Random Panel Partitions in Surf Competitions: Testing Nationality Bias using Exchangeability

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Contents

Abstract

We have collected the most complete data set on international surf competition publicly available. In this paper, we focus on the 2017, 2018, and 2019 seasons of the Men's Championship Tour because it offers the fullest set of waves that list judge scores along with their nationality. This study uses 15,990 waves surfed in 2017, 2018, and 2019 seasons to empirically analyze the role of judge nationalities in the Men's Championship Tour.

We present an empirical method for analyzing arrangement data in order to test for Nationality bias in International Surf Competitions. We present a test for exchangeability an empirical method for analyzing arrangement data in order to test for Nationality bias in International Surf Competitions.

1 Introduction

Editing Notes: I don't know where this should go. A wide range of non-performance based biases are documented and researched using various methods; for example, Price and Wolfers study own-race bias in NBA refereeing using [ ], [ ] and [ ] investigate nationality bias in synchronized swimming judging using [ ], and ordering bias of gymnastics judges. These are only a few examples in sports, but the social-cognitive underpinnings of biases are a key area of research in psychology and legal studies.

2 Motivation

Our goal in this paper is to construct a method to determine if judges have a tendency to give higher scores to surfers from their home countries compared to judges with nationalities different from the surfer. To achieve this we will identify the empirical distribution of arrangements of judges and test if this distribution is significantly different from its symmetric part.

Table 1: Orderability and Multiplicities in the Data

	Totally Ordered	Not Totally Ordered	Row Sum
5 Distinct Countries	248 (0.0179)	2830 (0.2040)	3078 (0.2219)
5 Non-Distinct Countries	891 (0.0642)	9903 (0.7139)	10794 (0.7781)
Col Sum	1139 (0.0821)	12733 (0.9179)	13872 (1.0000)

Our focus will be on arrangements or partial arrangements of judges, which we will analyze "by match" and "by country". An arrangement is simply a sequence and a partial arrangement is a sequence of arrays. We define an array as an unordered collection of things (in our case, country labels), and any thing may occur with any multiplicity in an array.

This algebraic structure of a partial arrangement was motivated by high proportion of panels with multiple judges from the same country and the high proportion of partial arrangements. More broadly, the literature on ordered statistics and rankings is focused on the case where each data point consists of a full or partial order on a set of distinct labels ([**?**],[**?**]).

Our ideas were inspired by those of Lebanon and Mao in [**?**] and Moyal in [**?**]. We have also made extensive use of the ideas in Persi Diaconis' book, Group Representations in Probability and Statistics, and direct the interested reader there [**?**]. Lebanon and Mao, and Diaconis provide a comprehensive approach to data which falls into the top row of table 1.

The surfing data evidently requires an approach that is suited to handle labels occurring with multiplicity (row 2), and particularly the case where these labels are not totally ordered (row 2, column 2). To address these needs, we will present two approaches. The first will treat an ordered partition of a judges as a vector-partition of the integer vector whose coordinates correspond to the number of judges from each country. This will omit some structure, leaving hypothesis testing procedures less enlightening. The second, is inspired by Moyal's approach to point processes when a population consists of distinguishable or indistinguishable members. We provide a similar, more explicit, algebraic construction which will account for multiplicities arising from unordered observations. In doing so, we will accommodate data with multiplicities and find remaining parts of the table to be important special cases.

More broadly, the methodology and associated computation presented is applicable to any situation in which the unit of analysis is a sequence of arrays of labels. With the surf-competition data, we consider the order statistic of a panel, which is a sequence where the ith entry is an array containing the nationalities of the judges that gave the ith highest score.

This means that order statistic is of a random length, depending on the number of distinct scores given my the judges. In applied settings we cannot expect labels to occur without multiplicity or be totally ordered or be of uniform length.

3 Background: Professional Surfing

3.1 Overview

The WSL Championship Tour (CT) is a series of events in which surfers are allocated points based on their placement in each of the constituent events and the surfer with the most points at the end of the season is crowned the WSL CT champion. While event formats and point allocation mechanisms have changed, surfers that advance though more rounds of an event are allocated more points. In addition to Championship Tour (CT), the WSL operates multiple Qualifying Series (QS): QS10000, QS6000, QS3000, QS1500, and QS1000. The number associated with a certain qualifying series indicates the number of points awarded to the winner of one of its constituent events. The specific breakdown of points awarded for various placement at QS events can be found in Appendix B of the WSL Rule Book.

3.2 Format of Men's Championship Tour

Each year, the 32 highest ranked (short board) surfers are invited to participate in the "Championship Tour" (CT), which consists of 10 or 11 surf competitions in 7 different countries. Each competition has 7 or 8 rounds, consisting of 1 to 16 heats, and each heat has 2 to 3 surfers and is 22 to 35 minutes long. Within a heat, a surfer may attempt to ride any number of waves, but their final heat score is the sum of their two highest scoring waves. The surfer with the highest heat score places 1st in the heat, the surfer with the next highest heat score places 2nd in the heat, and if there is another surfer, they place third.

4 Data

We collected data on every wave surfed in the 2017, 2018, and 2019 seasons of the Men's Championship Tour (CT). During a heat, a surfer may attempt to ride any number of waves. Anytime a surfer rides a wave, they receive a non-zero score determined by the scores of a judging panel. There are 5 judges on a panel for any given heat. They are visually separated and do not discuss scores. When a surfer takes a wave, they each observe the ride and write down a score which is some number between 0.1 and 10.0 and it is precise up to the tenths place. The highest and lowest scores given by the panel are dropped, and the surfer receives the mean of the 3 remaining scores for their ride, this is called the "wave score". These are rounded to 2 digits. At any given point in a heat, a surfer has a "heat score", which is the sum of their two highest scores, and simply the sum of their scores if they have surfed 0, 1, or 2 waves. When the time allocated for a heat elapses, a horn will sound, and the surfer with the highest heat score is awarded first place, the surfer with the second highest score is awarded second, and so on.

