Discretization of a Fourth-Order Butterworth Filter

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This is an example on how to design a filter in the analog domain, and then use the bilinear transform to transform it to the digital domain, while preserving the cut-off frequency.

We'll be using formulas derived on the Bilinear Transform and Butterworth Filters pages.

Design criteria

In this example, we'll design a digital fourth order Butterworth low-pass filter, with a sample frequency of $360\ \mathrm{Hz}$ and a cut-off frequency of $45\ \mathrm{Hz}$.

Frequency Pre-Warping

As discussed in the page on the Bilinear Transform, we have to apply pre-warping to the cut-off frequency before designing a filter. If we don't the cut-off frequency will shift to an incorrect frequency when we discretize the filter.

First, let's calculate the normalized digital frequency $\omega_{c,d}$, using the cut-off frequency f_c and the sample frequency f_s:

$$\begin{aligned}
f_c &= 45\ \mathrm{Hz} \\
f_s &= 360\ \mathrm{Hz} \\
T_s &= \frac{1}{f_s} \\
\omega_c &= 2\pi f_c \\
&\approx 282.7\ \mathrm{rad}\ s^{-1} \\
\omega_{c,d} &= \frac{\omega_c}{f_s} \\
&= \frac{\pi}{4} \frac{\mathrm{rad}}{\mathrm{sample}} \approx 0.7854 \frac{\mathrm{rad}}{\mathrm{sample}}
\end{aligned}$$

The Nyquist-Shannon sampling theorem tells us that we can never sample frequencies higher than $f_s/2$ without losing information. This also means that the cut-off frequency can never be higher than half of the sample frequency. Or in other words, all normalized frequencies will be in the interval $[0, \pi]$.

Next, we'll use the pre-warping formula we derived in the page on the Bilinear Transform, in order to calculate the analog design frequency $\omega_{c,a}$:

$$\begin{aligned}
\omega_{c,a} &= \frac{2}{T_s} \tan\left(\frac{\omega_{c,d}}{2}\right) \\
&= 720 \tan\left(\frac{\pi}{8}\right) \approx 298.2\ \mathrm{rad}\ s^{-1}
\end{aligned}$$

Note that this frequency is relatively close to ω_c, but it is *not* the same. The higher the cut-off frequency (relative to the sample frequency), the larger the error between ω_c and $\omega_{c,a}$.

Designing the Butterworth filter in the Analog Domain

Now that we know the pre-warped analog cut-off frequency, we can start designing the analog filter.
We'll use the formula for the Butterworth low-pass filter derived in the page on Butterworth Filters:

$$H_4(s') = \frac{1}{B_4(s')} \qquad \text{where } s' \triangleq \frac{s}{\omega_{c,a}} \tag{1}$$

$$B_4(s') = \left(s'^2 - 2\cos\left(2\pi\frac{4+1}{4\cdot 4}\right)s' + 1\right)\left(s'^2 - 2\cos\left(2\pi\frac{2+4+1}{4\cdot 4}\right)s' + 1\right)$$

Defining these constants will make the calculations much easier:

$$\begin{aligned}
\alpha &\triangleq -2\cos\left(\frac{5\pi}{8}\right) \\
&= \sqrt{2-\sqrt{2}} \qquad (2) \\
\beta &\triangleq -2\cos\left(\frac{7\pi}{8}\right) \\
&= \sqrt{2+\sqrt{2}} \qquad (3)
\end{aligned}$$

$$
\begin{aligned}
B_4(s') &= \left(s'^2 + \alpha s' + 1\right)\left(s'^2 + \beta s' + 1\right) \\
&= s'^4 + s'^3(\alpha+\beta) + s'^2(\alpha\beta+2) + s'(\alpha+\beta) + 1
\end{aligned} \tag{4}
$$

