Set Theory & Combinations

J. Alexander Branham

Fall 2016

Set Theory

Intro to Set Theory

- What is set theory?

- What is set theory?
- A branch of mathematics

- What is set theory?
- A branch of mathematics
- Collects objects into *sets* and studies the properties

What's a set?

- A **set** is a collection of objects

$$S = \{s_1, s_2, s_3, ...s_n\}$$

What's a set?

- A **set** is a collection of objects

$$S = \{s_1, s_2, s_3, ...s_n\}$$

- The objects can be anything

What's a set?

- A **set** is a collection of objects

$$S = \{s_1, s_2, s_3, ...s_n\}$$

- The objects can be anything
- We usually use variables or units of observation

- We can say whether an object is in a set:

$$s_{13} \in S$$

- We can say whether an object is in a set:

$$s_{13} \in S$$

- Or not:

$$q_1 \notin S$$

Subsets

- We can divide sets into *subsets*

$$M \subset S$$

Subsets

- We can divide sets into *subsets*

$$M \subset S$$

- M is a *proper subset* of S iff all elements of M are in S but not all elements of S are in M.

Subsets

- We can divide sets into *subsets*

$$M \subset S$$

- M is a *proper subset* of S iff all elements of M are in S but not all elements of S are in M.
- We can also define non-proper subsets:

$$L \subseteq S$$

Empty sets

- Some sets contain no elements

Empty sets

- Some sets contain no elements
- For example, cities on Earth with average temperatures of greater than 1000 degrees

Empty sets

- Some sets contain no elements
- For example, cities on Earth with average temperatures of greater than 1000 degrees
- Though Austin might be close...

$$Z = \{\emptyset\}$$

Set Universes

- We can talk about how subsets fill a certain **universe** of possibilities

Set Universes

- We can talk about how subsets fill a certain **universe** of possibilities
- For example:

$$R = \{1, 2, 3, 4, 5, 6\}$$

Set Universes

- We can talk about how subsets fill a certain **universe** of possibilities
- For example:

$$R = \{1, 2, 3, 4, 5, 6\}$$

- R represents all the possibilities of a (single) roll of a die

Set Universes

- We can talk about how subsets fill a certain **universe** of possibilities
- For example:

$$R = \{1, 2, 3, 4, 5, 6\}$$

- R represents all the possibilities of a (single) roll of a die
- We can define sets for the even possibilities and the odd possibilities

$$E = \{2, 4, 6\} \qquad O = \{1, 3, 5\}$$

Compliments

- A *compliment* is that together, they contain all the elements of the relevant universe

$$E = O^C \qquad O = E^C$$

Universe

- Board examples of how to draw sets

- Intersections: $A \cap B$ includes all the common elements of A and B

Intersection, Disjoint, unions

- Intersections: $A \cap B$ includes all the common elements of A and B
    - If $A \cap B = \emptyset$, then they are **disjoint**

Intersection, Disjoint, unions

- Intersections: $A \cap B$ includes all the common elements of A and B
    - If $A \cap B = \emptyset$, then they are **disjoint**
- Unions: $A \cup B$ contains all the elements of both A and B

- Intersections: $A \cap B$ includes all the common elements of A and B
  - If $A \cap B = \emptyset$, then they are **disjoint**
- Unions: $A \cup B$ contains all the elements of both A and B
  - Two sets unions can't be empty (unless both sets are empty to begin with)

Intersection, Disjoint, unions

- Intersections: $A \cap B$ includes all the common elements of A and B
    - If $A \cap B = \emptyset$, then they are **disjoint**
- Unions: $A \cup B$ contains all the elements of both A and B
    - Two sets unions can't be empty (unless both sets are empty to begin with)
- Size: $n(R) = 6$ tells us that there are 6 elements in R

Intersection, Disjoint, unions

- Intersections: $A \cap B$ includes all the common elements of A and B
  - If $A \cap B = \emptyset$, then they are **disjoint**
- Unions: $A \cup B$ contains all the elements of both A and B
  - Two sets unions can't be empty (unless both sets are empty to begin with)
- Size: $n(R) = 6$ tells us that there are 6 elements in R
- Sizes of unions

