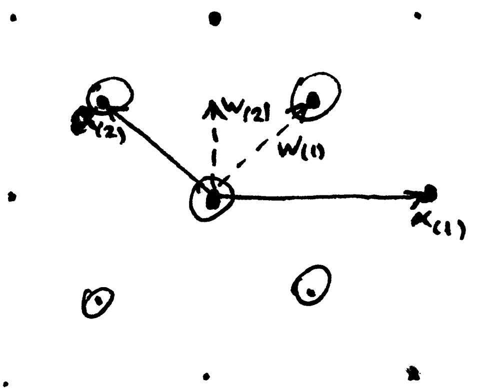
Symmetries, Fields and Particles
Example sheet 4

1. B_2 $\quad$ o⇒o $\quad A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\qquad (\alpha_{(1)}, \alpha_{(1)}) = 2(\alpha_{(2)}, \alpha_{(2)})$

$$\Phi = \{\pm\alpha_{(1)}, \pm\alpha_{(2)}, \pm(\alpha_{(1)} + \alpha_{(2)}), \pm(\alpha_{(1)} + 2\alpha_{(2)})\}$$

$$\alpha_{(i)} = \sum_j A^{ij} w_{(j)}$$

$$\begin{aligned} \alpha_{(1)} &= 2w_{(1)} - 2w_{(2)} \\ \alpha_{(2)} &= -w_{(1)} + 2w_{(2)} \end{aligned} \quad \Rightarrow \quad \begin{aligned} w_{(1)} &= \alpha_{(1)} + \alpha_{(2)} \\ w_{(2)} &= \tfrac{1}{2}(\alpha_{(1)} + 2\alpha_{(2)}) \end{aligned}$$



• roots (adjoint repn weights)
○ fundamental repn weights

Fundamental repn

$(\Lambda^1, \Lambda^2) = (1, 0)$, $\Lambda = w_{(1)}$

• $\Lambda = w_{(1)} \in S_f$
$\quad \Rightarrow \Lambda - \alpha_{(1)} = -w_{(1)} + 2w_{(2)} \in S_f$

• $\lambda = -w_{(1)} + 2w_{(2)} \in S_f$
$\quad \Rightarrow \lambda - \alpha_{(2)} = 0 \in S_f$
$\quad\quad \lambda - 2\alpha_{(2)} = w_{(1)} - 2w_{(2)} \in S_f$

• $\lambda' = w_{(1)} - 2w_{(2)} \in S_f$
$\quad \Rightarrow \lambda' - \alpha_{(1)} = -w_{(1)} \in S_f$

