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PRICE DISPERSION AND INCOMPLETE LEARNING IN THE LONG RUN*

Andrew McLENNAN

University of Toronto, Toronto, Canada

Cornell University, Ithaca, NY 14853, USA

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In each period a seller chooses a price, and the customer for that period buys either one unit or nothing. The relationship between price and purchase probability is one of two linear functions. It is shown that the seller, behaving optimally, does not necessarily learn which function is the actual relationship, since there can be a positive probability of sequences of prices converging to the price at which the two functions give the same purchase probability.

1. Introduction

The optimal control of infinite horizon stochastic systems with unknown parameters and learning is notoriously difficult, since different choices of control variables provide different amounts and types of information. In fact a common approach in the literature is, in each period, to choose control variables optimal for the problem with the random parameters replaced by their expectations [see Chow (1981)]. Very little seems to have been written about exact solutions to such problems.

This note provides a simple example establishing the possibility that exact solutions may entail a positive probability of incomplete learning in the long run. The example arose in connection with a theory of equilibrium price dispersion advanced by Rothschild (1974) in which the seller chooses in each period between finitely many prices with unknown expected returns. The theory of the 'two armed bandit' [Berry (1972)] shows that with probability one the seller charges only one price infinitely many times. Furthermore, under certain conditions there is a positive probability that the seller will 'settle' on a suboptimal price, since those finite sequences of outcomes that lead the seller to settle on a given price occur with positive probability even when the price is

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suboptimal. Thus identical sellers with identical initial information may charge different prices after their 'learning periods' are over.

Obviously this theory would be significantly strengthened if it could be shown that its conclusion held when the seller chooses from a continuum of prices. In this case the seller's initial information consists of a probability distribution over possible relationships between price and random demand. The example of this paper is the simplest structure of this type.

In each period one person enters the seller's store, the seller quotes a price, and the potential customer either buys one unit or leaves empty handed. There are two possible linear relationships between price and purchase probability, and the seller's initial information or beliefs give a probability distribution over these relationships. There is a price at which the two relationships give the same purchase probability, and of course nothing is learned when this price is charged. The optimal prices for the two relationships lie on different sides of this price, so there is a 'critical prior belief' with the property that expected profits in the first period are maximized by charging the price that yields no information.

It will be shown that this price may be optimal for the critical prior in the infinite horizon problem, so that a seller who begins with this prior charges this price in every period and never changes his beliefs. The space of possible priors is isomorphic to the unit interval, and it will be shown that it is possible that a seller with a prior on one side of the critical prior will never charge a price providing enough information to yield a posterior on the other side of the critical prior.

This means that if the proprietor starts out thinking that one relationship is very likely, he will never be able to learn that the other relationship is the actual one. It will be shown that even when the relationship considered likely at the beginning is the actual relationship, there is a positive probability that the sequence of posteriors converges to the critical prior. Thus identical sellers with the same initial beliefs and the same actual relationship between price and purchase probability can end up charging different prices in the long run.

Little is known about possible extensions of these results to, for instance, problems in which there are three possible relationships between price and purchase probability, with one exception. In McLennan (1982) it is shown that all results extend to the case of three possible relationships when the optimal price for one relationship is the price that does not distinguish between the other two.

2. The model and the results

The two possible relationships between price and purchase probability are

$$D_i: y \mapsto \max\{0, \min\{1, b - m_i(y-a)\}\}, \qquad i = 1,2,$$

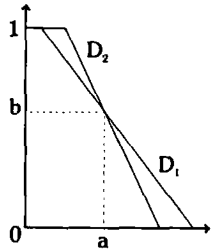

Fig. 1

where $a > 0$, $b \in (0,1)$, and

$$0 < m_1 < b/a < m_2.$$

The graphs of D_1 and D_2 are shown in fig. 1.

