PHY256 PS 2

1) a) $B|\psi\rangle = b_i|\psi\rangle$.

i) Expectation value of $B$ is $\langle B\rangle = b_i$.

ii) Uncertainty of $B$ is $\Delta B = 0$.

iii) $|\psi\rangle$ is a eigenstate of $B^2$:

$$B^2|\psi\rangle = B[B|\psi\rangle]$$
$$= B[b_i|\psi\rangle]$$
$$= b_i B|\psi\rangle$$
$$= b_i^2|\psi\rangle.$$

iv) Uncertainty of $B^2$ is $\Delta B^2 = 0$.

We are certain that $\langle B^2\rangle = b_i^2$.

b) $|\psi\rangle = \frac{1}{\sqrt{23}}\{i|4d\rangle - 2|3p\rangle + 4|2p\rangle + (1-i)|1s\rangle\}$.

i) Measure $|1s\rangle$ state:

$$\langle 1s|\psi\rangle = \frac{1}{\sqrt{23}}\{i\cancelto{0}{\langle 1s|4d\rangle} - 2\cancelto{0}{\langle 1s|3p\rangle} + 4\cancelto{0}{\langle 1s|2p\rangle} + (1-i)\cancelto{1}{\langle 1s|1s\rangle}\}$$
$$= \frac{1-i}{\sqrt{23}}$$

$$|\langle 1s|\psi\rangle|^2 = \left|\frac{1-i}{\sqrt{23}}\right|^2 = \frac{2}{23}$$

ii) Once measured & found in $|1s\rangle$, $|\psi\rangle \to |\psi_{new}\rangle = |1s\rangle$

state collapes into $|1s\rangle$ state.

A) You would never find the new state in $|3d\rangle$,

so $|\langle 3d|\psi_{new}\rangle|^2 = |\langle 3d|1s\rangle|^2 = 0$.

B) $\langle\psi|\psi_{new}\rangle = \langle\psi|1s\rangle = \langle 1s|\psi\rangle^* = \frac{1+i}{\sqrt{23}}$, so $|\langle\psi|1s\rangle|^2 = \frac{2}{23}$

Still.

PHY 256 PS2

2) a) $[S_z, S_y] = S_z S_y - S_y S_z$

$$= \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$

$$= -2i\frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$= -i\hbar S_x$$

b) $[S_y, S_y] = S_y S_y - S_y S_y$

$$= 0$$

c) $[S_z, S_y^2] = S_z S_y^2 - S_y^2 S_z$

By properties of Hermitian Metrices, $S_y^2 = I$.

$$= S_z I - I S_z$$

$$= S_z - S_z$$

$$= 0$$

d) $[M, D] = MD - DM$

$$= \begin{pmatrix}a & b\\ c & d\end{pmatrix}\underbrace{\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}}_{=2I} - \underbrace{\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}}_{=2I}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$= 2MI - 2IM$$

$$= 2M - 2M$$

$$= 0$$

PHY256 PS2

3) a) Know that $P_{tot} = P_1 + P_2 = 1$

Suppose $P_1 = |\sqrt{P_1}\, e^{i\phi_1}|^2$

and $P_2 = |\sqrt{P_2}\, e^{i\phi_2}|^2$

b) Final event occuring $P_{click} = |\sqrt{P_1}\, e^{i\phi_1} + \sqrt{P_2}\, e^{i\phi_2}|^2$.

Since path 1 and path 2 are indistinguishable.

$$P_{click} = (\sqrt{P_1}\, e^{i\phi_1} + \sqrt{P_2}\, e^{i\phi_2})(\sqrt{P_1}\, e^{-i\phi_1} + \sqrt{P_2}\, e^{-i\phi_2})$$

$$= P_1 \underbrace{\cancel{e^{i\phi_1} e^{-i\phi_1}}}_{1} + P_2 \underbrace{\cancel{e^{i\phi_2} e^{-i\phi_2}}}_{1} + \sqrt{P_1}\sqrt{P_2}\, e^{i\phi_1} e^{-i\phi_2} + \sqrt{P_1}\sqrt{P_2}\, e^{-i\phi_1} e^{i\phi_2}$$

$$= P_1 + P_2 + \sqrt{P_1 P_2}\left(e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)}\right)$$

