

**2c** Consider the function  $f(x, y) = 2 \sin(2x - 3y)$ , What is the maximum rate of change at  $(0, \pi)$  and in what direction does it occur?

$$\nabla f(x, y) = \langle 4 \cos(2x - 3y), -6 \cos(2x - 3y) \rangle \quad \text{Gradient of } f(x, y)$$

$$\nabla f(0, \pi) = \langle 4 \cos(-3\pi), -6 \cos(-3\pi) \rangle$$

Maximum rate of change occurs at the direction of

$$\nabla f(0, \pi) = \langle -4, 6 \rangle$$

$$\nabla f(0, \pi)$$

$$|\nabla f(0, \pi)| = \sqrt{(-4)^2 + 6^2}$$

Maximum rate of change is  $|\nabla f(0, \pi)|$

$$|\nabla f(0, \pi)| = 2\sqrt{13}$$

5 Evaluate  $\int_C (xz - y^2) ds$  where C is the line segment from  $(0, 1, 2)$  to  $(-3, 7, -1)$

$$x(t) = -3t$$

Reparameterize line segment

$$y(t) = 6t + 1$$

$$z(t) = -3t + 2$$

$$\frac{dx}{dt} = -3$$

Derivatives with respect to t

$$\frac{dy}{dt} = 6$$

$$\frac{dz}{dt} = -3$$

$$A = \int_C f(x, y, z) ds$$

Line integral with respect to arc length

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Arc length reparameterized in terms of t

$$A = \int_C f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Line integral with respect to t

$$A = \int_0^1 (-3t)(-3t + 2) - (6t + 1)^2 \sqrt{(-3)^2 + (6)^2 + (-3)^2} dt$$

Substitute

$$A = \int_0^1 ((-3t)(-3t + 2) - (6t + 1)^2) (3\sqrt{6}) dt$$

Integrate

$$A = \int_0^1 (-27t^2 - 18t - 1)(3\sqrt{6}) dt$$

$$A = 3\sqrt{6} \int_0^1 -27t^2 - 18t - 1 dt$$

$$A = 3\sqrt{6}(-9t^3 - 9t^2 - t)|_0^1$$

$$A = -57\sqrt{6}$$

- 6 Find the absolute maximum and minimum values of  $f(x, y) = x^4 + y^4 - 4xy + 2$  on the set  $D = \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 3\}$ .

$D = \{(x, y)   0 \leq x \leq 3, 0 \leq y \leq 3\}$	Region is rectangular
$\frac{\partial f}{\partial x} = 4x^3 - 4y$	1st partial derivative with respect to x
$\frac{\partial f}{\partial y} = 4y^3 - 4x$	1st partial derivative with respect to y
$0 = 4x^3 - 4y$	Set partials equal to 0 and evaluate system
$0 = 4y^3 - 4x$	
$x = y^3$	Solve for x
$0 = 4(y^3)^3 - 4y$	Substitute
$0 = 4y^9 - 4y$	Simplify
$0 = 4y(y^8 - 1)$	Factor
$0 = 4y(y^2 + 1)(y^2 - 1)$	Difference of 2 squares
$0 = 4y(y^2 + 1)(y + 1)(y - 1)$	Difference of 2 squares
$y = 0, 1, -1$	Real roots
$0 = 4(0)^3 - 4x$	Let y = 0
$x = 0$	
$0 = 4(1)^3 - 4x$	Let y = 1
$x = 1$	
$0 = 4(-1)^3 - 4x$	Let y = -1
$x = -1$	
$(x, y) = (0, 0), (1, 1), (-1, -1)$	Critical Points
$f(0, 0) = 2$	
$f(1, 1) = 0$	
$f(-1, -1) = 0$	$(-1, -1)$ out of bounds
$x = 0$	Left edge of region
$f(0, y) = y^4 + 2$	Let x = 0
$f'(0, y) = 4y^3$	Differentiate with respect to y
$0 = 4y^3$	Set to 0 and solve
$y = 0$	
$f(0, 0) = 2$	Bottom left corner, critical point (Local min)
$f(0, 2) = 18$	Top left corner (Local max)
$x = 3$	Right edge of region
$f(3, y) = 81 + y^4 - 12y + 2$	Let x = 3
$f'(3, y) = 4y^3 - 12$	Differentiate with respect to y
$0 = 4y^3 - 12$	Set to 0 and solve
$y = \sqrt[3]{3}$	

