1 Use a direct proof to show that the sum of two odd integers is even.

Proof. Assume a is even and b is odd.

$$\text{Let } a = 2k,\ \text{k is some integer}$$

$$\text{Let } b = 2k + 1,\ \text{k is some integer}$$

$b + b = n$	The sum of 2 odd integers	(1)
$2k + 1 + 2k + 1 = n$	Expand b	(2)
$4k + 2 = n$	Simplify	(3)
$2(2k + 1) = n$	Factor out 2	(4)
$2b = n$	Definition of an even number, b is some integer	(5)

■

3 Show that the square of an even number is an even number using a direct proof.

Proof. Assume a is even.

$$\text{Let } a = 2k,\ \text{k is some integer}$$

$a * a = n$	The square of two even numbers	(1)
$2k * 2k = n$	Expand a	(2)
$2 * 2k^2 = n$	Rearrange terms	(3)
Let $j = 2k^2$		(4)
$2j = n$	Definition of an even number, j is some integer	(5)

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4 Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

Proof. Assume a is even.

$$\text{Let } a = 2k,\ \text{k is some integer}$$

$a * -1 = n$	Additive inverse of an even number	(1)
$2k * -1 = n$	Expand	(2)
$2(-k) = n$	Rearrange terms	(3)
Let $j = -k$		(4)
$2j = n$	Definition of an even number, j is some integer	(5)

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6 Use a direct proof to show that the product of two odd numbers is odd.

Proof. Assume a is odd.

$$\text{Let } a = 2k + 1,\ \text{k is some integer}$$

$$\text{Let i be some integer}$$

$$\text{Let j be some integer}$$

$ai * aj = n$	Product of 2 odd numbers	(1)
$i(2k+1) * j(2k+1) = n$	Expand	(2)
$(2k+1)\,(i * j * (2k+1)) = n$	Rearrange terms	(3)
Let $m = (i * j * (2k+1))$		(4)
$am = n$	Definition of an odd number, m is some integer	(5)

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8 Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

Proof. We can define a as a perfect square

$$\text{Let } a = k^2,\ \text{k is some integer}$$

We can prove this by contradiction by assuming that if $n = i^2$ then $n + 2 = j^2$.

$n + 2 = j^2$	Assumption	(1)
$2 = j^2 - n$	Subtract n from both sides	(2)
$2 = j^2 - i^2$	Replace n with i^2	(3)
$2 = (j - i)(j + i)$	Factor	(4)
$\begin{array}{c} \text{j - i} = 1 \\ \underline{\text{+j + i} = 2} \\ 2j = 3 \end{array}$		(5)
$j = \dfrac{2}{3}$	j is not an integer so this is a contradiction	(6)

■

9 Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Proof. We can define a rational number as a fraction of 2 integers.

$$\text{Let } r = \frac{p}{q}, \text{ p and q are integers, } q \neq 0$$

Let x be an irrational number and let y be a rational number. To prove this by contradiction we will assume that the sum of x and y are rational. Using the definition of a rational number we can say

$$x + y = \frac{p}{q}, \text{ p and q are integers, } q \neq 0$$

and that

$$y = \frac{a}{b}, \text{ a and b are integers, } q \neq 0$$

such that

$$x + \frac{a}{b} = \frac{p}{q}$$

From here we can subtract $\frac{a}{b}$ from both sides

$$x = \frac{p}{q} - \frac{a}{b} = \frac{pb - aq}{qb}$$

a, b, p, and q are all nonzero integers. This would imply that x is rational, which contradicts our original assumption that x is irrational. Therefore our assumption that x + y is rational is false.

∎

13 Prove that if x is irrational, then 1/x is irrational.

Proof. We can define a rational number as a fraction of 2 integers.

$$\text{Let } r = \frac{p}{q}, \text{ p and q are integers, } q \neq 0$$

Let x be an irrational number. To prove this by contradiction we will assume that 1/x is rational. Using the definition of a rational number we can say

$$\frac{1}{x} = \frac{p}{q}, \text{ p and q are integers, } q \neq 0$$

Taking the reciprocal of both sides we get

$$x = \frac{q}{p}$$

This would imply that x is a rational number, which contradicts our assumption that 1/x is rational if x is irrational. Therefore our assumption that $\frac{1}{x}$ is rational is false

