MAT224 Linear Algebra II
Term Test 2

Instructions:

Please read the Term Test 2 Information document for details on submission policies, permitted resources, how to ask a question, test announcements, and more. You were expected to read them in detail in advance of the test.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit your solutions using only this template PDF**. Your submission should be a single PDF with your full written solutions for each question. If your solution is not written using this template PDF (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided.
3. **Show your work and justify your steps** on every question unless otherwise indicated. Put your final answer in the box provided, if necessary.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for the test.

Academic Integrity Statement:

Full Name: Jace Alloway

Student number: 1006940802

I confirm that:

- I have not communicated with any person about the test other than a MAT224 teaching team member.
- I have not used any resources other than those that are listed as permitted in the Term Test 2 Information document at any point during the test.
- I have not participated in or enabled any MAT224 group chat during the test.
- I have not viewed the answers, solutions, term work, or notes of anyone.
- I have read and followed all of the rules described in the Term Test 2 Information document.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated any of them while writing this assessment.

By signing this document, I agree that all of the statements above are true.

Signature: Jace Alloway

1. Decide if the following statement is True or False:

Let V and W be finite dimensional vector spaces, and let $T \in \mathfrak{L}(V, W)$. If $\dim \operatorname{im} T = \min\{\dim V, \dim W\}$, then T is either injective or surjective.

Note: $\min\{valueA, valueB\} =$ the minimum of the two values.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

I want to prove that if $\dim \operatorname{im} T = \min\{\dim V, \dim W\}$, then T is either injective or surjective.

Proof. I will break this proof into two cases: when $\dim V < \dim W$, $\dim V = \dim W$, and when $\dim V > \dim W$.

Suppose $\dim V < \dim W$. Then $\dim \operatorname{im} T = \min\{\dim V, \dim W\} = \dim V$. By **Corollary 2.4.5**, since V and W are finite-dimensional vector spaces and $\dim V < \dim W$, then T must be injective.

Now suppose $\dim V > \dim W$. Then $\dim \operatorname{im} T = \min\{\dim V, \dim W\} = \dim W$. By **Corollary 2.4.9**, since V and W are finite-dimensional vector spaces and $\dim V > \dim W$, then T must be surjective.

Lastly, supose $\dim V = \dim W$. Then $\dim \operatorname{im} T = \dim V = \dim W$, so by **Proposition 2.4.10**, T must be injective and surjective.

In all 3 cases, T is either injective, surjective, or both. Therefore this statement is true, which is what I needed to prove.

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2. Decide if the following statement is True or False:

Let U, V, and W be finite dimensional vector spaces, and let $S \in \mathfrak{L}(U, V)$, and $T \in \mathfrak{L}(V, W)$. If TS is surjective, then S is surjective.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

○ True
● False

This statement is false, and I will show it with a counterexample.

Let $\dim U = 3$, $\dim V = 4$, and $\dim W = 2$. We then have the mappings

$$U \xrightarrow{T \circ S} W \qquad \text{which is} \qquad U \xrightarrow{\text{S}} V \xrightarrow{\text{T}} W.$$

Since $\dim W = 2 < \dim U = 3$, then $T \circ S$ must be surjective by **Corollary 2.4.5**.
However, since $\dim V = 4 > 3 = \dim U$, then by **Corollary 2.4.8**, there is no surjective mapping $S : U \longrightarrow V$.

Therefore this statement is false.

3. Decide if the following statement is True or False:

Let V and W be n-dimensional vector spaces, and let $T \in \mathfrak{L}(V, W)$ be an isomorphism. Then there exist bases α for V, and β for W such that $[T]_\alpha^\beta = I_n$, where I_n denotes the $n \times n$ identity matrix.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

This statement is true. I want to prove that for any n-dimensional vector spaces V and W, and an isomorphism $T \in \mathfrak{L}(V, W)$, there exists bases α and β for V and W, respectively, such that the matrix $[T]_\alpha^\beta = I_{n\times n}$, the $n \times n$ identity matrix.

Proof. Initially, V and W may not be in the bases α and β, respectively. Let $\mathcal{E}$ be the initial basis for V and let $\mathcal{F}$ be the initial basis for W. Our matrix representation of the transformation $T : V \longrightarrow W$ would then be $[T]_\mathcal{E}^\mathcal{F}$.
Since V and W are both n-dimensional, there exists change of basis transformations $P : V \longrightarrow V$ and $Q : W \longrightarrow W$ such that the matrix for P is $[P]_\mathcal{E}^\alpha$ and the matrix for Q is $[Q]_\mathcal{F}^\beta$. Since P and Q are both change of basis transformations, then they are both invertible.

