Heisenberg Picture

We now have a spinor op @ each x^μ

$$\psi(x) = \psi(\underline{x}, t) \qquad \text{satisfying} \quad \frac{\partial \psi}{\partial t} = i[H, \psi]$$

which is solved by:

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^s u_p^s e^{-ip\cdot x} + c_p^{s\dagger} v_p^{s} e^{ip\cdot x} \right]$$

$$\Rightarrow \psi^\dagger(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^{s\dagger} u_p^{s\dagger} e^{ip\cdot x} + c_p^{s} v_p^{s\dagger} e^{-ip\cdot x} \right]$$

$$iS_{\alpha\beta}(x-y) = \{ \psi_\alpha(x), \bar{\psi}_\beta(y) \}$$ or dropping indices:

$$iS(x-y) = \{ \psi(x), \bar{\psi}(y) \}$$ where we should remember that S is a 4×4 matrix $\alpha\beta$

Substitute $\psi, \bar{\psi}$ in

$$iS(x-y) = \sum_{s,r} \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \Big(\{ b_p^s, b_q^{r\dagger} \} u_p^s \bar{u}_q^r e^{-i(p\cdot x - q\cdot y)}$$

$$+ \{ c_p^{s\dagger}, c_q^{r} \} v_p^s \bar{v}_q^r e^{i(p\cdot x - q\cdot y)} \Big)$$

since $\{ b^{(\dagger)}, c^{(\dagger)} \} = 0$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(u_p^s \bar{u}_p^s e^{-ip\cdot(x-y)} + v_p^s \bar{v}_p^s e^{ip(x-y)} \right)$$

use $\sum_s u_p^s \bar{u}_p^s = \not{p} + m$ etc

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[(\not{p} + m) e^{-ip\cdot(x-y)} + (\not{p} - m) e^{ip\cdot(x-y)} \right]$$

[cf. $\mathbb{R}$ scalar propagator $D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$]

$$= (i\not{\partial}_x + m) D(x-y) - (i\not{\partial}_x + m) D(y-x)$$

tells you the coordinate

Hence $iS(x-y) = (i\not{\partial}_x + m)[D(x-y) - D(y-x)]$

Comments

(i) $(x-y)^2 < 0$ ie spacelike separation $[D(x-y) - D(y-x)] = 0$

In bosonic theory we made a big deal of this, since it ensured that $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$ which we interpreted as showing the boson theory is causal.

For the spinor field, $\{\psi_\alpha(x), \bar{\psi}_\beta(y)\}$ for $(x-y)^2 < 0$, so what about causality?

The best we can say is that all our observables are bilinear in fermions, e.g.

$H = \int \psi^\dagger (-i\gamma^i \partial_i + m)\psi$, i.e. these observables will commute for spacelike separations.

(ii) At least away from singularities:

$$(i\not{\partial}_x - m) S(x-y) = 0 \quad \text{since} = (i\not{\partial}_x - m)(i\not{\partial} + m)[D(x-y) - D(y-x)]$$

$$= -\underbrace{(\partial_x^2 + m^2)}_{=0 \text{ using } p^2 = m^2}[\dots]$$

The Feynman Propagator

By a similar calculation to that above, we can determine the <u>vacuum expectation value</u> (vev).

$$\langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} + m)_{\alpha\beta} e^{-ip\cdot(x-y)}$$

and

$$\langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} - m)_{\alpha\beta} e^{ip\cdot(x-y)}$$

We may now define the Feynman propagator as the time ordered product

$$(S_F)(x-y) = \langle 0 | T \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle = \begin{cases} \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle & : x^0 > y^0 \\ -\langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle & : y^0 > x^0 \end{cases}$$

Note the minus sign! This is necessary when $(x-y)^2 < 0$:

$\nexists$ way to determine whether $x^0 \lessgtr y^0$. Then, $\{\psi(x), \bar{\psi}(y)\} = 0$ which means that

T as defined is Lorentz inv't -

the same occurs for strings of ops inside $T(\dots)$ - fermionic ops all anticommute (just as bosonic ops commute)

For normal ordered products, we have the same behaviour: fermionic ops anti-commute so

$:\psi_1\psi_2: = -:\psi_2\psi_1:$. Then the Feynman prop appears in Wick's theorem as the contraction

$$\overbrace{\psi(x)\bar{\psi}(y)} = T(\psi(x)\bar{\psi}(y)) - :\psi(x)\psi(y): = S_F(x-y)$$

Wick's theorem goes through as before, as long as we remember the minus signs, e.g.

$$:\overbrace{\psi_1 \psi_2 \psi_3} \psi_4: = -:\overbrace{\psi_1 \psi_3} \psi_2 \psi_4: = -\overbrace{\psi_1 \psi_3} :\psi_2 \psi_4:$$

For S_F, we have (again) a 4-momentum $\int$ expression

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \quad \text{which satisfies}$$

$$(i\not{\partial}_x - m) S_F(x-y) = i\delta^4(x-y) \text{, so } S_F \text{ is a Green's } f^n \text{ for the Dirac eq}^n.$$

Yukawa Theory

The interactions between a Dirac fermion and a $\mathbb{R}$ scalar field are governed by the Yukawa interaction:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\phi\bar{\psi}\psi$$

μ mass of ϕ $\qquad$ m mass of fermion

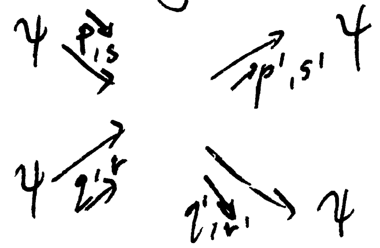
- the full version of the "baby scalar Yukawa" theory we looked at before. Mass dims:

kinetic term $\Rightarrow$ $[\psi] = [\bar{\psi}] = 3/2$ $\Rightarrow$ $[\lambda] = 0$

(aside: could have written $\mathcal{L}_{int} = -\lambda\phi\bar{\psi}\gamma^5\psi$)

Now we look at a proper treatment of "nuclear scattering from sec 3".

e.g. fermion scattering $\psi\psi \to \psi\psi$



$|i\rangle = \sqrt{2E_p}\sqrt{2E_q}\, b_p^{s\dagger}\, b_q^{r\dagger}|0\rangle$

$|f\rangle = \sqrt{2E_{p'}}\sqrt{2E_{q'}}\, b_{p'}^{s'\dagger}\, b_{q'}^{r'\dagger}|0\rangle$

where we must be careful since $\{b_p^{s\dagger}, b_q^{r\dagger}\} = 0$.

In particular, $\langle f| = +\sqrt{2E_{p'}}\sqrt{2E_{q'}}\,\langle 0|\, b_{q'}^{r'}\, b_{p'}^{s'}$

Then we can look at $O(\lambda^2)$ terms in $\langle f|(S-1)|i\rangle$. We have

$$\frac{(-i\lambda)^2}{2}\int d^4x_1\, d^4x_2\, T\left[\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\psi(x_2)\phi(x_2)\right]$$

- all fields in the int^n picture

The contribution to scattering comes from the contraction

$:\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2): \overline{\phi(x_2)\phi(x_1)}$ where the ψ annihilate $|i\rangle$ and $\bar{\psi}$ create $\langle f|$