Standard Model
Example sheet 2

1. $M \in \mathbb{C}^{N\times N}$

$$\mathcal{L} = \mathrm{Tr}(\partial^\mu M^\dagger \partial_\mu M) - \tfrac{1}{2}\lambda \mathrm{Tr}(M^\dagger M M^\dagger M) - k\,\mathrm{Tr}(M^\dagger M) \qquad , \lambda > 0$$

(i) Symmetry group $(U(N)\times U(N))/U(1)$

$M \mapsto A M B^{-1}$ $\qquad A, B \in U(N)$ $\qquad , \; U(1) : \; A = B = e^{i\theta} I$.

(U(1) is not a symmetry of the Lagrangian but instead leaves the field unchanged.)
Show using 1 and Schur's Lemma.

$$M^\dagger \mapsto (B^{-1})^\dagger M^\dagger A^\dagger = B M^\dagger A^{-1}$$

$$\mathcal{L} \mapsto \mathrm{Tr}(\partial^\mu (B M^\dagger A^{-1}) \partial_\mu (A M B^{-1})) - \tfrac{1}{2}\lambda \mathrm{Tr}(B M^\dagger A^{-1} A M B^{-1} B M^\dagger A^{-1} A M B^{-1})$$
$$- k\,\mathrm{Tr}(B M^\dagger A^{-1} A M B^{-1}) = \mathcal{L}$$

by using the fact that A, B are constant and cyclicity of Tr

If $k<0$,

$$V(M) = \tfrac{1}{2}\lambda \mathrm{Tr}((M^\dagger M)^2) + k\,\mathrm{Tr}(M^\dagger M) \longrightarrow \tfrac{1}{2}\lambda \mathrm{Tr}\left[(M^\dagger M - v^2 I)^2\right]$$

upto irrelevant constant term $\qquad v^2 = -\frac{k}{\lambda}$

Minimum using $\mathrm{Tr}(M^\dagger M) \geq 0$ and $\mathrm{Tr}(M^\dagger M M^\dagger M) \geq 0$

Hence the vacua (ground state) satisfy $\quad M_0^\dagger M_0 = v^2 I$.

~~Suppose $M_0 = vI$, then $M = vI + H$ and~~

$$\cancel{V(H) = \tfrac{1}{2}\lambda \mathrm{Tr}\left[((vI+H)^\dagger (vI+H) - v^2 I)^2\right]}$$
$$\cancel{= \tfrac{1}{2}\lambda \mathrm{Tr}\left[(v(H+H^\dagger) + H^\dagger H)^2\right]}$$

~~Define $S = \frac{1}{2}(H+H^\dagger)$, $S^\dagger = S$~~ $\qquad$ ~~$H = S+T$, $H^\dagger = S-T$~~

~~$T = \frac{1}{2}(H-H^\dagger)$, $T^\dagger = -T$~~

$$\cancel{V(S,T) = \tfrac{1}{2}\lambda \mathrm{Tr}\left[(2vS + (S-T)(S+T))^2\right]}$$
$$\cancel{= \tfrac{4v^2}{2}\lambda \mathrm{Tr}(S^2) + \tfrac{1}{2}\lambda \mathrm{Tr}(S^4) + \tfrac{1}{2}\lambda \mathrm{Tr}(T^4) + \tfrac{1}{2}\lambda \mathrm{Tr}([S,T]^2)}$$
$$\cancel{+ \tfrac{4v}{2}\lambda \mathrm{Tr}(S^3) - \tfrac{2v}{2}\lambda \mathrm{Tr}(ST^2 + T^2 S) + \tfrac{2v}{2}\lambda \mathrm{Tr}(S[S,T] + [S,T]S)}$$

$\cancel{-\frac{1}{2}\lambda \mathrm{Tr}(S^2T^2+T^2S^2)+\frac{1}{2}\lambda \mathrm{Tr}(S^2[S,T]+[S,T]S^2)-\frac{1}{2}\lambda \mathrm{Tr}(T^2[S,T]+[S,T]T^2)}$

~~Transformations where $B = e^{i\theta}I$ and $A\in U(N)$ transform between different vacua~~

For the invariant subgroup, $\quad G=(U(N)\times U(N))/U(1)$

$$H=\{(A,B)\ :\ AM_0B^{-1}=M_0\ ,\ (A,B)\in G\}$$

$$\Rightarrow AM_0 = M_0B$$

$$M_0^\dagger M_0 B = v^2 B = M_0^\dagger A M_0$$

$$B=\frac{1}{v^2}M_0^\dagger A M_0 \quad \text{or} \quad A=\frac{1}{v^2}M_0BM_0^\dagger$$

