Nambu - Goto action

$$I = -T \int dA \qquad T = \frac{1}{2\pi\alpha'}$$

α' - only length scale in string theory

T - string tension

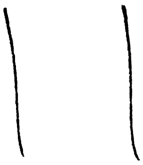
Take path integral

$$Z = \int D[\text{physical configurations}]\, e^{iI}$$

all two dimensional surfaces

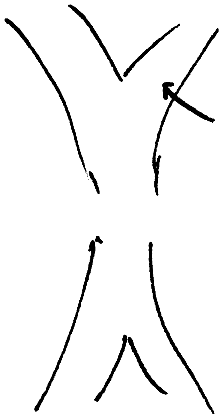
Contrast with quantum field theory - interactions are added in by hand; included from the outset

2d surfaces:

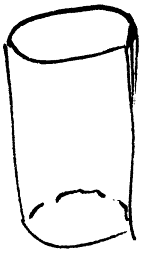


$\uparrow t$

open string

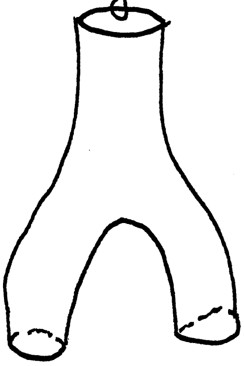


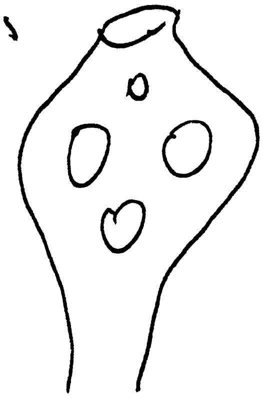
open string can split



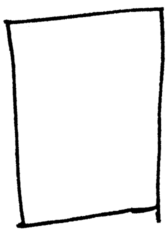
closed string

Closed string interactions





open strings can join to make closed strings



$X^a(\sigma, \tau)$

τ — time coordinate, σ — spatial coordinate } on the string world-sheet

Can always make this choice because the worldsheet is taken to be Lorentzian.

Induced metric is $H_{\mu\nu} = \partial_\mu X^a \partial_\nu X^b \eta_{ab}$ $\qquad \partial_\mu = \frac{\partial}{\partial \xi^\mu}$ $\qquad \xi^\mu = (\tau, \sigma)$

$$I = -T \int d^2\xi \sqrt{-\det H_{\mu\nu}} \qquad H_{\mu\nu} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}$$

$\partial_\tau \equiv \dot{}$

$\partial_\sigma \equiv {}'$

Y^2 means $Y^a Y^b \eta_{ab}$

$$I = -T \int d\sigma\, d\tau \sqrt{(\dot{X}\cdot X')^2 - X'^2 \dot{X}^2}$$

This is a desperately inconvenient form of the action.

Polyakov Action (invented by Howe + Tucker)

$$I = -\frac{1}{2} T \int d^2\xi \sqrt{-h}\, h^{\mu\nu} \partial_\mu X^a \partial_\nu X^b \eta_{ab}$$

Involves an auxiliary metric $h_{\mu\nu}$, X^a

Contains no derivatives of $h_{\mu\nu}$, so h is an auxiliary field

$$\frac{1}{\sqrt{-h}} \frac{4\pi}{\alpha'} \underbrace{\frac{\delta I}{\delta h^{\mu\nu}}} = T_{\mu\nu} \quad \Leftarrow \text{Energy-momentum tensor}$$

Equation of motion for $h_{\mu\nu}$: $\quad \delta I / \delta h^{\mu\nu} = 0$

$$T_{\mu\nu} = \left(\partial_\mu X^a \partial_\nu X^b - \frac{1}{2} h_{\mu\nu} h^{\rho\sigma} \partial_\rho X^a \partial_\sigma X^b\right) \eta_{ab} = 0$$

X^a are a set of world-sheet scalar fields on the world-sheet.

