Jojo

There are many beautiful connections between analysis and geometry. In finite dimensions, vector spaces, polynomial rings, manifolds, matrix groups and metric spaces are well understood.

In the last 100 years their "infinite" dimensional counterparts have been a focus of research. Banach Spaces & Hilbert Spaces are the principal metric & inner product spaces studied. The set of bounded linear operators on what. is analogous to the ring of $n \times n$ matricies (over some field), Both are given the structure of a Vector Space.

Developing tools / Methods to do analysis on these spaces is a more recent focus. This has benefited significantly from

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Advances in analysis on locally compact Abelian Groups, representations of groups (esp. "characters"), Lie theory, and the widespread use of "tensor" products.

My Goal is to expand the domain of analysis, Provide some analytical methods, and address geometric questions.

The space we consider is: $\mathbb{C}^S := \{x : S \to \mathbb{C} \mid x \text{ is a function}\}$

The group we consider is: $G := \{g : S \to S \mid g \text{ is one to one \& onto}\}$

With regard to functions on $\mathbb{C}^S$, we will focus on $\mathrm{Hom}(\mathbb{C}^S, \mathbb{C}_\times^S)$, and how $\mathrm{Aut}(\mathbb{C}^S)$ comes into play. Specifically we identify a Large collection of automorphisms of $\mathbb{C}^S$, and identify each with a function in $\mathrm{Hom}(\mathbb{C}^S, \mathbb{C}_\times^S)$ in a way that makes the map

$$\mathrm{Aut}(\mathbb{C}^S) \longrightarrow \mathrm{Hom}(\mathbb{C}^S, \mathbb{C}_\times^S)$$

a group homomorphism. This is a representation of $\mathrm{Aut}(\mathbb{C}^S)$ in $\mathrm{Hom}(\mathbb{C}^S, \mathbb{C}_\times^S)$. This is ~~obviously unnecessary~~ merely a parametrization of some subset of $\mathrm{Hom}(\mathbb{C}^S, \mathbb{C}_\times^S)$. The Subset will contain at least the functions nessisary for Fourier analysis. The Parametrization will make it possible to be explicit in describing functions invariant under Application of elements of G.

This set-up / approach resembles Klien's approach to geometry and the investigation of automorphic functions. For a pleasant introduction, see I. M. Gel'fand's paper, Automorphic functions and the theory of representations.

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We will depart from ~~the~~ ~~Klein's~~ ~~philosophy~~ Klein's Erlangen program in the following sense:

In Klein's Geometry one works "begins" with a Homogeneous or Principal Homogeneous Space. This is a set X, and a group H such that for every $x, y \in X$ there is a $h \in H$ satisfying $h \cdot x = y$. If ~~this~~ for all $x, y \in X$ the $h \in H$, taking x to y, is free-ness {unique then it is called a Principal Homogeneous Space.

~~The intuitive~~

The goal is then to determine properties of ~~X~~ or objects in invariant under H. It is not entirely clear how one proceeds from here ~~there~~:

If some $x \in X$ has some property invariant under H then so does every other point.

If some $x \in X$ is a representative of a subset of X with a property invariant under H, then every $y \in X$ is a representative of a subset with this property.

$$X/\sim_H = \{X\} = \{H \cdot x\}$$ ~~where~~ ~~x~~ ~~is~~ for any $x \in X$.

$$|X/\sim_H| = 1$$

These follow from the assumption that (X, H) is Homogeneous Space.

It then becomes more intuitive to consider the stabilizer group of ~~a~~ $x \in X$ and deduce properties of X by collecting together elements with the same stabilizer. We are now going backwards, describing ~~the~~ invariants of subgroups of H, and this is only possible if ~~the~~ (X, H) is not principal Homogeneous.

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This has evolved into the practice of taking a subgroup K of H, taking the "space" to be the <u>coset</u> space $K\backslash H$, and let H act on it, i.e. $(\underbrace{K\backslash H}_{\text{left coset decomposition}}, H)$. This is not principal homogeneous. The representation

$$h \to (Kx \to Kxh) \qquad (\text{where } x \in H)$$

is isomorphic to H. This is the structure of affine space.

A common theme in the literature is to let H be a matrix group and K an arithmetic subgroup; for example $PSL(2,\mathbb{R})$ and its subgroup "$PSL(2,\mathbb{Z})$" ~~consisting of elements of PSL(2,ℝ) with integer entries that have an inverse~~ $= SL(2,\mathbb{Z})/\{\pm I\}$. These can get very sophisticated, see Arithmetic Properties of Linear Algebraic Groups By Borel. These <u>Coset</u> decompositions also find their way into the literature on Automorphic functions and Forms.
cite Casselman.

