- Can prove that roots as eigenvalues are non-degenerate. (p87 of F&S)
(will assume this)

$\Rightarrow$ set of roots Φ consists of $(d-r)$ distinct elements of $\mathfrak{h}^*$

d = dim($\mathfrak{g}$), r = dim($\mathfrak{h}$) (number of zero eigenvalues)

- Define Cartan-Weyl basis for $\mathfrak{g}$ $\quad B = \{H^i, i=1,\dots r\} \cup \{E^\alpha, \alpha \in \Phi\}$

$[H^i, H^j] = 0$

$[H^i, E^\alpha] = \alpha^i E^\alpha$

α^i = one component of the root

$$[H, E^\alpha] = \alpha(H) E^\alpha \qquad \alpha(H) = \rho_i \alpha^i \qquad (H = \rho_i H^i) \qquad ③$$

$\alpha \in \mathfrak{h}^*$

- Define

$$K[X, Y] = \frac{1}{N} \mathrm{Tr}[ad_X \circ ad_Y] \qquad X, Y \in \mathfrak{g} \qquad ④$$

$\mathfrak{g}$ simple $\longrightarrow$ K non-degenerate
(where simplicity comes in)

<u>Evaluate K in Cartan-Weyl basis</u>

i) $\forall H \in \mathfrak{h}$, $\alpha \in \Phi$

$K(H, E^\alpha) = 0$

ii) $\forall \alpha, \beta \in \Phi$, $\alpha + \beta \neq 0$

$K(E^\alpha, E^\beta) = 0$

<u>Proof</u>

i) $\forall H' \in \mathfrak{h}$

$\alpha(H') K(H, E^\alpha)$

$\overset{③}{=\!=} K(H, [H', E^\alpha])$

$\overset{⑤}{=\!=} -K([H', H], E^\alpha) = -K(0, E^\alpha) = 0$

$\alpha \neq 0 \quad \alpha(H') \neq 0$ identically $\quad \Rightarrow K(H, E^\alpha) = 0$

ii) $\forall\, H' \in \mathfrak{h}$

$$(\alpha(H') + \beta(H'))\, K[E^\alpha, E^\beta]$$

$$\overset{③}{=} K([H', E^\alpha], E^\beta) + K(E^\alpha, [H', E^\beta])$$

$$\overset{③}{=} 0$$

hence $\forall\, \alpha, \beta \in \Phi \quad \alpha + \beta \neq 0$ ← at least one component is not zero

iii) $\forall\, H \in \mathfrak{h}, \ \exists\, H' \in \mathfrak{h}$ s.t. $K(H, H') \neq 0$

non-degeneracy within subalgebra (more strict condition than non-degeneracy on the full algebra)

<u>Proof</u>

For one $H \in \mathfrak{h}$ suppose that

$$K(H, H') = 0 \quad \forall\, H' \in \mathfrak{h}$$

from (i)

$$K(H, E^\alpha) = 0 \qquad \forall\, \alpha \in \Phi$$

$$\Rightarrow K(H, X) = 0 \quad \forall\, X \in \underset{\sim}{g}$$ ※ contradiction

<u>Consequence</u>

K started life as a non-degenerate inner product in $\underset{\sim}{g}$

iii) $\Rightarrow$ K is non-degenerate inner product on $\mathfrak{h}$

In components

$$H = e_i H^i, \quad H' = e_i' H^i \in \mathfrak{h}$$

$$\Rightarrow K(H, H') = K^{ij} e_i e_j'$$ ← $r \times r$ matrix (2-index tensor)

$$K^{ij} = K(H^i, H^j)$$

$K^{ij} = K(H^i, H^j)$ is <u>invertible</u> ← consequence of non-degeneracy

$\exists\, (K^{-1})_{ij}$ s.t. $(K^{-1})_{ij} K^{jk} = \delta_i{}^k$

K^{-1} non-degenerate $\Rightarrow$ n.d. inner product on $\mathfrak{h}^*$

$$[H^i, E^\alpha] = \alpha^i E^\alpha$$

$$[H^i, E^\beta] = \beta^i E^\alpha$$

$\alpha, \beta \in \mathfrak{h}^*$

$\alpha, \beta \in \Phi$ ← more strict

can define

$$\boxed{(\alpha, \beta) = (K^{-1})_{ij}\, \alpha^i \beta^j} \quad -(6)$$

K on $\mathfrak{h}$ defines an isomorphism $A: \mathfrak{h} \to \mathfrak{h}^*$

$$A: H \in \mathfrak{h} \longrightarrow A(H) \in \mathfrak{h}^* \qquad A(H) = K(H, \ )$$

