Butterworth Filters

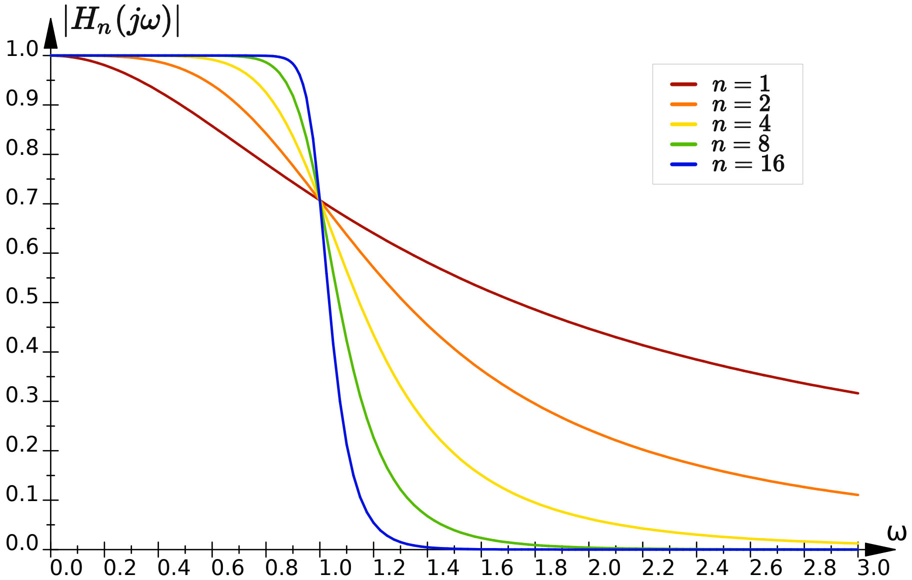
Pieter P

This page will cover the derivation of the transfer functions of low-pass and high-pass Butterworth filters. Butterworth filters are designed to have a very flat frequency response in the passband.

Definition

Normalized Butterworth filters are defined in the frequency domain as follows:

$$|H_n(j\omega)| \triangleq \frac{1}{\sqrt{1+\omega^{2n}}} \tag{1}$$



In order to determine the transfer function, we'll start from the frequency response squared. We'll assume that the transfer function $H_n(s)$ is a rational function with real coefficients. Therefore, $\overline{H_n(s)} = H_n(\overline{s})$.

$$\begin{aligned} |H_n(j\omega)|^2 &= H_n(j\omega)\overline{H_n(j\omega)} \\ &= H_n(j\omega)H_n(\overline{j\omega}) \\ &= H_n(j\omega)H_n(-j\omega) \\ &= \frac{1}{1+\omega^{2n}} \end{aligned}$$

We're looking for the transfer function $H_n(s)$, so we'll use the identity $s = j\omega \Leftrightarrow \omega = \frac{s}{j}$.

$$H_n(s)H_n(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$

Poles of $H_n(s)H_n(-s)$

The poles of this transfer function are given by:

$$\begin{aligned} & \left(\frac{s}{j}\right)^{2n} = -1 \\ \Leftrightarrow\quad & s^{2n} = -1(j)^{2n} \\ \Leftrightarrow\quad & s^{2n} = -1(-1)^{n} \\ \Leftrightarrow\quad & s^{2n} = (-1)^{n+1} \\ \Leftrightarrow\quad & s^{2n} = e^{j\pi(n+1)} \end{aligned}$$

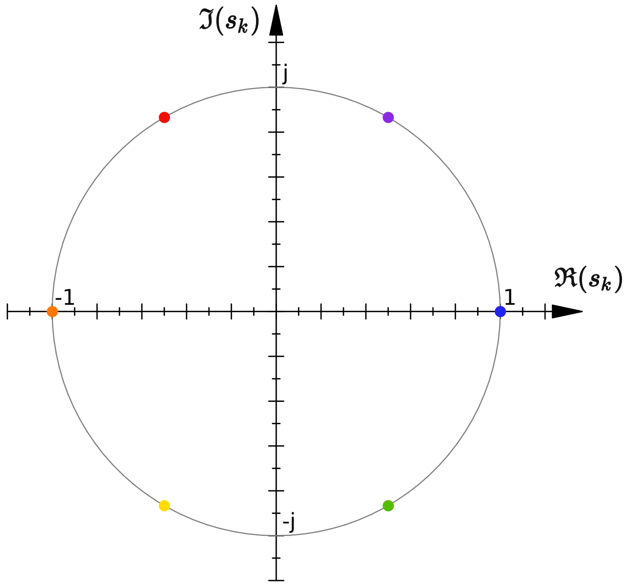
Keep in mind that this is a polynomial of order $2n$, so it has $2n$ complex roots.