As you can see, its quite easy to deviate from geometric considerations once we start considering group actions. I can not promise you any geometric results at this point, (How could I?). The idea behind taking a group of transformations of a space is <u>not</u> to watch them do stuff. It is hope that there is some set of functions out there that can help us identify equivalent things in our space and maybe be useful in the other things we pursue as mathematicians.

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A set, S. Assume S is not finite and countable.
The Space: $\mathbb{C}^S = \{x : S \to \mathbb{C} \mid x \text{ is a function}\}$
$\mathbb{C}$ is the set of complex numbers.

Why the assumptions on S? The case of a finite set is understood. The case of S not finite, less so. If you want to consider S uncountable, no one is stopping you. I might suggest:

$$\mathbb{C}^{\mathbb{C}} = \{x : \mathbb{C} \to \mathbb{C} \mid x \text{ is a function}\}$$

Why work with $\mathbb{C}^S$?
Let $\ell^p(\mathbb{C}^S) = \{x \in \mathbb{C}^S \mid \sum_{s \in S} |x(s)|^p < \infty\}$
Now, for $p \in \mathbb{R}$ ~~pos~~ & $p > 0$ $\ell^p(\mathbb{C}^S) \subset \mathbb{C}^S$
for $p \in \mathbb{C}$ $\ell^p(\mathbb{C}^S_x) \subset \mathbb{C}^S$.

Why work over $\mathbb{C}$? Imaginary numbers are useful. In particular, $\mathbb{C}$ has torsion. Also, about 2 months ago I was not familiar with complex analysis and figured I ought to familiarize myself.

$(\mathbb{C}^S, +)$ is an abelian group where $(x+y)(s) := x(s) + y(s)$

$(\mathbb{C}^S, \cdot, +)$ is a ring, where $(x \cdot y)(s) := x(s) \cdot y(s)$

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The Group $G = \{ g : S \to S \mid g$ is one to one & onto $\}$

The image of $s \in S$ under $g \in G$ is sg or $s(g)$

The product of $(g,h) \in G \times G$ is gh

The left application of $g \in G$ to $s \in S$ is defined by:

$$g(s) := sg \quad \text{or} \quad (g)(s) := s(g)$$

Applying definitions, $sgh = h(sg) = h(g(s)) = (h \circ g)(s)$

we find that $(h \circ g) = (gh)$.

From here we can define a representation of G in $\mathrm{Aut}(\mathbb{C}^S)$

by:
$$R : G \to (\mathrm{Aut}(\mathbb{C}^S), \circ)$$
$$g \mapsto (x \mapsto x \circ g)$$
$$x(s) \mapsto x(sg)$$

i.e. $R_g(x)(s) = x(sg) = x \circ g(s)$

This is an injective group homorphism.

$$R_{Id_S}(x) = x \circ Id_S = x = Id_{\mathbb{C}^S}(x)$$

$Id_{\mathbb{C}^S}(x) = x = x \circ g \circ g^{-1} = R_{g^{-1}}(R_g(x))$ hence $R_g^{-1} = R_{g^{-1}}$

$$(R_h \circ R_g)(x) = R_h(R_g(x)) = R_h(x \circ g) = x \circ g \circ h = x \circ (g \circ h) = x \circ (hg)$$
$$= R_{hg}(x).$$

One may also consider:

$T : (G, \circ) \to (\mathrm{Aut}(\mathbb{C}^S), \circ)$ given by $T_g(x) = x \circ g^{-1}$.

This is a group hom. in the sense that

$$(T_h \circ T_g)(x) = T_h(T_g(x)) = T_h(x \circ g^{-1}) = x \circ g^{-1} \circ h^{-1} = x \circ (h \circ g)^{-1} = T_{h \circ g}(x)$$

it is one of G with product given by $\circ$ (whereas R is a representation of G with product given by juxtaposition).

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$\mathbb{C}^S$ and Some Quotients

In this section, we'll identify G-equivalent elements of $\mathbb{C}^S$.

$x, y \in \mathbb{C}^S$ are G-equivalent ($x \sim_G y$) iff. there is a $g \in G$ s.t. $x \circ g = y$. Since the image of an $x \in \mathbb{C}^S$ is invariant under G (i.e. $x \circ g(S) = x(S)$) it is necessary that $x(S) = y(S)$ for $x \sim_G y$. Also, we need $g(\{s \in S \mid x(s) = v\}) = \{s \in S \mid y(s) = v\}$ for every $v \in x(S)$ and some $g \in G$. That is, the partitions of S given by the inverse images of x and y must be G-equivalent. We can try to go further by requiring that the "number" G-equivalent sets in the partition $x^{-1}(x(S))$ is the same as that of $y^{-1}(y(S))$, however the notion of cardinality is not precise enough.