Discretizing the Analog Filter

We can now just apply the Bilinear Transform to the analog transfer function, by substituting $s = \frac{2}{T_s}\frac{z-1}{z+1}$. Therefore:

$$
s' = \frac{2f_s}{\omega_{a,c}}\frac{z-1}{z+1}
$$

Again, we'll introduce a constant to simplify the expression:

$$
\begin{aligned}
\gamma &\triangleq \frac{2f_s}{\omega_{a,c}} = \frac{2f_s}{2f_s\tan\left(\frac{\omega_{c,d}}{2}\right)} = \cot\left(\pi\frac{f_c}{f_s}\right) \\
&= \cot\left(\frac{\pi}{8}\right) = 1+\sqrt{2}
\end{aligned} \tag{5}
$$

$$
s' = \gamma\frac{z-1}{z+1} \tag{6}
$$

What follows is just rearranging the expression of $B_4(s')$ from Equation $???$, using the substitution of Equation $???$.
Finally, we end up with an expression for the transfer function, using Equation $???$, and we can determine the coefficients using the constants defined in Equations $\mathbf{2}$, $\mathbf{3}$ & $???$.

$$
\begin{aligned}
B_4(s') &= s'^4 + s'^3(\alpha+\beta) + s'^2(\alpha\beta+2) + s'(\alpha+\beta) + 1 \\
B_4(z) &= \gamma^4 \frac{(z-1)^4}{(z+1)^4} \\
&\quad + \gamma^3 \frac{(z-1)^3(z+1)}{(z+1)^4}(\alpha+\beta) \\
&\quad + \gamma^2 \frac{(z-1)^2(z+1)^2}{(z+1)^4}(\alpha\beta+2) \\
&\quad + \gamma \frac{(z-1)(z+1)^3}{(z+1)^4}(\alpha+\beta) \\
&\quad + \frac{(z+1)^4}{(z+1)^4} \\
&= \frac{1}{(z+1)^4}\left[\begin{array}{cc}
 & \gamma^4(z-1)^4 \\
+ & \gamma^3(z-1)^3(z+1)(\alpha+\beta) \\
+ & \gamma^2(z-1)^2(z+1)^2(\alpha\beta+2) \\
+ & \gamma(z-1)(z+1)^3(\alpha+\beta) \\
+ & (z+1)^4
\end{array}\right] \\
&= \frac{1}{(z+1)^4}\left[\begin{array}{cccccccccccccc}
 & \gamma^4 & (& z^4 & - & 4z^3 & + & 6z^2 & - & 4z & + & 1 &) & \\
+ & \gamma^3 & (& z^4 & - & 2z^3 & & & + & 2z & - & 1 &) & (\alpha+\beta) \\
+ & \gamma^2 & (& z^4 & & & - & 2z^2 & & & + & 1 &) & (\alpha\beta+2) \\
+ & \gamma & (& z^4 & + & 2z^3 & & & - & 2z & - & 1 &) & (\alpha+\beta) \\
+ & & (& z^4 & + & 4z^3 & + & 6z^2 & + & 4z & + & 1 &) &
\end{array}\right] \\
&= \frac{1}{(z+1)^4}\left[\begin{array}{ccc}
 & \left(\gamma^4+\gamma^3(\alpha+\beta)+\gamma^2(\alpha\beta+2)+\gamma(\alpha+\beta)+1\right) & z^4 \\
+ & \left(-4\gamma^4-2\gamma^3(\alpha+\beta)+2\gamma(\alpha+\beta)+4\right) & z^3 \\
+ & \left(6\gamma^4-2\gamma^2(\alpha\beta+2)+6\right) & z^2 \\
+ & \left(-4\gamma^4+2\gamma^3(\alpha+\beta)-2\gamma(\alpha+\beta)+4\right) & z \\
+ & \left(\gamma^4-\gamma^3(\alpha+\beta)+\gamma^2(\alpha\beta+2)-\gamma(\alpha+\beta)+1\right) &
\end{array}\right] \\
H_4(z) &= \frac{1}{B_4(z)} \\
&= \frac{(z+1)^4}{\left[\begin{array}{ccc}
 & (\gamma^4+\gamma^3(\alpha+\beta)+\gamma^2(\alpha\beta+2)+\gamma(\alpha+\beta)+1) & z^4 \\
+ & (-4\gamma^4-2\gamma^3(\alpha+\beta)+2\gamma(\alpha+\beta)+4) & z^3 \\
+ & (6\gamma^4-2\gamma^2(\alpha\beta+2)+6) & z^2 \\
+ & (-4\gamma^4+2\gamma^3(\alpha+\beta)-2\gamma(\alpha+\beta)+4) & z \\
+ & (\gamma^4-\gamma^3(\alpha+\beta)+\gamma^2(\alpha\beta+2)-\gamma(\alpha+\beta)+1) &
\end{array}\right]} \\
&\triangleq \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4}}
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= 4 \\
b_2 &= 6 \\
b_3 &= 4 \\
b_4 &= 1
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \gamma^4+\gamma^3(\alpha+\beta)+\gamma^2(\alpha\beta+2)+\gamma(\alpha+\beta)+1 \quad \approx 97.95 \\
a_1 &= -4\gamma^4-2\gamma^3(\alpha+\beta)+2\gamma(\alpha+\beta)+4 \quad \approx -192.8 \\
a_2 &= 6\gamma^4-2\gamma^2(\alpha\beta+2)+6 \quad \approx 170.0 \\
a_3 &= -4\gamma^4+2\gamma^3(\alpha+\beta)-2\gamma(\alpha+\beta)+4 \quad \approx -70.96 \\
a_4 &= \gamma^4-\gamma^3(\alpha+\beta)+\gamma^2(\alpha\beta+2)-\gamma(\alpha+\beta)+1 \quad \approx 11.79
\end{aligned}
$$