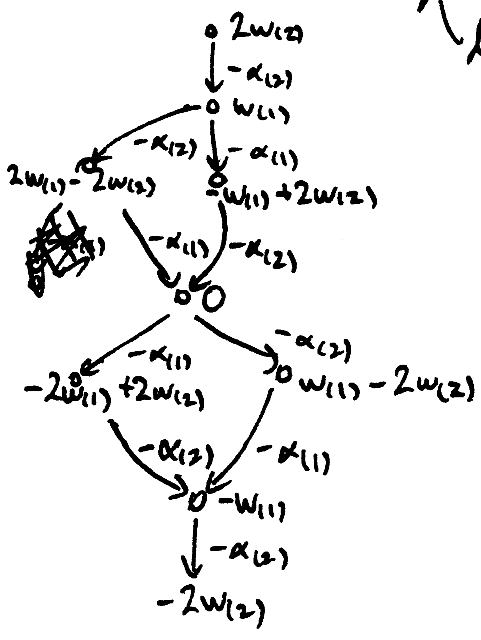
Terminates

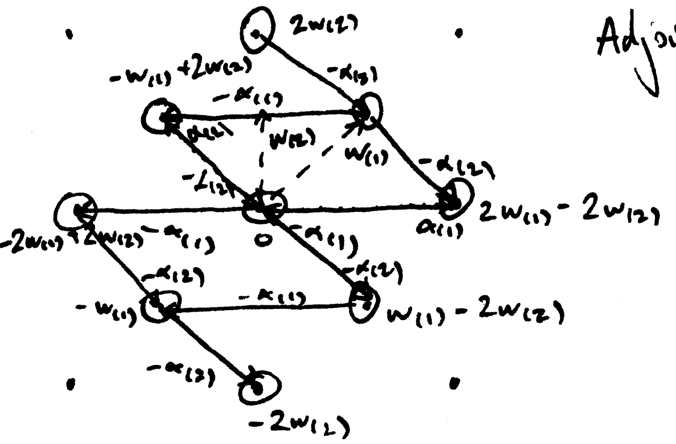
$$S_f = \{w_{(1)}, -w_{(1)} + 2w_{(2)}, 0, w_{(1)} - 2w_{(2)}, -w_{(1)}\}$$

Adjoint repn: roots $\alpha \in \Phi$ are weights

$$S = \{\pm(2w_{(1)} - 2w_{(2)}), \pm(-w_{(1)} + 2w_{(2)}), \pm w_{(1)}, \pm 2w_{(2)}, 0\}$$

↖ degeneracy 2





Adjoint repn

2. $H^1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $\quad H^2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ $\quad$ CSA of A_2

$I := \frac{1}{2} H^1 \quad , \quad Y := \frac{1}{3}(H^1 + 2H^2)$

Weight lattice has points

$\lambda = \lambda^1 w_{(1)} + \lambda^2 w_{(2)} = I w_I + Y w_Y \quad$ where $\lambda^1, \lambda^2 \in \mathbb{Z}$

$R(H^i) v = \lambda^i v \quad \forall v \in V_\lambda$

Similarly,

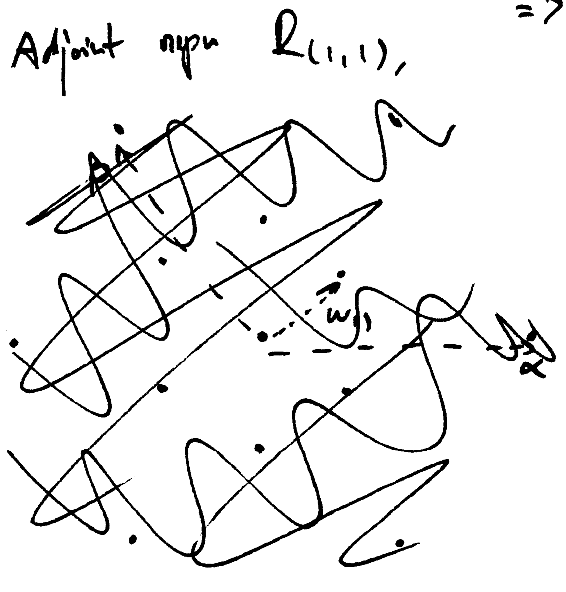
$R(I) v = I v \quad , \quad R(Y) v = Y v$