Intersection, Disjoint, unions

- Intersections: $A \cap B$ includes all the common elements of A and B
    - If $A \cap B = \emptyset$, then they are **disjoint**
- Unions: $A \cup B$ contains all the elements of both A and B
    - Two sets unions can't be empty (unless both sets are empty to begin with)
- Size: $n(R) = 6$ tells us that there are 6 elements in R
- Sizes of unions
    - $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

Intersection, Disjoint, unions

- Intersections: $A \cap B$ includes all the common elements of A and B
  - If $A \cap B = \emptyset$, then they are **disjoint**
- Unions: $A \cup B$ contains all the elements of both A and B
  - Two sets unions can't be empty (unless both sets are empty to begin with)
- Size: $n(R) = 6$ tells us that there are 6 elements in R
- Sizes of unions
  - $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$
  - Have to subtract the last term, otherwise we'd double-count elements in the intersection of the two sets

Properties

- $A \cup B = B \cup A$

Properties

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Properties

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cup B) \cup C = A \cup (B \cup C)$

Properties

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Properties

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Properties

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

You try!

1. Define the universe of cards in a standard 52-card deck $- \Omega$

You try!

1. Define the universe of cards in a standard 52-card deck — Ω
2. Define the set containing all the spades — S — the set containing the black cards — B — and the set of queens Q

You try!

1. Define the universe of cards in a standard 52-card deck — Ω
2. Define the set containing all the spades — S — the set containing the black cards — B — and the set of queens Q
3. What is $n(B)$

You try!

1. Define the universe of cards in a standard 52-card deck — Ω
2. Define the set containing all the spades — S — the set containing the black cards — B — and the set of queens Q
3. What is $n(B)$
4. What is $S \cup B$?

You try!

1. Define the universe of cards in a standard 52-card deck — Ω
2. Define the set containing all the spades — S — the set containing the black cards — B — and the set of queens Q
3. What is $n(B)$
4. What is $S \cup B$?
5. What is $B \cap Q$?

You try!

1. Define the universe of cards in a standard 52-card deck — Ω
2. Define the set containing all the spades — S — the set containing the black cards — B — and the set of queens Q
3. What is $n(B)$
4. What is $S \cup B$?
5. What is $B \cap Q$?
6. What is $B \cap (Q \cup S)$?

You try!

1. Define the universe of cards in a standard 52-card deck — Ω
2. Define the set containing all the spades — S — the set containing the black cards — B — and the set of queens Q
3. What is $n(B)$
4. What is $S \cup B$?
5. What is $B \cap Q$?
6. What is $B \cap (Q \cup S)$?
7. Is $Q \subset S$?

You try (answers)

1. Ω is a set with 52 elements, one for each card

You try (answers)

1. Ω is a set with 52 elements, one for each card
2. S is a set with 13 elements, one for each spade. B has 26 elements (the black cards) and Q has just four - the queens

You try (answers)

1. Ω is a set with 52 elements, one for each card
2. S is a set with 13 elements, one for each spade. B has 26 elements (the black cards) and Q has just four - the queens
3. $n(B) = 26$

You try (answers)

1. Ω is a set with 52 elements, one for each card
2. S is a set with 13 elements, one for each spade. B has 26 elements (the black cards) and Q has just four - the queens
3. $n(B) = 26$
4. $S \cup B = B$

You try (answers)

1. Ω is a set with 52 elements, one for each card
2. S is a set with 13 elements, one for each spade. B has 26 elements (the black cards) and Q has just four - the queens
3. $n(B) = 26$
4. $S \cup B = B$
5. $B \cap Q$ is the queen of spades and clubs

You try (answers)

1. Ω is a set with 52 elements, one for each card
2. S is a set with 13 elements, one for each spade. B has 26 elements (the black cards) and Q has just four - the queens
3. $n(B) = 26$
4. $S \cup B = B$
5. $B \cap Q$ is the queen of spades and clubs
6. $B \cap (Q \cup S) = B$

You try (answers)

1. Ω is a set with 52 elements, one for each card
2. S is a set with 13 elements, one for each spade. B has 26 elements (the black cards) and Q has just four - the queens
3. $n(B) = 26$
4. $S \cup B = B$
5. $B \cap Q$ is the queen of spades and clubs
6. $B \cap (Q \cup S) = B$
7. No

Combinatorics

Combinatorics - What is it?

- How to count things?

Combinatorics - What is it?