$$s_k = e^{j2\pi\frac{2k+n+1}{4n}} \qquad k \in \{0, 1, \ldots 2n-1\}$$

For example, for $n = 3$, the poles are:

$$\begin{aligned}
s_0 &= e^{j2\pi\frac{0+3+1}{12}} &= e^{j2\pi\frac{2}{6}} \\
s_1 &= e^{j2\pi\frac{2+3+1}{12}} &= e^{j2\pi\frac{3}{6}} \\
s_2 &= e^{j2\pi\frac{4+3+1}{12}} &= e^{j2\pi\frac{4}{6}} \\
s_3 &= e^{j2\pi\frac{6+3+1}{12}} &= e^{j2\pi\frac{5}{6}} \\
s_4 &= e^{j2\pi\frac{8+3+1}{12}} &= e^{j2\pi\frac{6}{6}} \\
s_5 &= e^{j2\pi\frac{10+3+1}{12}} &= e^{j2\pi\frac{1}{6}}
\end{aligned}$$

These are all points on the unit circle, $\pi/3 = 60^\circ$ apart.



The poles are stable if they are in the left half plane, if their complex argument is between 90° and 270°:

$$\begin{aligned}
& 2\pi\frac{2k+n+1}{4n} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\
\Leftrightarrow\quad & 2k+n+1 \in (n, 3n) \\
\Leftrightarrow\quad & k \in \left(-\frac{1}{2}, n-\frac{1}{2}\right) \\
\Rightarrow\quad & k \in \left(-\frac{1}{2}, n-\frac{1}{2}\right) \cup \{0, 1, \dots 2n-1\} \\
\Leftrightarrow\quad & k \in \{0, 1, \dots n-1\}
\end{aligned}$$

$$s_{k,stable} = e^{j2\pi\frac{2k+n+1}{4n}} \qquad k \in \{0, 1, \dots n-1\} \tag{2}$$

Poles of $H_n(s)$

We want our filter $H_n(s)$ to be stable, so we pick the poles in the left half plane to be the poles of $H_n(s)$. The unstable poles, for $k \in \{n, n+1, \dots 2n-1\}$ are the poles of $H_n(-s)$. They are the opposites of the poles of $H_n(s)$:

$$\begin{aligned}
s_{k,unstable} &= e^{j2\pi\frac{2k+n+1}{4n}} & k \in \{n, n+1, \dots 2n-1\} \\
& l \triangleq k-n & \\
&= e^{j2\pi\frac{2(l+n)+n+1}{4n}} & l \in \{0, 1, \dots n-1\} \\
&= e^{j\left(2\pi\frac{2l+n+1}{4n}+\pi\right)} & \\
&= e^{j\pi}\cdot e^{j2\pi\frac{2l+n+1}{4n}} & \\
&= -1\cdot e^{j2\pi\frac{2l+n+1}{4n}} & \\
&= -s_{l,stable} &
\end{aligned}$$

Butterworth Polynomials

We'll define the normalized Butterworth polynomial as follows:

$$B_n(s) \triangleq \prod_{k=0}^{n-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right) \tag{3}$$

