# 4 Tour into non-Riemannian geometries

## Newton-Cartan geometries

Trajectories in Newtonian physics:

$\ddot{\underline{x}} = -\nabla\phi$ , $\phi : \mathbb{R}^3 \to \mathbb{R}$ ; $\phi$ : gravitational potential (N)

Have absolute time.

Can we interpret N as a geodesic eq in some 4-dim spacetime w/ a torsion-free connection?

Let $x^\mu = (t, \underline{x}) = (x^0, \underline{x})$.

$d^2_{\tau^2} x^\mu + \Gamma^\mu{}_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0$ ; take $\tau = t = x^0$ , $d\tau = {}^\cdot$

Define $\nabla$ , $\Gamma_{00}{}^i = \delta^{ij} \partial_j \phi$ , other components vanish

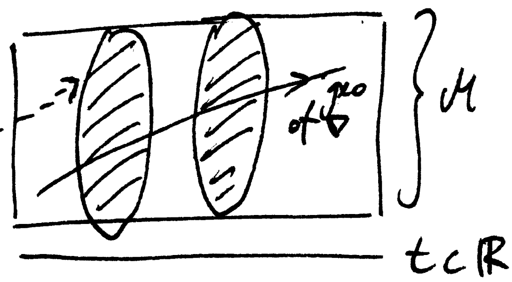
$d^2_{t^2} x^i + \Gamma^i{}_{00} \left(\frac{dt}{dt}\right)^2 = 0 \iff$ N.

Exercise: Show $\nabla$ not the Levi-Civita connection of any metric.

Def$^n$ A Newton-Cartan structure is a tuple $(h, \nabla, \theta)$ (on an $n$-mfd) s.t.

- $h$ a degenerate $(2,0)$ tensor (rank $n-1$)
- $\exists$ closed 1-form $\theta_a$ s.t. $h^{ab}\theta_a = 0$
- $\exists$ torsion-free connection $\nabla$ s.t. $\nabla_a \theta_b = 0$ , $\nabla_a h^{bc} = 0$

$\nabla_{[a}\theta_{b]} = 0 \xrightarrow{\text{locally}} \exists\, t : \mathcal{M} \to \mathbb{R}$ s.t. $\theta = dt$




$h$ non-degenerate on $t=$const slices

e.g. in 4D , $h = \mathrm{diag}(0,1,1,1)$ , $\theta = dt$

$\Gamma^i{}_{00} = h^{ij}\partial_j \phi$ , $\Gamma^i{}_{0j} = \frac{1}{2} F^i{}_j$ where

$F = \frac{1}{2} F_{ij}(x)\, dx^i \wedge dx^j + E_i\, dx^i \wedge dt$ , $dF = 0$

## Projective structures

Two torsion-free connections $\nabla, \hat{\nabla}$ are projectively equivalent if they share the same unparameterised geodesics.

$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = f \dot{x}^\mu$ , $x^\mu = (x^1, \dots, x^n)$

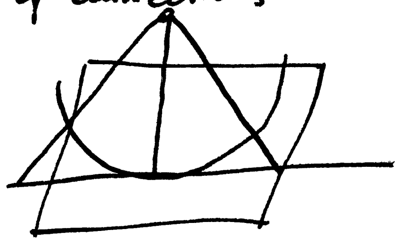
e.g. lines in $\mathbb{R}^2$ $(x,y)$ $\left.\begin{cases} \ddot{x} = 0 \Rightarrow x = \alpha t + \beta \\ \ddot{y} = 0 \Rightarrow y = \gamma t + \delta \end{cases}\right\} \Rightarrow d^2_{x^2} y = 0 \Rightarrow y = Ax + B$

Exercise : $\hat{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + \delta_\nu{}^\mu \omega_\rho + \delta_\rho{}^\mu \omega_\nu$ for some $\omega_\rho$ $\Leftrightarrow$ $\hat{\nabla} \sim \nabla$ :

projective equivalence classes

e.g. $\mathbb{R}^n$, $S^n$ w/ induced metric

Stereo projection takes great circle to lines $\Rightarrow$ equivalence of connections



<u>Magnetic geometries, Kaluza-Klein reductions</u>

<u>ex</u>: $(M, g)$ 3-dim Riemannian mfd, $(x, y, z)$ local chart

$g = dx^2 + dy^2 + (dz - x\,dy)^2$ (Heis metric)

geo Lagrangian: $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2\right)$.

E-L eqns:

$\partial_x L - d_t \partial_{\dot{x}} L = 0 \Rightarrow \ddot{x} = -\dot{y}(\dot{z} - x\dot{y})$

$d_t \partial_{\dot{y}} L = 0 \Rightarrow 0 = \dot{y} - x(\dot{z} - x\dot{y})$

$d_t \partial_{\dot{z}} L = 0 \Rightarrow c = \dot{z} - x\dot{y}$ ("charge")

$\ddot{x} = -c\dot{y}$, $\ddot{y} = c\dot{x}$ $\Rightarrow$ circle in $(x, y)$ plane (*)

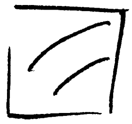
Compare with geos in magnetic field.

$(N,\ h = h_{ij}\,dx^i dx^j,\ F = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j)$

$\ddot{x}^i + \Gamma^i{}_{jk}\, \dot{x}^j \dot{x}^k = c F^i{}_j\, \dot{x}^j$

Take $N = \mathbb{R}^2$, $h_{ij} = \delta_{ij}$, $F_{ij} = -\epsilon_{ij}$, $F = -dx \wedge dy$

$\Gamma^i{}_{jk} = 0$, magnetic geos = (*)



$(M, g)$

In general, given $(N, h, F)$, $F = dA$, $A = A_i\, dx^i$ mag potential

$(N, h, F)$

Take $(M, g)$, $g = h + (dz + A)^2$ (special case $A = -x\,dy$).

E-L eqn for $g$ $\rightsquigarrow$ mag E-L for $(h, F)$.