$\phi_1 - \phi_2 = \Delta\phi$ is relative phase. Thus $P_{click}$ is only dependent on $\Delta\phi$.

$$= P_1 + P_2 + \sqrt{P_1 P_2}\left(\cos(\Delta\phi) + i\sin(\Delta\phi) + \cos(\Delta\phi) - i\sin(\Delta\phi)\right)$$

$$= P_1 + P_2 + 2\sqrt{P_1 P_2}\cos(\Delta\phi)$$

c) $$\overline{P_{click}(\Delta\phi)} = \int_0^{2\pi} \frac{d\Delta\phi}{2\pi}\left(P_{click}(\Delta\phi)\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(P_1 + P_2 + 2\sqrt{P_1 P_2}\cos(\Delta\phi)\right) d\Delta\phi$$

$$= \frac{1}{2\pi}\left[P_1\int_0^{2\pi} d\Delta\phi + P_2\int_0^{2\pi} d\Delta\phi + 2\sqrt{P_1 P_2}\int_0^{2\pi}\cos(\Delta\phi)\, d\Delta\phi\right]$$

$$= \frac{1}{2\pi}\left[2\pi P_1 + 2\pi P_2 + \cancel{2\sqrt{P_1 P_2}\left[\sin(\Delta\phi)\right]_0^{2\pi}}^{0}\right]$$

$$= P_1 + P_2$$

PHY 256 PS2

4) $|\psi\rangle = N\{3|-z\rangle - 4i|-x\rangle\}$

a) Need $\langle\psi|\psi\rangle = 1$ :

$$\langle\psi|\psi\rangle = N(3\langle -z| + 4i\langle -x|)\,N(3|-z\rangle - 4i|-x\rangle)$$

$$= N^2\left[9\langle -z|-z\rangle - 16i^2\langle -x|-x\rangle - \cancel{12i\langle -z|-x\rangle} + \cancel{12i\langle -x|-z\rangle}\right]$$

$$1 = N^2(9+16)$$

$$\Longrightarrow N^2 = \frac{1}{25} \Longrightarrow N = \pm\frac{1}{5}.$$ (More generally, $N = \frac{1}{5}$)

b) $\sigma_z$ in Dirac Notation is $|+z\rangle\langle +z| - |-z\rangle\langle -z|$.

Know $|-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$. Then $|\psi\rangle$ becomes

$$|\psi\rangle = \frac{1}{5}\left[3|-z\rangle - 4i\frac{1}{\sqrt{2}}\left[|+z\rangle - |-z\rangle\right]\right]$$

$$= \frac{1}{5}\left[-2\sqrt{2}i|+z\rangle + (3 + 2\sqrt{2}i)|-z\rangle\right]$$

Find $\langle +z|\psi\rangle$ : $\frac{1}{5}(-2\sqrt{2}i)\langle +z|+z\rangle + 0$

$$\langle +z|\psi\rangle = -\frac{2\sqrt{2}}{5}i, \quad \text{then} \quad \langle\psi|+z\rangle = \frac{2\sqrt{2}}{5}i.$$

Find $\langle -z|\psi\rangle$ : $\frac{1}{5}(3 + 2\sqrt{2}i)\langle -z|-z\rangle$

$$\langle -z|\psi\rangle = \frac{3+2\sqrt{2}i}{5}, \quad \text{then} \quad \langle\psi|-z\rangle = \frac{3-2\sqrt{2}i}{5}.$$

Evaluate $\langle\psi|\sigma_z|\psi\rangle$ :

$$= \langle\psi|\left[|+z\rangle\langle +z| - |-z\rangle\langle -z|\right]|\psi\rangle$$

$$= \langle\psi|\left[|+z\rangle\langle +z|\psi\rangle - |-z\rangle\langle -z|\psi\rangle\right] = \langle\psi|\left[|+z\rangle\left(-\frac{2\sqrt{2}i}{5}\right) - |-z\rangle\left(\frac{3+2\sqrt{2}i}{5}\right)\right]$$

$$= \langle\psi|+z\rangle\left(-\frac{2\sqrt{2}i}{5}\right) - \langle\psi|-z\rangle\left(\frac{3+2\sqrt{2}i}{5}\right) = \left(\frac{2\sqrt{2}}{5}i\right)\left(-\frac{2\sqrt{2}}{5}i\right) - \left(\frac{3-2\sqrt{2}i}{5}\right)\left(\frac{3+2\sqrt{2}i}{5}\right)$$

