

Latin Puzzles

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Abstract

Based on a previous generalization by the author of Latin squares to Latin boards, this paper generalizes partial Latin squares and related objects like partial Latin squares, completable partial Latin squares and Latin square puzzles. The latter challenge players to complete partial Latin squares, *Sudoku* being the most popular variant nowadays.

The present generalization results in *partial Latin boards*, *completable partial Latin boards* and *Latin puzzles*. Provided examples of Latin puzzles illustrate how they differ from puzzles based on Latin squares. The examples include *Sudoku Ripeto* and *Custom Sudoku*, two new *Sudoku* variants. This is followed by a discussion of methods to find Latin boards and Latin puzzles amenable to being solved by human players, with an emphasis on those based on constraint programming. The paper also includes an analysis of objective and subjective ways to measure the difficulty of Latin puzzles.

Keywords: asterism, board, completable partial Latin board, constraint programming, *Custom Sudoku*, Free Latin square, Latin board, Latin hexagon, Latin polytope, Latin puzzle, Latin square, Latin square puzzle, Latin triangle, partial Latin board, *Sudoku*, *Sudoku Ripeto*.

1 Introduction

Sudoku puzzles challenge players to complete a square board so that every row, column and 3×3 sub-square contains all numbers from 1 to 9 (see an example in Fig. 1). The simplicity of the instructions coupled with the entailed combinatorial properties have made *Sudoku* both a popular puzzle and an object of active mathematical research.

1		6			2	3		
	5				6		9	1
		9	5		1	4	6	2
	3	7	9		5			
5	8	1		2	7	9		
			4		8	1	5	7
			2	6		5	4	
	4	1	5		6		9	
9		8	7	4	2	1		

1	4	6	7	9	2	3	8	5
2	5	8	3	4	6	7	9	1
3	7	9	5	8	1	4	6	2
4	3	7	9	1	5	8	2	6
5	8	1	6	2	7	9	3	4
6	9	2	4	3	8	1	5	7
7	1	3	2	6	9	5	4	8
8	2	4	1	5	3	6	7	9
9	6	5	8	7	4	2	1	3

Figure 1. *Sudoku*

3	1	6	7	4	5		
6	3	7	4	5	2		
1	2	5	6	7	4		
2	5	4	1	6	3		
5	4	3	2	1	7		
4	7	1	3	2	6		

7	6	2	5	3	1	4	
3	1	6	7	4	5	2	
6	3	7	4	5	2	1	
1	2	5	6	7	4	3	
2	5	4	1	6	3	7	
5	4	3	2	1	7	6	
4	7	1	3	2	6	5	

Figure 2. Latin square puzzle and solution

Back in the 19th century, B. Meyniel created puzzles very similar to *Sudoku* that were featured in French newspapers [8]. The puzzle was rendered popular in Japan in the late 1980s by puzzle editor Maki Kaji, who coined the word “Sudoku” (*single number* in Japanese). Later, former Hong Kong judge Wayne Gould wrote a computer program to generate *Sudoku* puzzles and had them published in European and American newspapers. By the mid-2000s the puzzle was popular worldwide. It was later pointed out by puzzle author and editor Will Shortz that the creator of *Sudoku* was American architect Howard Garns (1905-1989), whose puzzles first appeared in 1979 in America under the name *Number Place* [19].

Based on the generalization of Latin squares to Latin boards proposed by the author in [42], this paper proposes a broad generalization of *Sudoku* regarding board topology, regions, symbols and inscriptions inside the board. This results in new categories of puzzles in addition to the existing one, to which *Sudoku* and current variants belong.

2 The Latin Square Puzzle

2.1 Latin Squares

A Latin square of order n is an $n \times n$ array of cells in which every row and every column holds all labels belonging to an n -set [10]. Latin squares are so named because the 18th century Swiss mathematician Leonhard Euler used Latin letters as labels in his paper *On Magic Squares* [13]. Fig. 2 (right) shows a Latin square of order seven with the set of labels $\{1, 2, 3, 4, 5, 6, 7\}$.

Latin squares are redundant structures: after removing some labels the result may still be completable to just the original Latin square. If presented with the challenge: “Complete the board in Fig. 2 (left) so that every row and every column contains all numbers from 1 to 7”, we can easily conclude that the *only* solution is the Latin square on its right¹.

¹For some historical facts about Latin squares including ludic ones see [2, 52, 53].

7				1	4	
3						
			1	5		1
1		5				
2	5		1	6		
5	6	2	2		7	6
5				1	5	

7	6	2	5	3	1	6
3	3	6	7	2	5	2
1	3	7	4	5	2	7
7	2	4	6	7	4	3
2	4	4	1	1	3	7
5	4	3	6	1	7	3
4	7	1	3	2	6	5

Figure 3. Partial magma and magma

2.2 Partial Objects Related to Latin squares

We list here definitions of classic objects related to Latin squares (see [26]).

Magmas. A *magma* of order n is a square grid of cells labeled with the elements of an n -set of labels². Fig. 3 (right) is an example with labels $\{1, 2, 3, 4, 5, 6, 7\}$, some of them repeated in rows and columns.

Partial Magmas. A *partial magma* is a magma with some labels missing. Fig. 3 (left) shows an example with labels $\{1, 2, 3, 4, 5, 6, 7\}$.

Partial Latin Squares. Fig. 4 shows four partial magmas with labels from $\{1, 2, 3, 4, 5, 6, 7\}$ in which no label occurs more than once in any row or column. Partial magmas fulfilling this condition are called *partial Latin squares* [26].

Completable Partial Latin Squares. We may wonder if each of the squares in Fig. 4 can be completable to either one or several Latin squares. It turns out that the top-left one can only be completable to any of the four Latin squares in Fig. 5, while the top-right one can't be completed to any at all. This is easy to see provided that the claim for the top-left square is true: the top-right one only differs from the former in the extra 2 on the first row, but none of the four squares in Fig. 5 has 2 in that position.

The two bottom squares in Fig. 4 can be completable to *just* the bottom-right square in Fig. 5. We say that all partial Latin squares in Fig. 4, except for the top-right one, are *completable partial Latin squares*.

Uniquely completable partial Latin squares. A partial Latin square that is completable to just one Latin square –like the two bottom ones in Fig. 4– is a *uniquely completable partial Latin square*.

Critical sets. If the removal of any label in a uniquely completable partial Latin square destroys its property of uniqueness of completion –the case of the bottom-right square in Fig. 4– then the square is a *critical set*.

²There is a more algebraical definition: “A *magma* is a set with an operation that sends any two elements in the set to another element in the set”. We use then the term in this paper to denote a magma’s multiplication table without its borders.

							4
3							
		4	5		1		
1		5					
2	5		1	6			
5	4	3	2		7	6	
7				6	5		

						2		4
3								
		4	5		1			
1		5						
2	5		1	6				
5	4	3	2		7	6		
7				6	5			

7					1	4
3						
		4	5		1	
1		5				
2	5		1	6		
5	4	3	2		7	6
7				6	5	

					1	4
3						
		4	5		1	
1		5				
2	5		1	6		
5	4	3	2		7	6
7				6	5	

Figure 4. Partial Latin squares

6	1	2	7	3	5	4
3	6	4	5	7	1	2
7	2	6	4	5	3	1
1	3	5	6	4	2	7
2	5	7	1	6	4	3
5	4	3	2	1	7	6
4	7	1	3	2	6	5

6	1	2	7	3	5	4
3	6	7	5	4	1	2
7	3	6	4	5	2	1
1	2	5	6	7	4	3
2	5	4	1	6	3	7
5	4	3	2	1	7	6
4	7	1	3	2	6	5

7	6	2	5	3	1	4
3	1	6	7	4	5	2
6	3	7	4	5	2	1
1	2	5	6	7	4	3
2	5	4	1	6	3	7
5	4	3	2	1	7	6
4	7	1	3	2	6	5

Figure 5. Latin squares

Latin square puzzles and critical Latin square puzzles. Uniquely completable partial Latin squares and critical sets are called respectively *Latin square puzzles* and *critical Latin square puzzles* in this paper.

3 Latin Puzzles

We reproduce for convenience in this section some definitions found in [42].

3.1 Boards

Definition 3.1. A *board* is a pair (P, C) where P is a set of geometric points and C is a set of subsets of P whose union is P .

We deduce from the definition that intersection among subsets is allowed. The set C is the *constellation* of the board, and every subset an *asterism*.

Definition 3.2. A *k-uniform board* is one whose asterisms have all size k .

3.1.1 Points and Cells in Boards

We will often represent a board with a grid of cells. To conciliate this with the formal definition, that uses points, we adopt the following convention: “cell” will mean either *centroid of the cell* or *point*, depending on the context. For example “write a label in the cell” will mean *label the centroid of the cell*, whereas “a board with n cells” will be *a board with n points*.

Example 3.3. If we remove the numbers inside the cells in Fig. 2 (left), the resulting object is a board whose points are the (centroids of) cells. Every row and column of (centroids of) cells is an asterism of this board.

3.2 Latin Boards

Definition 3.4. Let $B = (P, C)$ be a k -uniform board and L a k -multiset of *labels*. A *Latin board* is a tuple (B, L, F) where F is a bijective function $P \rightarrow L$.

In simpler terms: in a Latin board every asterism holds all labels in the multiset³. We also say that a Latin board is the result of *latinizing* or *labeling* a board with a multiset [42]. According to this definition, and in contrast with the one for Latin squares in Def. 2.1, Latin boards have no constraints on topology, number of asterisms or uniqueness of labels.

3.3 Partial Objects Related to Latin Boards

We define here for the first time partial objects related to Latin boards. We do this by generalizing partial objects related to Latin squares (see Sect. 2.2).

