Quantum Field Theory
Example sheet 2

1. $[q_n, q_m] = [p_n, p_m] = 0 \qquad [q_n, p_m] = i\delta_{nm}$

$$a_n = \sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n \qquad a_n^\dagger = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}}\, p_n$$

$$[a_n, a_m] = \frac{\sqrt{\omega_n\omega_m}}{2}\cancel{[q_n, q_m]}^{\,0} + i\frac{\sqrt{\omega_n/\omega_m}}{2}[q_n, p_m] + i\frac{\sqrt{\omega_m/\omega_n}}{2}[p_n, q_m]$$

$$+ (i)^2 \frac{1}{2\sqrt{\omega_n\omega_m}}\cancel{[p_n, p_m]}^{\,0} = i^2\frac{\sqrt{\omega_n/\omega_m}}{2}\delta_{nm} - i^2\frac{\sqrt{\omega_m/\omega_n}}{2}\delta_{mn} = 0$$

Similarly $[a_n^\dagger, a_m^\dagger] = 0$

$$[a_n, a_m^\dagger] = \frac{\sqrt{\omega_n\omega_m}}{2}\cancel{[q_n, q_m]}^{\,0} - i\frac{\sqrt{\omega_n/\omega_m}}{2}[q_n, p_m] + i\frac{\sqrt{\omega_m/\omega_n}}{2}[p_n, q_m]$$

$$+ \frac{1}{2\sqrt{\omega_n\omega_m}}\cancel{[p_n, p_m]}^{\,0} = \frac{\sqrt{\omega_n/\omega_m}}{2}\delta_{nm} + \frac{\sqrt{\omega_m/\omega_n}}{2}\delta_{mn}$$

$$= \delta_{nm}$$

$$\frac{1}{2}\omega_n(a_n a_n^\dagger + a_n^\dagger a_n) = \frac{1}{2}\omega_n([a_n, a_n^\dagger] + 2a_n^\dagger a_n) = \frac{\omega_n}{2} + \omega_n a_n^\dagger a_n$$

and

$$a_n^\dagger a_n = \frac{\omega_n}{2}q_n^2 - \frac{i}{2}p_n q_n + \frac{i}{2}q_n p_n + \frac{1}{2\omega_n}p_n^2$$

$$= \frac{1}{2}\left(\omega_n q_n^2 + \frac{1}{\omega_n}p_n^2 - 1\right)$$

$$\Rightarrow \frac{1}{2}\omega_n(a_n a_n^\dagger + a_n^\dagger a_n) = \frac{\omega_n}{2} + \omega_n a_n^\dagger a_n = \frac{1}{2}p_n^2 + \frac{1}{2}\omega_n^2 q_n^2$$

$$\therefore\ H = \sum_{n=1}^{\infty}\left(\frac{1}{2}p_n^2 + \frac{1}{2}\omega_n^2 q_n^2\right) = \sum_{n=1}^{\infty}\frac{1}{2}\omega_n(a_n a_n^\dagger + a_n^\dagger a_n)$$

As seen above can write

$$H = \sum_{n=1}^{\infty} \left(\frac{\omega_n}{2} + \omega_n a_n^{\dagger} a_n \right)$$

Hence the vacuum energy $H|0\rangle = \left(\sum_{n=1}^{\infty} \frac{\omega_n}{2} \right) |0\rangle$, $E_0 = \sum_{n=1}^{\infty} \frac{\omega_n}{2}$. Removing this term

$H = \sum_{n=1}^{\infty} \omega_n a_n^{\dagger} a_n$ this does not affect commutation relations as ω_n scalars.

$$[H, a_n^{\dagger}] = \left[\sum_m \omega_m a_m^{\dagger} a_m , a_n^{\dagger} \right] = \sum_m \omega_m \left[a_m^{\dagger} a_m , a_n^{\dagger} \right]$$

as $[a_n^{\dagger}, a_m^{\dagger}] = 0$

$$= \sum_m \omega_m \left(a_m^{\dagger} a_m a_n^{\dagger} - a_n^{\dagger} a_m^{\dagger} a_m \right) = \sum_m \omega_m a_m^{\dagger} \left(a_m a_n^{\dagger} - a_n^{\dagger} a_m \right)$$

$$= \sum_m \omega_m a_m^{\dagger} [a_m, a_n^{\dagger}] = \sum_m \omega_m a_m^{\dagger} \delta_{mn} = \omega_n a_n^{\dagger}$$

$$|l_1, l_2, \ldots, l_N\rangle = (a_1^{\dagger})^{l_1} (a_2^{\dagger})^{l_2} \ldots (a_N^{\dagger})^{l_N} |0\rangle$$

