Group Characters and Algebra

Author(s): D. E. Littlewood and A. R. Richardson

Source: *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 1934, Vol. 233 (1934), pp. 99-141

Published by: Royal Society

Stable URL: https://www.jstor.org/stable/91293

II. *Group Characters and Algebra.*

By D. E. LITTLEWOOD *and* A. R. RICHARDSON, *University College, Swansea.*

Communicated by H. W. TURNBULL, *F.R.S.*

(Received March 7—Revised December 12, 1933—Read January 18, 1934.)

§ 1. INTRODUCTION.

It has been known for some time* that the elements of a matrix of degree n may be arranged in sets which correspond to cycles of the symmetric group of order $n\,!$, and that there are relations connecting *permanents* and *determinants*, *e.g.*,†

$$ {}^{+}\!\begin{pmatrix} \alpha\ \beta\ \gamma\ \delta \\ \alpha\ \beta\ \gamma\ \delta \end{pmatrix}^{\!+} - \Sigma\ {}^{+}\!\begin{pmatrix} \alpha\ \beta\ \gamma \\ \alpha\ \beta\ \gamma \end{pmatrix}^{\!+}\!\begin{pmatrix} \delta \\ \delta \end{pmatrix} + \Sigma\ {}^{+}\!\begin{pmatrix} \alpha\ \beta \\ \alpha\ \beta \end{pmatrix}^{\!+}\!\begin{pmatrix} \gamma\ \delta \\ \gamma\ \delta \end{pmatrix} - \Sigma\ {}^{+}\!\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}^{\!+}\!\begin{pmatrix} \beta\ \gamma\ \delta \\ \beta\ \gamma\ \delta \end{pmatrix} + \begin{pmatrix} \alpha\ \beta\ \gamma\ \delta \\ \alpha\ \beta\ \gamma\ \delta \end{pmatrix} = 0. \qquad (1) $$

Further, MACMAHON and BRIOSCHI have pointed out the close analogy which exists between the threefold algebra of the symmetric functions a_n, h_n and s_n, and the theory of determinants, permanents, and the cycles of substitutions of the symmetric group.

Here we trace the analogy to its source by fixing attention on the *characters* of the irreducible representations of the symmetric group of linear substitutions, as the centre of the whole theory. By this means divers theories of combinatory analysis and algebra are seen to be merely different aspects of the same theory. For the symmetric group of order $n\,!$ the characters are all integers, and we associate with each partition of n both a character of the group and a cycle of substitutions. When substitutions affecting the rows (or columns) are applied to the diagonal term of the matrix of degree n we notice that the determinant corresponds to the partition 1^n, whilst the permanent corresponds to the partition n. Our new functions, termed *immanants*‡ are defined to correspond to the other partitions of n, and so fill the gap between the permanent and determinant. When this is done it is seen that many relations, including (1), which connect permanent and determinant can be found, and that there is a theory of *immanants* which includes the theory of determinants as a very special case. From such a wide choice of subjects we select that of the use of immanants in the calculation of group characters themselves. Although the subject-matter discussed is thus restricted

* MACMAHON, 'J. Lond. Math. Soc.,' p. 273 (1922).

† MUIR, "Theory of Determinants" vol. 4, p. 459.

‡ A name suggested to us by Professor A. R. FORSYTH, F.R.S.

it becomes clear that most, if not all existing tables of symmetric functions in combinatory analysis may be replaced by the tables of characters. Tables of characters of the symmetric groups up to order 9 ! are to be found on p. 137 *et seq.*

§ 2. Immanants of a Matrix.

Let $P_S = a_{1e_1}a_{2e_2} \dots a_{ne_n}$ be the product formed from the elements of the matrix $[a_{st}]$, in which a_{st} is the number in the sth row and tth column, by operating on the second suffixes of the diagonal element $a_{11}\, a_{22} \dots a_{nn}$ by the substitution S of the symmetric group. Further, let $\chi^{(\lambda)}$ be the character of the symmetric group which corresponds to the partition* (λ) of n, namely,

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_\mu \, (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\mu).$$

The number

$$|a_{st}|^{(\lambda)} = \sum_{S} \chi^{(\lambda)}(S^{-1})\, P_S$$

is termed the *Immanant* of the matrix $[a_{st}]$ corresponding to the partition (λ).

The permanent and determinant are special cases of immanants, for there is a set of characters, corresponding to the partition $n = n$, that takes the value $+ 1$ for every operation, and a set of characters corresponding to the partition $n = 1 + 1 + 1 \dots + 1 \equiv 1^n$, which takes the value $+ 1$ for operations of the alternating group and $- 1$ for the other operations.

Hence

$$|a_{st}|^{(n)} = \overset{+}{|}a_{st}\overset{+}{|}$$

$$|a_{st}|^{(1^n)} = |a_{st}|.$$

Thus the three immanants of the matrix of order 9 are the permanent $|a_{st}|^{(3)}$, the determinant $|a_{st}|^{(1^3)}$ and

$$|a_{st}|^{(21)} = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}.$$

Matrices of order 16, 25 and 36 have respectively 5, 7 and 11 immanants.

It is important to notice that the determinant is the only immanant which is invariant for a general transformation of the matrix, but *an immanant is invariant if the matrix is transformed by any permutation matrix*, that is by a matrix in which all the elements in some P_S are unity and the other elements zero.

In symbols, if A is a permutation matrix

$$|B|^{(\lambda)} = |A^{-1}BA|^{(\lambda)}.$$

The effect of transformation by a permutation matrix is to replace each product P_S

* The association between the partition of n and the character is dealt with in § 3.

by the product $P_{T^{-1}ST}$, where T is the permutation corresponding to the permutation matrix.

Since

$$\chi^{(\lambda)}(T^{-1}ST) = \chi^{(\lambda)}(S),$$

the immanant is unchanged.

§ 3. The Calculation of the Characters of the Symmetric Groups.

The chief methods are due to Frobenius* and Burnside,† and are laborious. Frobenius' formula is quoted here for reference.

Let S be an operation of the symmetric group on n symbols, and let it contain α_1 cycles on one symbol, α_2 cycles on two symbols and so on. Hence

$$n = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots.$$

Then the numbers $\alpha_1, \alpha_2, \alpha_3, \ldots$ define the class ρ to which S belongs, and the coefficient of $\pm x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$ in the product

$$(x_1 + x_2 \ldots + x_n)^{\alpha_1} (x_1^2 + x_2^2 \ldots + x_n^2)^{\alpha_2} (x_1^3 \ldots + x_n^3)^{\alpha_3} \ldots \Delta(x_1, x_2, \ldots, x_n)$$

is the character of S in an irreducible representation defined by the sequence of numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

The sign taken is $+$ or $-$ according as the numbers $\lambda_1, \ldots, \lambda_n$ form a positive or a negative permutation of the natural descending order, and

$$\Delta(x_1, \ldots, x_n) = \prod_{r, s} (x_r - x_s) \qquad (r < s).$$

Frobenius has also deduced formulæ for certain characters in terms of the cycles of substitutions which give the characters of the symmetric groups of orders up to 8 !, and some but not all the characters for groups of order > 8 !

The tables in this paper have been calculated by simpler methods.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n symbols which are permuted by the operations of the symmetric group of order n ! The matrix of transformation of the α's for any substitution S, is a permutation matrix, which will be represented by A_S. This forms a (reducible) representation of the group as a set of matrices and the spur of A_S is a compound character of the group which is the sum of the character $\chi^{(1)}(S)$ which is unity for every operation of the group, and a simple character $\chi^{(2)}$. We have thus two sets of characters.

Now consider the $\binom{n}{2}$ expressions of the form $\alpha_1^2\alpha_2$. These are permuted amongst

* 'SitzBer. Preuss. Akad. Wiss. Berl.,' p. 516 (1900), pp. 303–315 (1901).

† "The Theory of Groups," 2nd Ed., 1911.

themselves by the operations of the group. The matrix of transformation B_S is a (reducible) representation of the group, and its spur is a compound character which includes, besides the two known characters, two other simple characters that need to be separated.

Consider first those substitutions which interchange only the two symbols α_1 and α_2. These form a symmetric group of order 2. Consider also the two-rowed principal minor of the matrix B_S corresponding to these two symbols. The determinant and permanent of this minor are known to be invariant under transformations by permutation matrices.

This implies that the matrix B_S is reducible by transformation into the direct sum of two matrices, one of which is the matrix of transformation upon expressions of the form $\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2$, and the other upon expressions of the form $\alpha_1^2\alpha_2 - \alpha_1\alpha_2^2$.

The corresponding characters are the sum of the permanents of principal two-rowed minors of A_S, and the sum of the corresponding determinants. These give us two sets of compound characters $\phi^{(3)}$ and $\phi^{(4)}$.

The number of times each includes the known characters $\chi^{(1)}$, $\chi^{(2)}$, may be found very easily from the orthogonal properties of the characters,* as will be shown in the subsequent example, and the remainder is in each case a simple character.

Similarly the matrix of transformation of expressions of the form $\alpha_1^3\alpha_2^2\alpha_3$ is reducible, and we consider the matrices of transformation of expressions of the forms of the three immanants of the matrix

$$\begin{bmatrix} \alpha_1^3, & \alpha_1^2, & \alpha_1 \\ \alpha_2^3, & \alpha_2^2, & \alpha_2 \\ \alpha_3^3, & \alpha_3^2, & \alpha_3 \end{bmatrix}.$$

The corresponding characters are the sums of the corresponding immanants of the principal three-rowed minors of the matrix A_S.

These three characters will be compound, including multiples of the first four characters already obtained. When these have been removed, the remaining characters will be simple.

In general, the sum of the immanants of the principal r-rowed minors of the matrix A_S, is a compound character of the group. If these compound characters are evaluated in succession, the simple characters may be obtained by making use of the orthogonal properties of the characters.

It should be noticed that, if the principal minor is a permutation matrix, its immanant may be read from the table of characters of the symmetric group of order r ! If it is not a permutation matrix, its immanant is zero.

Example.—As an example we find the complete set of characters for the symmetric group of order 5 !

* BURNSIDE, *Op. cit.*, p. 291; FROBENIUS, 'SitzBer. Preuss. Akad. Wiss. Berl.,' p. 985 (1896).

There is a class of the group corresponding to every partition of 5. The classes may be represented by 1^5, the identical operation; $1^3 2$, the class of operations with just one cycle of order 2; $1^2 3$, the operations containing one cycle of order 3; 14, one cycle of order 4; 5, one cycle of order 5; 12^2, two cycles of order 2; and lastly 23, two cycles of orders 2 and 3 respectively.

The numbers of operations in the seven classes

(1) $$1^5,\quad 1^3 2,\quad 1^2 3,\quad 14,\quad 5,\quad 12^2,\quad 23$$

are respectively

(2) $$1,\quad 10,\quad 20,\quad 30,\quad 24,\quad 15,\quad 20.$$

The first character $\chi^{(1)}$ is equal to unity for each class

(3) $$1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1.$$

The spur of A_S is equal to the number of cycles of order 1, and for the seven classes will be respectively

$$5,\quad 3,\quad 2,\quad 1,\quad 0,\quad 1,\quad 0.$$

This is a compound character $\phi^{(2)}$. Denoting the class by ρ and the number of operations in the class by h_ρ, then

$$\Sigma\, h_\rho \chi_\rho^{(1)} \phi_\rho^{(2)} = 5 + 30 + 40 + 30 + 15 = 120 = h,$$

the order of the symmetric group. Hence the compound character $\phi^{(2)}$ contains the simple character $\chi^{(1)}$ exactly once. The remainder $\chi^{(2)} = \phi^{(2)} - \chi^{(1)}$ is, for the seven classes

(4) $$4,\quad 2,\quad 1,\quad 0,\quad -1,\quad 0,\quad -1.$$

and since

$$\Sigma\, h_\rho \chi_\rho^{(2)^2} = 120,$$

this is a simple character.*

The next compound character is the sum of the permanents of the principal two-rowed minors. For the class 1^5 there are ten principal minors each contributing 1 as its permanent. For the class $1^3 2$, there are three minors corresponding to the identical permutation and one corresponding to a simple interchange. The character is 4. For the class $1^2 3$ there is but one significant minor; for the classes 14 and 5, none; for the class 12^2, two, and for the class 23, one.

Hence we obtain the compound character

$$10,\quad 4,\quad 1,\quad 0,\quad 0,\quad 2,\quad 1.$$

* Frobenius, *Op. cit.*, 1896.

Representing this by $\phi^{(3)}$, we have

$$\Sigma\ h_\rho \chi_\rho^{(1)} \phi_\rho^{(3)} = 120,$$

$$\Sigma\ h_\rho \chi_\rho^{(2)} \phi_\rho^{(3)} = 120.$$

Hence $\chi^{(3)} = \phi^{(3)} - \chi^{(1)} - \chi^{(2)}$ takes the values

(5) $\qquad 5, \quad 1, \quad -1, \quad -1, \quad 0, \quad 1, \quad 1,$

and this character is evidently simple, since

$$\Sigma\ h_\rho \chi_\rho^{(3)^2} = 120.$$

Taking the determinants of the minors instead of the permanents, we obtain $\phi^{(4)}$

$$10, \quad 2, \quad 1, \quad 0, \quad 0, \quad -2, \quad -1.$$

Then

$$\Sigma\ h_\rho \chi_\rho^{(1)} \phi_\rho^{(4)} = 0,$$

$$\Sigma\ h_\rho \chi_\rho^{(2)} \phi_\rho^{(4)} = 120.$$

Hence we obtain $\chi^{(4)} = \phi^{(4)} - \chi^{(2)}$, taking the values

(6) $\qquad 6, \quad 0, \quad 0, \quad 0, \quad 1, \quad -2, \quad 0.$

It is not necessary to carry the calculation further, for there always exists one character which takes the value $+1$ for the operations of the alternating group, and -1 for the other operations. We have therefore $\chi^{(7)}$

(7) $\qquad 1, \quad -1, \quad 1, \quad -1, \quad 1, \quad 1, \quad -1.$

Further, we may multiply any simple character by $\chi^{(7)}$ and the result will be another simple character. Hence the complete table of characters is obtained as given on p. 138.

This set is complete, for there are exactly seven classes, and hence exactly seven characters. As a check

$$\sum_\lambda \chi_0^{(\lambda)^2} = 1 + 16 + 25 + 36 + 25 + 16 + 1 = 120 = h,$$

the order of the group.

This method presents no undue amount of labour for the symmetric groups of orders up to 10 !, and even beyond this. The large numbers of operations in the classes need not lead to heavy calculation in using the orthogonal properties of the characters, for where the answer must necessarily be a multiple of the order of the symmetric group, a very rough calculation is sufficient to determine which multiple it is, or even a calculation modulo some integer prime to the order of the group.

The proof that the character arrived at from each step, after the elimination of the known characters, is simple, follows from the next section.

Nomenclature.

This method of obtaining the characters gives a definite order to the characters. We say that the order of one character is less than that of another, if it is arrived at first by the above process. The most convenient nomenclature for the characters, however, is not based on the order. The number of characters is the same as the number of classes, and is therefore equal to the number of partitions of n. We shall associate each character with one of these partitions.