³This definition is for the so-called *k-unifit Latin boards*. See [42] for a more general definition of Latin boards.

3.3.1 Boards

As per Def. 3.1, boards generalize the underlying $n \times n$ array of cells in Latin squares and their distribution in rows and columns. Every row and column generalizes to an asterism; the set of all rows and columns generalizes to the constellation of the board.

3.3.2 Labeled Boards

Definition 3.5. Let $B = (P, C)$ be a k -uniform board and L a k -multiset of labels. A *k-labeled board* is a tuple (B, L, F) where F is a function $P \rightarrow L$.

As the function needs not be bijective, the count of each label in an asterism may not match its count in the multiset. Labeled boards generalize magmas.

3.3.3 Partial Labeled Boards

Definition 3.6. A *partial labeled board* is a labeled board with some missing labels.

Partial labeled boards generalize partial magmas.

3.3.4 Partial Latin Boards

Definition 3.7. A partial labeled board is a *partial Latin board* if and only if the count of each label in every asterism is not greater than its count in the multiset.

Partial Latin boards generalize partial Latin squares.

3.3.5 Completable Partial Latin Boards

As with partial Latin squares, a partial Latin board is not guaranteed to be completable, hence the following definition.

Definition 3.8. A partial Latin board is a *completable partial Latin board* if and only if it can be completed to one or several Latin boards.

Completable partial Latin boards generalize completable partial Latin squares.

3.3.6 Latin puzzles

In order to remove ambiguity when possible, we will follow hereafter the *puzzle-Puzzle name convention*: “a puzzle” (lower-case p) is an instance of a particular “Puzzle” (upper-case P), which is the general game –“a *Sudoku* puzzle” vs. “the *Sudoku* Puzzle” for example.

Definition 3.9. A completable partial Latin board is a *Latin puzzle* if and only if it can be completed to exactly one Latin board.

We call this Latin board the puzzle’s *solution*. The labels in the puzzle are called *clues*. Latin puzzles generalize uniquely completable partial Latin squares.

Like Latin squares, Latin boards are redundant structures: they contain more information than is strictly necessary to describe them, hence the possibility of Latin puzzles.

The board in Fig. 2 (left) is then a Latin puzzle, and also the two bottom ones in Fig. 4. The two top ones in the same figure are not Latin puzzles. Latin puzzles generalize Latin square puzzles in the sense that their solutions are Latin boards, themselves a generalization of Latin squares [42].

3.3.7 Latin Puzzles

Definition 3.10. A *Latin Puzzle* is the set of puzzles that results when we render variable both the multiset of labels and the set of clues in a given Latin puzzle.

We will also say that a particular Latin puzzle is a *member* or an *instance* of the corresponding Latin Puzzle. The three Latin puzzles in Fig. 4 belong then to the *Latin Square Puzzle*. Latin Puzzles generalize the *Latin square Puzzle*.

3.3.8 Inscripted Latin Puzzles

Multisets of labels open the door to Puzzles with inscriptions.

Definition 3.11. An *inscription* in a Latin Puzzle is a set of pairs (label, cell) always present among the clues of every Puzzle's puzzle.

We say that both the Puzzle and his puzzles are *inscripted*. Inscriptions make possible puzzles with generic words in any language and alphabet, see Sect. 4.5 and [43] for examples.

3.3.9 The Latin Puzzle

Definition 3.12. The *Latin Puzzle* is the set of puzzles that results when we render variable the board, the multiset of labels and the clues in a given puzzle.

3.3.10 Critical Latin puzzles

Definition 3.13. A Latin puzzle is *critical* if the removal of any of its clues destroys the uniqueness of the completion. An inscribed Latin puzzle is *critical* if the removal of any of its clues not in the inscription destroys the uniqueness of the completion (see Sect. 4.5).

3.3.11 Minimal Latin puzzles

Definition 3.14. A Latin puzzle is *minimal* if it has the smallest possible set of clues among the puzzles in its Latin Puzzle.

3.3.12 Corresponding Terms

Table 3.1 summarizes the correspondences between terms for Latin squares and Latin boards.

Latin Square Term	Latin Board Term
$n \times n$ array of cells	set of geometric points
cells grouped in rows and columns	board
row or column of cells	asterism
set of all rows and columns	constellation
magma	labeled board
partial magma	partial labeled board
partial Latin square	partial Latin board
completable partial Latin square	completable partial Latin board
uniquely completable partial Latin square	Latin puzzle
critical set	critical Latin puzzle
<i>Latin square</i> Puzzle	Latin Puzzle
Latin square	Latin board

Table 3.1 Correspondence between classic Latin squares terms and new Latin boards ones

3.4 Types of Latin Puzzles

In the light of Defs. 3.4 and 3.9 we may distinguish these main types of Latin Puzzles:

- *LS* if their solutions are Latin squares, *non-LS* otherwise
- *symmetric* if their solutions are Latin polytopes [42], *non-symmetric* otherwise
- *repeat* if the multiset has repeated labels, *non-repeat* otherwise
- *inscribed* if every puzzle in the Puzzle has an inscription (see Def. 4.5), *non-inscribed* otherwise

The *Sudoku* Puzzle for example is then LS, symmetric, non-repeat, non-inscribed.

4 Latin Puzzles Examples

The Latin puzzles and boards that follow have been created with the methods described in Sect. 5. Each puzzle is identified in bold typeface by its board name plus an adjective indicating whether or not inscriptions, sets or multisets are used. Except for the examples in Sect. 4.3.2, they are all guaranteed to be solvable in polynomial time in the number of cells (see Sect. 5.4).

Each puzzle is included in a section that illustrates a specific relaxation of the Latin square Puzzle (see Sect. 2). Unless otherwise stated, each puzzle is rated *medium* by the method described in Sect. 5.4.2. Additional Latin boards and puzzles can be found in [42, 43].

S	E			R		
P		E	N	O	S	G
N	G	R		O	A	P
O		S	I	N		
S	G	A	P	E	O	
		G	O	R	E	
R	I	E		S	O	N
O	P	A		G	R	I
E		R	N	G	S	P

S	E	A	P	G	R	N	O	I
P	A	E	N	O	I	S	G	R
N	G	R	I	S	O	A	P	E
G	O	P	S	I	N	E	R	A
R	S	N	G	A	P	I	E	O
I	N	G	O	R	E	P	A	S
A	R	I	E	P	S	O	N	G
O	P	S	A	E	G	R	I	N
E	I	O	R	N	A	G	S	P

Figure 6. Win One

				5			
		6	3	7	1	2	
	4						
	3	2	1				6
2		1		3			5
3			2		7	4	
5	6	4			3		

1	2	3	4	5	6	7	
4	5	6	3	7	1	2	
6	4	7	5	1	2	3	
7	3	2	1	4	5	6	
2	7	1	6	3	4	5	
3	1	5	2	6	7	4	
5	6	4	7	2	3	1	

Figure 7. Wave One

4.1 Extra Asterisms

By adding extra asterisms to the Latin square Puzzle we obtain *Latin square Puzzle variants*: those whose instances' solutions are Latin squares.

Win One (2014⁴). “Complete the board in Fig. 6 so that every row, column and highlighted region contains numbers 1 to 9”.

Wave One (2014). “Complete the board in Fig. 7 so that every row, column and highlighted region contains numbers 1 to 7”. In contrast to *Win One* puzzles, the extra asterisms here partition the set of cells. The solution in this case is a type of Latin square called *Gerechte square* [7]: a regular Latin square with an extra partition of the board in n regions (seven here) with n cells each.

Sudoku (1979, see [19]). “Complete the board in Fig. 8 so that every row, column and highlighted region contains numbers 1 to 9”⁵. *Sudoku* is arguably the most popular Latin square Puzzle variant, and is itself the base of other related Puzzles [45].

⁴Year of first publication (see [43]).

⁵*Sudoku* puzzles are then Gerechte squares too. Asterisms in *Sudoku* (rows, columns and sub-squares) are called *units* or *scopes*.

1	6		2	3				
	5			6		9	1	
	9	5		1	4	6	2	
	3	7	9		5			
5	8	1		2	7	9		
		4		8	1	5	7	
		2	6		5	4		
	4	1	5		6		9	
9		8	7	4	2	1		

1	4	6	7	9	2	3	8	5
2	5	8	3	4	6	7	9	1
3	7	9	5	8	1	4	6	2
4	3	7	9	1	5	8	2	6
5	8	1	6	2	7	9	3	4
6	9	2	4	3	8	1	5	7
7	1	3	2	6	9	5	4	8
8	2	4	1	5	3	6	7	9
9	6	5	8	7	4	2	1	3

Figure 8. Sudoku

		7	6		4	3	2	1
	6					5	4	
5		8						
					2	4	3	
			3		1			
	2		1		8	5		7
	5				4	3	8	
6	3	8	4		2	7	1	5
		4		8		9		2

9	8	7	6	5	4	3	2	1
1	6	2	9	7	3	8	5	4
5	4	3	8	2	1	6	7	9
8	7	1	5	6	9	2	4	3
4	9	5	2	3	7	1	8	6
3	2	6	1	4	8	5	9	7
2	5	9	7	1	6	4	3	8
6	3	8	4	9	2	7	1	5
7	1	4	3	8	5	9	6	2

Figure 9. Quadoku One

Quadoku One (2015). “Complete the board in Fig. 9 so that every row, column and highlighted region contains numbers 1 to 9”. This Puzzle combines *Sudoku* and *Win One* Puzzles into one⁶.

4.2 Board Shape

The next generalization of the Latin square Puzzle is just aesthetic: the board has a non-square shape, but is topologically equivalent to a Latin square.

