

# Introduction to Probability

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- There are other ways of thinking about probability, but we'll stick with this one

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  - The probability of disjoint (mutually exclusive) sets is equal to their sums

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  - $\pi$  represents the probability of success

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- PMF:

$$\binom{n}{k} p^k (1 - p)^{n-k}$$



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  - Kinda. . .



## Continuous distributions - Normal

```
ggplot(data.frame(x = c(-5, 5)), aes(x)) +  
  stat_function(fun = dnorm)
```

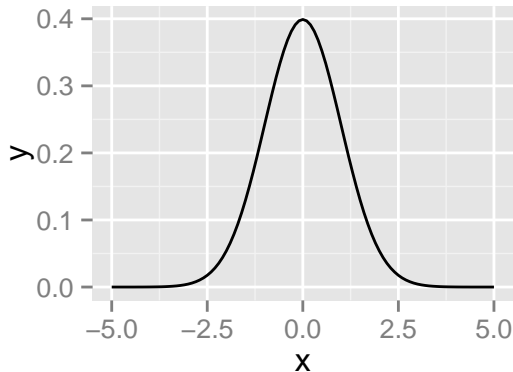


Figure 1:

## Continuous distributions - Uniform

```
ggplot(data.frame(x = c(-2, 2)), aes(x)) +  
  stat_function(fun = dunif)
```

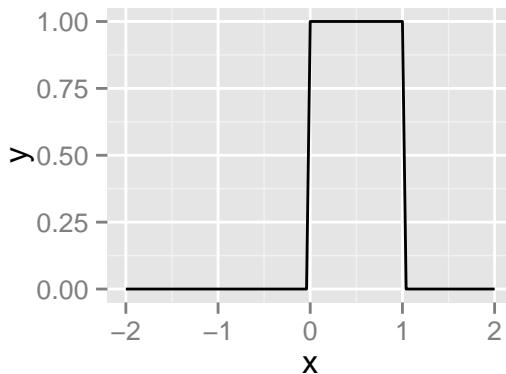


Figure 2:

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- For uniform from previous slide,  $Pr(0 < y < .5) = 0.5$

# CDF

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- $F_X(x) = Pr(X \leq x)$

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$$\sum_{y_i < y} \text{Pr}(Y = y_i)$$

$$Pr(y \leq 5)$$

$$\begin{aligned} & \binom{10}{1} .5^1 (1 - .5)^{10-1} + \\ & \binom{10}{2} .5^2 (1 - .5)^{10-2} + \\ & \binom{10}{3} .5^3 (1 - .5)^{10-3} + \\ & \binom{10}{4} .5^4 (1 - .5)^{10-4} + \\ & \binom{10}{5} .5^5 (1 - .5)^{10-5} \end{aligned}$$

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- $F_Y(y) = \int_{-\infty}^y f(y)dy$