

Convolutions

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Convolutions

The convolution of two signals f and g is defined as:

$$\begin{aligned} * : (\mathbb{Z} \rightarrow \mathbb{R}) \times (\mathbb{Z} \rightarrow \mathbb{R}) &\rightarrow (\mathbb{Z} \rightarrow \mathbb{R}) : \\ f[n], g[n] &\mapsto (f * g)[n] \triangleq \sum_{k=-\infty}^{+\infty} f[k] g[n - k] \end{aligned} \quad (1)$$

If no ambiguity exists, a shorthand can be used:

$$f * g \triangleq \sum_{k=-\infty}^{+\infty} f[k] g[n - k]$$

Properties

The convolution operator can be seen as a product of discrete functions, and it has many properties usually associated with multiplication.

Commutativity: $f * g = g * f$

Associativity: $(f * g) * h = f * (g * h)$

Distributivity: $(f + g) * h = f * h + g * h$

Identity

Convolution with the Kronecker delta function results in the original signal, thanks to the sifting property of the delta function:

$$f * \delta = f = \delta * f$$

Unilateral signals

If the first signal is unilateral (i.e. $\forall n < 0 : f[n] = 0$), the lower bound of the summation becomes zero instead of minus infinity:

$$f * g = \sum_{k=0}^{+\infty} f[k] g[n - k]$$

Signals as a sum of delta functions

Any discrete signal can be written as an infinite sum of scaled and shifted Kronecker delta functions.

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

You can easily see that all terms where $n \neq k$ are zero, because the Kronecker delta is zero in that case. Only the term for $n = k$ is non-zero, in which case the Kronecker delta is one, so the result is just $x[k]$. This is an application of the sifting property of the delta function, covered in the [previous page](#).

This once again shows that the Kronecker delta is the identity signal with respect to convolution operator, $x[n] = (x * \delta)[n]$.

DTLTI systems as convolutions with the impulse response

You can express the output of any discrete-time linear time-invariant system T with any input $x[n]$ as the convolution of the input with the impulse response of the system, $h[n] \triangleq T(\delta[n])$:

$$T(x[n]) = (x * h)[n] \quad (2)$$

Proof

The proof itself is very simple: We just decompose the input as a sum of delta functions, as described in a previous section, and then we use the linearity and time-invariance to bring the T operator inside of the summation.

$$\begin{aligned}
y[n] &= T(x[n]) \\
&= T\left(\sum_{k=0}^{\infty} x[k] \delta[n-k]\right) \\
&= \sum_{k=0}^{\infty} T\left(x[k] \delta[n-k]\right) \\
&= \sum_{k=0}^{\infty} x[k] T\left(\delta[n-k]\right) \\
&= \sum_{k=0}^{\infty} x[k] h[n-k] \\
&\triangleq (x * h)[n]
\end{aligned}$$

Because of the linearity of the system, T can be brought inside of the summation, and since $x[k]$ is a constant factor independent of the time step n , it can be moved outside of the T operator. T applied to the Kronecker delta is (by definition) the impulse response of T , $h[n]$. In this case, it is shifted by k time steps, which is allowed because of the time-invariance of T . \square

An important consequence is that every DTLTI transformation can be uniquely represented by its impulse response, in other words, there is a one-to-one correspondence between the definition of transformation T and its impulse response $h[n]$.