$\varphi : M \to N$ coords x^μ on M y^α on N $\varphi \leftrightarrow y^\alpha(x)$

S: $(0,s)$ tensor $\varphi^*(S)(X_1,\dots,X_s) = S(\varphi_*(X_1),\dots,\varphi_*(X_s))$

T: $(r,0)$ tensor $\varphi_*(T)(\eta_1,\dots,\eta_r) = T(\varphi^*(\eta_1),\dots,\varphi^*(\eta_r))$

$$\varphi^*(S)_{\mu_1\dots\mu_s} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}} S_{\alpha_1\dots\alpha_s}, \quad \varphi_*(T)^{\alpha_1\dots\alpha_s} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}} T^{\mu_1\dots\mu_r}$$

Diffeomorphisms & Lie derivative

Def $\varphi : M \to N$ is a diffeomorphism iff it is 1-1, onto, smooth & has a smooth inverse.

$\dim M = \dim N$

Def $\varphi : M \to N$ diffeo $T: (r,s)$ tensor on M. The push-forward of T is $\varphi_*(T)$ where

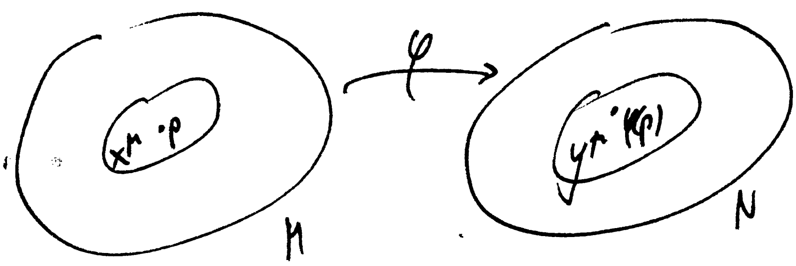
$$\varphi_*(T)(\underbrace{\eta_1,\dots,\eta_r}_{T^*_{\varphi(p)}(N)}, \underbrace{X_1,\dots,X_s}_{T_{\varphi(p)}(N)}) = T(\varphi^*(\eta_1),\dots,\varphi^*(\eta_r), (\varphi^{-1})_*(X_1),\dots,(\varphi^{-1})_*(X_s))$$

Ex 1. Show that φ_* commutes with contraction and outer products.

2. $r=s=1$. Show $[(\varphi_*(T))^\mu{}_\nu]_{\varphi(p)} = \left(\frac{\partial y^\mu}{\partial x^\rho}\right)_p \left(\frac{\partial x^\sigma}{\partial y^\nu}\right)_p (T^\rho{}_\sigma)_p$

generalize to (r,s) tensor.

Pull-back: $\varphi^* = (\varphi^{-1})_*$



pull back $y^\mu \Rightarrow$ coord chart y^μ near p

Def: $\varphi : M \to N$ diffeo. The push-forward of a covariant derivative ∇ on M is a cov. deriv. $\tilde\nabla$ on N defined by $\tilde\nabla_X T = \varphi_*(\nabla_{\varphi^*(X)} \varphi^*(T))$

GR: $M, g, F, \dots$

$\varphi : M \to N$ diffeo

$\Rightarrow (M, g, F, \dots)$ physically indistinguishable from $(N, \varphi_*(g), \varphi_*(F), \dots)$

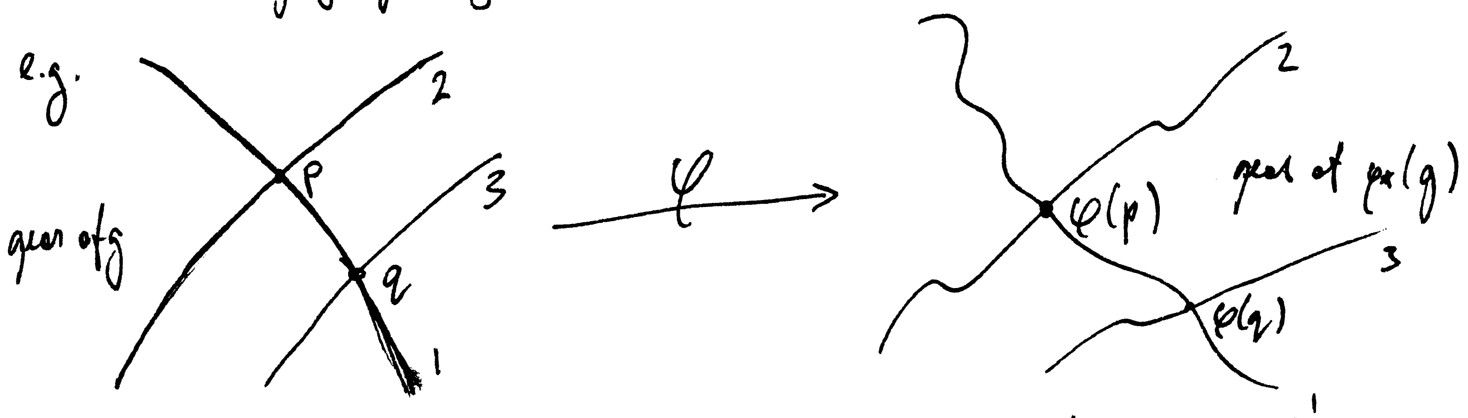
e.g. metric g has components $g_{\mu\nu}$ w.r.t. $\{e_\mu\}$ for $T_p(M)$

$\Rightarrow$ $\varphi_*(g)$ $g_{\mu\nu}$ $\{\varphi_*(e_\mu)\}$ for $T_{\varphi(p)}(N)$

$N = M$ $(M, g, F, \dots)$ physically indistinguishable from $(M, \varphi_*(g), \varphi_*(F), \dots)$

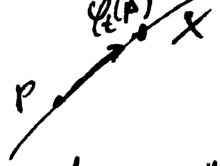
diffeos $\longleftrightarrow$ gauge symmetry of GR

e.g.