We can make this idea precise:

$$x \sim_G y \quad \text{iff.} \quad x^{-1}[x[S]] \in y^{-1}[y[S]]\, G$$

where

This is merely a restatement of the definition (consider $x \sim_G y$ iff. $x \in y \circ G$

A more Group theoretic way to state this is

$x \sim_G y$ iff. $G_x = \{g \in G \mid x \circ g = x\}$ is conjugate to G_y

i.e. $\exists\, h \in G$ s.t. $h^{-1} G_x h = G_y$.

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Its somewhat natural to wonder if we can get a smaller decomposition of $\mathbb{C}^S$ by including other invertible transformations in the group we quotient by. The set of multiplication operators $M_z : x \mapsto z \cdot x$ for $z \in \mathbb{C}_\times^S$, is an intuitive choice.
$(x(s) \mapsto z(s) \cdot x(s))$

Taking the quotient $\mathbb{C}^S / \{M_z \mid z \in \mathbb{C}_\times^S\}$, one arrives at the fact that $x \sim 1_{\mathrm{supp}(x)}$ (by finding some $z \in \mathbb{C}_\times^S$ s.t. $z(s) = x(s)^{-1}$ if $x(s) \neq 0$ & $z(s) \in \mathbb{C}_\times$ otherwise). Hence, the quotient amounts to $\{0,1\}^S$, and:

$$\{0,1\}^S / G = \bigsqcup_{K \in K} \{1_{Kg} \mid g \in G\}$$

where K is a ~~partially~~ ordered chain of subsets of S s.t. whenever N covers L ($L \subset N$ & $\neg \exists L' \in K$ s.t. $L \subset L' \subset N$) $|N \setminus L| = \mathrm{card}(N \setminus L) = 1$, <u>and</u> $\emptyset, S \in K$.

This provides some insight into the analytic structure of $\mathbb{C}^S$:

$$\mathbb{C}^S = \bigsqcup_{K \in K} \{x \in \mathbb{C}^S \mid \mathrm{supp}(x) \in K G\} \simeq \bigsqcup_{K \in K} \bigcup_{g \in G} \mathbb{C}_\times^{Kg}$$

$$= \bigsqcup_{K \in K} \bigcup_{g \in [G_{1_K} \backslash G]} \mathbb{C}_\times^{Kg}$$

where "$g \in [G_{1_K} \backslash G]$" means that g runs over a the set of coset representatives of the coset decomposition $G_{1_K} \backslash G$.

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Before Moving to function theory, I'd like to connect this group, G (or R_G), acting on the set $\mathbb{C}^S$ to the literature on group actions. Below is a list of adjectives describing a group H acting on a set X via $\cdot : H \times X \to X$

<u>Transitive</u>: $\forall x, y \in X \ \exists h \in H$ s.t. $h \cdot x = y$

<u>Faithful</u> (or effective): $\forall h \in H \ h \neq e \ \exists x \in X$ s.t. $h \cdot x \neq x$

<u>Free</u>: $[g, h \in H \ \& \ g \cdot x = h \cdot x] \Rightarrow g = h$

<u>Regular</u>: $\forall x, y \in X \ \exists! \ h \in H$ s.t. $h \cdot x = y$

If X is a topological space:

<u>Wandering</u>: $\forall x \in X \ \exists$ neighborhood U of x s.t. $|\{h \in H \mid h \cdot U \cap U \neq \phi\}|$ is finite.

<u>A covering space action</u>: $\forall x \in X \ \exists$ neighborhood U of x such that $\{h \in H \mid h \cdot U \cap U \neq \phi\} = \{id_H\}$

This is paraphrased from the wikipedia page on Group Actions. R_G acting on $\mathbb{C}^S$ is effective. It does not satisfy any other adjective from the above list describing a property.

A counter example to transitivity is $x \in \mathbb{C}^S$ & $x \neq 0_S$ with $x + x$. And not transitive implies not regular.

A counter example to Free and wandering is an $x \in \mathbb{C}^S$ that takes the same value on an infinite subset $M \subset S$, since the set of elements fixing x is then not finite. And not wandering $\Rightarrow$ not a covering space action

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This is mildly concerning given the choice of approach. Upon further consideration it might not be so bad.

Transitivity is a strong Requirement. In the previous section we covered some of the developments of research on Homogeneous spaces. These are defined as a group acting transitively on a set. This (Roughly speaking) brought us to coset decompositions. Such decompositions are useful, on the other hand we can always decompose a group this way & represent a group action via cosets. Regularity is simply a strengthening of transitivity to "unique transitivity".