Choose the basis α for V to be $\alpha = \{a_1, a_2, \ldots, a_n\}$. We want the identity transformation from $V \longrightarrow W$, and so we have

$$T(a_1) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, T(a_2) = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, T(a_n) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Then our basis β for W would be $\beta = \{Ta_1, Ta_2, \ldots, Ta_n\}$, which is also n-dimensional. The matrix $[T]_\alpha^\beta$ would be the $I_{n\times n}$ identity matrix.

By **Theorem 2.7.5**, we then have

$$I_{n\times n} = [T]_\alpha^\beta = [Q]_\mathcal{F}^\beta \cdot [T]_\mathcal{E}^\mathcal{F} \cdot [P^{-1}]_\alpha^\mathcal{E}.$$

Therefore there are bases α and β for finite-dimensional vector spaces V and W, respectively, such that $[T]_\alpha^\beta$ is the $n \times n$ identity matrix, which is what I needed to prove.

■

4. Decide if the following statement is True or False:

Let V be a finite dimensional vector space, and let $I \in \mathfrak{L}(V)$ be the identity transformation. Then it is possible to find bases α and β for V such that $\det[I]_\alpha^\beta = 0$.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

○ True
● False

This statement is false. I want to prove that it is impossible to find two bases α and β for V such that for the change of basis transformation $[I]_\alpha^\beta$, $\det[I]_\alpha^\beta = 0$.

Proof.
Suppose the finite-dimensional vector space V has dimension n. Then, bases α and β for V must both have n linearly independent elements.
It then follows that $[I]_\alpha^\beta$ would be an $n \times n$ identity matrix. By **Theorem 3.2.14**, the matrix $[I]_\alpha^\beta$ is invertible if and only if $\det[I]_\alpha^\beta \neq 0$.

We have reached a contradiction. If $\det[I]_\alpha^\beta = 0$, then the change of basis transformation I from bases α to β would not be invertible. This is impossible since α and β must both have n linearly independent elements, thus conserving the dimensionality of the vector space V.
$\det[I]_\alpha^\beta = 0$ violates this, and thus it must be that it is impossible to find bases α and β for V such that $\det[I]_\alpha^\beta = 0$.

Therefore this statement is false, which is what I needed to prove.

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5. Let V be a finite dimensional vector space, and let $R, T \in \mathfrak{L}(V)$. Suppose that $S \in \mathfrak{L}(V)$ is invertible, and that $T = SRS^{-1}$.

Prove that if λ is an eigenvalue of R, then λ is an eigenvalue of T. [4 marks]

I want to prove that if λ is an eigenvalue of the map $R \in \mathfrak{L}(V)$, then λ is also an eigenvalue of the map $T \in \mathfrak{L}(V)$ where $T = SRS^{-1}$.

Proof.
Suppose $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of the transformation R with eigenvalue λ. Then $R\mathbf{x} = \lambda\mathbf{x}$ by definition.
For $T = SRS^{-1}$, we have $TS = SRS^{-1}S \implies TS = SR$. Then

$$TS\mathbf{x} = SR\mathbf{x} = S(\lambda\mathbf{x}) = \lambda S\mathbf{x}.$$

Therefore $TS\mathbf{x} = \lambda S\mathbf{x}$, and thus the mappings R and T have the same eigenvalue λ, which is what I needed to prove.

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6. Let V be a 4-dimensional vector space, and let $R, T \in \mathfrak{L}(V)$. Suppose that the eigenvalues of R and T are -1, 0, 1, and 2.

Prove that there exists an invertible $S \in \mathfrak{L}(V)$ such that $T = SRS^{-1}$. [4 marks]

I want prove that there exists an invertible $S \in \mathfrak{L}(V)$ such that for operators $T, R \in \mathfrak{L}(V)$, $T = SRS^{-1}$

Proof.
Let α be a basis for V. Suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are the eigenvectors for the linear map $T \in \mathfrak{L}(V)$ each with distinct eigenvalues
$\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 2$, respectively. Then

$$T\mathbf{x}_1 = -\mathbf{x}_1,\ T\mathbf{x}_2 = \mathbf{0},\ T\mathbf{x}_3 = \mathbf{x}_3, \text{ and } T\mathbf{x}_4 = 2\mathbf{x}_4.$$

By **Corollary 4.2.8**, since T has $4 = \dim V$ distinct real eigenvalues and eigenvectors with algebraic multiplicity 1, then T is diagonalizable. Suppose R is the linear map of T with respect to the eigenbasis of T. Let me call this eigenbasis $\mathcal{E}$. Therefore T is diagonalizable, and thus by **Proposition 4.2.2**, there exists a diagonal matrix $[R]_{\mathcal{E}}$ and an invertible matrix $[S]_{\alpha}^{\mathcal{E}}$ such that $[T]_{\alpha} = [S]_{\mathcal{E}}^{\alpha}[R]_{\mathcal{E}}[S^{-1}]_{\alpha}^{\mathcal{E}}$. It is still true that T and R have the same eigenvalues, as proven in **Question 5)**.
Now let $S \in \mathfrak{L}(V)$ be the invertible change of basis operator from bases α to $\mathcal{E}$ defined by the invertible matrix $[S]_{\mathcal{E}}^{\alpha}$. We then have that $T = SRS^{-1}$.

