PHY483 Problem Set 4

Saturday, November 30, 2024 $\longrightarrow$ Monday, December 2, 2024

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Problem 1

(a) We begin with the line element

$$ds^2 = 2dx^+dx^- + W(x^+, x^2, x^3)(dx^+)^2 - (dx^2)^2 - (dx^3)^2 \tag{1.1}$$

where $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ is a coordinate change. To determine the metric, we may equivalently write

$$ds^2 = dx^+dx^- + dx^-dx^+ + W(x^+, x^2, x^3)(dx^+)^2 - (dx^2)^2 - (dx^3)^2 \tag{1.2}$$

by commutativity, hence working in a basis $\{x^+,\, x^-,\, x^2,\, x^3\}$, one finds

$$g_{\mu\nu} = \begin{pmatrix} W(x^+, x^2, x^3) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.3}$$

The inverse of the metric can be found by applying a row-reduction algorithm.

$$\begin{pmatrix} W & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(init)} \tag{1.4}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ W & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -W & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (r_2 \to r_2 \cdot W;\; r_1 \to r_1 - r_2) \tag{1.5}$$

$$\begin{pmatrix} W & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & W & 0 & 0 \\ 1 & -W & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (r_1 \leftrightarrow r_2;\; (r_3, r_4) \to (-r_3, -r_4)) \tag{1.6}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -W & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (r_1 \to r_1 \cdot W^{-1}) \tag{1.7}$$

hence

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -W(x^+, x^2, x^3) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.8}$$

(b) Let $x^2 \equiv y$ and $x^3 \equiv z$ for clarity. We use the Python code to compute the Christoffel symbols, the Riemann tensor, and the Ricci tensor. The only non-zero components are given as

$$\Gamma^1_{00} = \frac{1}{2}\frac{\partial W}{\partial x^+} \tag{1.9}$$

$$\Gamma^1_{02} = \Gamma^1_{20} = \frac{1}{2}\frac{\partial W}{\partial y} \tag{1.10}$$

$$\Gamma^1_{03} = \Gamma^1_{30} = \frac{1}{2}\frac{\partial W}{\partial z} \tag{1.11}$$

$$\Gamma^2_{00} = \frac{1}{2}\frac{\partial W}{\partial y} \tag{1.12}$$

$$\Gamma^3_{00} = \frac{1}{2}\frac{\partial W}{\partial z} \tag{1.13}$$

where W was defined as a Sympy symbol Function class. This makes sense, because the only surviving terms coming out of the metric are those consisting of W, as all the other 1's vanish upon differentiation. In the same way, the Riemann tensor was computed. The nonzero components were given as

$$R^1_{220} = -R^1_{202} = \frac{1}{2}\frac{\partial^2 W}{\partial y^2} \tag{1.14}$$

$$R^1_{230} = -R^1_{203} = R^1_{320} = -R^1_{302} = \frac{1}{2}\frac{\partial^2 W}{\partial y \partial z} \tag{1.15}$$

$$R^1_{330} = -R^1_{303} = \frac{1}{2}\frac{\partial^2 W}{\partial z^2} \tag{1.16}$$

$$R^2_{020} = -R^2_{002} = \frac{1}{2}\frac{\partial^2 W}{\partial y^2} \tag{1.17}$$

$$R^2_{030} = -R^2_{003} = \frac{1}{2}\frac{\partial^2 W}{\partial y \partial z} \tag{1.18}$$

$$R^3_{030} = -R^3_{003} = \frac{1}{2}\frac{\partial^2 W}{\partial z^2} \tag{1.19}$$

$$R^3_{020} = -R^3_{002} = \frac{1}{2}\frac{\partial^2 W}{\partial y \partial z} \tag{1.20}$$

which is again consistent with the above statement, including any additional symmetries of the Riemann. Upon contracting the indices, the Ricci tensor is

$$R_{00} = -\frac{1}{2}\left[\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right] \tag{1.21}$$

which is equivalently the Laplacian operator in y and z. This was also outputted from the Python code. From this and from the contravariant metric $g^{\mu\nu}$ as in (1.8), the only nonzero component is R_{00} contracted by $g^{00} = 0$, so the Ricci scalar is zero. I will not bother checking components by hand, I know my code works properly. Furthermore, the Laplacian is consistent with the wave behaviour propagating in the x^+ direction.

