

- 1 Reparametrize the curve $\vec{r}(t) = (4 \cos t)\vec{i} + (4 \sin t)\vec{j} + 3t\vec{k}$ with respect to the arc length measured from $(4, 0, 0)$ in the direction of increasing t .

$$\vec{r}(0) = \langle 4 \cos 0, 4 \sin 0, 3(0) \rangle = \langle 4, 0, 0 \rangle$$

Let $t = 0$

$$\vec{r}'(t) = \langle -4 \sin t, 4 \cos t, 3 \rangle$$

Derivative of $\vec{r}(t)$

$$s(t) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

Arc length formula

$$s(t) = \int_0^t \sqrt{(-4 \sin t)^2 + (4 \cos t)^2 + 3^2} dt$$

Substitute the components of $\vec{r}'(t)$

$$s(t) = \int_0^t \sqrt{25} dt$$

Evaluate the integral

$$s(t) = \sqrt{25} \Big|_0^t$$

$$s(t) = \sqrt{25}(t - 0)$$

$$t(s) = \frac{s}{5}$$

t in terms of arc length

$$\vec{r}(t(s)) = 4 \cos \frac{s}{5} \vec{i} + 4 \sin \frac{s}{5} \vec{j} + \frac{3s}{5} \vec{k}$$

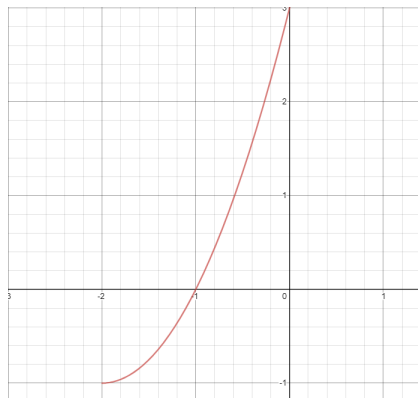
Curve reparametrized in terms of arc length

2c Given the plane curve $\vec{r}(t) = \langle t - 2, t^2 - 1 \rangle$

Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}'(t)$ at $t = 1$

t	x	y
0	-2	-1
1	-1	0
2	0	3
3	1	8
4	2	15
5	3	24

Table of values



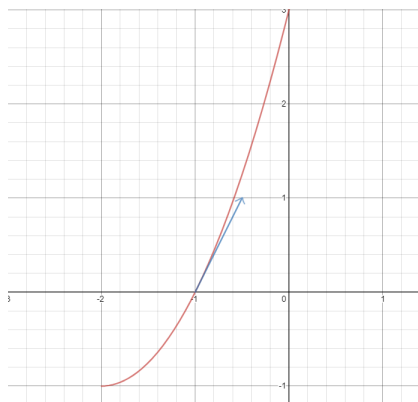
$$\vec{r}'(t) = \langle 1, 2t \rangle$$

$$\vec{r}'(1) = \langle 1, 2 \rangle$$

Graph of $\vec{r}(t)$

Take the derivative of $\vec{r}(t)$

Substitute $t = 1$



Graph $\vec{r}'(1)$ at $t = 1$

3a Given $9y^2 + 4z^2 = x^2 + 36$

a. Rewrite the equation in standard form

$$9y^2 + 4z^2 = x^2 + 36$$

Original equation

$$9y^2 + 4z^2 - x^2 = 36$$

Subtract x^2 on both sides

$$\frac{y^2}{4} + \frac{z^2}{9} - \frac{x^2}{36} = 1$$

Divide by 36 on both sides

$$-\frac{x^2}{36} + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

Rearrange terms

$$-\frac{x^2}{6^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$$

Rewrite denominators

b. Identify the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

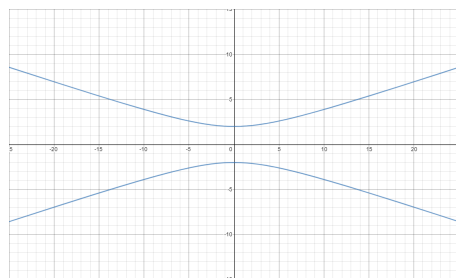
Standard form of a hyperboloid of 1 sheet

$$-\frac{x^2}{6^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$$

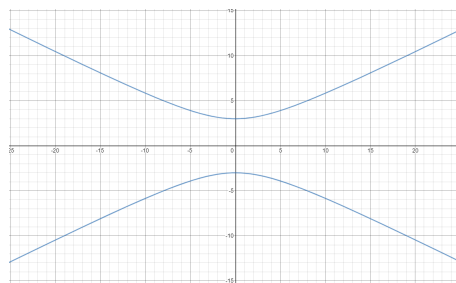
Equation matches the form

Hyperboloid of 1 sheet

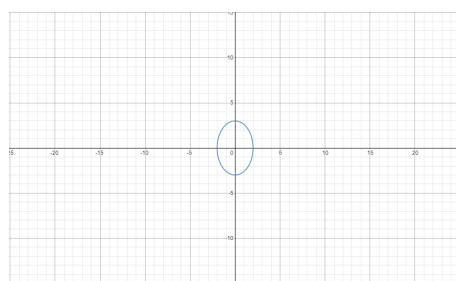
c. Draw each trace for the coordinate planes and describe the conic section. Label and scale your axes.



