PHY489 Problem Set 1

Monday, October 2, 2023

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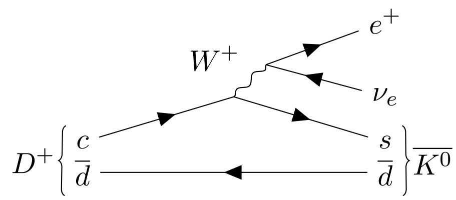
Problem 1

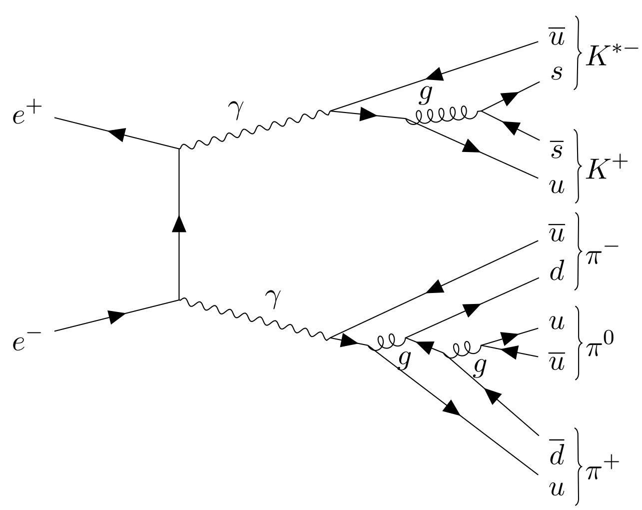
In this problem, determine whether or not the following particle interactions are possible according to the standard model (at tree level).

(a) $D^+ \to \overline{K^0} e^+ \nu_e$. This reaction is possible. Charge $(+1 \to +1)$ and lepton number $(0 \to +1 - 1 = 0)$ are both conserved, and there are no baryons in the interaction. The fact that an electron neutrino is produced implies that this takes place via the weak interaction (c quark emits W^+ into s quark):

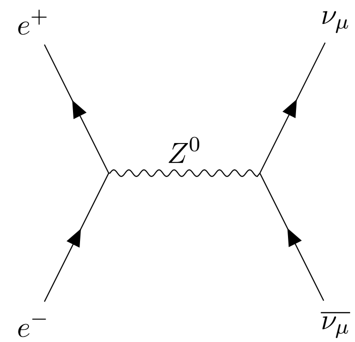


(b) $\mu^- \nu_\mu \to \tau^- \nu_\tau$. This interaction is not possible, as determined by the violation of lepton number conservation. On the reactants side, lepton number is $L_\mu = 2$ (1 for μ^- and 1 for ν_μ), and on the products side lepton number is $L_\tau = 2$ (by the same principle). Since neither of these are 0, L_μ and L_τ are not conserved, therefore the interaction is not possible.

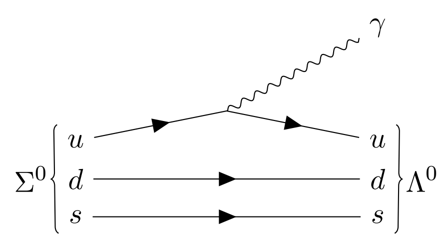
(c) $e^+ e^- \to K^+ K^{*-} \pi^+ \pi^- \pi^0$. This interaction is possible, and will predominately proceed via the electromagnetic interaction (to annihilate the fermions, meanwhile the strong proceeds the production of other quarks. It is possible for any process mediated by a γ to also be mediated by a Z^0 boson, however the electromagnetic interaction should dominate. Quark pair production must occur via the QCD fundamental vertex). Charge is conserved $(+1 - 1 = +1 - 1 + 1 - 1 + 0 = 0)$ and lepton number is conserved $(L_e = -1 + 1 = 0)$. There are no baryons so baryon number is also conserved. Quarks may produce in pairs, and therefore quark flavour is also conserved. A possible Feynman diagram, at tree level, is