Table 2: Completeness of Judging Observations Data

Year	Waves	0 Origins	5 Origins	Total Judge Origins	3 Scores	5 Scores	Total Judge Scores
2017	7328	5210	2118	10590	299	7029	36042
2018	6639	336	6303	31515	0	6639	33195
2019	7648	79	7569	37845	0	7648	38240
Total	21615	5625	15990	79950	299	21316	107477

4.1 Data Quality

4.2 Notions of "Origin", "Country", or "Nationality"

When collecting nationality associated with a particular athlete or a judge's score, we observed a set of labels. These were not limited to the following set of countries: AUS,BRA, ESP, FRA, PRT, USA, ZAF, FJI, IDN, ITA, JPN, NZL The original fields on the world surf league website also included Hawaii, French Polynesia, and Basque Country. None of these are globally recognized as independent nations. Hawaii is a state contained in the United States, French Polynesia is collective of France, and Basque Country is autonomous community in Spain. There is a less strict view of nationality, and one's particular definition may render all, some, or none of the 3 a nationality. "Origin" seems to be a more inclusive term to describe where someone comes from, or how they identify that for themselves.

The exact motivation for these non-Country labels is unclear. The rulebook states that no more than BLAH events should be held in one region and no more than 3 judges from any one country should be on the panel at once and it says ... " ". Regardless, we will present results for two sets of labels: C = AUS,BRA, ESP, FRA, PRT, USA, ZAF, FJI, IDN, ITA, JPN, NZL R = AUS, BAS, BRA, ESP, FRA, HAW, PRT, USA, ZAF, IDN, ITA, JPN, NZL Methods presented do not require you have notion or another, merely that one is consistent in this belief. Conclusions may differ depending on ones preferred definition. We encourage the reader to select one of these as their forgoing definition and observe how this choice may lead to different conclusions.

Much of the 2017 panel data includes judge scores and omits some of the judge nationalities. This is not a result of the web scraping process, rather the information available on the WSL site. A '-1' in a judge nationality field indicates it was not available and is unknown (to us). Precisely, '-1' should be interpreted as "any country" and is not restricted to the countries observed.

4.3 Intricacies of the Data Generating Process

The motivation for this publication is not only to present some analytical statistical methods to panel data but also to describe the intricacies of (this) surfing data. Explicitly, we have focused our efforts on one specific part of this data, the panel. There are many other aspects of this data that make it interesting from a statistical perspective.

4.3.1 On the Ordering of Waves

Each wave has a unique identifier, called a "Wave ID". These were valuable when collecting information and aggregating across procedures. Since Wave IDs are used in other WSL events, they are not linearly ordered (in our sample) nor do they increase in constant multiple. The IDs are strictly increasing from heat to heat, I.e. if waveid1 ¡ waveid2 then the wave given by waveid1 is in the same heat or prior heat to that of waveid2.

Within an a particular heat, Wave IDs do not exactly determine the order in which the waves were taken. However,the order determined by the Wave IDs could be considered as a close approximation. Unfortunately, we cannot provide a quantitative estimate for how close this is to the true ordering of waves (It is interesting to consider what form a quantitative estimate would take). On the other hand, taking waves within a heat to be exchangeable may be a poor reflection of the heat dynamics. It is a regular practice of the vibrant surf commentators to replay waves from earlier in the heat in order to speculate if a recent wave, that has yet to be scored, will receive a score higher or lower than a prior wave. That is, they assume judges are roughly consistent with their scoring, so an inferior ride receives a lower scores and a superior ride receives a higher score (with some small degree of error). This is an assumption of theirs which is up to personal interpretation. Another heat level dynamic noted by the commentators is that of "score taming" for rides early in a heat. The idea is that judges will resist giving especially high scores or 10s earlier on in a heat in case there is another ride later on that should be awarded a sufficient point premium over prior scores.

4.3.2 On Priority

4.3.3 On Interference Calls

4.3.4 On Judge Scores

After the long process of scraping, cleaning, and validating the data, we set out to perform some basic analyses.

5 Preliminary Analysis

Our goal is to determine if judges have a tendency to give higher scores to surfers that share their same nationality.

Throughout the remainder of this paper, we have used the 15990 waves that provide every judge's score and country. The wave in our sample with the smallest Wave Id ("1015077") is from Round 7 Heat 5 of the 2017 Trestles Event when Jadson Andre, a Brazilian surfer, is awarded a 5.83, from 5 judges

Table 3: Difference In Means t-test by Country (CC=True)

Country	AUS	BRA	FRA	PRT	USA	ZAF
N	3250	5234	769	203	2269	257
$\hat{\mu}$	0.015	0.009	0.043	0.022	0.034	0.044
t-val	2.703	2.302	3.632	0.854	5.082	2.072

who gave the scores, 5.5, 5.5, 6.0, 6.0, 6.5, and are from USA, AUS, USA, USA, BRA, respectively.

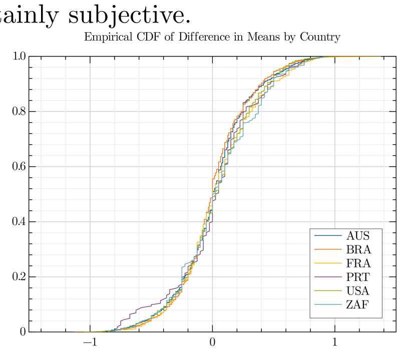
The simple approach is to test the differences in means between judges with the same nationality as the surfer and those with a different nationality. So we perform a hypothesis test where our hypotheses are:

$$H_0 : \mu_M - \mu_{\neg M} = 0 \quad H_1 : \mu_M - \mu_{\neg M} \neq 0$$

And the test statistic is $t = \frac{\bar{x}}{s/\sqrt{N}}$. Note that we are performing a one sample t-test on a series of data $(x_i)_{i=1}^{N}$ where $x_i := \mu_{M,i} - \mu_{\neg M,i}$. Not every panel is included in these tests because some panels have 0 Judges from the same country as the surfer riding the wave. In those cases the difference in means between Matching judges and Non-Matching judges would be undefined. Nonetheless, we are only interested in the difference in means exactly when there is a match between at least one judge's nationality and the surfer's nationality. Carrying out this test we obtain a t-value of 6.71160420326186, thus under the null hypothesis the probability we obtain this sample is less than 0.001.

therefore, we reject the null hypothesis

There is nothing fishy about this, however the picture isn't so compelling and this procedure doesn't seem to pass the scream-"the judges are biased in favor of surfers from their home country"-at-a-surf-competition test. The latter is certainly subjective.