Def A diffeo $M \to M$ is a <u>symmetry transformation</u> of a tensor field T iff $\varphi_*(T) = T$ everywhere. A sym transf^n of g_{ab} is an <u>isometry</u>.

Def X vec field. $\varphi_t : M \to M$ $\quad p \longmapsto$ point parameter distance t along $\int$curve of X thru p

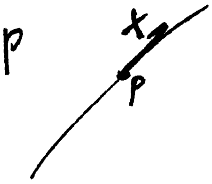


Can show φ_t is a diffeo.

φ_0 is identity, $\varphi_s \circ \varphi_t = \varphi_{s+t}$ $\therefore$ $\varphi_{-t} = (\varphi_t)^{-1}$

$\Rightarrow$ 1-parameter abelian group (if φ_t is defined $\forall\, t \in \mathbb{R} \Longleftrightarrow \int$curves of X are "complete").

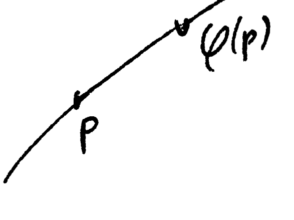
Conversely: given $\{\varphi_t\}$ define $X \in T_p(M)$ as tangent to curve $\varphi_t(p)$ thru p



$\Rightarrow$ vec field X s.t. $\int$curve of $X \to \varphi_t$.

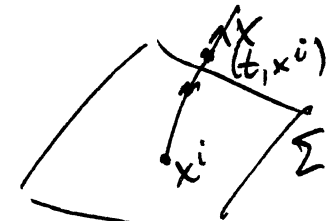
Def The <u>Lie derivative</u> of a tensor field T w.r.t vec field X is

$$(\mathcal{L}_X T)_p = \lim_{t \to 0} \frac{((\varphi_{-t})_* T)_p - T_p}{t}$$



$\mathcal{L}_X(\alpha S+\beta T) = \alpha\mathcal{L}_X(S)+\beta\mathcal{L}_X(T)$ (α,β const)

Σ: hypersurface, X nowhere parallel to Σ



x^i: coords on Σ

assign coords (t,x^i) to point parameter dist t along curve of X thru pt on Σ with coords x^i

$$\Rightarrow X=\frac{\partial}{\partial t} \qquad \varphi_t: \underbrace{(t_p,x^i_p)}_{x^\mu} \longmapsto \underbrace{(t_p+t,x^i_p)}_{y^\mu}$$

$$\Rightarrow \frac{\partial y^\mu}{\partial x^\nu}=\delta^\mu{}_\nu$$

$$\left[((\varphi_{-t})_*(T))^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\right]_{\varphi_{-t}(q)} = \left[T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\right]_q$$

Let $q=\varphi_t(p)$

$$\left[((\varphi_{-t})_*(T))^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\right]_p = \left[T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\right]_{\varphi_t(p)}$$

p coords (t_p,x^i_p)

$$= T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}(t_p+t,x^i_p)$$

$$(\mathcal{L}_X T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\Big|_p = \lim_{t\to 0}\frac{1}{t}\left(T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}(t_p+t,x^i_p) - T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}(t_p,x^i_p)\right)$$

$$(\mathcal{L}_X T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = \left(\frac{\partial}{\partial t}T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\right)(t_p,x^i_p)$$

$\mathcal{L}_X \leftrightarrow \frac{\partial}{\partial t}$ in this chart

$\nabla_X T \to \nabla T$, $\mathcal{L}_X T \not\to \mathcal{L}T$

Ex sheet 3: $(\mathcal{L}_X\omega)_\mu = X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu$ (in any coord basis)

$(\mathcal{L}_X\omega)_a = X^b\partial_b\omega_a + \omega_b\partial_a X^b$

$(\mathcal{L}_X g)_{\mu\nu} = X^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho$

$\Rightarrow \mathcal{L}_X(S\otimes T) = (\mathcal{L}_X(S))\otimes T + S\otimes\mathcal{L}_X(T)$

$\mathcal{L}_X$ commutes with contraction

$\mathcal{L}_X f = \frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(f) = X(f)$

Y: vec field $\qquad X^\mu=(1,0,\dots,0)$

$(\mathcal{L}_X Y)^\mu = \frac{\partial Y^\mu}{\partial t} \qquad [X,Y]^\mu = X^\nu Y^\mu{}_{,\nu} - \cancel{Y^\nu X^\mu{}_{,\nu}} = \frac{\partial Y^\mu}{\partial t} \qquad \mathcal{L}_X Y = [X,Y]$