xy axis, Let $z = 0$, Hyperbola

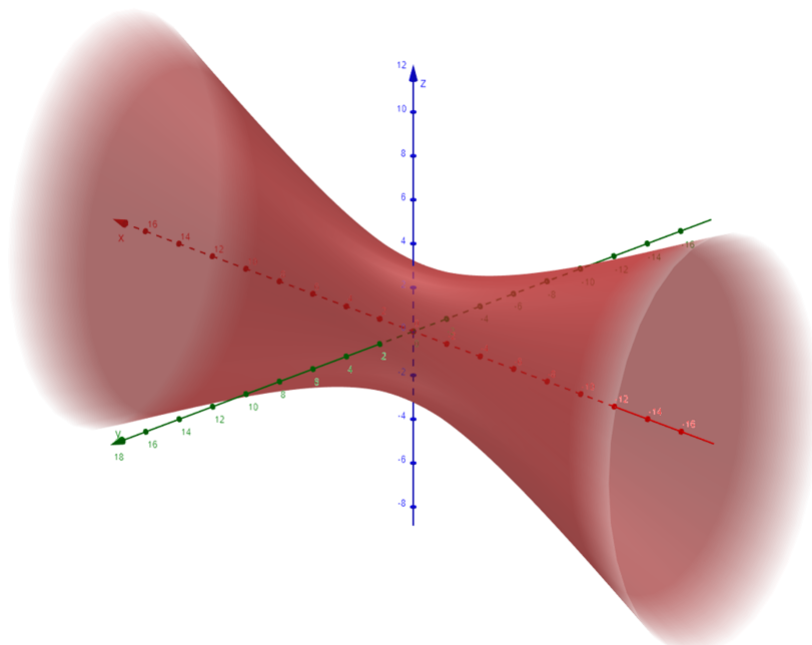


xz axis, Let $y = 0$, Hyperbola



yz axis, Let $x = 0$, Ellipse

d. Graph $9y^2 + 4z^2 = x^2 + 36$. Label and scale the axes.



4 Given $\vec{a}, \vec{b} \in \mathbb{R}^3$, show that $\vec{a} \times \vec{b}$ is orthogonal to \vec{a} and \vec{b} .

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

Angle between 2 vectors

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\left(\frac{\pi}{2}\right)$$

$$\text{Let } \theta = \frac{\pi}{2}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| * 0$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\therefore \text{When } \theta = \frac{\pi}{2},$$

the dot product is equal to 0

Given

$$\text{Let } \vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\text{Let } \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Take the cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1 \rangle$$

$$\langle a_1, a_2, a_3 \rangle \cdot \langle a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1 \rangle$$

Take the dot product with \vec{a}

$$= a_1(a_2b_3 - a_3b_2) + a_2(-a_1b_3 + a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1$$

$$= (a_1a_2b_3 - a_1a_2b_3) + (a_1b_2a_3 - a_1b_2a_3) + (b_1a_2a_3 - b_1a_2a_3)$$

Rearrange terms

$$= 0$$

\therefore the angle between

$$\vec{a} \text{ and } \vec{a} \times \vec{b} = \frac{\pi}{2}$$

$$\langle a_1, a_2, a_3 \rangle \cdot \langle a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1 \rangle$$

Take the dot product with \vec{b}

$$= b_1(a_2b_3 - a_3b_2) + b_2(-a_1b_3 + a_3b_1) + b_3(a_1b_2 - a_2b_1)$$

$$= b_1a_2b_3 - b_1a_3b_2 - b_2a_1b_3 + b_2a_3b_1 + b_3a_1b_2 - b_3a_2b_1$$

$$= (b_1a_2b_3 - b_1a_2b_3) + (a_1b_2b_3 - a_1b_2b_3) + (b_1b_2a_3 - b_1b_2a_3)$$

Rearrange terms

$$= 0$$

\therefore the angle between

$$\vec{b} \text{ and } \vec{a} \times \vec{b} = \frac{\pi}{2}$$

We know the angle between $\vec{a} \times \vec{b}$ and \vec{a} is $\frac{\pi}{2}$ and the angle between $\vec{a} \times \vec{b}$ and \vec{b} is $\frac{\pi}{2}$