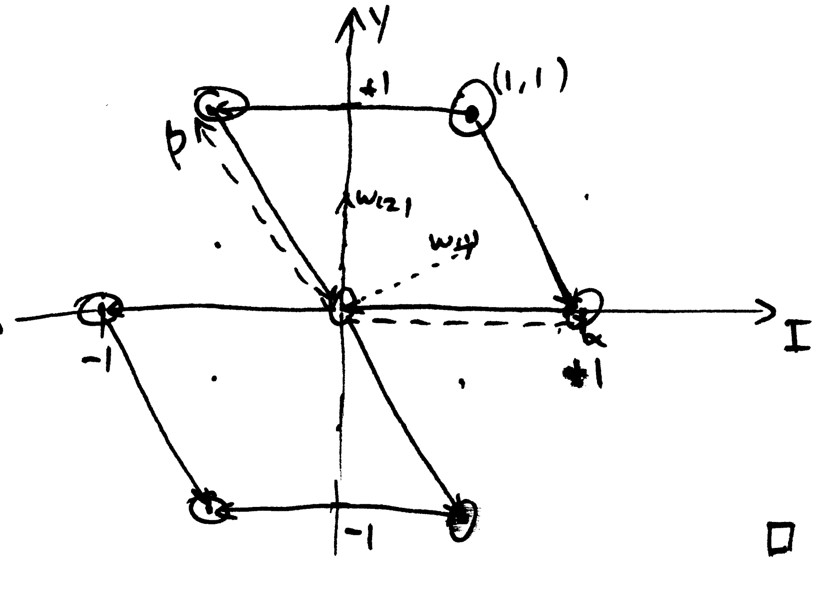
$\Rightarrow I v = R(I) v = R(\frac{1}{2} H^1) v = \frac{1}{2} \lambda^1 v \quad \Rightarrow I = \frac{1}{2}\lambda^1$

$\Rightarrow Y v = R(Y) v = R(\frac{1}{3}(H^1 + 2H^2)) v = \frac{1}{3}(\lambda^1 + 2\lambda^2) v \Rightarrow Y = \frac{1}{3}(\lambda^1 + 2\lambda^2)$

