Chapter 3 Exercise Solutions

Jörg Barkoczi

3

3.1

3.1-1

Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Given constants c_1, c_2 and n_0, a function $(m(n) = max(f(n), g(n))) \in \Theta(f(n)+g(n))$ if and only if $0 \leq c_1(f(n)+g(n)) \leq m(n) \leq c_2(f(n)+g(n))$. Since $m(n) \leq f(n) + g(n)$ we already have an upper bound to work with and can thus default c_2 to just 1.
For the lower bound we can just take the factor $min(\frac{f(n)}{f(n)+g(n)}, \frac{g(n)}{f(n)+g(n)})$, which shrinks the expression $f(n) + g(n)$ to $min(f(n), g(n))$ and therefore is always smaller or equal to $max(f(n), g(n))$.
And thus our function $m(n)$ satisfies $0 \leq min(\frac{f(n)}{f(n)+g(n)}, \frac{g(n)}{f(n)+g(n)})(f(n) + g(n)) \leq m(n) \leq f(n) + g(n), \forall n > 0$ and is indeed in the set of functions described by $\Theta(f(n) + g(n))$.

3.1-2

Show that for any real constants a and b, where $b > 0$,

$$(n + a)^b = \Theta(n^b).$$

Let us start by showing that $(n + a)^b = O(n^b)$. This requires us to find constants c, n_0 such that

$$0 \leq (n + a)^b \leq cn^b, \forall n \geq n_0.$$

Let $c = 2^b$

$$\Leftrightarrow (n + a)^b \leq (2n)^b$$

and thus we have $(n + a)^b \leq (2n)^b, \forall n \geq n_0 = |a|$ which implies

$$(n + a)^b = O(n^b).$$

Next we need to show that $(n+a)^b = \Omega(n^b)$, which again requires us to find constants c, n_0 such that

$$0 \leq cn^b \leq (n+a)^b, \forall n \geq n_0.$$

Let $c = (\frac{1}{2})^b$.

$$\Rightarrow (n+a)^b \geq (\frac{1}{2}n)^b$$

and thus we have $(n+a)^b \geq (\frac{1}{2}n)^b, \forall n \geq n_0 = 2|a|$ which implies

$$(n+a)^b = \Omega(n^b).$$

And therefore, by Theorem 3.1, $(n+a)^b = \Theta(n^b)$.

3.1-3

Explain why the statement, "The running time of algorithm A is at least $O(n^2)$", is meaningless.

Saying that the running time of algorithm A is at least $O(n^2)$ gives no information about the worst-case running time, because "at least" implies the best-case input. It gives no information on the best-case running time either, since the O-notation bounds a function from the above, not from below as the Ω-notation does. Therefore the statement is to be considered meaningless.

3.1-4

Is $2^{n+1} = O(2^n)$?

Inequality to prove:

$$0 \leq 2^{n+1} \leq c \cdot 2^n, \forall n \geq n_0.$$

Let $c = 2$, then

$$2^{n+1} \leq 2 \cdot 2^n = 2^{n+1}, \forall n \geq 0.$$

Therefore $2^{n+1} = O(n^2)$.

Is $2^{2n} = O(2^n)$?

Inequality to prove:

$$0 \leq 2^{2n} \leq c \cdot 2^n, \forall n \geq n_0.$$
$$2^{2n} \leq c \cdot 2^n$$
$$2^n \leq c$$

There is obviously no constant c that satisfies

$$\lim_{n \to \infty} 2^n \leq c,$$

therefore $2^{2n} \neq O(2^n)$.

3.1-5

Prove Theorem 3.1.

"For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$" (Theorem 3.1).

Per definition $\Theta(g(n))$ requires the existence of constants c_1, c_2, n_0 such that

$$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n), \forall n \geq n_0.$$

Therefore we can split of the inequalities as

$$0 \leq c_1 \cdot g(n) \leq f(n), \forall n \geq n_0,$$

which implies $f(n) = \Omega(n)$, and

$$0 \leq f(n) \leq c_2 \cdot g(n), \forall n \geq n_0,$$

which implies $f(n) = O(n)$.