(c) In the vacuum Einstein equations with $\Lambda = 0$, one has

$$R_{00} - \frac{1}{2}Rg_{00} + \Lambda g_{00} = \kappa T_{00} \tag{1.22}$$

$$\implies R_{00} = 0 \tag{1.23}$$

$$\implies \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0. \tag{1.24}$$

If $W(x^+, y, z) = \omega_{yy}(x^+)y^2 + 2\omega_{yz}(x^+)yz + \omega_{zz}(x^+)z^2$, one finds

$$0 = \frac{\partial^2}{\partial y^2}[\omega_{yy}(x^+)y^2 + 2\omega_{yz}(x^+)yz + \omega_{zz}(x^+)z^2] + \frac{\partial^2}{\partial z^2}[\omega_{yy}(x^+)y^2 + 2\omega_{yz}(x^+)yz + \omega_{zz}(x^+)z^2] \tag{1.25}$$

$$= 2\omega_{yy}(x^+) + 2\omega_{zz}(x^+) \tag{1.26}$$

$$\implies \omega_{yy}(x^+) = -\omega_{zz}(x^+) \tag{1.27}$$

hence $\omega_{MN}(x^+)$ satisfies a transverse traceless condition.

Problem 2

Section [20.2] from lecture notes. Consider the Schwarzchild geodesics of a photon, where $f(r) = 1 - \frac{2\mu}{r}$. The conservation of angular momentum yields a first integral in φ, so we call L the conserved quantity. We can choose to work in the equitorial plane $\theta = \frac{\pi}{2}$ so the θ equation is satisfied by rotational symmetry. The temporal and radial equations reproduce

$$\dot{r}^2 + \frac{L^2}{r^2} f(r) = E^2 \tag{2.1}$$

$$f(r)\dot{t} = E \tag{2.2}$$

where the tangent vector norm condition was used to obtain the energy-momentum relation (2.1). (2.2) was obtained by the first integral in t. The $(t - r)$ shape is a product of the chain rule,

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{dr}{dt}\frac{dt}{d\lambda}\right)^2 \tag{2.3}$$

$$= \left(\frac{dr}{dt}\right)^2 E^2[f(r)]^{-2} \tag{2.4}$$

where I used (2.2) in (2.4). Substituting the expression for $\dot{r}^2$ into (2.1), one finds

$$E^2 = \left(\frac{dr}{dt}\right)^2 E^2[f(r)]^{-2} + \frac{L^2}{r^2} f(r) \tag{2.5}$$

$$\implies [f(r)]^{-1} = \left(\frac{dr}{dt}\right)^2 [f(r)]^{-3} + \frac{E^2/L^2}{r^2} \tag{2.6}$$

At the distance of closest approach, called r_0, one has a constant radius (finding the minimum of $r(t)$):

$$\left.\frac{dr}{dt}\right|_{r_0} = 0 \tag{2.7}$$

At r_0, then, we find

$$[f(r_0)]^{-1} = \frac{b^2}{r_0^2} \tag{2.8}$$

where $b = \frac{E}{L}$ is the impact parameter. We proceed by solving for b:

$$b^2 = \frac{r_0^2}{(1 - 2\mu/r_0)} \tag{2.9}$$

$$= r_0^2 \frac{r_0}{(r_0 - 2\mu)} \tag{2.10}$$

$$\implies b = r_0 \left(\frac{r_0}{r_0 - 2\mu}\right)^{1/2} \tag{2.11}$$

$$= r_0 \left(\frac{r_0}{r_0 - 2G_N M/c^2}\right)^{1/2} \tag{2.12}$$

where the last line follows from re-dimensionalization of μ. Taking $M = 2 \times 10^3 0$ kg, $r_0 = 7 \times 10^8$ m, G_N and c in their respective SI units, we can compute the apparent size of the sun using (2.12):

