Physics 411: Midterm II

Tuesday March 15, 2016 before class (i.e. 10:10 AM)

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Note: questions get progressively more difficult. You do not need to solve the problems in order. Read through all questions before you start. Do not hesitate to ask questions on Piazza.

Shooting with friction

Calculating the trajectory of a projectile being thrown at an angle is a basic problem in classical mechanics. If air resistance is neglected, the differential equations can be solved analytically for given initial velocity v_0 (magnitude $|v_0|$ and angle α, $v_0 = |v_0|(\cos\alpha, \sin\alpha)$). In the presence of friction, however, one needs to use numerical methods.

The equation of motion for the position vector $r = \begin{pmatrix} r_x \\ r_z \end{pmatrix}$ is given by:

$$r_x'' = -\gamma|v|v_x, \tag{1}$$

$$r_z'' = -g - \gamma|v|v_z, \tag{2}$$

where γ is the friction coefficient and $g = 9.81m/s^2$ the gravitational constant for gravity acting in the -z direction.

1. Using the initial position $r_0 = (0, 3)$, initial angle $\alpha = \frac{\pi}{4}$ and $v_0 = 5m/s$, integrate the differential equation forward in time using the Forward Euler method and plot your trajectory for $\gamma = 1$.

2. Replace the Forward Euler method with the fourth order Runge Kutta method. Choose an initial angle of $\frac{3\pi}{4}$ and an initial velocity of $v_0 = 4m/s$ and $\gamma = 1$. Plot your trajectory. What is the maximum height your trajectory reaches, and when do you reach it?

3. Shoot a target six meters away, mounted at an elevation of 3.048 m (a basketball basket). Given a fixed initial velocity $v_0 = 20m/s$, a friction coefficient $\gamma = 0.2$, and an initial position of $r_x = 0, r_z = 2$, use the shooting method to find all possible αs that hit the target. List them and plot the trajectories.

The Poisson Problem

Consider the two-dimensional Poisson problem

$$\Delta\Phi(x, y) = -4\pi\rho(x, y) \tag{3}$$

in the interval $[0, 1]\times[0, 1]$, with boundary values $\Phi(x = 0, y) = 0$, $\Phi(x = 1, y) = 1$, and $\Phi(x, y = 0) = \Phi(x, y = 1) = x$, and with a charge distribution $\rho(x, y) = \sin(\pi x)\sin(2\pi y)$

1. Provide a $2d$ contour plot of the initial charge distribution.

2. Using a discretization of $\Delta x = \Delta y = \frac{1}{32}$, solve this problem using the relaxation method. Use a convergence cutoff of 10^{-5}. How many iterations are needed to reach convergence?

3. Provide a $2d$ contour plot of your solution.

For full credit: hand in a code that produces the solution from the relaxation method and contour plots of the solution and the initial charge distribution.

The Poisson Problem, II

1. Repeat the Poisson problem of the previous exercise for the multi-grid method described in class: Start with a discretization $\Delta x = 1/2 = \Delta y$.

The Poisson Problem, II continued on next page. . .

   (a) Linearly interpolate Φ, where not known, from the nearest neighbor points or the boundary.
   (b) Solve the Poisson problem using the relaxation method.
   (c) If $\Delta x = 1/1024 = \Delta y$, exit.
   (d) Otherwise set $\Delta x \leftarrow \Delta x/2$ and $\Delta y \leftarrow \Delta y/2$ and repeat from (a).

2. Plot the converged solution as a contour plot in each refinement step.

For full credit: hand in a code that produces the solution from the multi-grid method and a contour plot of the solution after each refinement step.

The Heat equation

A rod of length 1 has previously been heated in the middle, so that at time $t = 0$ its temperature distribution follows $T(x, t = 0) = 50e^{-80(x-0.5)^2}$. It is then brought into contact with a heat reservoir kept at fixed temperature of $T = 0$ at $x = 0$ and at another reservoir at fixed temperature of $T = 50$ at $x = 1$.

1. Plot the initial temperature distribution
2. Using a discretization $\Delta x = 100$ and a material constant $\alpha = \frac{\kappa}{c\rho} = 1$, solve the time evolution of the temperature profile using the Forward Euler method. Make sure to choose the time discretization such that the solution remains stable. Choose five representative times that show the evolution of the heat profiles to the steady state distribution at $t \to \infty$ and plot the temperature profile at those times.
3. Using the stability analysis discussed in class, show how the Forward Euler method becomes unstable if you choose your discretizations 'wrong'. Provide a sequence of plots that show the instability.
4. Implement the Crank-Nicolson method for the same problem. Show that it remains stable for the parameters you've chosen in the previous problem.