Orb One (2014). “Complete the board in Fig. 10 so that every parallel, meridian and highlighted region contains numbers 1 to 8”. The solution here is topologically equivalent also to a Gerechte square.

4.3 Board Symmetry

This generalization opens the door to Puzzles whose solutions are not Latin squares.

⁶The *Quadoku One* Puzzle is also called *Hyperdoku*.

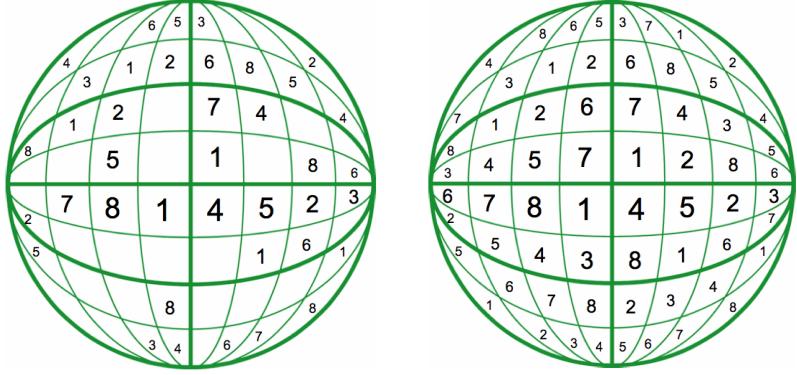


Figure 10. Orb One

4.3.1 Latin Polygons

Dihedral Groups

Any of the following eight geometric transformations leaves a square invariant:

- $0^\circ, 90^\circ, 180^\circ$ and 270° rotations
- 2 reflections across the two diagonals
- 2 reflections, one across the vertical symmetry axis, another across the horizontal one

Together with the operation of composition of transformations they form the so-called *dihedral group* D_4 [11], the structure that captures the symmetry of the square. Similarly, every regular polygon of n sides has an associated D_n dihedral group with n rotations and n reflections.

When a D_4 transformation is applied to a Latin square, another –not necessarily different– Latin square results. Similarly, a Latin board is a *Latin polygon* if every element of a particular dihedral group transforms it into another Latin board [42]. This is the case with the puzzle solutions that follow (see [42] for their specific definitions).

Canario One (2012). “Complete the board in Fig. 11 so that numbers 1 to 16 appear on every pair of triangle stripes pointed to by the same letter”. Solutions to *Canario* puzzles are *Latin triangles*.

Monthai One (2013). “Complete the board in Fig. 12 so that numbers 1 to 12 appear on every pair of triangle stripes pointed to by the same letter”. Solutions to *Monthai* puzzles are also *Latin triangles*.

Tartan One (2013). “Complete the board in Fig. 13 so that numbers 1 to 16 appear on every pair of rows pointed to by the same letter, every pair of columns pointed to by the same letter and every highlighted sub-square”. Solutions to *Tartan* puzzles are *free Latin squares*.

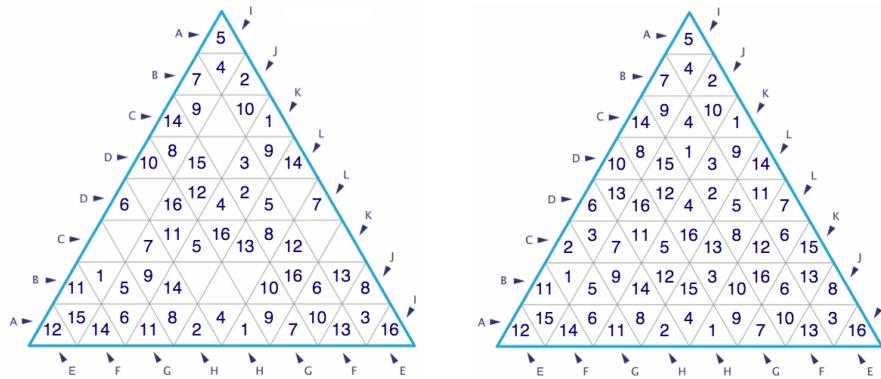


Figure 11. Canario One

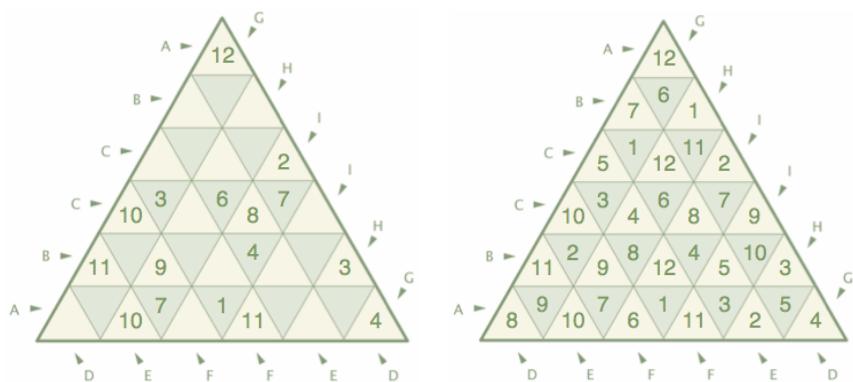


Figure 12. Monthai One

a	9	2	15		1	12	14
b	11	8		12	2	10	3
c	14	4		16	15		7
d	13	6	3	10	4	16	11
a	10	3	16	13	5	11	6
b	6	7		9	16		14
c	12	5		11	3		13
d	1	14	8		12	9	2
e	f	g	h	e	f	g	h

a	9	2	15	7	8	1	12	14
b	11	8	5	12	2	13	10	3
c	14	4	1	16	15	9	7	6
d	13	6	3	10	4	16	11	5
a	10	3	16	13	5	11	6	4
b	6	7	4	9	16	15	14	1
c	12	5	2	11	3	10	13	8
d	1	14	8	15	7	12	9	2
e	f	g	h	e	f	g	h	

Figure 13. Tartan One

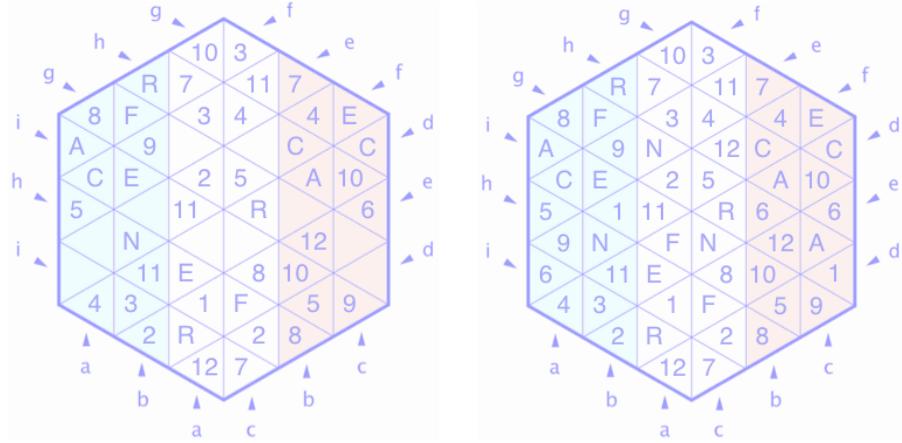


Figure 14. Douze France One

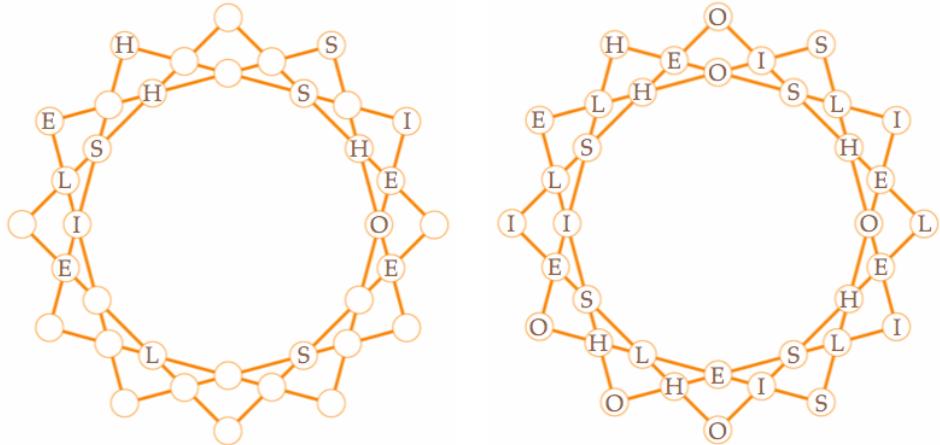


Figure 15. Helios One

Douze France One (2013). “Complete the board in Fig. 14 so that the numbers 1 to 12 and the letters F, R, A, N, C, E appear on every pair of triangle stripes pointed to by the same letter”. Solutions to *Douze France* puzzles are *Latin hexagons*.

Helios One (2013). “Complete the board in Fig. 15 so that the letters H, E, L, I, O, S appear on every line. Solutions to *Helios* puzzles are *Latin dodecagons*.