From the other side of the equivalence, if we have constants c_a, c_b, n_a, n_b such that

$$0 \leq c_a \cdot g(n) \leq f(n), \forall n \geq n_a,$$

and

$$0 \leq f(n) \leq c_b \cdot g(n), \forall n \geq n_b,$$

we can let $n_0 = max(n_a, n_b)$ and thus satisfy

$$0 \leq c_a \cdot g(n) \leq f(n) \leq c_b \cdot g(n), \forall n \geq n_0.$$

3.1-6

Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

If the worst-case running time of an algorithm is $O(g(n))$, the implication is that the running time is bounded from above (by some funtion $f(n) = O(g(n))$) for any input of size n.
Likewise if the best-case running time of an algorithm is $\Omega(g(n))$, the implication is that the running time is bounded from below (by some function $f(n) = \Omega(g(n))$) for any input of size n.
Therefore by Theorem 3.1, that algorithm's running time is $\Theta(g(n))$.

Written as an equivalence:

$$
\begin{aligned}
&f(n) \in \Theta(g(n)) \\
&\qquad\qquad \Leftrightarrow \exists c_1, c_2, n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \\
&\qquad\qquad \Leftrightarrow (0 \leq c_1 g(n) \leq f(n) \wedge 0 \leq f(n) \leq c_2 g(n)), \forall n \geq n_0 \\
&\qquad\qquad \Leftrightarrow f(n) \in \Omega(g(n)) \cap O(g(n))
\end{aligned}
$$

3.1-7

Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

For a function $f(n)$ to be considered a member of $o(g(n)) \cap \omega(g(n))$, the following equivalence must be true:

$$
\begin{aligned}
&f(n) \in \omega(g(n)) \cap o(g(n)) \\
&\Leftrightarrow 0 \leq c_1 g(n) < f(n) \wedge 0 \leq f(n) < c_2 g(n) \\
&\Leftrightarrow 0 \leq c_1 g(n) < f(n) < c_2 g(n)
\end{aligned}
$$

for all positive real constants c_1 and c_2, and for all n bigger than some positive constant n_0.
But since $c_1 g(n) < c_2 g(n)$ can't be true for all positive constants c_1 and c_2, $o(g(n)) \cap \omega(g(n))$ must be the empty set.

3.1-8

$$O(g(n,m)) = \{f(n,m) : \exists c, n_0, m_0 > 0 : 0 \leq f(n,m) \leq cg(n,m), \forall n \geq n_0 \vee \forall m \geq m_0\}$$

$$\Omega(g(n,m)) = \{f(n,m) : \exists c, n_0, m_0 > 0 : 0 \leq cg(n,m) \leq f(n,m), \forall n \geq n_0 \vee \forall m \geq m_0\}$$

$$\Theta(g(n,m)) = \{f(n,m) : f(n,m) \in \Omega(g(n,m)) \cap O(g(n,m))\}$$

3.2

3.2-1

Show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so are the functions $f(n) + g(n)$ and $f(g(n))$, and if $f(n)$ and $g(n)$ are in addition nonnegative, then $f(n) \cdot g(n)$ is monotonically increasing.

$f(n)$ and $g(n)$ being monotonically increasing implies that for $n \leq m$

$$\begin{aligned}
&f(n) \leq f(m) \wedge g(n) \leq g(m) \\
&\Leftrightarrow f(n) + g(m) \geq f(n) + g(n) \\
&\Leftrightarrow f(m) + g(m) \geq f(n) + g(n),
\end{aligned}$$

likewise it implies that $f(g(n)) \leq f(g(m))$.
If in addition both $f(n)$ and $g(n)$ are nonnegative, the following holds

$$\begin{aligned}
&f(n)g(n) = f(n)g(n) \\
&\Leftrightarrow f(n)g(n) \leq f(n)g(m) \\
&\Leftrightarrow f(n)g(n) \leq f(m)g(m)
\end{aligned}$$

3.2-2

Proof equation (3.16).

$$a^{\log_b c} = c^{\log_b a} \tag{3.16}$$

$$\begin{aligned}
&a^{\log_b c} = c^{\log_b a} \\
&\Leftrightarrow a^{\frac{\log_b c}{\log_b a}} = c \\
&\Leftrightarrow a^{\log_a c} = c \\
&\Leftrightarrow c = c
\end{aligned}$$

3.2-3

Prove equation 3.19 $\lg(n!) = \Theta(n \lg n)$. Also prove that $n! = \omega(2^n)$ and $n! = o(n^n)$.