$$H|l_1, l_2, \ldots, l_N\rangle = H (a_1^{\dagger})^{l_1} \ldots (a_N^{\dagger})^{l_N} |0\rangle$$

$$\Big[\; H (a_n^{\dagger})^{l} = \left([H, a_n^{\dagger}] + a_n^{\dagger} H \right) (a_n^{\dagger})^{l-1} = \omega_n (a_n^{\dagger})^{l} + a_n^{\dagger} H (a_n^{\dagger})^{l-1}$$

$$= l \omega_n (a_n^{\dagger})^{l} + (a_n^{\dagger})^{l} H \; \Big]$$

$$= \sum_{n=1}^{N} l_n \omega_n (a_1^{\dagger})^{l_1} \ldots (a_N^{\dagger})^{l_N} |0\rangle + (a_1^{\dagger})^{l_1} \ldots (a_N^{\dagger})^{l_N} H |0\rangle$$

$$= \left(\sum_{n=1}^{N} l_n \omega_n \right) |l_1, l_2, \ldots l_N\rangle$$

$$\Rightarrow E = \sum_{n=1}^{N} l_n \omega_n$$

2. $\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p e^{i\underline{p}\cdot\underline{x}} + a_p^\dagger e^{-i\underline{p}\cdot\underline{x}} \right]$

$\pi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left[a_p e^{i\underline{p}\cdot\underline{x}} - a_p^\dagger e^{-i\underline{p}\cdot\underline{x}} \right]$

$$[\phi(\underline{x}), \phi(\underline{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \Big([a_p, a_q] e^{i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}}$$
$$+ [a_p^\dagger, a_q] e^{-i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}} + [a_p, a_q^\dagger] e^{i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}}$$
$$+ [a_p^\dagger, a_q^\dagger] e^{-i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}} \Big) = 0 \qquad ①$$

$$[\pi(\underline{x}), \pi(\underline{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6} (-i)^2 \frac{\sqrt{E_pE_q}}{2} \Big([a_p, a_q] e^{i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}}$$
$$- [a_p^\dagger, a_q] e^{-i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}} - [a_p, a_q^\dagger] e^{i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}}$$
$$+ [a_p^\dagger, a_q^\dagger] e^{-i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}} \Big) = 0 \qquad ②$$

$$[\phi(\underline{x}), \pi(\underline{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{-i}{2} \Big([a_p, a_q] e^{i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}} + [a_p^\dagger, a_q] e^{-i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}}$$
$$- [a_p, a_q^\dagger] e^{i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}} - [a_p^\dagger, a_q^\dagger] e^{-i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}} \Big) = i\delta^3(\underline{x} - \underline{y}) \qquad ③$$

Consider the FT of these expressions

① $\int d^3x \, [\phi(\underline{x}) \phi(\underline{y})] e^{-i\underline{p}'\cdot\underline{x}}$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{d^3x}{\sqrt{4E_pE_q}} \Big([a_p, a_q] e^{i\underline{q}\cdot\underline{y}} e^{i(\underline{p}-\underline{p}')\cdot\underline{x}} + [a_p^\dagger, a_q] e^{-i(\underline{p}+\underline{p}')\cdot\underline{x}} e^{i\underline{q}\cdot\underline{y}}$$
$$+ [a_p, a_q^\dagger] e^{-i\underline{q}\cdot\underline{y}} e^{i(\underline{p}-\underline{p}')\cdot\underline{x}} + [a_p^\dagger, a_q^\dagger] e^{-i\underline{q}\cdot\underline{y}} e^{-i(\underline{p}+\underline{p}')\cdot\underline{x}} \Big)$$
$$= \int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_pE_q}} \Big([a_p, a_q] e^{i\underline{q}\cdot\underline{y}} \delta^3(\underline{p} - \underline{p}') + [a_p^\dagger, a_q] e^{i\underline{q}\cdot\underline{y}} \delta^3(\underline{p} + \underline{p}')$$
$$+ [a_p, a_q^\dagger] e^{-i\underline{q}\cdot\underline{y}} \delta^3(\underline{p} - \underline{p}') + [a_p^\dagger, a_q^\dagger] e^{-i\underline{q}\cdot\underline{y}} \delta^3(\underline{p} + \underline{p}') \Big)$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{p'}E_q}} \Big([a_{p'}, a_q] e^{iq\cdot y} + [a_{p'}^\dagger, a_q] e^{iq\cdot y} + [a_{p'}, a_q^\dagger] e^{-iq\cdot y} + [a_{-p'}^\dagger, a_q^\dagger] e^{-iq\cdot y} \Big)$$

