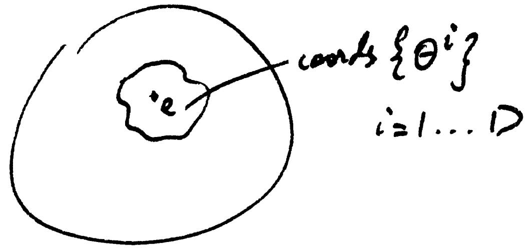
$G \subset Mat_n(F)$ $\quad F = \mathbb{R}$ or $\mathbb{C}$



$g = g(\theta) \in G$

$T_e(G) = \text{span}_{\mathbb{R}} \left\{ \frac{\partial g(\theta)}{\partial \theta_i} \right\}$

$\mathcal{L}(G) \overset{\text{def}^n}{=} (T_e(G), [\,,\,])$

C smooth curve on G $\qquad C: t \mapsto g(t) \in G \ , \ g(0) = \mathbb{1}_n$

$$\dot{g}(0) = \frac{d g(t)}{dt}\Big|_{t=0} = \dot{\theta}^i(\theta) \frac{\partial g(\theta)}{\partial \theta^i}\Big|_{\theta=0} \in T_e(G)$$

= tangent vector to C at e.

In general $\dot{g}(0)$ not $\in G$

- Near $t = 0$ we have,

$$g(t) = \mathbb{1}_n + Xt + O(t^2) \qquad \text{with } X = \dot{g}(0) \in \mathcal{L}(G)$$

- Given two elements $X_1, X_2 \in \mathcal{L}(G)$ we can find smooth curves

$$C_1 : t \mapsto g_1(t) \in G \qquad g_1(0) = g_2(0) = \mathbb{1}_n$$
$$C_2 : t \mapsto g_2(t) \in G \qquad \dot{g}_1(0) = X_1 \ , \ \dot{g}_2(0) = X_2$$

Near $t = 0$,

$$g_1(t) = \mathbb{1}_n + X_1 t + W_1 t^2 + O(t^3)$$
$$g_2(t) = \mathbb{1}_n + X_2 t + W_2 t^2 + O(t^3) \qquad \text{some } W_1, W_2 \in Mat_n(F)$$

Define,

$$h(t) = g_1^{-1}(t)\, g_2^{-1}(t)\, g_1(t)\, g_2(t) \in G$$

$$\Rightarrow g_1(t) g_2(t) = g_2(t) g_1(t) h(t) \qquad (*)$$

near $t = 0$

$$g_1(t) g_2(t) = \mathbb{1}_n + t(X_1 + X_2) + t^2(X_1 X_2 + W_1 + W_2) + O(t^3)$$
$$g_2(t) g_1(t) = \mathbb{1}_n + t(X_1 + X_2) + t^2(X_2 X_1 + W_1 + W_2) + O(t^3)$$

set $h(t) = \mathbb{1}_n + h_1 t + h_2 t^2 + O(t^3)$

$$(*) \Rightarrow h_1 = 0 \ , \ h_2 = X_1 X_2 - X_2 X_1 = [X_1, X_2]$$

Define new curve, $s = t^2$

$C_3 : s \longmapsto g_3(s) = h(\sqrt{s}) \in G$ parameter $s \in \mathbb{R}$ near $s=0$,

$g_3(s) = \mathbb{1}_n + s[X_1, X_2] + \mathcal{O}(s^{3/2})$

$\dot{g}_3(0) = [X_1, X_2] \in \mathcal{L}(G)$ 1st deriv exists at $s=0$, but not 2nd

$\Rightarrow \mathcal{L}(G) = (T_e(G), [\,,\,])$ real Lie algebra of dimension $D = \dim(G)$.

Examples

$G = SO(2)$ $g(t) = M(\theta(t)) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \in SO(2)$

$\dot{g}(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{\theta}(0)$

$\mathcal{L}(SO(2)) = \left\{ \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}, c \in \mathbb{R} \right\}$

$G = SO(n)$

$g(t) = R(t) \in SO(n)$ $R(0) = \mathbb{1}_n$

$\Rightarrow R^T(t) R(t) = \mathbb{1}_n$ $\forall t \in \mathbb{R}$

$\dot{R}^T(t) R(t) + R^T(t) \dot{R}(t) = 0$ $\forall t \in \mathbb{R}$

$X = \dot{R}(0)$ $\Rightarrow X^T + X = 0$

no further constraints from demanding $\det R = 1$

$\mathcal{L}(O(n)) = \mathcal{L}(SO(n)) = \{ X \in \mathrm{Mat}_n(\mathbb{R}) : X^T = -X \}$

$\dim \mathcal{L}(SO(n)) = \frac{1}{2} n(n-1)$ $(\dim(SO(n)))$

$G = SU(n)$ $g(t) = U(t) \in SU(n)$, $U(0) = \mathbb{1}_n$

$U^\dagger(t) U(t) = \mathbb{1}_n$

$\Rightarrow Z^\dagger + Z = 0$, $Z = \dot{U}(0) \in \mathcal{L}(SU(n))$

$\det(U(t)) = 1$ $\forall t \in \mathbb{R}$

$\det U(t) = 1 + \mathrm{tr}\, Z\, t + \mathcal{O}(t^2)$

hence $\det U(t) = 1$ $\forall t \in \mathbb{R}$ $\Rightarrow \mathrm{tr}\, Z = 0$ $\dim \mathcal{L}(SU(n)) = n^2 - 1$

$\mathcal{L}(SU(n)) = \{ \underbrace{Z \in \mathrm{Mat}_n(\mathbb{C})}_{2n^2}, \underbrace{Z^\dagger = -Z}_{n^2}, \underbrace{\mathrm{tr}\, Z = 0}_{1} \}$

$G = SU(2)$ in detail

$\mathcal{L}(SU(2)) = \{ 2\times 2 \text{ traceless anti-hermitian matrices} \}$

Pauli matrices σ_j $\quad j = 1, 2, 3$, $\quad \sigma_j^\dagger = \sigma_j$, $\operatorname{tr} \sigma_j = 0$

$\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i \epsilon_{ijk} \sigma_k$. $\quad T^a = -\frac{1}{2} i \sigma_j \quad a = j = 1, 2, 3$.

$[T^a, T^b] = -\frac{1}{4} [\sigma_a, \sigma_b] \quad (\sigma_a = \sigma_i \ \forall a = i)$

$= -\frac{1}{2} i \epsilon_{abc} \sigma_c = f^{ab}{}_c T^c$

$$\boxed{f^{ab}{}_c = \epsilon_{abc}}$$

$G = SO(3)$

$\mathcal{L}(SO(3)) = \{ 3\times 3 \text{ real anti-symmetric matrices} \}$

$$\tilde{T}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{T}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{T}^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(\tilde{T}^a)_{bc} = -\epsilon_{abc} \quad a, b, c = 1, 2, 3$

$[\tilde{T}^a, \tilde{T}^b] = f^{ab}{}_c \tilde{T}^c$ with $f^{ab}{}_c = \epsilon_{abc}$

$\mathcal{L}(SO(3)) \simeq \mathcal{L}(SU(2))$ although $SO(3) \not\simeq SU(2)$ $\begin{pmatrix} \pi_1(SO(3)) = \mathbb{Z}_2 \\ \pi_1(SU(2)) = \emptyset \end{pmatrix}$

$SO(3) \approx \dfrac{SU(2)}{\mathbb{Z}_2}$ (will see later) $\qquad \mathbb{Z}_2 = (\{ \mathbb{1}_2, -\mathbb{1}_2 \}, *)$

centre of $SU(2)$