4.3.2 Latin Polyhedra

Polyhedral Groups

Every regular polyhedron has an associated symmetry group whose elements leave the polyhedron invariant [11]. Table 4.1 lists the so called *Polyhedral groups* and the corresponding five platonic solids. A Latin board is a *Latin*

Group	Regular Polyhedron	Order
tetrahedral	tetrahedron	24
octahedral	cube, octahedron	48
icosahedral	dodecahedron, icosahedron	120

Table 4.1 Polyhedral symmetry groups of the five Platonic solids and their order

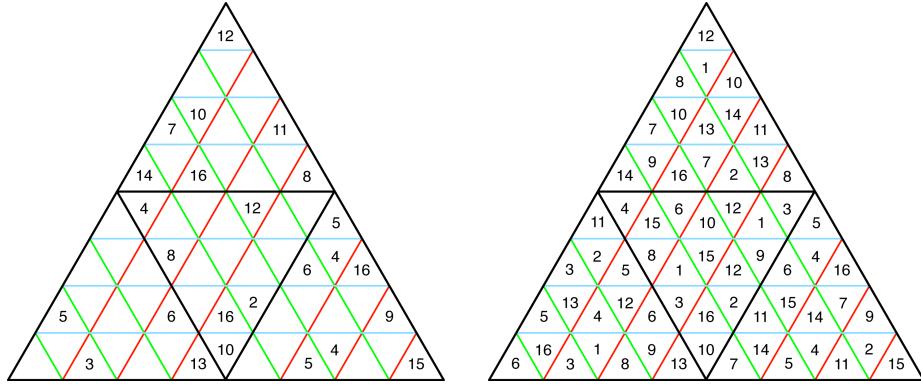


Figure 16. Latin Tetrahedron

polyhedron if every element of a particular polyhedral group transforms it into another Latin board [42]. This is the case with the puzzle solutions that follow.

Once folded along the inner black lines and glued along the outer ones, each of the pictures that follows becomes a polyhedral board with points and asterisms on its surface, each asterism being delimited by two lines of the same color. The corresponding puzzles are critical (see Def. 3.13), rated *difficult* (see Sect. 5.4.2) and have a Latin polyhedron as a solution.

Latin Tetrahedron Puzzle (2012). Fig. 16 shows a tetrahedron with twenty-four closed stripes of triangles on its surface. The challenge is to complete it so that every stripe holds all numbers from 1 to 16.

Free Latin Cube Puzzle (2014). Fig. 17 shows a cube with twelve closed stripes of squares on its surface. The challenge is to complete it so that every stripe holds all numbers from 1 to 16.

Latin Octahedron Puzzle (2014). Fig. 18 shows an octahedron with twelve closed stripes of triangles on its surface. The challenge is to complete it so that every stripe holds all numbers from 1 to 18.

Latin Icosahedron Puzzle (2014). Fig. 19 shows an icosahedron with twelve closed stripes of triangles on its surface. The challenge is to complete it so that every stripe holds all numbers from 1 to 20.

Latin Dodecahedron Puzzle (2014). Fig. 20 shows a dodecahedral board with points in the middle of every edge and asterisms indicated by lines of the same color. Every line here must hold all numbers from 1 to 6.

Figure 17. Latin Cube

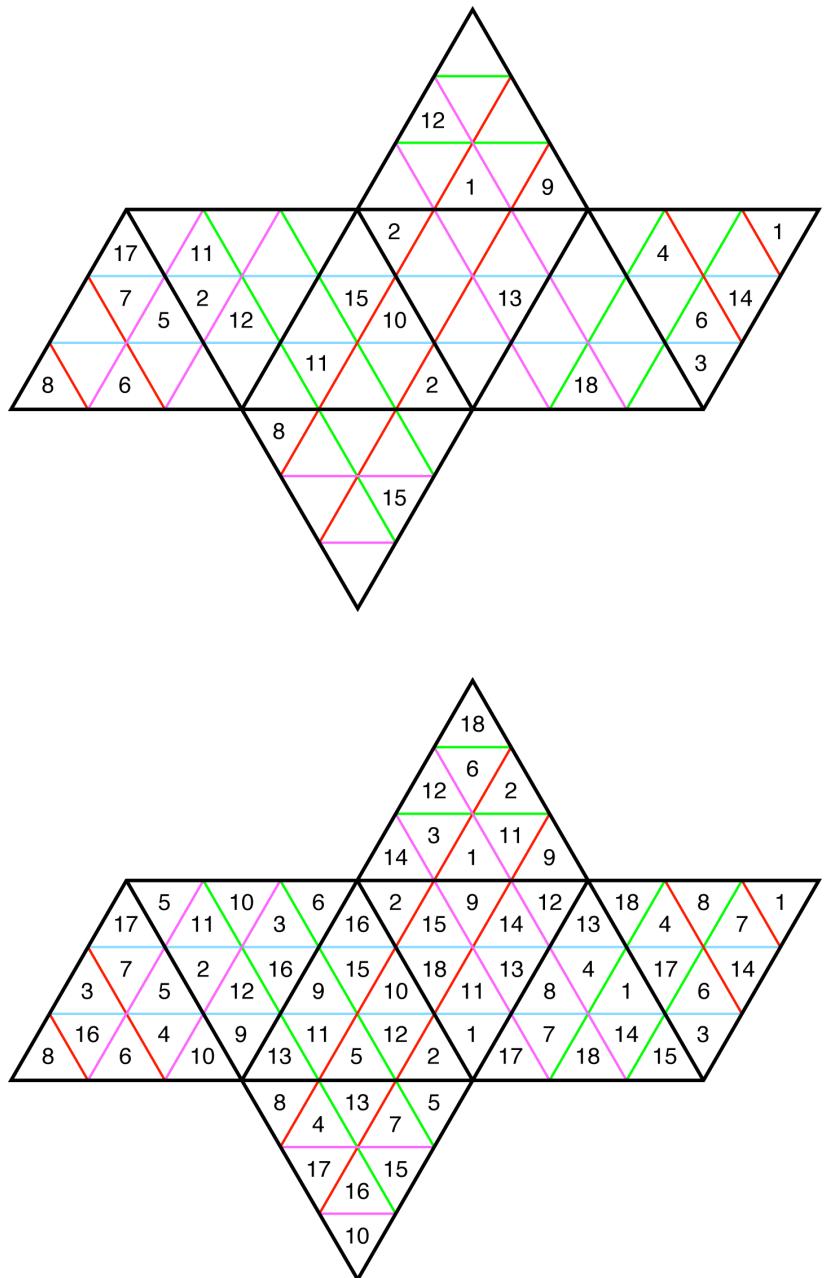


Figure 18. Latin Octahedron

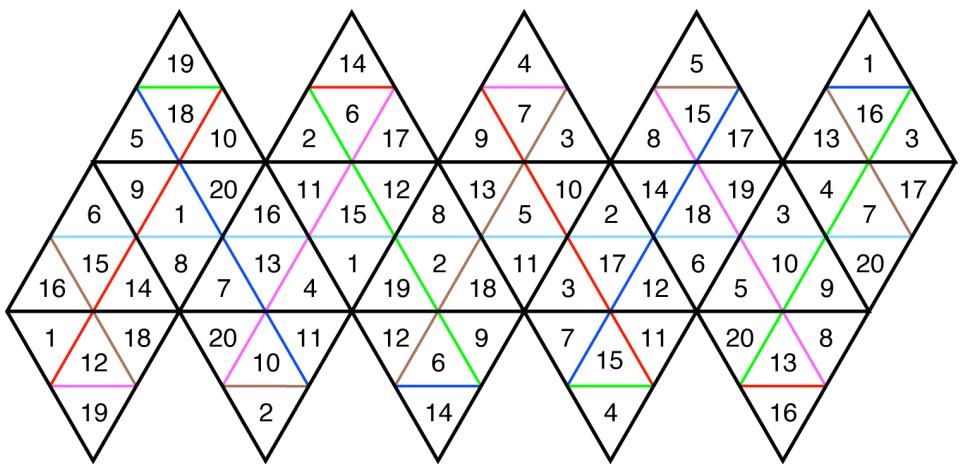
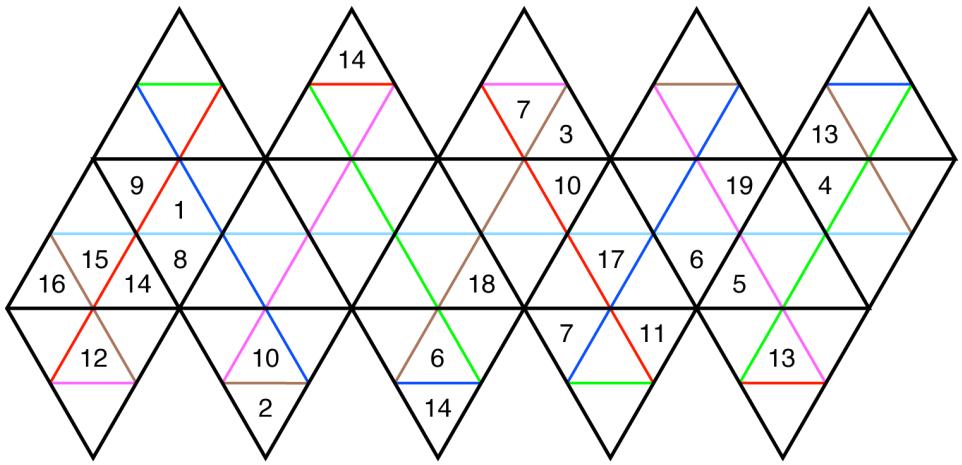


