$$S(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$\mathcal{Z}(m^2,\lambda) = \int_{\mathbb{R}} d\phi\, e^{-\frac{1}{\hbar}S(\phi)} \sim \frac{\sqrt{2\hbar}}{m}\sum_{n=0}^{N}\left(\frac{-\hbar\lambda}{3!\,m^4}\right)^n \frac{1}{n!}\,\Gamma(2n+1/2)$$

$$= \frac{\sqrt{2\hbar\pi}}{m}\sum_{n=0}^{N}\left(\frac{-\hbar\lambda}{m^4}\right)^n \frac{1}{(4!)^n\, n!}\,\frac{(4n)!}{4^n\,(2n)!} \quad \text{using the value of } \Gamma(2n+1/2)$$

$$= \frac{\sqrt{2\pi\hbar}}{m}\left[1 - \frac{\lambda\hbar}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \dots\right]$$

Remarks:

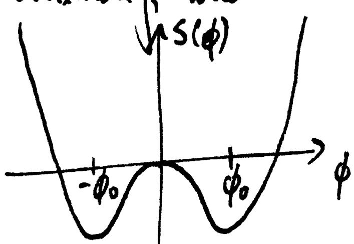
1) Upto the factor $\sqrt{\frac{2\pi\hbar}{m}}$ this series depends on $(\hbar, \lambda)$ only through $\hbar\lambda$ so we can equally view it as an asymp series in the coupling λ.

2) We can see the divergence of this series ~~from~~ for any finite $(\hbar\lambda) > 0$ using Stirling's approximation $n! \sim e^{n \log n}$ as $n \to \infty$ and so we find

$$\frac{1}{(4!)^n\, n!}\,\frac{(4n)!}{4^n\,(2n)!} \sim e^{n \ln n}$$ so the coefficients grow faster than exponentially.

$\Rightarrow$ the series has vanishing radius of convergence

3) If we continue to the region $m^2 < 0$, then $\mathcal{Z}(m^2,\lambda)$ still exists (so long as $\lambda > 0$). However, our asymp series is invalid in this region. Fundamentally, when $m^2 < 0$, the global minima of $S(\phi)$ are now at $\phi_0 = \pm\sqrt{\frac{6m^2}{\lambda}}$



The pt $\phi = 0$ is now a (local) maximum!
Fields with $m^2 < 0$ are called tachyons. These are always signs of some sort of instability. Actions where minima occur at non-zero field values are often associated with spontaneous symmetry breaking.

Feynman Diagrams

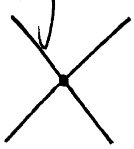
The powers of $\left(\frac{\hbar\lambda}{m^4}\right)^n$ are essentially fixed by dimensional analysis. The numerical coeffs are

$$\underbrace{\frac{1}{(4!)^n\, n!}}_{\text{came from expanding } e^{-\frac{\lambda}{4!}\phi^4} \text{ in } \lambda} \times \underbrace{\frac{(4n)!}{4^n\,(2n)!}}_{\#\text{ of ways of joining } 4n \text{ elements into inequivalent pairs}}$$

We can use these facts to give a diagrammatic rep^n of our asymptotic expansion of $\mathcal{Z}(m^2,\lambda)$

We have the following Feynman rules:

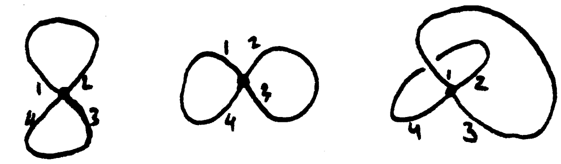
$\hbar/m^2$



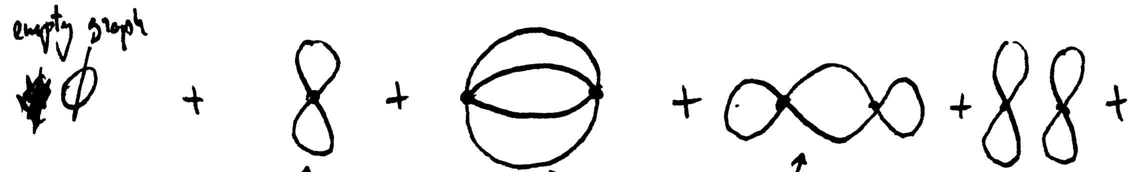
$-\lambda/\hbar$ (− sign since we have e^{-S} in Euclidean QFT)