Letting p denote the probability of D_1, the probability of purchase when y is quoted is

$$\mathrm{P}(p,y) = pD_1(y) + (1-p)D_2(y).$$

For the sake of simplicity is assumed that no marginal costs are incurred in the event of a sale, so

$$\pi(p,y) = \mathrm{P}(p,y)y$$

is the expected profit when D_1 has probability p and y is quoted.

When the seller observes whether or not a unit is sold, he updates his belief concerning the probability of D_1. Letting S and N denote the events 'sale' and 'no sale', the posterior probability of D_1 is given by

$$q\colon \{S,N\} \times [0,1] \times \boldsymbol{R}_+ \to [0,1],$$

where

$$q(S,p,y) = \frac{pD_1(y)}{\mathrm{P}(p,y)} \quad \text{and} \quad q(N,p,y) = \frac{p(1-D_1(y))}{1-\mathrm{P}(p,y)},$$

are given by Bayes' rules when it is well defined, and are arbitrary otherwise. It is easily checked that $q(S,p,a) = p = q(N,p,a)$ for all p, so that charging a yields no information.

The seller's objective is to maximize the expected sum of discounted profits. Let $\delta \in (0,1)$ denote the discount factor, so that profits in period n are multiplied by δ^n. Letting $V(p)$ denote the value of the optimally managed business when the initial probability of D_1 is p, V clearly satisfies the functional relationship

$$V(p) = \sup_{y \in \boldsymbol{R}_+} \{\pi(p,y) + \delta[\mathrm{P}(p,y)V(q(S,p,y)) + (1-\mathrm{P}(p,y))V(q(N,p,y))]\}.$$

Also, V has the following important property.

Lemma 1. *V is convex.*

A formal proof of Lemma 1 will not be given, since it would be technical, and the intuition is quite clear. If $p = \alpha p_1 + (1-\alpha)p_2$, p can be construed as the prior when the proprietor does not know whether a certain event has occurred, where the event occurs with probability α, p_1 is the appropriate prior if the event has occurred, and p_2 is appropriate otherwise. Knowing whether the event has occurred cannot, on average, lower the value of the business.

Lemma 1 implies that V is continuous, and consequently there must exist an optimal strategy, i.e., a function $x:[0,1] \to \boldsymbol{R}_+$ such that for all p,

$$V(p) = \pi(p,x(p)) + \delta[\mathrm{P}(p,x(p))V(q(S,p,x(p))) + (1-\mathrm{P}(p,x(p)))V(q(N,p,x(p)))].$$

This discussion will analyze an arbitrary optimal strategy x, which is henceforth fixed.

At this point it is convenient to analyze the price that maximizes one-period expected profits, and for this purpose certain algebraic expressions are simplified by setting

$$l(p) = pm_1 + (1-p)m_2 \quad \text{for} \quad p \in [0,1].$$

For y such that $D_i(y) \in (0,1)$ for both i it is possible to write

$$\begin{aligned}\pi(p,y) &= [p(b - m_1(y-a)) + (1-p)(b - m_2(y-a))]y \\ &= [b - l(p)(y-a)]y \\ &= l(p)\left[\frac{1}{4}\left(a + \frac{b}{l(p)}\right)^2 - \left(y - \frac{1}{2}\left(a + \frac{b}{l(p)}\right)\right)^2\right].\end{aligned} \tag{1}$$

Fig. 1 makes it obvious that if m_1 and m_2 are sufficiently close to b/a, one-period expected profits are maximized at a price $z(p)$ with $D_i(z(p)) \in (0,1)$ for both i, and the equation above shows that

$$z(p) = \frac{1}{2}\left(a + \frac{b}{l(p)}\right). \tag{2}$$

The critical prior p^* is now defined by $z(p^*)=a$ or $l(p^*)=b/a$. An explicit formula for p^* is

$$p^*=\frac{m_2-b/a}{m_2-m_1}.$$

For given a, b, δ, and $C>0$, one can choose m_1 and m_2 sufficiently close to b/a to insure that

$$|\pi(p,z(p))-\pi(p^*,a)|<C \quad \text{for all } p.$$

Clearly $C/(1-\delta)$ is an upper bound on the value of additional information and also on the costs, in terms of foregone expected profits in the immediate period, that can sensibly be incurred to obtain such information. This argument leads to the following result:

Lemma 2. Suppose a, b, δ, and $c>0$ are given. By choosing m_1 and m_2 sufficiently close to b/a, one can guarantee that

$$x(p),z(p)\in[a-c,a+c] \quad \textit{for all} \quad p\in[0,1].$$

It will be assumed throughout that the conclusion of Lemma 2 is satisfied for a fixed c such that

$$D_i(a-c),D_i(a+c)\in(0,1), \qquad i=1,2.$$

The next lemma develops certain inequalities that are used in the statement and proof of the main result. To avoid interrupting the flow of the argument, the algebra required to prove Lemma 3 has been placed in an appendix.

Lemma 3. If

$$F=\frac{a^2(m_2-m_1)^2}{4m_1},$$

$$G=\frac{a(m_2-m_1)^2}{4m_2}\left(\frac{b}{m_1}\max\left\{\frac{p^*}{1-p^*},\frac{1-p^*}{p^*}\right\}-a\right),$$

$$H=\frac{a(m_2-m_1)}{2m_1},$$

$$J=\frac{m_2-m_1}{4\min\{b-m_2c,1-b-m_2c\}},$$

then

(i) *for all* p,

$$0 \le \pi(p, z(p)) - \pi(p^*, a) \le F(p - p^*)^2;$$

(ii) *if* $|p - p^*| > \min\{p^*, 1 - p^*\}$, *then*

$$p^*\pi(1, z(1)) + (1 - p^*)\pi(0, z(0)) - \pi(p, z(p)) < G(p - p^*)^2;$$

(iii) *for all* p,

$$|z(p) - a| < H|p - p^*|;$$

(iv) *for all* p, $E \in \{S, N\}$, and $y \in [a - c, a + c]$,

$$|q(E, p, y) - p| \le J|y - a|.$$

The conclusion of the theorem below are those described in the Introduction, namely that $x(p^*) = a$ and that $q(E, p, x(p)) - p^*$ has the same sign as $p - p^*$. The conditions under which they hold, however, will seem mysterious. These conditions will be explained to some extent in the proof and the subsequent proposition.

Theorem. *Let* $K = \max\{F, G\}$. *If there exists* $s > 0$ *such that*

$$s = \frac{1}{1 + HJ}(Ks^2 + \delta)^{-\frac{1}{2}}\left(s - J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}\right),$$

then $x(p^*) = a$. *Furthermore, if* $HJ < 1$ *and*

$$\frac{2}{1 - HJ} J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}} \le s,$$

for some s *as above, then for all* p *and* $E \in \{S, N\}$,

$$|q(E, p, x(p)) - p| \le |p - p^*|.$$

Proof. The idea of the proof is to find a sequence $\{t_n\}_{n=0}^{\infty}$ of positive numbers such that

$$\mathrm{I}(n) \qquad q(E,p,x(p)) \in T_n \quad \text{and} \quad q(E,p,z(p)) \in T_n,$$

for all $p \in T_{n+1}$ and both $E \in \{S, N\}$, where for each n, T_n is the interval

$$(\max\{0, p^* - t_n\}, \min\{1, p^* + t_n\}).$$

If it can be shown that $t_n \to 0$, it will follow that

$$q(E,p^*,x(p^*)) \in \bigcap_{n=0}^{\infty} T_n = \{p^*\},$$

for both $E \in \{S, N\}$, and this clearly implies $x(p^*) = a$.