Freeness is intuitively useful, it is similar in essense to the requirement of a unique factorization domain. ~~In particular, in a group we have unique~~ The definition of freeness may be given as: $\forall h \in H \ \forall x \in X \ h \cdot x = x \Rightarrow h = Id_H$

Equivalently: $\forall x \in X \ \forall h \in H \ h \cdot x = x \Rightarrow h = Id_H$

The action of G on $\mathbb{C}^S$ is not free, the elements inhibiting this are those $x \in \mathbb{C}^S$ s.t. $|x(K)| = 1$ for some $K \subset S$ where $|K| > 1$, i.e. those elements that are not injective.

This is a relatively small subset of $\mathbb{C}^S$ compared to the subset of injective functions in $\mathbb{C}^S$, on which G is free. Moreover, the sequential closure of the set of injective elements of $\mathbb{C}^S$ (with respect to some function $d: \mathbb{C}^S \times \mathbb{C}^S \to \mathbb{R}_+$ that is symmetric & 0 only on the diagonal of $\mathbb{C}^S \times \mathbb{C}^S$), is all of $\mathbb{C}^S$.

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Now Some Function Theory

(This is largely ispired by character theory.
The goal of this section is to describe some "representations" of Automorphisms of $(\mathbb{C}^S, +)$ in $\mathrm{Hom}((\mathbb{C}^S, +), (\mathbb{C}^S_\times, \cdot))$.

I am certain this will initially seem superfluous.
It is a nessisary procedure because not all automorphisms of $(\mathbb{C}^S, +)$ have an analog in $\mathrm{Aut}((\mathbb{C}^S, \cdot))$, or even $\mathrm{Hom}((\mathbb{C}^S_\times, \cdot), (\mathbb{C}^S_\times, \cdot))$.

This is not a result of working with $(\mathbb{C}^S, +)$ insted of $(\mathbb{C}, +)$.

$$\mathbb{C} := \{z_0 i^0 + z_1 i^1 + z_2 i^2 + z_3 i^3 \mid z_0, z_1, z_2, z_3 \in \mathbb{R}_+\} / \{a i^0 + b i + d i^2 + b i^3 \mid a, b \in \mathbb{R}_+\}$$

<u>Step 1</u>: One dimensional Automorphisms & Reps.

$(\mathbb{C}, +)$ is an abelian group.
we have the map: $z \xmapsto{A_n} \underbrace{z + \dots + z}_{n \text{ times}}$. where $n \in \mathbb{N}_\times$

It is an automorphism because:

A_n is one to one and onto, (hence invertible).

$A_n(0) = 0$. because $(0+0) = 0$

$A_n(v+z) = A_n(v) + A_n(z)$ because $(v+z)+(v+z) = (v+v)+(z+z)$

$A_n(z^{-1}) = \underbrace{z^{-1} + \dots + z^{-1}}_{n \text{ times}} = (\underbrace{z + \dots + z}_{\text{semit } n})^{-1} = A_n(z)^{-1}$

We also have the map: $z_0 i^0 + z_1 i^1 + z_2 i^2 + z_3 i^3 \xmapsto{\sigma} z_0 i^1 + z_1 i^2 + z_2 i^3 + z_3 i^0$

This is also an automorphism.

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If we denote the Group generated by σ and A_n $(n \in \mathbb{N}_*)$ with $(M, \circ)$. [Note: We do not need to require that A_n^{-1} and σ^{-1} are generators because if H is a group that $\forall$ $h \in H$ there exists $g \in H$ satisfying $hg = 1_H$].

Pine Goodbos

We can construct an embedding of M,

$$\omega : (M, \circ) \longrightarrow \mathrm{Hom}((\mathbb{C}, +), (\mathbb{C}_*, \cdot))$$
$$a \longmapsto (z \mapsto \exp \circ a(z))$$

THIS IS NOT A GROUP HOMOMORPHISM

Trg. Addition: For $f \in \mathrm{Hom}((\mathbb{C}, +), (\mathbb{C}^\times, \cdot))$

$f + f = 2f$ and $2f(x+y) \neq 2f(x) \cdot 2f(y)$

$\omega(a \circ b) \neq \omega(a) \cdot \omega(b)$ because $\exp(a \circ b(x)) \neq \exp(a(x) + b(x))$

This does make it clear, however, that an extention of ω, $\tilde{\omega}$ on a group ring of $(M, \circ)$, would be a group Hom, but this seems Redundant/unappealling.

It seems more fruitful to consider the abelian group:

$\mathbb{C}[\omega(M)] = \{ \sum_{m \in M} c_m \omega(m) \mid c \in \mathbb{C}^M \}$ with addition.

- Scalar multiplication makes it a vector space, And $\mathrm{Log}(\mathbb{N}_*)$ This essentially contains fourier analysis on subgroups of $(\mathbb{C}, +)$.

