Then we can look at $O(\lambda^2)$ terms in $\langle f | S-1 | i \rangle$

we have

$$\frac{(-i\lambda)^2}{2} \int d^4x_1 \, d^4x_2 \, T\left[\bar{\psi}(x_1)\psi(x_1)\,\phi(x_1)\,\bar{\psi}(x_2)\psi(x_2)\,\phi(x_2)\right]$$

- all fields in the int^n picture. The contribution to scattering comes from the contraction

$$:\bar{\psi}(x_1)\,\psi(x_1)\,\cancel{\phi(x_1)}\,\bar{\psi}(x_2)\,\psi(x_2):\;\phi(x_2)\,\phi(x_1)$$

↳ annihilate the initial state

creates the final state

Nuclear Scattering

Put in $|i\rangle$ and we have (ignoring c ops as they don't contribute)

$$:\bar{\psi}(x_1)\,\psi(x_1)\,\bar{\psi}(x_2)\,\psi(x_2):\; b_p^{s\dagger}\, b_q^{s\dagger}\,|0\rangle$$

$$= -\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{2\sqrt{E_pE_q}}\left(\bar{\psi}(x_1)\,u_{k_1}^m\right)\left(\bar{\psi}(x_2)\,u_{k_2}^n\right)e^{-ik_1\cdot x_1 - ik_2\cdot x_2}$$

$$b_{k_1}^m\, b_{k_2}^n\, b_p^{s\dagger}\, b_q^{r\dagger}\,|0\rangle$$

$$= -\frac{1}{2\sqrt{E_pE_q}}\Big\{\left[\bar{\psi}(x_1)\,u_q^r\right]\left[\bar{\psi}(x_2)\cdot u_p^s\right]e^{-iq\cdot x_1 - ip\cdot x_2}$$

$$-\left[\bar{\psi}(x_1)\,u_p^s\right]\left[\bar{\psi}(x_2)\,u_q^r\right]e^{-ip\cdot x_1 - iq\cdot x_2}\Big\}$$

↑ crucial. Let's see what happens when we apply $\langle f|$

$$\langle 0|\, b_{q'}^{r'}\, b_{p'}^{s'}\,\left[\bar{\psi}(x_1)\,u_q^r\right]\left[\bar{\psi}(x_2)\cdot u_p^s\right]$$

$$= \frac{1}{2\sqrt{E_{p'}E_{q'}}}\Big\{\left[\bar{u}_{p'}^{s'}\cdot u_q^r\right]\left[\bar{u}_{q'}^{r'}\cdot u_p^s\right]e^{ip'\cdot x_1 + iq'\cdot x_2}$$

$$-\left[\bar{u}_{q'}^{r'}\cdot u_q^r\right]\left[\bar{u}_{p'}^{s'}\cdot u_p^s\right]e^{ip'\cdot x_2 + iq'\cdot x_1}\Big\}$$

The $[\bar{\psi}(x_1)\cdot u_p^s][\bar{\psi}(x_2)\cdot u_q^r]$ term double up with this, cancelling the $1/2$ from $\frac{(-i\lambda)^2}{2}$

Putting everything together, we have $\langle f | S-1 | i\rangle$

$\frac{1}{\sqrt{E}}$ cancel due to relativistic normalisation

$$(-i\lambda)^2 \int \frac{d^4x\, d^4x_2}{(2\pi)^8} \frac{d^4k}{(2\pi)^4} \frac{i e^{ik(x_1-x_2)}}{k^2-\mu^2+i\varepsilon} \Big\{ [\bar u^{s'}_{p'}\cdot u^s_p][\bar u^{r'}_{q'}\cdot u^r_q] \times e^{ix_1(q'-q)+ix_2(p'-p)} - [\bar u^{s'}_{p'}\cdot u^r_q][\bar u^{r'}_{q'}\cdot u^s_p]\, e^{ix_1(p'-q)+ix_2(q'-p)} \Big\}$$

$$= i(-i\lambda)^2 \int d^4k\, (2\pi)^4 \frac{1}{k^2-\mu^2+i\varepsilon} \Big\{ [\bar u^{s'}_{p'} u_p][\bar u^{r'}_{q'} u^r_q]\, \delta^4(q'-q+k)\, \delta^4(p'-p+k) - [\bar u^{s'}_{p'} u^r_q][\bar u^{r'}_{q'} u^s_p]\, \delta^4(p'-q+k)\, \delta^4(q'-p+k) \Big\}$$

