Probability

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- There are other ways of thinking about probability, but we'll stick with this one

Kolmogorov's Axioms

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- Third: $Pr\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} Pr(E_i)$
    - The probability of disjoint (mutually exclusive) sets is equal to their sums

Probability Distributions

Discrete

Discrete probabilities

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- More generally $Bernoulli(\pi)$
    - π represents the probability of success

Binomial distributions

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- PMF:

$$\binom{n}{k} p^k (1-p)^{n-k}$$

You try!

1. What’s the probability of getting dealt the ace of spades?

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2. What's the probability of rolling six five's in a row?

You try (answers)

1. $$\frac{1}{52}$$

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2. $$\binom{6}{6}\frac{1}{6}^{6}\left(1-\frac{1}{6}\right)^{6-6}\approx 0.00002$$

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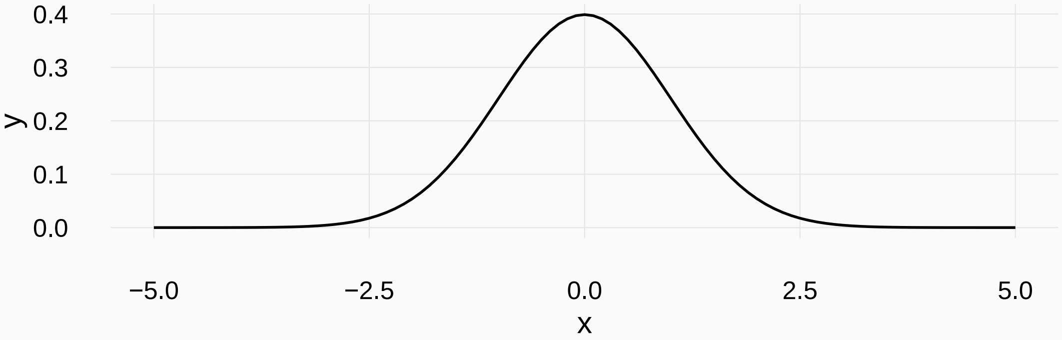
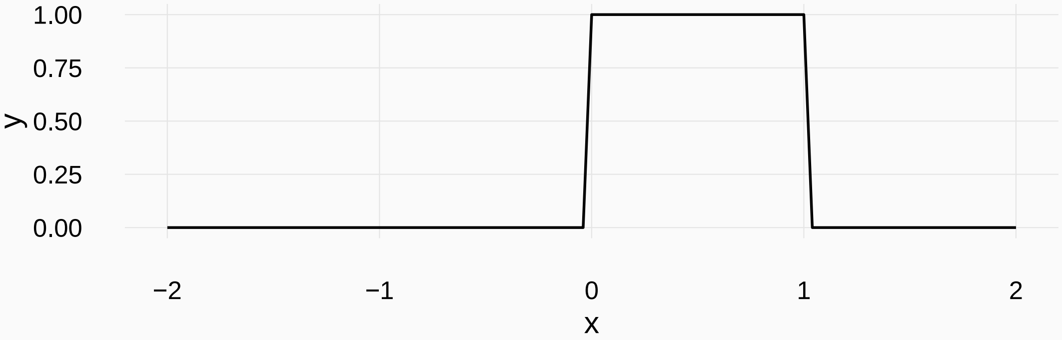
    - Kinda...

```r
ggplot(data.frame(x = c(-5, 5)), aes(x)) +
  stat_function(fun = dnorm)
```



Continuous distributions - Uniform

```r
ggplot(data.frame(x = c(-2, 2)), aes(x)) +
  stat_function(fun = dunif)
```



- $Pr(y = c) = 0$

Probabilies and continuous distributions

- $Pr(y = c) = 0$
- $Pr(0 < y < .5)$

$$= \int_0^{.5} f(y)dy$$

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- For uniform from previous slide, $Pr(0 < y < .5) = 0.5$

CDF

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- $F_X(x) = Pr(X \leq x)$

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- What's the probability that $y \leq 5$?
- $$\sum_{i=1}^{5} Pr(Y = y_i)$$

$Pr(y \leq 5)$

$$\binom{10}{1}.5^1(1-.5)^{10-1}+$$
$$\binom{10}{2}.5^2(1-.5)^{10-2}+$$
$$\binom{10}{3}.5^3(1-.5)^{10-3}+$$
$$\binom{10}{4}.5^4(1-.5)^{10-4}+$$
$$\binom{10}{5}.5^5(1-.5)^{10-5}$$

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- Solution: integration

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- $F_Y(y) = \int_{-\infty}^{y} f(y)dy$

Marginal probability

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- Probability of drawing a red card from a standard deck:

$$Pr(red) = \frac{1}{2}$$

Joint probability

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- $Pr(A \cap B)$
  - Probability of drawing a red card and a 4:
  - $Pr(red, 4) = \frac{2}{52} = \frac{1}{26}$