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The group of unit complex numbers is not a subgroup of $(\mathbb{C},+)$, but ~~the~~ motivation for ~~considering~~ the present method may be found in:

$$\mathbb{C}[\mathrm{Aut}(\mathbb{S}^1)] \supset \mathcal{C}(\mathbb{S}^1;\mathbb{C})$$

and

$$\mathrm{Dual}(\mathbb{C}^{\mathrm{Aut}(\mathbb{S}^1)}) \supset \mathrm{Dual}(\mathcal{C}(\mathbb{S}^1;\mathbb{C}))$$

[Recall we are working towards analysis on $\mathbb{C}^S$ where S is an arbitrary countable set. Note: $\mathrm{Aut}(\mathbb{S}^1) \simeq (\mathbb{Z},+)$]

$\mathrm{Aut}(\mathbb{S}^1)$ consists of the maps

$$z \longmapsto \underbrace{z\cdot\ldots\cdot z}_{n \text{ times}} \quad \text{where } n \in \mathbb{N}_*$$

and

$$z \longmapsto z^{-1}$$

These are also the automorphisms of $\mathbb{C}_\times$ (modulo conjugation, which i will discuss later, and some bizarre field automorphisms of $(\mathbb{C},\cdot,+)$ which I will essentially ignore) They are the images of $\omega(\underbrace{\{A_n\circ\sigma \mid n\in\mathbb{N}_*\}\cup\{A_n \mid n\in\mathbb{N}_*\}}_{})$ in the sense that we may lift these automorphisms of $(\mathbb{C},+)$ up from their embedding in $\mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_\times,\cdot))$. We cannot do this for the other elements of M. The benefit of starting with $(\mathbb{C},+)$ and constructing $\omega(M)$ is two fold. ① The polynomial ~~algebra~~ Ring of $\mathbb{C}_\times$ is contained in $\mathbb{C}[\omega(M)]$ excluding constant functions, i.e. $\left\{\sum_{n\in\mathbb{Z}_\times} c_n z^n \mid c\in\mathbb{C}^{\mathbb{Z}_\times}\right\} \subset \mathbb{C}[\omega(M)]$

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② We have a rigorous acceptable and useful notion of " z^v " for $z \in \mathbb{C}_\times$ and $v \in \mathbb{Q}_\times \cup \mathbb{Q}_\times i$

or

$\mathbb{Q}_* \cup \mathbb{Q}_* \sigma \cup \mathbb{Q}_* \sigma^2 \cup \mathbb{Q}_* \sigma^3$

I beleive the current concencus on the defn. of " z^v " is:

$$\text{"}z^v\text{"} := \exp(v \cdot (\mathrm{Log}(z) \bmod 2\pi i))$$

or

$$:= \exp(v \cdot (\mathrm{Log}(z) \underbrace{\bmod (-\pi, \pi] i}))$$

This doesn't entirely make sense:

$\pi i \bmod (-\pi i, \pi i] = 0$

$\frac{3\pi i}{2} \bmod (-\pi i, \pi i] = \frac{\pi}{2} i$ or $-\frac{\pi}{2} i$

It is particularly concerning that with these defns.

$$\text{"}(x \cdot y)^v\text{"} \neq \text{"}x^v \cdot y^v\text{"}$$

In contrast $\mathbb{Q}_\times \cup \mathbb{Q}_\times i$ parameterizes M

For $z \in \mathbb{C}_\times$ there is a $c \in \mathbb{C}$ s.t. $\exp(c) = z$

Define " z^v " $:= \exp(m_v(c))$.

Then $\forall a, b \in \mathbb{C}$ s.t. $a + b = c$ we have:

$$\exp(m_v(a+b)) \underset{\substack{\uparrow \\ m_v \text{ is an} \\ \text{Automorphism}}}{=} \exp(m_v(a) + m_v(b)) \underset{\substack{\uparrow \\ [m_v(a) + m_v(b)]_{2\pi i} = [m_v(a)]_{2\pi i} + [m_v(b)]_{2\pi i}}}{=} \exp(m_v(a)) \cdot \exp(m_v(b))$$

Hence $\forall x, y \in \mathbb{C}_\times$ s.t. $x \cdot y = z$ $\text{"}(x \cdot y)^v\text{"} = \text{"}x^v\text{"} \cdot \text{"}y^v\text{"}$

Now, we can remove quotes and write a general element of $\mathbb{C}[\omega(M)]$ as $\sum_{v \in \mathbb{Q}_\times \cup \mathbb{Q}_\times i} d_v z^v = \sum_v d_v \omega(m_v)(\vec{z}) = \sum_{m \in M} d_m \omega(m)(\vec{z})$

$= \sum_{m \in M} d_m \exp(m(\vec{z}))$ where $\vec{z} \in \{c \in \mathbb{C} \mid \exp(c) = z\}$.

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One more thing before we (finally) move on.

