PHY489 Problem Set 4

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Problem 1

Consider the Dirac equation, given (in natural units) by

$$(i\hbar\not{\partial} - mc)\psi = 0 \tag{1.1}$$

for a Dirac bispinor ψ, and $\not{\partial} \equiv \gamma^\mu \partial_\mu$ (Einstein summation convention is implied on repeated indices). Observe that by operating on (1.1) with the operator $i\not{\partial}$, we can observe that, under spinor component contraction, the components of ψ obey the Klein-Gordon equation

$$(-\hbar^2\Box - m^2c^2)\psi = 0 \tag{1.2}$$

where $\Box \equiv \partial^\mu\partial_\mu$ with Minkowski metric $g^{\mu\nu}$ with signature $(+\,-\,-\,-)$. We have that

$$\begin{aligned} 0 &= i\gamma^\nu\partial_\nu(i\hbar\gamma^\mu\partial_\mu - mc)\psi && (1.3)\\ &= -\hbar\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu\psi - imc\gamma^\nu\partial_\nu\psi. && (1.4)\end{aligned}$$

First, note that by the properties of the γ matrices, that the first term under the μ, ν contractions reduces to the box operator in (1.2):

$$\begin{aligned}\gamma^\nu\gamma^\mu\partial_\mu\partial_\nu = (\gamma^0)^2\partial_0^2 + (\gamma^1)^2\partial_1^2 + (\gamma^2)^2\partial_2^2 + (\gamma^3)^2\partial_3^2 \\ + (\gamma^0\gamma^1 + \gamma^1\gamma^0)\partial_0\partial_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0)\partial_0\partial_2 + \dots \end{aligned} \tag{1.5}$$

where we now require all of the anti-commutation $\mu \neq \nu$ terms to be zero, and $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, hence reinforcing the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Under this algebra, we have

$$\begin{aligned}\gamma^\nu\gamma^\mu\partial_\mu\partial_\nu &= g^{\mu\nu}\partial_\mu\partial_\nu \\ &= \Box.\end{aligned} \tag{1.6}$$

Secondly, observe that the Dirac equation can be substituted into the second term in (1.4), hence

$$\begin{aligned} imc\not{\partial}\psi &= imc\left(\frac{mc}{i\hbar}\psi\right) && (1.7)\\ &= \frac{m^2c^2}{\hbar}\psi. && (1.8)\end{aligned}$$

Putting (1.6) and (1.8) back into (1.4), we regain the Klein-Gordon equation acting on the individual components of the bispinor:

$$\begin{aligned} 0 &= -\hbar\Box\psi - \frac{m^2c^2}{\hbar}\psi && (1.9)\\ \implies& (-\hbar^2\Box - m^2c^2)\psi = 0 && (1.10)\end{aligned}$$

as desired.

Problem 2

For this problem, we invoke the Dirac algebra to prove some useful identities of γ matrices. The gamma matrices obey the Dirac algebra under the anticommutation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

(**a**) Before proceeding, note that any contraction of two gamma matrices yields 4:

$$\begin{aligned} \gamma^\mu \gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu & (2.1)\\ &= (1)(\gamma^0)^2 + (-1)(\gamma^1)^2 + (-1)(\gamma^2)^2 + (-1)(\gamma^3)^2 & (2.2)\\ &= (1)(1) + 3(-1)(-1) = 4 & (2.3) \end{aligned}$$

since $\beta^2 \equiv (\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, and any component $\mu \neq \nu$ is zero by the metric tensor. We have that, for any four vector a^μ,

$$\begin{aligned} \gamma^\mu \not{a} \gamma_\mu &= \gamma^\mu \gamma^\nu a_\nu \gamma_\mu & (2.4)\\ &= (2g^{\mu\nu} - \gamma^\nu\gamma^\mu) a_\nu \gamma_\mu & (2.5)\\ &= 2g^{\mu\nu} a_\nu \gamma_\mu - \gamma^\nu a_\nu \gamma^\mu \gamma_\mu & (2.6)\\ &= 2\not{a} - 4\not{a} & (2.7)\\ &= -2\not{a} & (2.8) \end{aligned}$$

since a_ν is just number (a component of a four-vector), and the second term in (2.7) follows from (2.3).