Let us start with equation **3.19** and by showing its membership to $O(n \lg n)$. Let c be some positive real constant, then

$$\begin{aligned}
&\lg(n!) \in O(n \lg n) \\
&\Leftrightarrow 0 \leq \lg(n!) \leq cn \lg n \\
&\Leftrightarrow \lg(n^n) \leq cn \lg n \\
&\Leftrightarrow n \lg n \leq cn \lg n
\end{aligned}$$

holds for all $n \geq n_0 = 1$ if $c >= 1$.

Now we will need to additionally show that $\lg(n!)$ is also a member of $\Omega(n \lg n)$. Again let c be some positive real constant, then

$$\begin{aligned}
&\lg(n!) \in \Omega(n \lg n) \\
&\Leftrightarrow 0 \leq cn \lg n \leq \lg(n!),\ \forall n > n_0.
\end{aligned}$$

By Sterling's approximation we have

$$\begin{aligned}
&cn \lg n \leq \lg(n!) \\
&\Leftrightarrow cn \lg n \leq \lg\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e\right) \\
&\Leftrightarrow cn \lg n \leq \frac{1}{2}(\lg 2\pi + \lg n) + n \lg n - n \lg e + \lg e.
\end{aligned}$$

Let $c = 0.5$, then we have

$$0 \leq \frac{1}{2}(\lg 2\pi + \lg n) + \frac{n}{2} \lg n - n \lg e + \lg e,$$

which is positive as long as $\frac{n}{2} \lg n - n \lg e$ is positive.

Therefore

$$
\begin{aligned}
&\frac{n}{2}\lg n - n\lg e \geq 0 \\
&\Leftrightarrow \frac{n}{2}(\lg n - 2\lg e) \geq 0 \\
&\Leftrightarrow \lg n - 2\lg e \geq 0 \\
&\Leftrightarrow n \geq 2^{2\lg e},
\end{aligned}
$$

and thus there exists $c = 0.5$ such that

$$0 \leq \frac{1}{2}(\lg 2\pi + \lg n) + \frac{n}{2}\lg n - n\lg e + \lg e,$$

for all $n \geq n_0 = 2^{2\lg e}$, and therefore $\lg(n!)$ is also element of $\Omega(n\lg n)$, which by Theorem 3.1 is equivalent to $\lg(n!) \in \Theta(n\lg n)$.

Prove $n! = o(n^n)$.
We need to show that

$$\lim_{n\to\infty} \frac{n^n}{n!} = \infty,$$

or alternatively

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

We can again make use of Sterling's approximation, such that we have

$$
\begin{aligned}
&\frac{n^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(\frac{1}{n})\right)} \\
=&\frac{e^n}{\sqrt{2\pi n}\left(1+\Theta\left(\frac{1}{n}\right)\right)} \\
=&\frac{e^n}{\sqrt{2\pi n}\Theta\left(\frac{\sqrt{2\pi n}}{n}\right)} \\
=&\frac{e^n}{\Theta\left(\frac{\sqrt{2\pi n}^2}{n}\right)} \\
=&\frac{e^n}{\Theta(2\pi)} \\
=&\frac{e^n}{\Theta(1)}
\end{aligned}
$$

now looking at the limit we have

$$\lim_{n\to\infty} \frac{e^n}{\Theta(1)} = \infty.$$

Prove $n! = w(n^n)$.
We need to show that

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0,$$

or alternatively

$$\lim_{n\to\infty} \frac{n!}{2^n} = \infty.$$

Using Sterling's approximation again, we get

$$\begin{aligned}
&\frac{2^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(\frac{1}{n})\right)}\\
=&\frac{(2e)^n}{\sqrt{2\pi n}\,n^n\left(1+\Theta(\frac{1}{n})\right)}\\
=&\frac{(2e)^n}{\sqrt{2\pi n}\,(n^n+\Theta(n^{n-1}))}\\
=&\frac{(2e)^n}{\sqrt{2\pi n}\Theta(n^{n-1})}\\
=&\frac{(2e)^n}{\sqrt{2\pi n}\Theta(n^n)}\\
=&\frac{(2e)^n}{\Theta(\sqrt{n})\Theta(n^n)}\\
=&\frac{(2e)^n}{\Theta(\sqrt{n}\,n^n)}\\
=&\frac{(2e)^n}{\Theta(n^{n+\frac{1}{2}})}\\
=&\frac{2e}{\Theta(n^{2n^{-1}+1})}
\end{aligned}$$