∎

17 Show that if n is an integer and $n^3 + 5$ is odd, then n is even

a Using proof by contraposition

Proof. We can define an even number as a and an odd number as b where

$$\text{Let } a = 2k, \text{ k is some integer}$$

$$\text{Let } b = 2k + 1, \text{ k is some integer}$$

To prove this by contraposition, we must show that that if n is **not** even then $n^3 + 5$ is **not** odd. In other words, if n is odd then $n^3 + 5$ is even. We will let n be an odd number such that

$$n = 2k + 1, \text{ k is some integer}$$

so we can write this statement as

$$m = (2k + 1)^3 + 5$$

Expanding this out gives us

$$m = 8k^3 + 12k^2 + 6k + 6$$

From here, we can factor a 2 from the entire polynomial

$$m = 2(4k^3 + 6k^2 + 3k + 3)$$

Since k is an integer, the result of $4k^3 + 6k^2 + 3k + 3$ will also be an integer, which we will define as c. We can then say

$$m = 2c$$

Which matches the definition of an even number. Thus we have proved that that if n is odd then $n^3 + 5$ is even

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b Using proof by contradiction

Proof. We can define an even number as a and an odd number as b where

$$\text{Let } a = 2k, \text{ k is some integer}$$

$$\text{Let } b = 2k + 1, \text{ k is some integer}$$

To prove this by contradiction we will assume that if $n^3 + 5$ is odd, then n is odd. We will let n be an odd number such that

$$n = 2k + 1, \text{ k is some integer}$$

and that the result of $(2k + 1)^3 + 5$ is odd such that so we can write this statement as

$$2j + 1 = (2k + 1)^3 + 5, \text{ j is some integer}$$

Expanding this out gives us

$$2j + 1 = 8k^3 + 12k^2 + 6k + 6$$

From here, we can factor a 2 from the right side

$$2j + 1 = 2(4k^3 + 6k^2 + 3k + 3)$$

Since k is an integer, the result of $4k^3 + 6k^2 + 3k + 3$ will also be an integer, which we will define as c. We can then say

$$2j + 1 = 2c$$

This is saying that an even number is equal to an odd number, which is a contradiction. Therefore if $n^3 + 5$ is odd, then n must be even

■

19 Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

Proof. The proposition states that if $n > 1$ then $n^2 > n$ so for $P(0)$, if $0 > 1$ then $0 > 0$. This is trivially true by vacuous proof.

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23 Show that at least ten of any 64 days chosen must fall on the same day of the week.

Proof. We will prove this using a contradiction. Assume that only 9 or less of any 64 days fall on the same day of the week. This means we can choose $9 * 7 = 63$ days, which is less than 64 days, so this is a contradiction.

■

27 Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd

Proof. Because this is a biconditional statement, our proof will have 2 parts. First we will prove that if n is odd then $5n + 6$ is odd using a direct proof. To do this we will need to define an odd number a and an even number b as such

$$\text{Let } a = 2k + 1, \text{ k is some integer}$$

$$\text{Let } b = 2k, \text{ k is some integer}$$

We will let n be some odd number such that

$$n = 2k + 1$$

and that

$$5(2k + 1) + 6 = 2j + 1, \text{ j is some integer}$$

Distributing the 5 to $2k + 1$ gives us

$$10k + 5 + 6 = 2j + 1$$

Which can then be rewritten as

$$10k + 10 + 1 = 2j + 1$$

We can now factor out a 2 from $10k + 10$

$$2(5k + 5) + 1 = 2j + 1$$

k is an integer so the result of $5k + 5$ will also be an integer which we will represent with c

$$2c + 1 = 2j + 1$$

Using the definition of an odd number, we have shown that if n is odd then $5n + 6$ is odd.

We now need to prove that if $5n + 6$ is odd then n is odd. This time we will use a proof by contradiction. Suppose if $5n + 6$ is odd, then n is not odd. Using the definitions of even and odd numbers, we will let n be some even number such that

$$n = 2k, \text{ k is some integer}$$

and we will let $5n + 6$ be an odd number such that

$$5(2k) + 6 = 2j + 1, \text{ j is some integer}$$

Multiplying out the left side gives us

$$10k + 6 = 2j + 1$$

We can factor out a 2 from $10k + 6$

$$2(5k + 3) = 2j + 1$$

And because k is an integer, the result of $5k + 3$ will also be an integer which we will represent with c

$$2c = 2j + 1$$

Using the definition of even and odd numbers, this contradicts our supposition that if $5n + 6$ is odd, then n is not odd. Therefore, we have shown that if $5n + 6$ is odd then n is odd.

■

35 Are these steps for finding the solutions of $\sqrt{x+3} = 3 - x$ correct?

$\sqrt{x+3} = 3 - x$	Given	(1)
$x + 3 = x^2 - 6x + 9$	Square both sides of (1)	(2)
$0 = x^2 - 7x + 6$	Subtract x+3 from both sides of (2)	(3)
$0 = (x-1)(x-6)$	Factor the right-hand side of (3)	(4)
$x = 1$ or $x = 6$,	Follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$	(5)

This is not correct because $3 - 6$ would yield a negative number which is not in the range of a square root