and $\int d^3x\, d^3y\, [\phi(x), \phi(y)]\, e^{-ip'\cdot x} e^{-iq'\cdot y}$

$$= \int \frac{d^3q}{2\sqrt{E_{p'}E_q}} \Big([a_{p'}, a_q]\, \delta^3(q-q') + [a_{p'}^\dagger, a_q]\, \delta^3(q-q') + [a_{p'}, a_q^\dagger]\, \delta^3(q+q') + [a_{-p'}^\dagger, a_q^\dagger]\, \delta^3(q+q') \Big)$$

$$= \frac{1}{2\sqrt{E_{p'}E_{q'}}} \Big([a_{p'}, a_{q'}] + [a_{-p'}^\dagger, a_{q'}] + [a_{p'}, a_{-q'}^\dagger] + [a_{-p'}^\dagger, a_{-q'}^\dagger] \Big) = 0$$

Similarly for ②, get 2 equations for all p, q (relabelling $p' \to p$),

① : $[a_p, a_q] + [a_{-p}^\dagger, a_q] + [a_p, a_{-q}^\dagger] + [a_{-p}^\dagger, a_{-q}^\dagger] = 0$

② : $[a_p, a_q] - [a_{-p}^\dagger, a_q] - [a_p, a_{-q}^\dagger] + [a_{-p}^\dagger, a_{-q}^\dagger] = 0$

or

$$[a_p, a_q] + [a_{-p}^\dagger, a_{-q}^\dagger] = 0 \qquad (1*)$$

$$[a_{-p}^\dagger, a_q] + [a_p, a_{-q}^\dagger] = 0 \qquad (2*)$$

From ③,

$$\int d^3x\, [\phi(x), \pi(y)]\, e^{-ip'\cdot x} = \int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{-i}{2} \Big([a_p, a_q] e^{iq\cdot y} \delta^3(p-p') + [a_p^\dagger, a_q] e^{iq\cdot y} \delta^3(p+p') - [a_p, a_q^\dagger] e^{-iq\cdot y} \delta^3(p-p') - [a_p^\dagger, a_q^\dagger] e^{-iq\cdot y} \delta^3(p+p') \Big)$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{-i}{2} \Big([a_{p'}, a_q] e^{iq\cdot y} + [a_{-p'}^\dagger, a_q] e^{iq\cdot y} - [a_{p'}, a_q^\dagger] e^{-iq\cdot y} - [a_{-p'}^\dagger, a_q^\dagger] e^{-iq\cdot y} \Big)$$

$$= \int d^3x\, i\delta^3(x-y)\, e^{-ip'\cdot x} = i e^{-ip'\cdot y}$$

$$\int d^3x\, d^3y\, [\phi(x), \pi(y)]\, e^{-ip'\cdot x} e^{-iq'\cdot y} = \frac{-i}{2} \Big([a_{p'}, a_{q'}] + [a_{-p'}^\dagger, a_{q'}] - [a_{p'}, a_{-q'}^\dagger] - [a_{-p'}^\dagger, a_{-q'}^\dagger] \Big)$$

$$= \int d^3y\, i e^{-ip'\cdot y} e^{-iq'\cdot y} = i (2\pi)^3 \delta^3(p'+q') \qquad (*)$$

Substituting in from above

$$\overset{(1^*,2^*)}{\Rightarrow} -\left([a_p, a_q] + [a_{-p}^\dagger, a_q]\right) = (2\pi)^3 \delta^3(p+q) \qquad (\dagger)$$

Switching labels $p \leftrightarrow q$ in $(*)$,

$$-\frac{1}{2}\left([a_p, a_q] + [a_{-p}^\dagger, a_q] - [a_p, a_{-q}^\dagger] - [a_{-p}^\dagger, a_{-q}^\dagger]\right) = (2\pi)^3 \delta^3(p+q) \qquad (*)$$

$$\Rightarrow \frac{1}{2}\left([a_p, a_q] + [a_p, a_{-q}^\dagger] - [a_{-p}^\dagger, a_q] - [a_{-p}^\dagger, a_{-q}^\dagger]\right) = (2\pi)^3 \delta^3(p+q)$$

Sum gives

$$\frac{1}{2}\left(-[a_{-p}^\dagger, a_q] + [a_p, a_{-q}^\dagger]\right) = (2\pi)^3 \delta^3(p+q)$$

$$\overset{(2^*)}{\Rightarrow} [a_p, a_{-q}^\dagger] = (2\pi)^3 \delta^3(p+q)$$

Relabelling, and from $(\dagger)$, and finally relabelling (1^*) gives

$$[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \qquad [a_p, a_q^\dagger] = (2\pi)^3 \delta^3(p-q) \quad \square$$

