PHY350 Problem Set 3 — Due November 21, 23:00

1006940802

4.5

I will begin this problem by imposing a cartesian coordinate system (x, y, z) with (x, y) as one normally would in a plane, and $\hat{z}$ pointing out of the page. The torque on a dipole is determined by the behaviour of the electric field at the center of the dipole (since we are working with ideal dipoles), and is given by $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$. Next, note that the electric field of a dipole in polar coordinates is given by

$$\boldsymbol{E}_i = \frac{p_i}{4\pi\varepsilon_0 r^3}[2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}],$$

where p_i is the i-th dipole in the configuration.

- First consider the electric field at $\boldsymbol{p}_2$ produced by the dipole $\boldsymbol{p}_1$. Here, $\boldsymbol{E}_1$ should be pointing in the $-\hat{\boldsymbol{y}}$ direction and should have no $\hat{\boldsymbol{x}}$ component. With respect to the dipole, $\theta = \frac{\pi}{2}$ here, so the electric field should be $\boldsymbol{E}_1 = \frac{p_1}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{\theta}}$, but with $\hat{\boldsymbol{\theta}} \to -\hat{\boldsymbol{y}}$, therefore the field produced by $\boldsymbol{p}_1$ is $\boldsymbol{E}_1 = -\frac{p_1}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{y}}$.
- Second, consider the field at $\boldsymbol{p}_1$ produced by the $\boldsymbol{p}_2$ dipole. Due to the 45−degree rotation of $\boldsymbol{p}_2$ dipole means that $\theta = \pi$ yields an electric field pointing purley in the $\hat{\boldsymbol{x}}$ direction, with no $\hat{\boldsymbol{y}}$ component. We obtain $\boldsymbol{E}_2 = \frac{p_2}{4\pi\varepsilon_0 r^3}[-2\hat{\boldsymbol{r}}]$, however with r^3 being positive in our coordinate system with the field pointing along the positive $\hat{\boldsymbol{x}}$ direction, then $\hat{\boldsymbol{r}} \to -\hat{\boldsymbol{x}}$, thus the electric field from dipole 2 at dipole 1 is $\boldsymbol{E}_2 = \frac{p_2}{2\pi\varepsilon_0 r^3}\hat{\boldsymbol{x}}$.

Now for the torques. In both cases, the electric fields $\boldsymbol{E}_i$ are perpendicular to their dipoles they pass over $\boldsymbol{p}_j$ with $i \neq j = 1, 2$, thus the cross product becomes just $\boldsymbol{E}_i \times \boldsymbol{p}_j = E_i p_j \hat{z}$. For $\boldsymbol{\tau}_1$, we have that

$$\begin{aligned}
\boldsymbol{\tau}_1 &= \boldsymbol{p}_1 \times \boldsymbol{E}_1 \\
&= \frac{p_1 p_2}{2\pi\varepsilon_0 r^3}[\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}}] \\
&= -\frac{p_1 p_2}{2\pi\varepsilon_0 r^3}\hat{\boldsymbol{z}},
\end{aligned}$$

which points into the page. Similarly,

$$\begin{aligned}
\boldsymbol{\tau}_2 &= \boldsymbol{p}_2 \times \boldsymbol{E}_1 \\
&= -\frac{p_1 p_2}{4\pi\varepsilon_0 r^3}[\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}] \\
&= -\frac{p_1 p_2}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{z}},
\end{aligned}$$

which also points into the page.

4.10

(a) The bound charges are defined as follows:

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}, \qquad\qquad \rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P}.$$

With a polarization $\boldsymbol{P}(\boldsymbol{r}) = k\boldsymbol{r}$, then the bound surface charge is given at the radius R, with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$, which is then just $\sigma_b = kr\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}|_{r=R} = kR$. The bound volume charge cam be found by taking the divergence of $\boldsymbol{P}$ in spherical coordinates. Since $\boldsymbol{P} = P\hat{\boldsymbol{r}}$, then the contribution to the divergence is just the $\hat{\boldsymbol{r}}$ component:

$$\begin{aligned} -\boldsymbol{\nabla} \cdot \boldsymbol{P} &= -\frac{1}{r^2}\frac{\partial}{\partial r}[kr^3] \\ &= -3k\frac{r^2}{r^2} = -3k. \end{aligned}$$

(b) As for the electric field inside and outside the sphere, we can apply Gauss's law $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \dfrac{Q_{enc}}{\varepsilon_0}$. The permittivity of free space is used here because the sphere does not have any designated dielectric inside it. Therefore:

(In) The enclosed charge is $\frac{4}{3}\pi r^3 \rho_b$, and the flux of $\boldsymbol{E}$ is parallel to $\hat{\boldsymbol{r}}$ which is just $E4\pi r^2$, so equating the two yields the electric field inside the sphere $\boldsymbol{E}(\boldsymbol{r}) = \dfrac{\rho_b}{3\varepsilon_0}r = -\dfrac{k}{\varepsilon_0}r$.

(Out) The enclosed charge is all of the total bound charge σ_b and ρ_b. Multiplying by their respective area and volume elements yields the total enclosed charge $Q_{enc} = \sigma_b 4\pi r^2 + \rho_b \frac{4}{3}\pi r^3 = 4\pi k r^2 R - \frac{3 \cdot 4}{3}kr^2R = 0$, which implies that there is no electric field outside the sphere.