We'll rearrange the product to group each pole with its complex conjugate. Then, using the identity $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, we can further simplify this expression:
Even order n:

$$\begin{aligned}
B_n(s) &= \prod_{k=0}^{n-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\prod_{l=\frac{n}{2}}^{n-1}\left(s - e^{j2\pi\frac{2l+n+1}{4n}}\right)\\
& l = n-k-1\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - e^{j2\pi\frac{2(n-k-1)+n+1}{4n}}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - \frac{e^{j2\pi\frac{-2k+3n-1}{4n}}}{1}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - \frac{e^{j2\pi\frac{-2k+3n-1}{4n}}}{e^{j2\pi}}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - e^{j2\pi(\frac{-2k+3n-1}{4n}-1)}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - e^{j2\pi\frac{-2k+3n-4n-1}{4n}}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - e^{j2\pi\frac{-2k-n-1}{4n}}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s - e^{j2\pi\frac{2k+n+1}{4n}}\right)\left(s - e^{-j2\pi\frac{2k+n+1}{4n}}\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s^2 - se^{j2\pi\frac{2k+n+1}{4n}} - se^{-j2\pi\frac{2k+n+1}{4n}} + 1\right)\\
&= \prod_{k=0}^{\frac{n}{2}-1}\left(s^2 - 2\cos\left(2\pi\frac{2k+n+1}{4n}\right)s + 1\right)
\end{aligned}$$

Odd order n:
In this case, $n-1$ is even, and you get a special pole for $k = \frac{n-1}{2}$:

$$\begin{aligned}
s_{\frac{n-1}{2}} &= e^{j2\pi\frac{2\frac{n-1}{2}+n+1}{4n}}\\
&= e^{j2\pi\frac{2n}{4n}}\\
&= e^{j\pi}\\
&= -1
\end{aligned}$$

After isolating this pole, we're left with an even number of complex conjugate poles, just like in the case where n was even.

In conclusion, the normalized Butterworth polynomial of degree n is given by:

$$B_n(s) = \begin{cases} \prod_{k=0}^{\frac{n}{2}-1}\left(s^2 - 2\cos\left(2\pi\frac{2k+n+1}{4n}\right)s + 1\right) & \text{even } n \\ (s+1)\prod_{k=0}^{\frac{n-1}{2}-1}\left(s^2 - 2\cos\left(2\pi\frac{2k+n+1}{4n}\right)s + 1\right) & \text{odd } n \end{cases} \tag{4}$$

Butterworth Transfer Function $H_n(s)$

The transfer function $H_n(s)$ has no zeros, so the numerator is a constant. The poles of $H_n(s)$ are given by Equation (2), so the denominator is given by Equation (3).

$$H_n(s) = \frac{c}{B_n(s)}$$

We wanted a DC gain of 1 $(= 0dB)$ for $\omega = 0$:

$$\begin{aligned}
& |H_n(0j)| &= 1 \\
\Leftrightarrow & \left|\frac{c}{B_n(0)}\right| &= 1 \\
\Leftrightarrow & \left|\frac{c}{\prod_{k=0}^{n-1}\left(0 - e^{j2\pi\frac{2k+n+1}{4n}}\right)}\right| &= 1 \\
\Leftrightarrow & \frac{|c|}{\prod_{k=0}^{n-1}\left|-e^{j2\pi\frac{2k+n+1}{4n}}\right|} &= 1 \\
\Leftrightarrow & \frac{|c|}{1} &= 1
\end{aligned}$$

If we want no phase offset for low frequencies, we can postulate that $\angle H_n(0j) = 0$:

$$\begin{aligned}
& \angle H_n(0j) &= 0 \\
\Leftrightarrow & \angle\left(\frac{c}{B_n(0)}\right) &= 0 \\
\Leftrightarrow & \angle c - \angle\left(\prod_{k=0}^{\frac{n}{2}-1}\left(0^2 - 2\cos\left(2\pi\frac{2k+n+1}{4n}\right)\cdot 0 + 1\right)\right) &= 0 \\
\Leftrightarrow & \angle c - \angle 1 &= 0
\end{aligned}$$

