1 Reparametrize the curve $\vec{r}(t) = (4\cos t)\vec{i} + (4\sin t)\vec{j} + 3t\vec{k}$ with respect to the arc length measured from (4,0,0) in the direction of increasing t.

$$\vec{r}(0) = \langle 4\cos 0, 4\sin 0, 3(0) \rangle = \langle 4, 0, 0 \rangle \qquad \text{Let } \mathbf{t} = 0$$

$$\vec{r}'(t) = \langle -4\sin t, 4\cos t, 3 \rangle \qquad \text{Derivative of } \vec{r}(t)$$

$$s(t) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt \qquad \text{Arc length formula}$$

$$s(t) = \int_0^t \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2} dt \qquad \text{Substitute the components of } \vec{r}'(t)$$

$$s(t) = \int_0^t \sqrt{25} dt \qquad \text{Evaluate the integral}$$

$$s(t) = \sqrt{25} \Big|_0^t$$

$$s(t) = \sqrt{25}(t-0)$$

$$t(s) = \frac{s}{5} \qquad \text{t in terms of arc length}$$

$$\vec{r}(t(s)) = 4\cos\frac{s}{5}\vec{i} + 4\sin\frac{s}{5}\vec{j} + \frac{3s}{5}\vec{k} \qquad \text{Curve reparametrized in terms of arc length}$$

2c Given the plane curve $\vec{r}(t) = \langle t-2, t^2-1 \rangle$ Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}'(t)$ at t=1

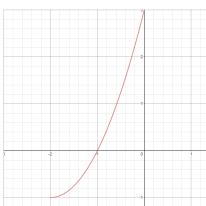
 $\begin{array}{cccc} t & x & y \\ 0 & -2 & -1 \end{array}$

 $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

 $\begin{array}{cccc} 2 & 0 & 3 \\ 3 & 1 & 8 \end{array}$

4 2 15

5 3 24



$$\vec{r}'(t) = \langle 1, 2t \rangle$$

$$\vec{r}'(1) = \langle 1, 2 \rangle$$

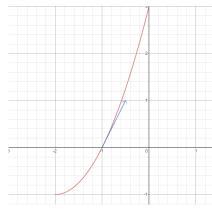


Table of values

Graph of $\vec{r}(t)$ Take the derivative of $\vec{r}(t)$ Substitute t = 1

Graph $\vec{r}'(1)$ at t=1

3a Given $9y^2 + 4z^2 = x^2 + 36$

a. Rewrite the equation in standard form

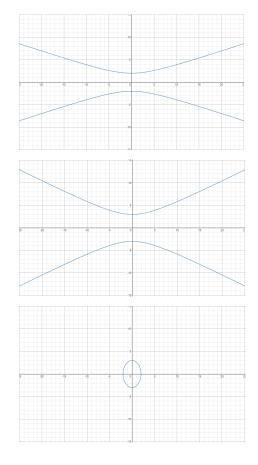
$$9y^2 + 4z^2 = x^2 + 36$$
 Original equation
 $9y^2 + 4z^2 - x^2 = 36$ Subtract x^2 on both sides
 $\frac{y^2}{4} + \frac{z^2}{9} - \frac{x^2}{36} = 1$ Divide by 36 on both sides
 $-\frac{x^2}{36} + \frac{y^2}{4} + \frac{z^2}{9} = 1$ Rearrange terms
 $-\frac{x^2}{6^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$ Rewrite denominators

b. Identify the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
 Standard form of a hyperboloid of 1 sheet
$$-\frac{x^2}{6^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$$
 Equation matches the form

Hyperboloid of 1 sheet

c. Draw each trace for the coordinate planes and describe the conic section. Label and scale your axes.

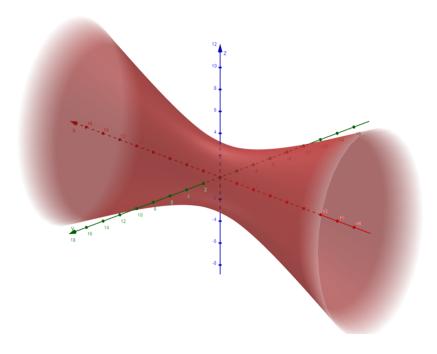


xy axis, Let z = 0, Hyperbola

xz axis, Let y = 0, Hyperbola

yz axis, Let x = 0, Ellipse

d. Graph $9y^2 + 4z^2 = x^2 + 36$. Label and scale the axes.



4 Given $\vec{a}, \vec{b} \in \mathbb{R}^3$, show that $\vec{a} \times \vec{b}$ is orthogonal to \vec{a} and \vec{b} .

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\theta)$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\frac{\pi}{2})$$

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$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \approx 0$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\therefore \text{ When } \theta = \frac{\pi}{2},$$
the dot product is equal to 0
Given

$$\text{Let } \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \vec{b} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} \vec{b} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1 \rangle$$

$$(a_1, a_2, a_3) \cdot \langle a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1 \rangle$$

$$= a_1 (a_2 b_3 - a_3 b_2) + a_2 (-a_1 b_3 + a_3 b_1) + a_3 (a_1 b_2 - a_2 b_1)$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_2 a_3 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_2 a_3 b_1$$

$$= (a_1 a_2 b_3 - a_1 a_3 b_2 - a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1)$$

$$= b_1 (a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1)$$

$$= b_1 (a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1)$$

$$= b_1 (a_2 b_3 - a_1 b_3 b_2 - b_2 a_1 b_3 + b_2 a_3 b_1 + b_3 (a_1 b_2 - a_2 b_1)$$

$$= b_1 (a_2 b_3 - b_1 a_3 b_2 - b_2 a_1 b_3 + b_2 a_3 b_1 + b_3 a_1 b_2 - b_3 a_2 b_1$$

