What about interactions?

$P: \; \bar\psi\psi(\underline{x},t) \longrightarrow \bar\psi\psi(-\underline{x},t)$ transforms as a scalar

$P: \; \bar\psi\gamma^\mu\psi(\underline{x},t)$ vector

$$\bar\psi\gamma^0\psi(\underline{x},t) \longrightarrow \bar\psi\gamma^0\psi(\underline{x},t)$$

$$\bar\psi\gamma^i\psi(x,t) \longrightarrow \bar\psi\gamma^0\gamma^i\gamma^0\psi(-\underline{x},t) = -\bar\psi\gamma^i\psi(-\underline{x},t)$$

$P: \; \bar\psi S^{\mu\nu}\psi(\underline{x},t)$ transforms as a tensor

$$P: \; \bar\psi\gamma^5\psi(\underline{x},t) \longrightarrow \bar\psi\gamma^0\gamma^5\gamma^0\psi(-\underline{x},t) = \underset{\substack{\uparrow \\ \text{a pseudoscalar}}}{-}\bar\psi\gamma^5\psi(-\underline{x},t)$$

$$P: \; \bar\psi\gamma^5\gamma^\mu\psi(\underline{x},t) \longrightarrow \begin{cases} \mu=0 & -\bar\psi\gamma^5\gamma^\mu\psi \\ \mu=i & +\bar\psi\gamma^5\gamma^\mu\psi \end{cases}$$ pseudo/axial vector

$$\text{Total \# bilinears } \underbrace{1}_{\text{scalar}} + \underbrace{4}_{\text{vector}} + \underbrace{\left(\frac{4\times 3}{2}\right)}_{\text{anti-symmetric tensor}} + 4 + 1 = 16$$

We can now add extra terms to $\mathcal{L}$ that use γ^5

Typically these terms break P inv. (not always: $\phi\bar\psi\gamma^5\psi$ where ϕ is pseudoscalar)

Nature uses this: e.g. a W boson is a vector field that couple only to LH fermions

$$\mathcal{L} = \frac{g}{2} W^\mu \bar\psi \gamma_\mu (1-\gamma^5)\psi$$

A theory which puts $\psi_\pm$ on equal footing are called vector-like. One in which ψ_+ and ψ_- appear differently is called chiral.

Plane wave solutions of Dirac eq$^{\underline{n}}$

$$(i\not\partial - m)\psi = 0$$

Start with simplest ansats $\psi = u_p e^{-ip\cdot\underline{x}}$

where u_p is a constant 4-component spinor which depends on p

$$(\gamma^\mu p_\mu - m)u_p = 0 \qquad - (*)$$

$$\begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & -m \end{pmatrix} u_p = 0 \qquad \sigma^\mu = (\mathbb{1}, \underline{\sigma}) \quad \bar\sigma^\mu = (\mathbb{1}, -\underline{\sigma})$$

<u>Claim</u>: sol is $u_p = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$ for any 2-cpt ξ s.t. $\xi^\dagger\xi = 1$

<u>Proof</u>: $u_p = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $(*)$ reads

$$\begin{aligned}(p\cdot\sigma)u_2 &= m u_1 \\ (p\cdot\bar\sigma)u_1 &= m u_2\end{aligned} \qquad - (**)$$

Either of these can be derived from the other since

$$(p\cdot\sigma)(p\cdot\bar\sigma) = p_0^2 - p_i p_j \sigma^i\sigma^j$$

Use $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ $\quad\downarrow$ relabelling i,j and commute p_i, p_j

$$= p_0^2 - p_i p_j \delta^{ij}$$

$$= p_\mu p^\mu = m^2$$

Try the ansats

$u_1 = (p\cdot\sigma)\xi'$ for a spinor ξ'

Use $(**)$ $\quad u_2 = \frac{1}{m}(p\cdot\bar\sigma)(p\cdot\sigma)\xi' = m\xi'$

So any vector of the form $u_p = A\begin{pmatrix} (p\cdot\sigma)\xi' \\ m\xi' \end{pmatrix}$ solves $(*)$

To make it more symmetric, $A = 1/m$ and $\xi' = \sqrt{p\cdot\bar\sigma}\,\xi$ with ξ constant

