SOMETHING

JOJO

Hi Professor Templier, I want your help. This is a brief note on automorphisms, arithmetic, and representations, that reflects some capabilities and interests I have developed over the last 6 or 7 months of exploration. I have been working hard on this. A few things are wrong (esp. near the end). Things in red are things I don't like, are wrong, or require explanation/justification. There are some off handed comments I am not proud of. I have relevant thoughts and ideas that are not here.

Contents

1. Introduction

Definitions and notations are explicitly stated after the introduction. Please read them. All choices are very intentional. Some are not traditional, it is unfortunate they may not feel familiar but I am not sorry.

The subject of the first three sections is fairly standard. Broadly, they are concerned with the automorphism group of the right $\mathbb{A}$ module $A^n_{\mathbb{A}}$. These sections contain observations, provided as results, with proofs, because they demonstrate faults with standard constructions and theorems. Remedies are provided. The main result in section 1 is the decomposition of $\mathrm{Aut}(A^n_{\mathbb{A}})$ as a direct sum of automorphism groups of $\mathbb{A}$ modules. Section 2 discusses the nature of tensor products on $A^n_{\mathbb{A}}$, `revealing basic problems with the ‘‘tensor product decomposition"` and popular theorems applied to the structure of characters. A description and decomposition of the tensor algebra $T(A^n_{\mathbb{A}})$ extends the results in section 1, particularly, in relation to symmetric and alternating characters of representaions which are essential in the theory of L-functions. Section 3 considers the "L^2 problem" which turns out not to be well-defined. So I provide a "solution" to a general problem concerning a larger function space an arbitrary subgroup of $\mathrm{Aut}(A^n_{\mathbb{A}})$. It concludes with a discussion of irreducibility for representations of and in $\mathrm{Aut}(A^n_{\mathbb{A}})$; touching on the concept of automorphy.

The last three sections are more constructive, playful, and exploratory. Section 4 continues with function theory by describing an explicit approach to class functions. It is simple and powerful. In the case of $\mathrm{Aut}(A^n_{\mathbb{A}})$, I pass through some group theory thats caused some unnecessary anxiety in the literature, so I offer some relevant research questions that could be useful in Galois theory. The next section returns to modules, but reformulates them in a functional language that relies on simpler group theory. The goal is to reformulate the representation theory, which is framed as homomorphisms into modules over rings, in terms of abelian group acted on by a subring of endomorphisms acted on by a (”multiplicative”) subgroup of automorphisms of the abelian group contained in the centralizer of the subring ... on which automorphisms of the subgroup may act. This imposes structure and adds flexibility. A choice of abelian group immediately determines the subrings, groups, and automorphism groups that can follow. Also, intertwining maps are then necessarily invertible and conjugation by one must be an isomorphism of automorphism groups. Groups with trivial center arise naturally this way which act naturally on projective spaces. Loosening the centralizer requirement to a normalizer requirement alters the structure, but this can also provide factors of automorphy which reflects it. The last section applies the results of sections 1-5 and contains some musings/questions.

I do not follow follow some conventions. $\mathbb{A}^n$ is a dependent product, the traditional ring of adeles (denoted as dependent product. is a dependent product of is a dependent product of Mathematically, the definition of $\mathbb{A}$ is not restrictive and I concentrate on $\mathrm{Aut}(\mathbb{A}^n)$ is not restrictive. Notationally, functions expand to the right. Frequently used notation and definitions are explicit below.

Essential: Traditional ring of adeles is $\mathbf{A}$, the one I

Notations. Notations Assume X is a set, Y is a set, Z is a Set, and I is a set

- $\prod_{x\in X} Y$ is a set satisfying $\forall \alpha \in \prod_{x\in X} Y \; \forall x \in X \; \exists! \, y \in Y$

  Not: $[X \neq \emptyset \wedge \alpha \in \prod_{x\in X} Y] \iff [X \neq \emptyset \wedge \alpha : X \to Y] \iff [X \neq \emptyset \wedge X \xrightarrow{\alpha} Y]$

  Note: If $X = \emptyset$, then $\forall \alpha \in \prod_{x\in X} Y \; \forall x \in X \; \exists! \, y \in Y$

  Notation: A symbol, such as $F(X, Y)$, is sometimes used to denote $\prod_{x\in X} Y$
- $\forall \alpha \in \prod_{x\in X}$

The domain of α is $(\alpha)\mathrm{Dom} = X$.

The image of α is $(\alpha)\mathrm{Img} = \{(x)\alpha \mid x \in X\}$.

The image of $x \in X$ under α is $(x)\alpha$

- If $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$ then $\gamma = \alpha\beta : X \to Z$ defined for $x \in X$ by $(x)\gamma = ((x)\alpha)\beta$
- If $\forall i \in I\ X_i$ is a set, then $\prod_{i\in I} X_i := \{\alpha : I \to \bigcup_{i\in I} X_i \mid \forall i \in I\ (i)\alpha \in X_i\}$
- $\times : \prod_{i\in I} \prod_{u\in Y_i} Z_i \to \prod_{y\in\prod_{i\in I} Y_i} \prod_{i\in I} Z_i$ is defined by $(i)(y)(f)\times := ((i)y)(i)f$
- $\times : \prod_{i\in I} F(Y_i, Z_i) \to F(\prod_{i\in I} Y_i, \prod_{i\in I} Z_i)$ is defined by $(i)(y)(f)\times := ((i)y)(i)f$
- If $\alpha \in F(X,Y)$ and α is one-to-one, then $\forall f \in F(X,X)\ (\alpha)c := (\alpha)^{-1} f\alpha$. So: $\forall f \in F(X,X)\ \forall x \in X\ ((x)\alpha)(f)(\alpha)c = (x)f\alpha = ((x)f)\alpha$.
- For $n \in \mathbb{N}, [n] := \{k \in \mathbb{N} \mid k < n\}$

Definitions.

Definition 1. A **Group** is an incredible concept, see Ch. 1 of Theory of Groups by Phillip Hall for three definitions **??**.

Certain function spaces: If X and Y be groups, then

- $\alpha \in \mathrm{Hom}(X,Y) \iff [(\alpha)\mathrm{Dom} \wedge (\alpha)\mathrm{Img} = Y \wedge \forall x_1, x_2 \in X (x_1x_2)\alpha = (x_1)\alpha(x_2)\alpha]$
- $\alpha \in \mathrm{End}(X,Y) \iff [(\alpha)\mathrm{Img} \subseteq Y \wedge \alpha \in \mathrm{Hom}(X, (\alpha)\mathrm{Img})]$
- $\mathrm{Iso}(X,Y) \subseteq \mathrm{End}(X,Y)$ such that $\forall \alpha \in \mathrm{Iso}(X,Y) \forall y \in Y \exists! x \in X$.
- $\mathrm{Hom}(X) := \mathrm{Hom}(X,X), \mathrm{End}(X) := \mathrm{End}(X,X)$, and $\mathrm{Aut}(X) := \mathrm{Iso}(X,X) = \mathrm{Hom}(X) \cap S_X$

Definition 2. A **Ring** is ${}_RR_R = (R, \cdot_R : R^{[2]} \to R)$ such that:
R is an abelian group,
$(R, \cdot_R : R^{[2]} \to R)$ has an identity
$\cdot_R : R^{[2]} \to R$ is bilinear and associates.

- $\alpha \in \mathrm{Hom}({}_TT_T, {}_RR_R) \iff [\alpha \in \mathrm{Hom}(T,R) \wedge \forall t_1, t_2 \in T\ ((t_1,t_2)\cdot_T)\alpha = ((t_1)\alpha, (t_2)\alpha)\cdot_R]$
- If X and Y are rings then $\mathrm{End}(X,Y), \mathrm{Iso}(X,Y), \mathrm{End}(X)$, and $\mathrm{Aut}(X)$ are analogous to the definition for groups but with the definition of Hom for rings.

Definition 3. Given a ring, R, a **right R-module**, M_R, is an abelian group, M, with module multiplication, $\cdot_{M_R} : \{(m,r) \mid m \in M, r \in R\} \to M$ such that $\forall m, m_0 \in M\ \forall r, r_0 \in R$

- $(m +_M m_0, r)\cdot_{M_R} = (m,r) \cdot_{M_R} +_M (m_0, r)\cdot_{M_R}$
- $(m, r +_R r_0)\cdot = (m,r) \cdot_{M_R} +_M (m, r_0)\cdot_{M_R}$
- $((m,r)\cdot_{M_R}, r_0)\cdot_{M_R} = (m, r \cdot_R r_0)\cdot$
- and $(m, 1_R)\cdot_{M_R} = m$

Definition 4. A ring,$_RR_R$, is a field when $(R - \{0_R\}, \cdot_R)$ is a group.