$$r_{\text{app}} = b - r_0$$

$$
\begin{aligned}
&= r_0 \left[\left(\frac{r_0}{r_0 - 2G_N M/c^2} \right)^{1/2} - 1 \right] \\
&= (7\text{e}5) \left[\left(\frac{7\text{e}5}{7\text{e}5 - 2(6.6743\text{e} - 11)(2\text{e}30)/(3\text{e}8)^2} \right)^{1/2} - 1 \right] \\
&\approx 2\,966.36\,\text{m}
\end{aligned}
$$

which is around 3km larger than the coordinate radius.

Problem 3

The Kerr metric in Boyd-Lindquist coordinates is given by

$$ds^2 = \left(1 - \frac{2\mu r}{\rho}\right) dt^2 + \frac{4\mu a r \sin^2\theta}{\rho^2} dt d\varphi - \frac{\rho^2}{\Delta} dr^2$$
$$- \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta d\varphi^2 \tag{3.1}$$

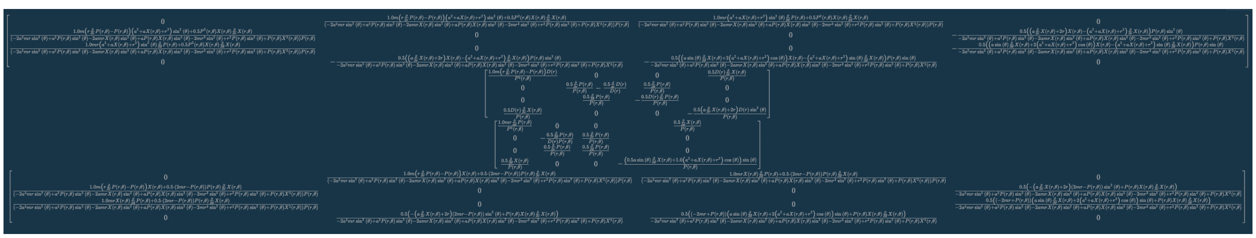
where

$$\rho^2 = [\rho(r,\theta)]^2 = r^2 + a^2\cos^2\theta \tag{3.2}$$
$$\Delta = \Delta(r) = r^2 - 2mur + a^2 \tag{3.3}$$

are coordinate changes. Since there are multiple nested functions within the metric, I tried my best to reduce the number of variables by defining variable functions in my Python code

$$P(r,\theta) \equiv [\rho(r,\theta)]^2 \tag{3.4}$$
$$D(r) \equiv \Delta(r) \tag{3.5}$$
$$X(r,\theta) = \frac{2\mu a \sin^2\theta}{P(r,\theta)} \tag{3.6}$$

All Christoffel symbols were computed using these functions, however we will only need the $\Gamma^r_{\mu\nu}$ connection, as we will soon find. In addition, the entries become indefinitely messy (no, I am not zooming in to show you):



The nonzero components were given as

```
(0, 0, 1)   (0, 0, 2)   (0, 1, 0)   (0, 1, 3)
(0, 2, 0)   (0, 2, 3)   (0, 3, 1)   (0, 3, 2)

(1, 0, 0)   (1, 0, 3)   (1, 1, 1)   (1, 1, 2)
(1, 2, 1)   (1, 2, 2)   (1, 3, 0)   (1, 3, 3)

(2, 0, 0)   (2, 0, 3)   (2, 1, 1)   (2, 1, 2)
(2, 2, 1)   (2, 2, 2)   (2, 3, 0)   (2, 3, 3)

(3, 0, 1)   (3, 0, 2)   (3, 1, 0)   (3, 1, 3)
(3, 2, 0)   (3, 2, 3)   (3, 3, 1)   (3, 3, 2)
```