It is also necessary to find positive numbers $\{M_n\}_{n=0}^{\infty}$ such that

$$\mathrm{II}(n) \qquad M_n \geq \Big(\sup_{p \in T_n} |V(p) - V(p^*)|\Big)^{\frac{1}{2}}.$$

Clearly I(0) and II(0) are satisfied if

$$M_0 = \Big(\max_{p \in [0,1]} |V(p) - V(p^*)|\Big)^{\frac{1}{2}},$$

$$t_0 = \max\{p^*, 1 - p^*\} \quad \text{and} \quad t_1 = M_0 s,$$

where s is arbitrary for the time being. Suppose M_{n-1}, t_{n-1}, and t_n have been chosen so that I($n-1$) and II($n-1$) are satisfied. It will be shown that I(n) and II(n) are satisfied if

$$M_n = \left(K t_n^2 + \delta M_{n-1}^2\right)^{\frac{1}{2}}, \tag{3}$$

and

$$t_{n+1} = \frac{1}{1 + HJ}\left(t_n - J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}} M_{n-1}\right). \tag{4}$$

To prove I(n), note that the defining properties of $x(p)$ and $z(p)$ imply that

$$\begin{aligned}&\pi(p,z(p))-\pi(p,x(p))\\&\le\delta[\mathrm{P}(p,x(p))V(q(S,p,x(p)))\\&\quad+(1-\mathrm{P}(p,x(p)))V(q(N,p,x(p)))\\&\quad-\mathrm{P}(p,z(p))V(q(S,p,z(p)))\\&\quad-(1-\mathrm{P}(p,z(p)))V(q(N,p,z(p)))].\end{aligned}$$

Replacing $V(q(S,p,x(p)))$ with $V(q(S,p,x(p)))-V(p^*)$, $V(q(N,p,x(p)))$ with $V(q(N,p,x(p)))-V(p^*)$, etc. does not change the value of the right-hand side, so for $p\in T_n$, I($n-1$) and II($n-1$) imply that

$$\pi(p,z(p))-\pi(p,x(p))\le 2\delta M_{n-1}^2.$$

From eqs. (1) and (2), however,

$$\pi(p,z(p))-\pi(p,x(p))=l(p)(x(p)-z(p))^2,$$

and $l(p)$ attains its minimum at $p=1$, so for $p\in T_n$,

$$|x(p)-z(p)|\le\left(\frac{2\delta}{l(p)}\right)^{\frac{1}{2}}M_{n-1}\le\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}M_{n-1}.$$

Applying Lemma 3(iii) and (iv), for $p\in T_n$ one has

$$\begin{aligned}|q(E,p,x(p))-p|&\le J(|x(p)-z(p)|+|z(p)-a|)\\&\le J\left(\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}M_{n-1}+H|p-p^*|\right).\end{aligned}\tag{5}$$

Adding $|p-p^*|$ to both sides shows that

$$|q(E,p,x(p))-p^*|\le J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}M_{n-1}+(1+HJ)|p-p^*|,$$

and it is easily seen from (4) that the right-hand side is less than t_n if $|p-p^*|<t_{n+1}$. Similarly,

$$\begin{aligned}|q(E,p,z(p))-p^*|&=|q(E,p,z(p))-p|+|p-p^*|\\&\le(1+HJ)|p-p^*|,\end{aligned}$$

and again the right-hand side is less than t_n when $p \in T_{n+1}$, so I(n) has been established.

To prove II(n), note that

$$\begin{aligned} V(p) &\le \pi(p, z(p)) + \delta\big[\mathrm{P}(p, x(p))V(q(S, p, x(p))) \\ &\quad + (1 - \mathrm{P}(p, x(p)))V(q(N, p, x(p)))\big] \\ &= (\pi(p, z(p)) - \pi(p^*, a)) + (\pi(p^*, a) + \delta V(p^*)) \\ &\quad + \delta\big[\mathrm{P}(p, x(p))(V(q(S, p, x(p))) - V(p^*)) \\ &\quad + (1 - \mathrm{P}(p, x(p)))(V(q(N, p, x(p))) - V(p^*))\big]. \end{aligned}$$

Clearly,

$$\pi(p^*, a) + \delta V(p^*) \le V(p^*),$$

so for $p \in T_n$, Lemma 3(i), I($n-1$), and II($n-1$) imply

$$\begin{aligned} V(p) - V(p^*) &\le F(p - p^*)^2 + \delta M_{n-1}^2 \\ &\le K t_n^2 + \delta M_{n-1}^2 = M_n^2. \end{aligned}$$