- How to count things?
- Countable *discrete* structures

- How to count things?
- Countable *discrete* structures
- Let's say we have the set R from above

$$R = \{1, 2, 3, 4, 5, 6\}$$

- How to count things?
- Countable *discrete* structures
- Let's say we have the set R from above

$$R = \{1, 2, 3, 4, 5, 6\}$$

- How many different combinations of 3 dice rolls are there?

Order

- First thing to ask is if order matters

- First thing to ask is if order matters
- Is rolling 1, 4, 2 the same as rolling 4, 2, 1?

Order

- First thing to ask is if order matters
- Is rolling 1, 4, 2 the same as rolling 4, 2, 1?
- If order matters, we talk about **permutations**

Order

- First thing to ask is if order matters
- Is rolling 1, 4, 2 the same as rolling 4, 2, 1?
- If order matters, we talk about **permutations**
- If not, it's a **combination**

Order

- First thing to ask is if order matters
- Is rolling 1, 4, 2 the same as rolling 4, 2, 1?
- If order matters, we talk about **permutations**
- If not, it's a **combination**
- think "permutation" = "position"

Repitition?

- Can we repeat things?

Repitition?

- Can we repeat things?
- Is rolling 3, 3, 3, permissable?

Repitition?

- Can we repeat things?
- Is rolling 3, 3, 3, permissable?
- If yes, we choose one die roll of six three times:

$$6 * 6 * 6 = 6^3$$

Repitition?

- Can we repeat things?
- Is rolling 3, 3, 3, permissable?
- If yes, we choose one die roll of six three times:

$$6 * 6 * 6 = 6^3$$

- more generally, n^r

Repitition?

- Can we repeat things?
- Is rolling 3, 3, 3, permissable?
- If yes, we choose one die roll of six three times:

$$6 * 6 * 6 = 6^3$$

- more generally, n^r
    - n is number of things to choose from

Repitition?

- Can we repeat things?
- Is rolling 3, 3, 3, permissable?
- If yes, we choose one die roll of six three times:

$$6 * 6 * 6 = 6^3$$

- more generally, n^r
    - n is number of things to choose from
    - choose r

Permutations without replacement

- We need to reduce the number of things from which to choose each time

- We need to reduce the number of things from which to choose each time
- How many different orderings of 9 horses in a race??

Permutations without replacement

- We need to reduce the number of things from which to choose each time
- How many different orderings of 9 horses in a race??
  - $9 * 8 * 7 ... = 9! = 362,880$

Permutations without replacement

- We need to reduce the number of things from which to choose each time
- How many different orderings of 9 horses in a race??
  - $9 * 8 * 7 ... = 9! = 362,880$
- Alternatively, maybe we just care about the first three:

Permutations without replacement

- We need to reduce the number of things from which to choose each time
- How many different orderings of 9 horses in a race??
  - $9 * 8 * 7 ... = 9! = 362,880$
- Alternatively, maybe we just care about the first three:
  - $9 * 8 * 7 = 504$

Permutations without replacement

- We need to reduce the number of things from which to choose each time
- How many different orderings of 9 horses in a race??
    - $9 * 8 * 7... = 9! = 362,880$
- Alternatively, maybe we just care about the first three:
    - $9 * 8 * 7 = 504$
- More generally:

$$\frac{n!}{(n-r)!}$$

- We use combinations when the order does **not** matter

Combinations

- We use combinations when the order does **not** matter
- So 4, 3, 1 is the same as 3, 1, 4

Repition not allowed

- Easiest way: do permutation without repitition, then get rid of all the “same” outcomes

Repition not allowed

- Easiest way: do permutation without repitition, then get rid of all the "same" outcomes
- We've already figured out the permutation part, so need to figure out the second part

$$\frac{n!}{(n-r)!} * \frac{1}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Repitition allowed

- These aren't used much

Repitition allowed

- These aren't used much
- Also the most difficult to calculate

Repitition allowed

- These aren't used much
- Also the most difficult to calculate
- Formula:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

You try!

Using a standard 52-card deck:

1. How many different 3-card hands exist?

You try!

Using a standard 52-card deck:

1. How many different 3-card hands exist?
2. How many ways are there to get dealt the queen of spades, then the king of diamonds, then the eight of spades?

You try (answers)

```r
# 1
choose(52, 3)
```

```
## [1] 22100
```

```r
# 2 - one
```