Ordinary Least Squares

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Introduction to Ordinary Least Squares

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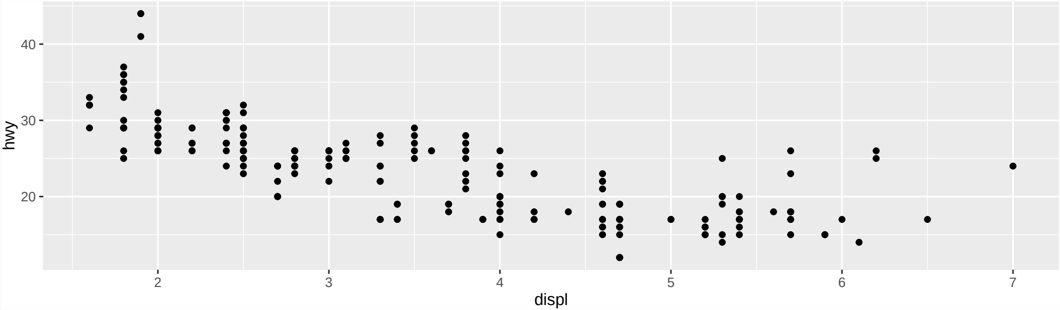
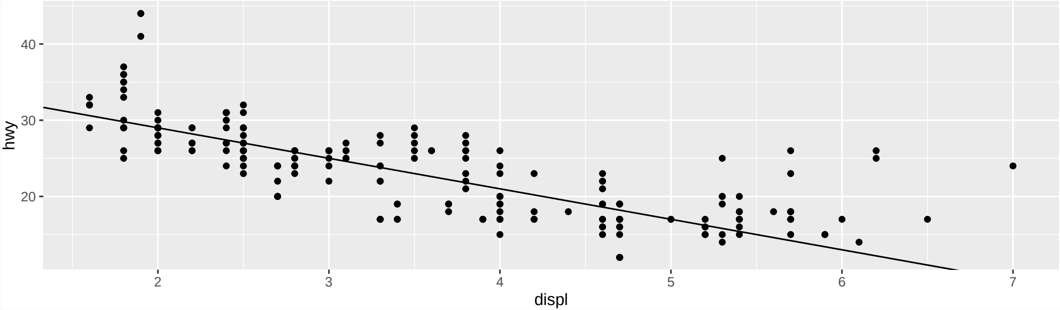
- At its core, it's all about drawing a line through data
- This allows us to asses the effect of x on y
- Dependent variable y must be continuous
    - OLS makes other assumptions you'll learn about in stats II

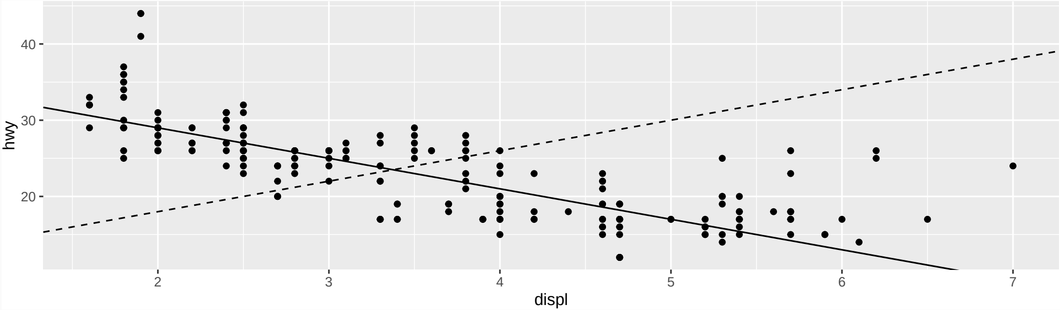

hwy
40
30
20
2
3
4
5
6
7
displ

How to decide on a line?

```
p <- p +  geom_abline(slope = -4, intercept = 37)
p
```

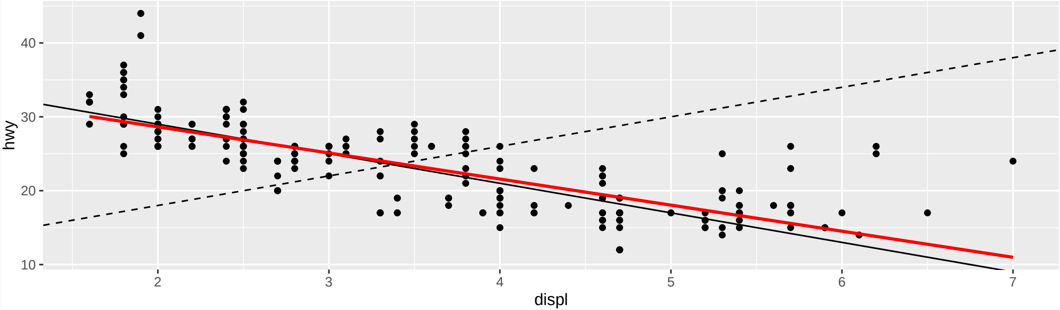


```r
p <- p +  geom_abline(slope = 4, intercept = 10, linetype = "dashed")
p
```