Consider the geodesic equations for a mass in a circular orbit moving in the equitorial plane. Using the non-zero entries given by the Python code, including the symmetries of the Christoffel entries, we have

$$0 = \ddot{x}^0 + 2\Gamma^0_{01}\dot{x}^0\dot{x}^1 + 2\Gamma^0_{02}\dot{x}^0\dot{x}^2 + 2\Gamma^0_{13}\dot{x}^1\dot{x}^3 + 2\Gamma^0_{23}\dot{x}^2\dot{x}^3 \tag{3.7}$$
$$0 = \ddot{x}^1 + \Gamma^1_{00}(\dot{x}^0)^2 + 2\Gamma^1_{03}\dot{x}^0\dot{x}^3 + \Gamma^1_{11}(\dot{x}^1)^2 + 2\Gamma^1_{12}\dot{x}^1\dot{x}^2 + \Gamma^1_{22}(\dot{x}^2)^2 + \Gamma^1_{33}(\dot{x}^3)^2 \tag{3.8}$$
$$0 = \ddot{x}^2 + \Gamma^2_{00}(\dot{x}^0)^2 + 2\Gamma^2_{03}\dot{x}^0\dot{x}^3 + \Gamma^2_{11}(\dot{x}^1)^2 + 2\Gamma^2_{12}\dot{x}^1\dot{x}^2 + \Gamma^2_{22}(\dot{x}^2)^2 + \Gamma^2_{33}(\dot{x}^3)^2 \tag{3.9}$$
$$0 = \ddot{x}^3 + 2\Gamma^3_{01}\dot{x}^0\dot{x}^1 + 2\Gamma^3_{02}\dot{x}^0\dot{x}^2 + 2\Gamma^3_{13}\dot{x}^1\dot{x}^3 + 2\Gamma^3_{23}\dot{x}^2\dot{x}^3 \tag{3.10}$$

In the equitorial plane, $\theta = x^2 =$ constant. Further, in a circular orbit, $r = x^1 =$ constant as well. This reduces the equations significantly, since $\dot{x}^1 = \dot{x}^2 = 0$. Left with equations in t and φ, we have

$$0 = \ddot{x}^0 \tag{3.11}$$
$$0 = \Gamma^1_{00}(\dot{x}^0)^2 + 2\Gamma^1_{03}\dot{x}^0\dot{x}^3 + \Gamma^1_{33}(\dot{x}^3)^2 \tag{3.12}$$
$$0 = \Gamma^2_{00}(\dot{x}^0)^2 + 2\Gamma^2_{03}\dot{x}^0\dot{x}^3 + \Gamma^2_{33}(\dot{x}^3)^2 \tag{3.13}$$
$$0 = \ddot{x}^3 \tag{3.14}$$

(3.11) and (3.14) imply $\dot{t}$ and $\dot{\varphi}$ are constants and are nonzero. Further, we find that the equation in θ is automatically satisfied by the equation in r. In the metric (3.1), note that

$$g_{11} = \frac{1}{\Delta(r)}g_{22} \implies g^{11} = \Delta(r)g^{22} \tag{3.15}$$

which implies, from the definition of the connection $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(g_{\beta\nu;\,\mu} + g_{\beta\mu;\,\nu} - g_{\mu\nu;\,\beta}\right)$, that

$$\Gamma^2_{\mu\nu} = \frac{1}{\Delta(r)}\Gamma^1_{\mu\nu}. \tag{3.16}$$

or that the satisfaction of the radial equation implies satisfaction of the θ equation. From (3.12), we find

$$\Gamma^r_{tt}\dot{t}^2 + 2\Gamma^r_{t\varphi}\dot{t}\dot{\varphi} + \Gamma^r_{\varphi\varphi}\dot{\varphi}^2 = 0 \tag{3.17}$$

