JPE395 Problem Set 3 March 31 2023

Problem 1

a) At time $t=0$, $e(0) = e_0$. At the instantaneous application of stress, the visco-elastic body has no instantaneous viscous response.

Under this assumption, we may neglect the viscosity term by taking $\eta \to \infty$ in the constitutive equation, therefore reducing it to $\dot{e} = \frac{\dot{\sigma}}{K}$.

Integrating, we find that $e_0 = \frac{\sigma_0}{K}$, or that the initial stress on the body is $\sigma(t=0) = K e_0$.

b) Now, consider when $t>0$. We may perform a variable separation to solve the constitutive equation for $\sigma(t)$:

$$\dot{e} = \frac{\dot{\sigma}}{K} + \frac{\sigma}{\eta}.$$

Since $e(t)$ is constant, $\dot{e} = 0$, and therefore

$$0 = \frac{\dot{\sigma}}{K} + \frac{\sigma}{\eta} \Rightarrow \frac{d\sigma}{dt} = -\frac{K}{\eta}\sigma.$$

$$\frac{d\sigma}{\sigma} = -\frac{K}{\eta}dt \Rightarrow \int_{\sigma_0}^{\sigma(t)} \frac{d\sigma'}{\sigma'} = -\frac{K}{\eta}\int_0^t dt'$$

Hence

$$\log|\sigma'|\Big|_{\sigma_0}^{\sigma(t)} = -\frac{K}{\eta}t$$

$$\log|\sigma(t)| - \log|\sigma_0| = -\frac{K}{\eta}t.$$

$$\Longrightarrow \quad \sigma(t) = \sigma_0 e^{-\frac{K}{\eta}t}$$

or $$\boxed{\sigma(t) = e_0 K e^{-\frac{K}{\eta}t}}$$

Problem 2

a) The restoring force on the mantle may be determined by comparing the forces in the two columns, such as by taking the difference $\Delta F = F_2 - F_1$.

On column 1 (centre column),

$$F_1 = F_{mantle} + F_{water}$$
$$= (D-d)\, g\, \rho_m\, \delta A + d g\, \rho_w\, \delta A$$

And on column 2,

$$F_2 = D g\, \rho_m\, \delta A. \qquad \text{(Just mantle force).}$$

Therefore

$$\Delta F = \cancel{D g\, \rho_m\, \delta A} - \cancel{D g\, \rho_m \delta A} + d g\, \rho_m\, \delta A - d g\, \rho_w\, \delta A$$
$$F_{res} = d g\, \delta A\, (\rho_m - \rho_w)$$

which is the restoring force on the mantle. Note that

$d = b(t)$, hence

$$\boxed{F_{res} = b(t)\, g\, \delta A\, (\rho_m - \delta_w)}$$

The net stress is then $\sigma(t) = \dfrac{F_{res}(t)}{\delta A}$, or

$$\boxed{\sigma(t) = b(t)\, g\, (\rho_m - \rho_w)}$$

where $\sigma(0) = d g (\rho_m - \rho_w)$.

b) To determine $b(t)$, we may apply the constitutive equation:

$$\dot{e} = \frac{\dot{\sigma}}{K} + \frac{\sigma}{\eta}.$$

Since the strein rate $\dot{e}$ is equivalent to the column lengthening rate, $-\frac{\dot{b}}{D}$, hence

$$-\frac{\dot{b}}{D} = \frac{\dot{b}\, g(\rho_m - \rho_w)}{K} + \frac{b\, g(\rho_m - \rho_w)}{\eta}.$$

Variable separation yields

$$-\dot{b}\left(\frac{1}{D} + \frac{g(\rho_m - \rho_w)}{K}\right) = b\,\frac{g(\rho_m - \rho_w)}{\eta}$$

$$\dot{b} = -\frac{g(\rho_m - \rho_w)}{\eta} \cdot \left(\frac{1}{D} + \frac{g(\rho_m - \rho_w)}{K}\right)^{-1} b$$

Which therefore yields

$$b(t) = b(t=0) \cdot \exp\left\{-\frac{g}{\eta}(\rho_m - \rho_w) \cdot \left(\frac{1}{D} + \frac{g(\rho_m - \rho_w)}{K}\right)^{-1} t\right\}.$$