$$= \frac{4\cdot 2}{25} - \frac{9 + 4\cdot 2}{25} = -\frac{9}{25}.$$

Thus $\langle \sigma_z \rangle = -\frac{9}{25}$, $\langle \sigma_z^2 \rangle = \langle I \rangle = 1$, so

$$\Delta \sigma_z = \sqrt{1 - \left(-\tfrac{9}{25}\right)^2}$$

$$= \pm \frac{4\sqrt{34}}{25} \quad \text{(approx.)}$$

c) $\sigma_x = |-z\rangle\langle +z| + |+z\rangle\langle -z|$. Using same results from part b),

Evaluate $\langle \psi | \sigma_x | \psi \rangle$:

$$= \langle \psi | \Big[ |-z\rangle\langle +z| + |+z\rangle\langle -z| \Big] |\psi\rangle$$

$$= \langle \psi | \Big[ |-z\rangle\langle +z|\psi\rangle + |+z\rangle\langle -z|\psi\rangle \Big]$$

$$= \langle \psi | -z\rangle\langle +z|\psi\rangle + \langle \psi | +z\rangle\langle -z|\psi\rangle$$

$$= \left(\frac{3-2\sqrt{2}\,i}{5}\right)\left(-\frac{2\sqrt{2}\,i}{5}\right) + \left(\frac{2\sqrt{2}\,i}{5}\right)\left(\frac{3+2\sqrt{2}\,i}{5}\right)$$

$$= \frac{-6\sqrt{2}\,i - 4\cdot 2}{25} + \frac{6\sqrt{2}\,i - 4\cdot 2}{25}$$

$$= -\frac{4\cdot 4}{25} = -\frac{16}{25}.$$

Thus $\langle \sigma_x \rangle = -\frac{16}{25}$, $\langle \sigma_x^2 \rangle = \langle I \rangle = 1$

$$\Delta \sigma_x = \sqrt{1 - \left(-\tfrac{16}{25}\right)^2}$$

$$= \pm \frac{3\sqrt{41}}{25} \quad \text{(approx.)}$$

PHY 256 PS2

5) Path 1 : $P_1 = \left| \frac{1}{3} e^{i\phi_1} \right|^2$

Path 2 : $P_2 = \left| \frac{1}{3} e^{i\phi_2} \right|^2$

Path 3 : $P_3 = \left| \frac{1}{3} e^{i\phi_3} \right|^2$ [Bomb explodes]

Since Path 1, 2 indistinguishable, probability of detection becomes

$$P_{detect} = \left| \frac{1}{3} e^{i\phi_1} + \frac{1}{3} e^{i\phi_2} \right|^2 + \underbrace{\left| \frac{1}{3} e^{i\phi_3} \right|^2}_{P_3}$$

$$= \left[\frac{1}{3}\right]^2 \left(e^{i\phi_1} + e^{i\phi_2}\right)\left(e^{-i\phi_1} + e^{-i\phi_2}\right) + \frac{1}{9}$$

$$= \frac{1}{9}\left(1 + 1 + e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)}\right) + \frac{1}{9}$$

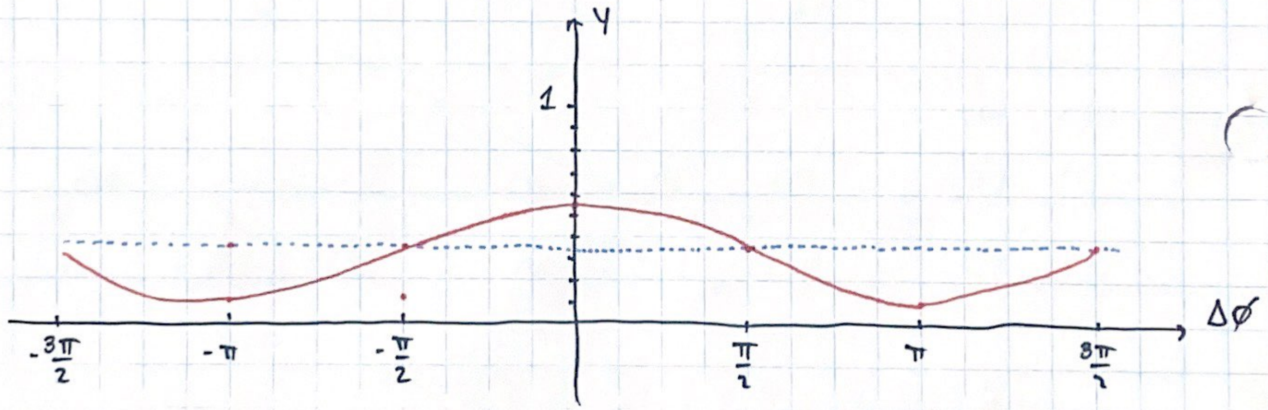
$$= \frac{1}{9}\left(2 + \cos(\phi_1-\phi_2) + i\sin(\phi_1-\phi_2) + \cos(\phi_1-\phi_2) - i\sin(\phi_1-\phi_1)\right) + \frac{1}{9}$$

