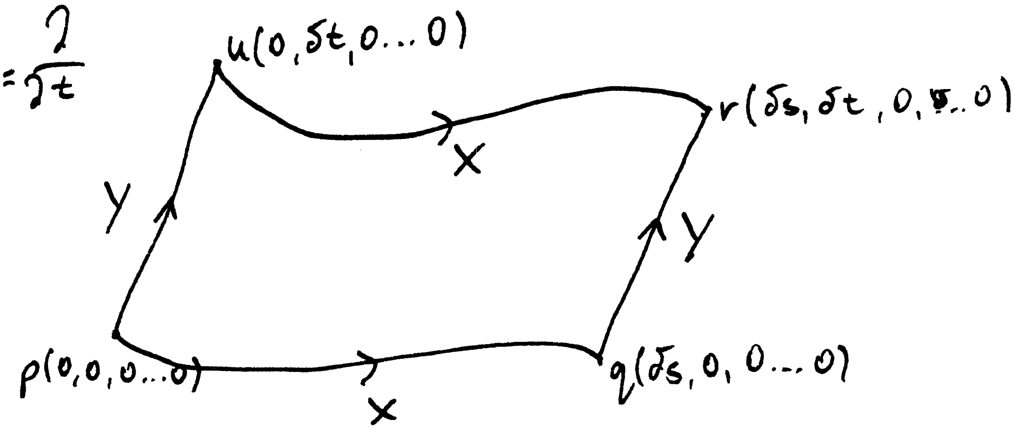
Relation between Riemann tensor and parallel transport

Let X, Y be the commuting vector fields ; $[X,Y] = 0$ on M

Frobenius thm $\Rightarrow \exists$ coordinate system $(s, t, \dots)$ on M such that

$$X = \frac{\partial}{\partial s} , \quad Y = \frac{\partial}{\partial t}$$



Let $Z_p \in T_pM$, parallel transport it twice

- along pqr to $Z_r \in T_r M$
- along pur to $Z'_r \in T_r M$

Compute the difference $\Delta Z_r = Z'_r - Z_r$ for a torsion-free connection.

[Taylor series . See notes]

$$(\Delta Z_r)^\mu = (Z'_r)^\mu - (Z_r)^\mu = \left(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma \right)\Big|_r \delta s\, \delta t + \mathcal{O}(\delta s^2, \delta t^2)$$

Rewrite as

$$R^a{}_{bcd} Z^b X^c Y^d \big|_r = \lim_{\delta s, \delta t \to 0} \frac{\Delta Z^a_r}{\delta s\, \delta t} \quad \Big\} \text{ allowed as LHS \& RHS are tensors}$$

Symmetries of the Riemann tensor

* From definition $R^a{}_{b(cd)} = 0$

* For torsion free ∇ , $R^a{}_{[bcd]} = 0$

Proof Use normal coordinates at p [enough, as this is an algebraic property of a tensor]

$$\Gamma^\mu_{\nu\rho}(p) = 0 \longrightarrow R^\mu{}_{\nu\rho\sigma}\big|_p = \left(\partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} \right)\big|_p$$

$$[\rho\nu\sigma] \longrightarrow 0 \quad (\text{symmetric } \Gamma^\mu_{\nu\rho}) \quad \square$$

* Bianchi identity : If ∇ is torsion free, then $R^a{}_{b[cd;e]} = 0$.

<u>Proof</u> (again) normal coordinates

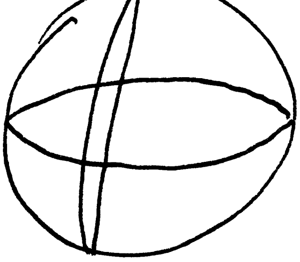
$$\nabla_\tau R^\mu{}_{\nu\rho\sigma}\big|_p = \partial_\tau R^\mu{}_{\nu\rho\sigma}\big|_p \qquad \left\}\begin{array}{l} \text{Formally} \\ R = \partial\Gamma - \Gamma\Gamma \xrightarrow{\text{normal}} \partial\Gamma \\ \partial R = \partial\partial\Gamma + 2\Gamma\partial\Gamma \xrightarrow{\text{normal}} \partial\partial\Gamma \end{array}\right.$$

so $$R^\mu{}_{\nu\rho\sigma;\tau}\big|_p = (\partial_\tau\partial_\rho\Gamma^\mu{}_{\nu\sigma} - \partial_\tau\partial_\sigma\Gamma^\mu{}_{\nu\rho})\big|_p$$

$$[\rho\sigma\tau] \to 0 \qquad \square$$

2.5 Geodesic deviation equations

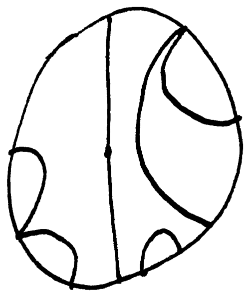
In $\mathbb{R}^n$ (flat) initially parallel lines remain parallel. For curved spaces no notion of parallel, so instead ask: given two geodesics initially close, do they remain close?



S^2

great circles intersect

Gauss curvature $K > 0$



$K < 0$

Poincare disc

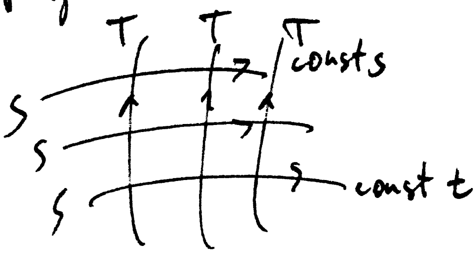
$$g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

$$x^2 + y^2 < 1$$

<u>Def</u> (M, ∇) mfd with torsion free connection. A 1-param family of geodesics in a map $\gamma : I \times I' \to M$, where

- I, I' open intervals in $\mathbb{R}$
- for fixed $s \in I$, $\gamma(s,t)$ is a geodesic with affine parameter t.
- Map $(s,t) \to \gamma(s,t)$ is a smooth 1:1 map, with a smooth inverse,

Family of geodesics $\equiv$ smooth surface $\Sigma \in M$



T = tangent to geodesics

So deviation vector y^μ coords on M, on Σ $x^\mu(s,t)$

$$S^\mu = \frac{\partial x^\mu}{\partial s} \quad ; \quad S = S^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial s}$$
$$T^\mu = \frac{\partial x^\mu}{\partial t} \quad ; \quad T = T^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}$$

extend (s,t) to a coordinate system on M, defined in a neighbourhood of $\Sigma \subset M$.

$[T,S] = 0$; $\delta s \cdot \nabla_T S$ = relative velocity

$\nabla_T \nabla_T S$ = — " — acceleration

<u>Prop</u> $\nabla_T \nabla_T S = R(T,S)T$ torsion free $\nabla_T S - \nabla_S T = [T,S] = 0$

<u>Proof</u> $\nabla_T \nabla_T S = \nabla_T \nabla_S T = \nabla_S \underbrace{\nabla_T T}_{0} + R(T,S)T$ } definition of R

Geodesic deviation eq.

0 T tangent to geodesics. $\square$

In abstract indices

$$T^c \nabla_c (T^b \nabla_b S^a) = R^a{}_{bcd} T^b T^c S^d$$

Try to "measure LHS $\to R^a{}_{(bc)d}$.

Claim: this determines the full curvature by [exercise]

$$R^a{}_{bcd} = \frac{2}{3}\left(R^a{}_{(bc)d} - R^a{}_{(bd)c}\right)$$

Riemann = tidal forces