This inequality holds with $p^* - (p - p^*)$ in place of p if $2p^* - p \in [0,1]$, and the convexity of V implies that

$$V(p^*) - V(p) \le V(2p^* - p) - V(p^*),$$

so that

$$|V(p) - V(p^*)| \le M_n^2.$$

If $2p^* - p \notin [0,1]$, i.e.,

$$|p - p^*| > \min\{p^*, 1 - p^*\},$$

consider the inequality

$$\begin{aligned} V(p) &\ge \pi(p, z(p)) + \delta V(p^*) \\ &\quad + \delta\big[\mathrm{P}(p, z(p))(V(q(S, p, z(p))) - V(p^*)) \\ &\quad + (1 - \mathrm{P}(p, z(p)))(V(q(N, p, z(p))) - V(p^*))\big]. \end{aligned}$$

Combining this with I$(n-1)$ and II$(n-1)$, it follows that

$$V(p^*)-V(p)\le \delta M_{n-1}^2+(1-\delta)V(p^*)-\pi(p,z(p)).$$

The convexity of V implies that

$$V(p^*)\le \frac{1}{1-\delta}(p^*\pi(1,z(1))+(1-p^*)\pi(0,z(0))),$$

so that, by Lemma 3(ii),

$$\begin{aligned}V(p^*)-V(p)&\le \delta M_{n-1}^2+p^*\pi(1,z(1))+(1-p^*)\pi(0,z(0))\\&\quad-\pi(p,z(p))\\&\le \delta M_{n-1}^2+G(p-p^*)^2\\&\le \delta M_{n-1}^2+Kt_n^2.\end{aligned}$$

This completes the proof of II(n).

It must still be shown that the numbers $\{t_n\}_{n=0}^{\infty}$ can be positive and tend toward 0. To do this, rewrite eqs. (3) and (4) as follows:

$$\frac{M_n}{M_{n-1}}=\left(K\left(\frac{t_n}{M_{n-1}}\right)^2+\delta\right)^{\frac{1}{2}}, \tag{6}$$

$$\frac{t_{n+1}}{M_n}=\frac{1}{1+HJ}\,\frac{M_{n-1}}{M_n}\left(\frac{t_n}{M_{n-1}}-J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}\right). \tag{7}$$

Combining these equations yields

$$\frac{t_{n+1}}{M_n}=\frac{1}{1+HJ}\left(K\left(\frac{t_n}{M_{n-1}}\right)^2+\delta\right)^{-\frac{1}{2}}\left(\frac{t_n}{M_{n-1}}-J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}\right),$$

so if $s=t_1/M_0$ satisfies the hypotheses, then $t_{n+1}/M_n=s$ for all n. In this case t_n is positive for all n, and since H and J are both positive, eq. (4) implies that the sequence $\{t_n\}_{n=0}^{\infty}$ tends to 0. As was stated before, this guarantees that $x(p^*)=a$, so the first assertion of the Theorem has been proved.

Now suppose that

$$\frac{2}{1-HJ}J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}\le s,$$

so that for $n = 1, 2, \ldots,$

$$\left(\frac{1+HJ}{1-HJ}+1\right)J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}} \le \frac{t_n}{M_{n-1}},$$

$$\frac{1}{1-HJ}J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}} \le \frac{1}{1+HJ}\left(\frac{t_n}{M_{n-1}} - J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}\right)$$

$$= \frac{t_{n+1}}{M_{n-1}} \quad \text{[from (4)]},$$

and

$$J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}} M_{n-1} \le (1-HJ)t_{n+1}.$$

If $p \in T_n - T_{n+1}$ for $n = 1, 2, \ldots,$ inequality (5) implies

$$|q(E, p, x(p)) - p| \le (1-HJ)t_{n+1} + HJ|p - p^*|$$
$$\le |p - p^*|.$$

If $p \in T_0 - T_1$, the argument leading to inequality (5) can be modified to show that

$$|q(E, p, x(p)) - p| \le J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}} M_0 + HJ|p - p^*|.$$