I did not require anything of the map

$$v \in \mathbb{Q}_* \cup \mathbb{Q}_* i \longmapsto m_v \in M$$

however we should require that this is a group hom. in order for the expression "z^v" to be defined as we wish (otherwise any element of $M^{(\mathbb{Q}_* \cup \mathbb{Q}_* i)}$ would work and the expression would still be well-defined).

A couple of Remarks

① The definition i provided for $\mathbb{C}$ could be cheekier:

Let $C = \{ z_0 i^0 + z_1 i^1 + z_2 i^2 + z_3 i^3 \mid z_0, z_1, z_2, z_3 \in \mathbb{R}_* \}$

If $\mathbb{R}_*$ is endowed with addition & multiplication and $\{i^0, i^1, i^2, i^3\}$ is endowed with multiplication then C has all of the arithmetic properties of the field of complex numbers, but no element is "inuitively" unique though it is, algebraically, by definition:

(i.e. $x_0 i^0 + x_1 i^1 + x_2 i^2 + x_3 i^3 = y_0 i^0 + y_1 i^1 + y_2 i^2 + y_3 i^3$ iff. $x_0 = y_0$ & $x_1 = y_1$ & $x_2 = y_2$ & $x_3 = y_3$)

We can resolve this by defining:

$$O_{\mathbb{C}} := \{ a i^0 + b i^1 + a i^2 + b i^3 \mid a, b \in \mathbb{R}_* \}$$

and taking the quotient $C / O_{\mathbb{C}}$. Now we have the multiplicative group of non-zero complex numbers. Why not set $\mathbb{C}_* := C / O_{\mathbb{C}}$ and then define:

$$\mathbb{C} := \mathbb{C}_* \cup \{ O_{\mathbb{C}} \}$$

(Note that $\mathbb{C}_*$ is only closed under multiplication, so we'd have to remove ~~that operation~~ addition and then reintroduce it to $\mathbb{C}$)

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② In building the group M, I adopted an opposite preference towards the abelian group $(\mathbb{C}, +)$. In particular we made 0 use of multiplication. This was deliberate. I'm honestly suprised how far it got us. M is obviously equivalent to $\mathbb{Q}_+ \cup \mathbb{Q}_+ i$, however it inherits its structure from the provided definitions ~~(I did use $\mathbb{N}$, though because I was lazy and did not want to reconstruct successive~~

The Prime elements are those that cannot be written as the composition of others except the identity.

③ This section was inspirited by "Representation Theory" and "character theory". I have ultimately used them to the extent that the notion of a group homomorphism is incredibly useful. Its interesting to note the most useful thing we did, embedding M in $\mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_+,\cdot))$, ~~was~~ is not a group Hom.

④ Walter Rudin has an interesting paper on "The Automorphisms and The Endomorphisms of The Group Algebra of The Unit Circle". He considers, for which $t: \mathbb{Z} \to \mathbb{Z}$ is it true that $\sum_{n\in\mathbb{Z}} a(t(n))\, e^{in\theta}$ is a fourier series whenever $\sum_{n\in\mathbb{Z}} a(n)\, e^{in\theta}$ is? Roughly speaking he determines some locally symmetric periodicity condition on t. I have a suspicion as to where this condition comes from, however it is difficult to connect his work with the present matter

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because it is quite ambigious as to how "is a fourier series" is defined. Also it is unclear what the domain of these series are, namely ~~the~~, where does θ live?

⑤ We could approach automorphisms of $\mathbb{C}[\omega(M)]$ ~~as~~ by treating it as ~~an~~ a set of countable sums of functions with complex coefficients over an arbitrary index set. This doesn't seem incredibly appropriate ~~of~~ fun. I propose:

Parameterize ~~[illegible]~~ M using a group hom from $\mathbb{Q}_x \cup \mathbb{Q}_x i$

Analyze the behaviour of functions in $\mathbb{C}[\omega(M)]$ w.r.t. automorphisms of $\mathbb{Q}_x \cup \mathbb{Q}_x i$.

$\mathrm{Aut}((\mathbb{Q}_x \cup \mathbb{Q}_x i, \cdot))$ includes the set $\{g : \mathbb{P} \to \mathbb{P} \mid g$ is one-to-one and onto$\}$ ~~[illegible]~~ where $\mathbb{P}$ is the set of prime elements of $\mathbb{Q}_x \cup \mathbb{Q}_x i$, if we take $\mathbb{Q}_x \cup \mathbb{Q}_x i$ as being generated by $\mathbb{P}$.

This is much more explicit than fiddling with arithmetic functions in other ways.

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Step 2:

Set $\mathbb{Q}_\times(i) := \mathbb{Q}_\times \cup \mathbb{Q}_\times i$

$\phi(a) := \omega(\varphi(a))$ where $\varphi(a)$ is the grp. hom from $\mathbb{Q}_\times(i)$ to M.