$\therefore \vec{a} \times \vec{b}$ is orthogonal to \vec{a} and \vec{b} ■

6e Given $\vec{b} = \langle 0, 4, 5 \rangle$ and $\vec{c} = \langle -1, 3, -2 \rangle$, find $proj_{\vec{b}}\vec{c}$.

$$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \right) \left(\frac{\vec{u}}{|\vec{u}|} \right)$$

Vector projection of \vec{v} onto \vec{u}

$$proj_{\vec{b}}\vec{c} = \left(\frac{\langle 0, 4, 5 \rangle \cdot \langle -1, 3, -2 \rangle}{|\langle 0, 4, 5 \rangle|} \right) \left(\frac{\langle 0, 4, 5 \rangle}{|\langle 0, 4, 5 \rangle|} \right)$$

Substitute

$$\langle 0, 4, 5 \rangle \cdot \langle -1, 3, -2 \rangle = 2$$

Dot Product of \vec{b} and \vec{c}

$$\sqrt{0^2 + 4^2 + 5^2} = \sqrt{41}$$

Magnitude of \vec{b}

$$proj_{\vec{b}}\vec{c} = \left(\frac{2}{\sqrt{41}} \right) \left(\frac{\langle 0, 4, 5 \rangle}{\sqrt{41}} \right)$$

Substitute

$$proj_{\vec{b}}\vec{c} = \left\langle 0, \frac{8}{41}, \frac{10}{41} \right\rangle$$

Vector projection of \vec{c} onto \vec{b}

- 7 Find the plane that goes through the points $(3, 0, -4)$, $(-2, -2, 5)$, and $(-1, 8, -3)$.

$$P = (3, 0, -4)$$

Define points

$$Q = (-2, -2, 5)$$

$$R = (-1, 8, -3)$$

$$\overrightarrow{PQ} = \langle -5, -2, 9 \rangle$$

Vector through points

$$\overrightarrow{PR} = \langle -4, 8, 1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -2 & 9 \\ -4 & 8 & 1 \end{vmatrix}$$

Take the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} -2 & 9 \\ 8 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -5 & 9 \\ -4 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -5 & -2 \\ -4 & 8 \end{vmatrix} \vec{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -74, -31, -48 \rangle$$

Normal vector

$$P = (3, 0, -4)$$

Point on plane

$$0 = \vec{n} \cdot (r - r_0)$$

Equation of a plane

\vec{n} : the normal vector

r_0 : the point on the plane

$$0 = \langle -74, -31, -48 \rangle \cdot (\langle x, y, z \rangle - \langle 3, 0, -4 \rangle)$$

Equation of the plane

$$0 = -74(x - 3) - 31y - 48(z - 4)$$

Simplified to scalar notation

8 Find the curvature of $\vec{r}(t) = \langle 0, t^3, t^2 \rangle$

$$\vec{r}'(t) = \langle 0, 3t^2, 2t \rangle$$

Derivative of \vec{r}

$$\vec{r}''(t) = \langle 0, 6t, 2 \rangle$$

2nd derivative of \vec{r}

$$|\vec{r}'(t)| = \sqrt{(0)^2 + (3t^2)^2 + (2t)^2}$$

Magnitude of $\vec{r}'(t)$

$$|\vec{r}'(t)| = \sqrt{0 + 9t^4 + 4t^2}$$

$$|\vec{r}'(t)| = \sqrt{t^2(9t^2 + 4)}$$

$$|\vec{r}'(t)| = t\sqrt{9t^2 + 4}$$

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Curvature formula

$$\kappa(t) = \frac{|\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle|}{(t\sqrt{9t^2 + 4})^3}$$

Evaluate curvature

$$\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix}$$

Cross Product

$$\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle = \begin{vmatrix} 3t^2 & 2t \\ 6t & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 2t \\ 0 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 3t^2 \\ 0 & 6t \end{vmatrix} \vec{k}$$

$$\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle = \langle 6t^2 - 12t^2, 0, 0 \rangle$$

$$|\langle 6t^2 - 12t^2, 0, 0 \rangle| = \sqrt{(6t^2 - 12t^2)^2} = \sqrt{(t^2(6 - 12))^2} = \sqrt{(t^2(-6))^2}$$

Magnitude

$$|\langle 6t^2 - 12t^2, 0, 0 \rangle| = 6t^2$$

$$\kappa(t) = \frac{6t^2}{t^3(9t^2 + 4)^{\frac{3}{2}}}$$

Curvature of $\vec{r}(t)$

$$\kappa(t) = \frac{6}{t(9t^2 + 4)^{\frac{3}{2}}}$$

Simplify