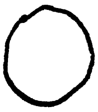
QFT in $d=1$ (aka QM)

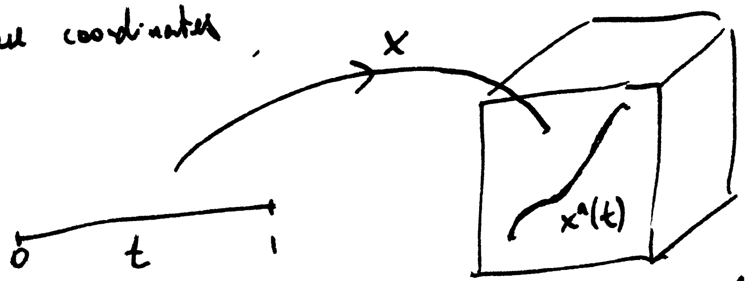
In $d=1$, there are two possible connected compact mfds M:

- $M = S^1$ 

- $M = I \equiv [0,1]$

when $M = I$ our universe has (two) boundaries which will need us to specify b.c. on the fields in the path integral.

I'll let $t \in [0,1]$ be a "worldline coordinate" and consider a field $x: I \to N$ where (N, g) is a Riemannian "target mfd". If $U \subseteq N$ has coordinates x^a, $a = 1, \dots, n \leftarrow \dim(N)$ then we usually write $x^a(t)$ for the values of the fields in these coordinates.



Euclidean worldline (!)

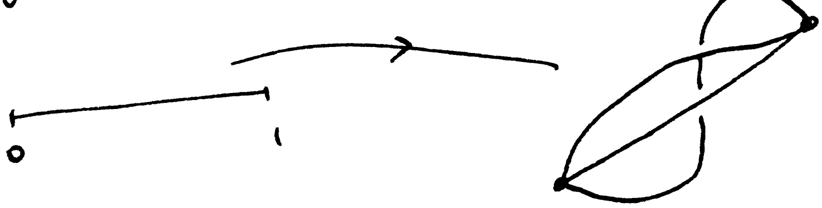
The standard choice of action is

$$S[x] = \int_I \left[\frac{1}{2} g_{ab}(x)\, \dot{x}^a \dot{x}^b + V(x)\right] dt \qquad \text{where} \quad \dot{x}^a = \frac{dx^a}{dt}$$

and this theory is called a non-linear sigma model. Extrema of $S[x]$ obey

$$\frac{d^2 x^a}{dt^2} + \Gamma^a_{bc}\, \dot{x}^b \dot{x}^c = g^{ab} \frac{\partial V}{\partial x^b}$$

and here the rhs describes some (non-gravitational) force



QM

The standard way to do QM on N is to pick a Hilbert space $\mathcal{H}$ (typically, $\mathcal{H} = L^2(N, d\mu)$) and also pick a Hamiltonian operator $H: \mathcal{H} \to \mathcal{H}$ with the usual choice being

$$H = \frac{1}{2}\Delta + V \qquad \text{where} \quad \Delta = \frac{1}{\sqrt{g}}\, \partial_a\left(\sqrt{g}\, g^{ab} \partial_b\right)$$

In this case, the amplitude for a particle to travel from $x \in N$ to $y \in N$ in Euclidean time T is $K_T(y,x) = \langle y | e^{-HT} | x \rangle$ (Heisenberg picture, or $\langle y, T | x, 0 \rangle$ in Schrödinger picture)

Here $K_t(y,x)$ is defined to be the unique soln of

$$\frac{\partial}{\partial t} K_t(y,x) + H K_t(y,x) = 0 \qquad \text{s.t.} \quad \lim_{t \to 0} K_t(y,x) = \delta^n(y-x)$$

(c.f. $t \to it$ $\quad i\partial_t K_t(y,x) = H K_t(y,x)$ which is the Schrödinger eq)

$K_t(y,x)$ is often called the <u>heat kernel</u> on (N,g,V).

e.g. $(N,g,V)=(\mathbb{R}^n,\delta,0)$ then $K_t(y,x)=\frac{1}{(2\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{2t}\right)$

e.g. $(N,g,V)=(N,g,0)$ then $\lim_{t\to 0}K_t(y,x)\sim\frac{a(x)}{(2\pi t)^{n/2}}\exp\left(-\frac{d(y,x)^2}{2t}\right)$

where $d(x,y)$ is the geodesic distance between (x,y) and $a(x)$ is some invariant of (N,g) built from (integrals of) polynomials in the Riemann curvature.

