Lemma The Ricci scalar of Σ is

$$R' = R \mp 2 R_{ab} n^a n^b \pm K^2 \mp K^{ab} K_{ab}$$

Lemma contracted Codacci equation

$$D_a K^a{}_b - D_b K = h_b{}^c R_{cd} n^d$$

The constraint equations

Take the Einstein equations

$$E_{ab} \equiv G_{ab} - 8\pi T_{ab} = 0 \implies R_{ab} - \frac{1}{2} g_{ab} R - 8\pi T_{ab} = 0$$

Contract E_{ab} with a timelike n_a (looking at spacelike Σ):

$$n^a n^b E_{ab} = 0 \implies R' - K^{ab} K_{ab} + K^2 = 16\pi\rho$$

where $\rho \equiv n^a n^b T_{ab}$ is the matter density measured by an observer with 4-velocity n_a.
Hamiltonian constraint.
Construct E_{ab} with n^a and $h^c{}_b$

$$D_b K^b{}_a - D_a K = 8\pi h_a{}^b T_{bc} n^c$$

Momentum constraint.
The remaining equations are evolution equations for (h_{ab}, K_{ab}).

Theorem (Choquet-Bruhat & Geroch 1969)
Let (Σ, h_{ab}, K_{ab}) be initial data satisfying the vacuum Hamiltonian and momentum constraints. Then there exists a unique (up to diffeos) spacetime (M, g_{ab}), called the maximal Cauchy development of (Σ, h_{ab}, K_{ab}):

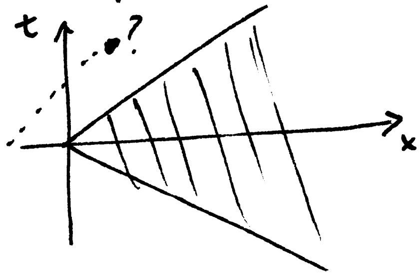
(i) (M, g_{ab}) satisfies the Einstein equations
(ii) is globally hyperbolic with Cauchy surface Σ
(iii) the induced metric and extrinsic curvature of Σ are h_{ab} and K_{ab} resp.
(iv) any other spacetime satisfying (i), (ii), (iii) is isometric to a subset of (M, g_{ab}).

It is possible that the maximal Cauchy development (M, g_{ab}) is extendible, i.e. it is isometric to a proper subset of another spacetime (M', g'_{ab}). Because of CBG theorem, Σ cannot be a Cauchy surface for (M', g'_{ab}). You cannot predict physics in $M' \setminus D(\Sigma)$.

Example 1 $\Sigma = \{(x, y, z) : x > 0\}$ with $h_{\mu\nu} = \delta_{\mu\nu}$ and vanishing $K_{\mu\nu} = 0$.

The maximal development $|t| < x$.

We will require our initial data to be inextendible.



Example 2

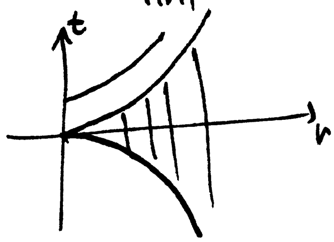
Take the Schwarzschild solution with $M<0$.

$$ds^2 = -\left(1+\frac{2|M|}{r}\right)dt^2 + \left(1+\frac{2|M|}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

There is no horizon but the metric is singular at $r=0$. Take Σ to be a $t=0$ hypersurface, h is the metric induced on such hypersurface. And with $K=0$ this initial data is singular since geodesics reach $r=0$ in finite proper time.

Outgoing geodesics (radial, null) satisfy

$$\frac{dt}{dr} = \left(1+\frac{2|M|}{r}\right)^{-1} \simeq \frac{r}{2|M|} \quad \text{for small } r \qquad \Rightarrow t(r) = t_0 + \frac{r^2}{4|M|}$$



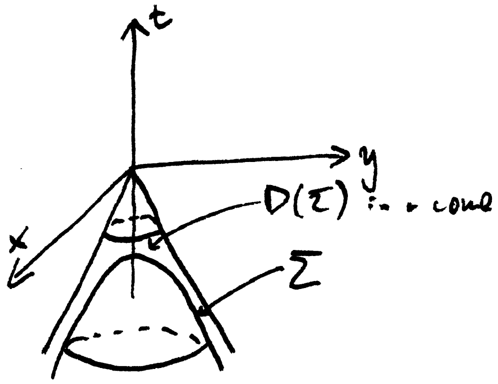
when $r \to 0$, $t(0) = t_0$.

We want initial data that is geodesically complete

Example 3

Even when geodesically complete initial data can be bad. Take the hyperboloid

$-t^2+x^2+y^2+z^2=-1$ with $t<0$ in Minkowski spacetime.



Def (Asymptotically flat)

(a) An initial data set (Σ, h_{ab}, K_{ab}) is an asymptotically flat end if

(i) Σ is diffeomorphic to $\mathbb{R}^3 \setminus B$ where B is a ball (closed) centred on the origin in $\mathbb{R}^3$.

(ii) if we pull-back the $\mathbb{R}^3$ coordinates to Σ

$$h_{ij} = \delta_{ij} + O(1/r), \quad K_{ij} = O(1/r^2) \quad \text{as } r\to\infty \quad \text{where } r=\sqrt{x^i x_i}$$

(iii) derivatives of (ii) also hold

$$h_{ij,k} = O(1/r^2) \quad \text{etc.}$$

(b) An initial data set is asymptotically flat with N ends if it is ~~asymptotically flat~~ the union of a compact set with N asymptotically flat ends.