(d) $e^+e^- \to \nu_\mu \overline{\nu_\mu}$. This interaction is possible, since Lepton number is conserved ($L_e = -1+1 = 0 = L_\mu = +1-1$) and charge is conserved (neutrinos are chargeless). The presence of the neutrinos in the interaction require the force to be mediated by a neutral current weak interaction:

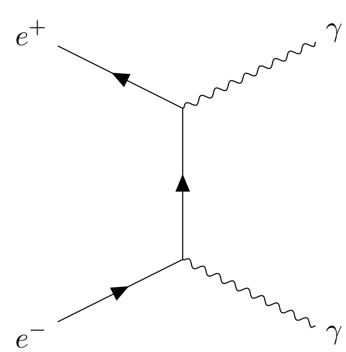


(e) $\Sigma^0 \to \Lambda^0\gamma$. This interaction is possible (and is principle for the Σ^0 baryon). Although the quark content of the two baryons is the same, their masses are different. Due to this, one quark may decay a photon and this process is mediated by the electromagnetic force. It is possible because baryon number is conserved (1 on each side), and the interaction charge is neutral with no leptons. There is no intergenerational quark flavour mixing, hence a possible Feynman diagram may be

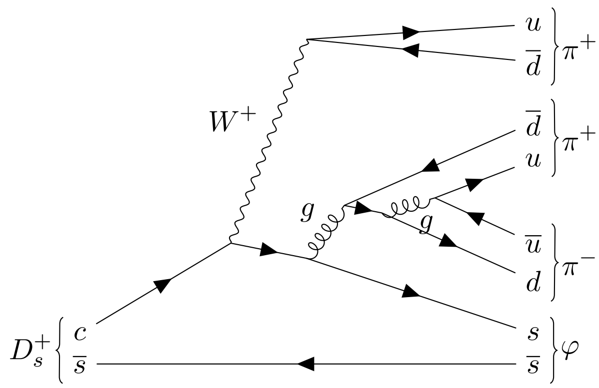


(f) $\tau^- \to \mu^-\gamma$. This interaction is not possible simply because lepton number is not conserved. On the left hand side, $L_\tau = 1$ but on the right hand side, $L_\mu = 1 \neq L_\tau$. Because of this, there can be no intergenerational lepton flavour mixing, and this interaction can not be observed.

(g) $e^+e^- \to \gamma\gamma$. This interaction is possible and is referred to as pair annihilation, which is goverened by the electromagnetic force carriers. In this interaction, lepton number ($L_e = -1+1 = 0$) and charge ($q = -1+1 = 0$) is conserved. A possible Feynman diagram is simply



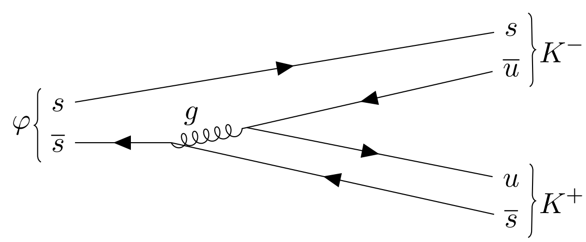
(h) $D_s^+ \to \varphi\pi^+\pi^+\pi^-$. This interaction is possible and is primarily goverened by the weak interaction (weak to produce a π^+ via a W^+ boson, but the strong to produce the other quarks.). First observe that charge is conserved ($q = +1 \to q' = 0 + 1 + 1 - 1 = 1$) and there are no baryon or lepton numbers to consider. Furthermore, quark flavour is conserved because, other than one π^+ meson, all other quarks can be produced in pairs via gluons. A possible Feynman diagram is



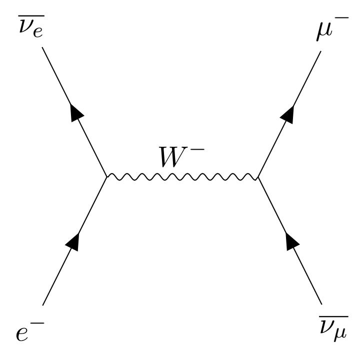
(i) $p\overline{p} \to K^- K^0 \pi^+ \pi^+ \pi^- \pi^0$. This interaction is possible, but not at tree-level. Charge ($+1 - 1 = -1 + 0 + 1 + 1 - 1 + 0 = 0$), baryon number ($+1 - 1 = 0$), and lepton number are all conserved. By the conservation of quark flavour, the only possible way to form the products is via gluon pair production, and the proton quark pairs can also be annihilated in this way. Since the interaction requires $3 \to 6$ gluons (3 for annihilation on reactants side, 6 for quark production on the products side), gluons must interact with themselves to produce more pairs, thus forming loops. Hence the governing force in this interaction is the strong force.