Feynman tells us to draw all possible vacuum graphs (i.e. graphs w/ 0 external edges, when computing $\mathcal{Z}(m^2,\lambda)$ using the propagator + vertex.

e.g. at order λ, can draw



$$\frac{Z(m^2,\lambda)}{Z(m^2,0)} \sim$$



$$1 + \frac{-\lambda\hbar}{m^4}\times\frac{1}{8} + \frac{\lambda^2\hbar^2}{m^8}\times\frac{1}{48} + \frac{\lambda^2\hbar^2}{m^8}\times\frac{1}{16} + \frac{\lambda^2\hbar^2}{m^8}\times\frac{1}{128}$$

The numerical factors have a combinatoric interpretation:

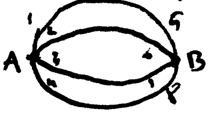
$\left(\text{e.g. } \frac{1}{8} = \frac{1}{\cancel{4!}}\frac{\cancel{4!}}{4\times 2!} \quad \checkmark\right)$

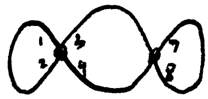
Let D_n be the set of all Feynman graphs with n vertices. Suppose there are $|D_n|$ of these. D_n is acted on by a group $G_n = (S_4)^n \rtimes S_n$ where S_4 permutes the 4 legs on a given vertex, and S_n the different vertices in our graph. $|G_n| = (4!)^n n!$, so $1/|G_n|$ is just the factor we get from expanding $\exp(-\frac{\lambda}{4!}\phi^4)$. Thus we have

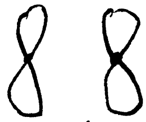
$$\frac{Z(m^2,\lambda)}{Z(m^2,0)} \sim \sum_{n=0}^{N} \frac{|D_n|}{|G_n|}\left(\frac{-\hbar\lambda}{m^4}\right)^n$$

There's another way to think about $|D_n|/|G_n|$. Let Γ be an <u>orbit</u> of G_n inside D_n, i.e. Γ is a topologically distinct graph and G_n just permutes labels. Let O_n be the set of all such orbits. Then the <u>orbit-stabilizer theorem</u> says $\frac{|D_n|}{|G_n|} = \sum_{\Gamma\in O_n}\frac{1}{|\mathrm{Aut}\,\Gamma|}$ where

$\mathrm{Aut}\,\Gamma$ is the stabilizer group of a <u>labelled</u> graph.

e.g. $\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 4!\times 2 = 48$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 2^3\times 2 = 16$ $\quad\left(\frac{1}{48}+\frac{1}{16}+\frac{1}{128} = \frac{35}{384}\right)$

$\Gamma =$  $|\mathrm{Aut}\,\Gamma| = 8\times 8\times 2 = 128$

More generally, we may have a theory with several fields and propagators of type $\frac{1}{P_i}$, and many different interactions with coupling constants λ_α. Then,

$$\frac{Z(\lambda_i)}{Z(0)} \sim \sum_{n=0}^{N}\sum_{\Gamma\in O_n}\frac{1}{|\mathrm{Aut}\,\Gamma|}\frac{\prod_\alpha \lambda_\alpha^{|v_\alpha(\Gamma)|}}{\prod_i P_i^{|e_i(\Gamma)|}}\hbar^E$$

where $e_i(\Gamma)$ ($v_\alpha(\Gamma)$) are # edges (vertices) of type i (α) in Γ.

If E, V are total # edges, vertices, then we have a factor,

$\hbar^{E-V} = \hbar^{L-C}$ (by Euler) where L is # loops in the graph and C is # connected components.

Remark: $\ln(Z/Z_0)$ is the sum of all <u>connected</u> graphs.