6 Summary Statistics

7 Random Panels

Each time a surfer rides a wave, there are 5 judges, j_1, j_2, j_3, j_4, j_5, giving the scores, $j(v) = j_1(v), j_2(v), j_3(v), j_4(v), j_5(v)$. The subscripts are arbitrary, and each j_i is a map $j_i : WAVEID \to \{0.1, 0.2, \ldots, 9.9, 10.0\}$. The set of judges will be denoted as j, and for a wave, v, the *set* of scores given by the judges is $j(v) = \{j_1(v), j_2(v), j_3(v), j_4(v), j_5(v)\}$. If each judge gives a different score for wave v, then $|j(v)| = 5$, otherwise $|j(v)| < 5$. The first order statistic of the scores is the minimum score, i.e. $\min(j(v))$. So we can define the first order statistic of the set of judges as the inverse image of the minimum score, $\{j_i \mid j_i(v) = \min(j(v))\}$, denoted as J^1. Likewise the last order statistic is $J^{|j(v)|}(v) = \{j_i \mid j_i(v) = max(j(v))\}$, denoted $J^{\#}$. So, for wave v, j has $|j(v)|$ order statistics. Since this number varies over waves, we define the order statistic of the judges for a given wave to be $|j(v)|$-tuple, $J_v = (J^1, \ldots, J^{\#})$.

We have co-variate information y_i for each j_i and wish to understand how the y_i behaves with the general J_v, and particularly the case when one or more of the y_i is the same as the judge country (i.e. $I_M = 1$), as well as other co-variate information. One approach is to pass the structure of J_v to the y_i by defining $Y_v = (\{y_i \mid j_i \in J^1\}, \ldots, \{y_i \mid j_i \in J^{\#}\})$. This seems like a good option. In the first wave in our sample, there were two judges that gave Jadson Andre a 6.0 and 6.0 for his ride and judge nationalities are USA and USA. So there exists a wave such that two or more co-variates of elements of some J^k are identical. Since these are sets, for some i we have $|Y^i| < |J^i|$ which does not $\sum_i |Y^i| < \sum_i |J^i|$, which does not seem like a good attribute of a statistical procedure, and $4462/15990 \approx 0.279$ of the panels have two or more judges from the same country giving the same score.

So it makes sense to count the number judges from each country in each pod. The literature suggests that one implement this using a "multiset", which is function from some set (in our case C) to the natural numbers $\{0, 1, 2, \ldots\}$ that returns the number of judges from a country, provided with a country label. We have implemented this by taking a vector,m, whose i-th entry corresponds to the i-th judge country (when arranged alphabetically). This is a sequence of bijections from the values of the entries $m^1, \ldots, m^{|\#C|}$ to the entry indices $1, \ldots, \#C$ to counties $AUS, \ldots, ZAF$. We have ensured consistency throughout the code in using this coordinate bases using julia's Enum type and the @enum macro, which defines ORIG as an Enum Type and the instances of ORIG, if c is a ORIG, then Int(c) is defined by instances(ORIG)[Int(c)] = c. This is the default relation, however a user may specify any other bijection from the instances of an Enum type to the integers.

This means that for a wave v, the co-variate Y_v is the $\#Y_v$-tuple entries are vectors of integers denoting the number of judges from a given ORIG, using the coordinates specified in the prior paragraph. This is quite useful. The sum of the entries of $Y^i$is the number of judges giving the ith lowest score. Then, $\hat{1}^T Y_v$

is an ordered partition of 5, and $\sum_i^{\#Y_v} Y^i$

$\{w \mid X(w) = x\}$ $\{x \mid w = X^{-1}(x)\}$

A panel of judge-scores may be considered a matrix of counts of the number of judges from country c giving score s: $Q = \overbrace{\begin{bmatrix} N_{(AUS,0.1)} & N_{(AUS,0.2)} & \cdots & N_{(AUS,9.9)} & N_{(AUS,10.0)} \\ N_{(BRA,0.1)} & N_{(BRA,0.2)} & \cdots & N_{(BRA,9.9)} & N_{(BRA,10.0)} \\ N_{(ESP,0.1)} & N_{(ESP,0.2)} & \cdots & N_{(ESP,9.9)} & N_{(ESP,10.0)} \\ N_{(FRA,0.1)} & N_{(FRA,0.2)} & \cdots & N_{(FRA,9.9)} & N_{(FRA,10.0)} \\ N_{(PRT,0.1)} & N_{(PRT,0.2)} & \cdots & N_{(PRT,9.9)} & N_{(PRT,10.0)} \\ N_{(USA,0.1)} & N_{(USA,0.2)} & \cdots & N_{(USA,9.9)} & N_{(USA,10.0)} \\ N_{(ZAF,0.1)} & N_{(ZAF,0.2)} & \cdots & N_{(ZAF,9.9)} & N_{(ZAF,10.0)} \end{bmatrix}}^{\text{Columns correspond to scores}}$,

So $Q1 = \begin{bmatrix} N_{AUS} \\ N_{BRA} \\ N_{ESP} \\ N_{FRA} \\ N_{PRT} \\ N_{USA} \\ N_{AUS} \end{bmatrix}$ and $1^t Q = \begin{bmatrix} N_{0.1} & N_{0.2} & \ldots & N_{9.9} & N_{10.0} \end{bmatrix}$.

Let Scores be the set of scores given by the judges, then:

- The first order statistic is $Z_{(1)} = Q[:, 10 * min(\text{scores})]$
- The last order statistic is $Z_{(|scores|)} = Q[:, 10 * max(\text{Scores})]$
- The order statistic of Q is: $Z = (Z_{(1)}, \ldots, Z_{(|\text{Scores}|)})$

Each wave we observe a panel of judges. In its full generality, it is a 2 dimensional array. Like this:

We can read off all of the information we want from this. The entry in the rth row and ith corresponds to the number of judges from origin r, that gave score i. Technically, the judges gave a score of i/10, but nothing is lost, mathematically, by considering this as some integer out of 100. Perhaps there is a psychological effect?

This conception of a panel is possible because there are a finite number of values a judge may give a surfer for their efforts. So it is quite strait forward to "bucket" the judges by the score they gave. These are the columns. Once we have a particular score in mind, we can see how many judges from each country gave that score. These are the rows in the column.

