$$[H^i, H^j] = 0 \qquad i,j = 1, \dots, r$$

$$[H^i, E^\alpha] = \alpha^i E^\alpha \qquad \forall \alpha \in \Phi$$

$$\begin{aligned}
[E^\alpha, E^\beta] &= N_{\alpha,\beta} E^{\alpha+\beta} && \alpha+\beta \in \Phi \\
&= K(E^\alpha, E^{-\alpha}) H^\alpha && \alpha+\beta = 0 \\
&= 0 && \text{otherwise} \qquad H^\alpha = (K^{-1})_{ij} \alpha^j H^i
\end{aligned}$$

~~The~~

Brackets of $H^\alpha \in \mathfrak{h}$, $\forall \alpha, \beta \in \Phi$

$\alpha, \beta \in \mathfrak{h}^*$

$(\alpha,\beta) = \alpha^i \beta^j (K^{-1})_{ij}$ ← inner product on $\mathfrak{h}^*$

$$[H^\alpha, E^\beta] = (K^{-1})_{ij} \alpha^i [H^j, E^\beta] = (K^{-1})_{ij} \alpha^i \beta^j E^\beta$$
$$= (\alpha,\beta) E^\beta$$

$\forall \alpha \in \Phi$ we define,

$$e^\alpha = \frac{\sqrt{2}}{\left((\alpha,\alpha) K(E^\alpha, E^{-\alpha})\right)^{1/2}} E^\alpha$$

* important that $(\alpha,\alpha) \neq 0 \quad \forall \alpha \in \Phi$

$$h^\alpha = \frac{2}{(\alpha,\alpha)} H^\alpha$$

roots $= d - r$ ($> r$ typically)

$\forall \alpha, \beta \in \Phi$

$$[h^\alpha, h^\beta] = 0$$

$$[h^\alpha, e^\beta] = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e^\beta$$

$$[e^\alpha, e^\beta] = \begin{cases} n_{\alpha,\beta} e^{\alpha+\beta} & \alpha+\beta \in \Phi \\ h^\alpha & \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{L}_{\mathbb{C}}(SU(2))$ subalgebras

$\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

For each pair $\pm\alpha \in \Phi$ we have an $\mathcal{L}_{\mathbb{C}}(SU(2))$ subalgebra with basis

$\{h^\alpha, e^\alpha, e^{-\alpha}\} \Rightarrow [h^\alpha, e^{\pm\alpha}] = \pm 2 e^{\pm\alpha}$

$[e^{+\alpha}, e^{-\alpha}] = h^\alpha$ ⟶ "$\mathfrak{sl}(2)_\alpha$"

consequence: <u>root strings</u>

For $\alpha, \beta \in \Phi$ define "α-string passing through β" as set of roots from $\beta + \rho\alpha$, $\rho \in \mathbb{Z}$

$$S_{\alpha,\beta} = \{\beta + \rho\alpha \in \Phi, \ \rho \in \mathbb{Z}\}$$

corresponding vector subspace

$$V_{\alpha,\beta} = \mathrm{span}_{\mathbb{C}} \{ e^{\beta+\rho\alpha} \; , \; \beta+\rho\alpha \in S_{\alpha,\beta} \}$$

consider action of $sl(2)_\alpha$ on $V_{\alpha,\beta}$

$$[h^\alpha , e^{\beta+\rho\alpha}] = \frac{2(\alpha,\beta+\rho\alpha)}{(\alpha,\alpha)} e^{\beta+\rho\alpha}$$

$$= \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2\rho \right) e^{\beta+\rho\alpha} \qquad (*)$$

$$[e^{\pm\alpha}, e^{\beta+\rho\alpha}] \propto e^{\beta+(\rho\pm1)\alpha} \quad \text{if } \beta+(\rho\pm1)\alpha \in \Phi \quad \in V_{\alpha,\beta}$$

$$= 0 \quad \text{otherwise}$$

$\Rightarrow$ $V_{\alpha,\beta}$ is invariant under $sl(2)_\alpha$

$\Rightarrow$ $V_{\alpha,\beta}$ is repn space for some repn R of $sl(2)_\alpha$

$\Rightarrow$ $(*)$ <u>weight set of R</u>

$$S_R = \left\{ \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2\rho \; , \; \beta+\rho\alpha \in \Phi \; , \; \rho \in \mathbb{Z} \right\} \qquad (\dagger)$$

$V_{\alpha,\beta}$ is finite dimensional, roots $\beta+\rho\alpha \in \Phi$ are non-degenerate,

- finite dimensional
- irreducible

$\Rightarrow R = R_\Lambda$, for some highest weight $\Lambda \in \mathbb{Z}$ $(\Lambda \geq 0)$

$$\Rightarrow S_R = \{ -\Lambda, -\Lambda+2, \; \dots, \; +\Lambda \} \qquad (\ddagger)$$

Allowed values of ρ (equating $\dagger$ and $\ddagger$)

$$\rho = n \in \mathbb{Z} \qquad n_- \leq n \leq n_+ \; , \; n_\pm \in \mathbb{Z}$$

with $\; -\Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_-$

$$+\Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+$$

adding

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-) \in \mathbb{Z} \qquad (\;)$$

Allowed roots for an ~~[illegible]~~ string

$$S_{\alpha,\beta} = \{ \beta + n\alpha , n \in \mathbb{Z} , n_- \leq n \leq n_+ \}$$

In Cartan - Weyl basis

$$[H^i , E^\delta] = \delta^i E^\delta \qquad \begin{matrix} i = 1, \dots, r \\ \forall \delta \in \Phi \end{matrix}$$

$$K^{ij} = K(H^i, H^j) = \frac{1}{N} \operatorname{Tr} [\mathrm{ad}_{H^i} \circ \mathrm{ad}_{H^j}]$$

$$= \frac{1}{N} \sum_{\delta \in \Phi} \delta^i \delta^j$$

$\forall \alpha, \beta \in \Phi$

$$(\alpha, \beta) = \alpha^i \beta^j (K^{-1})_{ij} = K^{ij} \alpha_i \beta_j \qquad \beta_j \overset{\text{def}^n}{=} (K^{-1})_{jk} \beta^k$$

$$= \frac{1}{N} \sum_{\delta \in \Phi} \alpha_i \delta^i \delta^j \beta_j = \frac{1}{N} \sum_{\delta \in \Phi} (\alpha, \delta)(\beta, \delta)$$

from ($\mathcal{S}$)

$$R_{\alpha,\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

$$\Rightarrow \frac{2}{(\beta, \beta)} R_{\alpha,\beta} = \frac{1}{N} \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta} \in \mathbb{R}$$

$$\Rightarrow (\alpha, \beta) \in \mathbb{R} \qquad \forall \alpha, \beta \in \Phi$$