Definition 5. A valuation of field, $_RR_R$, is a function, v, on $_RR_R$ taking values in an ordered multiplicative group with a zero element, $((H \cup \{0_H\}, <), \cdot_{H\cup\{0_H\}})$, **??** satisfying: $\forall a, b \in R$

- $(a)v = 0_H \iff a = 0_R$
- $(a \cdot_R b)v = (a)v \cdot_{H\cup\{0_H\}} (b)v$
- $(a +_R b)v \leq \{(a)v, (b)v\}\max$

Notation

- $n \in \mathbb{N}_\times$
- F is an abelian group and $\mathbb{F} = (F, \cdot_F) =$ is a field of characteristic 0.
- V := Places(F) := $\{v \mid$ v is a valuation of $\mathbb{F}\}/\sim_{orderisomorphism}$
- For $v \in V, F_v$ and $\mathbb{F}_v$ are the completions, with respect to v, of F and $\mathbb{F}$, respectively.
- $F_v^n := \prod_{i\in[n]} F_v$

Definition 6. $\mathbb{A}$,$\mathbb{A}^n$ (Globally)

- $\mathbb{A} = \prod_{v\in V} \mathbb{F}_v$
- $A^n := \prod_{v\in V} F_v^n = \prod_{v\in V} \prod_{i\in[n]} F_v = \prod_{i\in[n]} \prod_{v\in V} F_v$
- $\mathbb{A}^n = \prod_{v\in V} F_v^n\ {}_{\mathbb{F}_v} = A^n\ {}_{\mathbb{A}}$.

Definition 7. Ring of Adeles of $\mathbb{F}$ is $\mathbf{A} = \{x \in \prod_{v\in V} F_v \mid \{v \in V \mid 1 < (v)x\}$ is finite$\}$

Scope is End($\mathbb{A}^n$) and **focus** is Aut($\mathbb{A}^n$) for $n \in \mathbb{N}_>$. The traditional **Scope** is $\mathbf{A}$, $M(n, \mathbf{A})$, and **focus** is $GL(n, \mathbf{A})$ for $n \in \mathbb{N}_>$.

2. Automorphism Groups of Adelic Groups

Aut($\mathbb{A}^n$) is a direct sum. of automorphism groups of $\mathbb{A}$ modules.
Observe two consequences of the definitions:

- If $\alpha \in \text{Aut}(\mathbb{A}^n)$ then $\alpha \in \text{Aut}(A^n)$.
- Every $\alpha \in \text{Aut}(\mathbb{A}^n)$ commutes with each element of $\{(, r)\cdot \mid r \in \mathbb{A}\}$, that is: $\forall r \in \mathbb{A}\ (, r)\cdot\alpha = (()\alpha, r)\cdot$.

For each $v \in V$, define I_v by $(i)(w)I_v := \begin{cases} 1_{F_w} \text{w=v} \\ 0_{F_w} else \end{cases}$. Clearly, $I_v \in \mathbb{A}$. Now, for $W \subset V$, $1_W := \sum_{w\in W} I_w$.

For any $\alpha \in \text{Aut}(\mathbb{A}^n)$, $(((\ )\alpha, I_v)\cdot)\text{Img} = ((\ , I_v)\cdot)\text{Img}$ and $(\ , I_v) \cdot \alpha = ((\ )\alpha, I_v)\cdot$ implies that $((\ )\alpha, I_v)\cdot)\text{Img} = (((\ , I_v) \cdot \alpha)\text{Img}$. Since $(((\ , I_v) \cdot \alpha)\text{Img} = (Id_{((\ , I_v)\cdot\alpha)\text{Img}}\alpha)\text{Img}$, α fixes $((\ , I_v)\cdot)\text{Img}$, a subgroup of A^n denoted A_v^n. It is fixed by $\{(, r)\cdot \mid r \in \mathbb{A}\}$, hence $\mathbb{A}_v^n := (A_v^n, \cdot_{\mathbb{A}^n})$ is a right $\mathbb{A}$ module. (Analogously for $\beta \in \text{End}(\mathbb{A}^n)$, $(((\ , I_v) \cdot \beta)\text{Img} = (Id_{((\ , I_v)\cdot\beta)\text{Img}})\text{Img} \subseteq A_v^n)$.

This was for all places, v, and all automorphisms, α. It is now a matter of definition that $\mathbb{A}^n = \bigoplus_{v\in V}\mathbb{A}^n_v$. Algebraists restrict all direct sums with finite generation; I do not. A definition is below:

Definition 8. Let A be an abelian group.

$$A = \bigoplus_{j\in J} C_j \iff \begin{cases} \forall j \in J\ C_j \text{ is a subgroup of } A \\ \text{and } \exists\sigma : \prod_{j\in J} C_j \to A \text{ s.t.} \quad (c)\sigma = \sum_{j\in J}(j)c \text{ and } \sigma \text{ is one-one} \end{cases}$$

Definition 9. For an R-module, $B = A_R = (A, \cdot_{A_R}) = (A, \cdot_B)$.

$$B = \bigoplus_{j\in J} D_j \iff \begin{cases} A = \bigoplus_{j\in J} C_j \\ \text{and } \forall j \in J\ D_j \text{ is a module and } D_j = (C_j, \cdot_B) \end{cases}$$

Consequently, $\mathrm{End}(\mathbb{A}^n) = \bigoplus_{v\in V}\mathrm{End}(\mathbb{A}^n_v)$, and since $\mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{End}(\mathbb{A}^n)$, we have "$\mathrm{Aut}(\mathbb{A}^n) = \bigoplus_{v\in V}\mathrm{Aut}(\mathbb{A}^n_v)$". This is in the typical notation denoting a certain kind of situation that I'd like to clarify below:

$$\begin{tikzcd} \prod\limits_{v\in V}\mathrm{Aut}(\mathbb{A}^n_v) \arrow[r, "\times"] \arrow[dr, dotted, "\bigoplus"] & \mathrm{Aut}(\prod\limits_{v\in V}\mathbb{A}^n_v) \arrow[d, "(\sigma)c"] \\ & \mathrm{Aut}(\mathbb{A}^n) \end{tikzcd}$$

Here, $\times$ is a homomorphism ($\times$ is always injective) and $(\sigma)c$ is conjugation by σ (which is surjective), i.e $\alpha \mapsto (\sigma)^{-1}\alpha\sigma$ (where σ is a map satisfying the 2nd condition in Definition 9) . The subjectivity of $\times$ follows from the fact that each $\mathbb{A}^n_v$ is fixed by $\mathrm{Aut}(\mathbb{A}^n)$. Abstractly, Aut and $\bigoplus$, over R-modules contained in an R-module, mutually intertwine.

There's a lower half available too: For all $v \in V$ let $(v)\iota : F_v^{[n]} \to \mathbb{A}^n_v$ be a ring isomorphism, so ι is a function on V.

$$\begin{tikzcd} \prod\limits_{v\in V}\mathrm{Aut}(\mathbb{A}^n_v) & \mathrm{Aut}(\prod\limits_{v\in V}\mathbb{A}^n_v) \arrow[d, "((\iota)\times)c"] \\ \prod\limits_{w\in V}\mathrm{Aut}({}_{\mathbb{F}_w}\mathbb{F}_w^{[n]}) \arrow[u, "(\iota c)\times"] \arrow[r, "\times"] & \mathrm{Aut}(\mathbb{A}^n) \end{tikzcd}$$

The diagonal map should be $\bigoplus$ because for every $v \in PL(\mathbb{F})$ and $z \in \prod_{w\in PL(\mathbb{F})}\mathbb{A}^n_w$, $(v)(z)((\iota)\times)^{-1} = \sum_{w\in PL(\mathbb{F})}(v)(w)z$, i.e. $((\iota)\times)^{-1} = \sum : \prod\limits_{w\in PL(\mathbb{F})}\mathbb{A}^n_w \to \mathbb{A}^n$.

3. Multilinearity

If I is a set and $\forall i \in I\ L_i = ((M_i, +_i), \cdot_i)$ is a R_i-module, then $\prod_{i\in I} M_i$ is a set of functions, $\prod_{i\in I}(M_i, +_i) := (\prod_{i\in I} M_i, (+)\times)$ is an abelian group, and $\prod_{i\in I} L_i := ((\prod_{i\in I} M_i, (+)\times), (\cdot)\times)$ is a $\prod_{i\in I} R_i$ module. As usual, the abelian group $((\prod_{i\in I} M_i, (+)\times)$ may be given a T module structure for any ring $T \subseteq \prod_{i\in I} R_i$ containing the identity.

Tensor Products. The domain of a tensor product is the direct product of modules over the same ring.

The Tensor Product on a Direct Product of R-modules. $M_R \bigotimes_R N_R$

The Tensor Product on Direct Products $\mathbb{A}$ modules. For all $n \in \mathbb{N}_\times$:
$\mathbb{A}^n$ is not the direct product of $\mathbb{A}$-modules (because $\mathbb{A}^n$ is an $\mathbb{A}$-module).

So we consider some direct products of $\mathbb{A}$-modules containing $\mathbb{A}^n$.