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- The probability of some event happening **given** some other event having occurred
  - $Pr(A|B)$
- You've drawn a red card. What's the probability it's a four?
  - $Pr(4|red) = \frac{2}{26} = \frac{1}{13}$

$$Pr(A|B) = \frac{Pr(A, B)}{Pr(B)}$$

The conditional probability is equal to the joint probability over the probability of the condition

Bayes' Law (Theorem)

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)} = \frac{Pr(B|A)Pr(A)}{Pr(B|A)Pr(A) + Pr(B|\neg A)Pr(\neg A)}$$

Bayes Example

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- Person stopped and fails breathalyzer test
    - Test gives positive 98 percent of the time when person is drunk
    - Also gives positive 5 percent of the time when the person is actually sober
    - We think that 1 percent of people going through the checkpoint are drunk
- What is the probability that the man was actually drunk?

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- Test accuracy:
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    - $Pr(drunk|positive) = \frac{Pr(drunk)Pr(positive|drunk)}{Pr(positive)}$

DWI Example

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    - $Pr(drunk) = 0.01$
- Test accuracy:
    - $Pr(positive|drunk) = 0.98$
    - $Pr(positive|\neg drunk) = 0.05$
- Want to know:
    - $Pr(drunk|positive) = \dfrac{Pr(drunk)Pr(positive|drunk)}{Pr(positive)}$
- Everything known, except $Pr(positive)$

$Pr(positive)$

$$
\begin{aligned}
Pr(positive) &= Pr(positive|drunk)Pr(drunk) \\
&\quad + Pr(positive|\neg drunk)Pr(\neg drunk) \\
&= .98(.01) + .05(.99) \\
&\approx .0593
\end{aligned}
$$

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- $Pr(drunk|positive) = \frac{Pr(drunk)Pr(positive|drunk)}{Pr(positive)}$
- $Pr(drunk|positive) = \frac{0.01(.98)}{.0593}$
- .165
- So there is a 16.5 percent chance that the man is drunk given that he tested positive

Example, take two: Monty Hall

- *Let's Make a Deal* was a game show where contestants were asked to choose one of three doors (A, B, and C). Behind one is a car and behind the other two are goats

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- The contestant could then either stick with their original choice or switch to the unopened door
- Does switching doors increase your probability of winning?
    - YES!
    - Nearly 10,000 people wrote in refusing to believe this (including 1,000 PhDs!)

Bayes's solution to Monty Hall

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- Then: $Pr(B_{Monty}|A) = \frac{1}{2}$ and $Pr(B_{Monty}|B = 0)$ and $Pr(B_{Monty}|C) = 1$

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$$Pr(B_{Monty}, A) = Pr(B_{Monty}|A)Pr(A) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$$
$$Pr(B_{Monty}, B) = Pr(B_{Monty}|B)Pr(B) = 0 * \frac{1}{3} = 0$$
$$Pr(B_{Monty}, C) = Pr(B_{Monty}|C)Pr(C) = 1 * \frac{1}{3} = \frac{1}{3}$$

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$$Pr(B_{Monty}, A) = Pr(B_{Monty}|A)Pr(A) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$$

$$Pr(B_{Monty}, B) = Pr(B_{Monty}|B)Pr(B) = 0 * \frac{1}{3} = 0$$

$$Pr(B_{Monty}, C) = Pr(B_{Monty}|C)Pr(C) = 1 * \frac{1}{3} = \frac{1}{3}$$

- Note that $Pr(B_{Monty}) = Pr(B_{Monty}, A) + Pr(B_{Monty}, B) + Pr(B_{Monty}, C) = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$

$Pr(A|B_{Monty})$ AND $Pr(C|B_{Monty})$

- $Pr(A|B_{Monty}) = \frac{Pr(A)Pr(B_{Monty}|A)}{Pr(B_{Monty})} = \frac{\frac{1}{3} * \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$

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- $Pr(C|B_{Monty}) = \dfrac{Pr(C)Pr(B_{Monty}|C)}{Pr(B_{Monty})} = \dfrac{\frac{1}{3} * 1}{\frac{1}{2}} = \dfrac{2}{3}$

$Pr(A|B_{Monty})$ AND $Pr(C|B_{Monty})$

- $Pr(A|B_{Monty}) = \frac{Pr(A)Pr(B_{Monty}|A)}{Pr(B_{Monty})} = \frac{\frac{1}{3} * \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$
- $Pr(C|B_{Monty}) = \frac{Pr(C)Pr(B_{Monty}|C)}{Pr(B_{Monty})} = \frac{\frac{1}{3} * 1}{\frac{1}{2}} = \frac{2}{3}$
- So switch!