(**b**) For two four vectors a^μ and b^μ, we may also perform a similar contraction and anticommute the gamma matrices through twice:

$$\begin{aligned} \gamma^\mu \not{a}\not{b} \gamma_\mu &= \gamma^\mu \gamma^\lambda a_\lambda \gamma^\nu b_\nu \gamma_\mu & (2.9)\\ &= (2g^{\mu\lambda} - \gamma^\lambda\gamma^\mu) a_\lambda \gamma^\nu b_\nu \gamma_\mu & (2.10)\\ &= 2g^{\mu\lambda} a_\lambda \gamma^\nu b_\nu \gamma_\mu - \gamma^\lambda a_\lambda (2g^{\mu\nu} - \gamma^\nu\gamma^\mu) b_\nu \gamma_\mu & (2.11)\\ &= 2\gamma^\lambda a_\lambda \gamma^\nu b_\nu - 2\gamma^\lambda a_\lambda \gamma^\nu b_\nu + \gamma^\lambda a_\lambda \gamma^\nu b_\nu \gamma^\mu \gamma_\mu & (2.12)\\ &= 2\not{a}\not{b} - 2\not{a}\not{b} + 4\not{a}\not{b} & (2.13)\\ &= 4\not{a}\not{b}. & (2.14) \end{aligned}$$

Lastly, as in what was derived in (1.5), but now for two arbitrary four vectors a^μ and b^μ, the product of their Dirac contractions reduces to their dot product:

$$\begin{aligned} \gamma^\mu a_\mu \gamma^\nu b_\nu = (\gamma^0)^2 a_0 b_0 + (\gamma^1)^2 a_1 b_1 + (\gamma^2)^2 a_2 b_2 + (\gamma^3)^2 a_3 b_3 \\ + (\gamma^0\gamma^1 + \gamma^1\gamma^0) a_0 b_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0) a_0 b_2 + \dots \quad (2.15) \end{aligned}$$

which then reduces to $g^{\mu\nu} a_\mu b_\nu$, following the Dirac alegbra (any anticommutation goes to zero, and $\beta^2 = 1$, $(\gamma^i)^2 = -1$). This implies that (2.14) becomes

$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4a \cdot b \quad (2.16)$$

which is what I wanted to show.

(**c**) Define the "fifth gamma matrix" by the pseudoscalar product $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. I will show that $(\gamma^5)^2 = 1$ following the Dirac alegbra. Since for any $\mu \neq \nu$, the gamma matrices anticommute $\{\gamma^\mu, \gamma^\nu\} = 0$, we have that

$$(\gamma^5)^2 = i^2 \beta\gamma^1\gamma^2\gamma^3\beta\gamma^1\gamma^2\gamma^3 \quad (2.17)$$

requires an even number of commutations to group the matrices together. Allow me to show this explicitly:

$$
\begin{aligned}
(\gamma^5)^2 &= -(-1)^3\beta^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && (2.18)\\
&= -(-1)^3(1)(-1)^2(\gamma^1)^2\gamma^2\gamma^3\gamma^2\gamma^3 && (2.19)\\
&= -(-1)^3(-1)^2(-1)(-1)(\gamma^2)^2(\gamma^3)^2 && (2.20)\\
&= -(-1)^3(-1)^2(-1)^2(-1)^2\\
&= -(-1)^9\\
&= +1 && (2.21)
\end{aligned}
$$

since any interchange of two gamma matrices yields a factor of (-1).

(d) We may also use the similar anticommutation algebra as in (c) to show that $\{\gamma^5, \gamma^\mu\} = 0$. We have that

$$\gamma^5\gamma^\mu = i\beta\gamma^1\gamma^2\gamma^3\gamma^\mu \tag{2.22}$$

where $\mu = 0, \ldots, 3$. If $\mu = 0$, then we must anticommute β through $\gamma^3, \gamma^2, \gamma^1$ to yield a factor of $(-1)^3$. Similarly if $\mu = 1$, γ^1 anticommutes through γ^3, γ^2 and β to yield another factor of $(-1)^3$ to anticommute the matrix all the way to the other side. This is equivalent for $\mu = 2, 3$ as well, hence always yielding a factor of $(-1)^3$, thus

$$\gamma^5\gamma^\mu = (-1)^3\gamma^\mu\gamma^5 = -\gamma^\mu\gamma^5 \tag{2.23}$$

which implies that $\{\gamma^5\gamma^\mu\} = 0$.