We can also represent $K_T(x,y)$ as a path integral. Let $\Delta t = t/N$ for some large $N\in\mathbb{N}$. Then

$$K_T(y,x)=\langle y|e^{-TH}|x\rangle=\langle y|\overbrace{e^{-\Delta t H}\cdots e^{-\Delta t H}}^{N \text{ copies}}|x\rangle$$

$$=\int\prod_{i=1}^{N-1}d^n x_i\,\langle y|e^{-\Delta t H}|x_{N-1}\rangle\langle x_{N-1}|e^{-\Delta t H}|x_{N-2}\rangle\cdots\langle x_1|e^{-\Delta t H}|x\rangle$$

$$=\int\prod_{i=1}^{N}d^n x_i\,K_{\Delta t}(x_i,x_{i-1}) \qquad \text{where } x_0\equiv x,\ x_N\equiv y$$

This follows from the concatenation property of the heat kernel, i.e.

$$K_{t_1+t_2}(y,x)=\int d^n z\,K_{t_2}(y,z)\,K_{t_1}(z,x)$$ (for flat space, this is just convolution of Gaussians)

The purpose of introducing Δt is that we can use the asymptotic form of $K_{\Delta t}(y,x)$

$$\Rightarrow \langle y_1|e^{-HT}|y_0\rangle=\lim_{N\to\infty}\left(\frac{1}{2\pi\Delta t}\right)^{Nn/2}\int\prod_{i=1}^{N-1}d^n x_i\,a(x_i)\exp\left(\frac{-\Delta t}{2}\left(\frac{d(x_{i+1},x_i)}{\Delta t}\right)^2\right)$$

<u>If</u> we declare the path integral measure

$$Dx\overset{?}{\equiv}\lim_{N\to\infty}\left(\frac{1}{2\pi\Delta t}\right)^{Nn/2}\prod_{i=1}^{N-1}d^n x_i\,a(x_i)$$

and <u>if</u> our map $x(t)$ is at least once-ctsly differentiable, then

$$\lim_{N\to\infty}\prod_{i=1}^{N-1}\exp\left(-\frac{\Delta t}{2}\left[\frac{d(x_{i+1},x_i)}{\Delta t}\right]^2\right)=\exp\left(-\frac{1}{2}\int dt\,\left(g_{ab}\dot{x}^a\dot{x}^b\right)\right)$$

Then, $\langle y_1|e^{-HT}|y_0\rangle = \ldots = \exp(-S[x])$ here for $V(x)=0$.

$$=\int_{C_T[y_1,y_0]}Dx\;e^{-\int\frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b\,dt}$$ where $C_T[y_1,y_0]$ is the space of "all" maps from $I\to N$ s.t. $x(0)=y_0$, $x(1)=y_1$.

Notice that $|y_0\rangle\in\mathcal{H}$ and $\langle y_1|\in\mathcal{H}^*$ have arisen as boundary conditions on the map x.

The partition f^n

We also have $\mathrm{Tr}_{\mathcal{H}}(e^{-HT}) = \int d^n y \langle y | e^{-HT} | y \rangle$

$$= \int d^n y \int_{C_I[y,y]} Dx \; e^{-S} = \int_{C_{S^1}} Dx \; e^{-S}$$ where the circle S^1 has circumference T.

$$= \mathcal{Z}[S^1, (N, g, V)]$$

Correlation Functions

A local operator $\mathcal{O}(t)$ is one which depends on the values of the fields + finitely many derivatives just at one point $t \in M$. The simplest type come from functions on N, i.e. if $\mathcal{O}: N \to \mathbb{R}$, then by pull-back we get an operator $\mathcal{O}(x^a(t))$.

We have $\langle y_1 | \hat{\mathcal{O}}(t) | y_0 \rangle = \langle y_1 | e^{-H(T-t)} \mathcal{O} e^{-Ht} | y_0 \rangle$ where $\hat{\mathcal{O}} = \mathcal{O}(\hat{x})$

$$= \int d^n x \langle y_1 | e^{-H(T-t)} | x \rangle \langle x | \hat{\mathcal{O}} e^{-Ht} | y \rangle = \int d^n x \, \mathcal{O}(x) \langle y_1 | e^{-H(T-t)} | x \rangle \langle x | e^{-Ht} | y_0 \rangle$$

i.e. $$\langle y_1 | \hat{\mathcal{O}}(t) | y_0 \rangle = \int d^n x \, \mathcal{O}(x(t)) \int_{C_{[T,t]}[y_1, x_t]} Dx \; e^{-S[x]} \int_{C_{[t,0]}[x_t, y_0]} Dx \; e^{-S[x]}$$

$$= \int_{C_{[T,0]}[y_1, y_0]} Dx \; \mathcal{O}(x(t)) \, e^{-S[x]}$$