Calculus

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Limits

The limit of $f(x)$ as $x \longrightarrow a$ is the value that $f(x)$ approaches as x gets arbitrarily close to a

$$\lim_{x \longrightarrow a} f(x) = L$$

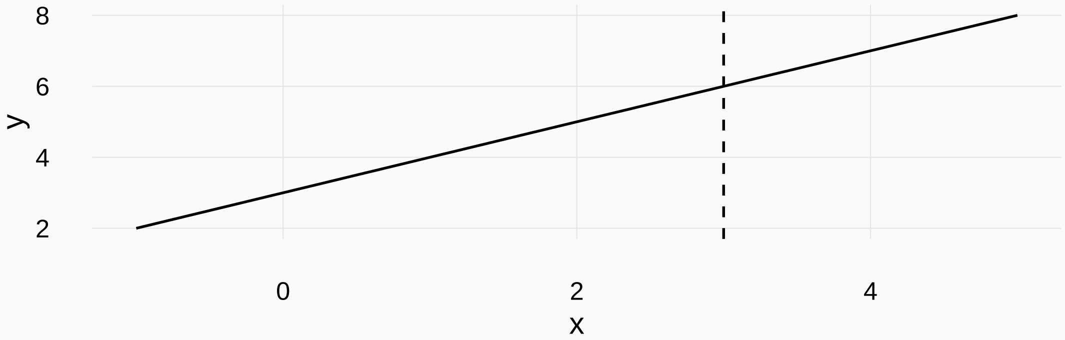
Limits

Let $f(x)$ be $x + 3$. Find:

$$\lim_{x \longrightarrow 3} f(x)$$

Limits

```
ggplot(data.frame(x = c(-1, 5)), aes(x)) +
  stat_function(fun = function(x) x + 3) +
  geom_vline(xintercept = 3, linetype = "dashed")
```



Limit example

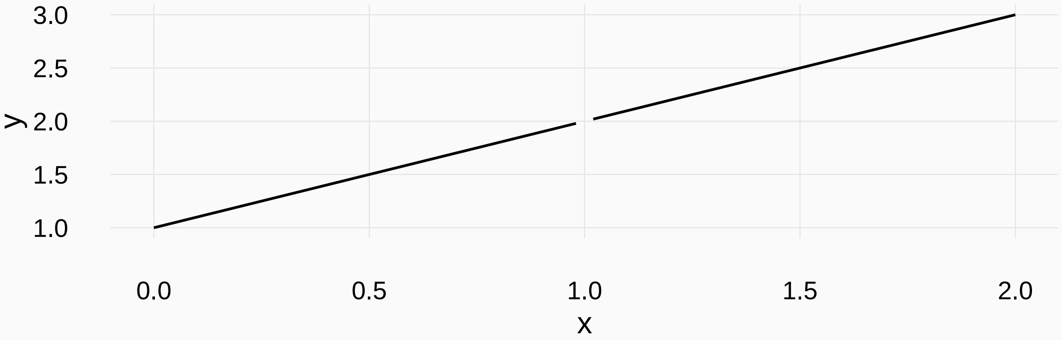
Consider the following function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

What is $f(1)$?

$f(1)$ is undefined

```
ggplot(data.frame(x = c(0, 2)), aes(x)) +
  stat_function(fun = function(x) (x ^ 2 - 1) / (x - 1))
```



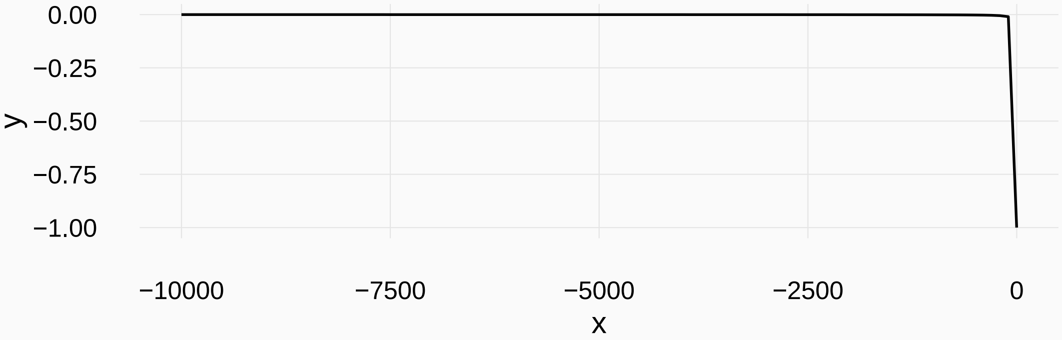
$$\lim_{x \to \infty} x^2$$

You try!