(j) $pp \to p\overline{n}\pi^+\pi^+\pi^+\pi^-\pi^-$. This interaction is not possible because baryon number is not conserved. On the left hand side, baryon number is 2 ($+1$ for each proton), but on the right hand side it is 0 ($+1$ for the proton, -1 for the anti-neutron).

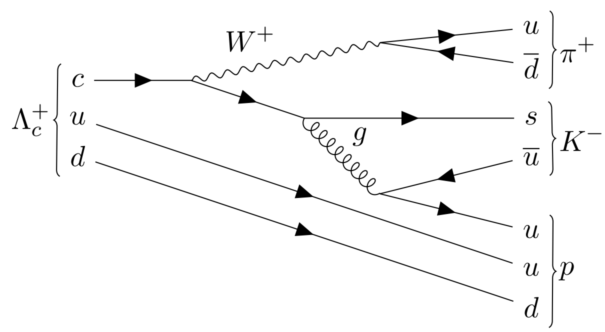
(k) $\varphi \to K^+K^-$. This interaction is possible, since charge is conserved ($-1 + 1 = 0$), lepton number is zero (hence conserved) and there are no baryons. Put simply, the production of K^- and K^+ mesons from a φ meson just involved a gluon emission and quark pair production, implying that the strong interaction governs this decay.



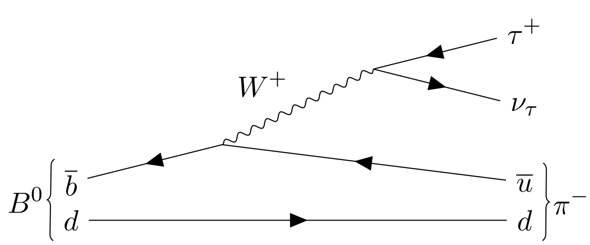
(l) $e^-\overline{\nu_e} \to \mu^-\overline{\nu_\mu}$. This interaction is possible. Charge is conserved (-1 for the whole reaction) and lepton number is conserved ($L_e = +1 - 1 = 0, L_\mu = +1 - 1 = 0$) and zero on both sides. The reaction contains no baryons or quarks, so baryon number and quark flavour are conserved. Since neutrinos are present in the interaction, it must be that this process proceeds via a charged current weak interaction, by emission of a W^- boson.



(m) $\Lambda_c^+ \to pK^-\pi^+$. This interaction is possible, since charge $(+1 \to +1+1-1 = +1)$ and baryon number $(+1 \to +1)$ is conserved. There is no lepton number. Since a π^+ is produced, the emission of charge must originate from the c quark to decay into the s, which is mediated by a charged current weak interaction. This usually occurs during hadron decays. As the weak interaction is predominate, the strong force (gluons) mediates the pair production of other quarks to produce the proton and the negative-kaon.

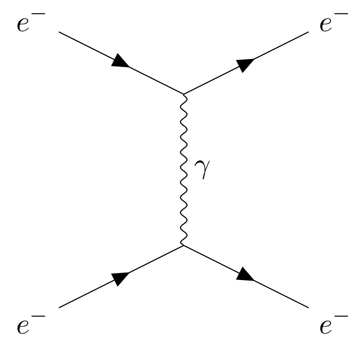


(n) $B^0 \to \pi^-\tau^+\nu_\tau$. This interaction is possible, since charge $(0 \to -1+1+0 = 0)$, lepton number $(0 \to -1+1+0 = 0)$ and baryon number (0) are conserved. The presence of the neutrino in the interaction implies that the force carrier must be a charged weak interaction.