Figure 19. Latin Icosahedron

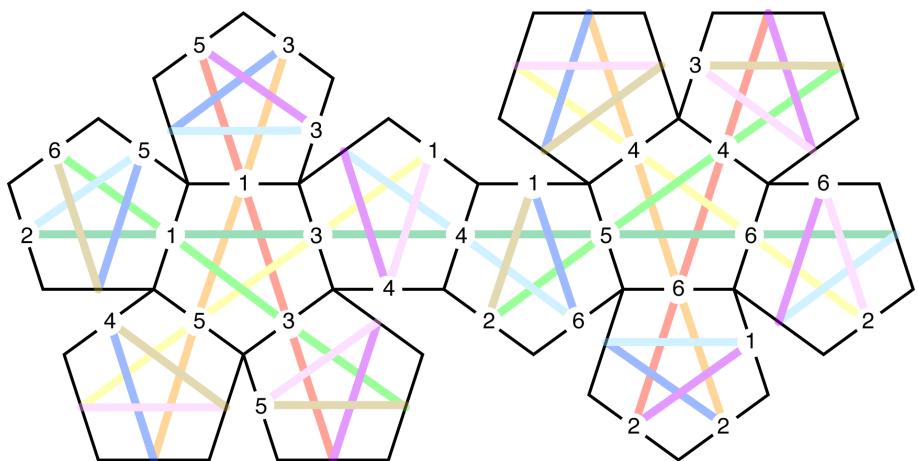
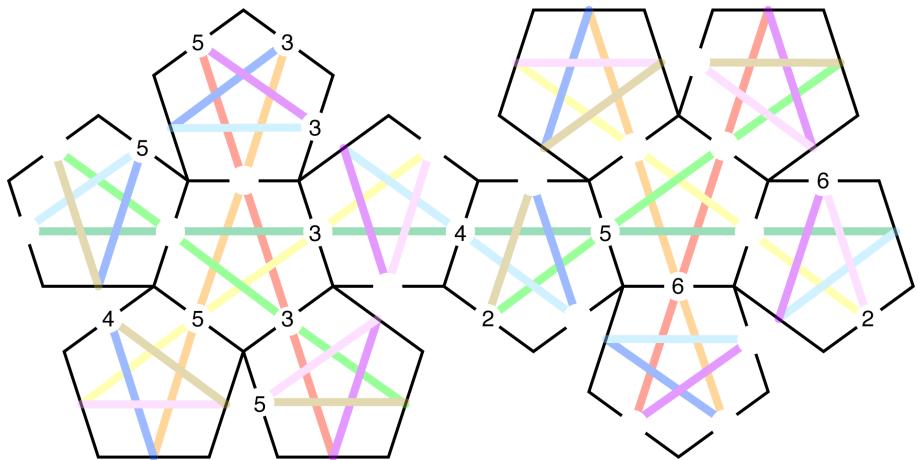


Figure 20. Latin Dodecahedron

3	2		1			3	1
3	2	1	2		3		1
	2	3		1		2	2
			1	2			2
		3		3		3	2
1	1	2	3	2	3		3
		1	3	1		2	2
1	1		2		2	1	2
2		2	2	3			1

3	2	1	1	2	2	3	3	1
3	2	1	2	1	3	2	3	1
1	2	3	3	1	3	2	2	1
3	3	2	1	2	1	3	1	2
2	1	3	1	3	2	3	1	2
1	1	2	3	2	3	1	3	2
2	3	1	3	1	1	2	2	3
1	1	3	2	3	2	1	2	3
2	3	2	2	3	1	1	1	3

Figure 21. Sudoku Ripeto

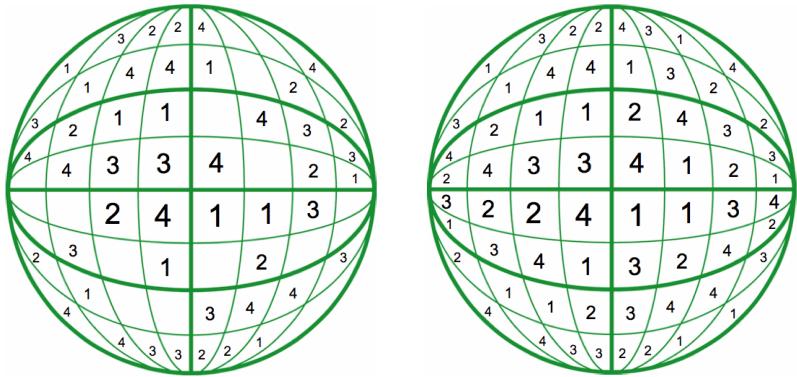


Figure 22. Orb Ripeto

4.4 Repeated Labels

This generalization allows asterisks to hold repeated labels (see Def. 3.4).

Sudoku Ripeto (2014). “Complete the board in Fig. 21 so that every row, column and highlighted region contains the labels {1, 1, 1, 2, 2, 3, 3, 3}”.

Orb Ripeto (2014). “Complete the board in Fig. 22 so that every parallel, meridian and highlighted region contains the labels {1, 1, 1, 2, 2, 3, 3, 3}”.

Quadoku Ripeto (2015). “Complete the board in Fig. 23 so that every row, column and highlighted region contains the labels {1, 1, 1, 2, 2, 3, 3, 3}”.

4.5 Inscriptions

Unless otherwise stated, in the examples that follow the challenge is “Complete the board in the figure so that every row, column and highlighted region contains the same letters as the completed row”. The inscription on the first puzzles is the name of a relevant figure in the history of *Sudoku* (see Sect. 1).

Custom Focus (2014). See Fig. 24.

3	1	2	2	1	3			
1	2			1	3			3
3			3	1		2	1	
2	1	3		3		2	3	
2		1	3		2	3		1
2			3		1	1	2	
	2	1	1		3	2		
1	2	2	1		1	3	3	2
3	3		2					1

3	1	2	2	1	3	1	2	3
1	2	2	2	1	3	3	1	3
3	3	1	3	1	2	2	1	2
2	1	3	1	3	1	2	3	2
2	1	1	3	2	2	3	3	1
2	3	3	3	2	1	1	2	1
1	2	1	1	3	3	2	2	3
1	2	2	1	3	1	3	3	2
3	3	3	2	2	2	1	1	1

Figure 23. Quadoku Ripeto

a	G	n	d	H	o	w	a	r
w		H	o	r	a	n	s	d
				r				r
						H	G	
			r	s	n	d	o	G
			d	G	r	H	r	a

a	r	G	n	d	H	o	w	a	r
w	a	H	o	G	r	a	n	s	d
									r
									G

Figure 24. Custom Focus

The figure consists of two separate 9x9 grids. The left grid has several letter constraints: 'B' at (1,2), 'M' at (1,3), 'e' at (1,4), 'y' at (1,5), 'n' at (1,6), 'i' at (1,7), 'l' at (1,8), 'e' at (1,9); 'n' at (2,1), 'M' at (2,2), 'B' at (2,3), 'y' at (2,4), 'e' at (2,5), 'i' at (2,6), 'n' at (2,7), 'M' at (2,8), 'l' at (2,9); 'y' at (3,1), 'e' at (3,2), 'n' at (3,3), 'i' at (3,4), 'l' at (3,5), 'e' at (3,6), 'i' at (3,7), 'n' at (3,8), 'e' at (3,9); 'I' at (4,1), 'e' at (4,2), 'i' at (4,3), 'n' at (4,4), 'M' at (4,5), 'n' at (4,6), 'e' at (4,7), 'I' at (4,8), 'e' at (4,9); 'n' at (5,1), 'l' at (5,2), 'e' at (5,3), 'i' at (5,4), 'e' at (5,5), 'i' at (5,6), 'e' at (5,7), 'l' at (5,8), 'e' at (5,9). The right grid has similar constraints: 'B' at (1,2), 'M' at (1,3), 'e' at (1,4), 'y' at (1,5), 'n' at (1,6), 'i' at (1,7), 'e' at (1,8), 'l' at (1,9); 'i' at (2,1), 'y' at (2,2), 'n' at (2,3), 'e' at (2,4), 'l' at (2,5), 'B' at (2,6), 'M' at (2,7), 'e' at (2,8), 'e' at (2,9); 'e' at (3,1), 'e' at (3,2), 'l' at (3,3), 'M' at (3,4), 'i' at (3,5), 'B' at (3,6), 'n' at (3,7), 'y' at (3,8), 'e' at (3,9); 'B' at (4,1), 'i' at (4,2), 'e' at (4,3), 'M' at (4,4), 'e' at (4,5), 'l' at (4,6), 'i' at (4,7), 'y' at (4,8), 'B' at (4,9); 'y' at (5,1), 'e' at (5,2), 'B' at (5,3), 'i' at (5,4), 'e' at (5,5), 'M' at (5,6), 'l' at (5,7), 'i' at (5,8), 'n' at (5,9); 'l' at (6,1), 'i' at (6,2), 'e' at (6,3), 'B' at (6,4), 'y' at (6,5), 'M' at (6,6), 'n' at (6,7), 'e' at (6,8), 'I' at (6,9); 'e' at (7,1), 'B' at (7,2), 'n' at (7,3), 'M' at (7,4), 'e' at (7,5), 'l' at (7,6), 'i' at (7,7), 'y' at (7,8), 'B' at (7,9); 'M' at (8,1), 'n' at (8,2), 'y' at (8,3), 'l' at (8,4), 'e' at (8,5), 'i' at (8,6), 'e' at (8,7), 'l' at (8,8), 'B' at (8,9).

Figure 25. Custom Sudoku

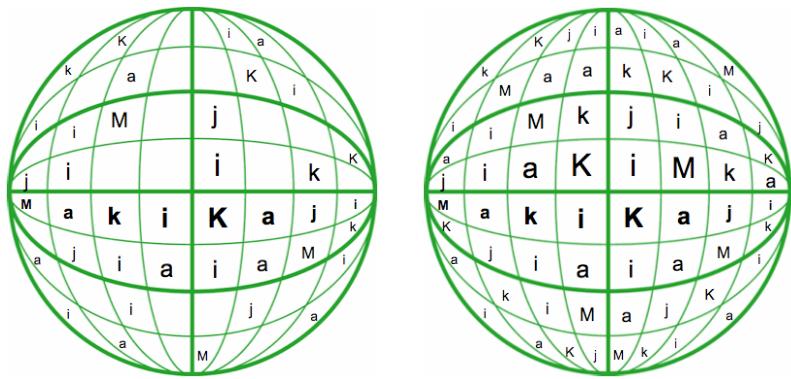


Figure 26. Custom Orb

Custom Sudoku (2014). See Fig. 25.