Using the Christoffel symbols found using Python, and our re-defined parameters in (3.4) - (3.6), the $\Gamma^r_{\mu\nu}$ connection components are given as

$$\begin{pmatrix} \frac{1.0m\left(r\frac{\partial}{\partial r}P(r,\theta) - P(r,\theta)\right)D(r)}{P^3(r,\theta)} & 0 & 0 & \frac{0.5D(r)\frac{\partial}{\partial r}X(r,\theta)}{P(r,\theta)} \\ 0 & \frac{0.5\frac{\partial}{\partial r}P(r,\theta)}{P(r,\theta)} - \frac{0.5\frac{d}{dr}D(r)}{D(r)} & \frac{0.5\frac{\partial}{\partial\theta}P(r,\theta)}{P(r,\theta)} & 0 \\ 0 & \frac{0.5\frac{\partial}{\partial\theta}P(r,\theta)}{P(r,\theta)} & -\frac{0.5D(r)\frac{\partial}{\partial r}P(r,\theta)}{P(r,\theta)} & 0 \\ \frac{0.5D(r)\frac{\partial}{\partial r}X(r,\theta)}{P(r,\theta)} & 0 & 0 & -\frac{0.5\left(a\frac{\partial}{\partial r}X(r,\theta) + 2r\right)D(r)\sin^2(\theta)}{P(r,\theta)} \end{pmatrix} \tag{3.18}$$

from which we can obtain

$$\Gamma^r_{tt} = \frac{m\Delta}{P^3}\left(r\frac{\partial P}{\partial r} - P\right)\bigg|_{\theta=\pi/2} \tag{3.19}$$
$$= \frac{m\Delta}{r^6}[2r^2 - r^2] \tag{3.20}$$
$$= \frac{m\Delta}{r^4} \tag{3.21}$$

$$\Gamma^r_{t\varphi} = \frac{\Delta}{2P}\frac{\partial X}{\partial r}\bigg|_{\theta=\pi/2} \tag{3.22}$$

$$= \frac{\Delta}{2r^2}\left(-\frac{2\mu a}{r^2}\right) \tag{3.23}$$

$$= -\frac{ma\Delta}{r^4} \tag{3.24}$$

$$\Gamma^r_{\varphi\varphi} = -\frac{\Delta\sin^2\theta}{2P}\left(a\frac{\partial X}{\partial r} + 2r\right)\bigg|_{\theta=\pi/2} \tag{3.25}$$

$$= -\frac{\Delta}{2r^2}\left(-\frac{2\mu a^2}{r^2} + 2r\right) \tag{3.26}$$

$$= \frac{\mu a^2\Delta}{r^4} - \frac{\Delta}{r} \tag{3.27}$$

In (3.17), this becomes

$$0 = \frac{\mu\Delta}{r^4}\dot{t}^2 - \frac{2\mu a\Delta}{r^4}\dot{t}\dot{\varphi} + \Delta\left(\frac{\mu a^2}{r^4} - \frac{1}{r}\right)\dot{\varphi}^2 \tag{3.28}$$

$$\implies 0 = \frac{\mu}{r^4}\dot{t}^2 - \frac{2\mu a}{r^4}\dot{t}\dot{\varphi} + \frac{\mu a^2 - r^3}{r^4}\dot{\varphi}^2 \tag{3.29}$$

$$\implies 0 = \mu\dot{t}^2 - 2\mu a\,\dot{t}\dot{\varphi} + (\mu a^2 - r^3)\dot{\varphi}^2 \tag{3.30}$$