Let $x, y \in \mathbb{A}^n$ and evaluate the tensor product

$$(x,y)\otimes = ((x, 1_{supp(x)})\cdot, (y, 1_{supp(y)})\cdot)\otimes = ((x, 1_{supp(x)}1_{supp(y)})\cdot, (y, 1_{supp(y)}1_{supp(x)})\cdot)\otimes$$
$$= ((x, 1_{supp(x)\cap supp(y)})\cdot, (y, 1_{supp(y)\cap supp(x)})\cdot)\otimes \quad = ((x,y)\otimes, 1_{supp(x)\cap supp(y)})\cdot$$

When $u, w \in V$ and $u \neq w$, we see that:

$$\begin{aligned}(\mathbb{A}^n_u \bigoplus \mathbb{A}^n_w) \bigotimes (\mathbb{A}^n_u \bigoplus \mathbb{A}^n_w) &= (\mathbb{A}^n_u \bigotimes \mathbb{A}^n_u) + (\mathbb{A}^n_u \bigotimes \mathbb{A}^n_w) + (\mathbb{A}^n_w \bigotimes \mathbb{A}^n_u) + (\mathbb{A}^n_w \bigotimes \mathbb{A}^n_w)\\ &= (\mathbb{A}^n_u \bigotimes \mathbb{A}^n_u) + \{(0_{\mathbb{A}}, 0_{\mathbb{A}})\otimes\} + \{(0_{\mathbb{A}}, 0_{\mathbb{A}})\otimes\} + (\mathbb{A}^n_w \bigotimes \mathbb{A}^n_w)\\ &= (\mathbb{A}^n_u \bigotimes \mathbb{A}^n_u) \bigoplus (\mathbb{A}^n_w \bigotimes \mathbb{A}^n_w)\end{aligned}$$

So: $\mathbb{A}^n \bigotimes \mathbb{A}^n = (\bigoplus_{v\in V} \mathbb{A}^n_v) \bigotimes (\bigoplus_{v'\in V} \mathbb{A}^n_{v'}) = \bigoplus_{\tilde{v}\in V} (\mathbb{A}^n_{\tilde{v}} \bigotimes \mathbb{A}^n_{\tilde{v}})$.

Let $T^1(X) = X$ and $T^d(X) = T^{d-1}(X) \bigotimes X, 1 < d \in \mathbb{N}$. Then by associativity (or induction if you want), the $\mathbb{A}^n$-tensor module is:

$$T(\mathbb{A}^n) = \bigoplus_{d\in\mathbb{N}_\times} T^d(\mathbb{A}^n) = \bigoplus_{d\in\mathbb{N}_\times} \bigoplus_{v\in V} T^d(\mathbb{A}^n_v) = \bigoplus_{v\in V} \bigoplus_{d\in\mathbb{N}_\times} T^d(\mathbb{A}^n_v) = \bigoplus_{v\in V} T(\mathbb{A}^n_v)$$

Clearly, the symmetric and alternating tensor algebras decompose similarly. And as in first section, the endomorphisms of $T(\mathbb{A}^n)$ fixes each $T(\mathbb{A}^n_v), v \in V$, so the group of endomorphisms decomposes as a direct sum and the group of automorphisms decompose likewise.

$$\mathrm{End}(T(\mathbb{A}^n)) = \mathrm{End}(\bigoplus_{v\in V} T(\mathbb{A}^n_v)) = \bigoplus_{v\in V} \mathrm{End}(T(\mathbb{A}^n_v)) \supset \bigoplus_{v\in V} \mathrm{Aut}(T(\mathbb{A}^n_v))$$

Characters!!! If ρ is a representation of $\mathrm{Aut}(\mathbb{A}^n)$, then $\rho = \oplus_{v\in V}\rho_v$ and $\rho\mathrm{Tr} = \chi_\rho = \sum_{v\in V} \chi_{r_v} = \sum_{v\in V} r_v \mathrm{Tr}$. For instance, if χ^d_σ and χ^d_α are the characters of ρ on the d^{th} symmetric and alternating power of $\mathbb{A}^n$, respectively, then $\chi^d_\sigma = \sum_{v\in V} \chi^d_{\sigma_v}$ and $\chi^d_\alpha = \sum_{v\in V} \chi^d_{\alpha_v}$, respectively. As a consequence

$$\chi_{\rho^{\otimes d}} = \chi^{\mathbf{d}}_\rho = \chi^d_\alpha + \chi^d_\sigma$$

A Word On The Use of Tensor Products. The domain of a tensor product is the direct product of the abelian groups of modules over the same ring.

Bump, among others, takes a restricted tensor product of representations $\rho_v : GL(n, F_v) \to GL(n, F_v)$, $v \in V$. Its image acts on the restricted tensor product of the "VECTOR SPACES", X_v, $v \in V$. So, a X_v can't be a F_v-vector space because there is not a tensor product of modules over different rings. Bump, unlike others, clarifies in the preface: "results in section 3.3 are complete only when the ground field is $\mathbb{Q}$

... in which case $\mathrm{Aut}(\overline{\mathbb{Q}})$ has at least continium-ly many representations.

Also, the first part of previous section renders Theorem 1.1 of Flath is false **??**.

I have searched many references and found no other mention of Bump's remark. Nor is there any explantion as to why we are developing a theory of finite dimensional matrix representations acting on such a massive vector space. Also, why even represent automorphisms as matrices? The essential property of field automorphisms is that they are linear, not only over addition but also over multiplication. In addition, matrix representations are not essential to carry out analysis.

4. The L^2 "Problem"

I will depart from the previous notation to solve a problem "fundamental in the theory of automorphic representations" [37]. The following is essentially stated on page 5 of "The Selberg-Arthur Trace Formula" by Shokranian [37], page 7 of "An Introduction to the Trace Formula" by Arthur [2] , and page 1 of "On the Functional Equations Satisfied by Eisenstein Series" Langlands [28]:

Let $\mathbf{A} = \{x \in \prod_{v\in V} F_v \mid \{v \in V \mid x_v \notin \mathbb{Z}_v\} \text{ is finite}\}$, $H = L^2(GL(n,\mathbb{Q})\backslash GL(n,\mathbf{A}))$ and define $R : GL(n,\mathbf{A}) \to \mathrm{Aut}(H)$ by $(R(g)\phi)(z) = \phi(zg)$ where $\phi \in H, z \in GL(n,\mathbf{A})$. **Problem:** Decompose R explicitly into irreducible representations.[1]

The group of units of $\mathbf{A}$ is $\{x \in \mathbf{A} \mid \{v \in V \mid |x_v|_v \neq 1\} \text{ is finite}\}$. So either $GL(n,\mathbf{A}) = \{m \in M_n(\mathbf{A}) \mid \forall v \in V det(m)(v) \neq 0_{F_v}\}$ and $GL(n,\mathbf{A})$ is not a group, or we accept $Z(GL(n,\mathbf{A})) \simeq \mathbf{A}_\times$.

I will solve a more general problem. The above problem will be answered as a consequence.

Assume:

- $\mathbb{C}$ carries a norm.
- U is a subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ and $U\backslash\mathrm{Aut}(\mathbb{A}^n)$ is a set of left cosets of U in $\mathrm{Aut}(\mathbb{A}^n)$.

Let:

- K be the $\mathbb{C}$ vector space $\prod_{B\in U\backslash\mathrm{Aut}(\mathbb{A}^n)} \mathbb{C} = \{f : U\backslash\mathrm{Aut}(\mathbb{A}^n) \to \mathbb{C} \mid \text{ f is a function}\}$.
- H be the $\mathbb{C}$ vector space $L^2(U\backslash\mathrm{Aut}(\mathbb{A}^n), \mathbb{C}; \mu) = \{f : U\backslash\mathrm{Aut}(\mathbb{A}^n) \to \mathbb{C} \mid |f|^2 \text{ integrated w.r.t } \mu \text{ is finite}\}$ where $U\backslash\mathrm{Aut}(\mathbb{A}^n) \subset (\mu)\mathrm{Dom}$.
- $r : \mathrm{Aut}(\mathbb{A}^n) \to \mathcal{S}_{U\backslash\mathrm{Aut}(\mathbb{A}^n)}$ given by $r_g(Ux) = Uxg, x \in \mathrm{Aut}(\mathbb{A}^n)$; this is anti-isomorphic to the permutation representation of $\mathrm{Aut}(\mathbb{A}^n)$ on $U\backslash\mathrm{Aut}(\mathbb{A}^n)$ (because we're applying functions on the left). (See p. 56-57 of [17] for insight)
- $R : \mathrm{Aut}(\mathbb{A}^n) \to (\mathrm{Aut}(K), \circ)$ by $R_g(f) = f \circ r_g$; this is a homomorphism into $(\mathrm{Aut}(K), \circ)$:
$$(R_h \circ R_g)(f) = R_h(R_g(f)) = R_h(f \circ r_g) = f \circ r_g \circ r_h = f \circ r_{hg} = R_{hg}(f).$$
- T be the image of R.

Important Note: If T is contained in $(\mathrm{Aut}(H), \circ)$, then H must be a T-invariant subspace of K. The original statement implicitly assumes that H is T-invariant, but this is not necessarily true. Assuming that Shokranian, Arthur, and Langlands wish H to be T-invariant, H must be a direct summand of K that is T-invariant. So it makes sense to investigate the T-invariance of K: **Problem:** Decompose R into irreducible representations.

[1] A problem is not exactly specified: 1) H is not well-defined until we specify a measure (it is meaningless to write $L^2(Y)$ for a set Y) and 2) ϕ is not in H so $R(g)$ can't be in $\mathrm{Aut}(H)$.

