irreps of $\mathcal{L}(SU(2))$

$R_\Lambda \quad \Lambda \in \mathbb{Z} \quad \Lambda \geq 0$

$S_\Lambda = \{-\Lambda, -\Lambda+2, \dots, \Lambda-2, \Lambda\} \subset \mathbb{Z}$

$R(H) \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \dim(R_\Lambda) = \Lambda + 1$

$SU(2)$ repns from $\mathcal{L}(SU(2))$ repns

smooth map:

$D : G \to GL(n, F) \qquad n \in \mathbb{N}, \; F = \mathbb{R} \text{ or } \mathbb{C}$

$D(g_1) D(g_2) = D(g_1 g_2) \qquad \forall g_1, g_2 \in G$

Locally parametrise group elements $A \in SU(2)$

starting from irreps of $\mathcal{L}(SU(2))$ $\qquad \mathrm{Exp}(\underline{n} \cdot \underline{\sigma}) = \cos|\underline{n}| \mathbb{1} + i(\underline{n} \cdot \underline{\sigma}) \sin|\underline{n}|$

$D_\Lambda(A) = \mathrm{Exp}(R_\Lambda(X)) \qquad \Lambda \in \mathbb{Z} \quad \Lambda \geq 0$

$\rightsquigarrow$ repn of $SU(2)$ /// $\qquad A \in SU(2)$

not ~~equal~~ in general a repn of $SO(3) \simeq \frac{SU(2)}{\mathbb{Z}_2} \qquad \{A, -A\}$

we require,

$D_\Lambda(-\mathbb{1}_2) = D_\Lambda(\mathbb{1}_2) \qquad -(*)$

$\Rightarrow D_\Lambda(-A) = D_\Lambda(A) \qquad \forall A \in SU(2)$

$-\mathbb{1}_2 = \mathrm{Exp}(i\pi H) \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow D_\Lambda(-\mathbb{1}_2) = \mathrm{Exp}(i\pi R_\Lambda(H))$

$R_\Lambda(H)$ has eigenvals $\lambda \in S_\Lambda$

$\Rightarrow D_\Lambda(-\mathbb{1}_2)$ has eigenvals $\exp(i\pi\lambda) = (-1)^\lambda = (-1)^\Lambda$

Two cases

- $\Lambda \in 2\mathbb{Z}$, $\Rightarrow$ D_Λ is a repn of $SU(2)$ and $SO(3)$
- $\Lambda \in 2\mathbb{Z}+1$ $\Rightarrow$ D_Λ is a repn of $SU(2)$ <u>not</u> $SO(3)$

"a spinor repn" of $SO(3)$

<u>New repns from old</u>

R is a repn of a real Lie algebra g,

<u>conjugate repn</u> ~~of~~ $\bar{R}$

$$\bar{R}(X) = R(X)^* \qquad \forall X \in g$$

sometimes $\bar{R} \simeq R$

R_1 and R_2 are repns of g, repn spaces V_1 and V_2 dimensions d_1 and d_2

	dim
$V_1 \oplus V_2$	d_1+d_2
$V_1 \otimes V_2$	$d_1 d_2$

- The direct sum $R_1 \oplus R_2$ act on

$$V_1 \oplus V_2 = \{ v_1 \oplus v_2 \ , \ v_1 \in V_1 \ , \ v_2 \in V_2 \}$$

$$(R_1 \oplus R_2)(X) \cdot (v_1 \oplus v_2) = (R_1(X) v_1) \oplus (R_2(X) v_2) \quad \forall X \in g, \ v_1 \in V_1, \ v_2 \in V_2$$

matrix corresponding to $(R_1 \oplus R_2)(X)$

$$= \left(\begin{array}{c|c} R_1(X) & 0 \\ \hline 0 & R_2(X) \end{array} \right) \begin{array}{l} \updownarrow d_1 \\ \updownarrow d_2 \end{array}$$

$$\dim(R_1 \oplus R_2) = \dim(R_1) + \dim(R_2) = d_1 + d_2 \quad ///$$

<u>Tensor product</u>

$R_1 \otimes R_2$ will act on

$V_1 \otimes V_2$ spanned by elements $v_1 \otimes v_2$; $v_1 \in V_1$, $v_2 \in V_2$

- Given two linear maps

$$M_1 : V_1 \to V_1$$
$$M_2 : V_2 \to V_2$$

define <u>tensor product map</u>

$$(M_1 \otimes M_2) : V_1 \otimes V_2 \to V_1 \otimes V_2 \quad \text{by}$$

$$(M_1 \otimes M_2)(v_1 \otimes v_2) = (M_1 v_1) \otimes (M_2 v_2) \quad \in V_1 \otimes V_2 \quad \forall v_1 \in V_1, v_2 \in V_2$$

Given R_1 and R_2, repn spaces V_1 and V_2 for $X \in g$

$$R_1(X) : V_1 \to V_1$$

$$R_2(X) : V_2 \to V_2$$

define tensor product repn $(R_1 \otimes R_2)$. For each $X \in g$

$$(R_1 \otimes R_2)(X) : V_1 \otimes V_2 \to V_1 \otimes V_2 \quad \text{linear map}$$

$$(R_1 \otimes R_2)(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X)$$ I_1, I_2 are identity maps on V_1, V_2

$$[(R_1 \otimes R_2)(X) \neq R_1(X) \otimes R_2(X)]$$

<u>Choosing bases</u>

$$B_1 = \{ v_1^j \,;\, j = 1, \dots, d_1 \} \qquad B_2 = \{ v_2^\alpha : \alpha = 1, \dots, d_2 \}$$

$(R_1 \otimes R_2)(X)$ becomes a "matrix" $\qquad i, j = 1, \dots, d_1$; $\alpha, \beta = 1, \dots, d_2$

$$(R_1 \otimes R_2)(X)_{i\alpha, j\beta} = R_1(X)_{ij} \mathbb{1}_{\alpha\beta} + \mathbb{1}_{ij} R_2(X)_{\alpha\beta}$$

$\dim(R_1 \otimes R_2) = \dim(R_1) \dim(R_2) = d_1 d_2$ /// <u>Exercise</u> show $R_1 \otimes R_2$ is a repn of g

An irrep R of g has no non-trivial invariant subspaces $\left\{ \begin{array}{l} R(X)u \in U \\ \forall X \in g \\ U \subset V \text{ is invariant subspace} \end{array} \right.$

If R takes the form

$$R(X) = \left(\begin{array}{c|c} A(X) & B(X) \\ \hline 0 & C(X) \end{array} \right) \quad \forall X \in g$$ vectors of the form $\begin{pmatrix} u \\ \hline 0 \end{pmatrix}$ invariant subspace

fully reducible

$$R(X) = \begin{pmatrix} R_1(X) & & & \\ & R_2(X) & & \\ & & \ddots & \\ & & & R_n(X) \end{pmatrix} \qquad R = R_1 \oplus R_2 \oplus \dots \oplus R_n$$