Dividing (3.30) by $\dot{t}\dot{\varphi}$, and letting $\Omega = \dfrac{\dot{\varphi}}{\dot{t}}$, we find

$$0 = \mu\frac{1}{\Omega} - 2\mu a + (\mu a^2 - r^3)\Omega \tag{3.31}$$

$$\implies 0 = \mu - 2\mu a\Omega + (\mu a^2 - r^3)\Omega^2 \tag{3.32}$$

from which we can apply the quadratic formula to solve for Ω. One finds

$$\Omega = \frac{2\mu a \pm \sqrt{4\mu^2a^2 - 4\mu(\mu a^2 - r^3)}}{2(\mu a - r^3)} \tag{3.32}$$

$$= \frac{2\mu a \pm \sqrt{4\mu^2a^2 - 4\mu^2a^2 + 4\mu r^3}}{2(\mu a^2 - r^3)} \tag{3.33}$$

$$= \frac{\mu a \pm \sqrt{\mu r^3}}{(\mu a^2 - r^3)} \tag{3.34}$$

$$= \frac{\mu a \pm \sqrt{\mu r^3}}{(\mu a^2 - r^3)} \cdot \frac{\mu a \mp \sqrt{\mu r^3}}{\mu a \mp \sqrt{\mu r^3}} \tag{3.35}$$

$$= \frac{\mu^2a^2 - \mu r^3}{(\mu a^2 - r^3)(\mu a \mp \sqrt{\mu r^3})} \tag{3.36}$$

$$= \frac{\mu}{(\mu a \pm \sqrt{\mu r^3})} \tag{3.37}$$

$$= \frac{\sqrt{\mu}}{a\sqrt{\mu} \pm r^{3/2}} \tag{3.38}$$

taking $\mu = GM$, we find the angular frequency to be

$$\Omega = \frac{\sqrt{GM}}{a\sqrt{GM} \pm r^{3/2}}. \tag{3.39}$$

Problem 4

(**a**) Consider the Schwarzchild metric with geometry $\{t, x, y, z\}$ in an orthonormal basis. Let $f(r) = 1 - \dfrac{2\mu}{r}$, and let the new basis coordinates be given by

$$(e^{\hat{0}})_\mu = \sqrt{f(r)}\delta^{\hat{0}}_\mu \tag{4.1}$$

$$(e^{\hat{1}})_\mu = \frac{1}{\sqrt{f(r)}}\delta^{\hat{1}}_\mu \tag{4.2}$$

$$(e^{\hat{2}})_\mu = r\delta^{\hat{2}}_\mu \tag{4.3}$$

$$(e^{\hat{3}})_\mu = r\sin\theta\delta^{\hat{3}}_\mu \tag{4.4}$$

which obey the orthogonality relations

$$(e_{\hat{\alpha}})^\nu(e^{\hat{\alpha}})_\mu = \delta^\nu_\mu, \qquad (e^{\hat{\alpha}})_\mu(e_{\hat{\beta}})^\mu = \delta^{\hat{\alpha}}_{\hat{\beta}} \tag{4.5}$$

and

$$g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}(e^{\hat{\alpha}})_\mu(e^{\hat{\beta}})_\nu, \qquad g^{\mu\nu} = \eta^{\hat{\alpha}\hat{\beta}}(e_{\hat{\alpha}})^\nu(e_{\hat{\beta}})^\mu. \tag{4.6}$$

Here, $\eta_{\hat{\alpha}\hat{\beta}}$ is the Minkowski metric in the orthonormal basis, diag$(1, -1, -1, -1)$, and $g_{\mu\nu}$ is the Schwarzchild metric. We note that, through equations (4.1) - (4.4), each μ index corresponds to it's respective hatted component. Hence $0 \to \hat{0}$, $1 \to \hat{1}$, and so on. The basis coordinates return the Schwarzchild metric via (4.6):

$$g_{00} = \eta_{\hat{0}\hat{0}}(e^{\hat{0}}_0)^2 = f(r) \tag{4.7}$$

$$g_{11} = \eta_{\hat{1}\hat{1}}(e^{\hat{1}}_1)^2 = \frac{1}{f(r)} \tag{4.8}$$

$$g_{22} = \eta_{\hat{2}\hat{2}}(e^{\hat{2}}_2)^2 = r^2 \tag{4.9}$$

$$g_{33} = \eta_{\hat{3}\hat{3}}(e^{\hat{3}}_3)^2 = r^2\sin^2\theta \tag{4.10}$$

which returns the Schwarzchild metric. Note that the inverse metric is given directly because it is diagonal,