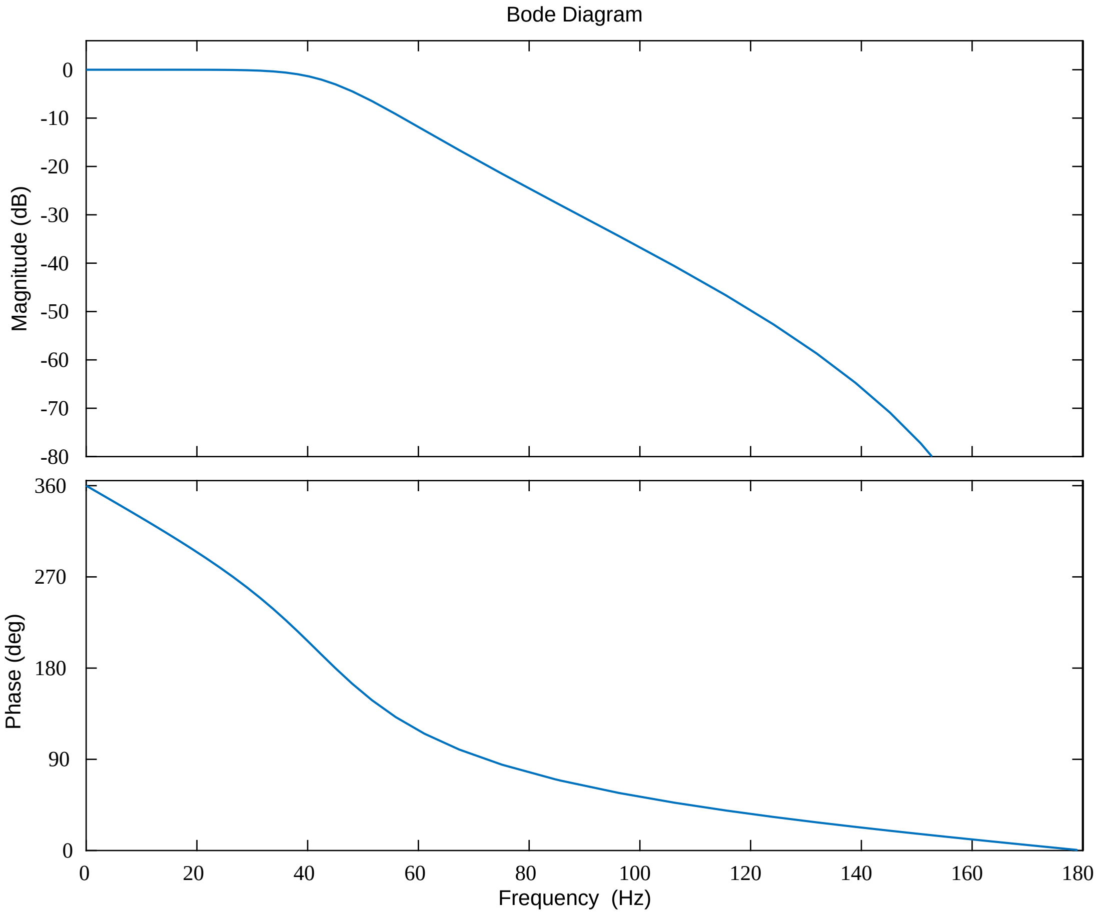
Frequency Response & Pole-Zero Map

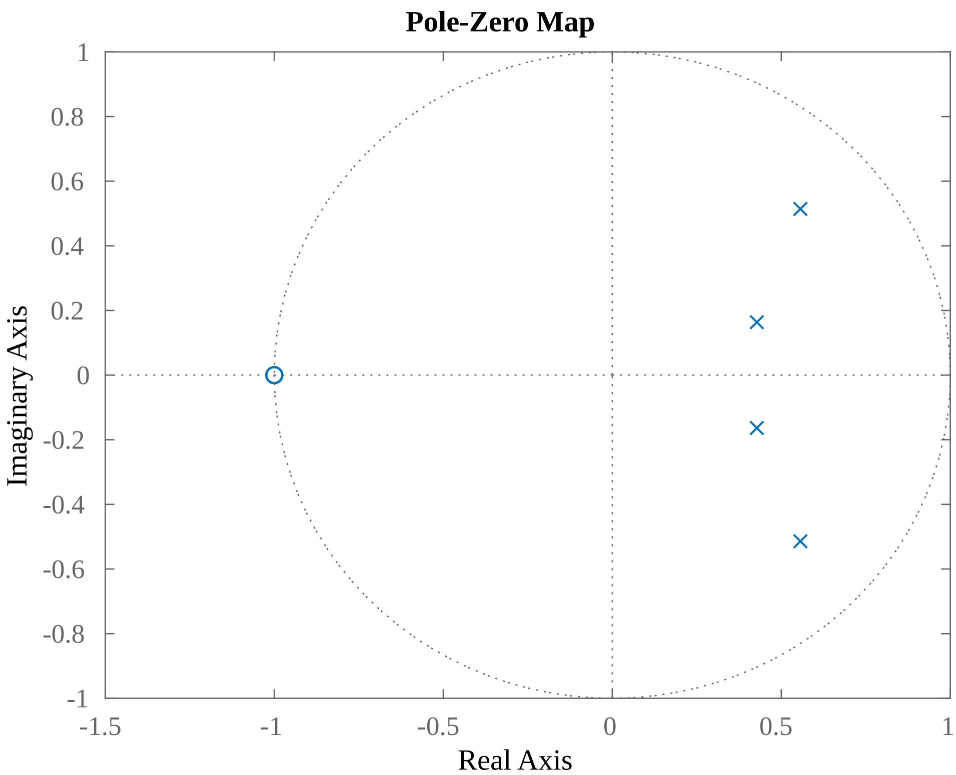
We can check the filter's frequency response to make sure that we didn't make any mistakes. As mentioned in other pages, the frequency response of a digital system can be a obtained by evaluating the transfer function $H(z)$ along the unit circle ($z = e^{j\omega}$). We'll plot the magnitude in decibels.

$$A(\omega) = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

We can also plot the phase angle of the response:

$$\phi(\omega) = \angle H\left(e^{j\omega}\right)$$





You can see that the corner frequency lies around $45\ \mathrm{Hz}$. We can check this mathematically:

$$A(\omega_{c,d}) = -3.01\mathrm{dB}$$

MATLAB & GNU Octave

If you have to design many different filters, you don't want to calculate them all by hand. Luckily, MATLAB and GNU Octave come with a command to calculate the coefficients of Butterworth filters.