Simplification of the term in the argument of the exponential, with noting that $b(t=0) = d$, we obtain the expression for $b(t)$:

$$\boxed{b(t) = d \exp\left\{\frac{-g(\rho_m - \rho_w) D}{\eta(K + gD(\rho_m - \rho_w))} t\right\}}$$

c) To determine t_{exp}, we wish to solve when

$$\exp\left\{ \frac{-g(\rho_m - \rho_w) D}{\eta(K + gD(\rho_m - \rho_w))} t \right\} = \exp\{-1\}$$

Hence

$$t_{exp} \cdot \frac{g(\rho_m - \rho_w) D}{\eta(K + g(\rho_m - \rho_w) D)} = 1,$$

which yields the exponential decay constant time

$$t_{exp} = \frac{\eta(K + g(\rho_m - \rho_w) D)}{g(\rho_m - \rho_w) D}$$

$$\boxed{t_{exp} = \frac{\eta K}{g(\rho_m - \rho_w) D} + \eta}$$

d) Upon rearranging the exponential decay term previously found, we find that

$$\eta = \frac{t_{exp} \cdot D K g(\rho_m - \rho_w)}{K + gD(\rho_m - \rho_w)},$$

With $t_{exp} \simeq 3'000$ yr $= 9{,}4608 \times 10^{10}$ s,

$\rho_m = 3'300 \ \frac{kg}{m^3}$, $\rho_w = 1'000 \ \frac{kg}{m^3}$, $g = 10 \ \frac{m}{s^2}$,

$D = 2'900'000$ m, and $K = 8 \times 10^{10}$ Pa, we find

$$\eta = \frac{(9.4608e10\ s)(2.9e6\ m)(8e10\ Pa)(10\ \frac{m}{s^2})(2300\ kg/m^3)}{(8e10\ Pa) + (2.9e6 m)(10\ \frac{m}{s^2})(2300\ kg/m^3)}$$

$$\boxed{\eta = 3.44 \times 10^{21}\ Pa\text{-}s}$$

e) The current rate of change is given by differentiation of $b(t)$,

$$\frac{db}{dt} = -\left(\frac{DKg(\rho_m - \rho_w)}{\eta(K + Dg(\rho_m - \rho_w))}\right) b(t) \Bigg|_{b(t)=30m}$$

$$\left|\frac{db}{dt}\right| = 30m \cdot \left(\frac{(2.9e6m)(8e10\ Pa)(10\frac{m}{s^2})(2300\ kg/m^3)}{(3.44e21\ Pa\text{-}s)\left[8e10\ Pa + (2.9e6m)(10\frac{m}{s^2})(2300\ \frac{kg}{m^3})\right]}\right)$$

$$= 3.17211 \times 10^{-10}\ \frac{m}{s}$$

Which, by converting to mm/yr, yields

$$\boxed{\dot{b} = 9.98\ mm/yr}$$

Compared with the Lecture literature value of 9 mm/yr, this value of lake-depth change is approximately equivalent. This is as expected.

Problem 3

a) If only radiogenic heat is being produced, the amount of heat produced in an infinitessimal time dt is

$$dq = H\, dm\, dt.$$

Integrated over the whole mass yields the total heat production,

$$Q = HM\, dt.$$

The heat flux Φ is given by heat/unit area · unit time, which implies that

$$\Phi = \frac{Q}{A\,dt} \implies \boxed{\Phi = \frac{HM}{A}}$$

b) This total heat flux may be represented by the relation

$$\Phi_{tot} = \frac{K(T-T_s)}{d} \cdot \left(\frac{Ra}{Ra_c}\right)^{1/3}$$

where K: thermal conductivity, d: mantle depth, T: internal temperature, T_s: surface temperature, Ra: Rayleigh number, and Ra_c: critical Rayleigh number for convection.

Noting that $Ra \equiv \frac{g\alpha d^3}{K\nu}\Delta T$, we find that

$$\frac{MH}{A} = \frac{K(T-T_s)}{d}\left(\frac{g\alpha d^3}{K\nu\, Ra_c}\right)^{1/3}(T-T_s)^{1/3}$$

with $\Delta T = T - T_s$.