T fixes each $x \in K$ that takes exactly one value. The set of such elements is denoted $\mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$. It is a subspace of K and T-invariant.

The action of $\mathrm{Aut}(\mathbb{A}^n)$ on $U\backslash\mathrm{Aut}(\mathbb{A}^n)$ is transitive and primitive. So either there is a T-invariant subspace M such that $K = M \bigoplus \mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$ or the representation is not reducible. If $U\backslash\mathrm{Aut}(\mathbb{A}^n)$ is finite, M is the subspace where every element sums to 0. The representation decomposes accordingly.

Now, assume $U\backslash\mathrm{Aut}(\mathbb{A}^n)$ is not finite. Let's suppose $\mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$ is a direct summand, i.e. there exists a subspace, M, such that $K = M \bigoplus \mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$. Then there must exist a homomorphism from K onto a submodule with kernel $\mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$.

Here is one: $D : x \mapsto x \circ \tau - x = (x(Ux) \mapsto x(\tau(Ux)) - x(Ux))$ where τ is a full cycle on $U\backslash\mathrm{Aut}(\mathbb{A}^n)$.[2] D is linear and commutes with $\mathbb{C}$-multiplication, so D is module homomorphism and $K = \mathrm{Img}(D) \oplus \mathrm{Ker}(D)$. From $\mathrm{Ker}(D) = \{x \in K \mid \forall B \in \mathcal{B} x(\tau(B)) - x(B) = 0\} = \mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$, we find that $M = \mathrm{Img}(D)$. It is straight forward to verify M is T-invariant. Since each summand is a T-invariant subspace, the representation decomposes into a direct sum $\mathrm{Aut}(\mathrm{Img}(D)) \oplus \mathrm{Aut}(\mathrm{Ker}(D)) = \mathrm{Aut}(M) \oplus \mathrm{Aut}(\mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)})$.

Does any summand decompose further? No. The set of $\mathbb{C}$-modules generated by additive closures of orbits of T on K is $\{\{0_K\}, \mathbb{C}_{U\backslash Aut(\mathbb{A}^n)}, M, K\}$. Other than the zero module, there is no additive closure of a T orbit contained in $C_{U\backslash Aut(\mathbb{A}^n)}$ or $M = \mathrm{Img}(D)$. So neither summand can decompose further into a direct sum of non-trivial T invariant $\mathbb{C}$-modules. Hence, $\mathrm{Aut}(\mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)})$ and $\mathrm{Aut}(M)$ are not reducible. As a consequence, a T-invariant subspace of K must be one of the following:

$$\{0_K\},\ \mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)},\ M, \text{ or } K = \mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)} \bigoplus M$$

If we're still requiring that $H = L^2(U\backslash\mathrm{Aut}(\mathbb{A}^n), \mathbb{C}; \mu)$ is T-invariant, then H must be one of the subspaces above. This imposes restrictions on μ:

- If μ is not a finite measure then $H = \{0_K\}$.
- If μ is a finite measure, but not finitely supported, then $H = \mathbb{C}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$.
- Otherwise μ is finitely supported and a finite measure, in which case $H = K$.

In the second case, the $\mathbb{C}$-linear subspace generated by the orbit of T on $L^2(U\backslash\mathrm{Aut}(\mathbb{A}^n), \mathbb{C}; \mu)$ is K. 1_U is in $L^2(U\backslash\mathrm{Aut}(\mathbb{A}^n), \mathbb{C}; \mu)$ and the $\mathbb{C}$ subspace generated by the T-orbit of 1_U is all of K. Any $x \in K$ equals $\sum_{B \in U\backslash \mathrm{Aut}(\mathbb{A}^n)} x(B) \cdot 1_B \in \langle c \cdot t(1_U) \mid c \in \mathbb{C}, t \in T\rangle$. If you're mightily concerned, the very first assumption is that $\mathbb{C}$ comes with a norm, so $\sum_{B \in U\backslash \mathrm{Aut}(\mathbb{A}^n)} x(B) \cdot 1_B \to x$ in the weak-* topology. With the definition of section 2, $K = \bigoplus_{B \in U\backslash \mathrm{Aut}(\mathbb{A}^n)} \mathbb{C}1_B$, both as an abelian group and as a $\mathbb{C}$-module.

So long as $H \subseteq K$, the "right regular representation" of $\mathrm{Aut}(\mathbb{A}^n)$ acts on H. This representation is not the same as one restricted to a subset of K, and necessarily so.

3

[2](Note: $\tau \in \mathcal{S}_{U\backslash \mathrm{Aut}(\mathbb{A}^n)}$ and τ is a commutator [32])

[3]Common choices on U include: "$GL(n, \mathbb{Q})$" and "$Z(\mathbb{A})GL(n, \mathbb{Q})$" where "$GL(n, \mathbb{Q})$" denotes the "diagonal embedding" of $GL(n, \mathbb{Q})$ in $GL(n, \mathbb{A})$. What is this choice trying to achieve? What does it achieve? Why not take $U = \prod_{v \in V} GL(n, Q_v)$ where Q_v is the field isomorphic to $\mathbb{Q}$ contained in $\mathbb{Q}_v$. I feel a little pedantic distinguishing between $\mathbb{Q}$ and Q_v, but its important: $\mathrm{Aut}(\mathbb{A}^n)$ would be radically different if $\mathbb{Q}$ was a subset of $\mathbb{Q}_v$ for all $v \in V$. Also, essentially all of Section 1 is a consequence of the fact that for distinct places v,w, $\mathbb{Q}_v \cap \mathbb{Q}_w = \emptyset$, which is a consequence of $0_{\mathbb{Q}_v} \neq 0_{\mathbb{Q}_w}$.

What Makes a Representation Automorphic? In 1979, according to Borel and Jacquet, nothing. For them, "automorphic" is an adjective for elements of the Hecke Algebra of something satisfying a slew of analytic desiderata. [8] (1979)

Langlands felt differently. For him, an automorphic representation is a constituent of a representation induced by a parabolic subgroup non-trivial on Levi-factors ([25], published in 1979). He frames this as a result; for the definition the reader is refered to the paper by Borel and Jacquet printed on the pages before his ([8]). Yet, he never uses their definition in his "result."

Two years later, Jacquet and Shalia describe an automorphic representation as a "sub-quotient" of a representation induced by a parabolic subgroup. This characterization differs from Langlands' because they Induce a representation from a parabolic subrgoup but require Ind to be trivial on the unipotent component. All of Langlands' automorphic representations are equivalent to one of Jacquet and Shalia. ([20], published 1981, p. 807-810).

A few years later, Gelbart says a representation of $\mathrm{Aut}(\mathbb{A}^n)$ in a Hilbert space is automorphic if it is equivalent to a sub-representation of the so called right regular representation. ([15], published 1984)

The "induction" in both of these papers is inconsistent with the typical definition of "induction" in representation theory (e.g. [**?**]) ...

- Let α be a representation of G in $\mathrm{Aut}(V)$, H a subgroup of G, and W **a** subspace of V fixed by the image of H under α.
- Let $\beta = I_H\alpha$
- α is induced by $\beta \iff \sum_{Hg\in H\backslash G} W\alpha_g = V = \bigoplus_{Hg\in H\backslash G} W\alpha_g$.
- If α is induced by β, then $\dim(V) = \dim(W)[G : H]$

The alleged induction operations in [25] and [20] do not satisfy $\dim(W)|\dim(V)$. Both papers also induce a tensor product of Parabolic groups, $\{P_{r_i} \in \mathrm{Aut}(V_{r_i})\}_{i=1}^k$ where $\dim(V_{r_i}) = r_i$ and $r \vdash n = \dim(V)$. This forces the fixed subspace of the tensor product of representations of the P_{r_i} to be the tensor product of the V_{r_i}. In which case, every $r_i|n$ and $\prod_{i=1}^k r_i < n$. More severely, these representations were obtained from α, and α acts on V which is not a tensor product of its subspaces.

5. Class Functions and Functions on Conjugacy Classes

Throughout this section G is a group

Class functions are functions on a group taking the same value on ab and ba, for all $a, b \in G$.

Class functions are incredibly powerful. Many people are aware of them, yet there's an aura that they arise through traces of representations when its the other way around. Also, this should be relevant to Adelic function theory because:

- Galois representations are only determined up to conjugacy.
- An L-function of a representation is a class function
- $\neg$ [An a class function is a L-function of a representation]
- The function taking a representation to the L-function of it, is a class function.

Facts:

- The number of irreducible representations (up to non-arithmetic isomorphism) equals the number of conjugacy classes
- A homomorphism from into a commutative group can't separate points.

Let G be a group and C a commutative group. The set of K-valued class functions is

$$\mathfrak{X}(G;K) = \{f : G \to K \mid \forall x, y \in G\ (xy)f = (yx)f\}$$

And $\widehat{\alpha} : \mathrm{Inn}(G) \to \mathrm{Aut}(\mathfrak{X}(G;K))$ be a representation of the automorphism group of G, $(x)(f)\widehat{\alpha_g} = (g^{-1}xg)f$. Now, $\mathfrak{X}(G;K)$ is invariant under the image of $\widehat{\alpha}$, so its isomorphic to the set of K-valued functions on conjugacy classes $K^{(G/\sim_{\mathrm{Inn}(G)})}$.

