2.6 Curvature of the Levi-Civita connection

$(M, g) \to \exists$ unique torsion-free ∇ s.t. $\nabla_a g_{bc} = 0$

<u>Proof:</u> $\underbrace{R_{abcd} = R_{cdab}}_{(1)}$ $\quad\underbrace{R_{(ab)cd} = 0}_{(2)}$,

(1) => (2) as $R_{ab(cd)} = 0$

Prove (1) in normal coordinates $R = \partial\Gamma - \Gamma\Gamma$

<u>Note</u> $\partial_\rho g_{\nu\rho}|_p = 0 \qquad \partial_\rho((g^{-1})^{\nu\rho} g_{\rho\sigma}) = \partial_\rho(\delta^\nu{}_\sigma) = 0$

$g_{\rho\sigma}\partial_\mu g^{\nu\rho}|_p = 0 \qquad \partial_\rho \Gamma^\tau_{\nu\sigma}|_p = \frac{1}{2} g^{\tau\mu}(g_{\mu\nu,\sigma\rho} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho})|_p$

$$R_{\mu\nu\rho\sigma}|_p = \frac{1}{2}(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma})|_p$$
$$= R_{\rho\sigma\mu\nu}|_p$$
, so (1) holds in normal coordinates □

<u>Prop</u> The Ricci tensor of LC connection is symmetric $R_{ab} = R_{ba}$

<u>Proof</u> $R_{ab} = g^{cd} R_{dacb} \overset{(1)}{=} g^{cd} R_{cbda} = R^d{}_{bda} = R_{ba}$ □

<u>Defn</u> Ricci scalar $R = g^{ab} R_{ab}$

<u>Defn</u> Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ $\qquad$ Exercise: $\nabla^a G_{ab} = g^{ac}\nabla_c G_{ab} = 0$

3 Einstein equations

$$G_{ab} = 8\pi G T_{ab}$$

T_{ab} = energy-momentum tensor of matter

LHS = Geometry (Marble palace)
RHS = (Wooden shed)

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \qquad \Lambda = \text{cosmological constant}$$

Vacuum equations $T_{ab} = 0 \rightsquigarrow G_{ab} = 0$, so

$$R_{ab} - \frac{1}{2} R g_{ab} = 0 \quad / \cdot g^{ab} \rightsquigarrow R - (\tfrac{1}{2}\times 4)R = -R = 0$$

so $R_{ab} = 0$

System of non-linear, 2^{nd} order coupled PDEs for g_{ab}.

In general, an n-dim Einstein mfd is a (pseudo) Riemannian mfd such that

$$R_{ab} = \frac{1}{n} R g_{ab} \qquad \text{(vacuum } R=0\text{)}$$

* $n=2$ Any metric is Einstein as $R_{ab} = \underbrace{\tfrac{1}{2}}_{} R g_{ab}$

Gaussian curvature

e.g. $g = \frac{dx^2+dy^2}{y^2}$ constant negative R

* $n=3$ Riemann tensor completely determined by Ricci tensor.

Einstein mfd = (locally) space of constant curvature S^3, H^3, $\mathbb{R}^3$

$R=1$, $R=-1$, $R=0$

[No propagating degrees of freedom $\to$ solvable 3D Quantum Gravity]

* $n \geq 4$ Riemann = Weyl + Ricci + scalar

$$R_{abcd} = \underbrace{C_{abcd}}_{\text{Weyl tensor}} + \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) - \frac{2}{(n-1)(n-2)} R\, g_{a[c}g_{d]b}$$

Weyl tensor = gravitational degrees of freedom, not fixed by Einstein's eq

- trace free on all pairs of indices, e.g. $g^{ac} C_{abcd} = 0 \ldots$
- conformal rescaling, $\Omega : M \to \mathbb{R}^+$, $\hat{g} = \Omega^2 \cdot g$

$\hat{\Gamma}^{\sigma}_{\mu\nu}$ = mess, $\hat{R}^a{}_{bcd}$ = mess, $\hat{C}^a{}_{bcd} = C^a{}_{bcd}$

Null geodesics of $\hat{g}$ = null geodesics of g

Say (M,g) is conformally flat if $\exists\, \Omega : M \to \mathbb{R}^+$ s.t. $\hat{g} = \Omega^2 g$ is flat.

g is conf flat iff $C^a{}_{bcd} = 0$

Examples $(n=4)$

- Schwarzschild metric, $R_{ab} = 0$
- $S^4 \subset \mathbb{R}^5$, metric induced by $|d\underline{x}|^2$ from $\mathbb{R}^5$ (Riemannian)

$C^a{}_{bcd} = 0$, $R_{ab} = \frac{1}{4} R g_{ab}$

positive constant

• pp wave . Local chart on M (x, y, u, v)

$$g = dx^2 + dy^2 + du\,dv + H(u,x,y)\,du^2$$

if $H = 0$, set $u = z + t$, $v = z - t$ Minkowski space

Say H is general, impose $R_{ab} = 0$ $\rightarrow H_{xx} + H_{yy} = 0$ $(H_x = \partial_x H)$

$$H(u,x,y) = \mathrm{Re}\left(f(\zeta, u)\right), \quad \zeta = x + iy, \quad f \text{ holomorphic}$$