If a character is obtained from the sum of the immanants of the principal minors of order $(n - \lambda_1)^2$, and the immanants correspond to the partition of $n - \lambda_1$

$$n - \lambda_1 = \lambda_2 + \lambda_3 + \ldots + \lambda_i,$$

then the partition of n

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_i$$

is that associated with the character, and also with the corresponding immanant. If the correspondence is obtained consecutively for the different degrees in their natural order, the logical sequence is retained.

The character defined by a partition is unique if

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_i.$$

We shall show that the character arrived at is independent of the order of the λ's, and that this partition of n used to define the character is the same as the partition used by FROBENIUS* and YOUNG.†

A compound character that is equal to the sum of a simple character taken once only, and other characters of order less than this simple character, is said to be *equivalent* to the simple character.

Now consider the partition of n

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_\mu.$$

To this partition of n corresponds a simple character $\chi^{(\lambda)}$. An equivalent compound character is obtained by taking the sum of the immanants of the $(n - \lambda_1)$-rowed principal minors of a permutation matrix. If the simple character used in these immanants is replaced by an equivalent compound character, the resulting compound character will still be equivalent to $\chi^{(\lambda)}$, for only characters of lesser order will have been added.

* *Op. cit.* (1900), (1901).

† 'Proc. Lond. Math. Soc.,' vol. 34, p. 361 (1902).

We can build up successively, compound characters corresponding to the partitions

$$\begin{aligned}&\lambda_\mu,\\&\lambda_{\mu-1}+\lambda_\mu,\\&\lambda_{\mu-2}+\lambda_{\mu-1}+\lambda_\mu,\\&\ldots\ldots\\&\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_\mu,\end{aligned}$$

the character corresponding to the first row being $+1$ for all classes of the symmetric group of order $\lambda_\mu\,!$, and the character of any subsequent row being obtained by taking the sum of the immanants of the principal minors of a permutation matrix, corresponding to the character of the preceding row.

The resulting compound character is equivalent to $\chi^{(\lambda)}$. We shall now find a mathematical expression for this compound character.

Consider the class corresponding to the partition of n

$$n=\alpha_1+\alpha_2+\ldots+\alpha_p.$$

The only significant $(n-\lambda_1)$-rowed principal minors of a corresponding permutation matrix will correspond to the separations* of this partition, such that the two separates are partitions of λ_1 and $n-\lambda_1$. The latter partition must again be separated to find the significant minors of the $(n-\lambda_1)$-rowed matrix. Similarly the partition of $n-\lambda_1-\lambda_2$ must be separated, and so on, until finally the character corresponding to λ_μ is unity.

Hence the compound character is equal to the number of ways in which the partition

$$n=\alpha_1+\alpha_2+\ldots+\alpha_p$$

can be separated into μ separates, such that the separates are partitions of $\lambda_1, \lambda_2, \ldots$ and λ_μ respectively.

That is to say, the compound character is the coefficient of $x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_\mu^{\lambda_\mu}$ in the product.

$$(x_1^{\alpha_1}+x_2^{\alpha_1}+\ldots+x_n^{\alpha_1})\,(x_1^{\alpha_2}+\ldots+x_n^{\alpha_2})\ldots(x_1^{\alpha_p}+\ldots+x_n^{\alpha_p}).$$

This is the compound character obtained by FROBENIUS† and shows that our partition

$$n=\lambda_1+\lambda_2+\ldots+\lambda_\mu$$

is the same as the partition used by FROBENIUS to define the character, and hence the same as that used by YOUNG.‡ Incidentally, this shows that the order of the λ's is immaterial. Again, since FROBENIUS' compound characters will yield the characters one at a time and simply, the proof follows, that the characters arrived at by the method described above, after elimination of the known characters, are simple.

* MACMAHON, "Combinatory Analysis," p. 45 (1915).

† *Op. cit.* (1901), (1903).

‡ 'Proc. Lond. Math. Soc.,' vol. 28, p. 255 (1927).

§ 4. Application to Symmetric Functions.

Schur-*Functions*.*—Consider the symmetric functions of the n-quantities

$$x_1, \quad x_2, \quad \ldots, \quad x_n.$$

The commonly defined symmetric functions are the following :

(1) $\mathrm{S}_r = \Sigma\, x_1{}^r$,

(2) a_r = the sum of the products of r different x's,

(3) h_r = the sum of the homogeneous products of weight r.†

The properties of these functions and the relations between them have been studied by MacMahon,‡ who also defines " new " symmetric functions.§ These have properties analogous to the Schur-functions which we shall define in this paper, and are in fact linear functions of the Schur-functions. The Schur-functions are, however, more fundamental and have considerable advantages over the functions of MacMahon, as will be shown.

Represent by $[\mathrm{Z}_r]$ the matrix

$$(4) \qquad [\mathrm{Z}_r] = \begin{bmatrix} \mathrm{S}_1, & 1, & 0, & 0, & \ldots & & & 0 \\ \mathrm{S}_2, & \mathrm{S}_1, & 2, & 0, & \ldots & & & 0 \\ \mathrm{S}_3, & \mathrm{S}_2, & \mathrm{S}_1, & 3, & 0, & \ldots & & 0 \\ \ldots\ldots & & & & & & & \\ & & & & & & & 0 \\ & & & & & \mathrm{S}_2, & \mathrm{S}_1, & r-1 \\ \mathrm{S}_r, & \mathrm{S}_{r-1} & \ldots & & & \mathrm{S}_3, & \mathrm{S}_2, & \mathrm{S}_1 \end{bmatrix}.$$

* These functions were first explicitly defined with reference to group characters by Schur in his Dissertation (" Ueber einer Klasse von Matrizen," Berlin (1901)). The fact that the quotient of $|x_s{}^{\lambda_t+n-t}|$ by $|x_s{}^{n-t}|$, which we shall show to be equal to the Schur-function, could be written as a determinant $|h_{\lambda_s-s+t}|$, was, however, known and used (Jacobi, see Muir, " Theory of Determinants, " vol. 1, p. 341 (1906), Trudi, see Muir, vol. 3, p. 135 (1920) ; Naegelsbach, see Muir, vol. 3, p. 144), and Kostka (Kostka, see Muir, vol. 4, p. 145 (1923)), calculated a series of tables connecting these functions with other symmetric functions. These tables may readily be deduced from tables of characters, as will be shown. Schur gives the formula for expressing them in terms of the products S_ρ, and conversely (p. 52), and expresses them as determinants with elements h_r (p. 47) and a_r (p. 50). He also demonstrates other properties, showing relations between S-functions and symmetric functions of the form $\Sigma\, x_1{}^{a_1}x_2{}^{a_2}\ldots$

† It is with some hesitation that we have used h_r in this sense, owing to the similarity to the symbol h_ρ used for an entirely different purpose. We have retained both symbols, however, because each is a used and accepted terminology, hoping that, as we remark upon the distinction here, confusion to the reader will be avoided. The suffix r of the symmetric function h_r denotes a number, the weight of the function, whilst for h_ρ, the order of a class, the suffix ρ denotes a class which would be represented, not by a number, but by the symbol of a partition of a number. For example, h_3 means a symmetric function of weight 3, $h_{(3)}$ means the order of the class (3), *i.e.*, 2. Hence no logical confusion arises.

‡ *Op. cit.* (1915).

§ *Ibid*, p. 203.

Of the many formulæ connecting the functions a_r, h_r and S_r, we consider especially two.

(5) $$r\,!\ a_r = |Z_r|,$$

(6) $$r\,!\ h_r = \overset{+}{|}Z_r\overset{+}{|}.$$

According to our previous generalization we define symmetric functions, which we shall call SCHUR-functions from the various immanants of this matrix.*

Let $r = \lambda_1 + \ldots + \lambda_p$, be a partition of r. Then the function which we shall designate by $\{\lambda_1, \ldots, \lambda_p\}$, is defined by the equation

(7) $$r\,!\,\{\lambda_1, \ldots, \lambda_p\} = |Z_r|^{(\lambda_1, \ldots, \lambda_p)}.$$

Let S be an operation of the class ρ of the symmetric group of order $r\,!$, containing α_1 cycles of order 1, α_2 of order 2, ... Denote by S_ρ the product

$$S_1^{\alpha_1} S_2^{\alpha_2} S_3^{\alpha_3} \ldots.$$

Then from the nature of the matrix $|Z_r|$ it is clear that the corresponding product P_S obtained from the matrix is either zero or a multiple of S_ρ.

From the known formula

$$a_r = \Sigma \pm h_\rho S_\rho,$$

h_ρ being the number of operations in the class ρ, together with the equation obtained from the determinant $|Z_r|$,

$$a_r = \Sigma \pm P_S,$$

it follows that

(8) $$\sum_\rho P_S = h_\rho S_\rho.$$

Of the alternative signs, + is taken for even permutations and − for odd permutations. The summation in (8) is taken over the permutations of the class ρ.

* In addition to the SCHUR-functions deduced from the matrix $[Z_r]$, the immanants of other special matrices are of interest. The threefold aspects of combinatory analysis suggest a study of the immanants of

$$\text{(i)} \begin{vmatrix} a_1, & 1, & 0, & \ldots & 0 \\ a_2, & a_1, & 1, & \ldots & 0 \\ \ldots\ldots & & & & \\ a_r, & a_{r-1}, & \ldots & \ldots & a_1 \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} a_1, & 1, & \ldots & 0 \\ 2a_2, & a_1, & 1, & 0 \\ \ldots\ldots & & & \\ ra_r, & a_{r-1}, & \ldots & a_1 \end{vmatrix} \quad \text{and of (iii)} \begin{vmatrix} S_0, & S_1, & S_2, & \ldots \\ S_1, & S_2, & S_3, & \ldots \\ S_2, & S_3, & S_4, & \ldots \\ \ldots\ldots & & & \end{vmatrix}.$$

For example, the threefold relations which express a_r, h_r and S_r in terms of one another, have their counterpart in the relations existing between the various immanants. Thus denoting the permanents of matrices of type (i) by H_r we have

$$H_r = a_1 H_{r-1} + a_2 H_{r-2} + \ldots + a_r$$

corresponding to

$$h_r = a_1 h_{r-1} - a_2 h_{r-2} + \ldots + (-1)^r a_r.$$

Hence we may read the values of the S-functions* from the table of characters.

$$r\,!\,\{\lambda_1, \ldots \lambda_p\} = \Sigma\, \chi_\rho^{(\lambda)} h_\rho S_\rho. \tag{9}$$

For example, taking the table of characters of the symmetric group of order 4 !, as given on p. 138, we obtain the equations

$$\begin{aligned} 4\,!\,h_4 = 4\,!\,\{4\} &= S_1^4 + 6S_1^2S_2 + 8S_1S_3 + 6S_4 + 3S_2^2, \\ 4\,!\,\{31\} &= 3S_1^4 + 6S_1^2S_2 - 6S_4 - 3S_2^2, \\ 4\,!\,\{2^2\} &= 2S_1^4 - 8S_1S_3 + 6S_2^2, \\ 4\,!\,\{21^2\} &= 3S_1^4 - 6S_1^2S_2 + 6S_4 - 3S_2^2, \\ 4\,!\,a_4 = 4\,!\,\{1^4\} &= S_1^4 - 6S_1^2S_2 + 8S_1S_3 - 6S_4 + 3S_2^2. \end{aligned} \tag{10}$$

The translation of S_r products into S-functions is even simpler. Owing to the orthogonal properties of the characters we may simply read down the columns of the table to obtain the coefficients. For example

$$\begin{aligned} S_1^4 &= \{4\} + 3\{31\} + 2\{2^2\} + 3\{21^2\} + \{1^4\}, \\ S_1^2S_2 &= \{4\} + \{31\} - \{21^2\} - \{1^4\}, \\ S_1S_3 &= \{4\} - \{2^2\} + \{1^4\}, \\ S_4 &= \{4\} - \{31\} + \{21^2\} - \{1^4\}, \\ S_2^2 &= \{4\} - \{31\} + 2\{2^2\} - \{21^2\} + \{1^4\}. \end{aligned} \tag{11}$$

There are exactly the correct number of S-functions to express the general symmetric function of weight r in terms of the S-functions of degree r, assuming that $n \geq r$. If $n < r$, the S_r products are linearly dependent, and some of the S-functions vanish identically. It will be shown later that the S-function $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ vanishes identically if $p > n$ and $\lambda_p \neq 0$.

§ 5. The Expression of Symmetric Products in Terms of S-Functions.

Expressions of the form $\Sigma\, x_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots$ may readily be expressed in terms of S-functions by a formula derived from Frobenius' equation,

$$S_1^{a_1}S_2^{a_2}S_3^{a_3}\ldots \Delta(x_1, \ldots, x_n) = \Sigma \pm \chi^{(\lambda)} x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \ldots x_n^{\lambda_n}. \tag{1}$$

We start with the equation

$$\begin{aligned} \Sigma\, x_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots &= \Sigma\, K_\lambda \{\lambda_1, \lambda_2, \lambda_3, \ldots\} \\ &= \frac{1}{h} \Sigma\, K_\lambda \chi_\rho^{(\lambda)} h_\rho S_\rho, \end{aligned} \tag{2}$$

and obtain a formula for the coefficients K_λ.

* We use the contraction S-function for Schur-function.

Multiply by $\Delta(x_1, \ldots, x_n)$.

$$\Sigma\, x_1^{a_1}x_2^{a_2}x_3^{a_3} \ldots \Delta(x_1, \ldots, x_n) = \frac{1}{h}\Sigma\, \mathrm{K}_\lambda \chi_\rho^{(\lambda)} h_\rho \mathrm{S}_\rho\, \Delta(x_1, \ldots, x_n),$$

$$= \frac{1}{h}\Sigma \pm \mathrm{K}_\lambda \chi_\rho^{(\lambda)} h_\rho \chi_\rho^{(\mu)} x_1^{\mu_1+n-1} x_2^{\mu_2+n-2} \ldots$$

$$= \Sigma \pm \mathrm{K}_\lambda x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} x_3^{\lambda_3+n-3} \ldots$$

since

$$\Sigma\, h_\rho \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)} = h, \qquad (\lambda) = (\mu),$$
$$= 0, \qquad (\lambda) \neq (\mu).$$

The negative sign is taken when the order of the x's in the product is a negative permutation of the natural order, $x_1, x_2, \ldots, x_n$. The numbers $\lambda_1, \lambda_2, \ldots$ and $\mu_1, \mu_2, \ldots$ are in descending order of magnitude.

The formula (3) thus gives K_λ as the coefficient of $x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2} \ldots x_n^{\lambda_n}$ in the product

$$\Sigma\, x_1^{a_1}x_2^{a_2} \ldots \Delta(x_1, \ldots, x_n).$$

Though the number of terms in $\Delta(x_1, \ldots, x_n)$ may be very large, the coefficients may be picked out very easily, as will be seen in the following example.

Example.—We now express $\Sigma\, x_1^3x_2x_3$ in terms of S-functions.

