Closed testing procedure:

Reject H_I iff $\forall\, J \supseteq I$, $\phi_J = 1$

Example

$$\underline{H_{1234}}$$

$$\underline{H_{123}} \quad \underline{H_{124}} \quad \underline{H_{134}} \quad \underline{H_{234}}$$

$$\underline{H_{12}} \quad \underline{H_{13}} \quad \underline{H_{14}} \quad \underline{H_{23}} \quad H_{24} \quad H_{34}$$

$$\underline{H_1} \quad \underline{H_2} \quad H_3 \quad H_4$$

Underlined means "rejected by local test"

- H_1 is rejected by closed testing
- H_2 is not rejected
- H_{23} is rejected

Theorem 38 The closed testing procedure makes no false rejections with probability $1-\alpha$. In particular, FWER $\leq \alpha$.

Proof In order for there to be a false rejection, we must have (~~rejected H_{I_0}~~) $\phi_{I_0} = 1$ but $\mathbb{P}(\phi_{I_0} = 1) \leq \alpha$:D

Different choices for the local tests will yield different multiple testing procedures. Holm's procedure uses ϕ_I as the Bonferroni test

$$\phi_I = \begin{cases} 1 & \text{if } \min_{i \in I} p_i \leq \frac{\alpha}{|I|} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}_{H_I}(\phi_I = 1) = \mathbb{P}_{H_I}\left(\bigcup_{i \in I} \left\{p_i \leq \frac{\alpha}{|I|}\right\}\right) \leq \sum_{i \in I} \underbrace{\mathbb{P}_{H_I}\left(p_i \leq \frac{\alpha}{|I|}\right)}_{\leq \alpha/|I|} \leq \alpha \quad \checkmark$$

This is equivalent to the following procedure, step-down procedure:

Order the p-values $p_{(1)} \leq p_{(2)} \leq \dots \leq p_{(m)}$

1. If $p_{(1)} \leq \frac{\alpha}{m}$, reject $H_{(1)}$ and go to step 2, else accept $H_{(1)}, \dots, H_{(m)}$.

i. If $p_{(i)} \leq \frac{\alpha}{m-i+1}$, reject $H_{(i)}$ and go to step $(i+1)$, else accept $H_{(i)}, \dots, H_{(m)}$.

m. If $p_{(m)} \leq \alpha$ reject $H_{(m)}$, else accept $H_{(m)}$.

4.2 False Discovery Rate

Many more modern multiple testing procedures attempt to control the false discovery rate (FDR).

$$\text{FDR} = \mathbb{E}(\text{FDP})$$

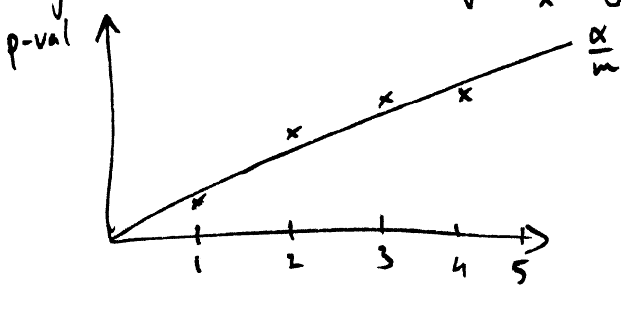
$$\text{FDP} = \frac{N_{01}}{\max(R, 1)} \quad (R = 0 \Rightarrow \text{FDP} = 0)$$

R - number of rejections

The Benjamini - Hochberg procedure (B-H) attempts to control the FDR at level α and works as follows.

Let $\hat{k} = \max\left\{i : p_{(i)} \leq \frac{i\alpha}{m}\right\}$

Reject $H_{(1)}, \dots, H_{(\hat{k})}$ or reject no hypotheses if $\hat{k}$ not defined.



$\longrightarrow$ Reject $H_{(1)}, \dots, H_{(4)}$

Theorem 39 Suppose that $p_{(i)} : i \in I_0$ are independent and independent of $\{p_i : i \notin I_0\}$.
Then the B-H procedure has

$$\mathrm{FDR} \le \frac{\alpha m_0}{m} \le \alpha.$$

Proof

$$\mathrm{FDR} = \mathbb{E}\left(\frac{N_{01}}{\max(R,1)}\right) = \sum_{r=1}^{m} \mathbb{E}\left(\frac{N_{01}}{r} \mathbb{1}_{\{R=r\}}\right)$$

$$= \sum_{r=1}^{m} \frac{1}{r} \mathbb{E}\left(\sum_{i\in I_0} \mathbb{1}_{\{p_i \le \frac{\alpha r}{m}\}} \mathbb{1}_{\{R=r\}}\right)$$

For each $i \in I_0$, let R_i the number of rejections of a modified B-H applied to

$$p^{\setminus i} = \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_m\}$$

with cut-off $\hat{k}_i = \max\left\{ j : p^{\setminus i}_{(j)} \le \frac{(j+1)\alpha}{m} \right\}$.

For $r = 1, \dots, m$ and $i \in I_0$ observe that

$$\left\{p_i \le \frac{\alpha r}{m}, R = r\right\} = \left\{ p_i \le \frac{\alpha r}{m}, p_{(r)} \le \frac{\alpha r}{m}, p_{(s)} > \frac{\alpha s}{m} \ \forall s > r \right\}$$

$$= \left\{ p_i \le \frac{\alpha r}{m}, p^{\setminus i}_{(r-1)} \le \frac{\alpha r}{m}, p^{\setminus i}_{(s-1)} > \frac{\alpha s}{m} \ \forall s > r \right\}$$

$$= \left\{ p_i \le \frac{\alpha r}{m}, R_i = r-1 \right\}$$

independent

$$\mathrm{FDR} = \sum_{r=1}^{m} \frac{1}{r} \cancel{\mathbb{E}} \sum_{i\in I_0} \underbrace{\mathbb{P}\left(p_i \le \frac{\alpha r}{m}\right)}_{\le \frac{\alpha r}{m}} \mathbb{P}(R_i = r-1)$$

$$\le \frac{\alpha}{m} \sum_{i\in I_0} \underbrace{\sum_{r=1}^{m} \mathbb{P}(R_i = r-1)}_{1} = \frac{\alpha m_0}{m} \le \alpha \qquad \square$$