---

iv) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

$$K(E^\alpha, E^{-\alpha}) \neq 0$$

Proof:

From i) $K(E^\alpha, H) = 0 \quad \forall\, H \in \mathfrak{h}$

ii) $K(E^\alpha, E^\beta) = 0 \quad \forall\, \beta \in \Phi \quad \alpha \neq -\beta$

Unless we have $-\alpha$ as a root and $K(E^\alpha, E^{-\alpha}) \neq 0$, we would conclude $K(E^\alpha, X) = 0 \ \forall\, X \in \mathfrak{g} \Rightarrow K$ degenerate ※

Therefore $-\alpha \in \Phi$, $K(E^\alpha, E^{-\alpha}) \neq 0$

Algebra in Cartan-Weyl basis

So far, $[H^i, H^j] = 0$, $\forall\, i,j = 1, \dots r$

$[H^i, E^\alpha] = \alpha^i E^\alpha \quad \forall\, \alpha \in \Phi$ (3)'

Remains to evaluate $[E^\alpha, E^\beta]$

$$[H^i, [E^\alpha, E^\beta]] \overset{(\alpha)}{=} -[E^\alpha, [E^\beta, H^i]] - [E^\beta, [H^i, E^\alpha]]$$
$$\overset{(3)'}{=} (\alpha^i + \beta^i)[E^\alpha, E^\beta] \quad \text{(swapping around commutators)}$$

$\alpha + \beta \neq 0$ we have $[E^\alpha, E^\beta] \begin{cases} = N_{\alpha,\beta} E^{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ = 0 & \text{if } \alpha+\beta \notin \Phi \end{cases}$

$\alpha + \beta = 0$, $\beta = -\alpha$

$$K([E^\alpha, E^{-\alpha}], H) \overset{(\gamma)}{=} K(E^\alpha, [E^{-\alpha}, H])$$
$$\overset{(3)}{=} \alpha(H)\, K(E^\alpha, E^{-\alpha}) \qquad ⊗$$

iv) $\Rightarrow K(E^\alpha, E^{-\alpha}) \neq 0$

def $H^\alpha = \dfrac{[E^\alpha, E^{-\alpha}]}{K(E^\alpha, E^{-\alpha})}$

$\circledast \Rightarrow K(H^\alpha, H) = \alpha(H) \quad \forall H \in \mathfrak{h}$

$H^\alpha = \rho^\alpha{}_i H^i \quad , \quad H = \rho_i H^i \quad \in \mathfrak{h}$

$K^{ij} \rho^\alpha{}_i \rho_j = \alpha^j \rho_j$ true for all ρ_j

$\Rightarrow K^{ij} \rho^\alpha{}_i = \alpha^j$

$\rho^\alpha{}_i = (K^{-1})_{ij} \alpha^j$

$H^\alpha = \rho^\alpha{}_i H^i = (K^{-1})_{ij} \alpha^j H^i$

$[H^i, H^j] = 0 \quad i,j = 1, \dots r$

$[H^i, E^\alpha] = \alpha^i E^\alpha \quad \forall \alpha, \Phi$

$[E^\alpha, E^\beta] = N_{\alpha\beta} E^{\alpha+\beta} \quad \alpha+\beta \in \Phi \quad , \quad \alpha+\beta \neq 0$

$[E^\alpha, E^{-\alpha}] = (K^{-1})_{ij} \alpha^j H^i$

$[E^\alpha, E^\beta] = 0 \quad \alpha+\beta \notin \Phi$