(e) Consider now two Dirac four-component bispinors ψ, and φ. We define the Dirac adjoint as the Hermitian spinor contracted with the β gamma matrix: $\overline{\psi} \equiv \psi^\dagger\beta$. Let $j = \overline{\psi}\gamma^\mu\varphi$ be the contraction of two spinors in product with a gamma matrix (note that j is a 1×1 matrix). Let us calculate the Hermitian conjugate of j, $j^\dagger$.

First observe the Hermitian conjugate of any γ matrix is simply the contraction with the metric tensor (lowered index): $(\gamma^\mu)^\dagger = \gamma_\mu$. Explicitly,

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies \beta^\dagger = \beta \tag{2.24}$$

$$\gamma^i = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix} \implies (\gamma^i)^\dagger = -\gamma^i \tag{2.25}$$

since the Pauli matrices are Hermitian, and each matrix entry is 2×2 (making 4×4 matrices), hence $(\gamma^\mu)^\dagger = g_{\mu\nu}\gamma^\nu$. Furthermore, notice that

$$
\begin{aligned}
(\gamma^\mu)^\dagger &= \beta\gamma^i\beta && (2.26)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (2.27)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -\alpha_i \\ -\alpha_i & 0 \end{pmatrix} && (2.28)\\
&= \begin{pmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{pmatrix} = -\gamma^\mu && (2.29)
\end{aligned}
$$

which allows us to write the Hermitian conjugate of the gamma matrices in the condensed form $(\gamma^\mu)^\dagger = \beta\gamma^\mu\beta$. All of this implies

$$j^\dagger = (\overline{\psi}\gamma^\mu\varphi)^\dagger \tag{2.30}$$
$$= \varphi^\dagger(\gamma^\mu)^\dagger\beta^\dagger(\psi^\dagger)^\dagger \tag{2.31}$$
$$= \varphi^\dagger\beta\gamma^\mu\beta\beta\psi \tag{2.32}$$
$$= \overline{\varphi}\gamma^\mu\psi \tag{2.32}$$

since $\beta^2 = 1$.

Problem 3

In this problem we consider the explicit $\mathbf{p} \neq \mathbf{0}$ plane wave constant four-component bispinor solutions to the Dirac equation $\psi = a u_p^{(r)} e^{ip\cdot x}$ (for particles, and $v_p^{(s)}$ for antiparticles). I will show that the relativistic mass-shell relation $E^2 = m^2c^4 + p^2c^2$ is recovered from the momentum space dirac equation $(\not{p} - mc)u_p^{(1)} = 0$ when $u_p^{(1)}$ (or in general, if any of the spinors) are substituted into the equation. In matrix form, we have that

$$u_p^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_3}{E+mc^2} \\ \dfrac{c(p_1+ip_2)}{E+mc^2} \end{pmatrix} \tag{3.1}$$

and the four-momentum gamma matrix contraction is

$$\not{p} = \gamma^\mu p_\mu = \beta\frac{E}{c} - \gamma^i p_i \tag{3.2}$$

$$= \begin{pmatrix} E/c & 0 & 0 & 0 \\ 0 & E/c & 0 & 0 \\ 0 & 0 & -E/c & 0 \\ 0 & 0 & 0 & -E/c \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & p_1 \\ 0 & 0 & p_1 & 0 \\ 0 & -p_1 & 0 & 0 \\ -p_1 & 0 & 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} 0 & 0 & 0 & -ip_2 \\ 0 & 0 & ip_2 & 0 \\ 0 & ip_2 & 0 & 0 \\ -ip_2 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & -p_3 \\ -p_3 & 0 & 0 & 0 \\ 0 & p_3 & 0 & 0 \end{pmatrix} \tag{3.3}$$

$$= \begin{pmatrix} E/c & 0 & -p_3 & -p_1+ip_2 \\ 0 & E/c & -p_1-ip_2 & p_3 \\ p_3 & p_1-ip_2 & -E/c & 0 \\ p_1+ip_2 & 0-p_3 & 0 & -E/c \end{pmatrix} \tag{3.4}$$