$\Rightarrow w_I = 2w_{(1)} - w_{(2)} \quad , \quad w_Y = \frac{3}{2} w_{(2)}$

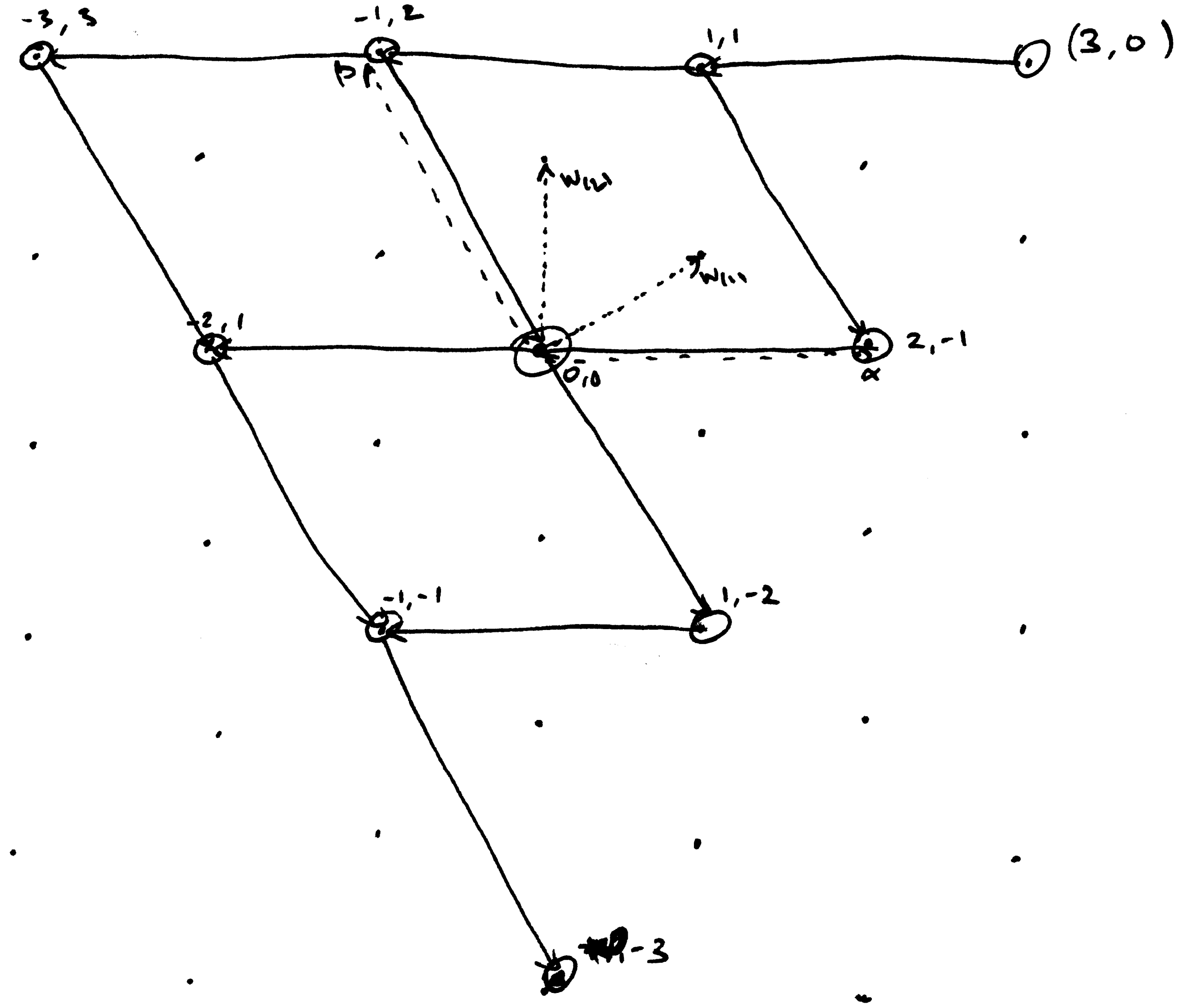
Adjoint repn $R_{(1,1)}$,



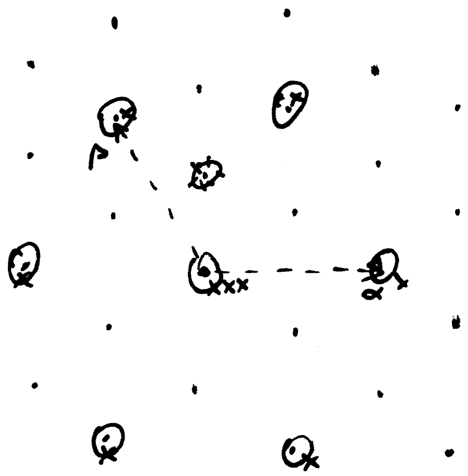


$\square$

3.

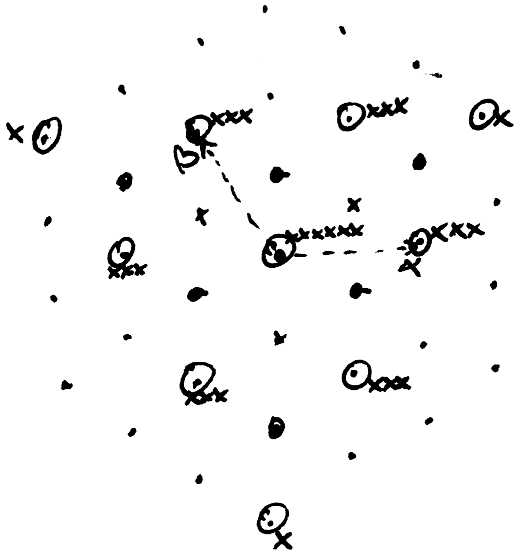


4.(i)