3. $$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\partial_{\sigma} \phi \partial^\sigma \phi \, \eta^{\mu\nu} + \frac{1}{2} m^2 \phi^2 \eta^{\mu\nu}$$

$$T^{0\mu} = \partial^0 \phi \partial^\mu \phi - \frac{1}{2}\partial_\sigma \phi \partial^\sigma \phi \, \eta^{0\mu} + \frac{1}{2} m^2 \phi^2 \eta^{0\mu}$$

$$T^{00} = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

$$T^{0i} = \pi \partial^i \phi = -\pi \nabla \phi$$

$$\int d^3x \, \frac{1}{2}\pi^2 = \frac{1}{2}\int \frac{d^3p \, d^3q}{(2\pi)^6} (-i)^2 d^3x \, \frac{\sqrt{E_p E_q}}{2} :\left[a_p a_q e^{i(p+q)\cdot x} - a_p^\dagger a_q e^{-i(p-q)\cdot x} - a_q^\dagger a_p e^{i(p-q)\cdot x} + a_p^\dagger a_q^\dagger e^{-i(p+q)\cdot x}\right]:$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} (-i)^2 \frac{E_p}{2}\left[a_p a_{-p} - 2 a_p^\dagger a_p + a_p^\dagger a_{-p}^\dagger\right]$$

$$\int d^3x \, \frac{1}{2}(\nabla\phi)^2 = \frac{1}{2}\int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{d^3x}{2\sqrt{E_p E_q}} i^2 \left[a_p a_q e^{i(p+q)\cdot x} - a_p^\dagger a_q e^{-i(p-q)\cdot x} - a_q^\dagger a_p e^{i(p-q)\cdot x} + a_p^\dagger a_q^\dagger e^{-i(p+q)\cdot x}\right]$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \frac{-1}{2E_p}\left[\overset{-p^2}{a_p a_{-p}} \;-2a_p^\dagger a_p \overset{p^2}{} + a_p^\dagger a_{-p}^\dagger \overset{-p^2}{}\right]$$

$$\int d^3x\, \frac{1}{2}m^2\phi^2 = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \frac{m^2}{2E_p}\left[a_p a_{-p} + 2a_p^\dagger a_p + a_p^\dagger a_{-p}^\dagger\right]$$

$$\Rightarrow P^0 = \int d^3x\, T^{00} = \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\left[\left(-E_p + \frac{p^2}{E_p} + \frac{m^2}{E_p}\right)a_p a_{-p} + 2\left(E_p + \frac{p^2}{E_p} + \frac{m^2}{E_p}\right)a_p^\dagger a_p \right.$$
$$\left. + \left(-E_p + \frac{p^2}{E_p} + \frac{m^2}{E_p}\right)a_p^\dagger a_{-p}^\dagger\right]$$
$$= \int \frac{d^3p}{(2\pi)^3}\, E_p\, a_p^\dagger a_p$$

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, \pi(\nabla\phi) = \cancel{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[-p\, a_p a_{-p} + p\, a_p^\dagger a_p - p\, a_p a_p^\dagger + p\, a_p^\dagger a_{-p}^\dagger\right]}$$
$$= -\int \frac{d^3p\, d^3q\, d^3x}{(2\pi)^6}(-i)\sqrt{\frac{E_p}{2}}\,\frac{i}{\sqrt{2E_q}}\left[q\, a_p a_q e^{i(p+q)\cdot x} - q\, a_p^\dagger a_q e^{i(q-p)\cdot x}\right.$$
$$\left. - q\, a_q^\dagger a_p e^{i(p-q)\cdot x} + q\, a_p^\dagger a_q^\dagger e^{-i(p+q)\cdot x}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(p\, a_p^\dagger a_p + \underbrace{p\, a_p a_{-p} + p\, a_p^\dagger a_{-p}^\dagger}_{\to 0 \text{ by relabelling } p \leftrightarrow -p}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\, p^i a_p^\dagger a_p$$

$$\therefore\; P^\mu = \int \frac{d^3p}{(2\pi)^3}\, p^\mu a_p^\dagger a_p \quad \text{where} \quad p^\mu = (E_p, \mathbf{p}).$$