$X := G/\sim_{\mathrm{Inn}(G)}$ partitions G.

- Every normal subgroup in G contains an element of X.
- X generates a topology $t_X = \{\cup_{y \in Y} y \mid Y \subset X\}$. So, every $c \in X$ is compact in (X, t_X).
- Accordingly, $\mathfrak{X}(G;K)$ has a K-basis $\{k_b \mid b \in X, (c)k_b \in K_\times \iff b = c\}$
- $\mathfrak{X}(G;K)$ injects into the set of K-valued measures on t_X.

Representation theory addresses and is credited for the theory of class functions. Most results are obtained through the use of characters, which are obtained as a trace of a representation. This works marvelously for finite groups. Representation theory becomes opaque for non-finite groups because there isn't always a trace function on the image of a representation and non-essential requirements are imposed on representations. For a popular example, it is stated that:

(1) The representations of the group of rotations in the plane, $C_\infty = \{r_\theta \mid \theta \in [0, 2\pi)\}$ are $\{\rho_n : C_\infty \to GL(1,\mathbb{C}) \mid n \in \mathbb{Z}\}$ where $(r_\theta)\rho_n := [e^{in\theta}]$

(2) The characters are $\chi_{\rho_n} = \rho_n \mathrm{Tr} : C_\infty \to \mathbb{C}_\times$

(3) And the characters $\{\chi_{\rho_n} \mid n \in \mathbb{N}\}$ is an *orthogonal* $\mathbb{C}$-basis for the space of class functions on T.

Let that sink in. Now, recall:

ALL $\mathbb{C}$-valued functions on C_∞ are $\mathbb{C}$-valued class functions on C_∞.

Assuming statements 1), 2), and 3), every $\mathbb{C}$-valued function on C_∞ is a $\mathbb{C}$-linear combination of the characters $\{\chi_{\rho_n} \mid n \in \mathbb{N}\}$. But the indicator function of each subgroup is a class function and not all of these are contained in the $\mathbb{C}$ span of $\{\chi_{\rho_n} \mid n \in \mathbb{N}\}$, e.g. the subgroup $\{r_0\}$ or a subgroup generated by $r_{\frac{2\pi}{n}}$ where $n \in \mathbb{N}_\times$.

This comes down to the fact that $\{\rho_n \mid n \in \mathbb{N}\}$ is a strict subset of $\mathrm{End}(C_\infty, GL(1,\mathbb{C}))$. With other groups, continuity and other topological requirements are imposed on representations (homomorphisms); this inhibits group theory and the resulting theory of characters is not a theory of class functions. It is deceptive and wrong to suggest otherwise, especially in light of the fact that all Fourier analysis is subsumed by class functions.

Consider $G = \mathrm{Aut}(\mathbb{A}^n)$. $G/\sim_{Inn(G)}$.

$$\begin{aligned}
\simeq \prod_{v\in V} \mathrm{Aut}(\overline{\mathbb{F}_v^n})/\sim_{Inn(\mathrm{Aut}(\overline{\mathbb{F}_v}^n))} &\simeq \prod_{v\in V}\{[T] \mid [T] \in \overline{\mathbb{F}_v^n}/\sim_{S_{[n]}}\} \\
&\simeq \prod_{v\in V}\{x : \overline{\mathbb{F}_v^\times} \to [n+1] \qquad \mid \sum_{q\in\overline{\mathbb{F}_v}} (q)x = n\} \\
&\simeq \{z : \prod_{v\in V} \overline{\mathbb{F}_v} \to [n+1] \qquad \mid \forall v \in V \sum_{q\in\overline{\mathbb{F}_v}} ((v)q)z = n\} \\
&\simeq \{z : \overline{\mathbb{A}} \to [n+1] \qquad \mid \forall v \in V \quad \sum_{q\in\overline{\mathbb{A}}} ((v)q)z = n\} \\
&\simeq \{z : \overline{\mathbb{A}} \to [n+1] \qquad \mid \sum_{q\in\overline{\mathbb{A}}} ((v)q)z = n\}
\end{aligned}$$

This is a space of divisors. On what and how so? is fairly flexible. Up to you. The last set offers one perspective of $G/\sim_{Inn(G)}$.[4]. This is not the only perspective. Alternatively,

$$\{v : N \to \{Q \subset \overline{\mathbb{A}} \mid Q\# \le n\} \mid \sum_{k=1}^{n} \sum_{q\in(k)v} k = \sum_{k=1}^{n} ((k)v)\# \cdot k = n\}$$

FOR $n > 1$, THE SUBGROUP OF $\mathbb{A}_\times^{[n]}$ WITH ALL ELEMENTS OF ORDER n IS A LATTICE OF RANK n-1? FOR $p|n$ THE SUBGROUP OF ORDER. $\sum_{p\in P, p<n}$ card of order p elements $* p * p - 1 = n!$????? It appears linear systems of divisors, Itsforward correspondence between the set on the last line and the set of divisors on $\mathbb{A}$ with degree n.

There is a distinguished set of subset of class functions, $\mathrm{Hom}(\mathrm{Aut}(\mathbb{A}^n); C)$. Suppose $h \in \mathrm{Hom}(\mathrm{Aut}(\mathbb{A}^n); C)$ and $m \in \mathrm{Aut}(\mathbb{A}^n)$, then m spits in $\mathrm{Aut}(\overline{\mathbb{A}}^n)$

$$\begin{aligned}
(m)h = (g^{-1}Diag(x_1,\dots,x_n)g)h &= (g^{-1})h \cdot (Diag(x_1,\dots,x_n))h \cdot (g)h \\
&= (Diag(x_1,\dots,x_n))h = (Diag(x_1,1,\dots,1)\dots Diag(1,\dots,1,x_n))h \\
&= (Diag(x_1,1,\dots,1)\dots Diag(x_n,\dots,1,1))h \\
&= (Diag(x_1\dots x_n,\dots,1,1))h \\
&= (Diag((m)det,\dots,1,1))h
\end{aligned}$$

So the set of class function homomorphisms on $\mathrm{Aut}(\mathbb{A}^n)$ is isomorphic to a function space over $\mathbb{A}^\times$. While we dipped our toe into the algebraic closure, $(m)det$ is in $\mathbb{A}^\times$ and the homomorphism does not depend on how the diagonal factors, just the value.

Arthur [3], Langlands, and other researchers seem to be anxious that two elements may be conjugate in $\mathrm{Aut}(\overline{\mathbb{A}}^n)$ but not $\mathrm{Aut}(\mathbb{A}^n)$. It should be this way because the normalizer, in G, of a subset Y is always contained in the normalizer, in H, of S when $G \subset H$. There is exactly one thing we learn in the case that $Y \subseteq \mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{Aut}(\overline{\mathbb{A}}^n)$: the normalizer of $\mathrm{Aut}(\mathbb{A}^n)$ in $\mathrm{Aut}(\overline{\mathbb{A}}^n)$ is not all of $\mathrm{Aut}(\overline{\mathbb{A}}^n)$. Anxiety about this fact seems to be occupying some research that I'd like to redirect:

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[4]The isomorphism of the last line is the reason I was a precise pain in the butt in section 2

Let K be a field. If L is an algebraic extension, then the minimal polynomial of L over K is characteristic polynomial of some element $g \in GL([L:K], K)$. As we know, all conjugates of g have the same characteristic polynomial. However, the elements of $GL([L:K], K)$ of with zeros on a set that is a K-basis for L are not all conjugate to g or in the normal subgroup generated by the conjugacy class of m. Question 1: Let $\sim_{ext}$ be the relation on $GL([L:K], K)$ that partitions it into subsets having characteristic polynomials with zeros that span the same extension of K. Is there a terminating algorithm that can determine if two elements in $GL([L:K], K)$ are related? If so, provide one. Question 2: Does the above relation partition $GL([M:L], M([L:K], K))$? If not, can you find one?

6. Special Functions

Early this year, the notion of an L-function drew my intrigue. I was enamored by the idea that a function defined by some automorphism group could reveal meaningful algebraic properties of its domain.

Functions of a complex variable, z, that are (limits of) algebraic functions of algebraic functions of q^z where $q \in \mathbb{Q}_>$[5].