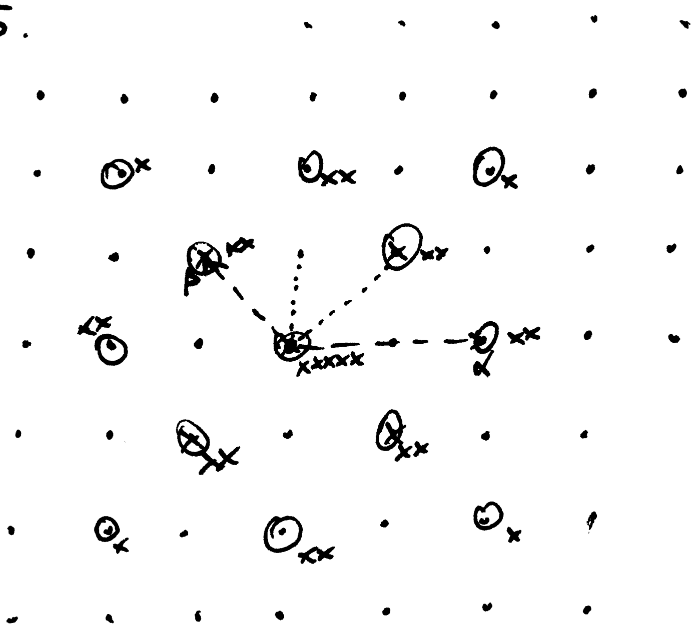
$$\underline{3}\otimes\underline{\bar{3}} = R_{(1,0)}\otimes R_{(0,1)}$$
$$= R_{(1,1)}\oplus R_{(0,0)}$$
$$= \underline{8}\oplus\underline{1}$$

(ii)



$$\underline{3}\otimes\underline{3}\otimes\underline{3}$$
$$= R_{(1,0)}\otimes R_{(1,0)}\otimes R_{(1,0)}$$
$$= R_{(3,0)}\oplus R_{(1,1)}\oplus R_{(1,1)}\oplus R_{(0,0)}$$
$$= \underline{10}\oplus\underline{8}\oplus\underline{8}\oplus\underline{1}$$

5.



Non-trivial rep of B_2 of smallest dim

$R_{(1,0)} = \underline{5}$ (or $R_{(0,1)}$)

& $\underline{5}\otimes\underline{5} = R_{(1,0)}\otimes R_{(1,0)}$
$$= R_{(2,0)}\oplus R_{(0,2)}\oplus R_{(0,0)}\oplus R_{(0,0)}\oplus R_{(0,0)}$$
$$= \underline{13}\oplus\underline{9}\oplus\underline{1}\oplus\underline{1}\oplus\underline{1}$$

6. $G = SU(N)$ $\qquad L(G) \ni A \Rightarrow \mathrm{tr} A = 0 \qquad A^\dagger = -A$

$$A_\mu'^\dagger = (g A_\mu g^{-1})^\dagger - [(\partial_\mu g) g^{-1}]^\dagger$$

$$= (g^{-1})^\dagger A_\mu^\dagger g^\dagger + [g(\partial_\mu g^{-1})]^\dagger \qquad \text{use } g^\dagger = g^{-1}$$

$$= g A_\mu^\dagger g^{-1} + (\partial_\mu g) g^{-1} = -A_\mu'$$

$$\mathrm{Tr}(A_\mu') = \mathrm{Tr}(g A_\mu g^{-1}) - \mathrm{Tr}[(\partial_\mu g) g^{-1}] = -\mathrm{Tr}[(\partial_\mu g) g^{-1}]$$

by cyclicity

As coordinate x^μ defines a curve on the group manifold in $g: \mathbb{R}^{3,1} \to G$

$$\partial_\mu g = \partial_\mu t^\nu \partial_t(\exp(tX_\nu)) = \partial_\mu t^\nu \exp(tX_\nu) X_\nu = \partial_\mu t^\nu g X_\nu \quad \text{where } X_\nu \in L(G)$$

$$\Rightarrow \mathrm{Tr}(A_\mu') = -\mathrm{Tr}[(\partial_\mu g) g^{-1}] = -\mathrm{Tr}[\partial_\mu t^\nu g X g^{-1}] = -\partial_\mu t^\nu \mathrm{Tr}[g X_\nu g^{-1}] = 0$$

as $X_\nu \in L(G)$

$A_\mu' \in L(G)$ $\square$

In general,

$$\partial_\mu g = \partial_\mu t^\nu g X_\nu \equiv g X_\mu'(x) \qquad X_\mu' : \mathbb{R}^{3,1} \to L(G)$$

$$A_\mu' = g A_\mu g^{-1} - g X_\mu' g^{-1} = g \underbrace{(A_\mu - X_\mu')}_{= Y_\mu \in L(G)} g^{-1}$$

$$= g Y_\mu g^{-1} = \exp(sX) Y_\mu \exp(-sX) \quad \text{at point } X \in L(G)$$

$$= \exp^{\mathrm{ad}(sX)} Y_\mu \in L(G) \quad \text{as Lie algebra is closed under } \mathrm{ad}X : \mathfrak{g} \to \mathfrak{g} \;\forall X \in \mathfrak{g}$$