—

$$[P^\mu, \phi(x)] = \left[\int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p,\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right)\right]$$
$$= \int \frac{d^3p\, d^3q}{(2\pi)^6}\,\frac{1}{\sqrt{2E_p}}\, q^\mu\left([a_q^\dagger a_q, a_p]e^{-ip\cdot x} + [a_q^\dagger a_q, a_p^\dagger]e^{+ip\cdot x}\right)$$
$$= \int \frac{d^3p\, d^3q}{(2\pi)^6}\,\frac{1}{\sqrt{2E_p}}\, q^\mu\left(-a_q e^{-ip\cdot x}(2\pi)^3\delta^3(q-p) + a_q^\dagger e^{+ip\cdot x}\delta^3(q-p)\right)$$
$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{-1}{\sqrt{2E_p}}\, p^\mu\left(a_p e^{-ip\cdot x} - a_p^\dagger e^{+ip\cdot x}\right)$$
$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{-i}{\sqrt{2E_p}}\,\partial^\mu\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right) = -i\,\partial^\mu\phi(x),$$

4. $\dot{\phi}(x) = i[H, \phi] = \frac{i}{2}\left[\int d^3y\left(\pi(y)^2 + (\nabla\phi(y))^2 + m^2\phi(y)^2\right), \phi(x)\right]$

$$= \frac{i}{2}\left[\int d^3y\ \pi(y)^2\ , \phi(x)\right] = i\int d^3y\ [\pi(y), \phi(x)]\,\pi(y)$$

$$= i\int d^3y\ \pi(y)\,(-i)\,\delta^3(\underline{x}-\underline{y}) = \pi(x)$$

$\dot{\pi}(x) = i[H, \pi] = \frac{i}{2}\left[\int d^3y\left(\pi(y)^2 + \nabla\phi(y)^2 + m^2\phi(y)^2\right), \pi(x)\right]$

$$= \frac{i}{2}\int d^3y\Big(\nabla_y\phi(y)\,\nabla_y[\phi(y), \pi(x)] + \nabla_y[\phi(y), \pi(x)]\,\nabla_y\phi(y)$$

$$+ 2m^2[\phi(y), \pi(x)]\,\phi(y)\Big)$$

$$= \frac{i}{2}\int d^3y\Big\{\nabla_y\phi(y)\ i\,\nabla_y\delta^3(\underline{x}-\underline{y}) + i\,\nabla_y\delta^3(\underline{x}-\underline{y})\,\nabla_y\phi(y)$$

$$+ 2m^2 i\,\delta^3(\underline{x}-\underline{y})\,\phi(y)\Big\}$$

$$= i\int d^3y\Big[\cancelto{0}{i\,\nabla_y\cdot\left(\nabla_y\phi(y)\,\delta^3(\underline{x}-\underline{y})\right)} - i\,\nabla_y^2\phi(y)\,\delta^3(\underline{x}-\underline{y})$$

$$+ m^2 i\,\delta^3(\underline{x}-\underline{y})\,\phi(y)\Big]$$

$$= \nabla^2\phi(x) - m^2\phi(x)$$

KG eq: $\partial_\mu\partial^\mu\phi + m^2\phi = \ddot{\phi} - \nabla^2\phi + m^2\phi = \dot{\pi} - \nabla^2\phi + m^2\phi = 0$.

5. $\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right)$

$|p\rangle = \sqrt{2E_p}\; a_p^\dagger |0\rangle$

$\langle 0|\phi(x)|p\rangle = \langle 0|\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\left(a_{p'}e^{-ip'\cdot x} + a_{p'}^\dagger e^{+ip'\cdot x}\right)\sqrt{2E_p}\; a_p^\dagger|0\rangle$

$= \langle 0|\int \frac{d^3p'}{(2\pi)^3}\sqrt{\frac{E_p}{E_{p'}}}\left(\left([a_{p'},a_p^\dagger] + a_p^\dagger a_{p'}\right)e^{-ip'\cdot x} + a_{p'}^\dagger a_p^\dagger e^{+ip'\cdot x}\right)|0\rangle$

$= \langle 0|\int \frac{d^3p'}{(2\pi)^3}\sqrt{\frac{E_p}{E_{p'}}}\,[a_{p'},a_p^\dagger]\,e^{-ip'\cdot x}|0\rangle$ as $a_p|0\rangle = 0$
$\langle 0|a_p^\dagger = 0$

$= \langle 0|\int \frac{d^3p'}{\cancel{(2\pi)^3}}\sqrt{\frac{E_p}{E_{p'}}}\,\cancel{(2\pi)^3}\,\delta^3(p'-p)\,e^{-ip'\cdot x}|0\rangle = \langle 0|e^{-ip\cdot x}|0\rangle = e^{-ip\cdot x}$

6. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$

$= \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + \eta^{\mu\nu}\frac{1}{2}m^2\phi^2$

$T^{0k} = \dot\phi\,\partial^k\phi - \eta^{0k}\mathcal{L} = \dot\phi\,\partial^k\phi = \pi\,\partial^k\phi$

