$M, \{\phi_\alpha, U_\alpha\}$ $\qquad \phi_\alpha : U_\alpha \xrightarrow{1-1} V_\alpha \subset \mathbb{R}^n$

$f : M \to \tilde{M}$, $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ smooth

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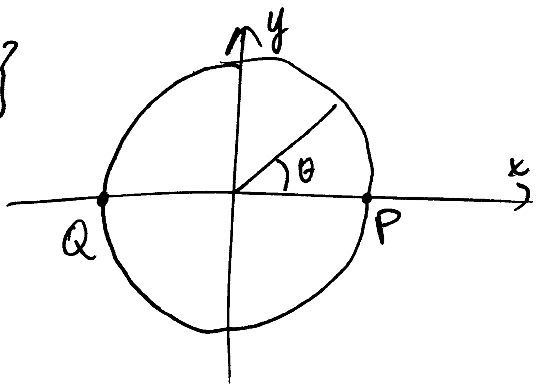
$f : M \to \mathbb{R}$ function (e.g. a scalar field in GR)

One way to specify a function. Let $F_\alpha : V_\alpha \to \mathbb{R}$. Set $f = F_\alpha \circ \phi_\alpha$ (no summation) must agree on $U_\alpha \cap U_\beta$.

E.g. $M = S^1$. $\{(x,y) = (\cos\theta, \sin\theta), \theta \in \mathbb{R}\}$



$\theta \in [0, 2\pi)$ no good (not an open set)

$\phi_1 : S^1 - \{P\} \to (0, 2\pi)$ $\qquad \phi_1(x,y) = \theta_1$

$\phi_2 : S^1 - \{Q\} \to (-\pi, \pi)$ $\qquad \phi_2(x,y) = \theta_2$

$U_1 \cap U_2$ = "upper semicircle" + "lower semicircle"

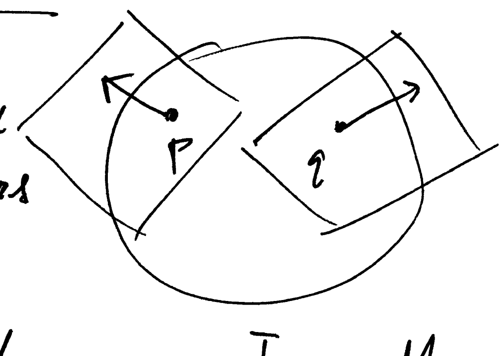
$\phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_2 = \theta_1 \qquad = \theta_2 = \theta_1 - 2\pi$

$f : \underset{S^1}{M} \to \mathbb{R}$, $\qquad F_1 : (0, 2\pi) \to \mathbb{R}$, $\theta_1 \to \sin(m\theta_1)$

$\qquad F_2 : (-\pi, \pi) \to \mathbb{R}$, $\theta_2 \to \sin(m\theta_2)$

$F_1 \circ \phi_1 = F_2 \circ \phi_2 \to$ need $m \in \mathbb{Z}$ $\qquad \sin(m\theta) = \sin(m(2\pi - \theta))$

1.2 Curves and vector fields



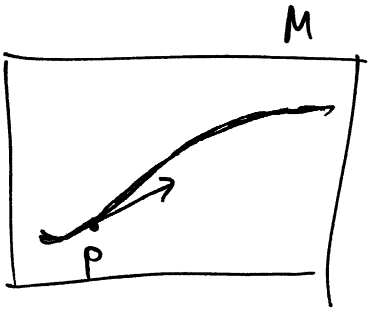
$S^2 \subset \mathbb{R}^3$

tangent planes at p, q are different. Can't add add vectors at p to vectors at q.

(e.g. $I = (0,1)$)

Def^n A smooth curve is a smooth map $\gamma : I \to M$, $I \subset \mathbb{R}$

First discuss vector fields on $M = \mathbb{R}^n$, then bring up charts etc.



$0 \in I$, $\gamma(0) = p$, $U \subset M$, $U \simeq \mathbb{R}^n$

$x^\mu = (x^1, \dots, x^n)$ local coordinates

$\varepsilon \to (x^1(\varepsilon), \dots, x^n(\varepsilon))$

Tangent vector X to γ at p

$$\left.\frac{d\gamma(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = X\big|_p \in T_pM$$

tangent space of all tangent vectors at p to all curves $= \mathbb{R}^n$

$$TM = \bigcup_{p \in M} T_pM = \text{tangent bundle}$$

A vector field assigns a tangent vector to each point $p \in M$.

Let $f : M \to \mathbb{R}$. Rate of change of f along γ:

$$\frac{d}{d\varepsilon} f(x^\mu(\varepsilon)) = \sum_{\mu=1}^{n} \dot{x}^\mu \frac{\partial f}{\partial x^\mu} = \underbrace{\sum_{\mu=1}^{n} X^\mu(x) \frac{\partial}{\partial x^\mu}}_{X} f$$

$$= X(f)$$

$X^\mu(x)$ are components of the vector field X in a basis $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ at p (coordinate basis of T_pM).

Vector fields = 1st order differential operators

<u>Def</u> An integral curve (or a flow) of a vector field X is defined by

$$\dot{\gamma}(\varepsilon) = X\big|_{\gamma(\varepsilon)} \quad \text{or} \quad \boxed{\frac{d}{d\varepsilon} x^\mu(\varepsilon) = X^\mu(x^\nu(\varepsilon))} \quad (*)$$

A system of n 1st order ODEs. Unique solution passing through $x^\mu(0)$.

$$x^\mu(\varepsilon, x^\nu(0)) = x^\mu(0) + \varepsilon \underbrace{X^\mu}_{\text{generator of a curve } \varepsilon \to x^\mu(\varepsilon)} + O(\varepsilon^2)$$

Example ~~the~~ $M = \mathbb{R}^2$, $x^\mu = (x, y)$, $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $X^\mu = (x, 1)$

$$(*) \to \begin{cases} \dot{x} = x \\ \dot{y} = 1 \end{cases} \to (x(\varepsilon), y(\varepsilon)) = (x(0) \cdot e^\varepsilon, y(0) + \varepsilon)$$

$x \exp(-y) = x(0)\, e^\varepsilon \exp(-y(0) - \varepsilon) = x(0) \exp(-y(0))$ ~~is~~ constant along γ

(constant along γ) $\equiv$ invariant of X

Def Invariant of a vector field X is a function constant along the flow.

$$f(x^\mu(0)) = f(x^\mu(\varepsilon)) \longleftrightarrow X(f) = 0 .$$

Example 1-parameter group of rotations in $\mathbb{R}^2$

$$(x(\varepsilon), y(\varepsilon)) = (x_0 \cos\varepsilon - y_0 \sin(\varepsilon), \; x_0 \sin(\varepsilon) + y_0 \cos(\varepsilon)) .$$

$$(*) \to X = \left(\frac{\partial y(\varepsilon)}{\partial \varepsilon} \frac{\partial}{\partial y} + \frac{\partial x(\varepsilon)}{\partial \varepsilon} \frac{\partial}{\partial x} \right) \Big|_{\varepsilon=0} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \text{ (generator of rotations)}$$

$r^2 = x^2 + y^2$ invariant. $X(r^2) = 0$