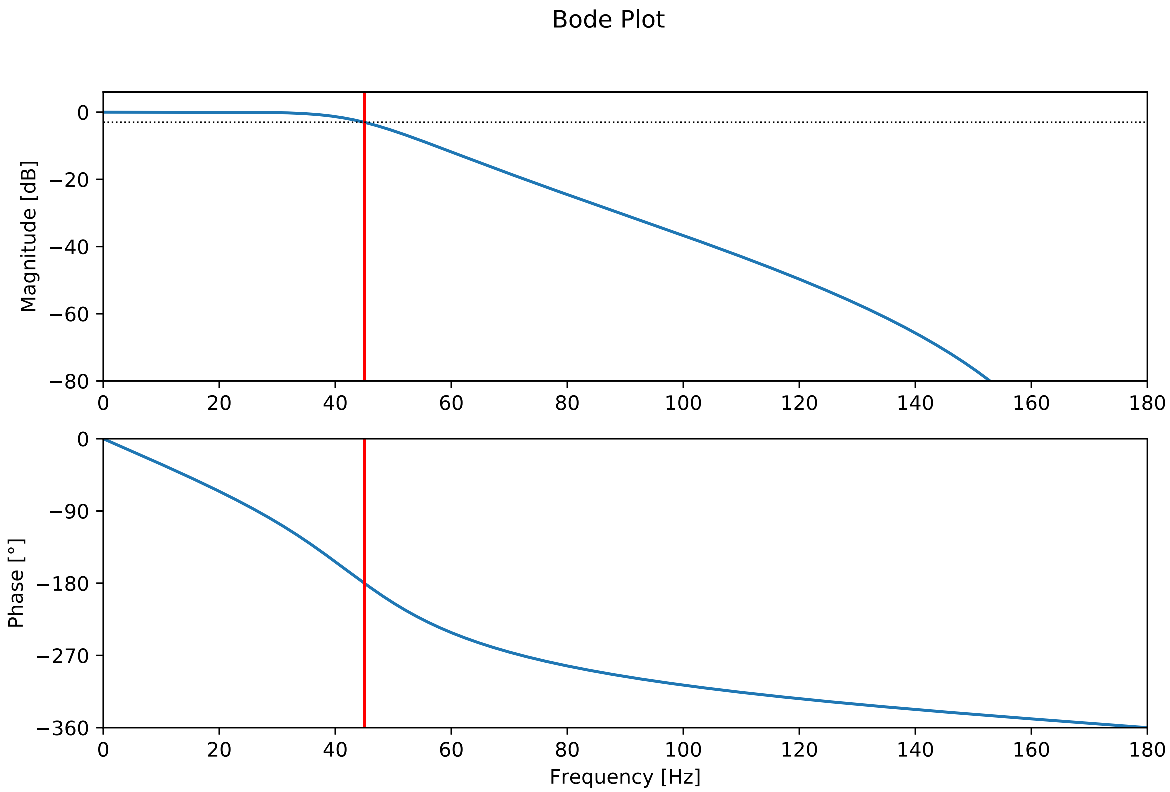
```
f_s = 360;    % Sample frequency in Hz
f_c = 45;     % Cut-off frequency in Hz
order = 4;    % Order of the butterworth filter

omega_c = 2 * pi * f_c;       % Cut-off angular frequency
omega_c_d = omega_c / f_s;    % Normalized cut-off frequency (digital)

[b, a] = butter(order, omega_c_d / pi);    % Design the Butterworth filter
disp('a = '); disp(a);                     % Print the coefficients
disp('b = '); disp(b);
H = tf(b, a, 1 / f_s);                     % Create a transfer function
bode(H);                                   % Show the Bode plot
```

Note that MATLAB expects the normalized frequency as a number from 0 to 1, so we have to divide by π before passing it to the `butter` function.

Python

A similar function is available in the SciPy signal package: `butter`.