HOW TO DECIDE

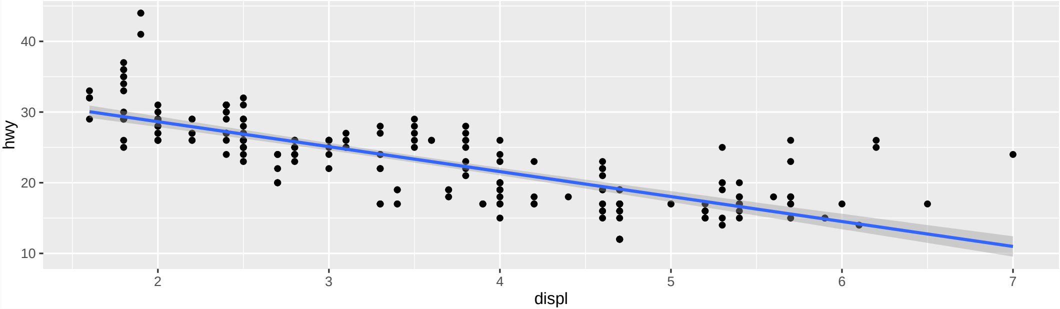
```
p +  geom_smooth(method = "lm", se = FALSE, color = "red")
```



OLS in R

```r
lm(hwy ~ displ, data = mpg)
```

```
##
## Call:
## lm(formula = hwy ~ displ, data = mpg)
##
## Coefficients:
## (Intercept)        displ
##      35.698       -3.531
```



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$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

Interpretation of coefficients

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- Intercept ($\hat{\beta}_0$) - predicted y when $x = 0$

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$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- Intercept ($\hat{\beta}_0$) - predicted y when $x = 0$
- Slope ($\hat{\beta}_1$) - a one unit change in x leads to a (slope) unit change in y, on average

- Notice that our line doesn’t fit our data perfectly - we always make some error with our prediction

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- That's referred to as the *residual*
- If we refer to our predicted value as $\hat{y}$, then we can calculate the residual for each observation with $e_i = y_i - \hat{y}_i$

Fitting the best line

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- How to find this?
- One option: Plug in all the values for the slope & intercept and calculate the sum of squared residuals for these infinity combinations
- That’s problematic...

How do we find the minimum sum of squared residuals?

Turns out we already know the solution - we learned it when we talked about *optimization*.

We just need to *minimize* the sum of squared residuals with respect to the two coefficients:

$$\sum_{i=1}^{n} e_i^2$$

Rearrange above equation in terms of e_i:

$$e_i = y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i$$

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Substitute:

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)^2$$

To find the minimum, we'll need to take the derivative with respect to $\hat{\beta}_0$ and $\hat{\beta}_1$. Starting with $\hat{\beta}_0$:

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$$\sum_{i=1}^{n}\left[\frac{\partial}{\partial \hat{\beta}_0}(y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)^2\right]$$

Derivatives

$$\sum_{i=1}^{n} \left[\frac{\partial}{\partial \hat{\beta}_0} (y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)^2 \right]$$

$$\sum_{i=1}^{n}\left[\frac{\partial}{\partial\hat{\beta}_0}(y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)^2\right]$$

The next step is to use the **chain rule** to take the derivative of the quantity in parentheses:

$$\sum_{i=1}^{n} \left[\frac{\partial}{\partial \hat{\beta}_0} (y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)^2 \right]$$

The next step is to use the **chain rule** to take the derivative of the quantity in parentheses:

$$\sum_{i=1}^{n} [-2(y_i - \hat{\beta}_0 - b_1 x_i)]$$

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$$\sum_{i=1}^{n}[-2(y_i - \hat{\beta}_0 - b_1 x_i)]$$

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Derivative for slope

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$$\sum_{i=1}^{n}\left[\frac{\partial}{\partial \hat{\beta}_1}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2\right]$$

Using the chain rule again, we get:

$$-2\sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)$$

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((Solutions on board))

Multiple variables

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lm(hwy ~ displ + year, data = mpg)
```

```
##
## Call:
## lm(formula = hwy ~ displ + year, data = mpg)
##
## Coefficients:
## (Intercept)        displ         year
##   -276.1544      -3.6110       0.1558
```

OLS in Matrix Form

Notation

- Let's pretend that we know the **true** model

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  - E is $nx1$ column vector
- Therefore, we have:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ 1 & \vdots & \vdots & \dots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$Y = X\beta + E$$

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- (show why on board)

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- First, what is sum of squared residuals?
- The residuals:

$$E = Y - X\hat{\beta}$$

- Sum of squared residuals:

$$E'E$$

- (show why on board)
- Alternatively,

$$\begin{aligned} E'E &= (Y - X\hat{\beta})'(Y - X\hat{\beta}) \\ &= Y'Y - \hat{\beta}X'Y - Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \\ &= Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}X'X\hat{\beta} \end{aligned}$$

To minimize the sum of squares, we take the derivative

- Remember:

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- To check that this is a minimum, we check to make sure that the second derivative is positive
- The second derivative is $2X'X$, which is positive definite so long as X is full rank

Solve for the estimator

- Here ya go:

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$$(X'X)^{-1}X'Y = (X'X)^{-1}(X'X)\hat{\beta}$$

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$$(X'X)^{-1}X'Y = (X'X)^{-1}(X'X)\hat{\beta}$$

- We know that $(X'X)^{-1}(X'X) = I$

$$(X'X)^{-1}X'Y = I\hat{\beta}$$

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- Move things around and divide by two:

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- We know that $(X'X)^{-1}(X'X) = I$

$$(X'X)^{-1}X'Y = I\hat{\beta}$$

- And I is (kinda) like multiplying by 1 so :

$$(X'X)^{-1}X'Y = \hat{\beta}$$