$\Phi(\alpha) := \bigotimes_{s\in S} \phi(\alpha(s))$

$$\Phi(\alpha) \in \bigotimes_{s\in S} \mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_\times,\cdot)) \subset \mathrm{Hom}((\mathbb{C}^S,+),(\mathbb{C}_\times^S,\cdot))$$

We need to do this because elements of the direct Products $\mathrm{Aut}(\mathbb{C},+)^S = \prod_{s\in S} \mathrm{Aut}(\mathbb{C},+)$

$$\mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_\times,\cdot))^S = \prod_{s\in S} \mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_\times,\cdot))$$

do not act on $(\mathbb{C}^S,+)$. The tensor map

$$\otimes : \mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_\times,\cdot))^S \longrightarrow \mathrm{Hom}((\mathbb{C}^S,+),(\mathbb{C}_\times^S,\cdot))$$

is just a canonical extention defined by:

$$\otimes(h)(z)(s) = \underbrace{(h(s))}_{\text{This is a function}}\underbrace{(z(s))}_{\text{its argument}} \quad \text{where } h \in \mathrm{Hom}((\mathbb{C},+),(\mathbb{C}_\times,\cdot))^S,\ z \in \mathbb{C}^S$$

little symbol

This is a nessisary procedure, because the direct Product of functions does not act on the direct Product of their domains.

It would be perfectly legitimate to use the Symbols $\bigoplus$ and $\oplus$.

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For $\alpha \in Q_s(i)^S$

$$\Phi(\alpha) : (\mathbb{C}^S, +) \longrightarrow (\mathbb{C}_\times^S, \cdot)$$

and it may be called a representation because it is a group homomorphism. (I should really say $\Phi(\alpha)$ is a representation of $(\mathbb{C}^S, +)$ in $(\mathbb{C}_\times^S, \cdot)$).

The character of $\Phi(\alpha)$ is determined by the characters of each "component function". i.e.

$$\chi_{\Phi(\alpha)} = \prod_{s \in S} \chi_{\phi(\alpha(s))} = \prod_{s \in S} \mathrm{Trace}(\phi(\alpha(s))) = \prod_{s \in S} \mathrm{Trace} \circ \phi(\alpha(s))$$

These are denoting multiplication

So for $z \in \mathbb{C}^S$ $\chi_{\Phi(\alpha)}(z) = \prod_{s \in S} \mathrm{Trace} \circ \phi(\alpha(s))(z(s)) = \prod_{s \in S} \phi(\alpha(s))(z(s))$

It is important to note that the codomain of $\Phi(\alpha)$ is $(\mathbb{C}_\times^S, \cdot)$, whereas the codomain of $\chi_{\Phi(\alpha)}$ is $\mathbb{C}$.

If we built $\tilde{\Phi}(\alpha)$ in the same way but using direct sums then $\chi_{\tilde{\Phi}(\alpha)} = \sum_{s \in S} \chi_{\phi(\alpha(s))}$.

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As algebraically sophisticated as this seems, its really not, and our parameterization of M has increased simplicity & added no baggage.

$$\Phi : Q_x(i)^S \times \mathbb{C}^S \longrightarrow \mathbb{C}_x^S$$

The equivalent map without the $Q_x(i)^S$ parameterization is $\otimes \circ \hat{\omega} : M^S \longrightarrow \mathbb{C}_x^S$ which we can extend to all of $\mathrm{Aut}(\mathbb{C},+)^S$: $\widehat{\otimes \circ \hat{\omega}} : \mathrm{Aut}(\mathbb{C},+)^S \longrightarrow \mathbb{C}_x^S$

Here, $\hat{\omega}$ is the pointwise application of ω to M^S

Regardless of ones preferences, the following vector space should be suitable for analysis:

$$\mathbb{C}[\{\Phi(\alpha) \mid \alpha \in Q_x(i)^S\}]$$

And! χ is a linear map from this space into $\mathbb{C}$.(ish)

I guess we should really define a new map on the vector space by $\Delta(c\,\Phi(\alpha)) := c\,\chi(\Phi(\alpha))$

and $\Delta(c\,\Phi(\alpha) + c'\,\Phi(\alpha')) = \Delta(c\,\Phi(\alpha)) + \Delta(c'\,\Phi(\alpha'))$.

χ and hence Δ are invariant under the transformation of $\Phi(\alpha)$ given by $\Phi(\alpha) \mapsto \Phi(\alpha) \circ R_g$ where R_g is the map from the earlier part of the paper. Neither, is invariant under the map $\Phi(\alpha) \mapsto \Phi(p(\alpha))$ where $p \in \mathrm{Aut}(Q_x(i)^S)$ for a general $\alpha \in Q_x(i)^S$

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The former suggests we can write

$$\Delta(\Phi(a)) = \int_G \chi(\Phi(a) \circ g \circ \pi)$$ where π is a projection onto a representative of $\mathrm{Injective}(\mathbb{C}^S)/G$.