Custom Orb (2014). “Complete the board in Fig. 26 so that every parallel, meridian and highlighted region has the same letters as the completed parallel”.

Custom Steps (2014). See Fig. 27.

Custom Sky (2015). See Fig. 28.

Custom Win (2014). “Complete the board in Fig. 29 so that every row, column and highlighted region contains the letters in bold typeface”. In this example the inscription is not confined in an asterism.

Custom Sudoku (2014). “Complete the board in Fig. 30 so that every row, column and highlighted region contains the letters in bold typeface”. This is a critical Latin puzzle (see Def. 3.13) with inscription: the removal of any label not in the central sub-square destroys the puzzle condition. This particular puzzle is rated *difficult* by the rating method described in Sect. 5.4.2.

n	G	o	l		d	y		e
e		a			G	I	d	
y	d				u	n	o	
				y	n	W	a	o
G		u	d		a		W	y
W	a	y	n	e	G	o	u	l
l	o	d			a	n	y	
o	y				W	G	u	l
					o			
		G				o	y	a

n	G	o	l	u	d	y	W	a	e
e	W	a	o	n	y	G	I	d	u
y	d	G	a	W	I	u	e	n	o
u	e	l	y	d	n	W	a	o	G
G	l	u	d	o	e	a	n	W	y
W	a	y	n	e	G	o	u	l	d
l	o	d	u	G	a	n	y	e	W
o	y				W	d	G	u	l
a	u	e	W	y	o	l	d	G	n
d	n	W	G	l	u	e	o	y	a

Figure 27. Custom Steps

i	S				I	I	W		
h	W				i	r	t	z	S
I		z	l	h				i	
W				S	z	l			
t			o	W				I	
W	i	I	I	S	h	o	r	t	z
S	z				I	I			
I	o		i	z	S				
z	I	l	t	S	h	o			
r									

i	S	t	r	z	I	I	W	o	h
o	h	W	I	I	i	r	t	z	S
I	t	S	z	l	r	h	o	W	i
h	W	i	o	r	I	S	z	I	t
t	z	r	h	o	W	I	i	S	I
W	i	I	I	S	h	o	r	t	z
S	r	z	t	h	o	I	I	I	W
I	l	l	o	W	i	t	z	S	h
z	I	I	l	t	S	W	h	r	o
r	o	h	S	W	z	t	I	I	I

Figure 28. Custom Sky

S	E			R					
P		E	N	O		S	G		
N	G	R			O	A	P		
O		S	I	N					
S		G	A	P		E	O		
		G	O	R	E				
R	I	E		S	O	N	G		
O	P	A		G	R	I	N		
E		R	N		G	S	P		

S	E	A	P	G	R	N	O	I	
P	A	E	N	O	I	S	G	R	
N	G	R	I	S	O	A	P	E	
G	O	P	S	I	N	E	R	A	
R	S	N	G	A	P	I	E	O	
I	N	G	O	R	E	P	A	S	
A	R	I	E	P	S	O	N	G	
O	P	S	A	E	G	R	I	N	
E	I	O	R	N	A	G	S	P	

Figure 29. Custom Win

K	S	D	O
O	N	U O	U
O	U	K	O
	U	N E O	
O		S U D	E
E		O K U N	D
U	S	K	D U
O	U		S O
D		U	

K	U	S	E	D	O	U	O	N
O	N	D	U	O	K	U	E	S
E	O	U	U	S	N	K	D	O
D	K	U	N	E	O	S	O	U
U	O	N	S	U	D	O	K	E
S	E	O	O	K	U	N	U	D
U	S	O	K	N	E	D	U	O
O	U	E	D	U	S	O	N	K
N	D	K	O	O	U	E	S	U

Figure 30. Critical Custom Sudoku

Term	Acronym
partial labeled board (Sect. 3.3.3)	PB
partial Latin board (Sect. 3.7)	PLB
completable partial Latin board (Sect. 3.3.5)	CPLB
Latin puzzle (Sect. 3.9)	LP
Latin board (Sect. 3.4)	LB

Table 5.1 Terms related to Latin boards and corresponding acronyms

5 Finding Latin Boards and puzzles

In this section we will use the shorthands indicated in Table 5.1. References to the term definitions appear in parentheses.

5.1 Algorithms

Algorithm `list_LBs_from_PB()` in Alg. 1 takes a PB and a positive integer $N \in \mathbb{N} \cup \{\infty\}$ as input. Given sufficient memory and time it returns a list with:

1. all LBs solutions to PB when $N = \infty$
2. N solutions when $0 < N < \infty$ and PB has a total number of solutions $\geq N$
3. a number of solutions $\leq N$ when $0 < N < \infty$ and PB has a total number of solutions $\leq N$

If the output list is empty in the first case, then either the input PB is a PLB but not a CPLB, or it is not a PLB at all. If it contains a single solution, the input is both a CPLB and an LP.

Suppose we have another algorithm that, using `list_LBs_from_PB()` with $N = \infty$, takes an LB as input and produces a list of LPs that have that LB as the solution. Alg. 2, that recursively looks for LPs of ever increasing difficulty, is an example.

Then, given a particular PB, and providing it is a PLB (a k -uniform empty board and a k -multiset for example), Alg. 3 will find N or less PBs and their corresponding LPs⁷.

⁷The algorithm could also stop the search when a quantity given as parameter is reached.

```

algorithm list_LBs_from_PB(PB, N):
    if PB is not a PLB:
        return empty_list
    elif PB is an LB:
        return [PB]
    else:
        find LBs for PB as per N and add them to list_LBs
        return list_LBs

```

Algorithm 1. Latin boards from a partial labeled board

```

algorithm list_LPfrom_LB(LB):
    list_LP = empty_list
    algorithm recur(PLB):
        for label in PLB:
            remove label from PLB
            l_LBs = list_LBs_from_PB(PLB, N = ∞)
            if l_LBs contains a single element:
                add a copy of PLB to list_LP
                recur(PLB)
            else:
                restore label in PLB
        recur(LB)
    return list_LP

```

Algorithm 2. Latin puzzles from a Latin board

```

algorithm list_LPfrom_PB(PB, N):
    list_LBs = list_LBs_from_PB(PB, N)
    list_all_LP = empty_list
    for LB in list_LBs:
        list_LP = list_LPfrom_LB(LB)
        list_all_LP += list_LP
    return list_all_LP

```

Algorithm 3. Latin puzzles from a partial labeled board

Algorithm `list_LBs_from_PB()` in Alg. 1 assists us in three ways:

1. to directly find some or all LBs for a PB
2. in particular, with $N = \infty$, to solve an LP given as input, as this is by definition a PB that is both a PLB and a CPLB with a single solution
3. with $N = \infty$ and embedded in another algorithm, to find LPs for an LB

In the first case we usually do not need all solutions. Moreover, the number of solutions here can be exceedingly large, so we can set N to a convenient finite value⁸. In the second case, and in contrast with $N = 1$, setting $N = \infty$ additionally allows us to check that the input is indeed an LP. In the third case the algorithm must run unbridled and give all solutions.

5.2 Latin Boards from Partial Labeled Boards

In this section we discuss ways to implement `list_LBs_from_PB()` in Alg. 1. As seen, finding LBs for a particular PB is intimately related to finding LPs. While checking whether a PB is a PLB is easy –and takes linear time in the number of cells– verifying whether a PLB is also a CPLB is hard in general. For one, verifying whether a generic partial Latin square is completable is an NP-complete decision problem [9], so it is unknown whether there is a general, polynomial time algorithm for the worst cases able to answer this last question.

Since partial Latin squares are a subset of PLBs, the problem of verifying PLBs’ completable is at least as hard. Consequently, a task equally difficult is deciding whether or not a PLB is an LP, since LPs are a subset of CPLBs.

Definition 5.1. The label that an empty cell in a CPLB has in a solution is a *solution label* for the cell.

Definition 5.2. Let x be an empty cell in a CPLB with multiset of labels L . Any set $D(x) \subseteq \{l, l \in L\}$ is a *domain* for x if and only if it includes all its solution labels.

The set $\{l, l \in L\}$ is obviously a domain for any empty cell, called the *initial domain* for the cell.

Definition 5.3. Let B be a CPLB with n empty cells x_1, \dots, x_n . Then the Cartesian product $\prod_{i=1}^n D(x_i)$, where $D(x_i)$ is a domain for x_i , is a *solution space* of the board.

The tuples (l_1, \dots, l_n) , with $l_i \in D(x_i)$, are called *label vectors*, whereas every $D = \{D(x_i)\}$, the set of all empty cells’ domains, is a *board domain*.

Definition 5.4. Let B be a CPLB with a set X of empty cells. Let D and D' be two board domains. We say that $D' \subseteq D$ if and only if $D'(x) \subseteq D(x), \forall x \in X$.

We say that D' is *more advanced* or *stronger* than D . Or equivalently, that D is *less advanced* or *weaker* than D' . This binary relation among board domains is clearly a partial order over the set of board domains.

⁸Other alternative or complementary option is to limit the algorithm’s running time.