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{1}{f(r)} & 0 & 0 & 0 \\ 0 & -f(r) & 0 & 0 \\ 0 & 0 & -\dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{4.11}$$

and that the unit vectors are square roots of each respective entries, since $\eta_{\hat{\alpha}\hat{\beta}}$ is also diagonal. This implies that, from (4.11),

$$(e_{\hat{0}})^\mu = \frac{1}{\sqrt{f(r)}}\delta^{\hat{0}}_\mu \tag{4.12}$$

$$(e_{\hat{1}})^\mu = \sqrt{f(r)}\delta^{\hat{1}}_\mu \tag{4.13}$$

$$(e_{\hat{2}})^\mu = \frac{1}{r}\delta^{\hat{2}}_\mu \tag{4.14}$$

$$(e_{\hat{3}})^\mu = \frac{1}{r\sin\theta}\delta^{\hat{3}}_\mu \tag{4.15}$$

are the inverse basis vectors.

(**b**) Using Sympy, the Riemann tensor components were computed using $f(r)$ as an input function in the Schwarzchild metric. The non-zero components were returned as

$$R^1_{001} = -R^1_{010} = -\frac{1}{2}f\frac{\partial^2 f}{\partial r^2} \tag{4.16}$$
$$R^2_{002} = -R^2_{020} = -\frac{1}{2}\frac{f}{r}\frac{\partial f}{\partial r} \tag{4.17}$$
$$R^3_{003} = -R^3_{030} = -\frac{1}{2}\frac{f}{r}\frac{\partial f}{\partial r} \tag{4.18}$$
$$R^2_{112} = -R^2_{121} = \frac{1}{2}\frac{1}{rf}\frac{\partial f}{\partial r} \tag{4.19}$$
$$R^3_{113} = -R^3_{131} = \frac{1}{2}\frac{1}{rf}\frac{\partial f}{\partial r} \tag{4.20}$$
$$R^3_{223} = -R^3_{232} = f - 1 \tag{4.21}$$

plus other terms related by the raising/lowering of indices and antisymmetry (all can be obtained from these). Subbing in $f(r)$ where derivatives lie and simplifying, we find

$$R^1_{001} = \frac{2\mu}{r^3}f(r) \tag{4.22}$$
$$R^2_{002} = R^3_{003} = -\frac{\mu}{r^3}f(r) \tag{4.23}$$
$$R^2_{112} = R^3_{113} = \frac{\mu}{r^3}\frac{1}{f(r)} \tag{4.24}$$
$$R^3_{223} = -\frac{2\mu}{r} \tag{4.25}$$

Consider now the transformation of these terms to the orthonormal frame. To simplify our lives, we note that since upper and lower indices match in each Riemann term, by the orthogonality condition we get 1 out of each of them, so the only terms that matter are the repeated index terms in the covariant slots:

$$R^{\hat{1}}_{\hat{0}\hat{0}\hat{1}} = R^1_{001}[(e_{\hat{0}})^0]^2 \cdot 1 = \frac{2\mu}{r^3}f(r)\frac{1}{f(r)} = \frac{2\mu}{r^3} \tag{4.26}$$
$$R^{\hat{2}}_{\hat{0}\hat{0}\hat{2}} = R^{\hat{3}}_{\hat{0}\hat{0}\hat{3}} = R^2_{002}[(e_{\hat{0}})^0]^2 \cdot 1 = -\frac{\mu}{r^3}f(r)\frac{1}{f(r)} = -\frac{\mu}{r^3} \tag{4.27}$$
$$R^{\hat{2}}_{\hat{1}\hat{1}\hat{2}} = R^{\hat{3}}_{\hat{1}\hat{1}\hat{3}} = R^2_{112}[(e_{\hat{1}})^1]^2 \cdot 1 = \frac{\mu}{r^3}\frac{1}{f(r)}f(r) = \frac{\mu}{r^3} \tag{4.28}$$
$$R^{\hat{3}}_{\hat{2}\hat{2}\hat{3}} = R^3_{223}[(e_{\hat{2}})^2]^2 \cdot 1 = -\frac{2\mu}{r}\frac{1}{r^2} = -\frac{2\mu}{r^3} \tag{4.29}$$

as desired.