**6** (cont.)

$$f(3, \sqrt[3]{3}) \approx 70.019$$

Critical point (Local min)

$$f(3, 0) = 83$$

Bottom right corner (Local max)

$$f(3, 2) = 75$$

Top right corner

$$y = 0$$

Bottom edge of region

$$f(x, 0) = x^4 + 2$$

Let  $y = 0$ 

$$f'(x, 0) = 4x^3$$

Differentiate with respect to  $x$ 

$$0 = 4x^3$$

Set to 0 and solve

$$x = 0$$

$$f(0, 0) = 2$$

Bottom left corner, critical point (Local min)

$$f(0, 2) = 18$$

Top left corner (Local max)

$$y = 2$$

Top edge of region

$$f(x, 2) = x^4 + 16 - 8x + 2$$

Let  $y = 2$ 

$$f'(x, 2) = 4x^3 - 8$$

Differentiate with respect to  $x$ 

$$0 = 4x^3 - 8$$

Set to 0 and solve

$$x = \sqrt[3]{2}$$

$$f(\sqrt[3]{2}, 2) \approx 10.44$$

Critical point (Local min)

$$f(3, 0) = 83$$

Absolute maximum

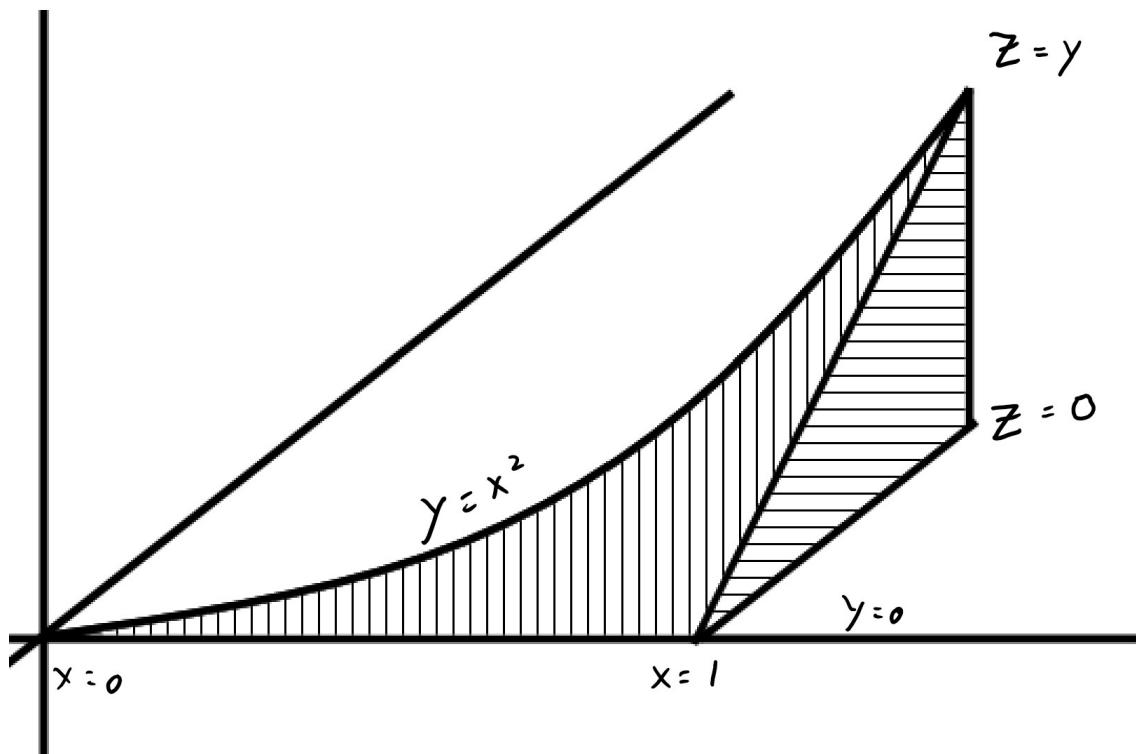
$$f(1, 1) = 0$$

Absolute minimum

- 8 Use the method of Lagrange multipliers to find the maximum and minimum values and their locations of  $f(x, y) = xy$  subject to the constraint  $x^2 + y^2 - xy = 9$