Now looking at the limit, we have

$$\lim_{n\to\infty} \frac{2e}{\Theta(n^{2n^{-1}+1})} = \frac{2e}{n} = 0.$$

3.2-4

Is the function $\lceil \lg n \rceil!$ polynomially bounded? Is the function $\lceil \lg \lg n \rceil!$ polynomially bounded?

3.2-5

Which is asymptotically larger: $\lg(\lg *n)$ or $\lg *(\lg n)$?

3.2-6

Show that the golden ratio ϕ and its $\widehat{\phi}$ both satify the equation $x^2 = x + 1$.
We can show this by using the quadratic formula:

$$\begin{aligned} x_{1,2} &= \frac{1}{2} \pm \sqrt{-\frac{1}{2}^2 + 1} \\ &= \frac{1}{2} \pm \sqrt{\frac{5}{4}} \\ &= \frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Therefore the first solution to the equation is $\frac{1-\sqrt{5}}{2} = \widehat{\phi}$ and the second is $\frac{1+\sqrt{5}}{2} = \phi$.

3.2-7

Prove by induction that the ith Fibonacci number satisfies the equality

$$F_i = \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}},$$

where ϕ is the golden ratio and $\widehat{\phi}$ is its conjugate.
First we show that it holds for $i = 1$ and $i = 2$.

$$\begin{aligned} F_1 &= \frac{\phi^1 - \widehat{\phi^1}}{\sqrt{5}} \\ &= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{1 + \sqrt{5} - (1 - \sqrt{5})}{2\sqrt{5}} \\ &= \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \end{aligned}$$

$$
\begin{aligned}
F_2 &= \frac{\phi^2 - \widehat{\phi^2}}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4\sqrt{5}} \\
&= \frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5)}{4\sqrt{5}} \\
&= \frac{4\sqrt{5}}{4\sqrt{5}} = 1
\end{aligned}
$$

Now assuming that the equality holds for F_{i-2} and F_{i-1}, we will show that it also holds for F_i by showing that $F_i = F_{i-2} + F_{i-1}$ as this is the property of the Fibonacci sequence.

$$
\begin{aligned}
F_i &= F_{i-2} + F_{i-1} \\
\Leftrightarrow \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-2} - \widehat{\phi}^{i-2}}{\sqrt{5}} + \frac{\phi^{i-1} - \widehat{\phi}^{i-1}}{\sqrt{5}} \\
\Leftrightarrow \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-2} - \widehat{\phi}^{i-2} + \phi^{i-1} - \widehat{\phi}^{i-1}}{\sqrt{5}} \\
\Leftrightarrow \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-1} + \phi^{i-2} - \widehat{\phi}^{i-1} - \widehat{\phi}^{i-2}}{\sqrt{5}} \\
\Leftrightarrow \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-2}(\phi^1 + 1) - \widehat{\phi}^{i-2}(\widehat{\phi}^1 + 1)}{\sqrt{5}} \\
\Leftrightarrow \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-2}(\phi^2) - \widehat{\phi}^{i-2}(\widehat{\phi}^2)}{\sqrt{5}} \\
\Leftrightarrow \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}} &= \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}}
\end{aligned}
$$

3.2-8

Show that $k \ln k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{\ln n}\right)$.

If $k \ln k = \Theta(n)$, then given positive constants c_1, c_2, n_0, we have that

$$0 \leq c_1 n \leq k \ln k \leq c_2 n,$$

is true for all $n \geq n_0$. Now suppose that $k = \frac{n}{\ln n} = \Theta(\frac{n}{\ln n})$. When substituting in the above equation we get

$$0 \leq c_1 n \leq \frac{n}{\ln n} \ln\left(\frac{n}{\ln n}\right) \leq c_2 n.$$

