MAT244 PS1

Q1) $\frac{dy}{dx} - \alpha x = (x^2+1)y + xy^2$

Find which α emits $y = c$ for a $c \in \mathbb{R}$.

If $y = c$, then $\frac{dy}{dx} = 0$.

$$\Longrightarrow \quad 0 = (x^2+1)y + xy^2 + \alpha x$$

Solve for y using quadratic formula:

$$y = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4\alpha x^2}}{2x}$$

- Only ways y can be constant is if numerator $= 0$.

→ Highest degree of 'x' in the numerator is 2 (x^2), and the highest degree of 'x' in the denomenator is 1 (x), so even if we tried to find a way to cancel the two out, y would still not be constant.

→ We could try to isolate the 'x^2' term in the square root, however manipulating this would not cancel out any x^2 term in the numerator.

→ Thus, $-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4\alpha x^2} = 0 = y(x)$.

$$\Rightarrow \quad (-x^2+1)^2 = \left(\mp\sqrt{(x^2+1)^2 - 4\alpha x^2}\right)^2$$

$$\Rightarrow \quad (x^2+1)^2 = (x^2+1)^2 - 4\alpha x^2$$

$$\Rightarrow \quad 4\alpha x^2 = 0 \quad \Rightarrow \quad \boxed{\alpha = 0}$$

Therefore $\alpha = 0$ is the only value of α which emits a constant solution $y = c$ for $c \in \mathbb{R}$ for the ODE.

2) $y'' - ty' = y$.

- Clearly, this equation is second order because the highest degree of derivative in the equation is 2.

- Let f, g be solutions to this ODE. Let $a \in \mathbb{R}$ be a constant. I want to prove that the equation $y'' - ty' = y$ is linear, or that
$$\frac{d^2}{dt^2}[af+g] - t\frac{d}{dt}[af+g] - [af+g]$$
$$= (a f'' - tf' - f) + (g'' - tg' - g) = 0.$$

<u>Proof</u>

- $\frac{d^2}{dt^2}[af+g] - t\frac{d}{dt}[af+g] - [af+g]$

$= a\frac{d^2}{dt^2}f + \frac{d^2}{dt^2}g - at\frac{d}{dt}f - t\frac{d}{dt}g - af - g$

$= af'' + g'' - atf' - tg' - af - g$

$= af'' - t(af') - af + g'' - tg' - g$

$= a(f'' - tf' - f) + (g'' - tg' - g) = 0$

Therefore $y'' - ty' = y$ is linear, which is what I wanted to prove. $\square$

3) a) $\frac{dF}{dt} = F^2 + t^2 + 1$.

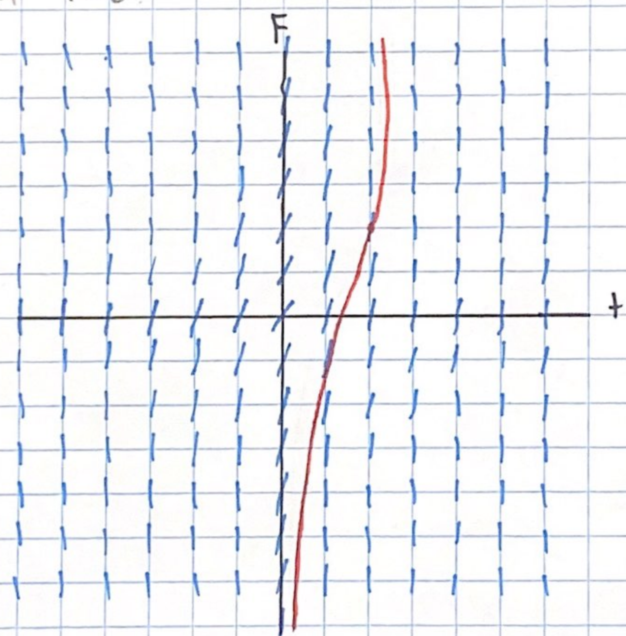
By monotonicity, a function g is increasing on an interval $[a,b]$ when $g' > 0$ on $[a,b]$.

Clearly $F^2 + t^2 + 1 > 0$, since $1 > 0$ and $F^2 \geq 0$, $t^2 \geq 0$,

So $F^2 + t^2 + 1 > 1 + 0 + 0$.

Therefore F is increasing.

b) Direction field.



4) a) If the slope at each point is x^2, then $y'(x) = x^2$. The curve will go through $(-2,2)$. Thus our initial value problem is

$$\frac{dy}{dx} = x^2 \quad , \quad y(-2) = 2.$$

b) To solve we can integrate both sides:

$$\int \frac{dy}{dx}\, dx = \int x^2\, dx$$

$$\Rightarrow \quad y = \frac{1}{3}x^3 + C.$$

Now we can solve our IVP for the constant C:

$$y(-2) = 2 = \frac{1}{3}(-2)^3 + C$$

$$\Rightarrow \quad C = 2 - \frac{1}{3}(-8) = 2 + \frac{8}{3} = \frac{14}{3}.$$

Thus our equation $y(x)$ is

$$y(x) = \frac{1}{3}x^3 + \frac{14}{3}.$$

5) $t^4 y^2 - y + (t^2 y^4 - t)\frac{dy}{dt} = 0.$

a) An ODE is not exact when for

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

$$\partial_y [M(x,y)] = \partial_x [N(x,y)].$$

Here we have that

$$M(t,y) = t^4 y^2 - y \quad \text{and}$$
$$N(t,y) = t^2 y^4 - t.$$

Then

$$\partial_y [M(t,y)] = 2t^4 y - 1 \neq 2t y^4 - 1 = \partial_t [N(t,y)].$$

Therefore this ODE is not exact.