$\therefore\ H=U(N)/U(1)$ is the unbroken symmetry group

Indeed, writing $M=M_0+P$, under H

$$M\longmapsto AMB^{-1}=AM_0B^{-1}+APB^{-1}=M_0+APB^{-1}$$

$$M^\dagger\longmapsto M_0^\dagger+BP^\dagger A^{-1}$$

$$V(P)=\frac{1}{2}\lambda\mathrm{Tr}\left[(P^\dagger M_0+M_0^\dagger P+P^\dagger P)^2\right]$$

$$\longmapsto \frac{1}{2}\lambda\mathrm{Tr}\left[(BP^\dagger A^{-1}M_0+M_0^\dagger APB^{-1}+BP^\dagger PB^{-1})^2\right]$$

$$=\frac{1}{2}\lambda\mathrm{Tr}\left[B(P^\dagger M_0+M_0^\dagger P+P^\dagger P)^2B^{-1}\right]=V(P) \qquad \text{by cyclicity of Tr}$$

and as $A^{-1}M_0=A^{-1}AM_0B^{-1}=M_0B^{-1}$, $M_0^\dagger A=BM_0^\dagger A^{-1}A=BM_0^\dagger$.

Number of Goldstone modes is then

$$(\dim G-\dim H)=(n^2+n^2-1)-(n^2-1)=n^2$$

(ii) $\mathcal{L}\to\mathcal{L}+\mathcal{L}'$, $\quad \mathcal{L}'=h(\det M+\det M^\dagger)$

Then under

$$M\longmapsto AMB^{-1}$$

must also have

$$\mathcal{L}'\longmapsto h(\det A\det M\det B^{-1}+\det B\det M^\dagger\det A^{-1})=\mathcal{L}'$$

$\Rightarrow \det(AB^{-1})=1 \qquad (\text{before } |\det(AB^{-1})|=1)$

$A=\alpha A'$, $\alpha\in U(1)$, $A'\in SU(N)$
$B=\beta B'$, $\beta\in U(1)$, $B'\in SU(N)$
$\Rightarrow \alpha^N=\beta^N \ \therefore\ G'=SU(N)\times SU(N) \times \mathbb{Z}_N$

Hence, symmetry group ~~$G'=(U(N)\times U(N))/(U(1)\times U(1))$~~. Unbroken symmetry group ~~$\times U(1)$~~ is still $H'=H$ as no new conditions from invariance of $\mathcal{L}'$. Goldstone modes

$$\dim G'-\dim H'=(n^2+n^2-1-1)-(n^2-1)=n^2-1 \text{ or something.}$$

2. $\phi_a \in \mathbb{R}$ $\quad \Phi = \sum_a \phi_a t_a \quad \mathrm{tr}\,\Phi = 0\,,\ \Phi^T = \Phi\,,\ \mathrm{tr}(t_a t_b) = \delta_{ab}$

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}(\partial^\mu \Phi \partial_\mu \Phi) - V(\Phi)\,, \quad V(\Phi) = g\left(\tfrac{1}{4}\mathrm{Tr}(\Phi^4) + \tfrac{1}{3}b\,\mathrm{Tr}(\Phi^3) + \tfrac{1}{2}c\,\mathrm{Tr}(\Phi^2)\right)$$

Symmetry $G = SO(3)$, for $O \in SO(3)$ $\qquad g > 0$

$\Phi \mapsto O \Phi O^{-1} \qquad O^{-1} = O^T$

$\therefore \mathcal{L} \mapsto \mathcal{L}$ by cyclicity of tr.