$$\lim_{x \to -\infty} \frac{1}{x}$$

You try!

```
ggplot(data.frame(x = c(-10000, -1)), aes(x)) +
  stat_function(fun = function(x) ifelse(x != 0, 1 / x, NaN))
```



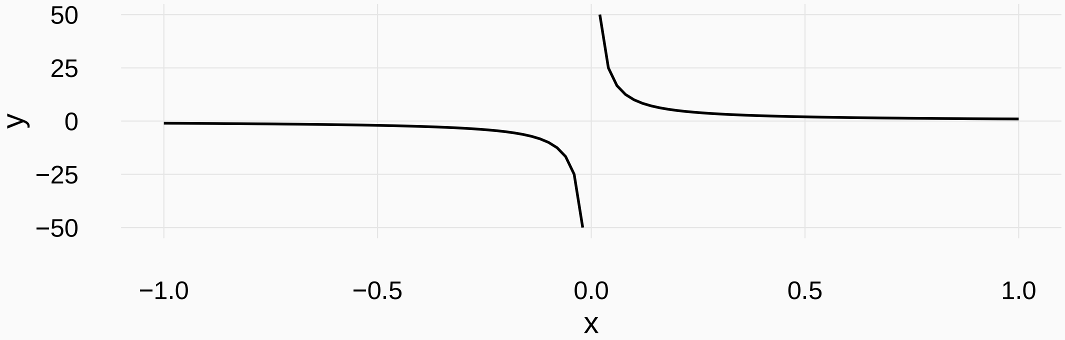
Sided limits

What is

$$\lim_{x \longrightarrow 0} \frac{1}{x}$$

Sided limits

```r
ggplot(data.frame(x = c(-1, 1)), aes(x)) +
  stat_function(fun = function(x) ifelse(x != 0, 1 / x, NaN))
```



Here, we can take two different limits: the limit as x approaches 0 from the left and then from the right

$$\lim x \to 0^- \qquad \text{from the left}$$

$$\lim x \to 0^+ \qquad \text{from the right}$$

Sided limits

The limit does not exist!

Sided limits

The limit does not exist!

For a limit to exist, both sides must exist *and* they must be equal

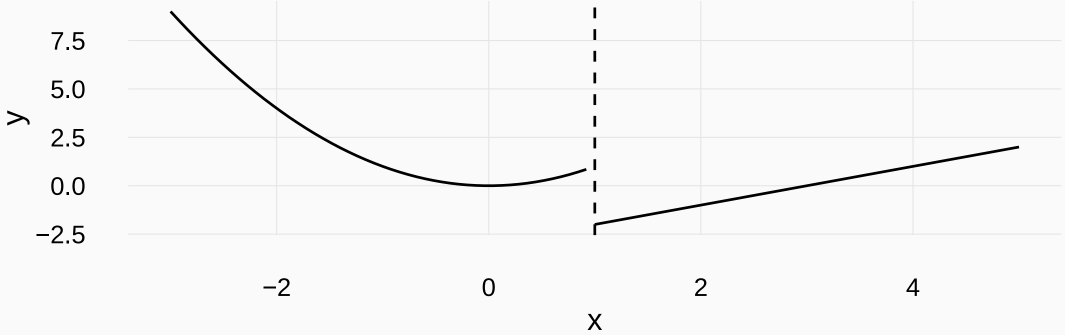
You try!

Find both one sided limits and the two sided limit as $x \longrightarrow 1$ of $f(x)$

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x - 3 & \text{if } x \geq 1 \end{cases}$$

You try (answers)

```
ggplot(data.frame(x = -3:5), aes(x)) +
  stat_function(fun = function(x) ifelse(x < 1, x ^ 2, NA)) +
  stat_function(fun = function(x) ifelse(x >= 1, x - 3, NA)) +
  geom_vline(xintercept = 1, linetype = "dashed")
```



Properties of limits

$$\lim_{x \to a} c = c$$

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$$\lim_{x \longrightarrow a} c = c$$

$$\lim_{x \longrightarrow a} x = a$$

$$\lim_{x \longrightarrow a} (f(x) \pm g(x)) = \lim_{x \longrightarrow a} f(x) \pm \lim_{x \longrightarrow a} g(x)$$

Properties of limits, continued

$$\lim_{x \to a} \left(\frac{\lim_{x \to a} f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{iff } \lim_{x \to a} g(x) \neq 0$$

Properties of limits, continued

$$\lim_{x \to a} \left(\frac{\lim_{x \to a} f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{iff } \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$$

Properties of limits, continued

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$$\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$$

$$\lim_{x \to a} (f(x)^n) = \left(\lim_{x \to a} f(x) \right)^n \qquad \text{iff } \lim_{x \to a} f(x) > 0$$

Derivatives

Derivatives calculate the instantaneous slope (rate of change) of a function at every point on its domain