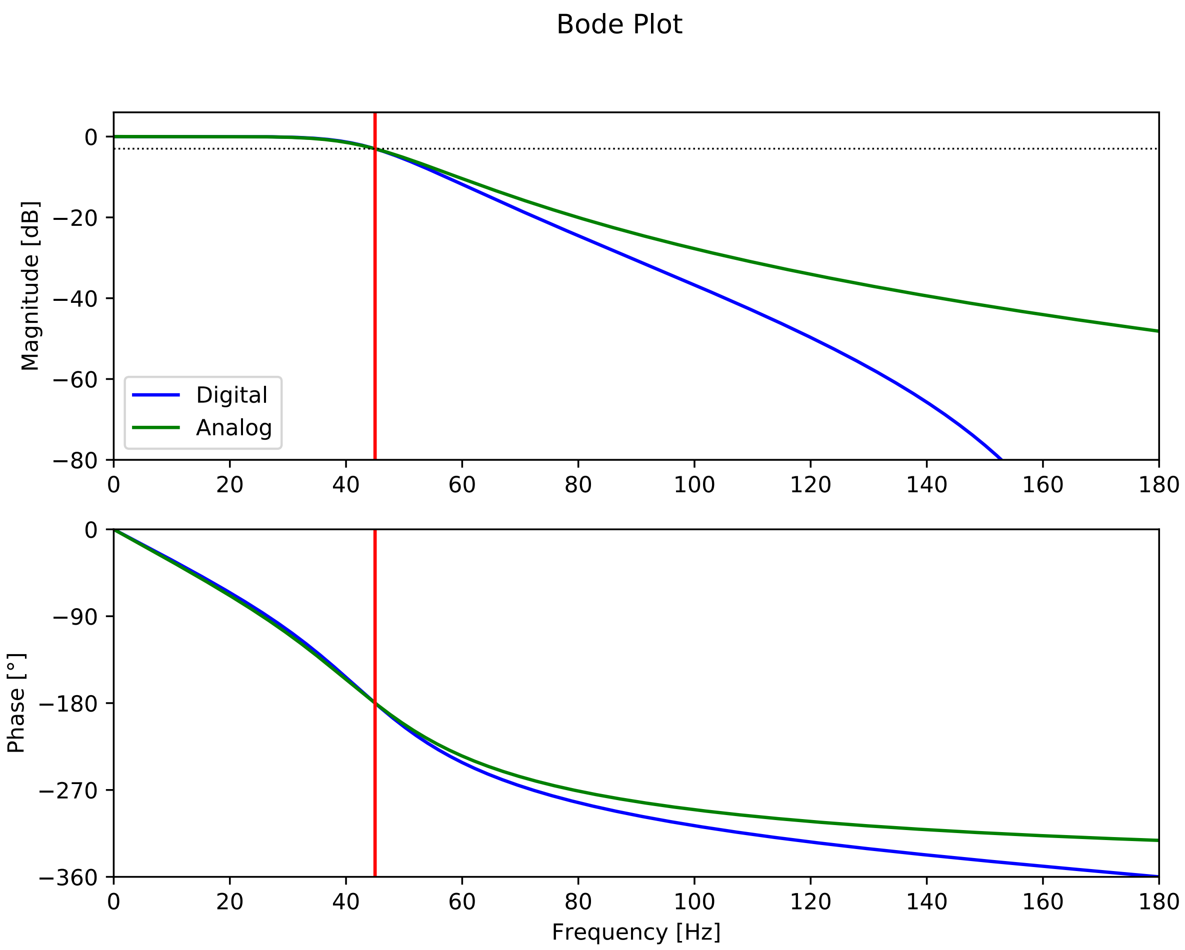
```
from scipy.signal import butter, freqz, freqs
import matplotlib.pyplot as plt
from math import pi
import numpy as np

f_s = 360    # Sample frequency in Hz
f_c = 45     # Cut-off frequency in Hz
order = 4    # Order of the butterworth filter

omega_c = 2 * pi * f_c       # Cut-off angular frequency
omega_c_d = omega_c / f_s    # Normalized cut-off frequency (digital)

# Design the digital Butterworth filter
b, a = butter(order, omega_c_d / pi)
print('Coefficients')
print("b =", b)                             # Print the coefficients
print("a =", a)

w, H = freqz(b, a, 4096)                    # Calculate the frequency response
w *= f_s / (2 * pi)                         # Convert from rad/sample to Hz

# Plot the amplitude response
plt.subplot(2, 1, 1)
plt.suptitle('Bode Plot')
H_dB = 20 * np.log10(abs(H))                # Convert modulus of H to dB
plt.plot(w, H_dB)
plt.ylabel('Magnitude [dB]')
plt.xlim(0, f_s / 2)
plt.ylim(-80, 6)
plt.axvline(f_c, color='red')
plt.axhline(-3, linewidth=0.8, color='black', linestyle=':')

# Plot the phase response
plt.subplot(2, 1, 2)
phi = np.angle(H)                           # Argument of H
phi = np.unwrap(phi)                        # Remove discontinuities
phi *= 180 / pi                             # and convert to degrees
plt.plot(w, phi)
plt.xlabel('Frequency [Hz]')
plt.ylabel('Phase [°]')
plt.xlim(0, f_s / 2)
plt.ylim(-360, 0)
plt.yticks([-360, -270, -180, -90, 0])
plt.axvline(f_c, color='red')

plt.show()
```