$\mathcal{M}_0 = \{\Phi_0 : V(\Phi_0) = V_{min}\}$

$SO(3)$ acts transitively on $\mathcal{M}_0$: $\forall \Phi_0, \Phi_0' \in \mathcal{M}_0\ \exists Q \in SO(3) : \Phi_0' = Q\Phi_0 Q^{-1} = Q\Phi_0 Q^T$

Suppose Φ_0 has real eigenvalues λ_i $i = 1,2,3$ as $\Phi \in \mathbb{R}^{3\times 3}$

i.e. $\Phi_0 e_i = \lambda_i e_i$

$\Rightarrow Q\Phi_0 e_i = \lambda_i Q e_i \Rightarrow Q\Phi_0 Q^T Q e_i = \lambda_i Q e_i$

$\therefore \Phi_0'(Qe_i) = \lambda_i (Qe_i)$ $\quad \Phi_0'$ has eigenvalues λ_i

This holds for any Φ_0 in $\mathcal{M}_0$. Real symmetric matrices are diagonalised by orthogonal matrices. Hence $\mathrm{diag}(\lambda_i) \in \mathcal{M}_0$ and see that $\sum_i \lambda_i = 0$ by traceless condition. Can choose $\Phi_0 = \mathrm{diag}(\lambda_i)$ w.l.o.g.

The unbroken subgroup must then satisfy $R\Phi_0 R^T = \Phi_0$

Write $R = (\underline{v}_1, \underline{v}_2, \underline{v}_3)$ for $\underline{v}_i \in \mathbb{R}^3$ vectors and as $R \in SO(3)$ $\quad \underline{v}_i^T \underline{v}_j = \delta_{ij}$

then

$$R\Phi_0 = (\underline{v}_1\ \underline{v}_2\ \underline{v}_3)\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix} = (\lambda_1\underline{v}_1\ \ \lambda_2\underline{v}_2\ \ \lambda_3\underline{v}_3)$$

$$= \Phi_0 R = \begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}(\underline{v}_1\ \underline{v}_2\ \underline{v}_3) = (\Phi_0\underline{v}_1\ \ \Phi_0\underline{v}_2\ \ \Phi_0\underline{v}_3)$$

$\Rightarrow v_i$ are eigenvectors of Φ_0

If all eigenvalues distinct, v_i are unit vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ as Φ_0 diagonal.

Then, unbroken subgroup is trivial. $H = \{I\}$.

If two eigenvals degenerate, then any $SO(2)$ in the corresponding subspace transforms to equivalent set of eigenvectors. Then, $H = SO(2)$.

(Three degenerate $\Rightarrow \lambda = 0$ and no broken symmetry.)

Φ_0 is the solution to minimising $\qquad (\Phi^T = \Phi)$

$V(\Phi) = \frac{1}{4} Tr(\Phi^4) + \frac{1}{3} b\, Tr(\Phi^3) + \frac{1}{2} c\, Tr(\Phi^2)$ subject to $Tr(\Phi) = 0$

Use Lagrange multiplier to instead minimise

$$\mathcal{L}(\Phi;\mu) = \frac{1}{4} Tr(\Phi^4) + \frac{1}{3} b\, Tr(\Phi^3) + \frac{1}{2} c\, Tr(\Phi^2) - \mu\, Tr(\Phi)$$

Using $\dfrac{\partial Tr \Phi}{\partial \Phi} = I$,

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \Phi^3 + b\Phi^2 + c\Phi - \mu I$$

Hence at Φ_0,

$$\Phi_0^3 + b\Phi_0^2 + c\Phi_0 = \mu I \quad \overset{Tr}{\Rightarrow} \quad Tr(\Phi_0^3) + b\,Tr(\Phi_0^2) = 3\mu$$

$(Tr(\Phi_0) = 0$ and $Tr\, I = 3)$

Taking $\Phi_0 = \mathrm{diag}(\lambda_i)$

$$\Rightarrow \begin{cases} \lambda_i^3 + b\lambda_i^2 + c\lambda_i = \mu \quad \forall i \\ \sum_i \lambda_i = 0 \end{cases}$$

For this cubic equation to have more than one root, need to have critical point(s)

$$\frac{\partial}{\partial \lambda_i}(\lambda_i^3 + b\lambda_i^2 + c\lambda_i - \mu) = 3\lambda_i^2 + 2b\lambda_i + c = 0$$

This has real solutions when $4b^2 - 12c > 0 \Rightarrow b^2 > 3c$

When the unbroken subgroup is $SO(2)$, have 2 eigenvals λ and one -2λ

$$\Rightarrow \begin{cases} \lambda^3 + b\lambda^2 + c\lambda = \mu \\ -8\lambda^3 + 4b\lambda^2 - 2c\lambda = \mu \end{cases} \Rightarrow 9\lambda^3 - 3b\lambda^2 + 3c\lambda = 0$$

$\lambda \neq 0 \quad \Rightarrow 3\lambda^2 - b\lambda + c = 0$ and get $b^2 > 12c$ to have a sol^n.