Since

$$M_0 = \frac{t_1}{s} \le t_1\left[\frac{2}{1-HJ}J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}\right]^{-1},$$

if follows that

$$|q(E, p, x(p)) - p| \le \frac{1-HJ}{2}t_1 + HJ|p - p^*| \le |p - p^*|.$$

It has now been shown that the desired inequality holds for all $E \in \{S, N\}$ and all

$$p \in \bigcup_{n=0}^{\infty}(T_n - T_{n+1}) = [0,1] - \{p^*\}.$$

Its validity for p^* is an immediate consequence of the first assertion of the Theorem, so the proof is complete.

At this point it has not quite been shown that the conclusions of the Theorem are possible, since it must be shown that the hypothesis can hold. The following result shows that the Theorem is not vacuous by giving three natural circumstances under which there is a number s with the desired properties.

Proposition. (a) *For fixed a, b, m_1, and m_2 such that $HJ<1$, there exists s satisfying the hypothesis of the Theorem for sufficiently small δ.*

(b) *Such an s also exists for fixed a, b, p^*, and δ provided that m_1 is sufficiently close to b/a and*

$$p^*m_1+(1-p^*)m_2=\frac{b}{a}.$$

(c) *Finally, fix a, δ, and the optimal prices $z(1)=\bar{z}_1$ and $z(0)=\bar{z}_2$ for D_1 and D_2, so that m_1 and m_2 are determined by b,*

$$\frac{1}{2}\left(a+\frac{b}{m_i}\right)=\bar{z}_i,\quad i.e.,\quad m_i=\frac{b}{2\bar{z}_i-a}.$$

Then an s satisfying the hypothesis of the Theorem exists if b is sufficiently small.

Proof. If the variable r is defined by

$$s=J\left(\frac{2\delta}{m_1}\right)^{\frac{1}{2}}r,$$

the goal is to prove the existence of

$$r\geq\frac{2}{1-HJ},$$

such that

$$r=\frac{\delta^{-\frac{1}{2}}}{1+HJ}\left(\frac{2J^2K}{m_1}r^2+1\right)^{-\frac{1}{2}}(r-1).$$

Computing the derivative of the right-hand side with respect to r yields

$$\frac{\delta^{-\frac{1}{2}}}{1+HJ}\left(\frac{2J^2K}{m_1}r^2+1\right)^{-\frac{3}{2}}\left(\frac{2J^2K}{m_1}r+1\right),$$

and this quantity tends to 0 as r becomes large. Consequently it suffices to show the existence of $r > 2/(1-HJ)$ such that

$$r \leq \frac{\delta^{-\frac{1}{2}}}{1+HJ}\left(\frac{2J^2K}{m_1}r^2+1\right)^{-\frac{1}{2}}(r-1).$$

Fixing a, b, m_1, and m_2 fixes H, J, and K. (Since only small δ are being considered, c of Lemma 2 can be regarded as a constant.) Therefore any $r > 2/(1-HJ)$ is satisfactory provided δ is sufficiently small, and (a) has been proved. Fixing a, b, p^*, and δ, and regarding c as a constant, H, J, and K can be made arbitrarily small by choosing m_1 and m_2 close to b/a with

$$p^*m_1 + (1-p^*)m_2 = \frac{b}{a}.$$

To prove (b) it suffices to consider any $r > 2$ such that

$$r < \delta^{-\frac{1}{2}}(r-1).$$

Finally, fixing a, δ, $\bar{z}_1$, and $\bar{z}_2$ (c can again be regarded as a constant), if m_1 and m_2 are functions of b as in (c) and b tends to 0, H is constant and J and K tend to 0, so again it suffices to consider $r > 2$ satisfying the inequality given above. The proof is complete.