Here n must be ≥ 5. We take the simplest case $n = 5$. We have to pick out the coefficients in the product

$$(\Sigma\, x_1^3x_2x_3)\,(\Sigma \pm x_1^4x_2^3x_3^2x_4),$$

the second summation covering all permutations of the suffixes 1, 2, 3, 4, 5, taking the negative sign for the negative permutations. In the product we need only pick out the terms with all the indices different.

We set out the calculation first and follow with the explanation.

4,	3,	2,	1,	0											
		3,	1,	1	4,	3,	5,	2,	1	1,	1,	1,	1,	1	+
1,			3,	1	5,	3,	2,	4,	1	1,	1,	1,	1,	1	+
1,	1,			3	5,	4,	2,	1,	3	1,	1,	1,	1,	1	+
	3,	1,	1,		4,	6,	3,	2,	0	2,	1,	1,	1,		—
3,	1,	1,			7,	4,	3,	1,	0	3,	1,	1,			+
1,	3,	1,			5,	6,	3,	1,	0	2,	2,	1,			—

We set down in the first row of the first column the indices of the right hand term in the product, namely, 4, 3, 2, 1, 0. We have to add to these the indices 3, 1, 1 in any order, so that the five indices resulting are all different. In the first case we may add any of the three to zero. In the second row we add 1 to zero. The sum is 1, and as this index must not be repeated, either the other 1 or the 3 must be added to 1. Following this principle we obtain the second and third rows, and if 3 is added to zero the fourth row. The other possible arrangements are given in the last three rows.

In the second column we give the corresponding sums of the indices, namely, $\lambda_1 + n - 1$, $\lambda_2 + n - 2$, . . ., λ_n in some order. In the third column these are rearranged in descending order, and the numbers 4, 3, 2, 1, 0 respectively subtracted, giving the numbers λ_1, λ_2, . . ., λ_n. In the fourth column we have a $+$ or a $-$ sign according as the order in the second column is a positive or a negative permutation of the descending order.

Hence

$$\Sigma\ x_1^3 x_2 x_3 = \{3, 1, 1\} - \{2, 1^3\} + 3\,\{1^5\} - \{2^2, 1\}.$$

There are of course other methods. HANKEL* shows that

$$n\,!\ \Sigma\ x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \ldots = \Sigma \begin{vmatrix} S_{\alpha}, & 1, & & & \\ S_{\alpha+\beta}, & S_{\beta}, & 2, & & \\ S_{\alpha+\beta+\gamma}, & S_{\beta+\gamma}, & S_{\gamma}, & 3, & \ldots \\ \ldots\ldots & & & & \end{vmatrix}$$

from which the appropriate formula could be deduced.

SCHUR also gives the following formula.† Using our notation let $\Sigma\ r_{n_1 n_2 n_3} \cdot x_1^{n_1} x_2^{n_2} \ldots$ be any homogeneous symmetric function of weight $n = n_1 + n_2 + \ldots$. Represent $r_{n_1 n_2 n_3} \ldots$ symbolically by $r_{n_1} r_{n_2} r_{n_3} \ldots$, and put $r_0 = 1$, $r_{-1} = r_{-2} = \ldots = 0$. Then

$$\Sigma\ r_{n_1 n_2 n_3} \ldots x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots = \Sigma\ \left| r_{\lambda_s - s + t} \right| \quad \{\lambda_1, \lambda_2, \ldots\}.$$

See also p. 114 of this paper.

§ 6. THE VALUES OF THE S-FUNCTIONS WHEN THE WEIGHT OF A SYMMETRIC FUNCTION OF n QUANTITIES EXCEEDS n.

If the weight exceeds n, the S_r products are not all linearly independent, hence the same is true of the S-functions. In the S-functions this linear dependency manifests itself simply.

An S-function of weight p, of n quantities x_1, x_2, . . ., x_n, which corresponds to a partition of p into more than n parts, is identically zero.

The number of symmetric functions of weight p is equal to the number of partitions of p into not more than n parts. Hence we have exactly the right number of non-zero S-functions to express these symmetric functions, and the non-zero S-functions are linearly independent.

* See MUIR, *op. cit.*, vol. 3, p. 220 (1920).

† SCHUR, *op. cit.*, p. 54.

Consider the equation

(1) $$\Sigma\, x_1^{a_1}x_2^{a_2}\ldots x_r^{a_r} = \Sigma g_\lambda\,\{\lambda\},$$

the symmetric functions being of weight p and (λ) being a partition of p,

(2) $$\lambda_1+\lambda_2+\ldots+\lambda_s = p. \qquad (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$$

g_λ is the coefficient of $x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\ldots x_n^{\lambda_n}$ in the product,

(3) $$\Sigma\, x_1^{a_1}x_2^{a_2}\ldots x_r^{a_r}\,\Delta\,(x_1,\ldots,x_n).$$

The only S-functions appearing on the right of (1) are those for which the number of parts, s, in the partition of p, is not greater than n. Hence all symmetric functions of weight p must be expressible in terms of these, which are linearly independent.

Now consider the symmetric functions of $n+1$ quantities $x_1, x_2, \ldots, x_n, x_{n+1}$, and put $x_{n+1}=0$.

The left-hand side of equation (1) is unchanged. The expression (3) is multiplied by $x_1x_2\ldots x_n$, and certain terms are added which are zero for $x_{n+1}=0$.

Hence the right-hand side of equation (1) is unaltered, save that certain S-functions are added which correspond to a partition of p into $n+1$ parts. Thus a certain linear function of the S-functions corresponding to a partition of p into $n+1$ parts must be zero. Corresponding to a different function $\Sigma\, x_1^{a_1}x_2^{a_2}\ldots$, we obtain a different linear function of these S-functions. Now there are exactly sufficient S-functions of weight p corresponding to partitions of p into not more than $n+1$ parts, to express the symmetric functions of $n+1$ quantities. Hence we may obtain in this way sufficient equations to define the values of these S-functions.

It follows, then, that the obvious solution of these equations, namely that for which all the S-functions corresponding to a partition of p into exactly $n+1$ parts are zero, is the only solution. Similarly the result follows for partitions of p into more than $n+1$ parts.

§ 7. The Expression of the S-Functions as a Determinant and as the Ratio of Determinants.

Let the class ρ of the symmetric group of order $h = n!$ contain α_1 cycles on 1 symbol, α_2 on 2 symbols, etc. Denote by S_ρ the product

$$S_\rho = S_1^{\alpha_1}S_2^{\alpha_2}\ldots$$

where

$$S_r = \sum_{s=1}^{m} x_s^r.$$

Then Frobenius' formula for the characters of the symmetric group may be expressed as follows :—

$$S_\rho\,\Delta\,(x_1,\ldots,x_m) = \Sigma \pm \chi_\rho^{(\mu)}x_1^{\mu_1+m-1}\ldots x_m^{\mu_m}.$$

Hence

$$\Sigma\, h_\rho \chi_\rho^{(\lambda)} \mathrm{S}_\rho\, \Delta\,(x_1, \ldots, x_m) = h\, \Sigma\, x_1^{\lambda_1+m-1} \ldots x_m^{\lambda_m},$$

since

$$\begin{aligned} \Sigma\, h_\rho \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)} &= h \qquad (\lambda) = (\mu) \\ &= 0. \qquad (\lambda) \neq (\mu) \end{aligned}$$

Hence

$$\frac{1}{h}\, \Sigma\, h_\rho \chi_\rho^{(\lambda)} \mathrm{S}_\rho = \frac{\Sigma \pm x_1^{\lambda_1+m-1} \ldots x_m^{\lambda_m}}{\Sigma \pm x_1^{m-1} \ldots x_{m-1}}.$$

The summations on the right are taken over all permutations of the suffixes of the x's ; of the alternative signs $+$ is taken for a positive and $-$ for a negative permutation.

Hence

$$\{\lambda\} = \frac{|x_s^{\lambda_t+m-t}|}{|x_s^{m-t}|},$$

s denoting the row and t the column of the matrix from which the element is taken, and the S-functions are expressed as the ratio of two determinants of degree m.

In this form some properties of the S-functions have actually been studied before the discovery of Group Characters by FROBENIUS.

In 1841 JACOBI, *loc. cit.*, and later, in 1864, TRUDI, *loc. cit.*, prove that the ratio of these determinants is equal to another determinant in which the elements are the symmetric functions h_r.

Again in 1871, NAEGELSBACH, *loc. cit.*, expressed the same ratio of the determinants as a determinant in which the elements are the symmetric functions a_r.

Later KOSTKA* proved TRUDI's and NAEGELSBACH's theorems for himself in a more elegant way, and obtains further properties of the functions. He calculates tables connecting these functions (S-functions) with other symmetric functions.

Recently AITKEN † has obtained a very illuminating generalization of KOSTKA's results.

The JACOBI–TRUDI formula may be written

$$\begin{aligned} \frac{|x_s^{\lambda_s+m-t}|}{|x_s^{m-t}|} &= \begin{vmatrix} h_{\lambda_1}, & h_{\lambda_1+1}, & h_{\lambda_1+2}, & \ldots & \\ h_{\lambda_2-1}, & h_{\lambda_2}, & h_{\lambda_2+1}, & \ldots & \\ \ldots\ldots & & & & \\ h_{\lambda_p-p+1}, & \ldots & & & h_{\lambda_p} \end{vmatrix} \\ &= |\ h_{\lambda_s-s+t}\ |, \end{aligned}$$

Hence

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} = |\ h_{\lambda_s-s+t}\ |.$$

Now assuming the number of x's to exceed the weight, any symmetric function may be expressed uniquely in terms of the functions S_1, S_2, S_3, etc. In any function ϕ,

* See MUIR, *op. cit.*, vol. 3, pp. 154, 158, (1920), vol. 4, p. 145 (1923).

† 'Proc. Edin. Math. Soc.,' vol. 1, p. 55 (1927) and vol. 2, p. 164 (1930).

replace $S_2, S_4, S_6, \ldots$ respectively by $-S_2, -S_4, -S_6, \ldots$. We term the function so obtained the conjugate of ϕ, and represent it by $\bar{\phi}$.

Clearly $\bar{a}_r = h_r$, $\bar{h}_r = a_r$. Again if $(\mu_1, \ldots, \mu_q)$ is the partition of n associated with $(\lambda_1, \ldots, \lambda_p)$

$$\overline{\{\mu_1, \ldots, \mu_q\}} = \{\lambda_1, \ldots, \lambda_p\}.$$

It immediately follows that

$$\begin{aligned}\{\lambda_1, \ldots, \lambda_p\} &= \overline{\{\mu_1, \ldots, \mu_q\}} \\ &= \overline{\left| \; h_{\mu_s - s + t} \; \right|} \\ &= \left| \; a_{\mu_s - s + t} \; \right| \\ &= \begin{vmatrix} a_{\mu_1}, & a_{\mu_1+1}, & \ldots & \\ a_{\mu_2-1}, & a_{\mu_2}, & \ldots & \\ \ldots\ldots & & & \\ a_{\mu_q-q+1}, & ,\,, & & a_{\mu_q} \end{vmatrix}.\end{aligned}$$

Alternatively this result could be deduced immediately from NAEGELSBACH's theorem.

KOSTKA's tables connect these functions with the functions $\Sigma x_1^{a_1} x_2^{a_2} \ldots$, and with the functions a_r. A typical table of KOSTKA, expressed in our terminology, is as follows :

—	$x_1x_2x_3x_4x_5$	$\Sigma x_1^2x_2x_3x_4$	$\Sigma x_1^2x_2^2x_3$	$\Sigma x_1^3x_2x_3$	$\Sigma x_1^3x_2^2$	$\Sigma x_1^4x_2$	Σx_1^5	—
$\{1^5\}$	**1**	-4	3	3	-2	-2	1	$\{5\}$
$\{21^3\}$	4	**1**	-2	-1	2	1	-1	$\{41\}$
$\{2^21\}$	5	2	**1**	-1	-1	1	0	$\{32\}$
$\{31^2\}$	6	3	1	**1**	-1	-1	1	$\{31^2\}$
$\{32\}$	5	3	2	1	**1**	-1	0	$\{2^21\}$
$\{41\}$	4	3	2	2	1	**1**	-1	$\{21^3\}$
$\{5\}$	1	1	1	1	1	1	**1**	$\{1^5\}$
	a_1^5	$a_1^3a_2$	$a_1a_2^2$	$a_1^2a_3$	a_2a_3	a_1a_4	a_5	

The table is read only as far as the leading diagonal, or from the leading diagonal onwards, either across the table or downwards, according to the equation required. For example

$$a_1^2a_3 = \{3, 1^2\} + \{2^2, 1\} + 2\{2, 1^3\} + \{1^5\},$$

$$\{3, 1^2\} = \Sigma\, x_1^3x_2x_3 + \Sigma\, x_1^2x_2^2x_3 + 3\Sigma\, x_1^2x_2x_3x_4 + 6x_1x_2x_3x_4x_5,$$

$$\Sigma\, x_1^3x_2^2 = \{3, 2\} - \{2^2, 1\} - \{3, 1^2\} + 2\{2, 1^3\} - 2\{1^5\},$$

$$\{3, 2\} = a_2a_3 - a_1a_4.$$

These relations could be obtained by methods already described in this paper, but the tables may also be deduced from the tables of characters as follows. Consider the table of characters of the symmetric group of order 5 !, the characters being in their usual order reversed, that it, commencing with $[1^5]$, and the classes being arranged with the partitions in the same order.

—	(1^5)	(21^3)	(2^21)	(31^2)	(32)	(41)	(5)
$[1^5]$	1	−1	1	1	−1	−1	1
$[21^3]$	4	−2	0	1	1	0	−1
$[2^21]$	5	−1	1	−1	−1	1	0
$[31^2]$	6	0	−2	0	0	0	1
$[32]$	5	1	1	−1	1	−1	0
$[41]$	4	2	0	1	−1	0	−1
$[5]$	1	1	1	1	1	1	1

First we show how to express a_1^5, $a_1^3a_2$, etc., in terms of S-functions.

Clearly a_1^5 may be read off directly, for the coefficients are the numbers in the first column.

$a_1^3a_2$ may be expressed in terms of S_1^5 and $S_1^3S_2$, and also in terms of $\{41\}$, $\{32\}$, . . . $\{1^5\}$. Hence the coefficients are obtained by taking that linear combination of the first two columns of the table of characters, which vanishes for the first row. Further, the coefficient in the leading diagonal must be unity. Clearly we must add columns (1) and (2) and divide by 2. We thus obtain the coefficients in the second column of the Kostka table below the leading diagonal.

For the third column of the Kostka table we take that linear combination of the first three columns of the table of characters, which vanishes for the first two rows, and is unity for the third row. Similarly we may complete that half of the Kostka table below the leading diagonal.

For the other half of the Kostka table, above the leading diagonal, we commence at the bottom right-hand corner, and proceed in a like manner ; for *e.g.*, $\Sigma x_1^3x_2x_3$ may be expressed in terms of $S_3S_1^2$ and terms of higher order, and also in terms of $\{311\}$ and S-functions of lower order.

