Set Theory & Combinations

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Set Theory

- What is set theory?

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- A branch of mathematics

Intro to Set Theory

- What is set theory?
- A branch of mathematics
- Collects objects into *sets* and studies the properties

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- The objects can be anything
- We usually use variables or units of observation

Elements in or not

- We can say whether an object is in a set:

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- We can say whether an object is in a set:

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- Or not:

$$q_1 \notin S$$

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- We can also define non-proper subsets:

$$L \subseteq S$$

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- Though Austin might be close...

$$Z = \{\emptyset\}$$

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- R represents all the possibilities of a (single) roll of a die
- We can define sets for the even possibilities and the odd possibilities

$$E = \{2, 4, 6\} \qquad O = \{1, 3, 5\}$$

Compliments

- A *compliment* is that together, they contain all the elements of the relevant universe

$$E = O^C \qquad O = E^C$$

Universe

- Board examples of how to draw sets

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  - $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$
  - Have to subtract the last term, otherwise we'd double-count elements in the intersection of the two sets

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7. Is $Q \subset S$?

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- How many different combinations of 3 dice rolls are there?

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- think "permutation" = "position"

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- More generally:

$$\frac{n!}{(n-r)!}$$

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- So 4, 3, 1 is the same as 3, 1, 4

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- We’ve already figured out the permutation part, so need to figure out the second part

$$\frac{n!}{(n-r)!} * \frac{1}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

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- Formula:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

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Using a standard 52-card deck:

1. How many different 3-card hands exist?

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1. How many different 3-card hands exist?
2. How many ways are there to get dealt the queen of spades, then the king of diamonds, then the eight of spades?

You try (answers)

```r
# 1
choose(52, 3)
```

```
## [1] 22100
```

```r
# 2 - one
```