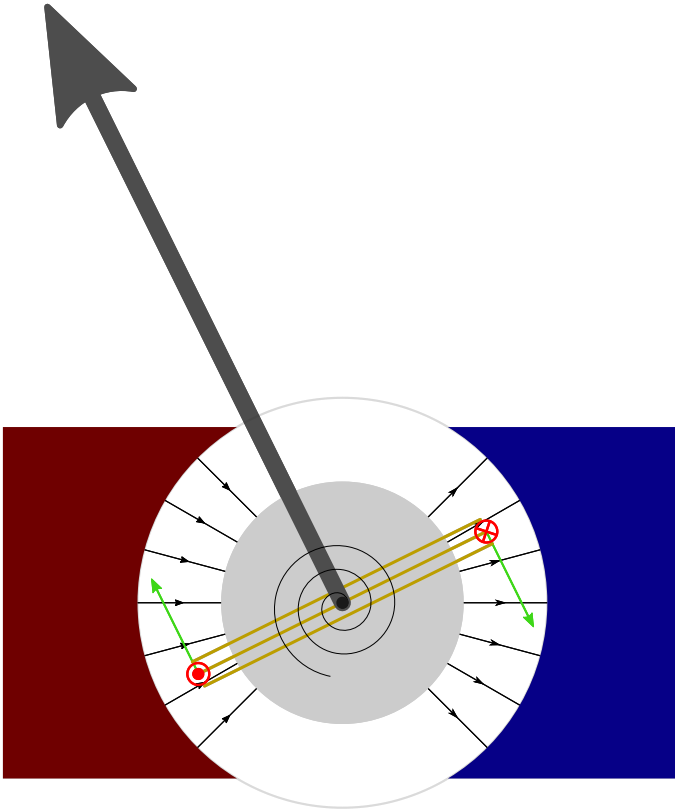


Model of a moving-coil galvanometer

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Construction

An analog VU meter is just a moving-coil galvanometer. The figure below shows a schematic representation:



Schematic representation of a moving-coil instrument.

The needle moves on a pivot and is kept in place using a spiral-shaped torsion spring. A rectangular coil is fixed to the needle. The coil moves in an air gap between the fixed permanent magnet and a cylindrical iron core. This configuration results in a strong, close to radial magnetic field in the air gap.

Law's of motion

Force on the coil

External current

When current flows through the coil, the force on one half of the coil is given by Lorentz's Force Law:

$$\vec{F}_L = Ni\vec{l} \times \vec{B},$$

where N is the number of windings, i is the current, \vec{l} is a vector that points along the vertical part of the wires of the coil, in the direction of the current, and with the same length as the height of the coil, and \vec{B} is the magnetic field vector.

In the figure above, \vec{B} is shown using black arrows, the vector $Ni\vec{l}$ is shown in red. The one on the left points out of the page, and the one on the right points into the page. The resulting force vectors are shown in green. You can verify their direction using the right-hand rule (middle finger is the magnetic field vector \vec{B} , index finger is the current vector $Ni\vec{l}$, thumb is the force vector \vec{F}_L).

The magnitude of the Lorentz force is then

$$F_L = \|\vec{F}_L\| = NilB \sin(\varphi),$$

where φ is the angle between the magnetic field vector and the direction of the current. Because the magnetic field in the air gap is radial, this angle is always 90° , so this simplifies to

$$F_L = NilB.$$

The horizontal part of the coil (parallel to the page) experiences no net force, since the current at the bottom and at the top of the coil flows in opposite directions and the magnetic field is symmetric.

Eddy currents

The coil is wound around a rectangular piece of metal. When the needle moves, the armature (coil and the rectangular piece of metal) rotates in the magnetic field. This rotation causes the magnetic flux through the armature to change, which according to Faraday's Law of Induction induces a current in the metal, resulting in a magnetic field that opposes the rotation (this is Lenz's Law). These induced currents are called Eddy currents or Foucault's currents, and the same principle is used for induction brakes in high-speed trains, for example.

The magnitude of this opposing force is proportional to the rate of change of the magnetic flux in the armature, $\frac{\partial \Phi_B}{\partial t}$. In a first order approximation, this is simply proportional to the angular velocity of the armature, $\dot{\theta} = \frac{\partial \theta}{\partial t}$:

$$F_d = -c\dot{\theta},$$

where c is a proportionality constant that takes into account the area of the armature, the resistance of the metal it's made of, the strength of the magnetic field, etc.

Torque on the coil

The torque on the coil caused by the current is

$$\tau_L = 2rF_L = 2rNilB = fi,$$

where r is the radius of the coil. The factor two accounts for the two halves of the coil (there are two green force vectors that have to be summed together). Since the number of windings, the dimensions of the coil, and the strength of the magnet are all constant, we introduce a constant $f = 2rNilB$.

The damping force F_d caused by induction in the armature causes an opposing torque, proportional to the velocity of the needle:

$$\tau_d = -rc\dot{\theta} = -d\dot{\theta}.$$

Again, a constant $d = rc$ is introduced to make future calculations easier.

When the needle moves away from its center position, there is a third torque, caused by the torsion spring. This torque can be approximated using Hooke's law:

$$\tau_H = -k\theta,$$

where k is the torsion coefficient (spring constant), and θ is the angle of the needle.

