Matrix Algebra

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```
##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

What is a matrix?

Scalars

- Let's start with something familiar, with a new word

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- One number (12, for example) is referred to as a *scalar*
- This can be thought of as a 1x1 matrix
    - More on that in a bit...

$$\begin{bmatrix} 12 \end{bmatrix} = c$$

Vectors

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- Since this is a column of numbers, we cleverly refer to it as a *column vector*

Row Vectors

If we take b and arrange it so that it it a row of numbers instead of a column, we refer to it as a *row vector*:

$$\begin{bmatrix} 12 & 14 & 15 \end{bmatrix} = d$$

Operators

The **summation operator** $\sum$ lets us perform an operation (sum) on a sequence of numbers (often but not always a vector):

$$\sum_{i=1}^{5} i$$

Let:

$$x_i = \begin{bmatrix} 12 & 7 & -2 & 0 & 1 \end{bmatrix}$$

Find:

$$\sum_{i=1}^{3} x_i$$

Summation operator

```r
x <- c(12, 7, -2, 0, 1)
sum(x[1:3])
```

```
## [1] 17
```

You try!

Let

$$y = \begin{bmatrix} 1 & 0 & -1 & 4 \end{bmatrix}$$

Find:

$$\sum_{i=1}^{\infty} y^2$$

You try (answer)

```
y <- c(1, 0, -1, 4)
sum(y ^ 2)
```

```
## [1] 18
```

We might want to multiply instead of add, in which case we can use the product operator $\prod$

Let:

$$z = \begin{bmatrix} 6 & -2 & 0 & 1 \end{bmatrix}$$

Find:

$$\prod_{i=1}^{2} z_i$$

Product operator

```r
z <- c(6, -2, 0, 1)
prod(z[1:2])
```

```
## [1] -12
```

Let:

$$a = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

Find:

$$\prod_{i=1}^{3} a^2 - 2a + 3$$

You try (answer)

```r
a <- c(1, 2, 3, 4, 5, 4, 3, 2, 1)
prod(a[1:3] ^ 2 - 2 * a[1:3] + 3)
```

```
## [1] 36
```

Matrices

We can put multiple vectors together to get a *matrix*:

$$\begin{bmatrix} 12 & 14 & 15 \\ 115 & 22 & 127 \\ 193 & 29 & 219 \end{bmatrix} = A$$

Matrices, cntd

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    - And sometimes bolded as well

ROW x COLUMN

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- Solution: come up with a clever indexing scheme
- Matrix A is an nxm matrix where $n = m = 3$.
- More generally, matrix B is an mxn matrix where the elements look like this:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix}$$

Matrix Operations

Addition and subtraction are EASY!

- Requirement: Must have *exactly* the same dimensions

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- To do the operation, just add or subtract each element with the corresponding element from the other matrix:

$$A \pm B$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

You try!

Let

$$A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -1 & 0 \\ 0 & -1 & 3 \end{bmatrix} B = \begin{bmatrix} 5 & 1 & 0 \\ 2 & -1 & 0 \\ 7 & 1 & 2 \end{bmatrix}$$

Find $A + B$

You try (answers)

```
A <- matrix(c(1, 4, 2, -2, -1, 0, 0, -1, 3),
            byrow = TRUE, nrow = 3)
B <- matrix(c(5, 1, 0, 2, -1, 0, 7, 1, 2),
            byrow = TRUE, nrow = 3)
A + B
```

```
##      [,1] [,2] [,3]
## [1,]    6    5    2
## [2,]    0   -2    0
## [3,]    7    0    5
```

Easy - just multiply each element of the matrix by the scalar

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{bmatrix}$$

Scalar multiplication

```r
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    4    2
## [2,]   -2   -1    0
## [3,]    0   -1    3
```

```r
3 * A
```

```
##      [,1] [,2] [,3]
## [1,]    3   12    6
## [2,]   -6   -3    0
## [3,]    0   -3    9
```

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- This means that the number of columns in the first matrix equals the number of rows in the second
- The resulting matrix will have the number of rows in the first, and the number of columns in the second!

Pop quiz

- Which can we multiply? What will the resulting dimensions be?

$$b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix} L = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 4 & 3 \end{bmatrix}$$

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- The dimensions will be 2x3

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```
L <- matrix(c(6, 5, -1, 1, 4, 3),
            nrow = 2, byrow = TRUE)
M <- matrix(c(1, 0, 2, 1, 2, 4, 2, 3, 2),
            nrow = 3, byrow = TRUE)
L %*% M
```

```
##      [,1] [,2] [,3]
## [1,]    9    7   30
## [2,]   11   17   24
```

Matrix Division

HAHAHA... NOPE

- Addition and subtraction

Properties of operators

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  - Associative $(A \pm B) \pm C = A \pm (B \pm C)$

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  - $A(B + C) = AB + AC$
  - $(A + B)C = AC + BC$

Transposition

- Switch the rows and columns

Qu'est-ce que c'est?

- Switch the rows and columns
- So a *nxm* matrix becomes *mxn*

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- So a *nxm* matrix becomes *mxn*
- Typically denoted L' or L^T

Transposition

```r
L
```

```
##      [,1] [,2] [,3]
## [1,]    6    5   -1
## [2,]    1    4    3
```

```r
t(L)
```

```
##      [,1] [,2]
## [1,]    6    1
## [2,]    5    4
## [3,]   -1    3
```

Properties of transposing

- Matrix is *always* conformable for multiplication with its transpose in both directions

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- Matrix is *always* conformable for multiplication with its transpose in both directions
- $(A \pm B)' = A' \pm B'$
- $A'' = A$
- $(AB)' = B'A'$
- $(cA)' = cA'$ where c is a scalar
- AA' and $A'A$ will always result in a symmetric matrix

You try!

Let:

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix}$$

Find:

1. $A'A$

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2. AB

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Find:

1. $A'A$
2. AB
3. BA

You try (answers)

```r
A <- matrix(c(2, 4, 3, 1), byrow = TRUE, nrow = 2)
B <- matrix(c(2, 3, 4, -1, 0, 1), byrow = TRUE, nrow = 2)
A %*% t(A)
```

```
##      [,1] [,2]
## [1,]   20   10
## [2,]   10   10
```

You try (answers)

```
A %*% B
```

```
##      [,1] [,2] [,3]
## [1,]    0    6   12
## [2,]    5    9   13
```

You try (answers)

```
B %*% A
```

```
## Error in B %*% A: non-conformable arguments
```

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- Then B is said to be the *inverse* of A
  - This is usually denoted A^{-1}
  - So $AA^{-1} = I_n$
- If B doesn't exist, then the matrix is *singluar*
- Finding inverses by hand is super hard (especially as n increases), so we let computers do this for us

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- $(A')^{-1} = (A^{-1})'$

Special matrices

Some matricies get more love than others

Square matrix

Any *nxn* matrix (same number rows and columns)

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix}$$

Symmetric matrix

A square matrix that is the same as its transpose

$$\begin{bmatrix} 2 & 5 & 7 \\ 5 & 9 & 6 \\ 7 & 6 & 7 \end{bmatrix}$$

Diagonal matrix

A symmetric matrix with zeros everywhere but the main diagonal

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Scalar matrix

A diagonal matrix with the same number all along the diagonal

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Identity matrix

A scalar matrix where the diagonal elements are 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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- This is a super important type of matrix.
- It gets its own notation: I_n where n is the number of rows and columns
- Note that $I_n A = A$ and also $A I_n = A$