When subtracted by $\mathbf{1}mc$, we have the explicit form of the Dirac equation in momentum space (where $\mathbf{1}$ is just the 4×4 identity matrix):

$$\not{p} - mc = \begin{pmatrix} E/c-mc & 0 & -p_3 & -p_1+ip_2 \\ 0 & E/c-mc & -p_1-ip_2 & p_3 \\ p_3 & p_1-ip_2 & -E/c-mc & 0 \\ p_1+ip_2 & 0-p_3 & 0 & -E/c-mc \end{pmatrix} \tag{3.5}$$

Matrix multiplication by (3.1) yields

$$(\not{p} - mc)u_p^{(1)} = \begin{pmatrix} E/c-mc & 0 & -p_3 & -p_1+ip_2 \\ 0 & E/c-mc & -p_1-ip_2 & p_3 \\ p_3 & p_1-ip_2 & -E/c-mc & 0 \\ p_1+ip_2 & 0-p_3 & 0 & -E/c-mc \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_3}{E+mc^2} \\ \dfrac{c(p_1+ip_2)}{E+mc^2} \end{pmatrix} \tag{3.6}$$

$$= \begin{pmatrix} E/c - mc \\ 0 \\ p_3 \\ p_1 + ip_2 \end{pmatrix} + \begin{pmatrix} \dfrac{-cp_3^2}{E + mc^2} \\ \dfrac{cp_3(-p_1 - ip_2)}{E + mc^2} \\ \dfrac{cp_3(-E/c - mc)}{E + mc^2} \\ 0 \end{pmatrix} + \begin{pmatrix} \dfrac{c(p_1 + ip_2)(-p_1 + ip_2)}{E + mc^2} \\ \dfrac{cp_3(p_1 + ip_2)}{E + mc^2} \\ 0 \\ \dfrac{c(p_1 + ip_2)(-E/c - mc)}{E + mc^2} \end{pmatrix} \tag{3.7}$$

$$= \begin{pmatrix} E/c - mc - \dfrac{cp_3^2}{E + mc^2} + \dfrac{c(p_1 + ip_2)(-p_1 + ip_2)}{E + mc^2} \\ \dfrac{cp_3(-p_1 - ip_2)}{E + mc^2} + \dfrac{cp_3(p_1 + ip_2)}{E + mc^2} \\ p_3 + \dfrac{cp_3(-E/c - mc)}{E + mc^2} \\ p_1 + ip_2 + \dfrac{c(p_1 + ip_2)(-E/c - mc)}{E + mc^2} \end{pmatrix}. \tag{3.8}$$

We can now consider each of the four rows individually. The second row immediately vanishes, as

$$-\frac{cp_3(p_1 + ip_2)}{E + mc^2} + \frac{cp_3(p_1 + ip_2)}{E + mc^2} = 0. \tag{3.9}$$

Similarly, the third row gives zero when the fraction is simplified:

$$p_3 + \frac{cp_3(-E/c - mc)}{E + mc^2} = p_3 - p_3 \frac{E + mc^2}{E + mc^2} \tag{3.10}$$

$$= p_3 - p_3 = 0. \tag{3.11}$$

The fourth row also gives zero following the exact same calculation as in (3.11). The first row, however, can be expanded and simplified:

$$\frac{E}{c} - mc - \frac{cp_3^2}{E + m^2} - \frac{c(p_1 + ip_2)(p_1 - ip_2)}{E + mc^2} = \frac{E}{c} - mc - \frac{c(p_1^2 + p_2^2 + p_3^2)}{E + mc^2} \tag{3.12}$$

$$= (E - mc^2)(E + mc^2) - c^2|\mathbf{p}|^2 \tag{3.13}$$

$$= E^2 - m^2c^4 - c^2p^2 \tag{3.14}$$

which must be set to zero, hence regaining the mass-shell energy relation

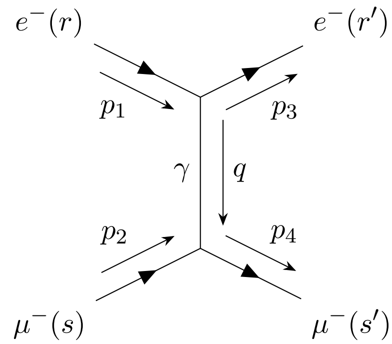
$$0 = E^2 - m^2c^4 - c^2p^2 \implies E^2 = m^2c^4 + p^2c^2 \tag{3.15}$$

as desired. The exact same calculation can be taken out for $(\not{p} - mc)u_p^{(2)} = 0$ as well as for the antifermion spinors $(\not{p} + mc)v_p^{(s)} = 0$.