Therefore $\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} -\dfrac{k}{\varepsilon_0} r\hat{\boldsymbol{r}} & \text{for } 0 \leq r < R \\ 0 & \text{for } r \geq R \end{cases}$ is the electric field in and out.

4.20

For this problem, due to the embedded uniform free charge ρ and dielectric constant ε_r, we may being by finding the electric field inside and outside the sphere. Once we have found the field, we may take the line integral, by the definition of the potential, from infinity to zero. For fields in matter, we have that $\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f,\,encl}$.

(In) For points inside the sphere, Gauss's law yields that $D4\pi r^2 = \frac{4}{3}\pi r^3 \rho$, which implies that $\boldsymbol{D}_{in} = \varepsilon \boldsymbol{E}_{in} = \frac{1}{3}\rho r \hat{\boldsymbol{r}}$.

(Out) For points outside the sphere, we obtain a similiar result $D4\pi r^2 = \frac{4}{3}\pi R^3 \rho$, hence $\boldsymbol{D}_{out} = \varepsilon_0 \boldsymbol{E}_{out} = \frac{1}{3}\rho \frac{R^3}{r^2}\hat{\boldsymbol{r}}$ (note the permittivity of free space here, since we are outside the dielectric).

Now to find the potential. From first year electrostatics,

$$\begin{aligned}
V &= -\int_{\infty}^{0} \boldsymbol{E} \cdot d\boldsymbol{\ell} \\
&= -\int_{\infty}^{R} \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}\, dr - \int_{R}^{0} \frac{\rho}{3\varepsilon_0 \varepsilon_r} r\, dr \\
&= \frac{\rho}{3\varepsilon_0}\frac{R^3}{r}\bigg|_{\infty}^{R} - \frac{\rho}{3\varepsilon_0\varepsilon_r}\frac{r^2}{2}\bigg|_{R}^{0} \\
&= \frac{\rho}{3\varepsilon_0}R^2 - \frac{\rho}{3\varepsilon_0}R^2\frac{1}{2\varepsilon_r} \\
&= \frac{\rho}{3\varepsilon_0 R^2}\left(1 + \frac{1}{2\varepsilon_r}\right),
\end{aligned}$$

which is thus the potential at the center of the sphere with infinity used as the reference point.

4.15

(a) First off, to locate the bound charges, we must consider the three regions where bound charge can inhabit: the inner surface at $r = a$, the volume $a < r < b$, and the outer surface $r = b$. With the polarization given by $\boldsymbol{P}(\boldsymbol{r}) = \frac{k}{r}\hat{\boldsymbol{r}}$, then the bound surface charges are given by $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$. For $r = a$, since $\hat{\boldsymbol{r}} = \hat{\boldsymbol{n}}$, then $\sigma_b = -\frac{k}{a}$ for the inner surface and $\sigma_b = \frac{k}{b}$ for the outer surface. The bound charge is determined by the negative gradient of the polarization $\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$, however since $\boldsymbol{P} = P_r\hat{\boldsymbol{r}}$, only the radial contribution matters. Thus $\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}[kr] = -\frac{k}{r^2}$, for $a < r < b$.

By applying Gauss's law, we can determine the field in each of the three regions $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = Q_{encl}$. For $r < a$, the enclosed charge is 0, so there is no field in that region. For $a < r < b$, we have that

$$\begin{aligned} Q_{encl} &= \sigma_{b,a} 4\pi a^2 + 4\pi \int_a^r \frac{-k}{r^2} r^2 \, dr \\ &= -4\pi k a - 4\pi k(r - a) \\ &= -4\pi k r. \end{aligned}$$

For outside the sphere, we have that

$$\begin{aligned} Q_{encl} &= -k4\pi a - 4\pi \int_a^b \frac{k}{r^2} r^2 \, dr + k4\pi b \\ &= 4\pi k(b - a) - 4\pi k(a - b) \\ &= 0, \end{aligned}$$

hence the field outside of the sphere is zero as well. However, for inside the dielectric, Gauss's law implies that $4\pi r^2 E = -\frac{4\pi k}{\varepsilon_0} r \implies \boldsymbol{E}(\boldsymbol{r}) = -\frac{k}{\varepsilon_0 r}\hat{\boldsymbol{r}}$.

(b) Applying a different method, Gauss's law for dielectrics, we have that $\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f,encl} = 0$ since there is no specified free charge on the sphere. This implies that $\boldsymbol{D} = 0$ in all space. Recall that $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = 0$, or that $\boldsymbol{E} = -\frac{\boldsymbol{P}(\boldsymbol{r})}{\varepsilon_0}$. Using the specified polarization $\boldsymbol{P}(\boldsymbol{r}) = \frac{k}{r}\hat{\boldsymbol{r}}$, one can easily verify that $\boldsymbol{E}((r)) = -\frac{k}{\varepsilon_0 r}\hat{\boldsymbol{r}}$, which is only specified throughout the region $a < r < b$, and is zero otherwise.

This result coincides with the result found in part (a), hence

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} 0 & \text{for } r < a \\ -\frac{k}{\varepsilon_0 r}\hat{\boldsymbol{r}} & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

as desired.