We can also carry out this "reading off" process in reverse order. Pick a country, and to see what scores judges from that country gave, move along the row and each entry corresponds to the number of judges from that country giving the score i/10.

If we look at one particular panel, there will be many zeros because we are observing t judges in an array with 7x100=700 entries. So at most we have 5/700 non-zero entries.

Conditioning on "which country judges are from" means removing the rows corresponding to countries not in that set. For example, in Round 1, Heat

1 of Gold Coast 2018, we can condition condition on the panel origins = AUS,BRA,ESP,USA,ZAF, and every wave in this heat will have a panel that is contained in this "shortend" table.

Similarly, we can condition on the panel origins in Round 1 Heat 2 of Gold Coast 2018, they are AUS,BRA,USA, in which case our "shortened panel" will look like:

Even though in both heats we see panels of 5 judges, we can have panels that are "shorter" than 5 because there are panels where multiple judges are from the same country. The WSL caps the number of judges from any single origin at 3, so in theory we will always have a panel that has at least 2 columns. (Though this rule is stated in the rule book, it appears that in the Round 3 Heat 14 of the 2019 Gold Coast pro, the panel consisted of 4 Australian judges and 1 Brazilian Judge. This is only one instance and we forgive the WSL).

Now when we observe some particular panel for a wave, there judges will be in some general vicinity of each other. Min=0 Max=2.0 Mean= 0.62 SD = 0.339

With the goal of testing nationality bias in mind, we'd like to abstract away from score, and focus on the arrangement of the judges. To do so, we eliminate the uninformative parts of a "shortened" panel. That is- given some "shortend panel" we eliminate the uninformative columns, which are those corresponding to scores that 0 of judges gave. For example:

In example 1, we've gone from a 7x100 panel to a more appropriate 5x100 array, and then condensed this information into an array that is 5x(number of distinct scores), so for any single wave, this is a 5x5 array or "skinnier". For the BLAHth wave of the heat, this is a 5x2 panel.

In example 2, we've gone from a 7x100 panel to a 3x100 one, and then condensed this to a 3x5 array or "skinnier". For the BLAHth wave of this heat, we obtain a 3x4 panel.

Regardless of where the 5 judges are from, their birthday, favorite color, or any other variables, they may be partitioned into various "buckets" a finite number of ways. [5] [4,1] [3,2] [3,1,1] [2,2,1] [2,1,1,1] [1,1,1,1,1]

These are unordered. We can write out all the ways to order these partitions of judges. (5) (4,1),(1,4) (3,2),(2,3) (3,1,1),(1,3,1),(1,1,3) (2,2,1),(2,1,2),(1,2,2) (2,1,1,1),(1,2,1,1),(1,1,2,1),(1,1,1,2) (1,1,1,1,1)

More generally, if we observe exactly 5 judges from any of the countries, they will end up in bucket(s) corresponding to scores, with bucket sizes that sum to 5. Below is the empirical distribution of the unordered partitions.

For each observed partition, we can arrange the buckets by the score that they correspond to. The empirical distribution over ordered partitions is below:

To test for nationality bias,we want to condition on a judge origin matching a surfer's origin, and then what? We shouldn't mandate that the distribution over partitions nor should we require this of ordered partitions. The question at hand is: do judges from the same country as the surfer end up towards the top more than we'd expect?

7.1 A "Graded" Log-Linear Model

Editing Notes:

- I don't know what to call this, its kind of a riff on a log-linear model, that is partitioned by dimension.
- Question: What is $int(\Delta_{k-1}) \setminus \exp(\log(h + rowspan(A)))$? What is $\exp(\log(h + rowspan(A))) \setminus int(\Delta_{k-1})$? See Log-linear model definition. This is why i do not enjoy interiors of simplicies.
- Idea: just use parition lattice.

The first wave of round 1, heats 1, of the Gold Coast event in 2018 and its order statistic are:

$$y_1 = ((BRA, ZAF), (AUS, ESP, USA)) = \begin{pmatrix} \begin{matrix} BRA \\ ZAF \end{matrix} & \begin{matrix} AUS \\ ESP \\ USA \end{matrix} \end{pmatrix}, z_1 = \hat{N_C} y_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Where the second equality follows from "how we'd like to typeset it". And the first wave of round 1, heat 2 of the Gold Coast event in 2018 and its order statistic are.

$$y_{13} = ((BRA), (AUS, AUS, USA), (BRA)) = \begin{pmatrix} BRA & \begin{matrix} AUS\,AUS \\ USA \end{matrix} & BRA \end{pmatrix}, z_{13} = \hat{N_C} x_{13} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

More generally we may decompose the series of panels $Y = (y_i)_{i=1}^M$ into $Y^{(1)}, Y^{(2)}, Y^{(3)}, Y^{(4)}, Y^{(5)}$ where $Y^{(k)} := (y_i \in Y \mid \#Y_i = k)$. For example, $y_1 \in Y^{(2)}$ and $y_{13} \in Y^{(3)}$.
Then $Z = (z_i)_{i=1}^M$ may be decomposed into $Z^{(1)}, \ldots, Z^{(5)}$.

$$Z^{(1)} = \begin{pmatrix} 474 \\ 642 \\ 155 \\ 235 \\ 110 \\ 214 \\ 140 \end{pmatrix} Z^{(2)} = \begin{pmatrix} 1874 & 2199 \\ 2661 & 2361 \\ 714 & 564 \\ 959 & 740 \\ 351 & 532 \\ 923 & 1017 \\ 472 & 498 \end{pmatrix} Z^{(3)} = \begin{pmatrix} 2014 & 2539 & 2230 \\ 2710 & 3234 & 2500 \\ 872 & 733 & 686 \\ 1062 & 1012 & 683 \\ 420 & 532 & 546 \\ 975 & 1198 & 1344 \\ 521 & 550 & 609 \end{pmatrix} Z^{(4)} = \begin{pmatrix} 1199 & 1274 & 1246 & 1086 \\ 1282 & 1602 & 1614 & 1341 \\ 467 & 409 & 395 & 413 \\ 553 & 534 & 455 & 336 \\ 243 & 203 & 250 & 301 \\ 534 & 619 & 685 & 728 \\ 252 & 256 & 251 & 332 \end{pmatrix}$$

$$Z^{(5)} = \begin{pmatrix} 322 & 302 & 304 & 307 & 259 \\ 322 & 358 & 368 & 341 & 369 \\ 115 & 108 & 110 & 87 & 108 \\ 135 & 114 & 122 & 101 & 92 \\ 64 & 54 & 44 & 64 & 59 \\ 121 & 156 & 140 & 176 & 171 \\ 60 & 47 & 51 & 63 & 81 \end{pmatrix}$$

The sum of all of the entries of every $Z^{(k)}$ is the number of observed judges. The i,j entry in $Z^{(k)}$ may be interpreted as the indicator of a judge from country i in the jth block when there are k blocks in the panel. We can aggregate this decomposition into one 7×5 matrix where the ith column corresponds to the ith order statistic if it exists.