Simple rearranging yields

$$\boxed{(T-T_s)^{4/3} = \frac{MH}{AK}\left(\frac{g\alpha}{\kappa\nu Ra_c}\right)^{1/3}}$$

c) With $H(t) = H_0 e^{-\lambda t}$, $\nu = \nu_0 e^{A_0/T}$, differentiation with respect to time yields

$$\dot{H} = -\lambda H \quad , \quad \dot{\nu} = -\frac{\nu A_0}{T^2}\frac{dT}{dt} .$$

Then, by chain/product rule,

$$\frac{d}{dt}(T-T_s)^{4/3} = \frac{4}{3}(T-T_s)^{1/3}\frac{dT}{dt} \qquad \text{and}$$

(Let $C_0 \equiv \frac{M}{AK}\left(\frac{g\alpha}{\kappa Ra_c}\right)^{1/3}$)

$$\frac{d}{dt}\left[C_0 H \nu^{1/3}\right] = C_0 \frac{dH}{dt}\nu^{1/3} + C_0 H \frac{1}{3}\nu^{-2/3}\frac{d\nu}{dt}$$

$$= -\lambda C_0 H \nu^{1/3} + C_0 H \frac{1}{3}\nu^{-2/3}\left(-\frac{\nu A_0}{T^2}\right)\frac{dT}{dt}$$

$$= C_0 H \nu^{1/3}\left[-\lambda - \frac{1}{3}\frac{A_0}{T^2}\frac{dT}{dt}\right].$$

Setting LHS = RHS, and noting that $C_0 H \nu^{1/3} = (T-T_s)^{4/3}$,

$$\frac{4}{3}\Delta T^{1/3}\dot{T} = -\Delta T^{4/3}\left(\lambda + \frac{A_0}{3T^2}\dot{T}\right)$$

$$\Rightarrow \dot{T}\left(1 + \frac{A_0}{4T^2}\Delta T\right) = -\frac{3\lambda}{4}\Delta T$$

$$\Longrightarrow \boxed{\frac{dT}{dt} = -\frac{3\lambda}{4}(T-T_s)\left[1 + \frac{A_0}{4T^2}(T-T_s)\right]^{-1}}$$

which is as required.

d) First, note that if $T_s = 300$ K and $T = 2300$ K, then $T - T_s = 2000$ K which is still in the order of magnitude of ~2000K. Therefore it is ok to neglect the $(T-T_s)$ term. This follows since

$T - T_s \ll A_0 = 80'000$ K.

The equation simplifies accordingly to

$$\frac{dT}{dt} = -\frac{3\lambda}{4}(T-T_s)\left[\frac{A_0}{4T^2}(T-T_s)\right]^{-1}$$

$$= -\frac{3\lambda T^2}{A_0},$$

Hence $\boxed{\dot{T} = -\frac{3\lambda T^2}{A_0}}$.

e) The cooling rate, given that $T = \sim 2'300$ K,

$\lambda = 1.42 \times 10^{-17}\ s^{-1}$, and $A_0 = 8 \times 10^4$ K, is therefore just

$$\dot{T}\Big|_{T=2'300\ K} = -\frac{3 \cdot (1.42 \times 10^{-17}\ s^{-1})}{8 \times 10^4\ K}(2'300\ K)^2$$

$$= -5.634 \times 10^{-15}\ \frac{K}{s}$$

or $\boxed{\dot{T} = 1.777 \times 10^{-7}\ \frac{K}{yr}}$ is the cooling rate.

f) We have the ODE $\frac{dT}{dt} = -\frac{3\lambda}{A_0} T^2$,

which, by variable separation, is

$$\int \frac{dT}{T^2} = -\frac{3\lambda}{A_0} \int dt$$

$$-\frac{1}{T} = -\frac{3\lambda}{A_0} t + T_0$$

$$\Longrightarrow \boxed{T(t) = \frac{A_0}{3\lambda t} + T_0'}$$

where T_0' is an initial condition.

To find T_0', we may invoke the condition that at $t = 4.54$ byr (current time), if $A_0 = 8 \times 10^4$ K, $\lambda = 1.42 \times 10^{-17}\ s^{-1}$, the current average mantle temperature is 2'300 K.

$$2'300\ K = \frac{A_0}{3\lambda(4.54\ byr)} + T_0'$$

$$\Longrightarrow T_0' = 2'300\ K - \frac{A_0}{3\lambda \cdot 4.54\ byr}$$

4.54 byr $\approx 1.4327 \times 10^{17}$ s, hence

$$T_0' = 2300\ K - \frac{80'000}{3(1.42 \times 10^{-17}\ s^{-1})(1.4327 \times 10^{17}\ s)}$$

$$\approx -10'807.5211\ K.$$

When $t \approx 0$ (earth formation), $T \to \infty$, which is not possible. This is why this may not be a good estimate for earths initial mantle temperature.