

Marginal, Joint, and Conditional Probabilities

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Marginal probability

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 - ▶ $Pr(A)$
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- ▶ Probability of drawing a red card from a standard deck:

$$Pr(red) = \frac{1}{2}$$

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Joint probability

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 - ▶ $Pr(A, B)$
- ▶ This is the intersection of the two sets
- ▶ $Pr(A \cap B)$
 - ▶ Probability of drawing a red card and a 4:
 - ▶ $Pr(\text{red}, 4) = \frac{2}{52} = \frac{1}{26}$

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 - ▶ $Pr(4|red) = \frac{2}{26} = \frac{1}{13}$

How to convert marginal, joint, and conditional probabilities

$$Pr(A|B) = \frac{Pr(A, B)}{Pr(B)}$$

- * The conditional probability is equal to the joint probability over the probability of the condition

Bayes' Law (Theorem)

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)} = \frac{Pr(B|A)Pr(A)}{Pr(B|A)Pr(A) + Pr(B|\neg A)Pr(\neg A)}$$

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 - ▶ Test gives positive 98 percent of the time when person is drunk
 - ▶ Also gives positive 5 percent of the time when the person is actually sober
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- ▶ What is the probability that the man was actually drunk?

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- ▶ Want to know:

- ▶ $Pr(drunk|positive) = \frac{Pr(drunk)Pr(positive|drunk)}{Pr(positive)}$

- ▶ Everything known, except $Pr(positive)$

Pr(positive)

$$\begin{aligned} Pr(positive) &= Pr(positive|drunk)Pr(drunk) \\ &\quad + Pr(positive|\neg drunk)Pr(\neg drunk) \\ &= .98(.01) + .05(.99) \\ &\approx .0593 \end{aligned}$$

$Pr(drunk|positive)$

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- ▶ .165
- ▶ So there is a 16.5 percent chance that the man is drunk given that he tested positive

Example, take two: Monty Hall

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- ▶ The contestant could then either stick with their original choice or switch to the unopened door
- ▶ Does switching doors increase your probability of winning?
 - ▶ YES!
 - ▶ Nearly 10,000 people wrote in refusing to believe this (including 1,000 PhDs!)

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$$Pr(B_{Monty}, A) = Pr(B_{Monty}|A)Pr(A) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$$

$$Pr(B_{Monty}, B) = Pr(B_{Monty}|B)Pr(B) = 0 * \frac{1}{3} = 0$$

$$Pr(B_{Monty}, C) = Pr(B_{Monty}|C)Pr(C) = 1 * \frac{1}{3} = \frac{1}{3}$$

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$$Pr(B_{Monty}, A) = Pr(B_{Monty}|A)Pr(A) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$$

$$Pr(B_{Monty}, B) = Pr(B_{Monty}|B)Pr(B) = 0 * \frac{1}{3} = 0$$

$$Pr(B_{Monty}, C) = Pr(B_{Monty}|C)Pr(C) = 1 * \frac{1}{3} = \frac{1}{3}$$

- ▶ Note that $Pr(B_{Monty}) =$

$$Pr(B_{Monty}, A) + Pr(B_{Monty}, B) + Pr(B_{Monty}, C) = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$$

$Pr(A|B_{Monty})$ and $Pr(C|B_{Monty})$

$$\blacktriangleright Pr(A|B_{Monty}) = \frac{Pr(A)Pr(B_{Monty}|A)}{Pr(B_{Monty})} = \frac{\frac{1}{3} * \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$$

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▶ So switch!