This includes every L-function I've encountered. Examples include:

- **Bernard Riemann** was interested in a function, which he wrote about and piqued a lot of interest from many humans. For each $s \in \{c \in \mathbb{C} \mid 1 < \Re(c)\}$ there is a unique $\zeta(s) \in \mathbb{C}$
$$\zeta(s) = \sum_{n \in \mathbb{N}_\times} (1/n)^s$$
- **Gustav Dirichlet** produced a set of functions to help a proof of his. For each ideal $J \subset \mathbb{Z}$, and homomorphism $\chi : (\mathbb{Z}/J)^\times \to \mathbb{C}_\times$, and $s \in \mathbb{C}$ there is a unique $L(s, \tilde{\chi}) \in \mathbb{C}$
$$L(s, \chi) = \sum_{n \in \mathbb{N}_\times} \tilde{\chi}(k + J)/n^s$$
Gustav Dirichlet's method was extended to other domains and functions on them. Examples of domains include number fields, ring of integers of number fields, the upper half plane, subsets of linear operators on automorphic forms, and elliptic curves over $\mathbb{Q}$. The functions used expanded to automorphic forms for subgroups of $SL(2, \mathbb{Z})$, eigenvectors for the Laplace operator that are automorphic functions for $SL(2, \mathbb{Z})$, eigenvalues for linear operators, and arithmetic information obtained by reducing curves modulo a prime number. (A function is an automorphic if it is a eigenvector for a linear operator on a function space representing the action of an automorphism of the domain. When the corresponding eigenvalue is 1, it is termed an automorphic function, otherwise it is an automorphic form.)
- **Emil Artin** introduced a generalization.
- [4] **Arthur and Gelbart**, in "Lectures on Automorphic L-functions", say for a number field F and representation $r : Gal(\overline{F}/F) \to GL(n, \mathbb{C})$ the local L-function is
$$L_v(s, r) := (\mathrm{Det}(I - \sigma_v(r) q_v^{-s}))^{-1}$$

[5] "algebraic functions of algebraic functions" is not a typo

and the global one is

$$L(s,r) := \prod_{v \in URV(r)} L_v(s,r)$$

where $URV(r)$ is a subset of V determined by the Frobenius conjugacy class in $Gal(\overline{\mathbb{F}}/F)$, $q_v = \#(\mathbb{Z}_v/B_{F_v}(0,1))$, and $\sigma_v(r) = Diag(q_v^{-z_{v,1}}, \ldots, q_v^{-z_{v,n}})$.

- [37] **Shokranian**, in "The Selberg-Arthur Trace Formula", gives a similar one for $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(n, \mathbb{C})$

$$L(s,\rho) := \prod_{p \in URP(\rho)} \mathrm{Det}((I - \phi_p(\rho)p^{-s}))^{-1}$$

where, similarly, $URP(r)$ is determined by the Frobenius conjugacy class in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\sigma_p(r) = Diag(p^{-z_{p,1}}, \ldots, p^{-z_{p,n}})$.

- Accompanying a string of conjectures outlining a research program, Robert Langlands introduced a notion of an L-function for automorphisms of a module over the ring of adeles of a number field. For each irreducible representation, ρ, of the group of square matrices with entries in the ring of adeles of a number field F, $GL(n, \mathbb{A}_F)$, and $s \in \{c \in \mathbb{C} \mid 1 < \Re(c)\}$ there is a $L(s,\rho) \in \mathbb{C}$

$$L(s,\rho) = \prod_{v \in Places(F)} L(s,\rho_v) = \prod_{v \in Places(F)} \prod_{j=1}^{deg(\rho_v)} \frac{1}{(1 - \frac{\phi_v, j(\rho)}{N(v)^s})}$$

- **Questions**:

  Is there are algorithm to determine $URV(r)$ and $URP(r)$

  General Form of an L-functions "of" algebraic structures over a field or actions on them

- **Challenge**: Define σ_p, explicitly.

Carl-Erik Froberg explores a paper inspired by Bernard Riemann's using methods evolving from Gustav Dirichlet's. For each $s \in \{c \in \mathbb{C} \mid 0 < \Re(c)\}$ there is a unique $P(s) \in \mathbb{C}$

$$P(s) = \sum_{k \in \mathbb{N}_\times} (\mu(k)/k) log\zeta(ks)$$

The underlying philosophy is that these functions (should) reflect certain arithmetic or geometric properties. It is evident that they (often) do. I have not seen an explanation as to *why* they should.

q^z is ambiguous. $q^z := \exp(z\log(q))$, so we must specify a logarithm of q. For distinct choices, $\hat{\log}(q)$ and $\tilde{\log}(q)$, $\exp(z\hat{\log}(q))$ and $\exp(z\tilde{\log}(q))$ are distinct analytic functions of z that agree on $\mathbb{Z}$. One cannot be continued into the other. Every instance of "unique" in the above list presumes a choice of $\log(n), \log(n), \log(q_v)$,or $\log(p)$ for each $n \in \mathbb{N}_x, n \in \mathbb{N}_x, q_v \in \{\#(\mathbb{Z}_v/B_{F_v}(0,1)) \mid v \in V\}$, or $p \in \mathbb{P}$, respectively.

For any branch of logarithm ℓ_p for each $p \in \mathbb{P}$, we let $x_p(q,z) = \exp(z\ell_p(q))$ have the equalities

$$\forall P \in \mathbb{P} \quad \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{x_p(1,z)}{x_p(p,z)}} = \sum_{n \in \mathbb{N}_\times} \frac{x_P(1,z)}{x_P(n,z)}$$

[6] in contrast to the following *equation*

$$\prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{x_p(p,z)}} = \sum_{n\in\mathbb{N}_\times} \frac{1}{x_n(n,z)}$$

.

The first is Euler's product formula for $\exp(z\log(q))$. The second ought be named "Euler's product functional equation."

I'll write "q^z" in place of "$\exp(z\ell_q(q))$ where ℓ_q is a branch of the logarithm", let log denote the principal branch of the logarithm, and set $x(q,z) := \exp(z\log(q))$. To be clear,

$$\prod_{p\in\mathbb{P}} \frac{1}{1-(\frac{1}{p})^z} = \sum_{n\in(\mathbb{N}^\mathbb{P})_0} \prod_{p\in\mathbb{P}} \exp(z\ell_p(1/p)n(p)) = \sum_{n\in(\mathbb{N}^\mathbb{P})_0} \exp(z\sum_{p\in\mathbb{P}}\ell_p(1/p)n(p))$$

and

$$\prod_{p\in\mathbb{P}} 1-(1/p)^z = \sum_{n\in(\{0,1\}^\mathbb{P})_0} \prod_{p\in\mathbb{P}} (-1)^{n(p)}\exp(z\ell_p(1/p)n(p)) = \sum_{n\in(\{0,1\}^\mathbb{P})_0} \mu(n)\exp(z\sum_{p\in\mathbb{P}}\ell_p(1/p)n(p))$$

These functions converge on $\mathbb{Q}_{>1}$, and may converge on $\mathbb{Q}_{>0}$. Certain choices of ℓ_p produce meaningful properties. Permuting one determination of ℓ_p with another determination has the effect of multiplication by a function in z, and occasionally this is true when permuting the determinations $\{\ell_p\}_{p\in\mathbb{P}}$ amongst each other. So it makes sense to start with $\ell_p = \log$ for all $p \in \mathbb{P}$.

One Case: $\forall p \in \mathbb{P}\ \ell_p = \log$. And we set $\zeta(z) = \sum_{n\in(\mathbb{N}^\mathbb{P})_0} \exp(z\sum_{p\in\mathbb{P}}\log(1/p)n(p))$ which is me being explicit and a bit pedantic about what we mean by the series representation of the commonly accepted zeta function on $\mathbb{Q}_{>1}$. This is worthwhile, we can be more explicit:

$$\frac{1}{p} = \frac{1}{\underbrace{1+\cdots+1}_{\text{p times}}} = \frac{1}{\prod_{w\in 1^{1/p}-\{1\}}(1-w)} = \prod_{w\in 1^{1/p}-\{1\}} \frac{1}{1-w}$$

so thanks to the imaginary part of the principal branch being divisible by $p-1$ and torsion.

$$\begin{aligned}\log(1/p) = \sum_{w\in 1^{1/p}-\{1\}} \log(1/(1-w)) &= \sum_{w\in 1^{1/p}-\{1\}} \log(1/(1-w)) = \sum_{w\in 1^{1/p}-\{1\}} \sum_{n\in\mathbb{N}_\times} w^n/n \\ &= \sum_{n\in\mathbb{N}_\times} \frac{\sum_{w\in 1^{1/p}-\{1\}} w^n}{n} = \sum_{n\in\mathbb{N}_\times} \frac{p1_{p|n}-1}{n} \\ &= \sum_{m\in\mathbb{N}} \frac{-1}{pn+1} + \cdots + \frac{-1}{pn+p-1} + \frac{p-1}{pn+p}\end{aligned}$$

So the term corresponding to n in the series is:

$$\exp(z\sum_{k\in\mathbb{N}_\times} \frac{(\sum_{p|k} n(p)p) - (\sum_{p\in\mathbb{P}} n(p))}{k}) = \exp(z\sum_{k\in\mathbb{N}_\times}\sum_{p\in\mathbb{P}} \frac{(p1_{p|k}-1)n(p)}{k}) = \exp(z\sum_{k\in\mathbb{N}_\times}\sum_{p\in\mathbb{P}} \frac{n(p)\sum_{w\in 1^{1/p}-\{1\}} w^k}{k})$$

which can be expressed many other ways. Geometrically (when p is prime and $p > 3$), the partial sums of $\log(1/p)$ develop as harmonic sums but every p integers it jumps to the harmonic curve that partial sums of $\log(1/q)$ just finished developing or are developing, where q is the largest prime less than p. I.e. at the same