Writing $g = \exp(\varepsilon X)$ $\quad \varepsilon \ll 1$ $\quad , g = 1 + \varepsilon X + \dots$

$$A_\mu' = (1 + \varepsilon X) A_\mu (1 - \varepsilon X) - (\partial_\mu(1 + \varepsilon X))(1 - \varepsilon X)$$

$$= A_\mu + \varepsilon X A_\mu - \varepsilon A_\mu X - \varepsilon(\partial_\mu X)$$

$$= A_\mu - \varepsilon \partial_\mu X + \varepsilon [X, A_\mu]$$

$$\Rightarrow \delta_X A_\mu = -\varepsilon \partial_\mu X + \varepsilon [X, A_\mu] \in L(G) \quad \square$$

7. $D_\mu \phi \to D'_\mu \phi = D_\mu \phi + \varepsilon R(X) D_\mu \phi$

where $R : G \to \mathrm{Mat}_N(\mathbb{C}) \quad X \in G$

$D_\mu \phi = \partial_\mu \phi + R(A_\mu)\phi$

$A_\mu \to A'_\mu = g A_\mu g^{-1} - (\partial_\mu g) g^{-1}$

$\phi_F \to \phi'_F = g\phi_F = (1+\varepsilon R(X))\phi_F \qquad R(X) = X$ in the fundamental repn

$D^{(F)}_\mu \phi_F = \partial_\mu \phi_F + R(A_\mu)\phi_F = \partial_\mu \phi_F + A_\mu \phi_F$

$D^{(F)}_\mu \phi'_F = \partial_\mu \phi'_F + A'_\mu \phi'_F = \partial_\mu (g\phi_F) + g A_\mu \phi_F - (\partial_\mu g)\phi_F$

$= g(\partial_\mu \phi_F + A_\mu \phi_F) = g D^{(F)}_\mu \phi_F$

$\Rightarrow (D^{(F)}_\mu \phi'_F, D^{\mu(F)}\phi'_F) = (D^{(F)}_\mu \phi_F, D^{\mu(F)}\phi_F) \qquad$ using $g^\dagger = g^{-1}$

$\mathcal{L} = (D^{(F)}_\mu \phi_F, D^{\mu(F)}\phi_F)$

$\phi_A \to \phi'_A = g\phi_A g^{-1} = (1+\varepsilon R(X))\phi_A \qquad R(X) = ad_X$ in the adjoint repn

$D^{(A)}_\mu \phi_A = \partial_\mu \phi_A + R(A_\mu)\phi_A = \partial_\mu \phi_A + [A_\mu, \phi_A]$

$D^{(A)}_\mu \phi_A \to D^{(A)}_\mu \phi'_A = \partial_\mu \phi'_A + [A'_\mu, \phi'_A]$

$= \partial_\mu (g\phi_A g^{-1}) + [gA_\mu g^{-1} - (\partial_\mu g)g^{-1}, g\phi_A g^{-1}]$

$= \partial_\mu (g\phi_A g^{-1}) + g[A_\mu, \phi_A]g^{-1} - (\partial_\mu g)\phi_A g^{-1} + g\phi_A g^{-1}(\partial_\mu g)g^{-1}$

$= g(\partial_\mu \phi_A + [A_\mu, \phi_A])g^{-1} + g\phi_A(\partial_\mu g^{-1}) + g\phi_A g^{-1}\underbrace{(\partial_\mu g)g^{-1}}_{= \partial_\mu(gg^{-1}) - g(\partial_\mu g^{-1})}$