Can think of this like so:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Derivatives - notation

$$\frac{dy}{dx} = \frac{d}{dx}f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Other derivatives

$$\frac{dy}{dx}c = 0$$

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$$\frac{dy}{dx}c = 0$$

$$\frac{d}{dx}e^x = e^x$$

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$$\frac{dy}{dx}c = 0$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\log_e(x) = \frac{1}{x}$$

The one derivative rule to rule them all

The **power rule**

$$\frac{dy}{dx}x^n = nx^{n-1}$$

Other derivative rules

$$\frac{d}{dx} cf(x) = cf'(x)$$

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$$\frac{d}{dx}cf(x) = cf'(x)$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

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$$\frac{d}{dx}cf(x) = cf'(x)$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

The chain rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

The "second derivative" is just the derivative of the derivative:

$$f''(x) = \frac{d^2}{dx}$$

You can take the third derivative and so on as well

You try!

Find the following:

1. $\frac{d}{dx}x^2$

You try!

Find the following:

1. $\frac{d}{dx}x^2$
2. $\frac{d}{dx}x^3 + x^2 + 2x - 1$

You try!

Find the following:

1. $\frac{d}{dx}x^2$
2. $\frac{d}{dx}x^3 + x^2 + 2x - 1$
3. $\frac{d^2}{dx}x^4 + x^3$

Sometimes we want to find the maximum (or minimum) value a function takes.

Optimization lets us do this!

Optimization

The derivative of a function gives rate of change, so when that is zero, the function has usually reached a (local) maximum or minimum. Why?

So we can find the local maxima and/or minima of a function by taking the derivative, setting it equal to zero, and solving for x (or whatever)

BUT we don't know if we've found a maximum or minimum.

The second derivative gives us the rate of change *of the rate of change* of the original function. So it tells us whether the slope is getting larger or smaller.

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$f''(x) = 2$$

So a *positive* second derivative indicates a *minimum* and a *negative* second derivative indicates a *maximum*

To find the max/min on some interval you have to compare the local max/min to the value of the function at the interval's endpoints. To find global max/min you must check the function's limits as it approaches positive and negative ∞

You try!

Find the global minimum for the following function:

$$f(x) = x^2 - 2x + 7$$

You try (answer)

Find first derivative:

$$f'(x) = 2x - 2$$

You try (answer)

Find first derivative:

$$f'(x) = 2x - 2$$

Set it equal to zero

$$2x - 2 = 0$$

$$x = 1$$

You try (answer)

Find first derivative:

$$f'(x) = 2x - 2$$

Set it equal to zero

$$2x - 2 = 0$$

$$x = 1$$

Check the second derivative to see if this is a maximum or minimum:

$$f''(x) = 2$$

It's positive, so this is a *minimum*.

You try (answer)

Find first derivative:

$$f'(x) = 2x - 2$$

Set it equal to zero

$$2x - 2 = 0$$

$$x = 1$$

Check the second derivative to see if this is a maximum or minimum:

$$f''(x) = 2$$

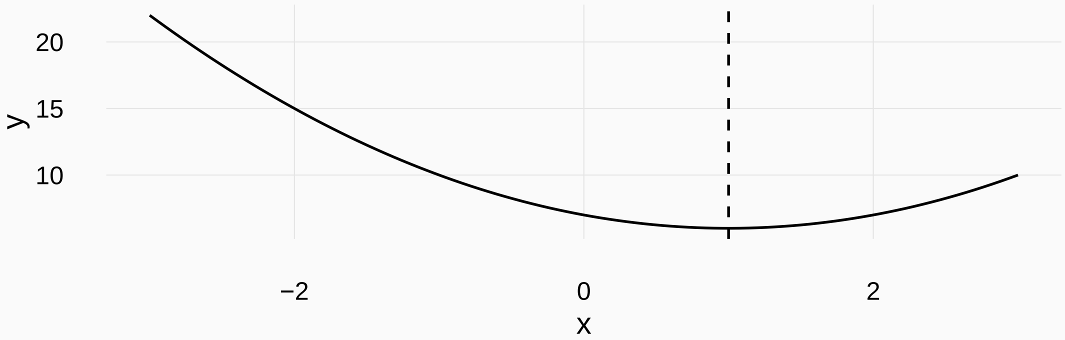
It's positive, so this is a *minimum*. Check the "endpoints"

$$\lim_{x \to -\infty} f(x) = \infty$$

$$\lim_{x \to \infty} f(x) = \infty$$

You try (answer)

```
ggplot(data.frame(x = c(-3, 3)), aes(x)) +
  stat_function(fun = function(x) x ^ 2 - (2 * x) + 7) +
  geom_vline(xintercept = 1, linetype = "dashed")
```



