$dt = dv - \frac{dr}{1-\frac{2M}{r}}$, $ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega_2^2$ (†)

We took the Schw. $r > 2M$, $(t, r, \theta, \phi) \to (v, r, \theta, \phi)$ by looking at (†), we see we can extend down to $r < 2M$. All metric components finite @ $r = 2M$ and $\det g_{\mu\nu} = -r^4 \sin^2\theta$

Ingoing geodesics:

$\frac{dr}{d\tau} = -1$ and constant v. This means that radial geodesics reach $r = 0$ in finite time.

First non-trivial scalar:

$R_{ab} - \frac{R}{2} g_{ab} = 0 \Rightarrow R_{ab} = 0$

$R = 0$

$R^{ab} R_{ab} = 0$

$R^{abcd} R_{abcd} = \frac{48M^2}{r^6}$ ← diverges as $r \to 0$

Curvature singularity.

Killing field for $r > 2M$

$K = \frac{\partial}{\partial t} = \frac{\partial x^\mu}{\partial v}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v}$ $\quad K^2 = -\left(1 - \frac{2M}{r}\right)$

$K^2 < 0$, $r > 2M$ (timelike)

$K^2 > 0$, $r < 2M$ (spacelike)

Finkelstein diagram

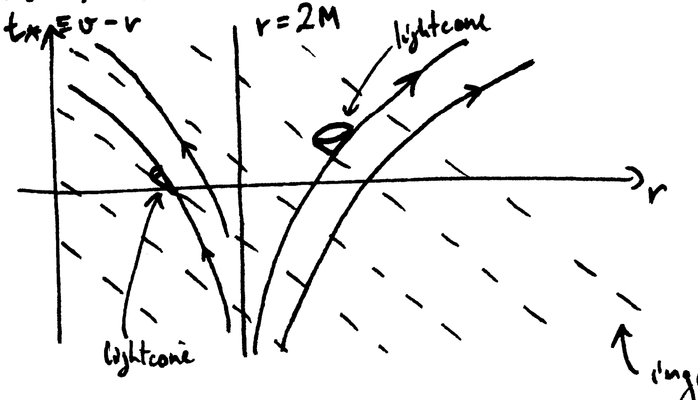
This is good for visualising what geodesics do.

- Ingoing : $\frac{dr}{d\tau} = -1$, v is const
- Outgoing: $t - r_* = \text{const}$

For $r > 2M$,

$$t - r_* = v - 2r_* = \text{const} \Rightarrow v = 2r + 4M \log|r/2M - 1| + \text{const}$$

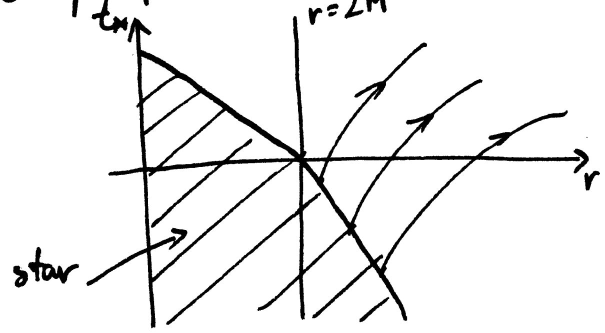
For $r < 2M$, have to do the calculation in (v, r, θ, ϕ), to get some result.



Black hole

- A region of spacetime for which it is impossible to send signals to infinity.

Collapse of star:



$\Delta t = \pi M$, $M = M_\odot \Rightarrow \Delta t = 10^{-5}$ s

Black hole region

Want to show that the region $r \le 2M$, (v, r, θ, ϕ) describes a black hole.

Def A vector is causal if it is null or timelike. (Assume the vector is non-zero.)
A curve is causal if its tangent vector is causal everywhere.

Def A spacetime is ~~orient~~ time-orientable if it admits a time-orientation: a causal vector field T^a.
Another vector field, X^a, is future-directed if it lies in the same lightcone as T^a.

Take Schwarzschild

$$K = \frac{\partial}{\partial t} = \frac{\partial}{\partial v} \qquad \begin{array}{ll} r > 2M : & K^2 < 0 \ \checkmark \\ r < 2M : & K^2 > 0 \ \times \end{array}$$

However, $\pm \partial/\partial r$, $g_{rr} = 0$ everywhere, $\pm \partial/\partial r$ is null, is causal.

$$K\left(-\frac{\partial}{\partial r}\right) = -1$$

Proposition Let $x^\mu(\lambda)$ be any future directed causal curve. Assume that $r(\lambda_0) \le 2M$, then $r(\lambda) \le 2M \ \forall \lambda > \lambda_0$.

Proof: the tangent vector is $V^\mu = \frac{dx^\mu}{d\lambda}$; $-\frac{\partial}{\partial r}$ and V^a are future directed causal vectors.

$$0 \ge \left(-\frac{\partial}{\partial r}\right)\cdot V = -\frac{\partial v}{\partial \lambda} \Rightarrow \frac{\partial v}{\partial \lambda} \ge 0 \qquad (*)$$

This means that v is increasing

$$V^2 \equiv V^a V_a = -\left(1 - \frac{2M}{r}\right)\dot{v}^2 + 2\dot{v}\dot{r} + r^2\left(\frac{d\Omega}{d\lambda}\right)^2 \quad \text{where } \left(\frac{d\Omega}{d\lambda}\right)^2 = \dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 .$$

From the above,

$$-2\dot{v}\dot{r} = -V^2 + \left(\frac{2M}{r} - 1\right)\dot{v}^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2 \qquad (**)$$

For $r < 2M$, everything on rhs is non-negative.
If $r < 2M$, then $\dot{v}\dot{r} \le 0$. Assume $\dot{r} \equiv \frac{dr}{d\lambda} > 0 \Rightarrow \dot{v} = 0$

From $(**)$, $V^2 = 0$ and $\left(\frac{d\Omega}{d\lambda}\right) = 0$

$V^\mu = \frac{dx^\mu}{d\lambda}$, the only non-zero component is $V^r = \frac{dr}{d\lambda} \ge 0$, V is a positive multiple of $\frac{\partial}{\partial r}$, is past directed. #

So we must have $\frac{dr}{d\lambda} \le 0$ if $r \le 2M$.