Comparison Between the Analog and Digital Filter

We can easily plot the Bode plots of the two filters on top of each other, in order to compare their properties.

```python
from scipy.signal import butter, freqz, freqs
import matplotlib.pyplot as plt
from math import pi
import numpy as np

f_s = 360    # Sample frequency in Hz
f_c = 45     # Cut-off frequency in Hz
order = 4    # Order of the butterworth filter

omega_c = 2 * pi * f_c       # Cut-off angular frequency
omega_c_d = omega_c / f_s    # Normalized cut-off frequency (digital)

# Design the digital Butterworth filter
b_d, a_d = butter(order, omega_c_d / pi)
print('Digital Coefficients')
print("b =", b_d)                         # Print the coefficients
print("a =", a_d)

w, H_d = freqz(b_d, a_d, 4096)            # Calculate the frequency response
w *= f_s / (2 * pi)                       # Convert from rad/sample to Hz

# Design the analog Butterworth filter
b_a, a_a = butter(order, f_c, analog=True)
print('Analog Coefficients')
print("b =", b_a)                         # Print the coefficients
print("a =", a_a)

w, H_a = freqs(b_a, a_a, w)               # Calculate the frequency response

# Plot the amplitude response
plt.subplot(2, 1, 1)
plt.suptitle('Bode Plot')
H_d_dB = 20 * np.log10(abs(H_d))          # Convert modulus of H to dB
H_a_dB = 20 * np.log10(abs(H_a))
plt.plot(w, H_d_dB, color='blue', label='Digital')
plt.plot(w, H_a_dB, color='green', label='Analog')
plt.legend()
plt.ylabel('Magnitude [dB]')
plt.xlim(0, f_s / 2)
plt.ylim(-80, 6)
plt.axvline(f_c, color='red')
plt.axhline(-3, linewidth=0.8, color='black', linestyle=':')

# Plot the phase response
plt.subplot(2, 1, 2)
phi_d = np.angle(H_d)                     # Argument of H
phi_a = np.angle(H_a)
phi_d = np.unwrap(phi_d) * 180 / pi       # Remove discontinuities
phi_a = np.unwrap(phi_a) * 180 / pi       # and convert to degrees
plt.plot(w, phi_d, color='blue')
plt.plot(w, phi_a, color='green')
plt.xlabel('Frequency [Hz]')
plt.ylabel('Phase [°]')
plt.xlim(0, f_s / 2)
plt.ylim(-360, 0)
plt.yticks([-360, -270, -180, -90, 0])
plt.axvline(f_c, color='red')

plt.show()
```



Discretization using Second Order Sections (SOS)

For higher order filters, small quantization errors on the transfer function coefficients can result in large errors on the pole and zero locations.
A solution is to factor the transfer function into second order factors or sections.