This makes me feel funny. I want something multidimensional. Try a distribution over partion lattice. Sullivant defines a Log linear model as $\mathcal{M}_{A,h} := \{p \in int(\Delta_{k-1}) \mid \log p \in \log h + rowspan(A)\}$, (p.122). For any $Z^{(k)}, rowspan(Z^{(k)}) = \mathbb{R}^k$ $colspan(z_t) \subseteq span\{e_c \mid c \in supp \sum z_t^i\}$

Further we may decompose the series of panels $Y = (y_i)_{i=1}^M$ into $Y_{(1)}, \ldots, Y_{(5)}$where $Y_{(k)} := (y_i \in Y \mid |\bigcap_{E_j \subset E_C | colspan(z_i) \subseteq span E_j} E_j| = k)$ where $E_C = \{e_c \mid c \in C\}$. $Y_{(k)}$may be interpreted as the panels that have judges from k distinct origins.

8 Random Arrangement Approach

Our goal is to account for the structure of the order statistic of a panel in a way that does not compromise an empirical distribution or interpretability.

8.0.1 Notations

Some Set will be denoted: S Natural numbers: $\mathbb{N} = \{0, 1, 2, \ldots\}$ Positive natural numbers: $\mathbb{N}_+ = \{1, 2, \ldots\}$ Natural Numbers up to a non-negative integer k: $\mathbb{N}_{[k]} = \{0, 1, \ldots, k-1\}$

The set of Countries: $C = \{AUS, BRA, ESP, FRA, PRT, USA, ZAF\}$

A finite dimensional vector space over $\mathbb{R}$ will be denoted as V. A vector space with the Country basis: $V_C = span\{e_{AUS}, e_{BRA}, e_{ESP}, e_{FRA}, e_{PRT}, e_{USA}, e_{ZAF}\}$ A vector space spanned by some subset, $K \subseteq C$, is: $V_K := span\{e_k \mid k \in K\}$

$\ell^k(S) = S^{\times k} = \{(s_1, \ldots, s_k) \mid \forall i s_i \in S\}$ Any monomial of numbers of judge countries: $\alpha \in \mathbb{N}^{\times 7}$ The set of all re-arrangements of n symbols: G_d Infinite Symmetric Group: $G_\infty := Iso(\mathbb{N}_+, \mathbb{N}_+)$

8.0.2 Definitions

$T^d(V) := V^{\otimes p} = \underbrace{V \otimes \cdots \otimes V}_{\text{d times}}$

$G : \bigcup_{n \in \mathbb{N}_+} C^{\times n} \to \bigcup_{n \in \mathbb{N}_+} C^{\times n}$ will denote $Res^{G_\infty}_{G_\#}$ and is given by $\bigcup_{n \in \mathbb{N}_+} k^{(n)} \mapsto \bigcup_{n \in \mathbb{N}_+} G_n k^{(n)}$ $\mathcal{A} : T \to T$ $\mathcal{S} : T \to T$

DEF: An Array with elements $a_1, a_2, \ldots, a_k$ is defined as $G(a_1, a_2, \ldots, a_k) = G_k(a_1, a_2, \ldots, a_k)$ An ordered array is a tuple. Computer scientists will have to cope.

A panel of judge-scores may be defined as Q = (Array of judges giving 0.1,..., Array of judges giving 10.0).

Then, the order statistic of Q would be: Z = (Array of judges giving min(Score),..., Array of judges giving max(Score)). We will call this a panel. Note that we could have just defined Z as the random variable.

The kind of panel we see usually is some Z s.t $\sum_i^{\#z} \#z^i = 5$. (chk if this is true in messy data, i dont think so. use messy data.)

8.1 Testing Exchangeability

Measuring Non-exchangeability. If you search something like that up a recent paper on arxiv says ppl havent done it. seems hard to beleive. anyways, topic seems relevant and I think useful in a very general context. Also its explainable and I think robust.

[?] Kolmogorov–Smirnoff statistic: $D_n = \sup_x |F_n(x) - F(x)|$. Glivenko Cantenelli says that $D_n \to 0$ if the sample is drawn from the distribution F. A random panel as consinious marginals (they are polynomials), so 3.1 applies. $\sqrt{n} sup_{t \in \mathbb{R}^k} |\mathbb{F}_n)(t) - F(t)| = \to^d \max_i K_i$ where $K \in \mathbb{R}^k$

$P(F_n(t) - \sqrt{\frac{-\log(2\alpha/k)}{2n}} < F(t) < F_n(t) + \sqrt{\frac{-\log(2\alpha/k)}{2n}}) \leq \alpha$ $P(|F_n(t) - F(t)| \leq \sqrt{\frac{-\log(2\alpha/k)}{2n}}) \leq \alpha$ This is true with the data.

The KS statistic really just bounds on max abs diff is really not compelling given the distribution of residuals. It might make sense to use this pointiwse a bunch.

Recall that $S(F_n(x))$ is the Uniformly minimum distance unbiased estimator. Therefore every concentration inequality that holds for max will hold for the norm defined in the prior section.

8.2 Distributions over measures

Let: $\Omega_C = \{(AUS)^{n_A} \otimes (BRA)^{n_B} \otimes (ESP)^{n_E} \otimes (FRA)^{n_F} \otimes (PRT)^{n_P} \otimes (USA)^{n_U} \otimes (ZAF)^{n_Z} \mid n = (n_A, n_B, n_E, n_F, n_P, n_U, n_Z) \in \mathbb{N}^7\} = \{Sort(w) \mid w \in l(C)\}$ μ be a random measure on $\mathcal{P}(\Omega_C)$, i.e. $\mu : \mathcal{P}(\Omega_C) \to [0, 1]$.