[6] I can't write this in symbols of the form p^z because $x_p(1,z)$ may vary with p.

integer in question, the partial sum of $\log(1/q)$ might itself jump to the harmonic curve of partial sums of the logarithm of the reciprocal of the largest prime less than q; or $\log(1/p)$ joins the harmonic development of $\log(1/q)$. And regardless of the prime, the sum of jump sizes is also a harmonic sum. Interestingly, I've seen the same structure of partial sums ... with real world multidimensional data ... when measuring the total variation distance between conditional empirical distributions and the symmetrization of the unconditional empirical distributions (The problem remains to provide a confidence set for the weak convergence of the conditional empirical distributions to the symmetrization of the unconditional empirical distributions and the weak convergence of symmetrization of the unconditional empirical distributions to a symmetric distribution. Assumptions are needed. One could also go after a confidence set for the rate a convergence.)

$$\zeta(z) = \sum_{n\in\mathbb{N}_\times} \exp(z \sum_{k\in\mathbb{N}_\times}\sum_{p\in\mathbb{P}} \frac{n(p)\sum_{w\in 1^{1/p}-\{1\}} w^k}{k})$$

and

$$\begin{aligned}
1/\zeta(z) &= \sum_{n\in\mathbb{N}_\times} \mu(n)\exp(z \sum_{k\in\mathbb{N}_\times}\sum_{p\in\mathbb{P}} \frac{n(p)\sum_{w\in 1^{1/p}-\{1\}} w^k}{k})\\
&= \sum_{A\subset\mathbb{P}} \exp(z \sum_{k\in\mathbb{N}_\times}\sum_{p\in A} \frac{\sum_{w\in 1^{1/p}-\{1\}} w^k}{k})
\end{aligned}$$

The structure of expansions of the logarithms of reciprocals of primes is not limited to the argument of exponentials.

$$\begin{aligned}
\text{Even prime, } [\mathbb{Q}_\times : \mathbb{Q}_>] : (1-2^{1-z})\zeta(z) &= \zeta(z) + -2^{1-z}\zeta(z) = \zeta(z) + -2^{-z}\zeta(z) + -(2-1)2^{-z}\zeta(z)\\
&= \sum_{n\in\mathbb{N}_\times} \frac{1}{(2n-1)^z} + \frac{1-2}{(2n)^z}\\
&= \sum_{n\in\mathbb{N}_\times} \frac{-\sum_{w\in 1^{1/2}-\{1\}} w^n}{n^z} = -\sum_{n\in\mathbb{N}_\times}\sum_{w\in 1^{1/2}-\{1\}} \frac{w^n}{n^z}\\
&= -\sum_{w\in 1^{1/2}-\{1\}}\sum_{n\in\mathbb{N}_\times} \frac{w^n}{n^z}\\
\text{Odd prime, p: } (1-p^{1-z})\zeta(z) &= \zeta(z) + -p^{1-z}\zeta(z) = \zeta(z) + -p^{-z}\zeta(z) + -(p-1)p^{-z}\zeta(z)\\
&= \sum_{n\in\mathbb{N}_\times} \frac{1}{(pn-(p-1))^z} + \cdots + \frac{1}{(pn-1)^z} + \frac{1-p}{(pn)^z}\\
&= \sum_{n\in\mathbb{N}_\times} \frac{-\sum_{w\in 1^{1/p}-\{1\}} w^n}{n^z} = -\sum_{n\in\mathbb{N}_\times}\sum_{w\in 1^{1/p}-\{1\}} \frac{w^n}{n^z}\\
&= -\sum_{w\in 1^{1/p}-\{1\}}\sum_{n\in\mathbb{N}_\times} \frac{w^n}{n^z}
\end{aligned}$$

Both functions are bounded on $\mathbb{Q}_>$ because they are finite sums of polylogarithms; $Li_s(t)$ is bounded when $s \in \mathbb{Q}_>$ and $|t| < 1$ or t is of finite order.

I'm fairly satisfied. Working with the expansion of a logarithm at primes is a delight, flexible, and more informative than manipulating symbols like $\log(1/p)$ (or more generally $\ell_p(1/p)$). A formula for the highest power of p dividing n would be nice too. You can do this via the development of the factorial of n and $n-1$ in the scale p. Alternatively,

$$v_p(n) = \frac{1}{p}(\sum_{w\in 1^{1/p}} w^n) + \frac{1}{p^2}(\sum_{w\in 1^{1/p^2}} w^n) + \dots + \frac{1}{p^k}(\sum_{w\in 1^{1/p^k}} w^n) + \dots$$

And with $1/\zeta(z)$ in mind, it might be nice to have a formula for μ that is independent of the representation of n as a function or as a set. One can "sum primitive n-th roots of 1"... which is easier said than done. And just for fun,

$$\mu(n) = \prod_{p\in\mathbb{P}|1^{1/p}\subset 1^{1/n}} \frac{1}{p^2}(\sum_{w\in 1^{1/p^2}} w^n) - \frac{1}{p}(\sum_{w\in 1^{1/p}} w^n)$$

and adding 1 to each term in the product and multiplying by zero yields, somewhat dramatically,

$$\forall n \in \mathbb{N} \quad 0 = \sum_{A\subset\mathbb{P}} \prod_{p\in A} \frac{1}{p^2}(\sum_{w\in 1^{1/p^2}} w^n) - \frac{1}{p}(\sum_{w\in 1^{1/p}} w^n)$$

Another Case: $\forall p \in \mathbb{P}\ \ell_p(s) := \log(s) + 2\pi i$ so $(1/p)^z = \exp(z\ell_p(1/p))$ is unambiguous. The resulting function is:

$$\prod_{p\in\mathbb{P}} \frac{1}{1-(1/p)^z} = \sum_{n\in\mathbb{N}} \exp(z\log(1/n) + z\Omega_0(n)2\pi i)$$

where $\Omega_t(n) := \sum_{p\in\mathbb{P}}\sum_{k\in\mathbb{N}_\times} 1_{p^k|n}p^t$. So $\Omega_0(n)$ is the number of positive powers of primes dividing n. If we use the factorization of $1/p$, as before, we get a different function:

$$\prod_{p\in\mathbb{P}} \frac{1}{1-\prod_{w\in 1^{1/p}-\{1\}}(\frac{1}{1-w})^z} = \sum_{n\in\mathbb{N}} \exp(z\log(1/n)+z(\Omega_1(n)-\Omega_0(n))2\pi i) = \sum_{n\in\mathbb{N}} \exp(z\sum_{p\in\mathbb{P}} v_p(n)(\log(1/p)+(p-1)2\pi i))$$

$\Omega_1(n) - \Omega_0(n)$ is the sum of terms of the form $v_p(n)(p-1)$. Applying the formula for, $v_p(n)$, the highest power of p dividing n,

$$\begin{aligned} v_p(n)(p-1) = (\sum_{k\in\mathbb{N}_\times} 1_{p^k|n})(p-1) &= (1(\sum_{w\in 1^{1/p}} w^n) + \frac{1}{p}(\sum_{w\in 1^{1/p^2}} w^n) + \dots + \frac{1}{p^{k-1}}(\sum_{w\in 1^{1/p^k}} w^n) + \dots) \\ &\quad + (\frac{-1}{p}(\sum_{w\in 1^{1/p}} w^n) + \dots + \frac{-1}{p^{k-1}}(\sum_{w\in 1^{1/p^{k-1}}} w^n) + \dots) \\ &= \frac{1}{p^0}(\sum_{w\in 1^{1/p}} w^n) + \frac{1}{p}(\sum_{w\in 1^{1/p^2}-1^{1/p}} w^n) + \dots \frac{1}{p^{k-1}}(\sum_{w\in 1^{1/p^k}-1^{1/p^{k-1}}} w^n) + \dots \end{aligned}$$

If $p^a|n$ and $p^{(a+1)} \not| n$, the final last sum becomes

$$p + \frac{p^2-p}{p} + \dots \frac{p^a-p^{a-1}}{p^{a-1}} + \frac{p^a}{p^a}(\sum_{w\in 1^{1/p}-1} w^{(n/p^a)}) + \frac{p^a}{p^{a+1}}(\sum_{w\in 1^{1/p^2}-1^{1/p}} w^{(n/p^a)}) + \dots + \frac{p}{p^k}(\sum_{w\in 1^{1/p^{k-1}}-1^{1/k}} w^{(n/p^a)}) + \dots$$

$$= \frac{p^2-p}{p} + \dots \frac{p^a-p^{a-1}}{p^{a-1}} + p + (\sum_{w\in 1^{1/p}-1} w^{(n/p^a)}) + \frac{1}{p^1}(\sum_{w\in 1^{1/p^2}-1^{1/p}} w^{(n/p^a)}) + \dots + \frac{1}{p^{k-1}}(\sum_{w\in 1^{1/p^{k-1}}-1^{1/k}} w^{(n/p^a)}) + \dots$$

$$= \underbrace{\frac{p^2-p}{p} + \dots \frac{p^a-p^{a-1}}{p^{a-1}} + p - 1}_{a(p-1)} + \underbrace{\sum_{w\in 1^{1/p}} w^{(n/p^a)}) + \frac{1}{p}(\sum_{w\in 1^{1/p^2}-1^{1/p}} w^{(n/p^a)}) + \dots + \frac{1}{p^{k-1}}(\sum_{w\in 1^{1/p^k}-1^{1/p^{k-1}}} w^{(n/p^a)}) + \dots}_{p(\sum_{m\in\mathbb{N}_\times} 1_{p^m|\frac{n}{p^a}})}$$

$$= a(p-1) + p(0) = a$$

One more relation We have been dealing with products of $1-(1/p)^z$ or $\frac{1}{1-(1/p)^z}$. The equality, $p=\prod_{w\in 1^{1/p}-\{1\}}(1-w)$, has been useful. Similarly, $1=\prod_{w\in 1^{1/p}-\{1\}}(-1-w)$. And also,

$$0=(\prod_{w\in 1^{1/p}|\Im(w)>0}(-1-w))(\prod_{w\in 1^{1/p}|\Im(w)<0}(1-w))+(\prod_{w\in 1^{1/p}|\Im(w)>0}(1-w))(\prod_{w\in 1^{1/p}|\Im(w)<0}(-1-w))$$