$\nabla f(x, y) = \langle y, x \rangle$	Gradient of objective function
$\nabla g(x, y) = \langle 2x - y, 2y - x \rangle$	Gradient of constraint function
$\nabla f(x, y) = \lambda \nabla g(x, y)$	Lagrange multiplier formula
$\langle y, x \rangle = \lambda \langle 2x - y, 2y - x \rangle$	
$E1 : y = \lambda(2x - y)$	Set up a system of equations
$E2 : x = \lambda(2y - x)$	
$E3 : 9 = x^2 + y^2 - xy$	
$E1 : xy = \lambda(2x^2 - xy)$	Multiply both sides of E1 by x
$E2 : xy = \lambda(2y^2 - xy)$	Multiply both sides of E2 by y
$\lambda(2x^2 - xy) = \lambda(2y^2 - xy)$	$\therefore E1 = E2$
$x^2 = y^2$	
$x = y, x = -y$	
$9 = x^2 + x^2 - x^2$	Substitute $x = y$ into E3
$\pm 3 = x$	
$9 = 9 + y^2 - 3y$	Substitute $x = 3$ into E3
$0 = y(y - 3)$	
$y = 0, y = 3$	
$(3, -3), (-3, 3)$	Local Extrema
$f(3, -3) = f(-3, 3) = -9$	
Let $x = 1$	Find a point within the constraint to determine if extrema are minimum or maximum
$9 = 1 + y^2 - y$	
$y \approx -2.37, y \approx 3.37$	
$f(2.37, -2.37) > -9$	$\therefore f(3, -3)$ and $f(-3, 3)$ are local minimums

## 8 Drawing of region for reference



- 8 Express the iterated integral  $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dz dy dx$  as a triple integral in two different orders:  $dx dz dy$  and  $dy dz dx$

Express as  $dx dz dy$

Rewrite x bounds in terms of y

$$x = \sqrt{y}$$

$$0 \leq x \leq \sqrt{y}$$

$$z = y$$

z bounds already in terms of y

$$0 \leq z \leq y$$

$$y = 1^2$$

Calculate the intersection between  $y = x^2$  and  $x = 1$

$$0 \leq y \leq 1$$

$$\int_0^1 \int_0^y \int_0^{\sqrt{y}} f(x, y, z) dx dz dy$$

Rewrite integral

Express as  $dy dz dx$

y bounds already in terms of x

$$y = x^2$$

$$0 \leq y \leq x^2$$

$$z = y = x^2$$

Rewrite z bounds in terms of x

$$z = x^2$$

$$0 \leq z \leq x^2$$

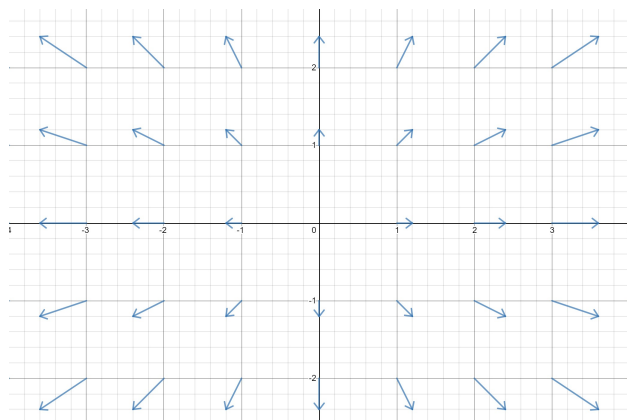
$$\int_0^1 \int_0^{x^2} \int_0^{x^2} f(x, y, z) dy dz dx$$

Rewrite integral

**10b** Given  $f(x, y) = x^2 + y^2$ , graph the gradient vector field

$$\nabla f(x, y) = \langle 2x, 2y \rangle$$

Calculate the gradient of the function



Graph vectors at points