The double use of the table may also be justified by such considerations, if we bear in mind the orthogonal properties of the table of characters.

§ 8. Multiplication of S-Functions.

We set out the rules for multiplication in the form of three theorems.

For the definition of Characteristic Unit, etc., see Frobenius.* Strictly, Frobenius uses the term Characteristic Unit (" Charakteristische Einheit ") for the set of h coefficients of the group elements in the idempotent element, rather than the idempotent

* ' SitzBer. Preuss. Akad. Wiss. Berl.,' pp. 328–358 (1903).

element itself, in which sense we use it, but the distinction is not thought to be sufficient to warrant the introduction of another name.

YOUNG* obtains certain Characteristic Units of the symmetric group thus. Consider the symmetric group of n symbols

$$\alpha_1, \alpha_2, \ldots, \alpha_n.$$

By the symmetric group of r of the symbols is meant the sum of the operations of the symmetric group on those r symbols. By negative symmetric group, is meant the same thing with a minus sign attached to negative substitutions.

Let $n = \lambda_1 + \ldots + \lambda_p$ be any partition of n. $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p)$. Build a tableau A with λ_1 of the symbols $\alpha_1, \ldots, \alpha_n$ in the first row, λ_2 in the second row, etc., and finally λ_p in the pth row. Each row commences at the first column, the other symbols being consecutively one column later. Take the product of the symmetric groups of the rows and the negative symmetric groups of the columns, the positive symmetric groups preceding (or succeeding) the negative symmetric groups, and denote this by (A). Then

$$\frac{\chi_0^{(\lambda)}}{h}\,(A)$$

is a Characteristic Unit corresponding to the simple character $\chi^{(\lambda)}$.

Further, there are $\chi_0^{(\lambda)}$ possible tableaux corresponding to (λ) in which the order of symbols in each row and each column follows the order $\alpha_1, \alpha_2, \ldots, \alpha_n$, and these "standard"† tableaux may be made to correspond to the diagonal elements of a corresponding matrix sub-algebra of the FROBENIUS Algebra. Hence *e.g.*, the modulus of the sub-algebra being a matrix with unity in each position in the leading diagonal and zero elsewhere, the modulus of the sub-algebra is

$$\frac{\chi_0^{(\lambda)}}{h}\,\Sigma\,(A)$$

summed over the $\chi_0^{(\lambda)}$ standard tableaux. The identity

$$1 = \Sigma\,\frac{\chi_0^{(\lambda)}}{h}\,(A)$$

follows, the summation being for all the $\Sigma\,\chi_0$ standard tableaux corresponding to all partitions of n.

The partition $n = \lambda_1 + \lambda_2 + \ldots + \lambda_p$ is the same partition as that associated with the character by FROBENIUS, and hence the one associated with the character in this paper.

The significant feature of the YOUNG tableaux is that if a tableau A contains two

* 'Proc. Lond. Math. Soc.,' vol. 33, p. 97 (1901); vol. 34, p. 361 (1902); vol. 28, p. 255 (1927); vol. 31, p. 253 (1931).

† YOUNG, 'Proc. Lond. Math. Soc.,' vol. 28, p. 258 (1927).

symbols α and β in the same row, and a tableau B contains the two symbols in the same column, then

$$(A)(B) = (B)(A) = 0,$$

since the symmetric group from (A) including α and β has a factor $[1 + (\alpha\beta)]$, and the negative symmetric group from (B) a factor $[1 - (\alpha\beta)]$, and $[1 + (\alpha\beta)][1 - (\alpha\beta)] = 0$.

Hence the product of any two different standard tableaux is zero.

We now proceed to the enunciation of our first theorem.

THEOREM I.—*The product of two* S*-functions of degrees r and s respectively is equal to the sum of integral multiples of the* S*-functions of degree* $r + s$.

The theorem is demonstrated by showing an isomorphism between the multiplication of S-functions and the multiplication of corresponding Characteristic Units.

LEMMA I.—*If* $\frac{1}{h}$ (A) *is a Characteristic Unit of the symmetric group of order* $h = r\,!$, *corresponding to the simple character* $\chi^{(\lambda)}$, *then* (A) *contains an aggregate of* $\chi_\rho^{(\lambda)} h_\rho$ *operations of the class* ρ.

All Characteristic Units corresponding to the same simple character must contain the same aggregate of operations from each class, for the Characteristic Units may be transformed into one another.

Again, the modulus of the invariant sub-algebra of the FROBENIUS algebra which corresponds to the character, is equal to the sum of $\chi_0^{(\lambda)}$ Characteristic Units. This modulus is also equal to

$$\frac{1}{h}\chi_0^{(\lambda)} \Sigma \chi_\rho^{(\lambda)} C_\rho,*$$

where C_ρ denotes the sum of the operations in the class ρ.

The Lemma follows.

LEMMA II.—*An isomorphism exists between the multiplication of* S*-functions and the multiplication of corresponding Characteristic Units involving different sets of symbols.*

Consider the symmetric group of substitutions on r symbols $\alpha_1, \ldots, \alpha_r$, and the symmetric group of substitutions on s symbols $\beta_1, \ldots, \beta_s$. The direct product of these groups is a sub-group of the symmetric group of substitutions on the $r + s$ symbols $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$. The product of two operations from these symmetric groups will have the sum of the cycles of the two operations. Hence, if the operations belong to classes ρ and ρ' respectively, and the product belongs to the class ρ''

$$S_\rho S_{\rho'} = S_{\rho''},$$

S_ρ being defined as in § 5.

Using this result together with Lemma I, the proof of Lemma II follows.

The proof of the theorem now follows immediately, for the product of two Characteristic Units, which are commutative, since they correspond to substitutions

* BURNSIDE, "The Theory of Groups," 2nd Ed., p. 316 (1911).

on different sets of symbols, must in the nature of the case be a Characteristic Unit. The Characteristic Unit of the product corresponds to a compound character which is the sum of integral multiples of simple characters.

Corollary.—The coefficient of the character $\chi^{(\nu)}$ *in the product of* $\{\lambda\}$ *and* $\{\mu\}$ *is not greater than*

$$\frac{\chi_0{}^{(\nu)}}{\chi_0{}^{(\lambda)}\chi_0{}^{(\mu)}}.$$

This follows from the multiplication of the Characteristic Units corresponding to $\chi_0{}^{(\lambda)}\{\lambda\}$ and $\chi_0{}^{(\mu)}\{\mu\}$.

THEOREM II.—*In the product of two* S-*functions* $\{\lambda_1, \ldots, \lambda_p\}$ and $\{\mu_1, \ldots, \mu_q\}$, *the* S-*functions appearing in the product may be obtained by the following rule.*

Corresponding to $\{\lambda_1, \ldots \lambda_p\}$ *form a tableau* A *with* λ_1 *symbols in the first row,* λ_2 *in the second row, etc., and corresponding to* $\{\mu_1, \ldots, \mu_q\}$ *form a tableau* B *with* μ_1 *symbols in the first row,* μ_2 *in the second row, etc. Construct new tableaux* C_1, C_2, . . ., *each containing all the symbols of tableaux* A *and* B, *and such that no two symbols in the same row, either in* A *or in* B, *are in the same column in a* C, *and no two symbols in the same column, either in* A *or in* B, *are in the same row in a* C.

Then the S-*functions which appear in the product correspond to the possible tableaux* C.

Form the product (A) of the symmetric groups of the rows of A and the negative symmetric groups of the columns. Then

$$\frac{1}{h}\chi_0{}^{(\lambda)}\,(A)$$

is a corresponding Characteristic Unit.

Similarly, form a Characteristic Unit

$$\frac{1}{g}\chi_0{}^{(\mu)}\,(B),$$

corresponding to the S-function $\{\mu_1, \ldots, \mu_q\}$, where (B) is the product of the symmetric groups of the rows of B and the negative symmetric groups of the columns, and $g = s\,!$

Now the product

$$\frac{\chi_0{}^{(\lambda)}\chi_0{}^{(\mu)}}{hg}\,(A)\,(B)$$

is a Characteristic Unit of the symmetric group of order $(r+s)\,!$, and may be expressed as the sum of simple Characteristic Units. As in Lemma II to Theorem I, if an S-function $\{\nu_1, \nu_2, \ldots\}$ appears in the product of $\{\lambda_1, \ldots \lambda_p\}$ and $\{\mu_1, \ldots, \mu_q\}$, then there will be a corresponding Characteristic Unit. If we multiply (A) (B) by the modulus of the sub-algebra of the FROBENIUS algebra corresponding to the character $(\nu_1, \nu_2, \ldots)$, we shall pick out from the terms of the product those which belong to the same sub-algebra. Hence if these terms exist, the product (A) (B) times the modulus of the sub-algebra is not zero.

But the modulus of the sub-algebra may be expressed as a multiple of the sum of $\chi_0^{(\nu)}$ tableaux corresponding to (ν). If two symbols appear in the same row (or column) either in A or in B, and in the same column (or row) in a tableau C, then the product (A) (B) (C) must be zero. Hence at least one tableau C must exist corresponding to $(\nu_1, \nu_2, \ldots)$, such that no two symbols in the same row either in A or in B appear in the same column of C, and no two symbols in the same column either of A or of B, appear in the same row of C. The theorem follows.

To find the product of an S-function by a second S-function of very small degree, Theorem II is often sufficient, the S-functions corresponding to possible compound tableaux appearing with coefficient unity. For example

$$\{3\}\{3\} = \{6\} + \{5, 1\} + \{4, 2\} + \{3, 3\}.$$

As a check we may evaluate the coefficient of S_1^6 on both sides.

It is not necessary to go to very high degrees, however, before the coefficients begin to differ from unity, and a more precise rule becomes necessary, *e.g.*,

$$\{2, 1\}\{2, 1\} = \{2^2 1^2\} + \{3, 1^3\} + \{2^3\} + \{4, 2\} + \{4, 1^2\} + \{3^2\} + 2\{3, 2, 1\}.$$

The following theorem, however, enables us to multiply S-functions of any degree.

THEOREM III.—*Corresponding to two* S-*functions* $\{\lambda_1, \ldots, \lambda_p\}$, $\{\mu_1, \ldots, \mu_q\}$ *build tableaux* A *and* B *as in Theorem II. Then in the product of these two functions, the coefficient of any* S-*function* $\{\nu_1, \nu_2, \ldots\}$ *is equal to the number of compound tableaux including all the symbols of* A *and* B, *and corresponding to* $\{\nu_1, \nu_2, \ldots\}$, *that can be built according to the following rules.*

Take the tableau A *intact, and add to it the symbols of the first row of* B. *These may be added to one row of* A, *or the symbols may be divided without disturbing their order, into any number of sets, the first set being added to one row of* A, *the second set to a subsequent row, the third to a row subsequent to this, and so on. After the addition no row must contain more symbols than a preceding row, and no two of the added symbols may be in the same column.*

Next add the second row of symbols from B, *according to the same rules, with this added restriction. Each symbol from the second row of* B *must appear in a later row of the compound tableau than the symbol from the first row in the same column.*

Similarly add each subsequent row of symbols from B, *each symbol being placed in a later row of the compound tableau than the symbol in the same column from the preceding row of* B, *until all the symbols of B have been used.*

No simple proof has been found that will demonstrate it in the general case. However, the only applications of the theorem used in this paper are included in the special case of multiplication by h_r, which we proceed to prove. Of course every multiplication could be reduced to this case, for every S-function can be expressed as a determinant with elements of the form h_r, but the procedure is not so simple as the direct use of the theorem. We next deduce the truth of the theorem for multiplication by an S-function corresponding to a partition of r into two parts. The method can be extended to partitions into three and more parts, but the explanation becomes lengthy and difficult, and this has not been attempted here. Exactly parallel methods would prove the theorem for multiplication by the associated S-functions $\{1^r\}$ and $\{2^p 1^q\}$.

Consider the product

$$\{\lambda_1, \ldots, \lambda_p\}\{r\}$$

where $\Sigma \lambda_1 = n$.

As in Theorem II, denote by (A) the product of the symmetric groups of the rows and the negative symmetric groups of the columns of a tableau A with λ_i symbols in the ith row. B is a tableau with one row containing r symbols. The corresponding Characteristic Units are

$$\frac{\chi_0^{(\lambda)}}{n\,!}(\mathrm{A}) \quad \text{and} \quad \frac{1}{r\,!}(\mathrm{B}).$$

Two numbers of the FROBENIUS algebra are said to be equivalent if one can be transformed into the other. The symbol $\sim$ denotes equivalence.

The theorem is proved for this case if we can demonstrate that

$$\frac{\chi_0^{(\lambda)}}{n\,!}(\mathrm{A}) \cdot \frac{1}{r\,!}(\mathrm{B}) \sim \Sigma \frac{\chi_0^{(\nu)}}{(r+n)\,!}(\mathrm{C}),$$

the summation on the right being taken over the tableaux C defined in the theorem, $\chi^{(\nu)}$ being the corresponding character.

The modulus of the invariant sub-algebra corresponding to $\chi^{(\nu)}$, of the FROBENIUS algebra of the symmetric group of order $(n+r)\,!$ is equal to

$$\frac{\chi_0^{(\nu)}}{(n+r)\,!}[\Sigma\ (\mathrm{D})],$$

the summation being taken over the $\chi_0^{(\nu)}$ standard tableaux D, corresponding to $\chi^{(\nu)}$. The assigned order of the symbols needed to decide which tableaux are standard, is any order of the n symbols of A that would make A a standard tableau, followed by the r symbols of B taken in the same order.

Since the identical element of the group, and also the modulus of the sub-algebra, are unaltered by transformations, we may multiply both sides of the above equivalence by this modulus and equate coefficients of the identical element. The right-hand side becomes equal to

$$\frac{g_{\lambda r \nu}\chi_0^{(\nu)}}{(n+r)\,!},$$

where $g_{\lambda r \nu}$ is the number of tableaux C which correspond to $\chi^{(\nu)}$. From the method of definition, each of the tableaux C is standard.

Now suppose that the row of r symbols in the tableau B is divided in the tableau C into t rows containing respectively $a_1, a_2, \ldots, a_t$ of these symbols. Corresponding to one tableau C, there will be $(r\,!)/(a_1\,!\ a_2\,!\ \ldots\ a_t\,!)$ standard tableaux D which can be obtained by the rearrangement of these symbols, the symbols in each row still being in the assigned order. The tableaux D that do not correspond in this way to a tableau C will make the product (A) (B) (D) zero, from Theorem II.

Consider the coefficient of the identical element in the product (A) (B) (D). The tableau A contains only the first n symbols, and the tableau B only the last r symbols. Hence the only significant terms from the tableau D are those that contain no cycles involving both sets of symbols. Thus the required coefficient will be unaltered if we replace the tableau D by that portion of it which contains the first n symbols, which portion must be identical with A if the product is not to be zero, and multiply by the symmetric groups on the $a_1, a_2, \ldots, a_t$ symbols from tableau B.