This would follow by definition if it weren't for non-injective elements of $\mathbb{C}^S$. However, the set $\mathrm{NotInjective}(\mathbb{C}^S)$ is a null set for any measure on $\mathbb{C}^S$ given as a product measure of measures on $\mathbb{C}$.

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Dealing with the fact that $\{c \in \mathbb{C}^S \mid |c(S)| = 1\}$ is a subspace fixed by the action of G

Representation theorists would call the action of G, on $\mathbb{C}^S$, reducible. I am not as agitated by this fact as they are, though it is an important consideration. Below are two approaches.

OPTION 1: Work with a projective version of $\mathbb{C}^S$.

Perhaps: $P'(\mathbb{C}^S) := \overline{\text{span}_{\mathbb{C}}\{1_{\{\kappa(n)\}} - 1_{\{\kappa(n+1)\}}\}}$ where $\kappa : \mathbb{N} \to S$ is an arbitrary bijection, and work with this.

The image of $\mathbb{C}_S := \{c \in \mathbb{C}^S \mid |c(S)| = 1\}$

$$= \{x \in \mathbb{C}^S \mid \exists z \in \mathbb{C} \text{ s.t. } \forall s \in S \ x(s) = z\}$$

under P' is $\{\sum_{n \in \mathbb{N}} o(1_{\{\kappa(n)\}} - 1_{\{\kappa(n+1)\}})\}$. I'd like to limit my use of formal sums to more meaningful spaces of functions. Lets consider:

i.e. the polynomial ring in differences of coordinates

or $F^* P' =$ where $d_{s,t}(x) = x(s) - x(t)$.

$$\mathbb{C}_\kappa[d_{\kappa(n),\kappa(n+1)} \mid n \in \mathbb{N}]$$

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We can say $f \in F^*P'$ is ~~these~~ antisymmetric on its S-Support if for every transposition τ in $\mathrm{Aut}(\{s \in S \mid f$ has a term including $d_{s,t}$ or $d_{t,s}$ for some $t \in S\})$ $f(x \circ \tau) = -f(x)$. This is only possible if every term $d_{s,t}$ is raised to an odd power, (this is an if and only if condition).

~~[illegible]~~

Polynomials satisfying this requirement are widely used / desired throughout differential geometry, though they are usually constructed using "non-commuting indeterminants".

I suppose ~~these~~ the elements of $f \in F^*P'$ that are antisymmetric on S could be used to define some notion of a differential form.

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The Beauty of the Group Algebra $\mathbb{C}[G]$.

Given any polynomial f with coefficients in $\mathbb{C}$ and powers in $\mathbb{Z}$ is well defined on G.

Any polynomial f with complex coefficients and integral powers (i.e. in $\mathbb{N}$) is well defined on $\mathbb{C}[G]$

If we work with $\mathbb{C}[R_g \mid g \in G]$ then:

Every R_g satisfies $\dfrac{\partial R_g}{\partial x_s^2} = 0 \quad \forall s \in S$.

By Linearity of $\dfrac{\partial}{\partial x_s^2}$, the above statement holds for every element of $\mathbb{C}[R_g \mid g \in G]$

Every $g \in G$ is periodic, many do not have finite order. Suppose $g \in G$ has finite order, p, then:

$$\exp(tg) = \sum_{n=0}^{\infty} \frac{(tg)^n}{n!} = \sum_{m=0}^{\infty} \sum_{r=0}^{p-1} \frac{(tg)^{mp+r}}{(mp+r)!} = \sum_{m=0}^{\infty} \sum_{r=0}^{p-1} \frac{t^{mp+r} g^r}{(mp+r)!}$$

$$= \sum_{r=0}^{p-1} \sum_{m=0}^{\infty} \frac{t^{mp+r} g^r}{(mp+r)!} = \sum_{r=0}^{p-1} g^r \left(\sum_{m=0}^{\infty} \frac{t^{mp+r}}{(mp+r)!} \right)$$

which is a finite sum in $\mathbb{C}[G]$. The derivative of $\exp(tg)$ w.r.t. t is $g\exp(tg) = \sum_{r=0}^{p-1} g g^r \left(\sum_{m=0}^{\infty} \frac{t^{mp+r}}{(mp+r)!} \right)$

more generally $\dfrac{d}{dt^n} \exp(tg) = \sum_{r=0}^{p-1} g^{r+n \bmod p} \left(\sum_{m=0}^{\infty} \frac{t^{mp+r}}{(mp+r)!} \right)$

I consider this one of the most beautiful things in mathematics.

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