5.2.1 Algorithm Specifications

In absence of general efficient methods to find LBs, the guiding principles to design `list_LBs_from_PB()` could be:

- accept a parameter for the number of solutions to be obtained
- verify first that the input is a PLB
- assume that this PLB is a CPLB
- perform *attrition*: find more advanced board domains for the supposed CPLB until solutions are found, either by reducing to singletons the domains of empty cells or by discarding entire label vectors. If the domain for an empty cell becomes empty in the process, then the initial assumption for the PLB being a CPLB was wrong. The algorithm should return an empty list as soon as this condition is detected
- optimize: perform attrition in the most efficient way possible, favoring at each step polynomial-time attrition over exponential attrition, and less complex polynomial attrition over more complex polynomial attrition
- use relevant heuristics to save space, time and complexity

When the input to `list_LBs_from_PB()` is an LP, this approach has the additional advantage of mimicking the way a human player would solve the puzzle, an important feature if we want to have compact written proofs of the solving process⁹.

5.2.2 Attrition Theorems

The reductions performed by `list_LBs_from_PB()` on board domains must be safe, in the sense that they must result in other domains.

Definition 5.5. Let B be a CPLB and D a board domain for it. An *attrition theorem* T is one that verifies $T(D) \subseteq D$.

Attrition theorems never map board domains to less advanced ones.

5.2.3 Attrition Algorithms

An *attrition algorithm* is a sequence of operations that iteratively searches relevant patterns on the board, checks if each one complies with the conditions to apply an attrition theorem, then effectively applies it.

5.2.4 Types of Attrition

The PB in Fig. 31 (left) has the same asterisms as *Sudoku*'s (nine rows, nine columns and nine 3×3 sub-squares) and multiset $\{2, 2, 3, 3, 4, 4, 4, 4\}$. The board is also a PLB since label counts in every asterism comply with the multiset. But it is unclear whether it has solutions¹⁰. We study next different attrition methods to find the possible solutions.

⁹As we will see later, this is also very useful to both puzzle authors and players.

¹⁰In fact it has only the one shown on its right (this is an instance of the *Sudoku Ripeto Puzzle* –see Sect. 4.4).

4	3			3	4			
		3			3	4		
	4		4	3	4	2		
	3	3		4	3	4		
			4			2		
	3	2		2	4	4		
2			4	3		4		3
3		4	4	3	4		3	2
4	4	3	2	3	3	4	2	1

4	4	3	2	2	4	3	3	4
3	4	2	3	4	4	2	3	4
3	2	4	3	4	3	4	2	4
2	3	3	4	2	4	3	4	4
4	4	4	4	3	3	2	2	3
4	3	2	3	4	2	4	4	3
2	3	4	4	3	2	4	4	3
3	2	4	4	3	4	4	3	2
4	4	3	2	4	3	3	4	2

Figure 31. Partial Latin board and a solution

5.2.5 Exponential Attrition by Exhaustive Search

With 37 empty cells, each with initial domain $\{2, 3, 4\}$, the initial solution space for the board in Fig. 31 (left) has 3^{37} vectors. After the initial checks, one way to find solutions is to form all possible vectors, write each one on the board and check compliance, including in the output list all solutions found along the way. The trivial attrition theorem behind this algorithm guarantees correctness and termination, but the search performed is exponential in the number of empty cells. Exhaustive search is the most clear example of exponential attrition.

5.2.6 Exponential Attrition by Backtracking

An alternative, although still exponential in nature, is for `list_LBs_from_PB()` to implement *backtracking* [56], a technique that systematically checks one, two, three, four, etc. components of label vectors instead of all components.

We start with the usual checks, then choose an empty cell's domain and a label from it¹¹, write it on the cell, then check if the resulting board is still a PLB. If it is, we inspect the board for completion, in which case we add it to the output list, as the board is now a solution by definition¹².

If the board is indeed a PLB but still contains empty cells, we choose a label from another cell domain, write it and perform the same checks. We proceed in this way until the board ceases to be a PLB, at which point we remove the just written label to restore the corresponding empty cell, then choose a different one from its domain and resume¹³.

When all labels in a cell domain have been tried, we backtrack to the precedent cell –if the current one wasn't the first– then, provided there are any left, choose a label that was not chosen before and resume. If after the full traversal we have rejected all labels in the domain we started with, we can safely conclude that the board is not a CPLB and return the empty list.

¹¹Heuristics may be used to pick domains (*branching strategy*) and labels (*instantiation strategy*) to optimize results. A common technique for branching is to choose a domain among those with the least number of labels.

¹²When easy ways to derive Latin boards from a given one are available –for example isotopic or conjugacy transformations in Latin squares, or geometric transformations in Latin polytopes– the algorithm could harvest instead canonical representatives of the corresponding equivalence classes [42, 46].

¹³This is known as *state restoration* [46].

A useful time saving device is *no-good recording* [46], a technique that remembers combinations of values for variables found not to lead to solutions, in order to avoid exploring again their associated search subtree¹⁴. In this case backtracking may not go back to the current variable domain but jump (*backjump*) to a different one [46].

Backtracking guarantees correct termination, so when instructed to find all solutions with $N = \infty$, `list_LBs_from_PB()` will return a list with either:

1. no solutions, if the input PB is not a CPLB or
2. all solutions, if the input is a CPLB with several solutions or
3. just one solution, if the input is a CPLB with just one solution¹⁵

5.2.7 Polynomial Attrition

As the maximum count for labels 2 and 3 has been reached in the bottom right sub-square of Fig. 31 (left), we can safely reduce the domain of each empty cell there from $\{2, 3, 4\}$ to $\{4\}$. Provided that the input PB is indeed a CPLB, 4 is now a solution label for both cells, and as such it can be used to trigger further reductions in other rows and columns the cells belong to¹⁶. An attrition algorithm that visits all empty cells and applies the theorem will be polynomial, as it will take quadratic time in the number of cells in the board.

Additional attrition theorems with associated polynomial algorithms may be considered¹⁷. Properly scheduled in `list_LBs_from_PB()` to maximize effects, algorithms could parse asterisks one at a time or tuples of them to profit from conditions arising at their intersections.

Again, as every reduction is backed up by theorems, any cell domain becoming empty in the process indicates that the initial assumption for the input PB being a CPLB was not right, so `list_LBs_from_PB()` should return an empty list in this case as specified.

As before, every time a singleton domain is produced, the board should be checked for completion. If that is the case, the solution –necessarily unique in this case– is returned as the only element in the output list.

A third outcome for `list_LBs_from_PB()` is to stop at a PLB that is not a solution. This will happen when the implemented attrition algorithms cannot fully reduce the board domain.

Polynomial attrition improves then exponential attrition at the expense of having to:

- prove attrition theorems
- implement attrition algorithms
- schedule the attrition algorithms
- put up in some cases with non-conclusive outputs

¹⁴Again, when easy ways to derive Latin boards from a given one are available, the algorithm could avoid exploring the corresponding symmetrical no-goods [42, 46].

¹⁵I.e.: an LP.

¹⁶The attrition theorem behind this reduction is immediate.

¹⁷Polynomial attrition algorithms for *Sudoku* may be found in [4].

5.2.8 Optimized Attrition

A way to finish the job in all cases is for `list_LBs_from_PB()` to perform exponential attrition just after polynomial attrition. An even better option is to *interleave* both techniques as we describe next.

We start by applying polynomial attrition after the initial checks. If this does not solve the board, we perform backtracking as described in Sect. 5.2.6.

Any time a label is written by backtracking, and provided the resulting board is indeed a PLB, we apply polynomial attrition again.

Every time a label is written by polynomial attrition, and in contrast to pure polynomial attrition, we have to check now the PLB condition. This is necessary because, once backtracking has been used, when a label is written by the ensuing polynomial attrition the resulting sets of label candidates for affected empty cells are no longer guaranteed to contain all solution labels, i.e.: the sets may have ceased to be cell domains. This has another consequence: when the board is no longer a PLB and we have to backtrack, we have to undo all changes made by polynomial algorithms to these *pseudo-domains* since the last label was chosen. This is known as *state restoration* [46].

Along the way, we check as always for board completion, adding to the output list every solution that may come up. As before, `list_LBs_from_PB()` is guaranteed to end with all the solutions if $N = \infty$, or part thereof otherwise.

5.3 Latin puzzles from Latin Boards

Once `list_LBs_from_PB()` has produced some LBs, we can find LPs for each of them with Alg. 2. We start by removing one label from the input LB, then check whether the resulting board –guaranteed now to be both a PLB and a CPLB– is an LP. If it is, we can try to find a more difficult puzzle by removing another label from the just found puzzle. If the first removal didn’t produce an LP we restore the label, remove a different one, then check again and so on. If the argument to the inner recursion sub-algorithm in Alg. 2 is passed as reference, just a single data structure will be needed for the different board domains and pseudo-domains generated in the process.

5.3.1 Single Pass puzzles

An interesting faster option to find LPs is to do so at the same time LBs are found. This may save considerable time at the expense of complicating `list_LBs_from_PB()`. We achieve this by both *instrumenting* the attrition process and *remembering* the intermediate PLBs produced.

Given an empty PB as input, we start by finding LBs as before. Then, any time a label is written –as a consequence of either polynomial or exponential attrition– the operation is uniquely tagged with a serial number or a time stamp. The tags and their corresponding PLBs are kept in a data structure created in advance that supports push and pop operations, that are carried out in sync with the attrition process.

Every PLB and LB found in the process will then have a unique *tag sequence* that identifies all its antecessor PLBs. If all possibilities for these boards have been explored, every tag in an LB’s tag sequence that does not show up in any other LB’s sequence corresponds necessarily to a puzzle. This puzzle is the stored PLB having a tagging sequence that ends with that very tag.