The coefficient of the identical element in the product

$$\frac{\chi_0^{(\lambda)}}{n\,!}\,(\mathrm{A})\,\frac{1}{r\,!}\,(\mathrm{B})\,\frac{\chi_0^{(\nu)}}{(n+r)\,!}\,(\mathrm{D})$$

must therefore be

$$\frac{\chi_0^{(\nu)}}{(n+r)\,!}\cdot\frac{1}{r\,!}\cdot a_1\,!\;a_2\,!\;\ldots\;a_t\,!$$

If we sum for the $(r\,!)/(a_1! \ldots a_t!)$ tableaux D which correspond to each tableau C, we obtain

$$\frac{\chi_0^{(\nu)}}{(n+r)\,!}$$

for each tableau C. The theorem follows for this case.

We next consider the case

$$\{\lambda_1, \ldots, \lambda_p\}\,\{q, r\}.$$

We use the equation

$$\{q, r\} = \{q\}\,\{r\} - \{q+1\}\,\{r-1\},$$

and seek the tableaux that can be formed corresponding to successive multiplication by $\{q\}$ and $\{r\}$, and take away those which correspond to tableaux formed in the multiplication by $\{q+1\}$ and $\{r-1\}$.

Corresponding to $\{q\}$ and $\{r\}$ form tableaux $(X_1, \ldots, X_q)$ and $(Y_1, \ldots, Y_r)$, which we call B and C. Let B_1 and C_1 denote the tableaux $(X_1, \ldots, X_{q+1})$ and $(Y_1, \ldots, Y_{r-1})$.

Let D be any tableau formed by successive multiplication by $\{q\}$ and $\{r\}$. Now if we read the letters added to the tableau A in forming D, beginning at the first row and reading right to left, omitting the suffixes of the X's and Y's, we form a permutation of X^qY^r.

If this permutation is a lattice permutation,* it corresponds to a tableau as defined in the theorem for multiplication by $\{q, r\}$. We shall show that to each non-lattice permutation there corresponds a tableau in the multiplication by $\{q+1\}\,\{r-1\}$ and conversely, which proves the theorem for this case.

* MacMahon, "Combinatory Analysis," p. 124 (1915). A lattice permutation of $X^aY^bZ^c\ldots$ is a permutation such that among the first r terms of it, the number of X's $\geq$ the number of Y's $\geq$ the number of Z's, etc., for all r.

In the permutation of X^qY^r number the X's and Y's in the order of their appearance. If Y_s succeed X_t and precede X_{t+1}, it is said to be of index $s - t$, and is said to be of positive, zero or negative index according as $s - t$ is positive, zero or negative.

For a lattice permutation, there is no Y of positive index.

Now in a non-lattice permutation of X^qY^r, take the first Y of greatest (positive) index and replace by an X. We obtain a permutation of $X^{q+1}Y^{r-1}$. Conversely from a permutation $X^{q+1}Y^{r-1}$, take the last Y of greatest (zero or positive) index, and replace the X immediately following it by a Y, unless all the Y's are of negative index, in which case replace the first X in the permutation by a Y. We thus form a one-one correspondence between the non-lattice permutations of X^qY^r and the permutations of $X^{q+1}Y^{r-1}$. Examination shows that if D is a permissible tableau in the multiplication by $\{q\}$ and $\{r\}$, and D_1 is the tableau obtained from this by replacing the Y by an X as above, then D_1 is a permissible tableau in the multiplication by $\{q+1\}$ and $\{r-1\}$. For the Y must precede an X in the permutation, and hence the replacing X in D_1 cannot follow a Y in the same row. Again the X cannot follow a Y in the same column in D_1 for this would mean that the tableau D had two Y's in the same column. Lastly the replacing X in D_1 cannot be in the same column as another X, for this would mean that an earlier Y in the permutation would have the same index as the replaced Y. Similar considerations prove the converse.

This completes the theorem for the multiplication by $\{q, r\}$.

Example.—To form the product of

$$\{4, 3, 1\} \times \{2^21\}.$$

We form the two tableaux

$$\begin{pmatrix} a, b, c, d \\ e, f, g \\ h \end{pmatrix}, \quad \begin{pmatrix} \alpha, \beta \\ \gamma, \delta \\ \varepsilon \end{pmatrix}.$$

The following compound tableaux may be built according to Theorem III. Since the positions of the symbols a, b, c, d, e, f, g, and h remain constant, we shall replace these symbols by 0's in their positions, as this brings to the eye more clearly, the variable positions of the other symbols

$$\begin{array}{llll}
0\,0\,0\,0\,\alpha\,\beta & 0\,0\,0\,0\,\alpha\,\beta & 0\,0\,0\,0\,\alpha\,\beta & 0\,0\,0\,0\,\alpha\,\beta \\
0\,0\,0\,\gamma\,\delta & 0\,0\,0\,\gamma\,\delta & 0\,0\,0\,\gamma & 0\,0\,0\,\gamma \\
0\,\varepsilon & 0 & 0\,\delta\,\varepsilon & 0\,\delta \\
 & \varepsilon & & \varepsilon \\
 & & & \\
0\,0\,0\,0\,\alpha\,\beta & 0\,0\,0\,0\,\alpha\,\beta & 0\,0\,0\,0\,\alpha\,\beta & 0\,0\,0\,0\,\alpha\,\beta \\
0\,0\,0\,\gamma & 0\,0\,0\,\gamma & 0\,0\,0 & 0\,0\,0 \\
0\,\varepsilon & 0 & 0\,\gamma & 0\,\gamma \\
\delta & \delta & \delta\,\varepsilon & \delta \\
 & \varepsilon & & \varepsilon
\end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0\ \alpha\ \beta \\ 0\ 0\ 0 \\ 0\ \gamma\ \delta \\ \varepsilon \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta\ \gamma \\ 0\ \delta\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta\ \gamma \\ 0\ \delta \\ \varepsilon \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta\ \gamma \\ 0\ \varepsilon \\ \delta \end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta\ \gamma \\ 0 \\ \delta \\ \varepsilon \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta \\ 0\ \gamma\ \delta \\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta \\ 0\ \gamma \\ \delta\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \beta \\ 0\ \gamma \\ \delta \\ \varepsilon \end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \gamma \\ 0\ \beta\ \varepsilon \\ \delta \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \gamma \\ 0\ \beta \\ \delta\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \gamma \\ 0\ \beta \\ \delta \\ \varepsilon \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0 \\ 0\ \beta\ \gamma \\ \delta\ \varepsilon \\ \ \end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0 \\ 0\ \beta\ \gamma \\ \delta \\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0 \\ 0\ \beta \\ \gamma\ \delta \\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \gamma \\ 0\ \varepsilon \\ \beta \\ \delta \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0\ \gamma \\ 0 \\ \beta \\ \delta \\ \varepsilon \end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0 \\ 0\ \gamma \\ \beta\ \varepsilon \\ \delta \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0\ \alpha \\ 0\ 0\ 0 \\ 0\ \gamma \\ \beta \\ \delta \\ \varepsilon \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ \alpha \\ 0\ \beta\ \gamma \\ \delta\ \varepsilon \\ \ \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ \alpha \\ 0\ \beta\ \gamma \\ \delta \\ \varepsilon \\ \ \end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ \alpha \\ 0\ \beta \\ \gamma\ \delta \\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ \alpha \\ 0\ \gamma \\ \beta\ \varepsilon \\ \delta \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ \alpha \\ 0\ \gamma \\ \beta \\ \delta \\ \varepsilon \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ \alpha\ \beta \\ \gamma\ \delta \\ \varepsilon \\ \ \end{array}$$

$$\begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ \alpha \\ \beta\ \gamma \\ \delta\ \varepsilon \\ \ \end{array} \qquad \begin{array}{l} 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ \alpha \\ \beta\ \gamma \\ \delta \\ \varepsilon \end{array}$$

Hence :

$$
\begin{aligned}
\{4, 3, 1\} \times \{2^2 1\} = &\{6, 5, 2\} + \{6, 5, 1^2\} + \{6, 4, 3\} \\
&+ 2\{6, 4, 2, 1\} + \{6, 4, 1^3\} + \{6, 3, 2^2\} \\
&+ \{6, 3, 2, 1^2\} + \{6, 3^2 1\} + \{5^2 3\} \\
&+ 2\{5^2 2, 1\} + \{5^2 1^3\} + 2\{5, 4, 3, 1\} \\
&+ 2\{5, 4, 2^2\} + 2\{5, 4, 2, 1^2\} + \{5, 3^2 2\} \\
&+ \{5, 3^2 1^2\} + \{5, 3, 2^2 1\} + \{5, 3, 2, 1^3\} \\
&+ \{4^2 3, 2\} + \{4^2 3, 1^2\} + 2\{4^2 2^2 1\}. \\
&+ \{4^2 2, 1^3\} + \{4, 3^2 2, 1\} + \{4, 3, 2^3\} \\
&+ \{4, 3, 2^2 1^2\}.
\end{aligned}
$$

This result may be checked by equating the coefficients of S_1^{13} on both sides of the equation. We obtain

$$\frac{\chi_0^{(\lambda)}}{8\,!} \cdot \frac{\chi_0^{(\mu)}}{5\,!} = \frac{\Sigma\, \chi_0^{(\nu)}}{13\,!},$$

(λ) and (μ) denoting the characters corresponding to $(4, 3, 1)$ and $(2^2 1)$, and $\chi^{(\nu)}$ being summed for all the characters corresponding to the S-functions on the right of the equation. In our example $\chi_0^{(\lambda)} = 70$, $\chi_0^{(\mu)} = 5$. Hence

$$\frac{70\,.\,5}{8\,!\;5\,!} = \frac{\Sigma\, \chi_0^{(\nu)}}{13\,!}$$

$$\Sigma\, \chi_0^{(\nu)} = \frac{70\,.\,5\,.\,13\,!}{8\,!\;5\,!}$$

$$= 450450.$$

Using the formula

$$\chi_0^{(\lambda)} = \frac{n\,!\; \prod_{rs} (\lambda_r - \lambda_s - r + s)}{\prod_r (\lambda_r + p - r)\,!},$$

this equation proves to be correct.

§ 9. Relation between S-Functions and MacMahon's New Symmetric Functions.

In his book on Combinatory Analysis, MacMahon (p. 203) defines New Symmetric Functions, which he denotes by $a_{p_1 p_2 \dots p_s}$ or $h_{q_1 q_2 \dots q_t}$, according to the multiplication law

$$a_{p_1 p_2 \dots p_s} a_{q_1 \dots q_t} = a_{p_1 \dots p_s + q_1, q_2 \dots q_t} + a_{p_1 \dots p_s, q_1 \dots q_t}$$

with a similar law connecting the functions $h_{q_1 \dots q_t}$. Further he shows that

$$a_{p_1 \dots p_s} = h_{q_1 \dots q_t},$$

if $(p_1 \dots p_s)$ and $(q_1 \dots q_t)$ are zigzag conjugate compositions of n.

These functions are very useful, and the multiplication law is very simple, but they have the serious defect of being linearly dependent. For example

$$a_{211} + a_{22} = a_{121} + a_{31}.$$

Further, without actual evaluation, the complete set of linear relations connecting the functions of a given weight would be very difficult to obtain.

By comparing the multiplication law of MacMahon's functions with the multiplication law of the S-functions, remembering in particular that the product of the two S-functions is the sum of multiples of S-functions with positive integral coefficients, it is clear that each of MacMahon's functions is a linear function of S-functions with positive integral coefficients.

Compared with MacMahon's functions, the S-functions have only one disadvantage. The multiplication law is not quite so simple. To offset this there are numerous advantages. The S-functions are linearly independent, and this fact alone more than counterbalances the possible disadvantage of the more complicated multiplication rules.

Further, as we have shown, the connections between the S-functions and the products of the functions S_r may be read immediately from the table of characters of the symmetric group.

Lastly, the more fundamental nature of the S-functions is illustrated by the formula connecting these with the symmetric product sums.

Example.—Express $\Sigma\, \alpha^3\beta^3\gamma$ in terms of S-functions, S_r products, and in terms of the functions a_r.

6	5	4	3	2	1	0		
		3			3	1	6 5 7 3 2 4 1	
		3	1			3	6 5 7 4 2 1 3	3{1111111}
1			3			3	7 5 4 6 2 1 3	
	3			3	1		6 8 4 3 5 2 0	− 2{211111}
	3	1			3		6 8 5 3 2 4 0	
3			3	1			9 5 4 6 3 1 0	2{31111}
3	1			3			9 6 4 3 5 1 0	
1	3			3			7 8 4 3 5 1 0	− {22111}
	3	3	1				6 8 7 4 2 1 0	{2221}
3		3	1				9 5 7 4 2 1 0	− {3211}
1	3		3				7 8 4 6 2 1 0	{2221}
3	3	1					9 8 5 3 2 1 0	{331}
3	1	3					9 6 7 3 2 1 0	− {322}

Hence

$$\Sigma\, \alpha^3\beta^3\gamma = \{3^2 1\} - \{3, 2^2\} - \{3, 2, 1^2\} + 2\,\{2^3 1\} - \{2^2 1^3\} + 2\,\{3, 1^4\} - 2\,\{2, 1^5\} + 3\,\{1^7\}.$$

We have expressed the symmetric product sum in terms of S-functions. To express in terms of the S_r's, we read from the table of characters of the symmetric group of order 7 !.