Problem 4

In this problem, we consider electron-muon scattering $e^- \mu^- \to e^- \mu^-$ with the modification that the photon propagator is now a massive spin-0 force carrier. At each vertex, we now include a factor of $ig_e\mathbf{1}$ where $\mathbf{1}$ is the four-by-four unit matrix, and g_e is the electromagnetic coupling constant. Since the photon is no longer a massless polarizable force carrier, the propagator goes into $\dfrac{-i}{q^2 - m_\gamma^2 c^2}$.

(**a**) To determine the Feynman amplitude for the $\mathcal{O}(g_e^2)$ matrix element, we must consider the diagrammatic perturbation of the scattering at tree level. There is only one contributing diagram, since the electron and muon are distinguishable particles:



where I have denoted the spins of the particles (r, s) before and (r', s') after. According to the Fynman rules, we work backwards from each fermion line, including each encountered vertex factor:

$$\overline{u}_{p_3}^{(r')}(ig_e\mathbf{1})u_{p_1}^{(r)} \quad \text{[electron]} \tag{4.1}$$

$$\overline{u}_{p_4}^{(s')}(ig_e\mathbf{1})u_{p_2}^{(s)} \quad \text{[muon]} \tag{4.2}$$

We also have two delta functions at each vertex, and we integrate over all internal momenta. (4.1), (4.2) in product then allows us to calculate the amplitude:

$$\begin{aligned} i\mathcal{A}_{fi} = \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 - m_\gamma^2 c^2} \left[\overline{u}_{p_3}^{(r')}(ig_e\mathbf{1})u_{p_1}^{(r)} \overline{u}_{p_4}^{(s')}(ig_e\mathbf{1})u_{p_2}^{(s)} \right] \\ \times (2\pi)^4 \delta^{(4)}(p_2 + q - p_4)(2\pi)^4 \delta^{(4)}(p_1 - q - p_3) \end{aligned} \tag{4.3}$$

Taking out the integral and bringing all constants outside of the integrand, also observing that for any two spinors $u_p^{(r)}$ and u_k^s, $\overline{u}_p^{(r)}\mathbf{1}u_k^s = \overline{u}_p^{(r)}u_k^s$ (the identity matrix does not act on the spinors at all, and the Dirac adjoints reduce to a single number), we have that

$$i\mathcal{A}_{fi} = \frac{ig_e^2}{(p_4 - p_2)^2 - m_\gamma^2 c^2} \left[\overline{u}_{p_3}^{(r')} u_{p_1}^{(r)} \overline{u}_{p_4}^{(s')} u_{p_2}^{(s)} \right] \tag{4.4}$$

where I have also cancelled the second delta function $\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ by energy-momentum conservation. Cancelling the second i (for brevity), we obtain the Feynman amplitude at $\mathcal{O}(g_e^2)$,

$$\mathcal{A}_{fi} = \frac{g_e^2}{(p_4 - p_2)^2 - m_\gamma^2 c^2} \left[\overline{u}_{p_3}^{(r')} u_{p_1}^{(r)} \overline{u}_{p_4}^{(s')} u_{p_2}^{(s)} \right]. \tag{4.5}$$

(**b**) We now take the spin-average square of the Feynman amplitude $|\mathcal{A}_{fi}|^2$. First, we write the amplitude in terms of seperate spinor contractions for both the electron and muon:

$$\mathcal{A}_{fi} = \frac{g_e^2}{(p_4 - p_2)^2 - m_\gamma^2 c^2} \left[\overline{u}_{p_3}^{(r')} u_{p_1}^{(r)}\right] \left[\overline{u}_{p_4}^{(s')} u_{p_2}^{(s)}\right]. \tag{4.6}$$

Taking the complex square then yields

$$|\mathcal{A}_{fi}|^2 = \frac{g_e^4}{[(p_4 - p_2)^2 - m_\gamma^2 c^2]^2} \left[\overline{u}_{p_3}^{(r')} u_{p_1}^{(r)}\right] \left[\overline{u}_{p_3}^{(r')} u_{p_1}^{(r)}\right]^\dagger \left[\overline{u}_{p_4}^{(s')} u_{p_2}^{(s)}\right] \left[\overline{u}_{p_4}^{(s')} u_{p_2}^{(s)}\right]^\dagger \tag{4.7}$$