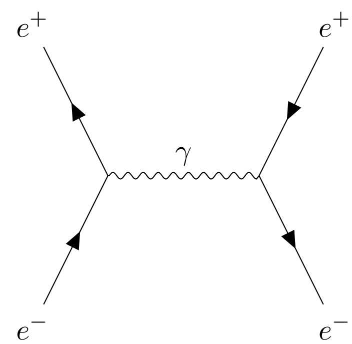


Problem 2

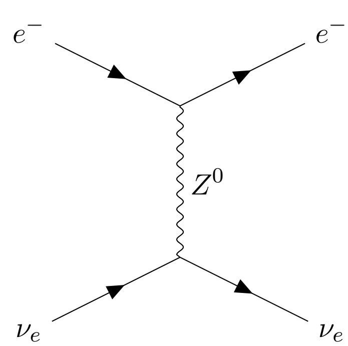
(a) $e^- e^- \to e^- e^-$. This interaction proceeds via the electromagnetic interaction, and can only be represented with one diagram (crossing symmetry creates different interactions):

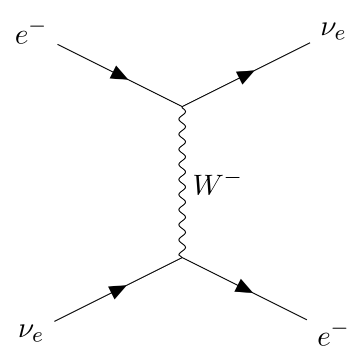


(b) $e^+ e^- \to e^+ e^-$. Similarly, the electromagnetic interaction dominates in this reaction and crossing symmetry creates different processes. Once again, there is only one diagram:



(c) $e^- \nu_e \to e^- \nu_e$. This process contains two diagrams, both representing weakly interacting cases. First, the neutral current interaction:





while the second involves the virtual transmission of the -1 charge via a W^- boson.

Problem 3

We now consider a fixed target pp scattering experiment, where in the lab frame, one proton is at rest while the other is accelerated.

(**a**) To determine the proton energy such that the CM energy is 14TeV, the expression $s = (p_1 + p_2)^2$ can be used, where s is the square of the CM energy and p_1, p_2 are the respective energy-momentum 4-vectors of the two protons. Note first that $p_1 = (E_p, \mathbf{p})$ and $p_2 = (m_p, \mathbf{0})$ since the first proton has been accelerated and the second proton is at rest (in the lab frame). Then

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ &= (E_p + m_p, \mathbf{p})^2 \\ &= (E_p + m_p)^2 - |\mathbf{p}|^2 \\ &= E_p^2 + m_p^2 + 2E_p m_p - |\mathbf{p}|^2 \\ &= m_p^2 + |\mathbf{p}|^2 + m_p^2 + 2E_p m_p - |\mathbf{p}|^2 \\ &= 2m_p^2 + 2E_p m_p. \end{aligned} \tag{3.1}$$

Solving for the energy, we obtain

$$E_p = \frac{s - 2m_p^2}{2m_p}, \tag{3.2}$$

and with $s = 14^2\text{TeV}^2$ and $m_p = 938.272\text{MeV} \cdot \dfrac{1\text{TeV}}{1 \times 10^{-6}\text{MeV}}$, we find that $E_p = 104'447.324\text{TeV}$, the (very large) energy of the proton beam. Using the expression $E = \gamma(v)m$ (which implies $v = \sqrt{1 - m^2/E^2}$), the velocity of the beam is approximately $\sqrt{1 - 8.0698 \times 10^{-17}}\, c \approx c$. Physically, this is impossible to achieve with a proton, which is why the LHC uses two proton beams each of 7 TeV to achieve the 14 TeV CM energy, $s = 14^2\text{TeV}^2 = 4(7^2\text{TeV}^2)$.