Will Show: μ induces a distribution on $T(V_C)$. $G^{(w)} := \frac{1}{(N_C w)!} 1_{\mathfrak{S} w} = \frac{1}{\prod_{c \in C} (N_c w)!} 1_{\mathfrak{S} w} =$

Equivalently we can let

- $M^{(d)} = \{Sort(w) \mid w \in l^d(\{e_c \mid c \in C\})\}$
- $F^{(d)} = \mathcal{P}(M^{(d)})$, where $\mathcal{P}$ denotes the power set.
- $\mathbf{M} = \bigcup_{d \in \mathbb{N}} M^{(d)}$,
- $\mathbf{F} = \bigcup_{d \in \mathbb{N}} F^{(d)}$

Then:

- For each $d \in \mathbb{N}$, $(M^{(d)}, F^{(d)})$ is a measurable space
- $(\mathbf{M}, \mathbf{F})$ is also measurable

And the random measure is $\mu : \mathbf{F} \to [0,1]$, which could induces the random measures:

- A Symmetric Measure: $\mu_S : T(V_C) \to [0,1]$ given by $\mu_S(h) = \frac{1}{(\#h)!} \sum_{g \in \mathfrak{S}} 1_{gh=h} \mu(h)$
- A Measure on Dimension: $p_1, p_2, \ldots, p_d, \ldots$ where $p_d = \sum_{f \in F^{(d)}} \mu(f)$

We may represent μ_S^* by: $\mathcal{S} \sum_{f \in \mathbf{F}} \mu(f) e_f = \sum_{f \in \mathbf{F}} \mu(f) \mathcal{S} e_f$.

MacEachern's Criterion for distributions over collections of measures. (Foti and Williamson, A survery of Non-Exchangeable Priors for Bayesian Non-Parametric Models)

- Support of prior on $\{G^{(x)} \mid x \in \mathcal{X}\}$ for any particular $\mathcal{X}$ should be large
- The posterior should be easy to get
- $G^{(x)}$ should follow a particular distribution for a specific x.
- If $(x_n)_{n \in \mathbb{N}} \to_n \tilde{x}$ then $G^{(x_n)} \to_n G^{(\tilde{x})}$.

$\mathcal{S} f = \frac{1}{(\#f)!} \sum_{g \in \mathfrak{S}_{\#f}} gf = \int_{\mathfrak{S}_{\#f}} gf dg$ $\mathcal{A} f = \frac{1}{(\#f)!} \sum_{g \in \mathfrak{S}_{\#f}} (-1)^g gf$

A complex poisson? If $\lambda := \mathfrak{S} w$, then λ is an $\#w!$ covering of the torus $\mathbf{T}^{\#w} = S^1_{w^1} \times \cdots \times S^1_{w^{\#}}$... right? Find the character.

8.3 Definitions

Graded Sequence Space: $l(A) := \bigcup_{n \in \mathbb{N}} A^{\times n}$ where $\times$n denotes the nth Cartesian power. Note that this is a disjoint union. Length of a Sequence: For $x \in l(A)$, let $\#x := n$ s.t. $x \in A^{\times n}$, the length of x. Last element of a Sequence: For $x \in l(A)$, let $x^{\#} := x^{\#x}$. This will make notation so much easier and it shouldn't cause much confusion.

$\mathfrak{S}_d$ denotes the symmetric group on d letters. $\mathfrak{S}_\infty$ is the infinite symmetric group. We do not constrain the infinite symmetric group to finite permutations. E.g. $(1\,2)(3\,4)(5\,6)... \in \mathfrak{S}_\infty$.

$\mathfrak{S} : \bigcup_{n \in \mathbb{N}} A^{\times n} \to \bigcup_{n \in \mathbb{N}} A^{\times n}$ is given by $\mathfrak{S} x = Res^{\mathfrak{S}_\infty}_{\mathfrak{S}_{\#x}} x = \{gw \mid g \in \mathfrak{S}_{\#w}\}$, i.e. action on x of the restriction of the infinite symmetric group to symmetric group on $\#x$.

Multi-set Space: $M(A) := l(A)/\mathfrak{S}$, so $x \sim y$ if and only if $x \in \mathfrak{S} y$, i.e. $\#x = \#y$ and $\exists \tau \in \mathfrak{S}_{\#x}$ s.t. $\tau x = y$.

$\otimes : l(A) \times l(B) \to l(A \cup B)$ by $\otimes(x,y) := x \otimes y$, i.e. $\otimes((x^1, \ldots, x^m), (y^1, \ldots, y^n)) = (x^1, \ldots, x^m) \otimes (y^1, \ldots, y^n) = (x^1, \ldots, x^m, y^1, \ldots, y^n)$. This is well defined because $x \otimes y \in (A \cup B)^{\times(m+n)} \subset l(A \cup B)$. This may be interpreted as concatenation. $\otimes : M(A) \times M(B) \to M(A \cup B)$ by $\otimes(mA, mB) := \{\otimes(x,y) \mid (x,y) \in mA \times mB\} = \{x \otimes y \mid (x,y) \in mA \times mB\} = mA \otimes mB$

Given a partition of mA, $\lambda \in \Pi_{mA}$, $\lambda = (\lambda^1, ..., \lambda^\#) = (\mathfrak{S}\lambda^1, ..., \mathfrak{S}\lambda^\#) = (\mathfrak{S}_{\#\lambda^1}\lambda^1, ..., \mathfrak{S}_{\#\lambda^\#}\lambda^\#)$ So $\otimes\lambda = \mathfrak{S}_{\#\lambda^1}\lambda^1 \otimes ... \otimes \mathfrak{S}_{\#\lambda^\#}\lambda^\# = \mathfrak{S}_{\#\lambda^1,...,\#\lambda^\#}\lambda^1 \otimes ... \otimes \lambda^\# = \mathfrak{S}_{\hat{\#}\lambda}\lambda^1 \otimes ... \otimes \lambda^\#$, where $\mathfrak{S}_{\hat{\#}\lambda}$ is the young subgroup of $\mathfrak{S}_{\#\lambda}$ determined by the ordered lengths of parts of the partition. Abusing notation, we will write the young subgroup defined by $\hat{\#}\lambda$ as $\mathfrak{S}_\lambda$.