3. SU(2) gauge theory $\qquad \phi \in \mathbb{C}^2 \qquad \underline{\tau} = \frac{1}{2}\underline{\sigma}$

$$\mathcal{L} = -\frac{1}{4}\underline{F}^{\mu\nu}\cdot\underline{F}_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \frac{1}{2}\lambda\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)^2$$

$$\underline{F}_{\mu\nu} = \partial_\mu\underline{A}_\nu - \partial_\nu\underline{A}_\mu - g\underline{A}_\mu\times\underline{A}_\nu \qquad D_\mu\phi = \partial_\mu\phi + ig\underline{A}_\mu\cdot\underline{\tau}\phi$$

Can choose $\phi = (0, v+h)^T/\sqrt{2}$:

Choose vacuum ~~at~~ $\phi_0 = (0, v)^T/\sqrt{2}$ w.l.o.g as different vacua related by SU(2) transforms.

~~Hence, generally $\phi = (f, v+h)^T/\sqrt{2}$ but can make a gauge transform to~~
~~writing $f \in \mathbb{C}$, $2k \in \mathbb{R}$, eliminate f.~~

~~$\phi' = \ldots \begin{pmatrix} f \\ v+he^{i\theta}\end{pmatrix}$ $f, h \in \mathbb{C}$~~

~~$\phi' = \begin{pmatrix} & \\ & 1\end{pmatrix}\begin{pmatrix} f \\ v+h\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h\end{pmatrix}$~~

Then $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi e^{i\theta} \\ (v+h)e^{i\xi}\end{pmatrix}$ near the VEV but by performing a SU(2) gauge transformation can eliminate χ, θ, ξ.

$$\Rightarrow \phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h\end{pmatrix}$$

In this gauge,

$$\mathcal{L} = -\frac{1}{4}\underline{F}^{\mu\nu}\cdot\underline{F}_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \frac{1}{2}\lambda\left(\frac{1}{2}(v+h)^2 - \frac{1}{2}v^2\right)^2$$

$$D_\mu\phi = \frac{1}{\sqrt{2}}\partial_\mu\begin{pmatrix}0\\h\end{pmatrix} + \frac{i}{\sqrt{2}}g\underline{A}_\mu\cdot\underline{\tau}\begin{pmatrix}0\\v+h\end{pmatrix}$$

$$= -\frac{1}{4}\underline{F}^{\mu\nu}\cdot\underline{F}_{\mu\nu} + \frac{1}{2}\partial^\mu h\,\partial_\mu h - \frac{ig}{2}(0 \;\; v+h)(\underline{A}^\mu\cdot\underline{\tau})^\dagger\partial_\mu\begin{pmatrix}0\\h\end{pmatrix}$$

$$+ \frac{ig}{2}\partial^\mu(0 \;\; h)\underline{A}_\mu\cdot\underline{\tau}\begin{pmatrix}0\\v+h\end{pmatrix} + \frac{1}{2}g^2(0 \;\; v+h)\underbrace{(\underline{A}^\mu\cdot\underline{\tau})^\dagger(\underline{A}_\mu\cdot\underline{\tau})}_{=\frac{1}{4}\underline{A}^{\mu *}\cdot\underline{A}_\mu + \frac{i}{2}(\underline{A}^{\mu *}\times\underline{A}_\mu)\cdot\underline{\tau}}\begin{pmatrix}0\\v+h\end{pmatrix}$$

$$-\frac{1}{2}\lambda\left(vh + \frac{1}{2}h^2\right)^2$$

~~[crossed out]~~. Only σ^3 contributes

$$= -\frac{1}{4}\underline{F}^{\mu\nu}\cdot\underline{F}_{\mu\nu} + \frac{1}{2}\partial^\mu h\,\partial_\mu h - \frac{ig}{2}(v+h)\partial^\mu h\left(A_{3\mu} - A_{3\mu}^*\right)/2 + \frac{1}{8}g^2\underline{A}^{\mu *}\cdot\underline{A}_\mu(v+h)^2$$

$$\pm \frac{i}{4} g^2 \left(\underline{A}^{\mu +} \times \underline{A}_\mu\right)_3 (v+h)^2 - \frac{1}{2} \lambda \left(vh + \frac{1}{2} h^2\right)^2 \quad \ldots$$

4.