When all the conclusions of the Theorem hold it is possible to calculate the probability of incomplete learning. First note that if a sequence of posteriors generated by optimal behavior does not converge to p^*, there must be infinitely many prices outside some interval $(a-\varepsilon, a+\varepsilon)$. Such sequences of prices generate complete learning (convergence of the sequence of posteriors to 0 or 1) with probability 1. It follows that the probability that optimal behavior will generate a divergent sequence of posteriors is 0, and the only possible limits with positive probability are 0, p^*, and 1.

If the initial prior is, for instance, $p < p^*$, and the conclusions of the Theorem hold, the only possible limits are 0 and p^*; let

$$w_0(p) \text{ and } w_*(p) = 1 - w_0(p)$$

denote the probabilities of these limits. An elementary property of Bayesian updating is that the expectation of the posterior after any observation or sequence of observations is equal to the prior. Extended to the limit this implies that

$$w_0(p)\cdot 0 + w_*(p)\cdot p^* = p,$$

so

$$w_0(p) = 1 - \frac{p}{p^*} < 1 - p.$$

Let $w_0(p|D_i)$ be the probability that the sequence of posteriors converges to 0, conditional on the actual relationship being D_i, $i = 1,2$. Clearly $w_0(p|D_1) = 0$, and

$$w_0(p) = pw_0(p|D_1) + (1-p)w_0(p|D_2),$$

so one can compute that, when $p < p^*$,

$$w_0(p|D_2) = \frac{w_0(p)}{1-p} = \frac{p^* - p}{p^*(1-p)}.$$

The last term is less than 1 when $p > 0$, so incomplete learning occurs with positive probability even when it is not precluded by the 'barrier' p^*. In particular, different sellers may end up charging different prices in the long run even when they begin with the same prior.

Appendix: Proof of Lemma 3

(i) Eqs. (1) and (2) imply that

$$\begin{aligned}\pi(p, z(p)) - \pi(p^*, a) &= \frac{1}{4}\left[l(p)\left(a + \frac{b}{l(p)}\right)^2 - l(p^*)\left(a + \frac{b}{l(p^*)}\right)^2\right] \\ &= \frac{1}{4}\left[a^2(l(p) - l(p^*)) + b^2\left(\frac{1}{l(p)} - \frac{1}{l(p^*)}\right)\right] \\ &= \frac{1}{4}\left(a^2 - \frac{b^2}{l(p)l(p^*)}\right)(l(p) - l(p^*)).\end{aligned}$$

Since $l(p^*) = b/a$,

$$\begin{aligned}\pi(p, z(p)) - \pi(p^*, a) &= \frac{a}{4}\left(\frac{al(p) - b}{l(p)}\right)(l(p) - l(p^*)) \\ &= \frac{a^2}{4l(p)}(l(p) - l(p^*))^2,\end{aligned}$$

and the desired inequality follows from $l(p) \geq l(1) = m_1$ and

$$l(p) - l(p^*) = (m_1 - m_2)(p - p^*).$$

(ii) Let

$$Q = p^*\pi(1, z(1)) + (1 - p^*)\pi(0, z(0)) - \pi(p^*, z(p^*)).$$

Eq. (2) implies that

$$
\begin{aligned}
Q &= \frac{p^* l(1)}{4}\left(a+\frac{b}{l(1)}\right)^2 + \frac{(1-p^*)l(0)}{4}\left(a+\frac{b}{l(0)}\right)^2 \\
&\quad - \frac{l(p^*)}{4}\left(a+\frac{b}{l(p^*)}\right)^2 \\
&= \frac{a^2}{4}\left(p^* l(1) + (1-p^*)l(0) - l(p^*)\right) + \frac{b^2}{4}\left(\frac{p^*}{l(1)} + \frac{l-p^*}{l(0)} - \frac{1}{l(p^*)}\right).
\end{aligned}
$$