5.4 Fair Latin puzzles

A puzzle designer would like to guarantee players that her puzzles can be solved by “reasoning”, “logic”, “without trial and error” or “with a pen, not a pencil”. In our terms, she would like to produce puzzles solvable just by polynomial attrition, without resorting to exponential methods. How can she go about it?

Definition 5.6. Given a Latin puzzle, a *polynomial attrition sequence* (PAS) is a sequence of applications of polynomial attrition algorithms (see Def. 5.2.3). A *full polynomial attrition sequence* (FPAS) is one that solves the puzzle.

Let’s assume our designer has solved a puzzle with a particular FPAS. Now, when someone else tries to solve the puzzle he may well start using a different strategy. One involving perhaps different polynomial algorithms, or the designer’s but in a different order, i.e: he may start using a PAS that is not a prefix of the designer’s FPAS. The question now arises: will our player be able to solve the puzzle polynomially? Or equivalently, can any PAS be part of a FPAS in a Latin puzzle?

Definition 5.7. A Latin puzzle is *fair* if and only if any PAS is part of an FPAS.

Fair Latin puzzles are then what our designer is after. So an equivalent question would be: under which condition is a Latin puzzle fair? To answer, let’s define first a particular class of attrition theorems: those that achieve at least the same reduction in a domain than in a less advanced one (see Def. 5.2):

Definition 5.8. Let D, D' be two board domains for a CPLB verifying $D' \subseteq D$. An attrition theorem T is *monotonic* if and only if $T(D') \subseteq T(D)$.

We also say that an attrition algorithm (see Def. 5.2.3) is *monotonic* if it implements a monotonic attrition theorem, and that a PAS is *monotonic* if all their intervening algorithms are monotonic.

Theorem 5.9. *A Latin puzzle is fair if it admits a monotonic FPAS.*

Proof. Let S' and S be respectively a PAS –not necessarily monotonic– and a monotonic FPAS for a given Latin puzzle. Let $D(x_i), i = 1 \dots n$, be the initial domains for the n empty cells in the puzzle. After S' has been applied, we necessarily have $D'(x_i) \subseteq D(x_i), i = 1 \dots n$, where $D'(x_i)$ is the new domain for cell i . We apply now the first attrition algorithm in S . Although the cell domains intervening in its premises may have been reduced by S' , and since this algorithm is monotonic, we are certain to get a reduction at least as strong as the one achieved if S' had not been applied. This in turns enables successive, similar application of the rest of monotonic attrition algorithms in S until the puzzle is solved. \square

Our designer solves then her problem by making sure that all algorithms intervening in her own FPAS are monotonic¹⁸.

¹⁸For more information to players, the guarantee given by the puzzle designer could mention the set –not the sequence!– of attrition theorems used in her FPAS.

5.4.1 Finding Fair Latin puzzles

A way of finding fair Latin puzzles is to instrument `list_LBs_from_PB()` as described in Sect. 5.3.1 and make it perform optimized attrition (see Sect. 5.2.8). This time though, the data structure in the instrumentation will support push and pop operations on attrition algorithm names instead.

Now, any time a label is written we push the name of either the polynomial algorithm applied, or “backtrack” if this was the cause of the writing. The structure is kept in sync with the attrition process as before, so name pops will also occur when labels are removed from the board.

At the end, every puzzle found will have an associated sequence of logical steps towards its solution, i.e: a *proof*. The fair puzzles among them are simply those that do not include “backtrack” in their proofs.

5.4.2 Rating Fair Latin puzzles

Another advantage of puzzle proofs is that we can derive from them objective measures of difficulty, for example as a function of the complexity of the attrition algorithms present in the proof and the number of times they have been applied.

This information may be used also to design a subjective scale of difficulty, much more useful to players. To this end, we select first a sample of players, give them samples of fair puzzles to solve and ask for their perceived difficulty¹⁹. From these objective and subjective measurements, a subjective metric for difficulty can be statistically designed. Related methods to rate *Sudoku* puzzles may be found in [24, 29, 44]²⁰.

5.5 Constraint Programming

It turns out that the problem of finding Latin boards (LBs) from a partial labeled board (PB) is a special case of a much more general one [3, 46]:

Definition 5.10. A *Constraint Satisfaction Problem* (CSP) P is a triple $P = (X, D, C)$ where X is an n -tuple of variables $X = (x_1, x_2, \dots, x_n)$, D is a corresponding n -tuple of *domains* $D = (D(x_1), D(x_2), \dots, D(x_n))$, with x_i ranging over $D(x_i)$, and C is a t -tuple of *constraints* $C = (C_1, C_2, \dots, C_t)$, where a constraint C_j is a pair (R_{S_j}, S_j) in which R_{S_j} is a relation on the variables in $S_i = \text{scope}(C_i)$.

A constraint is then a subset of the Cartesian product of its variables. If we identify the tuple of empty cells in a partial labeled board with X , a board domain with D , the cell domains with $D(x_i)$, and the need for every asterism in the board to hold all labels in the multiset with a constraint, then the problem to find solution Latin boards for a partial labeled board can be formulated as a Constraint Satisfaction Problem (CSP).

¹⁹In the scale *very easy*, *easy*, *medium*, *difficult*, *very difficult* for example

²⁰For Latin Puzzles akin to *Sudoku*, we may also collect published *Sudoku* puzzles, solve them with instrumented attrition, relate the proofs to their featured difficulties, and use the results as an additional ingredient in the subjective scale of difficulty.

5.5.1 Constraint Programming Systems (CPS)

This new way of putting things does spare many efforts when specifying and implementing algorithm `list_LBs_from_PB()` (see Alg. 1), as it also turns out that many of the services it should provide are readily available in *Constraint Programming Systems*²¹ (CPS) [3, 46].

The main feature of a CPS is that the user simply *declares* the constraints in some language –usually a subset of first-order logic– then lets the system run to find the answers. This programming paradigm differs notably from imperative ones, as the focus here is not the solving algorithm, but rather the specification of the properties of the solution.

A CPS can be used to tackle a wide range of combinatorial optimization problems, for example that of completing partial Latin squares [22, 51]. A suitable CPS can also be used to solve *Sudoku* puzzles, both alone [12, 28, 38, 54] or in conjunction with other techniques [14, 32, 55, 57].

In a CPS, attrition theorems (see Def. 5.2.2) are called *rules* or *proof rules*. Implemented polynomial attrition algorithms (see Def. 5.2.3) are called *propagators*; their application, *constraint propagation* or *inference*, and exponential attrition (see Sect. 5.2.4), *search*. When scheduled by a *propagation engine*, a CPS performs propagation by maintaining *network consistency*, which means carrying out safe attrition in all its forms. CPSs come with propagators that enforce *local consistency* for individual variables (*node consistency*), pairs of variables (*arc consistency*) and tuples of variables (*path consistency*) with the aim of obtaining *global network consistency* [46].

CPSs are implemented either as autonomous systems or libraries [46], and frequently allow users to plug in their own propagators and search strategies. In fact, the author used a custom CPS to generate the example Latin boards and puzzles featured in Sect. 4.

5.6 Alternatives to Constraint Programming

A CPS equipped with suitable propagators is a good tool to find and solve Latin puzzles, but there are alternatives. We could in principle consider other combinatorial optimization techniques known to solve partial Latin squares and *Sudoku* puzzles, and adapt them to work with the multisets and arbitrary topologies proper to Latin puzzles. Some of these techniques are²²:

- propositional satisfiability inference (SAT) [31]
- Gröbner bases [5, 15, 18]
- rewriting logic [49]
- dancing links [27]
- another solution problem [60]
- graphs and hypergraphs coloring [59]
- ant colony optimization [48, 50]

²¹Or more specifically, as our variable domains are finite, in *Finite Domain CPSs* [46].

²²For related techniques specific to combinatorial designs see Sect. *Computational Methods in Design Theory* in [10].

- simulated annealing [30]
- tabu search [21]
- genetic algorithms [25, 34, 36, 37]
- bee colony simulation [39]
- harmony search [20]
- flower pollination [1]
- particle swarm optimization [35]
- entropy measurement [23, 61]

Deterministic techniques like SAT, Gröbner bases or dancing links, which are able in principle to find all solutions to a partial labeled board²³, could be used to find Latin puzzles, while stochastic methods would be suitable to solve them.

6 Conclusions

The generalization of Latin squares to Latin boards proposed in [42] prompts new lines of research. In this paper we have suggested a corresponding generalization of Latin square puzzles, together with methods to obtain Latin puzzles and examples thereof. Future work could proceed along the following lines:

- new Latin boards and Puzzles, in addition to the ones featured in this paper and those in [42, 43]
- monotonic attrition theorems and propagators to solve partial Latin boards (see Sect. 5.2)
- techniques to solve partial Latin boards other than constraint programming (see Sect. 5.6)
- applications in scheduling, timetabling and resource allocation problems when solutions are Latin boards
- generalization of existing results for partial Latin squares and quasi-groups in Algebra:
 - notions of isotopism and conjugacy in partial Latin boards (for partial Latin squares see [16])
 - connection with other branches of discrete mathematics [6]
 - enumeration of particular Latin boards (for *Sudoku* see [17])
 - number of clues in minimal Latin puzzles, with and without inscription (for *Sudoku* see [33, 47])
 - NP-completeness of the problem of completing specific partial Latin boards, like partial Latin triangles, partial Latin hexagons, etc. (for partial Latin squares see [9])

²³We saw in Sect. 5.3 that this feature is critical to find Latin puzzles.

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The author is a telecommunication engineer from the Polytechnical University of Madrid. He has created several mathematical Puzzles like *Sudoku Ripeto* [43], *Custom Sudoku* [43], *Konseku* [40] and *Moshaiku* [41].