We can expand the Hermitian conjugate expressions and use (2.26), then invoke the completeness relations of the Dirac bispinors when the initial and final spins are summed over. By (2.32), we have

$$|\mathcal{A}_{fi}|^2 = \frac{g_e^4}{[(p_4 - p_2)^2 - m_\gamma^2 c^2]^2} \left[\overline{u}_{p_3}^{(r')} u_{p_1}^{(r)} \overline{u}_{p_1}^{(r)} u_{p_3}^{(r')}\right] \left[\overline{u}_{p_4}^{(s')} u_{p_2}^{(s)} \overline{u}_{p_2}^{(s)} u_{p_4}^{(s')}\right] \tag{4.8}$$

We aim to invoke the completeness relations (for particles, and antiparticles respectively)

$$\sum_{r=1,2} \overline{u}_p^{(r)} u_p^{(r)} = \not{p} + mc \tag{4.9}$$

$$\sum_{s=1,2} \overline{v}_p^{(s)} v_p^{(s)} = \not{p} - mc \tag{4.10}$$

into (4.8), however we notice that only two of the four spinor pairs are grouped in the $u\overline{u}$ form (namely, the incoming pairs) in (4.10). This is an easy fix however, because we have already defined the Dirac adjoint contraction of a spinor ψ as $\overline{\psi}\psi = \psi\overline{\psi}$, which is just a scalar bilinear form, and hence can commute through other spinor contractions when we take its trace (the trace is invariant under cyclic permutations of factors). Then

$$|\mathcal{A}_{fi}|^2 = \frac{g_e^4}{[(p_4 - p_2)^2 - m_\gamma^2 c^2]^2} \text{tr}\left[u_{p_1}^{(r)} \overline{u}_{p_1}^{(r)} \overline{u}_{p_3}^{(r')} u_{p_3}^{(r')}\right] \text{tr}\left[u_{p_2}^{(s)} \overline{u}_{p_2}^{(s)} \overline{u}_{p_4}^{(s')} u_{p_4}^{(s')}\right] \tag{4.11}$$

and hence, summing over the spins r, r', s, s',

$$\frac{1}{4} \sum_{r,r',s,s'=1,2} |\mathcal{A}_{fi}|^2 = \frac{1}{4} F(p_2, p_4) \sum_{r=1,2} \sum_{r'=1,2} \text{tr}\left[u_{p_1}^{(r)} \overline{u}_{p_1}^{(r)} \overline{u}_{p_3}^{(r')} u_{p_3}^{(r')}\right] \sum_{s=1,2} \sum_{s'=1,2} \text{tr}\left[u_{p_2}^{(s)} \overline{u}_{p_2}^{(s)} \overline{u}_{p_4}^{(s')} u_{p_4}^{(s')}\right] \tag{4.12}$$

$$= \frac{1}{4} F(p_2, p_4) \text{tr}\left[(\not{p}_1 + m_1 c)(\not{p}_3 + m_3 c)(\not{p}_2 + m_2 c)(\not{p}_4 + m_4 c)\right] \tag{4.13}$$

where I have defined the function $F(p_2, p_4) = \dfrac{g_e^4}{[(p_4 - p_2)^2 - m_\gamma^2 c^2]^2}$ to be a function of the outgoing four momenta. To obtain an answer in terms of the four-momenta of the particles, we must expand and simplify the trace of the matrix product in (4.13):