Recall from the previous sections, and define α_k, $H_{2,k}$ and H_1 as follows:

$$
\begin{aligned}
\gamma &\triangleq \cot\left(\pi \frac{f_c}{f_s}\right) && (7)\\
s' &\triangleq \frac{s}{\omega_{c,a}} = \gamma \frac{z-1}{z+1} && (8)\\
\alpha_k &\triangleq 2\cos\left(2\pi \frac{2k+n+1}{4n}\right) && (9)\\
H_{2,k}(s') &\triangleq \frac{1}{B_{2,k}(s')} \triangleq \frac{1}{s'^2 - \alpha_k s' + 1} && (10)\\
H_1(s') &\triangleq \frac{1}{s'+1} && (11)\\
H_n(s') &= \begin{cases} \prod_{k=0}^{\frac{n}{2}-1} H_{2,k}(s') & \text{even } n \\ H_1(s') \prod_{k=0}^{\frac{n-1}{2}-1} H_{2,k}(s') & \text{odd } n \end{cases} && (12)
\end{aligned}
$$

Second Order Sections

We'll use the same technique as before to substitute s' into $B_{2,k}(s')$ using the pre-warped bilinear transform relation 8 to get the discrete-time Butterworth polynomial $B_{2,k}(z)$:

$$
\begin{aligned}
B_{2,k}(s') &\triangleq s'^2 - \alpha_k s' + 1 \\
B_{2,k}(z) &= \gamma^2 \left(\frac{z-1}{z+1} \right)^2 - \alpha_k \gamma \frac{z-1}{z+1} + 1 \\
&= \gamma^2 \frac{(z-1)^2}{(z+1)^2} - \alpha_k \gamma \frac{(z-1)(z+1)}{(z+1)^2} + \frac{(z+1)^2}{(z+1)^2} \\
&= \frac{\gamma^2 (z-1)^2 - \alpha_k \gamma (z-1)(z+1) + (z+1)^2}{(z+1)^2} \\
&= \frac{\left(\gamma^2 - \alpha_k \gamma + 1\right) z^2 - \left(2\gamma^2 - 2\right) z + \left(\gamma^2 + \alpha_k \gamma + 1\right)}{(z+1)^2} \\
H_{2,k}(z) &= \frac{z^2 + 2z + 1}{(\gamma^2 - \alpha_k \gamma + 1) z^2 - (2\gamma^2 - 2) z + (\gamma^2 + \alpha_k \gamma + 1)} \\
&= \frac{1 + 2z^{-1} + z^{-2}}{(\gamma^2 - \alpha_k \gamma + 1) - (2\gamma^2 - 2) z^{-1} + (\gamma^2 + \alpha_k \gamma + 1) z^{-2}}
\end{aligned}
$$

The coefficients of the k-th factor of the discrete-time transfer function are thus:

$$
\begin{aligned}
b_{k,0} &= 1 \\
b_{k,1} &= 2 \\
b_{k,2} &= 1
\end{aligned}
$$

$$
\begin{aligned}
a_{k,0} &= \gamma^2 - \alpha_k \gamma + 1 \\
a_{k,1} &= 2\left(1 - \gamma^2\right) \\
a_{k,2} &= \gamma^2 + \alpha_k \gamma + 1
\end{aligned}
$$

First Order Section

For odd orders n, we need $\frac{n-1}{2}$ second order sections and a single first order section.

Again, we'll use the the pre-warped bilinear transform relation 8:

$$
\begin{aligned}
H_1(s') &\triangleq \frac{1}{s' + 1} \\
&= \frac{1}{\gamma \frac{z-1}{z+1} + 1} \\
&= \frac{z+1}{\gamma(z-1) + (z+1)} \\
&= \frac{z+1}{(\gamma+1) z + (1-\gamma)} \\
&= \frac{1 + z^{-1}}{(\gamma+1) + (1-\gamma) z^{-1}}
\end{aligned}
$$

This gives us the coefficients of the first order factor of the discrete-time transfer function:

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= 1
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \gamma + 1 \\
a_1 &= 1 - \gamma
\end{aligned}
$$