5. $\mathcal{L} = -\frac{1}{4} F_{\mu\nu a} F^{\mu\nu}_{a} + \frac{1}{2}(D^\mu \phi)\cdot(D_\mu \phi) - V(\phi)$

- $\phi \mapsto g\phi = e^{i\omega^a \theta_a} \phi = (1 + i\omega^a \theta_a + \mathcal{O}(\omega^2))\phi$

$V(\phi) \longmapsto V(g\phi) = V((1 + i\omega^a \theta_a)\phi) = V(\phi) + \frac{\partial V(\phi)}{\partial \phi_i} \cdot (i\omega^a \theta_a \phi)_i + \mathcal{O}(\omega^2)$

$D^\mu \phi \mapsto g D^\mu \phi$

$(D^\mu \phi)\cdot(D_\mu \phi) \longmapsto ((1 + i\omega^a \theta_a) D^\mu \phi)\cdot((1 + i\omega^a \theta_a) D_\mu \phi)$.

$$= (D^\mu \phi)\cdot(D_\mu \phi) + (i\omega^a \theta_a D^\mu \phi)\cdot(D_\mu \phi) + \underbrace{(D^\mu \phi)\cdot(i\omega^a \theta_a D_\mu \phi)}_{= -(i\omega^a \theta_a D^\mu \phi)\cdot(D_\mu \phi)} + \mathcal{O}(\omega^2)$$

Assuming $\tilde{\phi}\cdot(\theta_a \phi) = -(\theta_a \tilde{\phi})\cdot \phi$ [illegible]

6. $\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - \frac{1}{2} g (\phi^* \phi - \frac{1}{2} v^2)^2$ VEV $v \in \mathbb{R}$

$$\phi = \frac{1}{\sqrt{2}} (v + f + i\alpha)$$

$$\mathcal{L} = \frac{1}{2} \partial^\mu (f - i\alpha) \partial_\mu (f + i\alpha) - \frac{1}{2} g \left(v f + \frac{1}{2} f^2 + \frac{1}{2} \alpha^2 \right)^2$$

$$= \frac{1}{2} \partial^\mu f \partial_\mu f + \frac{1}{2} \partial^\mu \alpha \partial_\mu \alpha - \frac{1}{2} g v^2 f^2 - \frac{1}{8} g f^4 - \frac{1}{8} g \alpha^4 - \frac{1}{2} g v f^3$$

$$- \frac{1}{2} g v f \alpha^2 - \frac{1}{4} g f^2 \alpha^2$$

No α^2 term, α massless

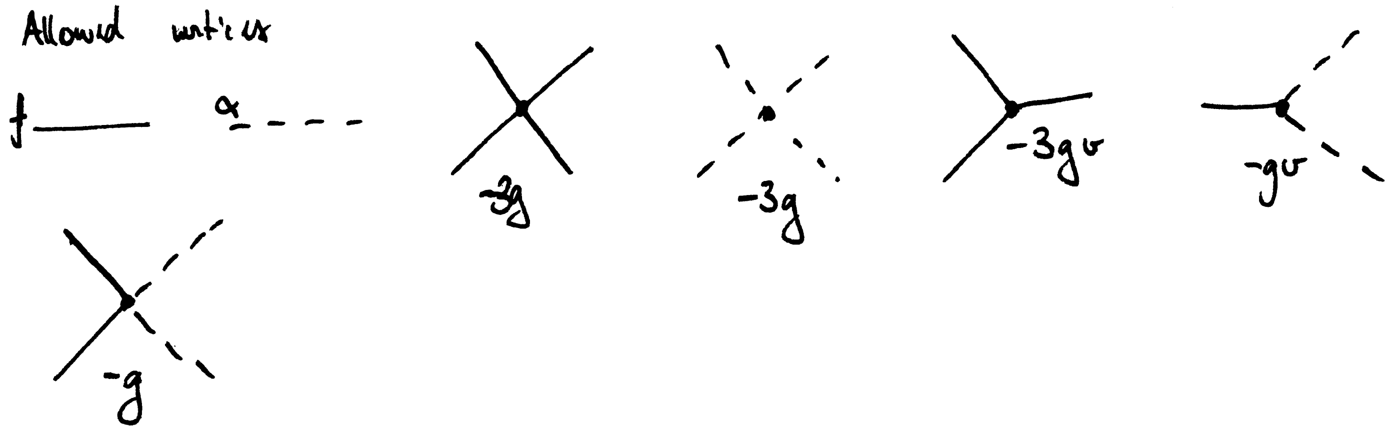
$$m_f^2 = g v^2$$

$$\mathcal{L}_{int} = - \frac{1}{8} g f^4 - \frac{1}{8} g \alpha^4 - \frac{1}{2} g v f^3 - \frac{1}{2} g v f \alpha^2 - \frac{1}{4} g f^2 \alpha^2$$