(**b**) We now consider the collision process $pp \to ppp\bar{p}$. The threshold energy can be determine by evaluating the invariant mass (which was calculated in part (a)), before and after the collision. Since the frame of reference remains invariant for the invariant mass, we can determine that in the CM frame after the collision, all produced particles with momenta $\mathbf{p}_i$ sum to net momentum $\mathbf{0}$, else violating conservation of energy/momentum. The net squared momentum 4-vector (invariant mass) should then be $(4m_p, \mathbf{0}) \cdot (4m_p, \mathbf{0}) = 16m_p^2$. Setting this equal to our expression (3.1), which is just the invariant mass evaluated in the lab frame prior to the collision, we find

$$\begin{aligned} 2m_p^2 + 2E_p m_p &= 16m_p^2 \\ \implies 2E_p m_p &= 14m_p^2 \\ \implies E_p &= 7m_p, \end{aligned} \tag{3.3}$$

which is approximately $6'567.9$ MeV, and thus this is the threshold energy for producing the anti-proton.

Problem 4

Consider the decay $\alpha \to 1 + 2$. First, note that each particle contains a 4-momentum which must be conserved (this accounts for energy and $\mathbf{p}$ conservation):

$$p_\alpha^\mu = p_1^\mu + p_2^\mu. \tag{4.1}$$

In the CM frame, $\mathbf{p}_\alpha = \mathbf{0}$ and $\mathbf{p_1} = -\mathbf{p_2}$, although the masses m_1 and m_2 may be different. First, note that the square of any energy-momentum 4-vector is equivalent to the squared mass of the particle:

$$\begin{aligned} p_\mu p^\mu &= E^2 - |\mathbf{p}|^2 \\ &= m^2 + |\mathbf{p}|^2 - |\mathbf{p}|^2 \\ &= m^2, \end{aligned} \tag{4.2}$$

hence by squaring both sides of (4.1), we obtain

$$\begin{aligned} m_\alpha^2 &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ &= m_1^2 + m_2^2 + 2(E_1 E_2 - |\mathbf{p_1}||\mathbf{p_2}| \cos\theta) \end{aligned} \tag{4.3}$$

where θ is the (opening) angle in between $\mathbf{p_1}$ and $\mathbf{p_2}$. Since the general expressions for E and $\mathbf{p}$ are $E = \gamma(v)m$ and $\mathbf{p} = \gamma m \mathbf{v}$, respectively, dividing out the $E_1 E_2$ in (4.3) yields

$$\begin{aligned} m_\alpha^2 &= m_1^2 + m_2^2 + 2E_1 E_2 \left(1 - \frac{\gamma(v_1) m_1 \gamma(v_2) m_2 v_1 v_2}{\gamma(v_1) m_1 \gamma(v_2) m_2} \cos\theta\right) \\ &= m_1^2 + m_2^2 + 2E_1 E_2 \left(1 - v_1 v_2 \cos\theta\right) \end{aligned} \tag{4.4}$$

where we have used natual units ($\hbar = c = 1$) and v_1, v_2 are the velocities of the scattered particles.

Problem 5

Consider the Mandelstam variables for the interaction process $1+2 \to 3+4$:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{5.1}$$

By invoking the energy-momentum 4-vector square (determine in equation (4.2) in problem 2), $p^2 = m^2$, and using the fact that momentum is conserved ($p_1^\mu = -p_2^\mu + p_3^\mu + p_4^\mu$), we have that

$$\begin{aligned} s+t+u &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 + p_1^2 + p_3^2 - 2p_1 \cdot p_3 + p_1^2 + p_4^2 - 2p_1 \cdot p_4 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot (p_1 + p_2 - p_3 - p_4) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot (0) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned} \tag{5.2}$$

which is the appropriate value of the Mandelstam variable sum.

Problem 6

Consider the decay $\Lambda \to \pi^- p$ with $p_\pi = 0.75$ GeV, $p_p = 4.25$ GeV and an opening angle of 9°.