	Class.	1^7	$1^5 2$	$1^4 3$	$1^3 4$	$1^2 5$	16	7	$1^3 2^2$	$1^2 2 3$	124	25	13^2	34	12^3	$2^2 3$
	Order.	1	21	70	210	504	840	720	105	420	630	504	280	420	105	210
1	$\{3^2 1\}$	21	1	−3	−1	1	0	0	1	1	−1	1	0	−1	−3	1
−1	$\{32^2\}$	21	−1	−3	1	1	0	0	1	−1	−1	−1	0	1	3	1
2	$\{2^3 1\}$	14	−4	−1	2	−1	0	0	2	−1	0	1	2	−1	0	1
−1	$\{321^2\}$	35	−5	−1	1	0	−1	0	−1	1	1	0	−1	1	−1	−1
−1	$\{2^2 1^3\}$	14	−6	2	0	−1	1	0	2	0	0	−1	−1	0	−2	2
2	$\{31^4\}$	15	−5	3	−1	0	0	1	−1	1	−1	0	0	−1	3	−1
−2	$\{21^5\}$	6	−4	3	−2	1	0	−1	2	−1	0	1	0	1	0	−1
3	$\{1^7\}$	1	−1	1	−1	1	−1	1	1	−1	1	−1	1	−1	−1	1
		0	0	0	0	0	−3	7	0	0	0	0	9	−12	0	0
		0	0	0	0	0	$-\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	−1	0	0

Add the numbers of the rows corresponding to the S-functions. We thus obtain the first row below the table. Multiply by h_ρ/h to obtain the second row below the table. Hence

$$\Sigma\, \alpha^3 \beta^3 \gamma = -\tfrac{1}{2} S_1 S_6 + S_7 + \tfrac{1}{2} S_1 S_3^2 - S_3 S_4.$$

Lastly,

$$\Sigma\, \alpha^3 \beta^3 \gamma = \{3^2 1\} - \{3, 2^2\} - \{3, 2, 1^2\} + 2\,\{2^3 1\} - \{2^2 1^3\}$$
$$+ 2\,\{3, 1^4\} - 2\,\{2\ 1^5\} + 3\,\{1^7\}$$

$$= \begin{vmatrix} a_3, & a_4, & a_5 \\ a_1, & a_2, & a_3 \\ a_0, & a_1, & a_2 \end{vmatrix} - \begin{vmatrix} a_3, & a_4, & a_5 \\ a_2, & a_3, & a_4 \\ & a_0, & a_1 \end{vmatrix} - \begin{vmatrix} a_4, & a_5, & a_6 \\ a_1, & a_2, & a_3 \\ & a_0, & a_1 \end{vmatrix}$$

$$+ 2 \begin{vmatrix} a_4, & a_5 \\ a_2, & a_3 \end{vmatrix} - \begin{vmatrix} a_5, & a_6 \\ a_1, & a_2 \end{vmatrix} + 2 \begin{vmatrix} a_5, & a_6, & a_7 \\ a_0, & a_1, & a_2 \\ & a_0, & a_1 \end{vmatrix}$$

$$- 2 \begin{vmatrix} a_6, & a_7 \\ a_0, & a_1 \end{vmatrix} + 3a_7$$

$$= a_3 a_2^2 - a_3^2 a_1 + a_4 a_3 - a_4 a_2 a_1 + a_5 a_1^2 - a_5 a_2 - a_3^2 a_1$$
$$+ a_4 a_3 + a_4 a_2 a_1 - a_5 a_2 - a_4 a_2 a_1 + a_4 a_3 + a_5 a_1^2 - a_6 a_1$$
$$+ 2a_4 a_3 - 2a_5 a_2 - a_5 a_2 + a_6 a_1 + 2a_5 a_1^2 - 2a_5 a_2 - 2a_6 a_1$$
$$+ 2a_7 - 2a_6 a_1 + 2a_7 + 3a_7$$
$$= a_3 a_2^2 - 2a_3^2 a_1 - a_4 a_2 a_1 + 5a_4 a_3 + 5a_4 a_1^2 - 7a_5 a_2 - 4a_6 a_1 + 7a_7$$

A similar procedure would express the function in terms of the h_r's.

§ 10. Construction of Tables of Characters of the Symmetric Groups.

Second Method.

By using the rules for the multiplication of S-functions, and a formula* connecting the characters of a group with those of a sub-group, the characters of a symmetric group can be constructed easily.

Let H be the symmetric group of order $h = n + 1\,!$, and G the symmetric group of order $g = n\,!$, of substitutions on the first n symbols.

The group G is a sub-group of H, which leaves one symbol α_{n+1}, unaltered. Denote any class by ρ, and let this class have h_ρ substitutions in H, and g_ρ in G.

Now consider the equation

$$(1)\quad \{\lambda_1, \lambda_2, \ldots, \lambda_p\}\{1\} = \{\lambda_1 + 1, \lambda_2, \ldots, \lambda_p\} + \{\lambda_1, \lambda_2 + 1, \ldots, \lambda_p\} + \ldots$$
$$\ldots + \{\lambda_1, \lambda_2, \ldots, \lambda_p + 1\} + \{\lambda_1, \lambda_2, \ldots, \lambda_p, 1\},$$

those terms on the right for which the numbers in the brackets are not in descending order being taken as zero ; *e.g.*,

$$\{4, 4\}\{1\} = \{5, 4\} + \{4, 4, 1\}$$

the term $\{4, 5\}$ being taken as zero.

$$(2)\qquad \{\lambda_1, \ldots, \lambda_p\}\,S_1 = \frac{1}{g}\sum_p \phi_\rho^{(\lambda)} g_\rho S_\rho,$$

$\phi_\rho^{(\lambda)}$ being a character of the group G, and S_ρ being that product of the S_r's which corresponds to the class ρ of the group H. Denote any of the terms on the right of (1) by

$$(3)\qquad \{\mu_1, \ldots, \mu_q\} = \frac{1}{h}\sum_p \chi_\rho^{(\mu)} h_\rho s_\rho,$$

$\chi_\rho^{(\mu)}$ being a character of H.

Hence

$$(4)\qquad \phi_\rho^{(\lambda)} = \frac{gh_\rho}{hg_\rho}\sum_\mu \chi_\rho^{(\mu)}.$$

We now use Frobenius' formula connecting the characters of a group with those of a sub-group, namely, in the simple case in which each class of the group corresponds to a single class only (of possible order zero) of the sub-group,

$$(5)\qquad \chi_\rho^{(i)} = \Sigma\, g_{ij}\phi_\rho^{(j)},$$

and

$$\phi_\rho^{(j)} = \sum_i \frac{gh_\rho}{hg_\rho} g_{ij}\chi_\rho^{(i)},$$

the positive integral coefficients g_{ij} being the same in both sets of equations (5).

Hence from (4)

$$(6)\qquad \chi_\rho^{(\mu)} = \Sigma\,\phi_\rho^{(\lambda)},$$

* Frobenius, 'SitzBer. Preuss. Akad. Wiss. Berl.,' p. 501 (1898).

the summation being taken over all those partitions (λ) of n, such that in the product $\{\lambda\}\ \{1\}$, $\{\mu\}$ appears as one of the terms.

Equation (6) enables us to construct the greater part of the table of characters of the symmetric group of order $n + 1$! from the table of characters of the group of order n !, namely, the characteristics* of all those classes of substitutions which leave one symbol unchanged.

For a class of H that has no substitution in G, equation (4) becomes

$$\sum_{\mu} \chi_{\rho}^{(\mu)} = 0. \tag{7}$$

This set of equations is not quite sufficient to determine the characteristics of the remaining classes, additional equations being required. These equations could be obtained by using examples of the multiplication rule other than (1), but in general the use of the orthogonal properties of the characters, together with (7), is sufficient to complete the table.

Example.—The table of characters of the symmetric group of order 5 ! is given on p. 138. We will construct the table of characters of the symmetric group of order 6 !. Using equation (6) we may construct the table of characters of the group so far as the classes with unitary substitutions are concerned. The classes (6), (3^2), 2^3) and (24) remain.

The characters corresponding to [6] and [1^6] are known. The self-conjugate character [321] must be zero for the negative classes. The characteristics of the class (6) are given by FROBENIUS,† these being alternately $+ 1$ and $- 1$ for the characters [6], [51], [41^2], [31^3], [21^4] and [1^6], and zero otherwise.

These results, together with equation (7), now allow us to fill in the table completely if we use five undetermined numbers, a, b, c, d, e.

The top half of the table for the last four classes becomes

—	—	6	3^2	2^3	24
[6]		1	1	1	1
[51]		-1	-1	-1	-1
[42]		0	a	b	c
[41^2]		1	$1 - a$	$1 - b$	$1 - c$
[3^2]		0	d	$- b$	e
[321]		0	$- a - \mathrm{d}$	0	$- c - e$

* We use the word "character" to represent the complete set of numbers which are the spurs of the matrices in a given matrix representation of the group. The value of the character for a given operation or class we call the "characteristic" of the operation or class. Hitherto, "character" has been used in both senses.

† 'SitzBer. Preuss. Akad. Wiss. Berl.,' p. 516 (1900); p. 303 (1901).

The lower half of the table is the same as the top half with the signs changed for the negative classes.

From the orthogonal properties of the characters,

$$\Sigma\, \chi_{\rho}^{(\lambda)}\chi_{\rho'}^{(\lambda)} = 0 \qquad (\rho \neq \rho')$$
$$= h/h_{\rho}. \quad (\rho = \rho').$$

For the classes (6) and (2^3) we obtain

$$1 + 1 + 1 - b = 0.$$

Hence

$$b = 3.$$

For the classes (15) and (3^2)

$$1 - a - a - d - a + 1 = 0.$$

Hence

$$d = 2 - 3a.$$

For the class (3^2)

$$1 + 1 + a^2 + 1 - 2a + a^2 + d^2 + (a + d)^2 + d^2 + 1 - 2a + a^2$$
$$+ a^2 + 1 + 1 = 6\,!/40 = 18$$
$$18 - 36a + 26a^2 = 18.$$

Hence $a = 0$ is the only integral root.

Lastly, for the classes (15) and (24)

$$1 - c - c - e - c + 1 = 0,$$

and for the classes (3^2)and (24)

$$1 + 1 + 1 - c + 2e + 2c + 2e + 2e + 1 - c + 1 + 1 = 0.$$

Hence

$$3c + e = 2$$
$$-6e = 6$$
$$e = -1$$
$$c = 1.$$

The table may now be completed.

Another fact that may be useful in completing the table, is the following. If two classes are similar, save that operations of the one class contain a cycle of order p, and the corresponding operations of the other class leaves the p symbols unaltered, then, p being prime, the characteristics of the two classes are congruent mod. p.

For example, the characteristics of the class (3^2) are congruent to the characteristics of the class ($1^3 3$), and these in turn are congruent to those of the class (1^6), mod. 3. In finding the characters of the symmetric group of order 8 !, for example, when these values of the characters for the first 15 classes have been found, as has been shown, there remain seven more classes of which one is the class (35). The characteristics of this class are congruent to those of the class $(1^3 5)$ mod. 3, and congruent to those of the class $(1^5 3)$ mod. 5. Hence we obtain the residues of the characteristics mod. 15, and this is sufficient to determine the values absolutely, since for this class

$$\Sigma \chi^{(\lambda)^2} = 15.$$

By this method the residues of the characteristics of the class (23^2) may be obtained mod. 6, and the absolute characteristics easily deduced.

The proof of this property depends on FROBENIUS' expression for the characters as coefficients in the product

$$S_1^{\alpha_1} S_2^{\alpha_2} \ldots \Delta (x_1, \ldots, x_n).$$

Since $S_p \equiv S_1^p$ (mod. p), the expressions for the two classes will be congruent, and hence the coefficients will be congruent, and these coefficients are the characteristics themselves.

§ 11. EVALUATION OF THE CHARACTERS OF SYMMETRIC GROUPS.

Third Method.

If (λ) represents the partition of n

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_i,$$

then YOUNG* has shown that $\chi_0^{(\lambda)}$ is the number of standard tableaux that can be formed with λ_1 symbols in the first row, λ_2 in the second row, etc. If any given order be assigned to the symbols, then a standard tableau is one in which the symbols in each row and each column follow the assigned order.

From the definition of MACMAHON,† it is clear that this is equivalent to saying that $\chi_0^{(\lambda)}$ is equal to the number of lattice permutations of

$$x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \ldots$$

We shall now give another proof of this result, deducing it from FROBENIUS' formula for the characters of the symmetric group,

$$S_1^{\alpha_1} S_2^{\alpha_2} S_3^{\alpha_3} \ldots \Delta (x_1, \ldots, x_n) = \Sigma \pm \chi_\rho^{(\lambda)} x_1^{\lambda_1+n-1} \ldots x_n^{\lambda_n}.$$

We shall then generalize the result for the other classes, and show a further connection between the characters and lattices, obtaining a third method of evaluating the characters of the symmetric groups.

From FROBENIUS' formula, $\chi_0^{(\lambda)}$ is the coefficient of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \ldots x_n^{\lambda_n}$ in the product

$$S_1^n \Delta (x_1, \ldots, x_n).$$

Now consider $\Delta (x_1, \ldots, x_n)$ and multiply the expression n times by S_1. At each step the coefficient of any term in which two indices are equal, is zero, for a simple interchange of two of the x's will change the sign, and yet leave the term unchanged.

* 'Proc. Lond. Math. Soc.,' vol. 31, p. 253 (1930).

† "Combinatory Analysis," p. 124 (1915).

Since at each step we multiply by a linear function of the x's, the order of magnitude of the indices can only be altered if at some step two indices are equal, and in this case the coefficient is zero.

Hence at each step we need only consider the coefficients of the terms

$$x_1^{\mu_1}x_2^{\mu_2}\ldots \times x_1^{n-1}x_2^{n-2}\ldots x_{n-1},$$

where $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots$.

We obtain the terms at the next step by multiplying these terms by $(x_1 + x_2 + \ldots + x_n)$, and considering in the product, only those terms which still satisfy

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots.$$

It is clear that each of the $\chi_0^{(\lambda)}$ terms $x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\ldots x_n^{\lambda_n}$ in the product

$$S_1^n \Delta(x_1, \ldots, x_n)$$

must be arrived at by this process, and for each of these terms we obtain a lattice permutation of

$$x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_n^{\lambda_n}.$$

Hence $\chi_0^{(\lambda)}$ is the number of lattice permutations of

$$x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_n^{\lambda_n}.$$

Now for a class containing a cycle of order r, S_1^r is replaced by S_r. It is clear that the order of magnitude of the indices may now be changed without passing through a stage in which two indices are equal. Consider the term

$$x_1^{\alpha_1}\ldots x_p^{\alpha_p}x_{p+1}^{\alpha_{p+1}}\ldots x_{p+i}^{\alpha_{p+i}}\ldots(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}) \qquad \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$

Multiplying by S_r we consider that term obtained by multiplying by x_{p+i}^r. If two indices are the same, the coefficient will be zero as before, hence we assume them all to be different. Suppose

$$\alpha_p + n - p > \alpha_{p+i} + n - p - i + r > \alpha_{p+1} + n - p - 1.$$

We obtain the term

$$x_1^{\alpha_1}\ldots x_p^{\alpha_p}x_{p+i}^{\alpha_{p+i}+r-i+1}x_{p+1}^{\alpha_{p+1}+1}\ldots x_{p+i-1}^{\alpha_{p+i-1}+1}x_{p+i+1}^{\alpha_{p+i+1}}\ldots$$
$$\times (x_1^{n-1}\ldots x_p^{n-p}x_{p+i}^{n-p-1}x_{p+1}^{n-p-2}\ldots x_{n-1}).$$

There will be a corresponding term in which the order of the suffixes of the x's are in natural order, a negative sign being attached if i is even. This rearranged term is the term we require.

This may be expressed in terms of lattices as follows.

Consider the product

$$\Delta(x_1, \ldots, x_n) S_{\alpha_1}S_{\alpha_2}S_{\alpha_3}\ldots$$

corresponding to a class with cycles of orders a_1, a_2, a_3, etc. Consider the term derived as has been shown above, taking from S_{a_1}, S_{a_2}, S_{a_3}, etc., the terms $x_{b_1}^{a_1}$, $x_{b_2}^{a_2}$, $x_{b_3}^{a_3}$, etc., respectively. We obtain a succession of regular graphs, according to MACMAHON'S definition,* by the following steps.

For the term $x_{b_1}^{a_1}$, place one node at the beginning of the b_1th row, one above that, etc., until we place a node at the beginning of the first row. Successive nodes are then placed in the first row until a_1 nodes have been placed.