$$= (b_1 a_2 b_3 - b_1 a_3 b_2 - b_2 a_1 b_3 + b_2 a_3 b_1 + b_3 a_1 b_2 - b_3 a_2 b_1$$

$$= (b_1 a_2 b_3 - b_1 a_3 b_2 - b_2 a_1 b_3 + b_2 a_3 b_1 + b_3 a_1 b_2 - b_3 a_2 b_1$$

$$= (b_1 a_2 b_3 - b_1 a_2 b_3) + (a_1 b_2 b_3 - a_1 b_2 b_3) + (b_1 b_2 a_3 - b_1 b_2 a_3)$$
Rearrange terms
$$\therefore \text{ the angle between}$$

$$\vec{b} \text{ ind } \vec{a} \times \vec{b} = \frac{\pi}{2}$$

We know the angle between $\vec{a} \times \vec{b}$ and \vec{a} is $\frac{\pi}{2}$ and the angle between $\vec{a} \times \vec{b}$ and \vec{b} is $\frac{\pi}{2}$

 $\vec{a} \times \vec{b}$ is orthogonal to \vec{a} and \vec{b}

6e Given $\vec{b} = \langle 0, 4, 5 \rangle$ and $\vec{c} = \langle -1, 3, -2 \rangle$, find $proj_{\vec{b}}\vec{c}$.

$$\begin{split} proj_{\vec{u}}\vec{v} &= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}\right) \left(\frac{\vec{u}}{|\vec{u}|}\right) \\ proj_{\vec{b}}\vec{c} &= \left(\frac{\langle 0,4,5\rangle \cdot \langle -1,3,-2\rangle}{|\langle 0,4,5\rangle|}\right) \left(\frac{\langle 0,4,5\rangle}{|\langle 0,4,5\rangle|}\right) \\ \langle 0,4,5\rangle \cdot \langle -1,3,-2\rangle &= 2 \\ \sqrt{0^2 + 4^2 + 5^2} &= \sqrt{41} \\ proj_{\vec{b}}\vec{c} &= \left(\frac{2}{\sqrt{41}}\right) \left(\frac{\langle 0,4,5\rangle}{\sqrt{41}}\right) \\ proj_{\vec{b}}\vec{c} &= \left\langle 0,\frac{8}{41},\frac{10}{41}\right\rangle \end{split}$$

Vector projection of \vec{v} onto \vec{u}

Substitute

Dot Product of \vec{b} and \vec{c}

Magnitude of \vec{b}

Substitute

Vector projection of \vec{c} onto \vec{b}

7 Find the plane that goes through the points (3,0,-4), (-2,-2,5), and (-1,8,-3).

$$P = (3, 0, -4)$$

$$Q = (-2, -2, 5)$$

$$R = (-1, 8, -3)$$

Define points

$$\overrightarrow{PQ} = \langle -5, -2, 9 \rangle$$

$$\overrightarrow{PR} = \langle -4, 8, 1 \rangle$$

Vector through points

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -5 & -2 & 9 \\ -4 & 8 & 1 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} -2 & 9 \\ 8 & 1 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} -5 & 9 \\ -4 & 1 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} -5 & -2 \\ -4 & 8 \end{vmatrix} \overrightarrow{k}$$

Take the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -74, -31, -48 \rangle$$

$$P = (3, 0, -4)$$

Normal vector Point on plane

$$0 = \vec{n} \cdot (r - r_0)$$

Equation of a plane \vec{n} : the normal vector r_0 : the point on the plane

$$0 = \langle -74, -31, -48 \rangle \cdot (\langle x, y, z \rangle - \langle 3, 0, 5 \rangle)$$

$$0 = -74(x - 3) - 31y - 48(z - 4)$$

Equation of the plane Simplified to scalar notation

8 Find the curvature of $\vec{r}(t) = \langle 0, t^3, t^2 \rangle$

$$\vec{r}''(t) = \langle 0, 3t^2, 2t \rangle \qquad \text{Derivative of } \vec{r}$$

$$\vec{r}'''(t) = \langle 0, 6t, 2 \rangle \qquad 2\text{nd derivative of } \vec{r}$$

$$|\vec{r}'(t)| = \sqrt{(0)^2 + (3t^2)^2 + (2t)^2} \qquad \text{Magnitude of } \vec{r}'(t)$$

$$|\vec{r}'(t)| = \sqrt{0 + 9t^4 + 4t^2}$$

$$|\vec{r}'(t)| = \sqrt{9t^2 + 4}$$

$$|\vec{r}'(t)| = t\sqrt{9t^2 + 4}$$

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \qquad \text{Curvature formula}$$

$$\kappa(t) = \frac{|\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle|}{(t\sqrt{9t^2 + 4})^3} \qquad \text{Evaluate curvature}$$

$$\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} \qquad \text{Cross Product}$$

$$\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle = \begin{vmatrix} 3t^2 & 2t \\ 6t & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 2t \\ 0 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 3t^2 \\ 0 & 6t \end{vmatrix} \vec{k}$$

$$\langle 0, 3t^2, 2t \rangle \times \langle 0, 6t, 2 \rangle = \langle 6t^2 - 12t^2, 0, 0 \rangle$$

$$|\langle 6t^2 - 12t^2, 0, 0 \rangle| = \delta(t^2 - 12t^2, 0, 0) = \delta(t^2 - 12t^2, 0, 0)$$

$$|\langle 6t^2 - 12t^2, 0, 0 \rangle| = 6t^2$$

$$\kappa(t) = \frac{6t^2}{t^3(9t^2 + 4)^{\frac{3}{2}}} \qquad \text{Curvature of } \vec{r}(t)$$
 Simplify