$Q_i = \epsilon_{ijk}\int d^3x\left(x^j T^{0k} - x^k T^{0j}\right) = \epsilon_{ijk}\int d^3x\;\pi\left(x^j\partial^k\phi - x^k\partial^j\phi\right)$

$= \epsilon_{ijk}\int d^3x \int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{(-i)\sqrt{E_q}}{2\sqrt{E_p}}\Big[x^j\left(a_q e^{-iq\cdot x} - a_q e^{+iq\cdot x}\right)\left(-ip^k a_p e^{-ip\cdot x} + ip^k a_p^\dagger e^{+ip\cdot x}\right)$

$- x^k\left(a_q e^{-iq\cdot x} - a_q e^{+iq\cdot x}\right)\left(-ip^j a_p e^{-ip\cdot x} + ip^j a_p e^{+ip\cdot x}\right)\Big]$

$:Q_i: = :\epsilon_{ijk}\int d^3x\,\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{(-i)}{2}\sqrt{\frac{E_q}{E_p}}\left(a_q e^{-iq\cdot x} - a_q^\dagger e^{+iq\cdot x}\right) i$

$\left(-a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right)\left(x^j p^k - x^k p^j\right):$ $\cancel{\left(p^k\frac{\partial}{\partial p_j} - p^j\frac{\partial}{\partial p^k}\right)}$

$= \cancel{\epsilon_{ijk}\int d^3x\,\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{i}{2}\sqrt{\frac{E_q}{E_p}}\left(a_q e^{iq\cdot x} - a_q^\dagger e^{-iq\cdot x}\right)\left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)}:$

$$= \epsilon_{ijk} \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{-i}{2} \sqrt{\frac{E_q}{E_p}} \left(a_q e^{-iq\cdot x} - a_q^\dagger e^{+iq\cdot x} \right) \left(a_p \left(p^k \partial/\partial p_j - p^j \partial/\partial p_k \right) e^{-ip\cdot x} + a_p^\dagger \left(p^k \partial/\partial p_j - p^j \partial/\partial p_k \right) e^{+ip\cdot x} \right)$$

$$= \epsilon_{ijk} \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{-i}{2} \sqrt{\frac{E_q}{E_p}} \left(a_p \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) e^{-i(p+q)\cdot x} a_q - a_p \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) e^{-i(p-q)\cdot x} a_q^\dagger \right.$$

$$\left. + a_p^\dagger \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) e^{-i(q-p)\cdot x} a_q - a_p^\dagger \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) e^{+i(p+q)\cdot x} a_q^\dagger \right)$$

$$= \epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2} \left(\cancel{a_p \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) a_{-p}}^{\to 0} - a_p \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) a_p^\dagger \right.$$

$$\left. + a_p^\dagger \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) a_p - \cancel{a_p^\dagger \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) a_{-p}^\dagger}^{\to 0} \right)$$

← integrate by parts and normal order

relabelling $p \leftrightarrow -p$

$$= -i\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a_p^\dagger \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) a_p$$

$$Q_i |p\rangle = -i\epsilon_{ijk} \int \frac{d^3p'}{(2\pi)^3} a_{p'}^\dagger \left(p'^k \frac{\partial}{\partial p'_j} - p'^j \frac{\partial}{\partial p'_k} \right) a_{p'} \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$= -i\epsilon_{ijk} \int \frac{d^3p'}{(2\pi)^3} a_{p'}^\dagger \left(p'^k \frac{\partial}{\partial p'_j} - p'^j \frac{\partial}{\partial p'_k} \right) [a_{p'}, a_p^\dagger] \sqrt{2E_p} |0\rangle$$

$$= -i\epsilon_{ijk} \int d^3p' \sqrt{2E_p}\, a_{p'}^\dagger \left(p'^k \frac{\partial}{\partial p'_j} - p'^j \frac{\partial}{\partial p'_k} \right) \delta^3(p'-p) |0\rangle$$

$$-i\epsilon_{ijk} \int d^3p' \sqrt{2E_p}\, a_{p'}^\dagger \delta^3(p'-p) \cancel{\left(\frac{\partial p'^k}{\partial p'_j} - \frac{\partial p'^j}{\partial p'_k} \right)}^{\to 0} |0\rangle = 0$$

$$L_i = \left(p^k \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_k} \right) \epsilon_{ijk} i, \quad Q_i |p\rangle = L_i |p\rangle$$