For the next step a_2 nodes may be placed in the b_2th row, if this leaves the graph regular, *i.e.*, if the number of nodes in this row is not brought in excess of the number in the preceding row. Otherwise add nodes to the b_2th row until the number of nodes in this row exceeds by one the number of nodes in the preceding row. Then add nodes to the preceding row until the number exceeds by one the number in the preceding row, continuing thus until a_2 nodes have been placed. When the last node is added the graph must be regular.

This procedure is repeated for the succeeding steps. If at any step nodes are added to an even number of rows, a minus sign is attached.

For each graph that can be built in this way there is a corresponding term in

$$\Delta\,(x_1, \ldots, x_n)\, S_{a_1} S_{a_2} S_{a_3} \ldots .$$

and $\pm$ 1 is contributed to the corresponding character.

Example.—We obtain the values of all the characteristics for that class of the symmetric group of order 7 ! which has two cycles of order 3, one symbol being unaltered. We build graphs from the product $S_3 S_3 S_1$. Instead of nodes, to make the composition clear in one graph, we use a 1, 2 or 3, according as the node would be added at the first, second or third step. We obtain the following graphs :

$$\begin{array}{l} 1\ 1\ 1\ 2\ 2\ 2\ 3 \end{array} \qquad \begin{array}{l} 1\ 1\ 1\ 2\ 2\ 2 \\ 3 \end{array} \qquad \begin{array}{l} 1\ 1\ 1\ 3 \\ 2\ 2\ 2 \end{array} \qquad \begin{array}{l} 1\ 1\ 1 \\ 2\ 2\ 2 \\ 3 \end{array}$$

$$\begin{array}{l} \overline{1\ 1\ 1\ 3} \\ 2\ 2 \\ 2 \end{array} \qquad \begin{array}{l} \overline{1\ 1\ 1} \\ 2\ 2\ 3 \\ 2 \end{array} \qquad \begin{array}{l} \overline{1\ 1\ 1} \\ 2\ 2 \\ 2\ 3 \end{array} \qquad \begin{array}{l} \overline{1\ 1\ 1} \\ 2\ 2 \\ 2 \\ 3 \end{array}$$

$$\begin{array}{l} 1\ 1\ 1\ 3 \\ 2 \\ 2 \\ 2 \end{array} \qquad \begin{array}{l} 1\ 1\ 1 \\ 2\ 3 \\ 2 \\ 2 \end{array} \qquad \begin{array}{l} 1\ 1\ 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{array}$$

* *Op. cit.*, p. 124 (1915). A regular graph would be the same as a YOUNG tableau with nodes in the place of symbols.

$$\begin{array}{l}\overline{1\;1\;2\;2\;2\;3}\\ 1\end{array}\qquad\begin{array}{l}\overline{1\;1\;2\;2\;2}\\ 1\;3\end{array}\qquad\begin{array}{l}\overline{1\;1\;2\;2\;2}\\ 1\\ 3\end{array}$$

$$\begin{array}{l}1\;1\;2\;3\\ 1\;2\;2\end{array}\qquad\begin{array}{l}1\;1\;2\\ 1\;2\;2\\ 3\end{array}\qquad\begin{array}{l}1\;1\;3\\ 1\;2\\ 2\;2\end{array}\qquad\begin{array}{l}1\;1\\ 1\;2\\ 2\;2\\ 3\end{array}$$

$$\begin{array}{l}\overline{1\;1\;3}\\ 1\\ 2\\ 2\\ 2\end{array}\qquad\begin{array}{l}\overline{1\;1}\\ 1\;3\\ 2\\ 2\\ 2\end{array}\qquad\begin{array}{l}\overline{1\;1}\\ 1\\ 2\\ 2\\ 2\\ 3\end{array}\qquad\begin{array}{l}1\;2\;2\;2\;3\\ 1\\ 1\end{array}\qquad\begin{array}{l}1\;2\;2\;2\\ 1\;3\\ 1\end{array}\qquad\begin{array}{l}1\;2\;2\;2\\ 1\\ 1\\ 3\end{array}$$

$$\begin{array}{l}\overline{1\;2\;2\;3}\\ 1\;2\\ 1\end{array}\qquad\begin{array}{l}\overline{1\;2\;2}\\ 1\;2\;3\\ 1\end{array}\qquad\begin{array}{l}\overline{1\;2\;2}\\ 1\;2\\ 1\;3\end{array}\qquad\begin{array}{l}\overline{1\;2\;2}\\ 1\;2\\ 1\\ 3\end{array}$$

$$\begin{array}{l}1\;2\;3\\ 1\;2\\ 1\;2\end{array}\qquad\begin{array}{l}1\;2\\ 1\;2\\ 1\;2\\ 3\end{array}\qquad\begin{array}{l}1\;3\\ 1\\ 1\\ 2\\ 2\\ 2\end{array}\qquad\begin{array}{l}1\\ 1\\ 1\\ 2\\ 2\\ 2\\ 3\end{array}$$

A bar has been placed over those graphs which contribute -1.

It follows that for this class the values of the characters are

$$\chi_\rho^{(7)} = 1,\quad \chi_\rho^{(61)} = 0,\quad \chi_\rho^{(52)} = -1,\quad \chi_\rho^{(51^2)} = 0,\quad \chi_\rho^{(43)} = 2,$$

$$\chi_\rho^{(421)} = -1,\quad \chi_\rho^{(3^21)} = 0,\quad \chi_\rho^{(41^3)} = 2,\quad \chi_\rho^{(32^2)} = 0,\quad \chi_\rho^{(321^2)} = -1,$$

$$\chi_\rho^{(2^31)} = 2,\quad \chi_\rho^{(31^4)} = 0,\quad \chi_\rho^{(2^21^3)} = -1,\quad \chi_\rho^{(21^5)} = 0,\quad \chi_\rho^{(1^7)} = 1.$$

This method may be found useful if, in the computation of tables of characters by the second method, one of the residual classes presents exceptional difficulty. Again, if the characteristic of a given class is required, rather than the complete table of characters, the method is particularly useful.

For example, for the symmetric group of order 20 ! we shall find the characteristic

$\chi^{(16,2,2)}$ of the class of substitutions containing a cycle of order 15, two cycles of order 2, one symbol being unchanged. The possible graphs are

$$\begin{array}{llllllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\
2 & 2 \\
3 & 3
\end{array}$$

$$\begin{array}{llllllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\
2 & 3 \\
2 & 3
\end{array}$$

$$\begin{array}{llllllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 \\
1 & 3 \\
1 & 3
\end{array}$$

$$\begin{array}{llllllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 4 \\
1 & 2 \\
1 & 2
\end{array}$$

Hence for this class

$$\chi_\rho^{(16,2,2)} = 1 + 1 - 1 - 1 = 0.$$

§ 12. Relations between Immanants deduced from Relations between S-Functions.

We use the notation $(1^\alpha 2^\beta 3^\gamma \ldots)$ to denote the class ρ of products P_S corresponding to permutations composed of α cycles of period 1, β cycles of period 2, γ cycles of period 3, and so on. The number of elements in this class is evidently

$$\frac{n\,!}{1^\alpha 2^\beta 3^\gamma \ldots \alpha\,!\ \beta\,!\ \gamma\,! \ldots},$$

which is precisely the coefficient of the term $S_1{}^\alpha S_2{}^\beta S_3{}^\gamma \ldots$ in the corresponding S-function.

Hence, if in the development of an S-function

$$n\,!\ \{1^a 2^b 3^c \ldots\} = \Sigma\, \chi_\rho \frac{n\,!}{1^\alpha 2^\beta 3^\gamma \ldots \alpha\,!\ \beta\,!\ \gamma\,!} S_1{}^\alpha S_2{}^\beta S_3{}^\gamma \ldots,$$

we replace $S_1{}^\alpha S_2{}^\beta S_3{}^\gamma \ldots$ by $1^\alpha 2^\beta 3^\gamma \ldots \alpha\,!\ \beta\,!\ \gamma\,! \ldots (1^\alpha 2^\beta 3^\gamma \ldots)$ we get

$$n\,!\ \Sigma\, \chi_\rho\, (1^\alpha 2^\beta 3^\gamma \ldots)$$

which is equal to

$$n\,!\ [1^a 2^b 3^c \ldots],$$

the square bracket denoting the immanant.

Conversely, from the expansion of an immanant in classes we can deduce the expansion of an S-function in S_r's.

We shall now prove that in a relation between S-*functions we may replace the* S-*functions by the corresponding immanants if at the same time we replace the multiplication sign by a suitable sign of summation.*

It will be sufficient to prove the result true for the product of two S-functions. Let

$$n!\,\{1^{\alpha_1}2^{\beta_1}3^{\gamma_1}\ldots\} = \Sigma\,\chi_\rho\,\frac{n!\;S_1^{a_1}S_2^{b_1}S_3^{c_1}\ldots}{1^{a_1}2^{b_1}3^{c_1}\ldots a_1!\;b_1!\;c_1!\ldots}$$

$$m!\,\{1^{\alpha_2}2^{\beta_2}3^{\gamma_2}\ldots\} = \Sigma\,\phi_{\rho'}\,\frac{m!\;S_1^{a_2}S_2^{b_2}S_3^{c_2}\ldots}{1^{a_2}2^{b_2}3^{c_2}\ldots a_2!\;b_2!\;c_2!\ldots},$$

thus

$$\{1^{\alpha_1}2^{\beta_1}3^{\gamma_1}\ldots\}\,\{1^{\alpha_2}2^{\beta_2}3^{\gamma_2}\ldots\}$$
$$= \Sigma\,\chi_\rho\phi_{\rho'}\,\frac{S_1^{a_1+a_2}S_2^{b_1+b_2}S_3^{c_1+c_2}\ldots}{1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots a_1!\;a_2!\;b_1!\;b_2!\;c_1!\;c_2!}.$$

By the previous work, if in this we replace

$$S_1^{a_1+a_2}S_2^{b_1+b_2}S_3^{c_1+c_2}\ldots$$

by

$$1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots(a_1+a_2)!\;(b_1+b_2)!\ldots(1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots),$$

then the result will be the same as if in the expression of the right-hand side as a sum of S-functions of degree $(n+m)$ we had replaced the S-functions by the corresponding immanants.

The result is

$$\Sigma\,\chi_\rho\phi_{\rho'}\,\frac{(a_1+a_2)!\;(b_1+b_2)!\;(c_1+c_2)!\ldots}{a_1!\;a_2!\;b_1!\;b_2!\;c_1!\;c_2!\ldots}\,(1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots). \quad . \quad . \quad . \quad . \qquad (2)$$

Next consider

$$\Sigma\,[1^{\alpha_1}2^{\beta_1}3^{\gamma_1}\ldots]\,[1^{\alpha_2}2^{\beta_2}3^{\gamma_2}\ldots]$$

where the Σ means that the immanants are taken from complementary coaxial principal minors of the matrix of degree $(n+m)$, of degrees n and m respectively and the sum taken. The number of elements of the form $(1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots)$ arising from sums of products of elements in $(1^{a_1}2^{b_1}3^{c_1}\ldots)\,(1^{a_2}2^{b_2}3^{c_2}\ldots)$ is

$$\frac{(n+m)!}{n!\;m!}\,\frac{n!\;m!}{1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots a_1!\;a_2!\;b_1!\;b_2!\;c_1!\;c_2!\ldots}.$$

Now every element of the class $(1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots)$ appears equally often in this product and it contains

$$\frac{(n+m)!}{1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots(a_1+a_2)!\;(b_1+b_2)!\;(c_1+c_2)!\ldots}$$

terms. It follows that the class $(1^{a_1+a_2}2^{b_1+b_2}3^{c_1+c_2}\ldots)$ occurs exactly

$$\frac{(a_1+a_2)!\,(b_1+b_2)!\,(c_1+c_2)!\ldots}{a_1!\,a_2!\,b_1!\,b_2!\,c_1!\,c_2!\ldots}\text{ times.}$$

Hence, from (2) we see that the result of the substitution on the left-hand side is to replace the product of S-functions

$$\{1^{\alpha_1}2^{\beta_1}3^{\gamma_1}\ldots\}\,\{1^{\alpha_2}2^{\beta_2}3^{\gamma_2}\ldots\}$$

by

$$\Sigma\,[1^{\alpha_1}2^{\beta_1}3^{\gamma_1}\ldots]\,[1^{\alpha_2}2^{\beta_2}3^{\gamma_2}\ldots]$$

summed over all possible $(n+m)!/n!\,m!$ complementary sets of principal coaxial minors of degrees n and m in the matrix of degree $(n+m)$; and on the right-hand side to replace a linear combination of S-functions by the same linear combination of the corresponding immanants.

Thus from

$$\{3\}\,\{1^2\} = \{41\} + \{31^2\},$$

follows

$$\Sigma\,[3]\,[1^2] = [41] + [31^2],$$

the summation being taken over the complementary sets of coaxial minors of degrees 3 and 2 in the matrix of degree 5.

Some particular cases of this theorem are known* *e.g.*, from the relation

$$a_4 - h_1a_3 + h_2a_2 - h_3a_1 + h_4 = 0$$

we immediately deduce the relation between the corresponding determinants and permanents, viz.:†

$$\begin{pmatrix}\alpha\,\beta\,\gamma\,\delta\\ \alpha\,\beta\,\gamma\,\delta\end{pmatrix} - \Sigma\begin{pmatrix}\alpha\,\beta\,\gamma\\ \alpha\,\beta\,\gamma\end{pmatrix}\overset{+\;\;+}{\begin{pmatrix}\delta\\ \delta\end{pmatrix}} + \Sigma\begin{pmatrix}\alpha\,\beta\\ \alpha\,\beta\end{pmatrix}\overset{+\;\;\;+}{\begin{pmatrix}\gamma\,\delta\\ \gamma\,\delta\end{pmatrix}} - \Sigma\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\overset{+\;\;\;\;\;+}{\begin{pmatrix}\beta\,\gamma\,\delta\\ \beta\,\gamma\,\delta\end{pmatrix}} + \overset{+\;\;\;\;\;\;\;+}{\begin{pmatrix}\alpha\,\beta\,\gamma\,\delta\\ \alpha\,\beta\,\gamma\,\delta\end{pmatrix}} = 0.$$

§ 13. Application to Computation of Characters.

A term such as $S_1^{\alpha}S_2^{\beta}S_3^{\gamma}\ldots$ is expressible as a linear function of S-functions, the coefficients being the characteristics of the class $(1^{\alpha}2^{\beta}3^{\gamma}\ldots)$. These may be calculated very simply if we express $S_1^{\alpha}S_2^{\beta}S_3^{\gamma}\ldots$ as the product of two similar functions, each of which can be expressed as a linear function of S-functions of lower degree. The use of the multiplication theorem then gives the required characters. The simplest case is that in which we write the product as

$$S_1\,.\,S_1^{\alpha-1}S_2^{\beta}S_3^{\gamma}\ldots$$

* MacMahon, 'J. Lond. Math. Soc.,' p. 273 (1922).

† Muir, "Theory of Determinants," vol. 4, p. 459 (1923).

This is in effect the method on p. 127; it enables us to compute the characteristics of any class containing a unitary substitution very readily.

