\psi^* \sim b^\dagger + c$ destroys an anti-nucleon or creates a nucleon

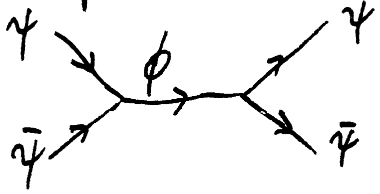
Part^c # not conserved but $Q = N_c - N_b$ <u>is</u>.

At 1st order in perturbation theory (PT), we have terms $b^\dagger c^\dagger a$. This destroys a meson but produces a $\psi\bar{\psi}$ pair. It contributes to meson decay: $\phi \to \psi + \bar{\psi}$.



At 2nd order in PT, $\exists$ more complicated terms $\sim (c^\dagger b^\dagger a)(c b a^\dagger)$ giving rise to

$\psi + \bar{\psi} \to \phi \to \psi + \bar{\psi}$



To calculate amplitudes, we make an important assumption:

<u>initial and final states behave like free part^{le}</u>

This means that the initial state $|i\rangle$ at $t \to -\infty$ and the final state $|f\rangle$ at $t \to \infty$ are e'states of the free Hamiltonian H_0. It's plausible that at $t = \pm\infty$, states are well separated and don't feel the effect of each other and are e'states of part^c # N, which commutes with H_0 (but not H_{int}).

As the part^{les} approach each other, they interact departing again, each going its own way.

The amplitude to go from $|i\rangle$ to $|f\rangle$ is

$$\lim_{t_\pm \to \pm\infty} \langle f | O(t_+, t_-) | i \rangle \equiv \langle f | S | i \rangle$$

unitary operator known as the S (scattering)-matrix

(Bound states show up as poles in the S-matrix).