Kruskal extension

We will introduce (yet) another set of coordinates Kruskal-Szekeres (U, V, θ, ϕ)

$$U = -e^{-\frac{u}{4M}} , \quad V = e^{\frac{v}{4M}} \qquad (u = t - r_* , \ v = t + r_*)$$

For $r > 2M$, $U < 0$ and $V > 0$, look at

$$UV = -e^{\frac{r_*}{2M}} = -e^{\frac{r}{2M}}\left(\frac{r}{2M} - 1\right) \qquad \text{monotonic } f^n \text{ of } \frac{r}{2M}$$

So we have defined implicitly $r(U,V)$.

$$V/U = -e^{t/2M} \qquad \text{monotonic } f^n \text{ of } \frac{t}{2M}$$

This also defines $t(U,V)$

Now look at $dU = \frac{1}{4M} e^{-u/4M} du$, $dV = \frac{1}{4M} e^{v/4M} dv$

Take $dU\, dV = \frac{1}{16M^2} e^{r_*/2M} \left[dt^2 - \frac{dr^2}{(1-\frac{2M}{r})^2}\right]$

This means

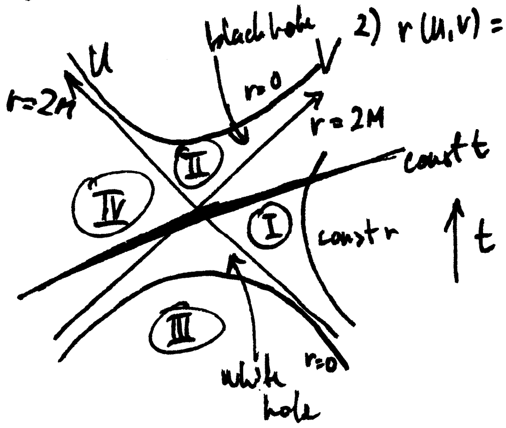
$$ds^2 = \underbrace{-\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}}_{} + r^2 d\Omega^2$$

$$= -\frac{32M^3}{r(U,V)} \exp\left[-\frac{r(U,V)}{2M}\right] dU\,dV + r(U,V)^2 d\Omega_2^2 \quad //$$

Because the resulting metric is non-singular for any $U, V : r(U,V) \neq 0$. We can take U, V to have either sign.

Consider the condition 1) $r(U,V) = 0 \Rightarrow UV = 1$ (is singular)

2) $r(U,V) = 2M \Rightarrow$ corresponds to 2 surfaces $U = 0$ or $V = 0$



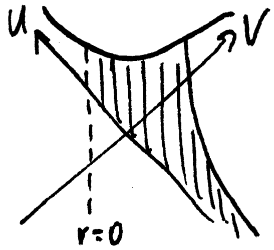
(I) $U < 0$, $V > 0$

(II) Ingoing Eddington-Finkelstein

(III) Outgoing "

(IV) Isometric to (I) $(U,V) \to -(U,V)$

For spherical collapse of a star



$(K \equiv \partial_t)$

If you take ∂_t, $r > 2M$, and write it as a f^n of U and V,

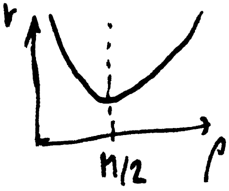
$$K = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right) \qquad \text{in I, IV is timelike}$$

$$K^2 = -\left[1 - \frac{2M}{r(U,V)}\right] \qquad \text{and in II, III is spacelike}$$

Bifurcation sphere $U, V = 0$

Let us look at a line of constant t. Introduce

$$r = \rho + M + \frac{M^2}{4\rho}$$



For $\rho > M/2$, we are in region I

For $0 < \rho < M/2$, we are in region IV

There is an isometry $\rho \to \dfrac{M^2}{4\rho}$.

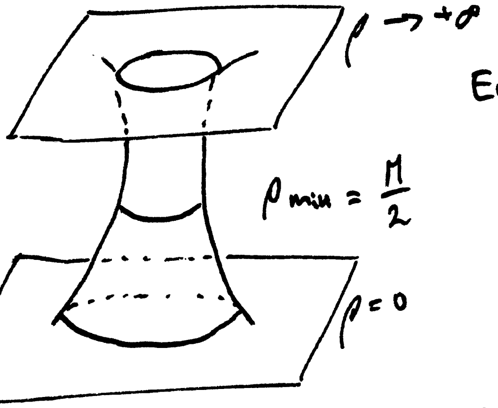
In these coordinates,

$$ds^2 = -\left[\frac{1-\frac{M}{2\rho}}{1+\frac{M}{2\rho}}\right]^2 dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \underbrace{\left(d\rho^2 + \rho^2 d\Omega^2\right)}_{dE_3^2} \quad , \rho \in (0, +\infty)$$

These are called isotropic coordinates.

You can embed $t=$ const surface in E_4

$$ds^2_\Sigma = \left(1+\frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right)$$



Einstein-Rosen bridge

<u>Extendability</u> A spacetime (M, g) is <u>extendible</u> if it is isometric to a proper subset of another spacetime (M', g'). (M', g') is an extension of (M, g).

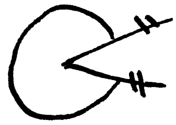
The Kruskal spacetime is inextendible.

<u>Singularity</u> • coordinate singularity ($r = 2M$ in Schwarzschild coordinates) not physical

• curvature singularities
  - Scalar curvature singularities, $r = 0$ in Kruskal
  - Curvature singularities <u>but</u> all scalars are finite

• There are singularities which the curvature does not see.

$$g = dr^2 + r^2\lambda^2 d\phi^2 \quad , \quad \phi \sim \phi + 2\pi$$



If I define $\tilde{\phi} = \lambda\phi$, $g = dr^2 + r^2 d\tilde{\phi}^2$ all $R_{abcd} = 0$

Conical singularity $\dfrac{\text{circumference}}{\text{radius}} = \dfrac{2\pi\lambda\varepsilon}{\varepsilon} = 2\pi\lambda$ as $\varepsilon \to 0$ this is not going to 2π!

Curve is a map $\gamma : (a, b) \to M$

<u>Def</u> $p \in M$ is a future end point of a f directed causal curve $\gamma : (a, b) \to M$, iff for any neighbourhood O of p there exists a t_0 such that $\gamma(t) \in O \ \forall t > t_0$

If there is no future endpoint, it is called future-inextendible.