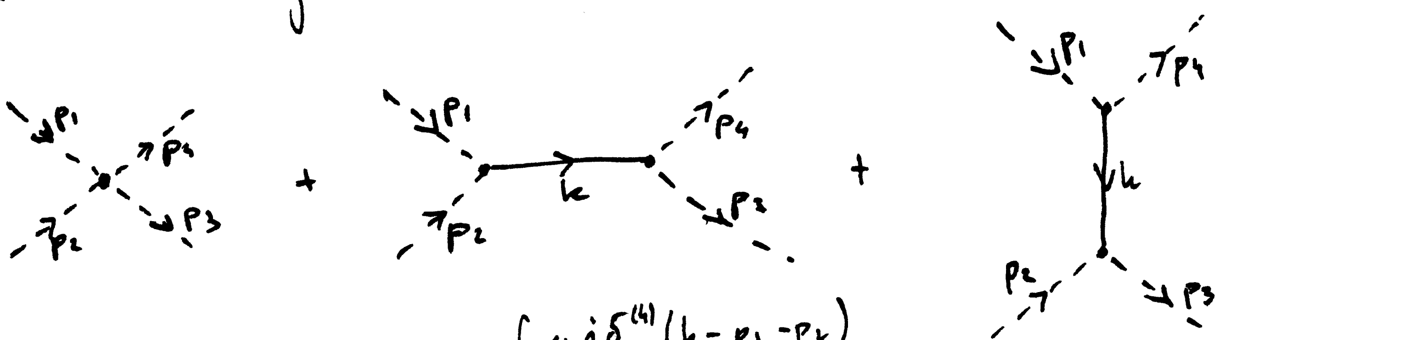
$$\langle \alpha(p_3) \alpha(p_4) | T | \alpha(p_1) \alpha(p_2) \rangle = (2\pi)^4 \delta^{(4)} (p_3 + p_4 - p_1 - p_2) \mathcal{M}$$

$$S = 1 - iT$$

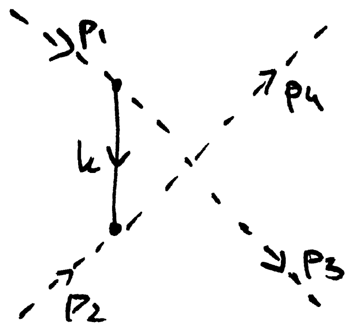
Allowed vertices



$\alpha\alpha \to \alpha\alpha$ diagrams:



$$-i3g + (-i)^2 g^2 v^2 \int d^4k \frac{i \delta^{(4)}(k - p_1 - p_2)}{k^2 - g v^2} + (-i)^2 g^2 v^2 \int d^4k \frac{i \delta^{(4)}(k + p_4 - p_1)}{k^2 - g v^2}$$

+ 

$$(-i)^2 g^2 v^2 \int d^4k \, \frac{i\delta^{(4)}(k+p_3-p_1)}{k^2 - gv^2}$$

$$\Rightarrow \mathcal{M} = g^2 v^2 \left(\frac{1}{s-gv^2} + \frac{1}{t-gv^2} + \frac{1}{u-gv^2} \right) + 3g$$

$$s=(p_1+p_2)^2 \quad , \quad t=(p_3-p_1)^2 \quad , \quad u=(p_4-p_1)^2$$

$$s+t+u = p_1^2 + p_2^2 + 2p_1\cdot p_2 + p_3^2 + p_1^2 - 2p_1\cdot p_3 + p_1^2 + p_4^2 - 2p_1\cdot p_4$$

$$= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1\cdot \underbrace{(p_2-p_3-p_4)}_{=-p_1}$$

$$= \sum p_i^2 = 0 \quad \text{as } \alpha \text{ massless}$$

$$\mathcal{M} = \frac{g^2v^2}{gv^2}\left(-3 - \frac{\overset{=0}{s+t+u}}{gv^2} - \frac{s^2+t^2+u^2}{g^2v^4} + \dots \right) + 3g$$

$$= -\frac{s^2+t^2+u^2}{gv^4} + \dots = \mathcal{O}(E^4) .$$