(**c**) Now consider particles moving radially inward. How do the geodesic deviation equations

$$\frac{d^2}{d\lambda^2}S^{\hat{\alpha}} = R^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}}T^{\hat{\beta}}T^{\hat{\gamma}}S^{\hat{\delta}} \tag{4.30}$$

affect their shape / relative paths? In the Instantaneous Inertial Rest Frame (IIRF), the velocity of any stationary particle in that frame is zero, which implies $T^{\hat{1}} = T^{\hat{2}} = T^{\hat{3}} = 0$, and time remains ticking at the proper rate, so $T^{\hat{0}} = 1$. Then, using (4.30) and the orthonormal Riemann components (4.26) - (4.29), we have

$$\frac{d^2}{d\lambda^2}S^{\hat{1}} = R^{\hat{1}}_{\hat{0}\hat{0}\hat{1}}(T^{\hat{0}})^2 S^{\hat{1}} = \frac{2\mu}{r^3}S^{\hat{1}} \tag{4.31}$$

$$\frac{d^2}{d\lambda^2}S^{\hat{2}} = (R^{\hat{2}}_{\hat{0}\hat{0}\hat{2}}(T^{\hat{0}})^2 + R^{\hat{2}}_{\hat{1}\hat{1}\hat{2}}(T^{\hat{1}})^2)S^{\hat{2}} = -\frac{\mu}{r^3}S^{\hat{2}} \tag{4.32}$$

$$\frac{d^2}{d\lambda^2}S^{\hat{3}} = (R^{\hat{3}}_{\hat{0}\hat{0}\hat{3}}(T^{\hat{0}})^2 + R^{\hat{3}}_{\hat{1}\hat{1}\hat{3}}(T^{\hat{1}})^2 + R^{\hat{3}}_{\hat{2}\hat{2}\hat{3}}(T^{\hat{2}})^2)S^{\hat{3}} = -\frac{\mu}{r^3}S^{\hat{3}} \tag{4.33}$$

where any other $T^{\hat{\alpha}}$ is zero if $\hat{\alpha} \neq 0$. The $S^{\hat{0}}$ equation is zero because the Riemann tensor terms must contain nonzero elements, but $R^{\hat{0}}_{\hat{0}\hat{0}\hat{0}} = 0$ and any non-zero index yields $T^{\hat{\alpha}=1,2,3} = 0$, so there is no deviation in the temporal geodesics.

Physically, the radial seperation of geodesics increases exponentially as one approaches the singularity, which will stretch you. Similarly, along the polar and azimuthal axes, an inverse-cube oscillatory behaviour is observed, which will stretch and compress any infalling object. These correspond to tidal forces arising from the black hole.

For human tissues, we can take an acceleration gradient of approximately $40\,g/\text{m}$, where g is earth gravity at sea level ($\approx 9.8\text{m/s}^2$), we can apply the weak-field limit (for a radius $r > r_s$) to the acceleration gradient from (4.31),

$$\frac{40 * 9.8}{S^{\hat{1}}} \lesssim \frac{2\Phi}{c^2} = \frac{2G_N M}{c^2} \tag{4.34}$$

$$= \frac{2(6.67\text{e} - 11)M}{(3\text{e}\,8)^2} \tag{4.35}$$

$$\implies M \lesssim 5.289\text{e}\,29 \approx 10^5 M_E \tag{4.36}$$

hence any black hole heavier than $10^5 M_E$ would kill an infalling astronaut prior to reaching the event horizon.