7. $\mathcal{L} = i\psi^* \partial_0 \psi - \frac{1}{2m} \nabla\psi^* \nabla\psi$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) = \frac{\partial \mathcal{L}}{\partial \psi} : \quad \partial_0(i\psi^*) + \frac{1}{2m}\partial_i(\nabla\psi^*) = 0$$

$$\Rightarrow i\dot{\psi}^* + \frac{1}{2m}\nabla^2\psi^* = 0$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\right) = \frac{\partial \mathcal{L}}{\partial \psi^*} : \quad + \frac{1}{2m}\partial_i(\nabla\psi) = i\partial_0\psi$$

$$\Rightarrow -i\dot{\psi} + \frac{1}{2m}\nabla^2\psi = 0$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\partial^\nu\psi^* - \eta^{\mu\nu}\mathcal{L}$$

$$= \delta_0^\mu i\psi^*\partial^\nu\psi + \frac{1}{2m}\delta_i^\mu\partial^i\psi^*\partial^\nu\psi + \frac{1}{2m}\delta_i^\mu\partial^i\psi^*\partial^\nu\psi^* - \eta^{\mu\nu}\mathcal{L}$$

$$\psi \to e^{i\alpha}\psi = \psi + i\alpha\psi$$

$$j^\mu = \delta_0^\mu i\psi^*(i\psi) + \frac{1}{2m}\delta_i^\mu\partial^i\psi^*(i\psi) + \frac{1}{2m}\delta_i^\mu\partial^i\psi(-i\psi^*)$$

$$= -\delta_0^\mu\psi^*\psi + \frac{i}{2m}\delta_i^\mu(\psi\partial^i\psi^* - \psi^*\partial^i\psi) \qquad \partial_\mu j^\mu = 0 \text{ consistent w/ eom}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial\dot{\psi}} = i\psi^* \qquad \pi^* = \frac{\partial \mathcal{L}}{\partial\dot{\psi}^*} = 0$$

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^*\dot{\psi} - i\psi^*\dot{\psi} + \frac{1}{2m}\nabla\psi^*\nabla\psi = \frac{1}{2m}\nabla\psi^*\nabla\psi$$

Quantizing with (analogously from Poisson brackets):

$$[\psi(\underline{x}), \psi(\underline{y})] = [\psi^\dagger(\underline{x}), \psi^\dagger(\underline{y})] = [\pi(\underline{x}), \pi(\underline{y})] = [\pi^\dagger(\underline{x}), \pi^\dagger(\underline{y})] = 0$$

and $[\psi(\underline{x}), \pi(\underline{y})] = i\delta^{(3)}(\underline{x}-\underline{y})$, $[\psi(\underline{x}), \pi^\dagger(\underline{y})] = 0$

From above

$$[\psi(\underline{x}), \psi^\dagger(\underline{y})] = -i[\psi(\underline{x}), \pi(\underline{y})] = \delta^{(3)}(\underline{x}-\underline{y})$$

is self consistent when ignoring $\pi, \pi^\dagger$.

$$\psi(\underline{x}) = \int\frac{d^3p}{(2\pi)^3} a_p e^{i\underline{p}\cdot\underline{x}} \quad , \quad \psi^\dagger(\underline{x}) = \int\frac{d^3p}{(2\pi)^3} a_p^\dagger e^{-i\underline{p}\cdot\underline{x}}$$

$$[\psi(x), \psi(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}[a_p, a_q]\, e^{ip\cdot x + iq\cdot y} = 0$$

$$\int e^{-ip'\cdot x} d^3x \Rightarrow \int \frac{d^3p\, d^3q}{(2\pi)^3}[a_p, a_q]\, e^{iq\cdot y}\, \delta^{(3)}(p-p') = 0$$

$$\int \frac{d^3q}{(2\pi)^3}[a_p, a_q]\, e^{iq\cdot y} = 0$$

$$\int e^{-iq'\cdot y} d^3y \Rightarrow \int d^3q\, [a_p, a_q]\, \delta^{(3)}(q-q') = 0$$

$$\therefore [a_p, a_q] = 0 \quad , \quad [a_p^\dagger, a_q^\dagger] = 0$$

$$[\psi(x), \psi^\dagger(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^6}[a_p, a_q^\dagger]\, e^{ip\cdot x - iq\cdot y} = \delta^3(x-y)$$

$$\int e^{-ip'\cdot x} d^3x \Rightarrow \int \frac{d^3p\, d^3q}{(2\pi)^3}[a_p, a_q^\dagger]\, e^{-iq\cdot y}\, \delta^{(3)}(p-p') = e^{-ip'\cdot y}$$

$$\int \frac{d^3q}{(2\pi)^3}[a_p, a_q^\dagger]\, e^{-iq\cdot y} = e^{-ip\cdot y}$$

$$\int e^{-iq'\cdot y} d^3y \Rightarrow \int d^3q\, [a_p, a_q^\dagger]\, \delta^{(3)}(q+q') = (2\pi)^3 \delta^{(3)}(p+q')$$