I Need you to figure out some flow here:

Example: $y_1 = ((BRA, ZAF), (AUS, ESP, USA)) = \begin{pmatrix} BRA & AUS \\ ZAF, & ESP \\ & USA \end{pmatrix}$ This, we can consider to be $(e_{\{BRA,ZAF\}}, e_{\{AUS,ESP,USA\}})$

We extend this definition to any $w = \{w_k \mid k \in K\} \subseteq \bigcup_{k\in\mathbb{N}} C^k$ by $N_C w = \sum_{k\in K} N_C w_k$

$N_C : C^k \to V_C$ given by $w \mapsto \begin{bmatrix} N_{AUS}w \\ N_{BRA}w \\ \vdots \\ N_{ZAF}w \end{bmatrix}$ and for each $c \in C, N_c w = \sum_{i=1}^{\#w} 1_{w^i=c}$

8.3.1 A Random Arrangement

Let Y be the random variable corresponding to a random tuple, y, of length $\#y$, satisfying $|\hat{N_C}y| = \sum_{i=1}^{\#y} N_C y^i =$ Monomial of Panel Origins, and $\sum_{i=1}^{\#y} \#y^i =$ Number of Judges . This is all we require of a random panel.

It will be convenient to consider a similar random variable, Z, corresponding to a random tuple z (consisting of vectors in V_C), of length $\#z$, that satisfies $\sum_{i=1}^{\#z} z^i =$ Monomial of Panel Origins and $|\sum_{i=1}^{\#z} z^i| =$ Number of Judges. In either case, the ith entry of the tuple identifies the origins of the judges who gave the ith lowest score, which is the same thing as the $(\# - i)$th highest score.

$Z = \hat{N_C}Y$ is the expression of Z in terms of Y. In particular, Z is the Order Statistic of Y. This follows from the fact that Z^i is the ith order statistic, so $Z^1, \ldots, Z^i, \ldots, Z^\#$ are the order statistics (plural). Typically, one considers the order statistics up to some integer n, however that approach is not applicable here because $Z^i = \hat{N_C}Y^i$, which need not exist for every observation if $1 < i$. It may be more appropriate in our setting to think of the "ith Order Statistic" as the "ith Statistic of (sort observation by score)"

For any parametric model in which the parameter is the monomial of Panel Origins, Z is sufficient.

The above gives a nice description of the panel, but what is random about it? We can break this down into a few different parts.

In our setting we have 5 judges The judges may be partitioned into 7 different buckets (indicating their national origin) We then can obtain any ordered partition of this vector.

More generally Get k judges Judges may be partitioned into n different buckets obtain some ordered partition of this vector

9 Appendix

9.1 A Probability Algebra

Editing Notes:

- Can we call it the l_1 norm before we show it?
- See defn for Probability Measure on T, whats the best way to define it?

We have used an l_1-esque norm here and there on $T^5(V_C)$, but haven't given it a full discussion. Throughout this section we will assume V is a finite dimensional vector space over a field with characteristic 0.

First, we define a norm, $||\cdot||$, on $T^d(V) = V^{\otimes d}$ for some arbitrary $d \in \mathbb{N}$ in the same way we did for the concrete setting with our data, i.e. $||Y|| := \sum_{w \in C^5} |Y_w|$ where $Y \in T^d(V)$.

Claim. *: $||\cdot||$ is a norm.*

Proof. First, we show $\forall d \in \mathbb{N}, \forall Y \in T^d(V(\mathbb{C})), ||Y|| \geq 0$ and $||Y|| = 0 \iff Y = 0$. $||Y|| \geq 0$ follows from the definition. $||Y|| = 0 \iff \sum_{w \in C^5} |Y_w| = 0 \iff \forall w \in \{1, \ldots, \dim(V)\}^d,\ Y_w = 0 \iff Y = \sum_w 0e_w = 0$Second, we show $\forall X, Y \in T^d(V), ||X + Y|| \leq ||X|| + ||Y||. ||X + Y|| = \sum_w |X_w + Y_w| \leq \sum_w |X_w| + |Y_w| = \sum_w |X_w| + \sum_w |Y_w| = ||X|| + ||Y||$. Third, we show $\forall a \in \mathbb{C}, \forall Y \in T^d(V), ||aY|| = |a|||Y||$. $||aY|| = \sum_w |aY_w| = \sum_w |a||Y_w| = |a| \sum_w |Y_w| = |a|||Y_w||$ □

Claim. *$(T^d(V(\mathbb{C})), *, +)$ is a vector space over $\mathbb{C}$, where $*$ and $+$ are point-wise operations. Proof is left to the reader.*

Claim. *$(T^d(V(\mathbb{C})), ||\cdot||)$ is complete.*

Proof. To prove this we will show that every Cauchy sequence converges in $T^d(V)$. Let $(X^{(n)} \in T^d(V))_{n \in \mathbb{N}}$ be a Cauchy Sequence. So we know: $\forall \epsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n, m > N, ||X^{(n)} - X^{(m)}|| \leq \epsilon$. Now, we will find an $L \in T^d(V)$ such that $\lim_n ||X^{(n)} - L|| = 0$. For each $w \in \{1, \ldots, \dim(V)\}^{\times d}$, $(X_w^{(n)})_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{C}$ by assumption, and it converges to some $L_w \in \mathbb{C}$ since $(\mathbb{C}, |\cdot|)$ is complete. Thus we have: $\lim_n X^{(n)} = \lim_n \sum_w X_w^{(n)} e_w = \sum_w (\lim_n X_w^{(n)}) e_w = \sum_w L_w e_w$. Let $L := \sum_w L_w e_w$, and $L \in T^d(V)$. □