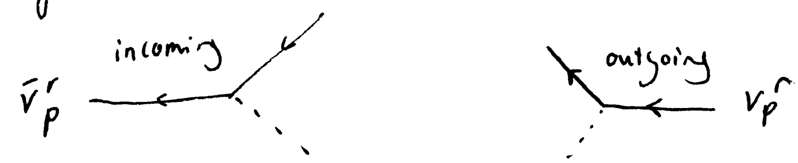
$$\langle f|S-1|i\rangle = i\mathcal{A}\, (2\pi)^4 \delta^4(p+q-q'-p')$$

$$\mathcal{A} = (-i\lambda)^2 \Big\{ \frac{[\bar u^{s'}_{p'} u^s_p][\bar u^{r'}_{q'} u^r_q]}{(p'-p)^2-\mu^2+i\varepsilon} - \frac{[\bar u^{s'}_{p'} u^r_q][\bar u^{r'}_{q'} u^s_p]}{(q'-p)^2-\mu^2+i\varepsilon} \Big\}$$

Feynman rules for fermions

- Incoming / outgoing fermion

- Antifermions



- Each vertex gets a factor $(-i\lambda)$
- Internal lines get a factor

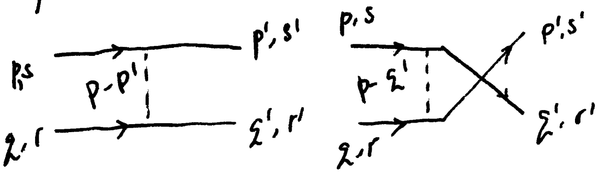
$p \rightarrow$ - - - - scalar $\qquad \dfrac{i}{p^2-\mu^2+i\varepsilon}$ $\qquad\qquad$ $\xrightarrow{p\,\rightarrow}$ $\qquad \dfrac{i(\not p + m)}{p^2-m^2+i\varepsilon}$

The arrows on fermion lines must flow consistently, ensuring fermion number conservation.

N.B. The fermion propagator is a 4×4 matrix, indices are contracted at each vertex, either with propagators or external spinors ($\bar{u}, u, \bar{v}, v$)

- Impose energy-mom conservation at each vertex
- $\int \frac{d^4k}{(2\pi)^4}$ over ~~all~~ undetermined loop momenta
- Add extra minus sign for a loop of fermion

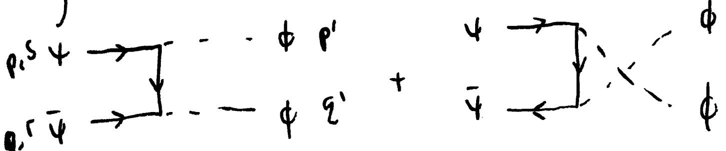
e.g. Nuclear Scattering



} two fermions are swapped, i.e. a relative minus sign

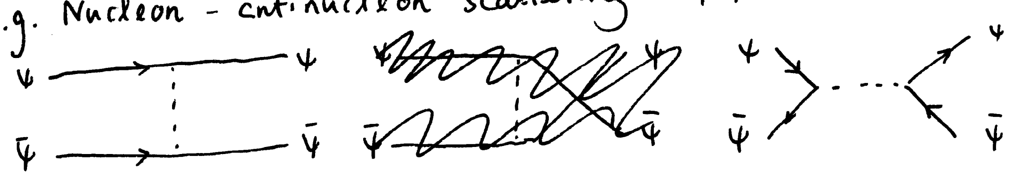
$$A = (-i\lambda)^2 \left\{ \frac{[\bar{u}^{s'}_{p'} u^s_p][\bar{u}^{r'}_{q'} u^r_q]}{(p-p')^2 - \mu^2 + i\varepsilon} - \frac{[\bar{u}^{s'}_{p'} u^r_q][\bar{u}^{r'}_{q'} u^s_p]}{(p-q')^2 - \mu^2 + i\varepsilon} \right\}$$