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(p-q)$$

Can only extract 3 equations from commutation relations. Further operators would be redundant.

$$H = \frac{1}{2m}\int d^3x\, \nabla\psi^* \nabla\psi = \frac{1}{2m}\int \frac{d^3p\, d^3q}{(2\pi)^6} d^3x\; p\cdot q\; a_q^\dagger a_p\, e^{i(p-q)\cdot x} = \frac{1}{2m}\int \frac{d^3p}{(2\pi)^3} p^2 a_p^\dagger a_p$$

Single particle state

$$|p\rangle = a_p^\dagger |0\rangle$$

$$H|p\rangle = \frac{1}{2m}\int \frac{d^3q}{(2\pi)^3} q^2 a_q^\dagger a_q a_p^\dagger |0\rangle = \frac{1}{2m}\int \frac{d^3q}{(2\pi)^3} q^2 a_q^\dagger [a_q, a_p^\dagger] |0\rangle = \frac{1}{2m} p^2 a_p^\dagger |0\rangle = \frac{p^2}{2m}|p\rangle$$

$\therefore E = \frac{p^2}{2m}$: non-relativistic part^c of mass m.

8. $T(\phi(x_1)\phi(x_2)) = \begin{cases} \phi(x_1)\ \phi(x_2) & x_1^0 > x_2^0 \\ \phi(x_2)\ \phi(x_1) & x_2^0 > x_1^0 \end{cases}$

$\rightarrow T(\phi(x_2)\phi(x_1)) = \begin{cases} \phi(x_2)\phi(x_1) & x_2^0 > x_1^0 \\ \phi(x_1)\ \phi(x_2) & x_1^0 > x_2^0 \end{cases} = T(\phi(x_1)\phi(x_2))$

$:\phi(x_1)\ \phi(x_2): = \phi(x_1)\ \phi(x_2) - \langle 0|\phi(x_1)\phi(x_2)|0\rangle$

$:\phi(x_2)\ \phi(x_1): = \phi(x_2)\ \phi(x_1) - \langle 0|\phi(x_2)\phi(x_1)|0\rangle$

$= \phi(x_1)\ \phi(x_2) - [\phi(x_1), \phi(x_2)] - \langle 0|\phi(x_1)\phi(x_2)|0\rangle + \langle 0|[\phi(x_1), \phi(x_2)]|0\rangle$

$= :\phi(x_1)\ \phi(x_2): - \underbrace{[\phi(x_1), \phi(x_2)] + \langle 0|[\phi(x_1), \phi(x_2)]|0\rangle}_{=0}$

$= :\phi(x_1)\ \phi(x_2):$

or could argue that creation and annihilation operators commute respectively.

$\Delta_F(x_1-x_2) = \langle 0|T\phi(x_1)\ \phi(x_2)|0\rangle = \langle 0|T\ \phi(x_2)\phi(x_1)|0\rangle = \Delta_F(x_2-x_1)$

9. Have ~~$\Delta_F(x_1-x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = T\phi(x_1)\phi(x_2)$~~

$T\phi(x_1)\phi(x_2) = :\phi(x_1)\ \phi(x_2): + \Delta_F(x_1-x_2)$

$T\phi(x_1)\phi(x_2)\phi(x_3) = T\phi(x_1)\ T\phi(x_2)\ \phi(x_3)$

$= T\ \phi(x_1) :\phi(x_2)\phi(x_3): + T\ \phi(x_1)\Delta_F(x_2-x_3)$

$= (\phi^+(x_1) + \phi^-(x_1)) :\phi_2\phi_3: + \phi_1\Delta_F(x_2-x_3)$ w.l.o.g.

$= \phi_1^- :\phi_2\phi_3: + \phi_1^+(\phi_2^+\phi_3^+ + \phi_2^-\phi_3^+ + \phi_3^-\phi_2^+ + \phi_2^-\phi_3^-) + \phi_1\Delta_F(x_2-x_3)$

$= \phi_1^- :\phi_2\phi_3: + \cancel{\phi_2^+\phi_3^+} :\phi_2\phi_3:\phi_1^+ + \phi_3^+\Delta_F(x_1-x_2) + \phi_2^+\Delta_F(x_1-x_3)$

$+ \phi_2^-\Delta_F(x_3-x_1) + \phi_3^-\Delta_F(x_2-x_1) + \phi_1\Delta_F(x_2-x_3)$

$= :\phi_1\phi_2\phi_3: + \phi_1\Delta_F(x_2-x_3) + \phi_2\Delta_F(x_1-x_3) + \phi_3\Delta_F(x_1-x_2)$