Since $l(p) = pl(1) + (1-p)l(0)$, the first term vanishes, and

$$
\begin{aligned}
Q &= \frac{b^2}{4l(1)l(0)l(p^*)}\left(p^* l(0) l(p^*) + (1-p^*)l(1)l(p^*) - l(1)l(0)\right) \\
&= \frac{b^2}{4l(1)l(0)l(p^*)}\left(p^* l(0)(l(p^*) - l(1))\right. \\
&\quad \left. + (1-p^*)l(1)(l(p^*) - l(0))\right),
\end{aligned}
$$

Now $l(p_1) - l(p_2) = (p_1 - p_2)(m_1 - m_2)$, $l(1) = m_1$, $l(0) = m_2$, and $l(p^*) = b/a$, so

$$
\begin{aligned}
Q &= \frac{ab}{4m_1 m_2}\left(p^* m_2(p^*-1)(m_1-m_2) + (1-p^*)m_1 p^*(m_1-m_2)\right) \\
&= \frac{abp^*(1-p^*)(m_2-m_1)^2}{4m_1 m_2}.
\end{aligned}
$$

Using the equality derived in the proof of (i) it now follows that

$$
\begin{aligned}
& p^*\pi(1, z(1)) + (1-p^*)\pi(0, z(0)) - \pi(p, z(p)) \\
&= Q + \pi(p^*, z(p^*)) - \pi(p, z(p)) \\
&= \frac{abp^*(1-p^*)(m_2-m_1)^2}{4m_1 m_2} - \frac{a^2}{4l(p)}(l(p) - l(p^*))^2 \\
&\le \frac{abp^*(1-p^*)(m_2-m_1)^2}{4m_1 m_2} - \frac{a^2}{4m_2}(m_2-m_1)^2(p-p^*)^2 \\
&= \frac{a(m_2-m_1)^2}{4m_2}\left(\frac{b}{m_1}p^*(1-p^*) - a(p-p^*)^2\right).
\end{aligned}
$$

If $p^* < \frac{1}{2}$ and $p > 2p^*$, then

$$\frac{b}{m_1}p^*(1-p^*) - a(p-p^*)^2 < \left(\frac{b}{m_1}\frac{1-p^*}{p^*} - a\right)(p-p^*)^2,$$

while if $p^* > \frac{1}{2}$ and $p < 1-2(1-p^*)$, then

$$\frac{b}{m_1}p^*(1-p^*) - a(p-p^*)^2 < \left(\frac{b}{m_1}\frac{p^*}{1-p^*} - a\right)(p-p^*)^2,$$

and in either case the result follows.

(iii) Eq. (2) implies that

$$\begin{aligned}|z(p)-a| &= \left|\frac{1}{2}\left(\frac{b}{l(p)} - a\right)\right| = \left|\frac{1}{2}\left(\frac{b}{l(p)} - \frac{b}{l(p^*)}\right)\right| \\ &= \left|\frac{b}{2l(p)l(p^*)}(l(p^*) - l(p))\right| \\ &= \frac{a(m_2-m_1)}{2l(p)}|p-p^*| \le \frac{a(m_2-m_1)}{2m_1}|p-p^*|.\end{aligned}$$

(iv) For $y \in [a-c, a+c]$ one has

$$\begin{aligned}|q(S,p,y)-p| &= \left|\frac{pD_1(y)}{P(p,y)} - p\right| = \left|p\left(\frac{D_1(y)-P(p,y)}{P(p,y)}\right)\right| \\ &= \left|p\left(\frac{b-l(1)(y-a)-b+l(p)(y-a)}{b-l(p)(y-a)}\right)\right| \\ &\le \frac{p(m_2-m_1)(1-p)}{b-m_2c}|y-a|,\end{aligned}$$

and similarly,

$$|q(N,p,y)-p| \le \frac{p(1-p)(m_2-m_1)}{1-b-m_2c}|y-a|.$$

Since $p(1-p) \leq \frac{1}{4}$, for both E it follows that

$$|q(E,p,y)-p| \leq \frac{m_2 - m_1}{4\min\{b - m_2 c, 1 - b - m_2 c\}}|y-a| = J|y-a|.$$

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