Marginal, Joint, and Conditional Probabilities

J. Alexander Branham

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Marginal probability

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  - $Pr(A)$

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- Probability of drawing a red card from a standard deck:

$$Pr(red) = \frac{1}{2}$$

Joint probability

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  - Probability of drawing a red card and a 4:
  - $Pr(red, 4) = \frac{2}{52} = \frac{1}{26}$

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- You've drawn a red card. What's the probability it's a four?
  - $Pr(4|red) = \frac{2}{26} = \frac{1}{13}$

How to convert marginal, joint, and conditional probabilities

$$Pr(A|B) = \frac{Pr(A, B)}{Pr(B)}$$

* The conditional probability is equal to the joint probability over the probability of the condition

Bayes' Law (Theorem)

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)} = \frac{Pr(B|A)Pr(A)}{Pr(B|A)Pr(A) + Pr(B|\neg A)Pr(\neg A)}$$

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  - Also gives positive 5 percent of the time when the person is actually sober
  - We think that 1 percent of people going through the checkpoint are drunk
- What is the probability that the man was actually drunk?

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- Want to know:
  - $Pr(drunk|positive) = \dfrac{Pr(drunk)Pr(positive|drunk)}{Pr(positive)}$
- Everything known, except $Pr(positive)$

$Pr(positive)$

$$\begin{aligned} Pr(positive) &= Pr(positive|drunk)Pr(drunk) \\ &\quad + Pr(positive|\neg drunk)Pr(\neg drunk) \\ &= .98(.01) + .05(.99) \\ &\approx .0593 \end{aligned}$$

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- $Pr(drunk|positive) = \frac{0.01(.98)}{.0593}$
- .165
- So there is a 16.5 percent chance that the man is drunk given that he tested positive

Example, take two: Monty Hall

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- The contestant could then either stick with their original choice or switch to the unopened door
- Does switching doors increase your probability of winning?
  - YES!
  - Nearly 10,000 people wrote in refusing to believe this (including 1,000 PhDs!)

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- Then: $Pr(B_{Monty}|A) = \frac{1}{2}$ and $Pr(B_{Monty}|B = 0)$ and $Pr(C_{Monty}|C) = 1$

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$$Pr(B_{Monty}, A) = Pr(B_{Monty}|A)Pr(A) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$$

$$Pr(B_{Monty}, B) = Pr(B_{Monty}|B)Pr(B) = 0 * \frac{1}{3} = 0$$

$$Pr(B_{Monty}, C) = Pr(B_{Monty}|C)Pr(C) = 1 * \frac{1}{3} = \frac{1}{3}$$

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$$Pr(B_{Monty}, A) = Pr(B_{Monty}|A)Pr(A) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$$
$$Pr(B_{Monty}, B) = Pr(B_{Monty}|B)Pr(B) = 0 * \frac{1}{3} = 0$$
$$Pr(B_{Monty}, C) = Pr(B_{Monty}|C)Pr(C) = 1 * \frac{1}{3} = \frac{1}{3}$$

- Note that $Pr(B_{Monty}) =$ $Pr(B_{Monty}, A) + Pr(B_{Monty}, B) + Pr(B_{Monty}, C) = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$

$Pr(A|B_{Monty})$ and $Pr(C|B_{Monty})$

- $Pr(A|B_{Monty}) = \dfrac{Pr(A)Pr(B_{Monty}|A)}{Pr(B_{Monty})} = \dfrac{\frac{1}{3} * \frac{1}{2}}{\frac{1}{2}} = \dfrac{1}{3}$

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- $Pr(C|B_{Monty}) = \dfrac{Pr(C)Pr(B_{Monty}|C)}{Pr(B_{Monty})} = \dfrac{\frac{1}{3} * 1}{\frac{1}{2}} = \dfrac{2}{3}$

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- $Pr(A|B_{Monty}) = \dfrac{Pr(A)Pr(B_{Monty}|A)}{Pr(B_{Monty})} = \dfrac{\frac{1}{3} * \frac{1}{2}}{\frac{1}{2}} = \dfrac{1}{3}$
- $Pr(C|B_{Monty}) = \dfrac{Pr(C)Pr(B_{Monty}|C)}{Pr(B_{Monty})} = \dfrac{\frac{1}{3} * 1}{\frac{1}{2}} = \dfrac{2}{3}$
- So switch!