As a result,

$$1+p=\left(((\prod_{w\in 1^{1/p}|\Im(w)>0}(-1-w))+(\prod_{w\in 1^{1/p}|\Im(w)>0}(1-w))\right)\left(((\prod_{w\in 1^{1/p}|\Im(w)<0}(-1-w))+(\prod_{w\in 1^{1/p}|\Im(w)<0}(1-w))\right)$$

Multiplying the second term in both factors by i, yields $1-p$. The same identities hold if every term is taken to the z power defined by the principal branch of the logarithm.

Actually I'm not done In "another case" 6, we didn't discuss the convergence of the sums resulting from the choice $\forall p\, \ell_p(s):=\log(s)+2\pi i$. We had:

$$\prod_{p\in\mathbb{P}}\frac{1}{1-(1/p)^z}=\sum_{n\in\mathbb{N}}\exp(z\log(1/n)+z\Omega_0(n)2\pi i)$$

and

$$\prod_{p\in\mathbb{P}}\frac{1}{1-\prod_{w\in 1^{1/p}-\{1\}}(\frac{1}{1-w})^z}=\sum_{n\in\mathbb{N}}\exp(z\log(1/n)+z(\Omega_1(n)-\Omega_0(n))2\pi i)=\sum_{n\in\mathbb{N}}\exp(z\sum_{p\in\mathbb{P}}v_p(n)(\log(1/p)+(p-1)2\pi i))$$

The rate of convergence have meaning.

Definition 10. $\mathbb{P}=\{(G)\#\mid G \text{ is a group with one subgroup with more than one element}\}$

$\mathbb{Q}=(\mathbb{P},/)$ is a group, see [17]. Equivalently, $(\mathbb{P},\cdot)$ is a group, see [17].

7. Considerations re L-functions

These functions bear tremendous beautiful significance throughout mathematics.

"Time and again it has turned out that the crux of a problem lies in the theory of these functions. At some level it is not entirely clear to us why L-functions should enter decisively, through in hindsight one can give reasons." - H. Iwaniec and Peter Sarnak

L-functions of Automorphisms of $\mathbb{A}^n$. The current theory of L-functions of automorphisms is absurd. They are determined by a semi-simple conjugacy class corresponding to a homomorphism into $\mathrm{Aut}(\mathbb{C}^n)$. the restricted product structure passes into the matrix ring, which both produces more singularities than appropriate but not all singularities it should. Also, the function is determined by some map which

If V is a R-module, $\alpha\in\mathrm{Aut}(V)$, then $(It-\alpha)\mathrm{Det}$ is the characteristic polynomial. The function, $((I-\alpha T)\mathrm{Det})^{-1}$, is obtained by inverting the polynomial, multiplying by $t^n=(It)\mathrm{Det}$, and defining $T:=t^{-1}$. The singular points in T correspond to the inverses of the zeros in t, which is a fine correspondence because α is invertible so all zeros are invertible.

In the case that $V=\mathbb{A}^n$ and $\alpha\in\mathrm{Aut}(\mathbb{A}^n)$, $(I-\alpha T)\mathrm{Det})^{-1}$ decomposes as the direct **sum** of local functions of the same form. With the definition of $\mathbb{A}^n$ we've been using, if the global function is singular on $\mathbb{A}$, then the set of singularities is the intersection of the singular points of the local functions. Do note that the transformation of the characteristic polynomial into the function of L type does is not a rational map if the restricted product definition for the Adele ring, $\mathbf{A}$ was in use (almost every point is not invertible, and in particular, locally-invertible does not imply globally invertible).

$\mathbb{A}$ is not algebraically closed, so the sum of the orders of $(I - \alpha T)\mathrm{Det})^{-1}$ over singular points may be less than n. If we embed $\mathbb{A}$ in $\overline{\mathbb{A}}$, then α conjugated by this embedding splits in $\mathrm{Aut}(\overline{\mathbb{A}})$ so we can diagonalize it easily determine its singular points (... as in the previous section).

Likewise, with the restricted product definition of the Adele ring, $\mathbf{A}$ embedds in $\overline{\mathbf{A}}$ and by conjugation $\mathrm{Aut}(\mathbf{A}^n)$ embeds in $\mathrm{Aut}(\overline{\mathbf{A}^n})$. The image of the L-function of some $a \in \mathrm{Aut}(\mathbf{A}^n)$ (obtained sketchy methods) under these embeddings has a larger set of singularities. I don't think this is a good direction if we wish to deduce information from function.

The conclusion of section 3 encapsulates my perspective on L-functions of representations in $\mathrm{Aut}(\mathbb{A}^n)$. There are a few exceptions: 1) Characters are functions on a group.

A non-complicated approach

For instance: F_F is an algebraic number field and $G = \mathrm{Aut}(F_F)$. So, $F_{\mathbb{Q}}$ is a $\mathbb{Q}$ module of dimension $[F : \mathbb{Q}]$. Hence, you can compute the matrix $m_\alpha \in M_{[F:\mathbb{Q}]}(\mathbb{Q})$ of any $\alpha \in \mathrm{Aut}(F_F)$ with respect to a basis. The symmetric character in the indeterminate, T, of the representation m_α of α is $\frac{1}{1-\alpha T}$.

We have the tools to treat the case when F is countable over $\mathbb{Q}$. On the other hand I do not understand the non-abelian aspects.

8. Questions/Musings

I have not clearly describe a philosophy motivating these explorations or the associated methods. For that, I am sorry. I hope the following retrospective summary will do.

Mathematics is complicated. We should not further complicate it. And we should develop sophisticated methods/approaches. Clearly addressing simple questions is a good start. Simplifying methods when possible is another. Both of these are overlooked.

Simple Question: $\{z \in ? \mid p \in \mathbb{Q}[X], ev(p, z) = 0\}$.

-
- What makes (one of) the trace formula a formula? Both sides are just numbers?
- Galwhat? (could replace $\mathbb{Q}$ with F.

    So, $\bar{\mathbb{Q}} = \mathbb{Q}[?]$.

    And, $G := \mathrm{Aut}(\bar{\mathbb{Q}}) \subseteq \mathrm{Aut}(Q \cup Q[?]) = Sym$

    H acting on $\mathbb{Q}(x)$
- Why do people: Justify stuff by "we want harmonic analysis" then impose topological requirements and then fiddle with group theory?
- Considering automorphism groups of abelian groups that contain all p-iteratations. (Hence, the domain of the automorphism group, i.e. the abelian group in question, is divisible and torsion free). (Hence, the abelian group under the group containing all p-iterations is isomorphic to a rational

vector space).

- $\text{End}(M_R) \subset \text{End}(M)$. In the fairly common case that $\{(\ ,r)\cdot \mid r \in R\} = Z(\text{End}(M))$. If we let $T = \{(\ ,r)\cdot \mid r \in R\}$, then $\text{End}(M_R) = C_{\text{End}(M)}(T) =$ " Centralizer of T in $\text{End}(M)$". E.g.

  By definition, $\text{Aut}(\mathbb{A}^n) \subseteq \text{Aut}(A^n) \subset \text{End}(A^n)$. Any element of $\text{Aut}(A^n)$ commutes with $Z(\text{End}(A^n)) = \{(\ ,a)\cdot \mid a \in \mathbb{A}\}$, hence is an element of $\text{Aut}(\mathbb{A}^n)$. So $\text{Aut}(\mathbb{A}^n) = \text{Aut}(A^n)$.

One Quote.

> "An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of Adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations." – Robert P. Langlands

Eisenstein series and automorphic representations. https://arxiv.org/pdf/1511.04265.pdf

9. Acknowledgments

My family and friends. Nicholas Templier. Martin T. Wells. Marshall Hall. Cornell University Library. Wikipedia.

10. Annotations to the Bibliography

The most under-cited paper I am aware of is Oystein Ore's paper, *Some Remarks on Commutators* [32]. It is 5 pages, read it. This remarks are remarkable and central to mathematics. Few make use of the paper.

The Prime zeta function

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