b) $\mu(t,y)$ is an integrating factor when

$$\partial_y [\mu M] = \partial_x [\mu N].$$

We have $\mu = \frac{1}{t^2 y^2}$, so

$$\partial_y \left[t^2 - \frac{1}{t^2 y}\right] = \frac{1}{t^2 y^2} = \partial_t \left[y^2 - \frac{1}{t y^2}\right].$$

Therefore $\mu(t,y) = \frac{1}{t^2 y^2}$ is an integrating factor for this equation.

c) Multiplying $\mu(t,y) = \frac{1}{t^2 y^2}$ through, we have

$$t^2 - \frac{1}{t^2 y} + \left(y^2 - \frac{1}{t y^2}\right)\frac{dy}{dt} = 0.$$

We now need to find a function $\Psi(t,y)$ such that

$$\partial_t \Psi(t,y) + \partial_y \Psi(t,y) \frac{dy}{dt} = 0.$$

Our terms are

$$\Psi_t(t,y) = t^2 - \frac{1}{t^2 y} \quad \text{and} \quad \Psi_y = y^2 - \frac{1}{ty^2}.$$

For the first term, it is easy to see that the first term of $\Psi(t,y)$ must be $\frac{t^3+y^3}{3}$. Similarly with the second term, it must be $+\frac{1}{ty}$, since $\frac{\partial}{\partial t}\left[\frac{1}{ty}\right] = -\frac{1}{t^2 y}$, which is just from single variable calculus.

Therefore $\Psi(t,y)$ becomes

$$\Psi(t,y) = \frac{t^3+y^3}{3} + \frac{1}{ty} + c.$$

Therefore the solutions for this equation are given implicity by

$$C = \frac{t^3+y^3}{3} + \frac{1}{ty}, \text{ where } C \in \mathbb{R}.$$

6) We can begin by finding a solution for

$$\frac{dT}{dt} = -\alpha(T-C).$$

a) $T' = -\alpha T + \alpha C$

$T' + \alpha T = \alpha C \qquad I(t) = e^{\int \alpha dt} = e^{\alpha t}$

$e^{\alpha t} T' + \alpha e^{\alpha t} T = \alpha e^{\alpha t} C$

$\frac{d}{dt}[e^{\alpha t} T] = \alpha e^{\alpha t} C$

$\int \frac{d}{dt}[e^{\alpha t} T] dt = C \int \alpha e^{\alpha t} dt$

$e^{\alpha t} T = Ce^{\alpha t} + C_0$, $C_0 \in \mathbb{R}$ is constant

$\Rightarrow \boxed{T(t) = C + \frac{C_0}{e^{\alpha t}}}$

is our general solution. Now we can solve the IVP:

$T(0) = 200° \qquad T(20) = 120° \qquad C = 100°$

$\Rightarrow 200 = 100 + \frac{C_0}{e^{\alpha(0)}} = 100 + C_0$

$\Rightarrow C_0 = 100°$

Now our equation becomes

$T(t) = 100 + \frac{100}{e^{\alpha t}}$

(Here I denote 'log' as '$\log_e x$' instead of 'ln')

So now we can find α.

$T(20) = 120 = 100 + \frac{100}{e^{\alpha(20)}}$

$\Rightarrow 20 = \frac{100}{e^{\alpha(20)}} \rightarrow e^{\alpha(20)} = 5$, thus

$20 \cdot \alpha = \log 5 \Rightarrow \boxed{\alpha = \frac{\log 5}{20}}$

Thus

$$\boxed{T(t) = 100 + 100\, e^{-\frac{\log 5}{20} \cdot t}}$$

A more general solution could be

$$\boxed{T(t) = C + [T(t_0) - 100] \cdot 5^{-\frac{t}{20}}},$$

since $C_0 = T(t_0) - 100$ and $e^{-\frac{\log 5}{20} t} = 5^{-\frac{t}{20}}$. C is starting medium temp.

b) If $T_0 - C > 0$ (T_0 hotter than medium temp. C)

then $[T_0 - C]\, 5^{-\frac{t}{20}}$ is positive and decreasing to 0, since

$$\lim_{t \to \infty} [T_0 - C]\, 5^{-\frac{t}{20}} = 0.$$

Then $T(\infty) = C$, which is intuitively correct.

Similarly if $T_0 - C < 0$ (T_0 colder than medium temp. C)

then $[T_0 - C]\, 5^{-\frac{t}{20}}$ is negative and decreasing to 0, since

$$\lim_{t \to \infty} [T_0 - C]\, 5^{-\frac{t}{20}} = 0.$$

Then again, $T(t \to \infty) = C$, which is intuitively correct.