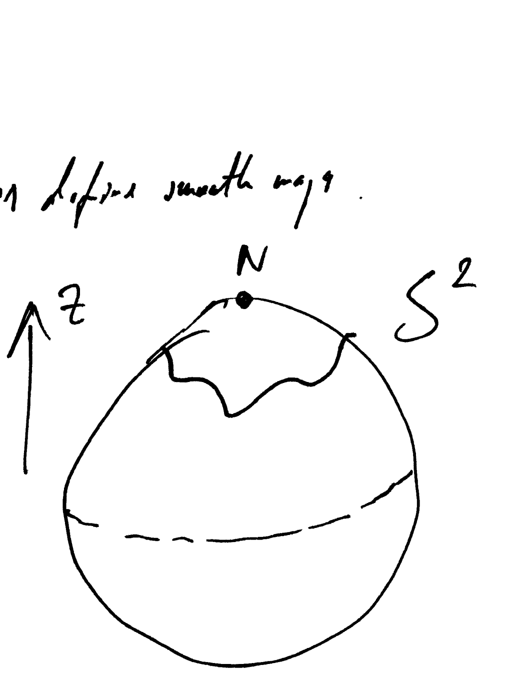
Lie groups

A Lie group G is a group and a manifold. Grp operations define smooth maps.

$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$



coordinate patch near N $(0, 0, +r)$

solve for $z = \pm\sqrt{r^2 - x^2 - y^2}$

(x,y) good coordinates near N

- dimension of G, $\dim(G)$, is dimension of manifold $\mathcal{M}(G)$

introduce coordinates $\{\theta^i\}$ $i = 1, \dots, D = \dim(G)$

in patch P containing e

group elements (near e)

$g = g(\theta) \in G$ set $g(0) = e$

- group multiplication

$g(\theta)g(\theta') = g(\varphi) \in G$ corresponds to a smooth map $G \times G \to G$

in coordinates

$\varphi^i = \varphi^i(\theta, \theta')$ continuous, differentiable $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$

- group inversion

also defines a smooth map $G \to G$

$\forall\, g(\theta) \in G$, $\exists\, g^{-1}(\theta) = g(\tilde\theta) \in G$

$g(\theta)g(\tilde\theta) = g(\tilde\theta)g(\theta) = e$

in coordinates $\tilde\theta^i = \tilde\theta^i(\theta)$ continuous, differentiable

Example $G = (\mathbb{R}^D, +)$

"multiplication" $\underline{x}'' = \underline{x} + \underline{x}' \quad \forall\ \underline{x}, \underline{x}' \in \mathbb{R}^D$

"inverse" $\underline{x}^{-1} = -\underline{x} \quad \forall\ \underline{x} \in \mathbb{R}^D$

Matrix groups

Let $Mat_n(F)$ denote set of $n \times n$ matrices in $F = \mathbb{R}$ or $\mathbb{C}$

- matrix multiplication is closed and associative

  unit element $e = \mathbb{1}_n \in Mat_n(F)$

$Mat_n(F)$ not a group as M^{-1} does not exist for all $M \in Mat_n(F)$

- Define general linear group

$$GL(n, F) = \{ M \in Mat_n(F) \,;\, \det M \neq 0 \}$$

special linear groups

$$SL(n, F) = \{ M \in GL(n, F) : \det M = 1 \}$$

Closure follows from, $\det(M_1 M_2) = \det(M_1) \det(M_2) \quad \forall M_1, M_2 \in GL(n, F)$

- Less obvious $GL(n, F)$ and $SL(n, F)$ are Lie groups

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}$$

$e = \mathbb{1} \longleftrightarrow a = d = 1, \quad b = c = 0$

near e solve $d = \frac{1 + bc}{a}$ $\quad a, b, c$ good coordinates near e

$SL(2, \mathbb{R})$ Lie group $\quad D(SL(2, \mathbb{R})) = 3$

$D(SL(n, \mathbb{R})) = n^2 - 1, \quad D(SL(n, \mathbb{C})) = 2n^2 - 2$

$\dim(GL(n, \mathbb{R})) = n^2 \quad \dim(GL(n, \mathbb{C})) = 2n^2$

- Subgroup H of G is a subset that is also a group

If H is also a smooth manifold, H is a Lie subgroup of G