Then $u_1 = \frac{1}{m}(p\cdot\sigma)(p\cdot\bar\sigma)\xi = \sqrt{p\cdot\sigma}\,\xi$

$$\Rightarrow u_p = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

e.g. a stationary particle of mass m

$$\psi = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \text{ for any 2-cpt spinor } \xi$$

Under spacial rotations, $\xi \longrightarrow e^{i\underline{\phi}\cdot\underline{\sigma}/2}\,\xi$ which rotates ξ

The 2 cpt object ξ defines the spin of the particle.

e.g. $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ spin $\uparrow$ along z - axis

N.B. Solving Dirac eq$^{\underline{n}}$ has reduced d.o.f $4 \longrightarrow 2$.

Consider a part$^{\underline{c}}$ boosted along x^3 - direction.

For $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ sol. to Dirac equation is

$$u_p = \begin{pmatrix}\sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xrightarrow[m\to 0]{} \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad -(m1)$$

For $\downarrow$ field, $\xi = \begin{pmatrix}0\\1\end{pmatrix}$

$$u_p = \begin{pmatrix}\sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \xrightarrow[m\to 0]{} \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad -(m2)$$

Helicity

The helicity op is a projection of $\not{x}$ mom. along the direction of momentum

$$h = \hat{\underline{p}}\cdot\underset{\not{x}}{\underline{J}} = \frac{1}{2}\hat{p}_i\cdot\begin{pmatrix}\sigma^i & 0 \\ 0 & \sigma^i\end{pmatrix}$$

$\not{x}$ mom. $J_i = \frac{i}{2}\varepsilon_{ijk}S^{jk}$

m1 : massless spin $\uparrow$ part$^{\underline{c}}$ has $h = +\frac{1}{2}$ RH

m2 : massless spin $\downarrow$ part$^{\underline{c}}$ has $h = -\frac{1}{2}$ LH

Negative energy sol

Ansats $V_p e^{ip\cdot x}$ $\uparrow$ negative freq.

$$V_p = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta\end{pmatrix} \text{ with } \eta^\dagger\eta = 1 \qquad \begin{pmatrix}\sqrt{p\cdot\sigma}\,v_1 = -m\,v_2 \\ \sqrt{p\cdot\bar{\sigma}}\,v_2 = -m\,v_1\end{pmatrix}$$

Quantising the Dirac field

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = i\bar\psi\gamma^0 = i\psi^\dagger$$

E.o.m. are 1st order in t. All we have to do is specifying $\psi^\dagger$ in some initial time slice to dictate the full evolution.

Imposing the canonical commutation relations.

$$\left.\begin{aligned}[\psi_\alpha(x), \psi_\beta(y)] &= [\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)] = 0 \\ [\psi_\alpha(x), \psi_\beta(y)] &= \delta_{\alpha\beta}\,\delta^3(\underline{x}-\underline{y})\end{aligned}\right\} \text{Naively}$$

Write quantum operators as

$$\psi(\underline{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b_p^s u_p^s e^{ip\cdot x} + c_p^{s\dagger} v_p^s e^{-ip\cdot x}\right]$$

$$\psi^\dagger(\underline{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b_p^{s\dagger} u_p^{s\dagger} e^{-ip\cdot x} + c_p^{s} v_p^{s\dagger} e^{ip\cdot x}\right]$$

<u>Claim</u>:

$$[b_p^r, b_q^{s\dagger}] = (2\pi)^3\delta^{rs}\delta^3(p-q)$$

$$[c_p^r, c_q^{s\dagger}] = -(2\pi)^3\delta^{rs}\delta^3(p-q)$$

Others vanish.

Hamiltonian $\mathcal{H} = \pi\dot\varphi - \mathcal{L} = \cancel{i\psi^\dagger\dot\psi} - \cancel{i\bar\psi\gamma^0\dot\psi} - i\bar\psi\gamma^i\partial_i\psi + m\bar\psi\psi$

$$= \bar\psi(-i\gamma^i\partial_i + m)\psi$$

$$H = \int d^3x\,\mathcal{H}$$