$$\begin{aligned}(\not{p}_1 + m_1 c)&(\not{p}_3 + m_3 c)(\not{p}_2 + m_2 c)(\not{p}_4 + m_4 c) \\ &= (\not{p}_1\not{p}_3 + \not{p}_1 m_3 c + \not{p}_3 m_1 c + m_1 m_3 c^2) \\ &\qquad \times (\not{p}_2\not{p}_4 + \not{p}_2 m_4 c + \not{p}_4 m_2 c + m_2 m_4 c^2)\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
&= \not{p}_1\not{p}_3\not{p}_2\not{p}_4 + \not{p}_1\not{p}_3\not{p}_2 m_4 c + \not{p}_1\not{p}_3\not{p}_4 m_2 c + \not{p}_1\not{p}_3 m_2 m_4 c^2 \\
&\quad + \not{p}_1\not{p}_2\not{p}_4 m_3 c + \not{p}_1\not{p}_2 m_3 m_4 c^2 + \not{p}_1\not{p}_4 m_3 m_2 c^2 + \not{p}_1 m_3 m_2 m_4 c^3 \\
&\quad + \not{p}_3\not{p}_2\not{p}_4 m_1 c + \not{p}_3\not{p}_2 m_1 m_4 c^2 + \not{p}_3\not{p}_4 m_1 m_2 c^2 + \not{p}_3 m_1 m_2 m_4 c^3 \\
&\quad + \not{p}_2\not{p}_4 m_1 m_3 c^2 + \not{p}_2 m_1 m_3 m_4 c^3 + \not{p}_4 m_1 m_2 m_3 c^3 + m_1 m_2 m_3 m_4 c^4. \qquad (4.15)
\end{aligned}
$$

Although this expression may appear complicated, it simplifies significantly once we take it's trace. This is because the trace of any odd number of gamma matrices is zero (this is easy to verify) (4.16). Furthermore, we may use the metric identities (4.17), (4.18)

$$
\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0 \qquad (4.16)
$$

$$
\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \qquad (4.17)
$$

$$
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}). \qquad (4.18)
$$

For instance, for any three 'slashed' four-vectors, it's trace is zero:

$$
\mathrm{tr}(\not{a}\not{b}\not{c}) = a_\mu b_\nu c_\lambda \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0 \qquad (4.19)
$$

and similarly for any single slashed four-vector. Taking the trace of (4.15) then reduces the expression to

$$
\begin{aligned}
&\mathrm{tr}[(\not{p}_1 + m_1 c)(\not{p}_3 + m_3 c)(\not{p}_2 + m_2 c)(\not{p}_4 + m_4 c)] \\
&\quad = \mathrm{tr}[\not{p}_1\not{p}_3\not{p}_2\not{p}_4] + m_2 m_4 c^2 \mathrm{tr}(\not{p}_1\not{p}_3) + m_3 m_4 c^2 \mathrm{tr}(\not{p}_1\not{p}_2) + m_2 m_3 c^2 \mathrm{tr}(\not{p}_1\not{p}_4) \\
&\quad\quad + m_1 m_4 c^2 \mathrm{tr}(\not{p}_3\not{p}_2) + m_1 m_2 c^2 \mathrm{tr}(\not{p}_3\not{p}_4) + m_1 m_3 c^2 \mathrm{tr}(\not{p}_2\not{p}_4) + m_1 m_2 m_3 m_4 c^4 \mathrm{tr}(\mathbf{1}). \qquad (4.19)
\end{aligned}
$$

The last trace simply reduces to 4. The traces of any two four-momenta i, j go to

$$
\begin{aligned}
\mathrm{tr}(\not{p}_i\not{p}_j) &= p_{i\mu} p_{j\nu} \mathrm{tr}(\gamma^\mu\gamma^\nu) &(4.20)\\
&= 4 p_{i\mu} p_{j\nu} g^{\mu\nu} &(4.21)\\
&= 4 p_i \cdot p_j. &(4.22)
\end{aligned}
$$

We may similarly expand the first term in the trace, utilizing (4.18), to find

$$
\begin{aligned}
\mathrm{tr}(\not{p}_1\not{p}_3\not{p}_2\not{p}_4) &= p_{1\mu} p_{3\nu} p_{2\lambda} p_{4\sigma} \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) &(4.23)\\
&= 4 p_{1\mu} p_{3\nu} p_{2\lambda} p_{4\sigma} (g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) &(4.24)\\
&= 4[(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]. &(4.25)
\end{aligned}
$$