$$= \frac{1}{9}\left(2 + 2\cos(\Delta\phi)\right) + \frac{1}{9}$$

$$= \frac{2}{9}\left(1 + \cos(\phi_1-\phi_2)\right) + \frac{1}{9} = \frac{2}{9}\cos(\Delta\phi) + \frac{3}{9}$$

Let $\Delta\phi = \phi_1 - \phi_2$ : Sketch $P_{detect}(\Delta\phi)$

$$y = \frac{2}{9}\cos(\Delta\phi) + \frac{3}{9}$$



PHY256 PS 2

6) Bases:

$|H\rangle, |V\rangle$

$|D\rangle, |A\rangle$ : $|D\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|V\rangle$ , $|A\rangle = \frac{1}{\sqrt{2}}|H\rangle - \frac{1}{\sqrt{2}}|V\rangle$

$|L\rangle, |R\rangle$ : $|L\rangle = \frac{1}{\sqrt{2}}|H\rangle - \frac{i}{\sqrt{2}}|V\rangle$ , $|R\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{i}{\sqrt{2}}|V\rangle$.

$$\underbrace{H\ H\ R\ V}_{\substack{H/V \\ ①}} \quad \underbrace{D\ V\ L\ L}_{\substack{D/A \\ ②}} \quad \underbrace{L\ H\ D\ R}_{\substack{R/L \\ ③}}$$

| | | | Bob Remeasures | chance of error (%) |
|---|---|---|---|---|
| ① | $|\langle H|H\rangle|^2 = 1$ | $|\langle V|H\rangle|^2 = 0$ | H | 0 |
| | $|\langle H|H\rangle|^2 = 1$ | $|\langle V|H\rangle|^2 = 0$ | H | 0 |
| | $|\langle H|R\rangle|^2 = \frac{1}{2}$ | $|\langle V|R\rangle|^2 = \frac{1}{2}$ | 50% R, L | 50% |
| | $|\langle H|V\rangle|^2 = 0$ | $|\langle V|V\rangle|^2 = 1$ | V | 0% |
| ② | $|\langle D|D\rangle|^2 = 1$ | $|\langle A|D\rangle|^2 = 0$ | D | 0% |
| | $|\langle D|V\rangle|^2 = \frac{1}{2}$ | $|\langle A|V\rangle|^2 = \frac{1}{2}$ | 50% H, V | 50% |
| | $|\langle D|L\rangle|^2 = \frac{1}{2}$ | $|\langle A|L\rangle|^2 = \frac{1}{2}$ | 50% R, L | 50% |
| | $|\langle D|L\rangle|^2 = \frac{1}{2}$ | $|\langle A|L\rangle|^2 = \frac{1}{2}$ | 50% R, L | 50% |
| ③ | $|\langle R|L\rangle|^2 = 0$ | $|\langle L|L\rangle|^2 = 1$ | L | 0% |
| | $|\langle R|H\rangle|^2 = \frac{1}{2}$ | $|\langle L|H\rangle|^2 = \frac{1}{2}$ | 50% H, V | 50% |
| | $|\langle R|D\rangle|^2 = \frac{1}{2}$ | $|\langle L|D\rangle|^2 = \frac{1}{2}$ | 50% D, A | 50% |
| | $|\langle R|R\rangle|^2 = 1$ | $|\langle L|R\rangle|^2 = 0$ | R | 0% |

In total, Bob has a $\frac{6 \cdot 0.5}{12} = 25\%$ chance of being in disagreement with Alice.