(a) The baryon mass can easily be determined by invoking equation (4.3) and using the π^-, p rest energies: $m_\pi = 139.570$ MeV, $m_p = 938.272$ MeV. Plugging in numbers, and taking $\theta = 9^\circ$ as the final momenta seperation angle and plugging in the digits,

$$
\begin{aligned}
m_\Lambda^2 &= m_\pi^2 + m_p^2 + 2\left[\sqrt{m_\pi^2 + p_\pi^2}\sqrt{m_p^2 + p_p^2} - p_\pi p_p \cos\theta\right] \qquad (6.1)\\
\implies m_\Lambda &= \Bigg[(139.57)^2 + (938.272)^2 + 2\big[\sqrt{139.57^2 + (0.75\cdot 10^3)^2}\sqrt{938.272^2 + (4.25\cdot 10^3)^2} \\
&\qquad - (0.75)(4.25)\cdot 10^6\cos(9^\circ)\big]\Bigg]^{1/2} \quad [\text{MeV}]\\
&= 1115.307 \text{ MeV}
\end{aligned}
$$

which is also within the textbook value of the Λ mass (see Griffiths).

(b) For a production length of 0.35m, the lifetime can be determined by first finding the velocity of the Λ (in this decay experiment) and then transforming back into the proper time frame τ. First, as in problem (4), the velocity can be determined via $\mathbf{v} = \dfrac{\mathbf{p}}{E} = \dfrac{\gamma(v)m\mathbf{v}}{\gamma(v)m}$, thus if we take the velocity to be along some axis $\hat{s}$,

$$
\begin{aligned}
v_s &= \frac{p_\Lambda}{E_\Lambda} \qquad (6.2)\\
&= \frac{\left[(\sqrt{m_\pi^2 + p_\pi^2} + \sqrt{m_p^2 + p_p^2})^2 - m_\Lambda^2\right]}{\sqrt{m_\pi^2 + p_\pi^2} + \sqrt{m_p^2 + p_p^2}}\\
&= \dots\\
&= 0.975940444154 \quad \left(\cdot c\,\frac{\text{m}}{\text{s}}\right),
\end{aligned}
$$

(I'm sorry, I really don't want to waste my time copying numbers into LaTeX when this can just be done in a calculator) where the units $c = 1$ are still invoked. To determine the lifetime in the lab frame, however, the factors of c will need to be re-introduced:

$$
\begin{aligned}
t_L &= \frac{d_L}{v_L}\\
\implies \tau_\Lambda &= \frac{d}{\gamma(v)vc} \qquad (6.3)\\
&= \sqrt{1 - v_s^2}\cdot\frac{0.38\text{m}}{v_s\cdot c}\\
&= \dots\\
&= 2.60828\times 10^{-10}\text{ s}
\end{aligned}
$$

which is the lifetime of the Λ as measured in its rest frame. Comparing this with the textbook value for τ_Λ (again, see Griffiths) this value is within the correct order of magnitude.

Problem 7

Consider the decay of the neutral pion $\pi^0 \to \gamma\gamma$. We wish to determine the minimum opening angle for a π^0 of 10 GeV. Note first that since the decay products have no mass (and equal mass), and in the CM frame of the pion, both products must have equal and opposite momenta.

For the pion at rest in the CM frame with some arbitrary axis $\hat{s}$, suppose that the photons decay oppositely along the $\pm\hat{s}$ direction. This means that the boost of the π^0 from the $\hat{s}$ direction into the lab frame cannot overcome the momenta difference created by the backwards-moving γ travelling at speed c, which means that this decay mode violates causality (in the lab frame). The opening angle is then 180° in the CM frame, which constitutes a maximum value. Thus, for a minimum opening angle, both photons must decay along an axis $\hat{s}'$ in the CM frame, where $|\hat{s} \times \hat{s}'| \neq 0$ ($\hat{s}'$ is not parallel to $\hat{s}$), which constitutes a symmetric decay.

Under this symmetry requirement, both photons must have an equal energy and momentum for it to be conserved throughout the decay process as a whole. We then have that

$$E_\pi = 2E_\gamma. \tag{7.1}$$

By using the mass relation (4.3)/(6.1), and letting $m_\gamma = 0$ (which implies $E_\gamma = p_\gamma$ since $\hbar = c = 1$), the equation reduces to

$$\begin{aligned} m_\pi^2 &= 2E_\gamma^2(1 - \cos\theta) \\ &= E_\pi^2 \sin^2 \frac{\theta}{2} \\ \implies \theta &= 2 \arcsin\left(\frac{m_\pi}{E_\pi}\right) \end{aligned} \tag{7.3}$$

