Schwarzschild solution:

$$ds^2 = \left(1-\frac{2M}{R}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{R}\right)} + r^2 d\Omega_2^2 \qquad (*)$$

Theorem: Birkhoff theorem

Any spherically symmetric solution of the vacuum-Einstein is isometric to (*).

Proof:

$$ds^2 = -f\,(dt + \chi\, dr)^2 + \frac{dr^2}{g} + \Phi\, d\Omega_2^2 \quad , \quad f, g, \chi, \Phi \text{ depend on } (t, r)$$

This is the most general line element that admits complete reparametrization invariance in (t, r).

Fix this $\chi = 0$, $\Phi = r^2$

$$ds^2 = -f\,dt^2 + \frac{dr^2}{g} + r^2 d\Omega_2^2 \qquad t \to F(t)$$

From (t,r): $\quad -\frac{\partial_t g}{rg} = 0 \quad \Rightarrow \quad g = g(r)$

From (t,t): $\quad \frac{f}{r^2}(1 - g - rg') = 0 \quad \Rightarrow \quad g(r) = 1 - \frac{2M}{r}$

From (r,r): $\quad \left[1 - \frac{1}{g} + \frac{rf'}{f}\right]\frac{1}{r^2} = 0 \quad \Rightarrow \quad f(t,r) = C(t)\left(1 - \frac{2M}{r}\right)$

$$ds^2 = -\left(1-\frac{2M}{r}\right)C(t)\,dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_2^2$$

$$= -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_2^2 \qquad \square$$

spherically symmetric $\Rightarrow$ Killing field ∂_t

Gravitational red shift

Consider two observers Alice (A) and Bob (B). Let these observers be at constant (θ, ϕ), $r_B > r_A$. Alice will send two photons to Bob, separated by a time Δt (Schwarzschild time). From perspective of ~~proper~~ proper time, for Alice for Bob

$$\Delta\tau_A = \sqrt{1 - \frac{2M}{r_A}}\,\Delta t \qquad \Delta\tau_B = \sqrt{1 - \frac{2M}{r_B}}\,\Delta t$$

So $\frac{\Delta\tau_B}{\Delta\tau_A} > 1$ because $r_B > r_A \Rightarrow \frac{\lambda_B}{\lambda_A} > 1$.

If $r_B \gg 2M$, $\quad 1 + z = \frac{\lambda_B}{\lambda_A} \approx \frac{1}{\sqrt{1 - \frac{2M}{r_A}}}$. For stars $R_{min} = \frac{9M}{4}$, $z_{max} = 2$.

Geodesics of the Schwarzschild solution

Let $x^\mu(\tau)$ be an affinely parametrized geodesic, $u^\mu = \frac{dx^\mu}{d\tau}$, $u^\mu \nabla_\mu u^\nu = 0$

Since Schwarzschild metric has two Killing fields: $k = \frac{\partial}{\partial t}$, $m = \frac{\partial}{\partial \phi}$

$$\begin{cases} E = -k \cdot u = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \\ h = m \cdot u = r^2 \sin^2\theta \,\frac{d\phi}{d\tau} \end{cases}$$

$$\frac{d}{d\tau}(k \cdot u) = 0 \Rightarrow u^a \nabla_a (k^b u_b) = \underbrace{u^a u^b \nabla_a k_b}_{\to 0} + u^a k^b \nabla_a u_b = 0$$

For timelike geodesics (if you pick τ to be proper time)

$$\begin{cases} E & \text{energy per unit mass} \\ h & \text{angular mom per unit mass} \end{cases}$$

For null, $\left|\frac{h}{E}\right| = b$ (impact parameter)

The action $$S = \int d\tau\, \dot{x}^a \dot{x}^b g_{ab} = \int d\tau \left[-\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2M}{r}\right)} + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2 \right]$$

For θ: $$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow r^2 \frac{d}{d\tau}\left(r^2 \dot{\theta}^2\right) - \frac{\cos\theta}{\sin^3\theta} h^2 = 0$$

$$\theta(0) = \pi/2, \quad \dot{\theta}(0) = 0 \Rightarrow \ddot{\theta}(0) = 0 \Rightarrow \theta(\tau) = \pi/2$$

Because we are looking at geodesics

$$\dot{x}^a \dot{x}^b g_{ab} = \sigma \qquad \sigma = (\underset{\text{timelike}}{-1}, \overset{\text{null}}{0}, \underset{\text{spacelike}}{1})$$

$$\frac{\dot{r}^2}{2} + V(r) = \frac{E^2}{2} \qquad V(r) = \left(\sigma + \frac{h^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)$$

Eddington-Finkelstein coordinates:

Radial null geodesics $\sigma = 0$, $h = 0$. We can choose $E = 1$: $\dot{r} = \pm 1$, $\dot{t} = \dfrac{1}{1 - \frac{2M}{r}}$

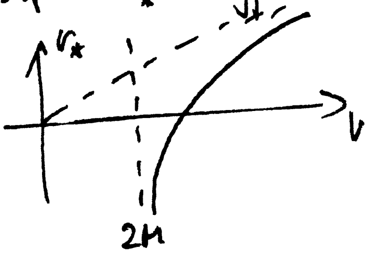
Outgoing: $\dfrac{\dot{t}}{\dot{r}} > 0$, Ingoing: $\dfrac{\dot{t}}{\dot{r}} < 0$

$\dot{r} = -1 \Rightarrow$ will cross $r = 2M$ (in finite proper time)

In either case

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$$

Define r_* (Regge-Wheeler coord) $dr_* = \dfrac{dr}{1 - \frac{2M}{r}} \Rightarrow r_* = r + 2M \log\left|\dfrac{r}{2M} - 1\right| + c$



$$\boxed{\frac{dt}{dr_*} = \pm 1} \Rightarrow t = \pm r_* + c \Rightarrow t \mp r_* = c$$

We can define $v = t + r_*$ constant for ingoing radial geodesics.

Can rewrite Schw. in (v, r, θ, ϕ), $dt = dv - dr_* = dv - \dfrac{dr}{1 - \frac{2M}{r}}$

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega_2^2}$$

$$\begin{pmatrix} -1 + \frac{2M}{r} & 1 & & \\ 1 & 0 & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$