3.3 Charge conj

$$\psi^c(x) \equiv \hat{C}\psi(x)\hat{C}^{-1} = \eta_c C\bar{\psi}^T(x)$$
$$\bar{\psi}^c(x) \equiv \hat{C}\bar{\psi}(x)\hat{C}^{-1} = -\eta_c^* \psi^T(x) C^{-1}$$

Fermion bilinears

e.g. $j^\mu(x) = \bar{\psi}\gamma^\mu\psi$

$$\hat{C} j^\mu(x)\hat{C}^{-1} = \hat{C}\bar{\psi}\hat{C}^{-1}\gamma^\mu\hat{C}\psi\hat{C}^{-1} = -\eta_c^*\psi^T C^{-1}\gamma^\mu C\bar{\psi}^T\eta_c$$
$$= \bar{\psi}(C^{-1}\gamma^\mu C)\psi \quad \text{[fermions anticommute]}$$
$$= \bar{\psi} C^T \gamma^{\mu T} C^{-1T}\psi$$
$$= -\bar{\psi}\gamma^\mu\psi = -j^\mu(x)$$

$\therefore$ $A_\mu(x) j^\mu(x)$ is invariant under $\hat{C}$

Similarly, $\bar{\psi}\gamma^\mu\gamma^5\psi \rightarrow +\bar{\psi}\gamma^\mu\gamma^5\psi$ under $\hat{C}$

3.4 Time reversal

Under $\hat{T}$: $x^\mu = (x^0, x^i) \rightarrow x_T^\mu = (-x^0, x^i)$

$p^\mu = (p^0, p^i) \rightarrow p_T^\mu = (p^0, -p^i)$

Theories that are invariant under $\hat{T}$: physics unchanged if time runs backwards.

Boson field

$\hat{T} a(p)\hat{T}^{-1} = \eta_T a(p_T)$, $\hat{T} c^\dagger(p)\hat{T}^{-1} = \eta_T c^\dagger(p_T)$ $\qquad (p_T\cdot x = -p\cdot x_T)$

[relative phases are fixed using same argument as for $\hat{P}$, $\hat{C}$]

From the decomposition, (remember $\hat{T}$ is anti-linear)

$$\hat{T}\phi(x)\hat{T}^{-1} = \sum_p\left[\hat{T}a(p)\hat{T}^{-1}e^{+ip\cdot x} + \hat{T}c^\dagger(p)\hat{T}^{-1}e^{-ip\cdot x}\right]$$
$$= \cdots = \eta_T\sum_p\left[a(p)e^{-ip\cdot x_T} + c^\dagger(p)e^{+ip\cdot x_T}\right]$$
$$= \eta_T\phi(x_T)$$

Dirac field

$\hat{T}$ flips the sign of angular momentum. The creation/annihilation operators can be taken to transform as

$\hat{T} b^s(p)\hat{T}^{-1} = \eta_T(-1)^{1/2-s} b^{-s}(p_T)$

$\hat{T} d^{s\dagger}(p)\hat{T}^{-1} = \eta_T(-1)^{1/2-s} d^{-s\dagger}(p_T)$

It can be shown $(-1)^{1/2-s} u^{-s*}(p_T) = -B u^s(p)$, $(-1)^{1/2-s} v^{-s*}(p_T) = -B v^s(p)$

where $B = -\gamma^5 C = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$ in our rep.

Then
$$\hat{T}\psi(x)\hat{T}^{-1} = \eta_T \sum_{p,s} (-1)^{1/2-s}\left[b^{-s}(p_T)\, u^{s*}(p)\, e^{+ip\cdot x} + d^{-s\dagger}(p_T)\, v^{s*}(p)\, e^{-ip\cdot x}\right]$$

[same steps as usual, $s \leftrightarrow -s$, $p_T\cdot x = -p\cdot x_T$]
$$= \dots = \eta_T \sum_{p,s} (-1)^{1/2-s+1}\left[b^{s}(p)\, u^{-s*}(p_T)\, e^{-ip\cdot x_T} + d^{s\dagger}(p)\, v^{-s*}(p_T)\, e^{+ip\cdot x_T}\right]$$

using results above.
$$= +\eta_T \sum_{p,s}\left[b^{s}(p)\, B u^{s}(p)\, e^{-ip\cdot x_T} + d^{s\dagger}(p)\, B v^{s}(p)\, e^{+ip\cdot x_T}\right]$$
$$= \eta_T B \psi(x_T)$$

Similarly,
$$\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \eta_T^* \bar{\psi}(x_T) B^{-1}$$

Bilinears

$$\bar{\psi}(x)\psi(x) \to \bar{\psi}(x_T)\psi(x_T)$$
$$\bar{\psi}(x)\gamma^\mu\psi(x) \to \bar{\psi}(x_T) B^{-1}\gamma^{\mu *} B\psi(x_T)$$
$$= -T^\mu{}_\nu\, \bar{\psi}(x_T)\gamma^\nu\psi(x_T)$$

$$\left[\begin{array}{l} B^{-1}\gamma^{\mu *}B = -T^\mu{}_\nu\gamma^\nu \\ T^\mu{}_\nu = \mathrm{diag}(-1,1,1,1) \end{array}\right]$$

S-matrix

$$\langle p_1, p_2, \dots | S | k_A, k_B, \dots\rangle = {}_{out}\langle p_1, p_2, \dots | k_A, k_B, \dots\rangle_{in}$$

S: scattering S-matrix (unitary op)

$$= \lim_{T\to\infty} \langle p_1, p_2, \dots | e^{-iH2T} | k_A, k_B, \dots\rangle$$

[see QFT course]

We can write
$$S = \mathcal{T}\exp\left(-i\int_{-\infty}^{\infty} V(t)\,dt\right) \quad \text{where} \quad V(t) = -\int d^3x\, \mathcal{L}_I(x)$$
($\mathcal{T}$: time ordering)

E.g. QED, $\mathcal{L}_I = -e\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x)$.

	P	C	T
$\mathcal{L}_I(x)$	$\mathcal{L}_I(x_P)$	$\mathcal{L}_I(x)$	$\mathcal{L}_I(x_T)$
$V(t)$	$V(t)$	$V(t)$	$V(-t)$
S	S	S	?

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\, V(t_1)V(t_2)\dots V(t_n)$$

$$S_T = \hat{T}S\hat{T}^{-1} = \sum_{n=0}^{\infty}(+i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\, V(-t_1)V(-t_2)\dots V(-t_n)$$

$$= \sum_{n=0}^{\infty}(+i)^n \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n\, V(\tau_n)V(\tau_{n-1})\dots V(\tau_1)$$

where $\tau_i = -t_{n+1-i}$.