Rotational version of Newton's Second Law

Newton's Second Law ($F = ma$) has a rotational equivalent that relates the net torque to the angular acceleration of an object:

$$\tau = I\alpha,$$

where τ is the net torque, I is the moment of inertia, and $\alpha = \ddot{\theta}$ is the angular acceleration.

Applying this law to the three torques derived above:

$$\begin{aligned}\tau_L + \tau_d + \tau_H &= I\alpha \\ fi - d\dot{\theta} - k\theta &= I\ddot{\theta} \\ I\ddot{\theta} + d\dot{\theta} + k\theta &= fi.\end{aligned}$$

Both the angle θ and the current i vary over time. This results in the differential equation

$$I\ddot{\theta}(t) + d\dot{\theta}(t) + k\theta(t) = fi(t). \quad (1)$$

Recall that I , d , k and f are physical constants, $\theta(t)$ is the angle of the needle, and $i(t)$ is the current applied to the coil. Overdots indicate derivatives with respect to time.

The damped harmonic oscillator

You might recognize the previous equation (1) as the equation for the damped harmonic oscillator

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = f(t). \quad (2)$$

In this equation, $x(t)$ is the displacement, m is the mass (cfr. moment of inertia I), d is the damping factor, k is the spring constant, and $f(t)$ is an external driving force.

The damped harmonic oscillator is a well-known problem, so in the remainder of this document, we'll use the formulation in equation (2), because it uses the symbol names you'll usually find in the literature. Apart from naming conventions, equations (1) and (2) are completely equivalent.

To make solving the equation easier, we'll define two more constants:

$$\begin{aligned}\omega_n &\triangleq \sqrt{\frac{k}{m}} \\ \zeta &\triangleq \frac{d}{2\sqrt{km}}\end{aligned}\tag{3}$$

Equation (2) then becomes:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{1}{m}f(t)\tag{4}$$

Solution of the damped harmonic oscillator

Equation (4) is a linear ordinary differential equation with constant coefficients, so the general solution is the sum of the homogeneous solution and a particular solution:

$$x(t) = x_h(t) + x_p(t)$$

In this section, we'll derive the solution of the step response, that is, the solution for $x(t)$ if $f(t) = u(t)$, the Heaviside step function:

$$u(t) \triangleq \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Homogeneous solution

The homogeneous solution $x_h(t)$ is the solution to

$$\ddot{x}_h(t) + 2\zeta\omega_n\dot{x}_h(t) + \omega_n^2x_h(t) = 0.\tag{5}$$

The characteristic equation is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0,$$

with the solutions

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

The homogeneous solution is then

$$x_h(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t},\tag{6}$$

where c_1 and c_2 are two arbitrary real constants of integration.

Particular solution for the step response

The particular solution $x_p(t)$ is a (preferably simple) solution to

$$\ddot{x}_p(t) + 2\zeta\omega_n\dot{x}_p(t) + \omega_n^2x_p(t) = \frac{1}{m}u(t).\tag{7}$$

The Heaviside step function $u(t)$ is flat everywhere, so its derivatives are zero, except in $t = 0$, where they are undefined:

$$\begin{aligned}\forall t \neq 0 : \\ \dot{u}(t) &\equiv 0 \\ \ddot{u}(t) &\equiv 0\end{aligned}$$

A guess for the particular solution could be $x_p(t) = c_3u(t)$. You can substitute this into equation (7) to verify that this is indeed a solution, if $c_3 = \frac{1}{m\omega_n^2} = \frac{1}{k}$:

$$\cancel{c_3\ddot{u}(t)} + \cancel{2c_3\zeta\omega_n\dot{u}(t)} + c_3\omega_n^2u(t) = \frac{1}{m}u(t)$$

General solution for the step response

The general solution is the sum of the homogeneous and the particular solution:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \frac{1}{k} u(t) \quad (8)$$

Determining the constants of integration

To find c_1 and c_2 , we use the following initial conditions:

$$\begin{cases} x(0^+) = 0 \\ \dot{x}(0^+) = 0 \end{cases}$$

They are specified at $t = 0^+$, which should be understood as a limit for t approaching zero from the right, because the derivatives are undefined at $t = 0$.

$$\dot{x}(0^+) = 0 \quad \Leftrightarrow \quad \lim_{t \rightarrow 0^+} \dot{x}(t) = 0$$

For the practical computation c_1 and c_2 , this doesn't matter, but the discontinuity of the \dot{x} does imply that we cannot extend the solution for $t < 0$.

In order to be able to evaluate the second boundary condition, we need an expression for $\dot{x}(t)$. Luckily, $x(t)$ consists of just exponentials and a flat Heaviside step (recall that $\dot{u}(t) \equiv 0$).

$$\dot{x}(t) = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}$$

Evaluating for $\lim_{t \rightarrow 0^+}$:

$$\begin{cases} x(0^+) = c_1 + c_2 + \frac{1}{k} = 0 \\ \dot{x}(0^+) = c_1 \lambda_1 + c_2 \lambda_2 = 0 \end{cases}$$

This is simply a system of two equations and two unknowns, c_1 and c_2 . The solutions are:

$$\begin{cases} c_1 = \frac{1}{k(\lambda_1/\lambda_2 - 1)} \\ c_2 = \frac{1}{k(\lambda_2/\lambda_1 - 1)} \end{cases}$$