

# SET THEORY & COMBINATIONS

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J. Alexander Branham

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# SET THEORY

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- What is set theory?

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- A branch of mathematics
- Collects objects into *sets* and studies the properties

# WHAT'S A SET?

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- The objects can be anything
- We usually use variables or units of observation



- We can say whether an object is in a set:

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- Or not:

$$q_1 \notin S$$

# SUBSETS

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- We can also define non-proper subsets:

$$L \subseteq S$$

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- Though Austin might be close...

$$Z = \{\emptyset\}$$



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- $R$  represents all the possibilities of a (single) roll of a die
- We can define sets for the even possibilities and the odd possibilities

$$E = \{2, 4, 6\} \quad O = \{1, 3, 5\}$$

- A *compliment* is that together, they contain all the elements of the relevant universe

$$E = O^c \qquad O = E^c$$

- Board examples of how to draw sets

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- Sizes of unions
  - $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$
  - Have to subtract the last term, otherwise we'd double-count elements in the intersection of the two sets

- $A \cup B = B \cup A$

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7. Is  $Q \subset S$ ?

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# COMBINATORICS

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- How many different combinations of 3 dice rolls are there?

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- think “permutation” = “position”

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  - $n$  is number of things to choose from
  - choose  $r$

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- More generally:

$$\frac{n!}{(n-r)!}$$



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- So 4, 3, 1 is the same as 3, 1, 4

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- We’ve already figured out the permutation part, so need to figure out the second part

$$\frac{n!}{(n-r)!} * \frac{1}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

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- Formula:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

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1. How many different 3-card hands exist?



## YOU TRY!

Using a standard 52-card deck:

1. How many different 3-card hands exist?
2. How many ways are there to get dealt the queen of spades, then the king of diamonds, then the eight of spades?

## YOU TRY (ANSWERS)

```
# 1
```

```
choose(52, 3)
```

```
## [1] 22100
```

```
# 2 - one
```