Physics 411: Homework II

Tuesday Jan 26 before class (i.e. 10:10 AM)

Emanuel Gull

Convergence of Newton Cotes rules

In computational physics we often need to integrate functions that are discretized on equidistant intervals. This problem will illustrate the convergence of some of the commonly used integration routines. For simplicity we focus on the integral of $\sin(x)$ between 0 and π.

1. Write a function that implements the rectangular rule to compute

$$I = \int_a^b f(x)dx \approx h \sum_{n=0}^{N-1} f(x_n) = I_N, \tag{1}$$

   $h = \frac{b-a}{N}, x_n = a + nh$. Your function should take a, b, N, and the function f as arguments and return the approximation I_N.

2. Write a program to plot the error $I - I_N$ against h, using $\sin(x)$ between 0 and π. Use a log-log scale. Label your axes and curve.

3. Write a function that implements the trapezoidal rule for the same function and add a curve for the error to the plot of task 2.

4. Write a function that implements the Simpson rule for the same function and add a curve for the error to the plot of task 2.

5. Write a function that implements the Boole's rule for the same function and add a curve for the error to the plot of task 2. Choose a regime of parameters for which you can illustrate that the different integration routines converge differently. Label all of your curves clearly.

6. Write a program that computes the power law for the convergence of the different algorithms.

For full credit hand in a program that implements all integration routines and generates a plot with log-log axes. Also hand in a program that computes the power law of the convergence. Make sure all curves are clearly visible and labeled, and that all axes are labeled.

Lagrange Interpolation

Polynomial interpolation using high order polynomials on equidistant points is often a very bad idea. This exercise will illustrate why.

1. Consider a function for which the following values are known:

x	0	1	3	4
f(x)	0	3	3	4

   Write a program to construct the Lagrange interpolation polynomial for the data.

2. Plot the polynomial using a blue line and the function values using red circles. Label your axes, the curve, and the data.

3. Consider the function

$$f(x) = \frac{1}{1 + 25x^2}, \tag{2}$$

   for x in $[-1, 1]$. Construct the 5th order and the 10th order interpolating polynomial using an equidistant discretization in the interval $[-1, 1]$.

Lagrange Interpolation continued on next page. . .

4. Plot the function and the two interpolating polynomials. Label your curves and axes.

For full credit hand in the interpolation programs and the two plots.

Romberg Integration

Romberg integration often gives very precise results for comparatively few function evaluations. In this exercise you will implement the integration method and illustrate its precision. As an example we use the function

$$\ln 2 = \int_1^2 \frac{1}{x} dx \approx 0.69314718056. \tag{3}$$

1. Implement a function that computes a sequence of trapezoidal integrals with step size h_i for i up to m, $h_{i-1} = 2h_i$, as described in the lecture. Make sure you reuse data from $i-1$ for obtaining the ith integral. Your function should return the estimates $I_1, \ldots, I_m$.

2. Implement a function that computes all Romberg coefficients $R_{p,q}$ with $p \leq m$, $q \leq p$, given the output of task 1.

3. Make a table in which entry (p, q) corresponds to the Romberg estimate $R_{p,q}$ for the integral mentioned above. use $m = 5$. Show in a plot how the relative error of the integration converges as a function of h (use the best Romberg estimate and plot it on a log-log scale).

For full credit hand in the program, the table of coefficients, and a plot of the relative errors. Make sure your curves and axes are properly labeled and the data is clearly visible.