3.3 - Charge conjugation

(particles $\leftrightarrow$ antiparticles)
($\hat{C}$ is a unitary operator)

Scalar field

Lorentz sym again constrains the phases

$$\hat{C}\, a(p)\, \hat{C}^{-1} = \eta_c\, c(p)$$
$$\hat{C}\, c(p)\, \hat{C}^{-1} = \underbrace{\eta_c^*\, a(p)}_{\text{from } [\phi(x),\ \hat{C}\phi(y)\hat{C}^{-1}] \text{ for spacelike } x-y}$$

$$\hat{C}\,|\text{particle}, p\rangle = \hat{C}\, a^\dagger(p)|0\rangle = \eta_c^*\, c^\dagger(p)\hat{C}|0\rangle = \eta_c^*\,|\text{antiparticle}, p\rangle$$

assume vacuum is invariant under $\hat{C}$

From the decomposition

$$\hat{C}\,\phi(x)\,\hat{C}^{-1} = \eta_c\,\phi^\dagger(x)$$
$$\hat{C}\,\phi^\dagger(x)\,\hat{C}^{-1} = \eta_c^*\,\phi(x)$$

- If $\phi(x)$ is real field, $\phi^\dagger = \phi$, then $\eta_c^* = \eta_c = \pm 1$. This is called the intrinsic C-parity of the field. Consequences:
  - Will see later that photon field transforms like $\hat{C} A_\mu(x) \hat{C}^{-1} = -A_\mu(x)$

    $\therefore \pi^0$ ($\eta_c^\pi = +$) ~~can decay to 2 photons (but not 1 or 3)~~

    Experimentally π^0 decays to 2 photons (not 1 or 3)
- For a complex scalar field, η_c is arbitrary. However we can do a global $U(1)$ rotation $\phi \to \phi' = e^{-i\beta}\phi$ s.t. $\eta_c' = 1$ (we can redefine ϕ s.t. $\eta_c = 1$)

Dirac field

(4x4 in spinor space)

First we define the matrix C s.t. $(C\gamma^\mu)^\dagger = C\gamma^\mu$. In our rep where $\gamma^{0T} = \gamma^0$, $(\gamma^2)^T = \gamma^2$, $(\gamma^1)^T = -\gamma^1$, $(\gamma^3)^T = -\gamma^3$, a suitable choice (not unique) is $C = -i\gamma^0\gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$

One can check: $C = -C^T = -C^\dagger = -C^{-1}$ and $(\gamma^\mu)^T = -C\gamma^\mu C^{-1}$

$$(\gamma^5)^T = +C\gamma^5 C^{-1}$$

Similarly to bosons, $\hat{C}\, b^s(p)\, \hat{C}^{-1} = \eta_c\, d^s(p)$ ← note the spin in unchanged

$$\hat{C}\, d^{s\dagger}(p)\, \hat{C}^{-1} = \eta_c\, b^{s\dagger}(p)$$

$$\underbrace{\hat{C}\, b^s(p)\, \hat{C}^{-1}}_{\text{in } \psi} \qquad \underbrace{d^{s\dagger}(p)}_{\text{in } \bar{\psi}}$$

Now compare

$$\hat{C}\,\psi(x)\,\hat{C}^{-1} = \eta_c \sum_{p,s} \left[d^s(p)\, u^s(p)\, e^{-ipx} + b^{s\dagger}(p)\, v^s(p)\, e^{ipx} \right]$$

with

$$\bar{\psi}^T = \sum_{p,s} \left[b^{s\dagger}(p) (\bar{u}^s(p))^T e^{ip\cdot x} + d^s(p)\, (\bar{v}^s(p))^T e^{-ip\cdot x} \right]$$

Considering those spinors, if we take $\eta^s = i\sigma^2 \xi^{s*}$ as basis for spinors $\xi^s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

We can write $v^s(p) = C(\bar{u}^s(p))^T$ $\qquad i\sigma^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

and $u^s(p) = C((\bar{v}^s(p))^T$

$$\psi^c(x) \equiv \hat{C}\,\psi(x)\,\hat{C}^{-1} = \boxed{\eta_c\, C\, \bar{\psi}^T(x)}$$ ← matrix

Similarly $\boxed{\bar{\psi}^c(x) = \hat{C}\,\bar{\psi}(x)\,\hat{C}^{-1} = -\eta_c^*\, \psi^T(x)\, C^{-1}}$

Note :

1) If $\psi(x)$ satisfies the Dirac equation, so does $\psi^c(x)$
2) Majorana fermions have $b^s(p) = d^s(p) \Rightarrow$ particle is its own antiparticle, $\psi^c(x) = \psi(x)$
3) It is not known whether the only neutral fermions in the SM, the neutrinos are Majorana fermions. (Neutrinoless double beta decay)

Fermion bilinears under $\hat{C}$

E.g. $j^\mu = \bar{\psi}(x)\, \gamma^\mu\, \psi(x)$

$$\hat{C}\, j^\mu\, \hat{C}^{-1} = \hat{C}\,\bar{\psi}\, \hat{C}^{-1} \gamma^\mu\, \hat{C}\,\psi\,\hat{C}^{-1}$$
$$= -\eta_c^*\, \psi^T C^{-1} \gamma^\mu\, \eta_c\, C\, \bar{\psi}^T$$