Therefore there exists an invertible $S \in \mathfrak{L}(V)$ such that for $T, R \in \mathfrak{L}(V)$, $T = SRS^{-1}$, which is what I needed to prove.

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7. Let V be a finite dimensional vector space, and let $T \in \mathfrak{L}(V)$. Suppose that the only eigenvalues of T are 0,1, and 2.

7(a) Prove that $\dim E_1(T) + \dim E_2(T) \leq \dim \operatorname{im} T$. [4 marks]

I want to prove that for a linear map $T \in \mathfrak{L}(V)$ with eigenvalues $\lambda_0 = 0$, $\lambda_1 = 1$, and $\lambda_2 = 2$, that $\dim E_1(T) + \dim E_2(T) \leq \dim \operatorname{im} T$.

Proof.
Let $\dim V = n$ By the Fundamental Theorem of Algebra, the sum of the algebraic multiplicities of each eigenvalue in the Characteristic Polynomial of T must be less than or equal to the dimension of V:

$$m_0 + m_1 + m_2 \leq \dim V = n.$$

Furthermore, by **Proposition 4.2.6**, we have that the dimension each eigenspace is less than or equal to the multiplicity of that respective eigenvalue:

$$1 \leq \dim E_i(T) \leq m_i \quad \text{for} \quad i = 0, 1, 2.$$

It then follows that

$$1 + 1 + 1 = 3 \leq \dim E_0(T) + \dim E_1(T) + \dim E_2(T) \leq m_0 + m_1 + m_2 \leq n = \dim V.$$

This implies that $\dim E_0 + \dim E_1 + \dim E_2 \leq \dim V$.
The eigenspace $E_0(T)$ is the Kernel of T, since any vector $\mathbf{x} \in E_0(T)$ is taken to $\mathbf{0}$ when T is applied: $T\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. Therefore $E_0(T) = \ker T$.
By the dimension theorem, we have that $\dim V = \dim \operatorname{im} T + \dim \ker T$. Our inequality is then

$$\dim E_0(T) + \dim E_1(T) + \dim E_2(T) = \dim \ker T + \dim E_1(T) + \dim E_2(T) \leq \dim V = \dim \operatorname{im} T + \dim \ker T.$$

Therefore

$$\dim \ker T + \dim E_1(T) + \dim E_2(T) \leq \dim \operatorname{im} T + \dim \ker T \implies \dim E_1(T) + \dim E_2(T) \leq \dim \operatorname{im} T.$$

Therefore $\dim E_1(T) + \dim E_2(T) \leq \dim \operatorname{im} T$, which is what I needed to prove.

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7. Let V be a finite dimensional vector space, and let $T \in \mathfrak{L}(V)$. Suppose that the only eigenvalues of T are 0,1, and 2.

7(b) Prove that if $\dim E_1(T) + \dim E_2(T) = \dim \operatorname{im} T$, then T is diagonalizable. [4 marks]

I want to prove that if $\dim E_1(T) + \dim E_2(T) = \dim \operatorname{im} T$, then T is diagonalizable.

Proof.
Assume $\dim E_1(T) + \dim E_2(T) = \dim \operatorname{im} T$. As previously stated in **Question 7a)**, the eigenspace $E_0(T)$ is the Kernel of T, since every vector $\mathbf{x} \in E_0(T)$ is brought to $\mathbf{0}$ when T is induced: $T\mathbf{x} = 0\mathbf{x} = \mathbf{0}$.

Because each eigenvalue is distinct, then each eigenvector associated with each of the 3 eigenvalues is also distinct, which implies that $E_0(T) \cap E_1(T) = E_0(T) \cap E_2(T) = E_1(T) \cap E_2(T) = \{\mathbf{0}\}$. By the dimension theorem, we have that $\dim V = \dim \operatorname{im} T + \dim \ker T = \dim E_1(T) + \dim E_2(T) + \dim E_0(T)$.

By **Corollary 4.2.9** (or alternatively **Quiri Li's Diagonalization Slides, Definition 7**), we have that

$$V = E_0(T) \oplus E_1(T) \oplus E_2(T),$$

and therefore T is diagonalizable, which is what I needed to prove.

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