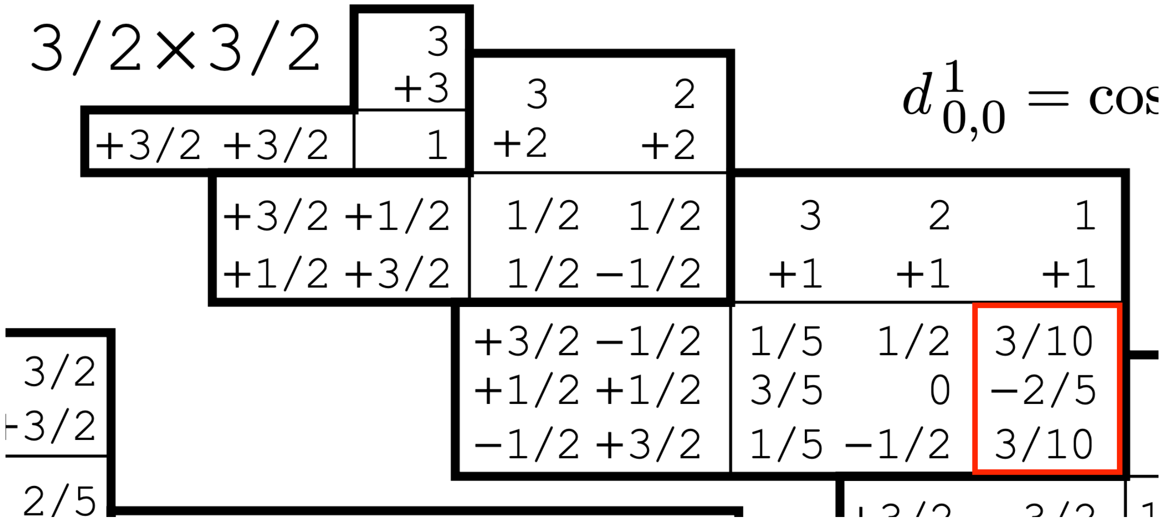
which is the minimum opening angle for this energy value of E_π. Taking $m_\pi = 134.977$ MeV and $E_\pi = 10$ GeV = $10'000$ MeV, the opening angle calculates to

$$\theta = 2 \arcsin\left(\frac{134.977}{10'000}\right) = 1.546^\circ \tag{7.4}$$

as desired.

Problem 8

Consider two spin-$\frac{3}{2}$ particles whose spin state is given by $|1,1\rangle$ with no relative orbital angular momentum. To determine the probability of measuring two identical-spin particles via S_z, we can consult the Clebshe-Gordon coefficient table for $\frac{3}{2} \times \frac{3}{2}$ particles:



We require $m_1 + m_2 = 1$ by the z-addition of the spins, and the probability expansion coefficients can be found to be $\sqrt{\frac{3}{10}}, -\sqrt{\frac{2}{5}}$, and $\sqrt{\frac{3}{10}}$ respectively for the various values of m_1, m_2. Thus we have:

$$
\begin{aligned}
|j,m\rangle &= \sum_{j=|j_1-j_2|}^{|j_1+j_2|} C^{j\,j_1\,j_2}_{m\,m_1,m_2} |j,m_1\rangle\,|j,m_2\rangle \\
|1,1\rangle &= C^{1\,3/2\,3/2}_{1,m_1,m_2} \left|\frac{3}{2},m_1\right\rangle \left|\frac{3}{2},m_2\right\rangle \\
&= \sqrt{\frac{3}{10}}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{3}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{3}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{3}{2},\frac{3}{2}\right\rangle.
\end{aligned}
$$

This implies that the measurement of the two identical spin states yields the probability

$$
\begin{aligned}
\left|\left(\left\langle\frac{3}{2},\frac{1}{2}\right|\left\langle\frac{3}{2},\frac{1}{2}\right|\right)|1,1\rangle\right|^2 &= \left|-\sqrt{\frac{2}{5}}\right|^2 \\
&= \frac{2}{5}
\end{aligned}
$$

which is 40%.