$h^{\mu\nu} T_{\mu\nu} = 0 \Leftarrow$ trace free energy-momentum tensor

Hence $T_{\mu\nu}$ (3 indep components in general) only has two.

$$h_{\mu\nu} = \frac{2 \partial_\mu X^a \partial_\nu X^b \eta_{ab}}{\partial_\rho X^c \partial_\sigma X^d \eta_{cd} h^{\rho\sigma}} = \frac{2 H_{\mu\nu}}{\Phi} \quad \text{—} \quad \Phi = \partial_\rho X^c \partial_\sigma X^d \eta_{cd} h^{\rho\sigma}$$

conformal transformation : $h_{\mu\nu} \longrightarrow h'_{\mu\nu} = e^{\lambda(\xi)} h_{\mu\nu}$

L Weyl transformation

$H_{\mu\nu}$ is conformal to $h_{\mu\nu}$. Since $h_{\mu\nu}$ is an auxiliary field, substituting into the action changes nothing

$$I = -\frac{1}{2} T \int d^2\xi \; \Phi \; \sqrt{-H} \; 2/\Phi = -T \int d^2\xi \sqrt{-H} \qquad \text{as } \sqrt{-h} = \sqrt{-H}\,\frac{2}{\Phi}$$

So solving for $h_{\mu\nu}$ regenerates the Nambu-Goto action.

$$\delta I = -T \int d^2\xi \sqrt{-h}\, h^{\mu\nu} \partial_\mu \delta X^a \partial_\nu X^b \eta_{ab} \qquad \text{Integrate by parts, call world-sheet } \Sigma$$

$$= -T \Big[\int_\Sigma d^2\xi \; \delta X^a \partial_\mu (\sqrt{-h}\, h^{\mu\nu} \partial_\nu X^b) \eta_{ab} + \int_{\partial\Sigma} d\tau \; \delta X^a \, n^\mu \partial_\mu X^b \eta_{ab} \Big]$$

$\uparrow$ unit normal to $\partial\Sigma$

Bulk term: $\frac{1}{\sqrt{-h}} \partial_\mu \left(\sqrt{-h}\, h^{\mu\nu} \partial_\nu X^a\right) = 0$

D_μ is the covariant derivative wrt (the symmetric metric connection) of $h_{\mu\nu}$

$D^2 X^a = 0 \Leftarrow$ Wave equation for X^a

$\delta X^a n^\mu \partial_\mu X^b \eta_{ab} = 0$ Two essential ways of doing this:

$n^\mu \partial_\mu X^b = 0 \quad \leftarrow$ Neumann bc. Gradient of $X \perp$ the boundary must vanish. N

$\delta X^a = 0 \quad \leftarrow$ Dirichlet bc. Boundary is fixed at X^a. D

Boundary conditions need to be applied at

1) each boundary – the two possible boundaries of the open string
2) choice of N/D in every direction on each boundary

Symmetries of the Polyakov action

$$I = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \underbrace{\sqrt{-h}\, h^{\mu\nu} \partial_\mu X^a \partial_\nu X^b \eta_{ab}}$$

$h_{\mu\nu}$ is a metric

X^a, η_{ab} are world-sheet scalars then this is a scalar density of Σ

$\xi^\mu \rightarrow \xi^\mu + \eta^\mu \quad \eta^\mu$ infinitesimal (coord transf)

Invariance of the line element of Σ gives

$$h_{\mu\nu} \longmapsto h_{\mu\nu} - \underline{D_\mu \eta_\nu - D_\nu \eta_\mu}$$

$$X^a \longmapsto X^a + \eta^\mu D_\mu X^a$$

$$\eta_{ab} \longmapsto \eta_{ab}$$

Polyakov action invariant under world-sheet coordinate transformations.

In variance under Weyl transformations

$$h_{\mu\nu} \mapsto e^{\lambda} h_{\mu\nu}$$

$$h^{\mu\nu} \mapsto e^{-\lambda} h^{\mu\nu}$$

$$\sqrt{-h} \mapsto e^{\lambda} \sqrt{-h}$$

$$X^{a} \mapsto X^{a}$$

$$\eta_{ab} \mapsto \eta_{ab}$$

Invariant under conformal transformations