Taking the preceding 3 lemmas together, for every $d \in \mathbb{N}, (T^d(V(\mathbb{C})), *, +, ||\cdot||)$ is a complete normed vector space. Hence, it is a Banach space. Since the data contains a few panels have 3 judges and the remaining panels have 5 judges, the empirical panel distribution is not contained in $T^3(V_C)$ or $T^5(V_C)$, rather it is some $F = F_3 + F_5 \in T^3(V_C) \oplus T^5(V_C)$. Also, some heats have 2 surfers while others have 3, so the empirical distribution of heat results of is an element of $T^2(V_A) \oplus T^3(V_A)$. There are many other settings where the dimension of the observed data is not concentrated on one value. This motivates the space: $T := \{f \in \bigoplus_{d=0}^{\infty} T^d(V) \mid ||f|| < \infty\}$ where there norm is induced by the norms of the subspaces, i.e. $||f|| = ||\sum_d f_d|| = \sum_d ||f_d||$. Since T is the (internal) direct sum of Banach Spaces, it is also Banach. Multiplication and addition on T is defined by those operations on each subspace: for $f, g \in T$, $f + g = \sum_m f_m + \sum_n g_n := \sum_d f_d + g_d$ and $f * g = (\sum_m f_m) * (\sum_n g_n) := \sum_d f_d * g_d$. These are two operations are intuitive and useful when we restrict our attention to binary operations on $T^d(V)$. Now that we have a more general space that is not restricted to exactly d copies of V, it makes sense to endow T with $\otimes$ given by: $f \otimes g = (\sum_m f_m) \otimes (\sum_n g_n) := \sum_d \sum_{(m,n)|m+n=d} f_m \otimes g_n$.

The norm on T is also multiplicative:

Claim. $\forall A, B \in T, ||A \otimes B|| = ||A||||B||$

I Need to fix this proof, the norm is not defined to extend to the basis elements, it can then I don't think bounded coefficients would be sufficient for bounded norm. Counter example is take some basis element to be a vector of numbers above 2. If norm is strictly applied to coefficients, any element with bounded coefficients is norm able which is what we want. Additionally, leveraging this behaviour could be helpful.

Proof. $||(\sum_w A_w) \otimes (\sum_u B_u)|| = ||(\sum_w A_w) \otimes (\sum_u B_u)|| = ||\sum_w \sum_u A_w \otimes B_u|| = \sum_w \sum_u |A_w \otimes B_u| = \sum_w \sum_u |a_w w \otimes b_u u| = \sum_w \sum_u |a_w b_u w \otimes u| = \sum_w \sum_u |a_w b_u||w \otimes u| = \sum_w \sum_u |a_w||b_u||w^1 \otimes \cdots \otimes w^\# \otimes u^1 \otimes \cdots \otimes u^\#| = \sum_w \sum_u |a_w||b_u||w^1| \otimes \cdots \otimes |w^\#| \otimes |u^1| \otimes \cdots \otimes |u^\#|$ This follows from the fact that every w^i,u^j is a number. $= \sum_w \sum_u |a_w||b_u||w^1 \otimes \cdots \otimes w^\#| \otimes |u^1 \otimes \cdots \otimes u^\#|$ $= \sum_w \sum_u |a_w||w^1 \otimes \cdots \otimes w^\#| \otimes |b_u||u^1 \otimes \cdots \otimes u^\#| = (\sum_w |a_w||w^1 \otimes \cdots \otimes w^\#|) \otimes (\sum_u |b_u||u^1 \otimes \cdots \otimes u^\#|) = (\sum_w |A_w|) \otimes (\sum_u |B_u|) = ||\sum_w A_w|| \otimes ||\sum_u B_u|| = ||A|| \otimes ||B||$ $\square$

T admits a unit w.r.t $\otimes$, $1 \in \mathbb{C}$. $\forall t \in T, t \otimes 1 = (\sum_d t_d) \otimes 1 = \sum_d t_d \otimes 1 = \sum_d t_d 1 = \sum_d t_d = t$. T also admits a unit w.r.t. $*$, $\mathbf{1} = \sum_d \vec{1}^{\otimes d}$, where $1_0 = 1 \in \mathbb{C}$.

Definitiown (A Probability Measure on T). *is the total variation of some* $\mu \in T$ *that satisfies* $||\mu|| = 1$, *i.e.* $\sup_{E \vdash l(A)} \sum_i |\mu * 1_{E_i}| = || \, |\mu| \, ||$

IMPORTANT NOTE: A better definition could be $\mu \in T^* = \bigoplus_k L(V^{\otimes k}, \mathbb{C})$ s.t. $\sum_{w \in l(C)} \mu_w = 1$

Definitiown (The Expectation). *of a random variable, X, w.r.t. a measure* μ *is* $||\mu * X|| = ||\mu * X^+|| - ||\mu * X^-|| = \sum_d \sum_{w \in l^d([1:n])} \mu_w X_w$

I am satisfied with the generality of T. We may embedded the labels of the variable of interest in V and account for data in which a data point is any number of these labels with any additional structure, e.g. being ordered or partly ordered or a mixture of the two. Taking limits in the number of observations is classical approach, which I WOULD LOVE TO DO/DISCUSS IN THE NEXT SECTION.

Taking limits in the dimension of V is a big business theses days. Consider how we end up with an infinite number of distinct things: we observe some data $\{Y_i\}_{i=1}^n$where Y_i has any finite dimensional structure, $l(n)$ is the set of labels observed, and $V_{l(n)} := span\{e_L \mid L \in l(n)\}$. The empirical distribution of our data is $F_n = \frac{1}{n}\sum_{i=1} Y_i$ and $F_n \in T(V_{l(n)})$. This has an asymptotic distribution $\lim_{n\to\infty} F_n$, which is well defined if $\lim_{n\to\infty} |l(n)|$ is bounded, i.e. $\exists M s.t. \forall n, |l(n)| < M$. Taking limits in the dimension of V corresponds precisely to the limit case of $T(V_{l(n)})$ where $l(n)$ is unbounded; what happens?

In exchange for assuming the number of labels to be finite, we get a probability algebra that is really accommodating to multi-dimensional data! At a high level we can consider how much more or less ”dimension-accommodating” the vector-space-limit-approach gets us by considering:

$$\lim_n \frac{|l(n)|}{\dim(F_n)} = \lim_n \frac{\dim(V_{l(n)})}{\dim(F_n)}$$

This is bounded above by 1 and below by 0. ...Any discrete distribution should work fine on T, or polynomial function, and weighted multivariate distributions work nicely too ($\mathbf{X}^w = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}^w$ where $w : \mathbf{X} \to A \subseteq \mathbb{R}^+$, then $Q(x) = \frac{w(x)P(x)}{E[w(X)]}$)