e.g. Nucleons to mesons $\psi\bar{\psi} \to \phi\phi$



$$A = (-i\lambda)^2 \left\{ \frac{\bar{v}^r_q [\gamma^\mu (p_\mu - p'_\mu) + m] u^s_p}{(p-p')^2 - m^2 + i\varepsilon} \underset{\substack{\uparrow \\ \text{boson}}}{+} \frac{\bar{v}^r_q [\gamma^\mu (p_\mu - q'_\mu) + m] u^s_p}{(p-q')^2 - m^2 + i\varepsilon} \right.$$

e.g. Nucleon - antinucleon scattering $\psi\bar{\psi} \to \psi\bar{\psi}$



$$A = (-i\lambda)^2 \left\{ \frac{-[\bar{u}^{s'}_{p'} u^s_p][\bar{v}^r_q \cdot u^{r'}_{q'}]}{(p-p')^2 - \mu^2 + i\varepsilon} + \frac{[\bar{v}^r_q \cdot u^s_p][\bar{u}^{s'}_{p'} v^{r'}_{q'}]}{(p+q)^2 - \mu^2 + i\varepsilon} \right\}$$

$|i\rangle \propto b^{s\dagger}_p c^{r\dagger}_q |0\rangle$

ordering is crucial

$|f\rangle \propto b^{s'\dagger}_{p'} c^{r'\dagger}_{q'} |0\rangle$

Examine the operators

$$\psi \sim \cancel{b} \; b + c^{\dagger}$$

$$\bar{\psi} \sim \cancel{b} \; b^{\dagger} + c$$

$$\langle f | : \bar{\psi} \psi \bar{\psi} \psi : b_p^{s\dagger} b_q^{r\dagger} | 0 \rangle$$

$$\approx \langle f | \, [\bar{v}_{k_1}^{m} \psi] [\bar{\psi} u_{k_2}^{n}] \, b_{k_1}^{m} b_{k_2}^{n} b_p^{s\dagger} b_q^{r\dagger} | 0 \rangle$$

$$\sim + \langle f | \, [\bar{v}_q^{m} \psi] [\bar{\psi} u_p^{s}] | 0 \rangle$$

$$\sim \langle 0 | c_{q'}^{r'} b_{p'}^{s'} c_{l_1}^{m\dagger} b_{l_2}^{n\dagger} [\bar{v}_q^{r} \cdot v_{l_1}^{m}] [\bar{u}_{l_2}^{m} u_p^{s}] | 0 \rangle$$

$$\sim - [\bar{v}_q^{r} \cdot v_{q'}^{r'}] [\bar{u}_{p'}^{s'} \cdot u_p^{s}]$$

Follow similar contraction for the other diagram.

Quantum Electrodynamics

Will start as before, quantising the EM field

Maxwell eq^n

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Rightarrow \text{e.o.m.} \quad \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 \qquad \partial_\mu F^{\mu\nu} = 0$$

Bianchi Identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (\text{from } \mathbf{antisym})$$

We must be very careful about the definition of 3-vectors

$\underline{E}$ and $\underline{B}$ were defined with no minuses from $\eta^{\mu\nu}$

We have $\underline{E} = -\underline{\nabla}\phi - \dot{\underline{A}}$, $\underline{B} = \underline{\nabla} \times \underline{A}$

$$\underline{\nabla} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \partial_i$$

$A^\mu = (\phi, \underline{A})$ i.e. $A = (A^1, A^2, A^3)$ in vector language

Finally $\underline{E} = (F_{01}, F_{02}, F_{03}) = (-F^{01}, -F^{02}, F^{03})$

Look at F_{0i}, get $\underline{E} = -\underline{\nabla}\phi - \dot{\underline{A}}$

For magnetic field, $\underline{B} = \underline{\nabla} \times \underline{A}$

$$\Rightarrow B_z = \partial_1 A^2 - \partial_2 A^1 = -\partial_1 A_2 + \partial_2 A_1 = -F_{12}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Our Bianchi identity reads

$(\lambda = 3, \mu = 1, \nu = 2)$

$$\dot{\underline{B}} = -\underline{\nabla} \times \underline{E}$$

e.o.m. gives $\underline{\nabla} \cdot \underline{E} = 0$

$$\dot{\underline{E}} = \underline{\nabla} \times \underline{B}$$

The massless vector field has 4 d.o.f. but the photon (γ) only has two polarisation states.