Some functions have more than one variable (e.g. $f(x, z) = x^2 + 2xz + z^2$). In this case, we have to be very clear with respect to which we are differentiating. We can only do one at a time and when we do so w treat all the other variables as constants. The notation is:

$$\frac{\partial f(x, z)}{\partial x}$$

Example: $y = 3x^2w - 4w$

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$$\frac{\partial y}{\partial x} = 6xw$$

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$$\frac{\partial y}{\partial x} = 6xw$$

$$\frac{\partial y}{\partial w} = -4$$

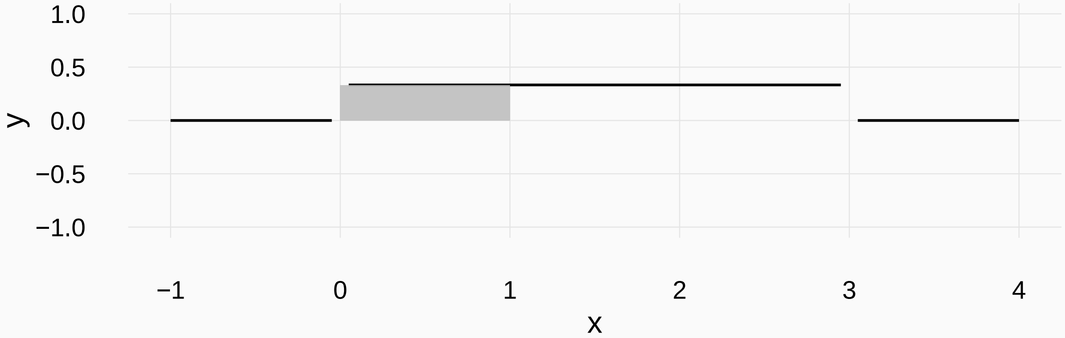
Integrals

Sometimes (pretty often, turns out) we might be interested in obtaining the **area under a curve**

This is sometimes super easy. What is the area under the curve between $x = -1$ and $x = 1$?

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x \in [0, 3] \\ 0 & \text{otherwise} \end{cases}$$

$$A = \ell w$$



This is the basic question motivating integration. If we wanted to find the area under the curve of $f(x)$ from $x = 0$ to $x = 4.5$, this is the notation:

$$\int_{x=0}^{4.5} f(x)\, dx$$

This is the basic question motivating integration. If we wanted to find the area under the curve of $f(x)$ from $x = 0$ to $x = 4.5$, this is the notation:

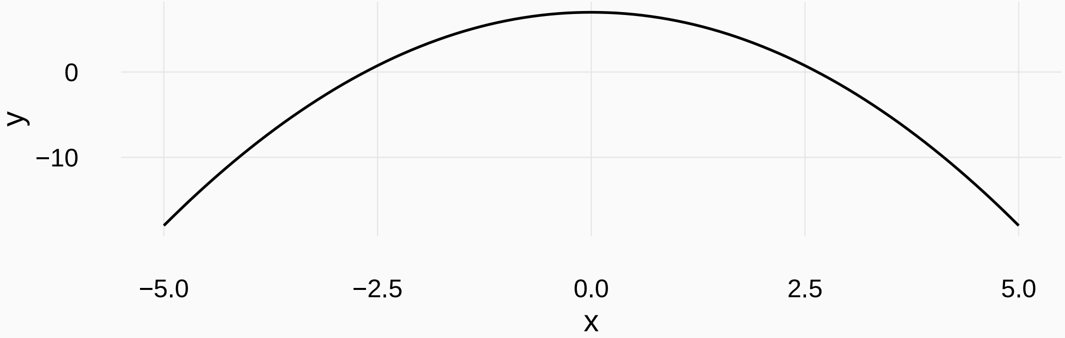
$$\int_{x=0}^{4.5} f(x)\, dx$$

How do we do this?

Finding integrals

Consider: $f(x) = -x^2 + 7$

```
ggplot(data.frame(x = -5:5), aes(x)) +
  stat_function(fun = function(x) -x ^ 2 + 7)
```



How to find the area between 0 and 2?

((Board examples))

Integration rules

Where a is a constant:

$$\int a \, dx = ax + C$$

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$$\int a \, dx = ax + C$$

$$\int af(x) \, dx = a \int f(x) \, dx$$

Where a is a constant:

$$\int a\,dx = ax + C$$

$$\int af(x)\,dx = a\int f(x)\,dx$$

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \forall n \neq 0$$

$$\int x^{-1} \, dx = \ln|x| + C$$

You try!

Find the integral:

1. $\int x^2 \, dx$

You try!

Find the integral:

1. $\int x^2 \, dx$
2. $\int 3x^2 \, dx$

You try!

Find the integral:

1. $\int x^2 \, dx$
2. $\int 3x^2 \, dx$
3. $\int 3x^2 + 2x - 7 \, dx$