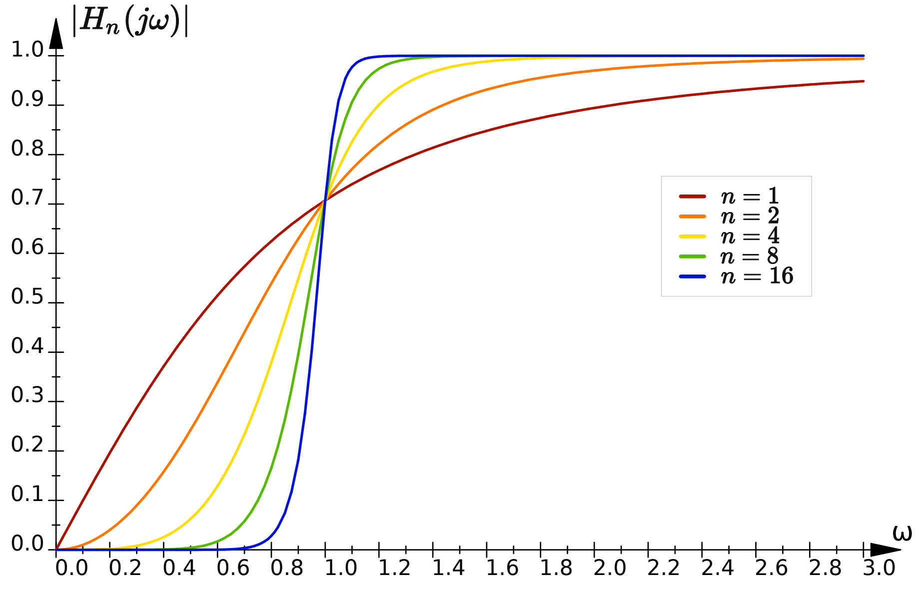
The derivation is analogous for odd n.
Therefore, $c = 1$, and we've eliminated all unknown parameters from the transfer function:

$$H_n(s) = \frac{1}{B_n(s)} \tag{5}$$

High-Pass Butterworth filters

Up until now, we only looked at the low-pass Butterworth filter. There's also a high-pass version:

$$|H_{n,hp}(j\omega)| \triangleq \frac{1}{\sqrt{1+\omega^{-2n}}} \tag{6}$$



We can just multiply the numerator and the denominator by ω^n to get a more familiar form:

$$|H_{n,hp}(j\omega)| = \frac{\omega^n}{\sqrt{1+\omega^{2n}}}$$

As you can see, the poles will be the same as for the low-pass version. On top of that, there now are n zeros for $s = 0$.
So the transfer function becomes:

$$H_{n,hp}(s) = \frac{s^n}{B_n(s)} \tag{7}$$

Non-normalized Butterworth Filters

Up until now, we only looked at normalized Butterworth filters, that have a corner frequency of $1\ rad/s$. To get a specific corner frequency ω_c, we can just scale ω, so the definitions become:

$$|H_{n,lp}(j\omega)| \triangleq \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} \tag{8}$$

$$|H_{n,hp}(j\omega)| \triangleq \frac{1}{\sqrt{1+\left(\frac{\omega_c}{\omega}\right)^{2n}}} \tag{9}$$

If you start recalculating the transfer functions, you'll quickly realize that this just scales everything by a factor of ω_c. The poles no longer lie on the unit circle, but on a circle with radius $|s_k| = \omega_c$.
This results in the following transfer functions:

$$H_{n,lp}(s) = \frac{1}{B_n\left(\frac{s}{\omega_c}\right)} \tag{10}$$

$$H_{n,hp}(s) = \frac{s^n}{\omega_c^n\ B_n\left(\frac{s}{\omega_c}\right)} \tag{11}$$

The gain at the corner frequency can easily be determined from the definitions:

$$\begin{aligned}
|H_{n,lp}(j\omega_c)| \ = \ |H_{n,hp}(j\omega_c)| &= \frac{1}{\sqrt{1+\left(\frac{\omega_c}{\omega_c}\right)^{2n}}} \\
&= \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2} \\
&\approx 0.707 \\
20\log_{10}|H_n(j\omega_c)| &= 20\log_{10}\left(\frac{\sqrt{2}}{2}\right) \\
&= 10\log_{10}\left(\frac{1}{2}\right) \\
&\approx -3.01\ dB
\end{aligned}$$

This is often called the $-3\ dB$-point or the half-power point, because a sinusoidal input signal at that frequency will result in an output signal that has only half of the power of the input signal: $|H_n(j\omega_c)|^2 = \frac{1}{2}$.