$= g(\partial_\mu \phi_A + [A_\mu, \phi_A])g^{-1} = g(D^{(A)}_\mu \phi_A)g^{-1}$

$\Rightarrow (D^{(A)}_\mu \phi'_A, D^{\mu(A)}\phi'_A) = (D^{(A)}_\mu \phi_A, D^{\mu(A)}\phi_A)$

$\mathcal{L} = (D^{(A)}_\mu \phi_A, D^{\mu(A)}\phi_A)$

8. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad \in \quad \mathcal{L}(G)$

$D_\mu^{(A)} = \partial_\mu + ad_{A_\mu}$

$\Rightarrow F_{\mu\nu} = D_\mu^{(A)} A_\nu - \partial_\nu A_\mu = \partial_\mu A_\nu - D_\nu^{(A)} A_\nu$

$[D_\mu^{(A)}, D_\nu^{(A)}] = [\partial_\mu + ad_{A_\mu}, \partial_\nu + ad_{A_\nu}]$

$= [\partial_\mu, ad_{A_\nu}] + [ad_{A_\mu}, \partial_\nu] + [ad_{A_\mu}, ad_{A_\nu}]$

$= \partial_\mu [A_\nu, \cdot] - [A_\nu, \partial_\mu \cdot] + [A_\mu, \partial_\nu \cdot] - \partial_\nu [A_\mu, \cdot] + [ad_{A_\mu}, ad_{A_\nu}]$

$= [\partial_\mu A_\nu, \cdot] - [\partial_\nu A_\mu, \cdot] + [A_\mu, [A_\nu, \cdot]] - [A_\nu, [A_\mu, \cdot]]$

$= [\partial_\mu A_\nu - \partial_\nu A_\mu, \cdot] + [[A_\mu, A_\nu], \cdot]$

$= ad_{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]} = ad_{F_{\mu\nu}}$

$\Rightarrow R(F_{\mu\nu}) = [D_\mu^{(A)}, D_\nu^{(A)}]$

$R(F_{\mu\nu}) \longrightarrow R(F'_{\mu\nu}) = [D_\mu^{(A)\prime}, D_\nu^{(A)\prime}] = [g D_\mu^{(A)} g^{-1}, g D_\nu^{(A)} g^{-1}]$

$= g [D_\mu^{(A)}, D_\nu^{(A)}] g^{-1} = g R(F_{\mu\nu}) g^{-1}$

$\mathcal{L} = \frac{1}{4g^2} \mathrm{Tr}_N [R(F_{\mu\nu}) R(F^{\mu\nu})]$

$\mathcal{L} \to \mathcal{L}' = \frac{1}{4g^2} \mathrm{Tr}_N [R(F'_{\mu\nu}) R(F^{\mu\nu\prime})] = \frac{1}{4g^2} \mathrm{Tr}_N [g R(F_{\mu\nu}) R(F^{\mu\nu}) g^{-1}]$

$= \frac{1}{4g^2} \mathrm{Tr}_N [R(F_{\mu\nu}) R(F^{\mu\nu})] = \mathcal{L}$ by cyclicity

$\mathcal{L} = \frac{1}{4g^2} \mathrm{Tr}_N [R(F_{\mu\nu}) R(F^{\mu\nu})] = \frac{1}{4g^2} \mathrm{Tr}_N [ad_{F_{\mu\nu}} \circ ad_{F^{\mu\nu}}] = \cancel{\frac{1}{4g^2} K(F_{\mu\nu}, F^{\mu\nu})}$

9. $K(X, Y) = \mathrm{Tr}[ad_X \circ ad_Y] \quad \forall \, X, Y \in \mathcal{L}(G)$

$\Rightarrow \mathcal{L} = \frac{1}{4g^2} K(F_{\mu\nu}, F^{\mu\nu})$