For example, we can compute the characteristics of $(1^2 3)$.

$$\begin{aligned} S_1^2 S_3 &= S_1 \cdot S_1 S_3 \\ &= \{1\} [\{4\} - \{2^2\} + \{1^4\}] \\ &= \{5\} + \{41\} - \{32\} - \{2^2 1\} + \{21^3\} + \{1^5\}, \end{aligned}$$

and the characteristics are therefore

$$1, \quad 1, \quad -1, \quad 0, \quad -1, \quad 1, \quad 1.$$

(See Table on p. 128.)

In this way the characteristics of any class can easily be obtained.

Similar considerations enable us to deduce a number of identities between immanants, including the expression of an immanant as a polynomial in determinants.

§ 14. Conclusion.

The methods of this paper may be applied to many other problems. For example, the structure of a group and many of the properties of its sub-groups may be read off from the character tables. Also, although we have restricted this investigation to the symmetric group, there is a corresponding theory for any group leading to a development of the theory having applications to the theory of equations. Further, the immanants of special matrices can be used to give results relating to the numbers of Bernoulli, Stirling and Euler.

Finally, we would point out that the rational fractions used by Padé and Hadamard have coefficients which can be expressed in terms of S-functions, which suggests that the study of the essential singularities of functions can be carried out by means of rational fractions corresponding to the self-conjugate partitions.

§ 15. Tables of Characters of the Symmetric Groups.

Degree 2.

Class . .	1^2	2
Order . .	1	1
[2]	1	1
[1^2]	1	−1

Degree 3.

Class . .	1^3	12	3
Order . .	1	3	2
[3]	1	1	1
*[21]	2	0	−1
[1^3]	1	−1	1

* Denotes a self-associated partition and character.

Degree 4.

Class	1^4	1^22	13	4	2^2
Order	1	6	8	6	3
[4]	1	1	1	1	1
[31]	3	1	0	−1	−1
*$[2^2]$	2	0	−1	0	2
$[21^2]$	3	−1	0	1	−1
$[1^4]$	1	−1	1	−1	1

Degree 5.

Class	1^5	1^32	1^23	14	12^2	23	5
Order	1	10	20	30	15	20	24
[5]	1	1	1	1	1	1	1
[41]	4	2	1	0	0	−1	−1
[32]	5	1	−1	−1	1	1	0
*$[31^2]$	6	0	0	0	−2	0	1
$[2^21]$	5	−1	−1	1	1	−1	0
$[21^3]$	4	−2	1	0	0	1	−1
$[1^5]$	1	−1	1	−1	1	−1	1

Degree 6.

Class . .	1^6	1^42	1^33	1^24	1^22^2	123	15	6	24	2^3	3^2
Order . .	1	15	40	90	45	120	144	120	90	15	40
[6]	1	1	1	1	1	1	1	1	1	1	1
[51]	5	3	2	1	1	0	0	−1	−1	−1	−1
[42]	9	3	0	−1	1	0	−1	0	1	3	0
$[41^2]$	10	2	1	0	−2	−1	0	1	0	−2	1
$[3^2]$	5	1	−1	−1	1	1	0	0	−1	−3	2
*[321]	16	0	−2	0	0	0	1	0	0	0	−2
$[2^3]$	5	−1	−1	1	1	−1	0	0	−1	3	2
$[31^3]$	10	−2	1	0	−2	1	0	−1	0	2	1
$[2^21^2]$	9	−3	0	1	1	0	−1	0	1	−3	0
$[21^4]$	5	−3	2	−1	1	0	0	1	−1	1	−1
$[1^6]$	1	−1	1	−1	1	−1	1	−1	1	−1	1

Degree 7.

Class . .	1^7	1^52	1^43	1^34	1^32^2	1^223	1^25	16	124	12^3	13^2	25	2^23	34	7
Order . .	1	21	70	210	105	420	504	840	630	105	280	504	210	420	720
[7]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[61]	6	4	3	2	2	1	1	0	0	0	0	−1	−1	−1	−1
[52]	14	6	2	0	2	0	−1	−1	0	2	−1	1	2	0	0
$[51^2]$	15	5	3	1	−1	−1	0	0	−1	−3	0	0	−1	1	1
[43]	14	4	−1	−2	2	1	−1	0	0	0	2	−1	−1	1	0
[421]	35	5	−1	−1	−1	−1	0	1	1	1	−1	0	−1	−1	0
$[3^21]$	21	1	−3	−1	1	1	1	0	−1	−3	0	1	1	−1	0
*$[41^3]$	20	0	2	0	−4	0	0	0	0	0	2	0	2	0	−1
$[32^2]$	21	−1	−3	1	1	−1	1	0	−1	3	0	−1	1	1	0
$[321^2]$	35	−5	−1	1	−1	1	0	−1	1	−1	−1	0	−1	1	0
$[2^31]$	14	−4	−1	2	2	−1	−1	0	0	0	2	1	−1	−1	0
$[31^4]$	15	−5	3	−1	−1	1	0	0	−1	3	0	0	−1	−1	1
$[2^21^3]$	14	−6	2	0	2	0	−1	1	0	−2	−1	−1	2	0	0
$[21^5]$	6	−4	3	−2	2	−1	1	0	0	0	0	1	−1	1	−1
$[1^7]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

*Denotes a self-associated partition and character.

Degree 8.

Class . . .	1^8	1^62	1^53	1^44	1^42^2	1^323	1^35	1^26	1^224	1^22^3	1^23^2	125	12^23	134	17	8	4^2	2^24	26	23^2	35	2^4
Order . . .	1	28	112	420	210	1120	1344	3360	2520	420	1120	4032	1680	3360	5760	5040	1260	1260	3360	1120	2688	105
[8]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[71]	7	5	4	3	3	2	2	1	1	1	1	0	0	0	0	−1	−1	−1	−1	−1	−1	−1
[62]	20	10	5	2	4	1	0	−1	0	2	−1	0	1	−1	−1	0	0	2	1	1	0	4
[61^2]	21	9	6	3	1	0	1	0	−1	−3	0	−1	−2	0	0	1	1	−1	0	0	1	−3
[53]	28	10	1	−2	4	1	−2	−1	0	2	1	0	1	1	0	0	0	−2	−1	1	1	−4
[521]	64	16	4	0	0	−2	−1	0	0	0	−2	1	0	0	1	0	0	0	0	−2	−1	0
[51^3]	35	5	5	1	−5	−1	0	0	−1	−3	2	0	1	1	0	−1	−1	1	0	2	0	3
[4^2]	14	4	−1	−2	2	1	−1	0	0	0	2	−1	−1	1	0	0	2	2	0	−2	−1	6
[431]	70	10	−5	−4	2	1	0	1	0	−2	1	0	−1	−1	0	0	−2	0	1	1	0	−2
[42^2]	56	4	−4	0	0	−2	1	1	0	4	−1	−1	0	0	0	0	0	0	−1	1	1	8
*[421^2]	90	0	0	0	−6	0	0	0	2	0	0	0	0	0	−1	0	2	0	0	0	0	−6
*[3^22]	42	0	−6	0	2	0	2	0	−2	0	0	0	2	0	0	0	2	0	0	0	−1	−6
[3^21^2]	56	−4	−4	0	0	2	1	−1	0	−4	−1	1	0	0	0	0	0	0	−1	−1	1	8
[32^21]	70	−10	−5	4	2	−1	0	−1	0	2	1	0	−1	1	0	0	−2	0	1	−1	0	−2
[2^4]	14	− 4	−1	2	2	−1	−1	0	0	0	2	1	−1	−1	0	0	2	−2	0	2	−1	6
[41^4]	35	− 5	5	−1	−5	1	0	0	−1	3	2	0	1	−1	0	1	−1	−1	0	−2	0	3
[321^3]	64	−16	4	0	0	2	−1	0	0	0	−2	−1	0	0	1	0	0	0	0	2	−1	0
[2^31^2]	28	−10	1	2	4	−1	−2	1	0	−2	1	0	1	−1	0	0	0	2	−1	−1	1	−4
[31^5]	21	−9	6	−3	1	0	1	0	−1	3	0	1	−2	0	0	−1	1	1	0	0	1	−3
[2^21^4]	20	−10	5	−2	4	−1	0	1	0	−2	−1	0	1	1	−1	0	0	−2	1	−1	0	4
[21^6]	7	−5	4	−3	3	−2	2	−1	1	−1	1	0	0	0	0	1	−1	1	−1	1	−1	−1
[1^8]	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	1

* Denotes a self-associated partition and character.

Degree 9.

Class	1^9	1^72	1^63	1^54	1^52^2	1^423	1^45	1^36	1^324	1^32^3	1^33^2	1^225	1^22^23	1^234	1^27	18	14^2	12^24	126	123^2	135	12^4	45	3^3	36	234	27	2^25	2^33	9
Order	1	36	168	756	378	2520	3024	10080	7560	1260	3360	18144	7560	15120	25920	45360	11340	11340	30240	10080	24192	945	18144	2240	20160	15120	25920	9072	2520	40320
[9]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[81]	8	6	5	4	4	3	3	2	2	2	2	1	1	1	1	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1
[72]	27	15	9	5	7	3	2	0	1	3	0	0	1	−1	−1	−1	−1	1	0	0	−1	3	0	0	0	1	1	2	3	0
[71^2]	28	14	10	6	4	2	3	1	0	−2	1	−1	−2	0	0	0	0	−2	−1	−1	0	−4	1	1	1	0	0	−1	−2	1
[63]	48	20	6	0	8	2	−2	−2	0	4	0	0	2	0	−1	0	0	0	0	2	1	0	0	3	1	0	−1	−2	−2	0
[621]	105	35	15	5	5	−1	0	−1	−1	−1	−3	0	−1	−1	0	1	1	1	1	−1	0	1	0	−3	−1	−1	0	0	−1	0
[61^3]	56	14	11	4	−4	−1	1	0	−2	−6	2	−1	−1	1	0	0	0	0	0	2	1	0	−1	2	0	1	0	1	3	−1
[54]	42	14	0	−4	6	2	−3	−1	0	2	3	−1	0	2	0	0	2	0	−1	−1	0	2	1	−3	−1	0	0	1	2	0
[531]	162	36	0	−6	6	0	−3	0	0	0	0	1	0	0	1	0	−2	−2	0	0	0	−6	−1	0	0	0	1	1	0	0
[52^2]	120	20	0	0	0	−4	0	1	0	4	−3	0	0	0	1	0	0	0	−1	−1	0	8	0	3	1	0	−1	0	4	0
[521^2]	189	21	9	1	−11	−3	−1	0	1	−3	0	1	1	1	0	−1	1	1	0	0	−1	−3	1	0	0	1	0	−1	−3	0
[4^21]	84	14	−6	−6	4	2	−1	1	0	−2	3	−1	−2	0	0	0	0	2	1	−1	−1	4	−1	3	1	0	0	−1	−2	0
[432]	168	14	−15	−4	4	−1	3	2	−2	2	0	−1	1	−1	0	0	0	0	0	2	0	0	1	−3	−1	1	0	−1	−1	0
[431^2]	216	6	−9	−4	−4	3	1	0	2	−6	0	1	−1	−1	−1	0	0	0	0	0	1	0	1	0	0	−1	−1	1	3	0
*[51^4]	70	0	10	0	−10	0	0	0	−2	0	4	0	2	0	0	0	−2	0	0	0	0	6	0	−2	0	−2	0	0	0	1
*[3^3]	42	0	−6	0	2	0	2	0	−2	0	0	0	2	0	0	0	2	0	0	0	−1	−6	0	6	0	−2	0	2	0	0
[42^21]	216	−6	−9	4	−4	−3	1	0	2	6	0	−1	−1	1	−1	0	0	0	0	0	1	0	−1	0	0	−1	1	1	−3	0
[3^221]	168	−14	−15	4	4	1	3	−2	−2	−2	0	1	1	1	0	0	0	0	0	−2	0	0	−1	−3	1	1	0	−1	1	0
[32^3]	84	−14	−6	6	4	−2	−1	−1	0	2	3	1	−2	0	0	0	0	−2	1	1	−1	4	1	3	−1	0	0	−1	2	0
[421^3]	189	−21	9	−1	−11	3	−1	0	1	3	0	−1	1	−1	0	1	1	−1	0	0	−1	−3	−1	0	0	1	0	−1	3	0
[3^21^3]	120	−20	0	0	0	4	0	−1	0	−4	−3	0	0	0	1	0	0	0	−1	1	0	8	0	3	−1	0	1	0	−4	0
[32^21^2]	162	−36	0	6	6	0	−3	0	0	0	0	−1	0	0	1	0	−2	2	0	0	0	−6	1	0	0	0	−1	1	0	0
[2^41]	42	−14	0	4	6	−2	−3	1	0	−2	3	1	0	−2	0	0	2	0	−1	1	0	2	−1	−3	1	0	0	1	−2	0
[41^5]	56	−14	11	−4	−4	1	1	0	−2	6	2	1	−1	−1	0	0	0	0	0	−2	1	0	1	2	0	1	0	1	−3	−1
[321^4]	105	−35	15	−5	5	1	0	1	−1	1	−3	0	−1	1	0	−1	1	−1	1	1	0	1	0	−3	1	−1	0	0	1	0
[2^31^3]	48	−20	6	0	8	−2	−2	2	0	−4	0	0	2	0	−1	0	0	0	0	−2	1	0	0	3	−1	0	1	−2	2	0
[31^6]	28	−14	10	−6	4	−2	3	−1	0	2	1	1	−2	0	0	0	0	2	−1	1	0	−4	−1	1	−1	0	0	−1	2	1
[2^21^5]	27	−15	9	−5	7	−3	2	0	1	−3	0	0	1	1	−1	1	−1	−1	0	0	−1	3	0	0	0	1	−1	2	−3	0
[21^7]	8	−6	5	−4	4	−3	3	−2	2	−2	2	−1	1	−1	1	0	0	0	0	0	0	0	1	−1	1	−1	1	−1	1	−1
[1^9]	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

§ 16 *Summary.*

MacMahon has pointed out the close analogy which exists between the threefold algebra of the symmetric functions a_r, h_r and S_r and the theory of determinants, permanents and cycles of substitutions of the symmetric group. In this paper we trace the analogy to its source by fixing attention on the characters of the irreducible representations of the symmetric group, as the centre of the whole theory. By this means divers theories of combinatory analysis and algebra are seen to be merely different aspects of the same theory. For the symmetric group of order $n\,!$ the characters are all integers, and we associate with each partition of n both a character of the group and a cycle of substitutions, and new functions termed immanants, are defined corresponding to each partition of n. Of these the permanent of a matrix of degree n corresponds to the partition (n) whilst the determinant corresponds to the partition (1^n), the other immanants filling the gap between. There is thus a theory of immanants which includes that of determinants as a very special case.

In this paper immanants are used to develop simple methods which suffice for the calculation of characters of symmetric groups of orders up to, say, 20 !. Incidentally, it is shown that most of the existing tables of symmetric functions may be replaced by the tables of characters. In particular those of Kostka are shown to be merely simple linear combinations of the characters. Tables of characters of symmetric groups of orders up to 9 ! are calculated.