If we can show that this equation holds for the given constants, we see that $k \ln k = \Theta(n)$ does indeed imply $k = \Theta\left(\frac{n}{\ln n}\right)$.
Let $f(n) := \frac{n}{\ln n} \ln\left(\frac{n}{\ln n}\right)$. We will first show that $f(n) \in O(n)$. To show this we need constants c_2, n_0 such that

$$0 \leq f(n) \leq c_2 n,$$

for all $n \geq n_0$.

$$\begin{aligned}
& 0 \leq f(n) \leq c_2 n \\
\Leftrightarrow \quad & \frac{n}{\ln n} \ln\left(\frac{n}{\ln n}\right) \leq c_2 n \\
\Leftrightarrow \quad & \frac{n}{\ln n} \cdot (\ln n - \ln \ln n) \leq c_2 n \\
\Leftrightarrow \quad & n - \frac{n}{\ln n} \ln \ln n \leq c_2 n
\end{aligned}$$

This holds for $c_2 = 1$ as long as $\frac{n}{\ln n} \ln \ln n$ is nonnegative. So we either must have that both $\frac{n}{\ln n}$ and $\ln \ln n$ are negative, or that both are positive. Since n is always nonnegative, $\ln n$ must be negative for $\frac{n}{\ln n}$ to be negative, and $\ln n$ is negative for $n \in (0, 1)$. On the other hand we have that $\ln \ln n$ is negative precisely when $0 < \ln n < 1$, which is true for $n \in (1, e)$. Thus both expressions can't be negative at the same time and both must therefore be positive, which is true for $n \geq e$.
Thus $\frac{n}{\ln n} \ln\left(\frac{n}{\ln n}\right) \leq c_2 n$ for $c_2 = 1$ and $n \geq n_0 = e$ and therefore $f(n) \in O(n)$.
Now we will show that $f(n) \in \Omega(n)$ is true aswell. To show this we need constants c_1, n_0 such that

$$0 \leq c_1 n \leq f(n),$$

for all $n \geq n_0$.

$$\begin{aligned}
& 0 \leq c_1 n \leq f(n)n \\
\Leftrightarrow \quad & c_2 n \leq n - \frac{n}{\ln n} \ln\ln n \\
\Leftrightarrow \quad & c_2 \leq 1 - \frac{1}{\ln n} \ln\ln n \\
\Leftrightarrow \quad & c_2 - 1 \leq -\frac{1}{\ln n} \ln\ln n \\
\Leftrightarrow \quad & \frac{1}{\ln n} \ln\ln n \leq -c_2 + 1 \\
\Leftrightarrow \quad & \frac{\ln\ln n}{\ln n} \leq -c_2 + 1 \\
\Leftrightarrow \quad & e^{\left(\frac{\ln\ln n}{\ln n}\right)} \leq e^{(-c_2+1)} \\
\Leftrightarrow & (e^{(\ln\ln n)})^{\frac{1}{\ln n}} \leq e^{(-c_2+1)} \\
\Leftrightarrow \quad & (\ln n)^{\frac{1}{\ln n}} \leq e^{(-c_2+1)} \\
\Leftrightarrow \quad & \ln n \leq \left(e^{(-c_2+1)}\right)^{\ln n} \\
\Leftrightarrow \quad & \ln n \leq e^{(-c_2+1)\cdot\ln n} \\
\Leftrightarrow \quad & \ln n \leq e^{-c_2 \ln n + \ln n} \\
\Leftrightarrow \quad & \ln n \leq e^{-c_2 \ln n} n \\
\Leftrightarrow \quad & \ln n \leq n^{-c_2} n \\
\Leftrightarrow \quad & n \leq e^{n^{-c_2} n} \\
\Leftrightarrow \quad & n \leq e^{\frac{n}{n^{c_2}}}
\end{aligned}$$

We have that

$$\lim_{c_2 \to 0} e^{\frac{n}{n^{c_2}}} = e^n,$$

which shows that the inequality holds for small choices of c_2. It is not difficult to show it for concrete values of c_2 by comparing an initial value and the gradients. Therefore $f(n) = \frac{n}{\ln n} \ln\left(\frac{n}{\ln n}\right) \in \Omega(n)$. Since it also a member of $O(n)$ as we showed before, it follows that $f(n) \in \Theta(n)$ is true, which implies that $k = \frac{n}{\ln n} \in \Theta(\frac{n}{\ln n})$ is also true.