Implementing these into (4.19), we find the trace to be

$$
\begin{aligned}
&\mathrm{tr}[(\not{p}_1 + m_1 c)(\not{p}_3 + m_3 c)(\not{p}_2 + m_2 c)(\not{p}_4 + m_4 c)] \\
&\quad = 4[(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] + 4 m_2 m_4 c^2 p_1 \cdot p_3 \\
&\quad\quad + 4 m_3 m_4 c^2 p_1 \cdot p_2 + 4 m_2 m_3 c^2 p_1 \cdot p_4 + 4 m_1 m_4 c^2 p_3 \cdot p_2 \\
&\quad\quad + 4 m_1 m_2 c^2 p_3 \cdot p_4 + 4 m_1 m_3 c^2 p_2 \cdot p_4 + 4 m_1 m_2 m_3 m_4 c^4 \qquad (4.26)
\end{aligned}
$$

which is indeed a scalar value. (4.13) then gives the appropriate squared Feynman amplitude, using (4.26):

$$\frac{1}{4}\sum_{r,r',s,s'=1,2}|\mathcal{A}_{fi}|^2 = \frac{1}{4}\frac{4g_e^2}{[(p_4-p_2)^2-m_\gamma^2c^2]}\Big[(p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)$$
$$+m_2m_4c^2(p_1\cdot p_3)+m_3m_4c^2(p_1\cdot p_2)+m_2m_3c^2(p_1\cdot p_4)+m_1m_4c^2(p_2\cdot p_3)$$
$$+m_1m_2c^2(p_3\cdot p_4)+m_1m_3c^2(p_2\cdot p_4)+m_1m_2m_3m_4c^4\Big] \tag{4.27}$$

where the momenta of the incoming/outgoing particles are depicted in the above Feynman diagram (these will eventually be rescaled in the calculation of the differential cross-section by the energy-momentum conserving delta function), and $m_1 = m_3 = m_e$, $m_2 = m_4 = m_\mu$.

(c) If we consider a high-energy limit where $m_e = m_\mu = 0$, the squared amplitude (4.27) reduces significantly, and we may calculate the differential cross-section $\frac{d\sigma}{d\Omega}$, given by

$$d\sigma = \frac{1}{\sqrt{(p_1\cdot p_2)^2-(m_1m_2c^2)^2}}\int_{\text{FINAL STATES}}|\mathcal{A}_{fi}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}. \tag{4.28}$$

In the massless limit, any energy $E_i = |\mathbf{p}_i|c$. In the center of mass frame, $\mathbf{p}_1 = -\mathbf{p_2}$, so with these simplifications the first term (outside the integrand) goes into (see problem set 3, question 4b)

$$\sqrt{(p_1\cdot p_2)^2-(m_1m_2c^2)^2} = (E_1+E_2)\frac{|\mathbf{p}_1|}{c} = 2|\mathbf{p}_1|^2. \tag{4.29}$$

I will also break the energy-momentum conserving four-delta function into energy and momentum components seperately:

$$\delta^{(4)}(p_1+p_2-p_3-p_4) = \delta(p_1c+p_2c-p_3c-p_4c)\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) \tag{4.30}$$
$$= \delta(2p_1c-p_3c-p_4c)\delta^{(3)}(\mathbf{p}_3+\mathbf{p}_4) \tag{4.31}$$

since both particles are massless and $\mathbf{p}_1 = -\mathbf{p}_2$ in the center of mass frame. Due to the massless high-energy limit, the delta functions in (4.31) enforce that $\mathbf{p}_3 = -\mathbf{p}_4$ as well. At this point, it should really be argued whether or not the particles are still distiguishable or not, because both have the same mass and momenta. In this case we would need to include a statistical factor of $\frac{1}{2}$ in (4.28) as well, but I will still ride the assumption that no degeneracies occur following from the previous parts of this problem. The differential cross section is then

$$d\sigma = \frac{1}{8(2\pi)^2|\mathbf{p}_1|^2}\int|\mathcal{A}_{fi}|^2\frac{d^3p_3d^3p_4}{|\mathbf{p}_3||\mathbf{p}_4|c^2}\delta(2p_1c-p_3c-p_4c)\delta^{(3)}(\mathbf{p}_3+\mathbf{p}_4) \tag{4.32}$$
$$= \frac{1}{8(2\pi)^2|\mathbf{p}_1|^2}\int\left[\frac{g_e^2}{[(p_4+p_1)^2-m_\gamma^2c^2]^2}[(p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)]\right]$$
$$\times\frac{d^3p_3d^3p_4}{|\mathbf{p}_3||\mathbf{p}_4|c^2}\delta(2p_1c-p_3c-p_4c)\delta^{(3)}(\mathbf{p}_3+\mathbf{p}_4) \tag{4.33}$$
$$= \frac{1}{8(2\pi)^2|\mathbf{p}_1|^2}\frac{g_e^2}{[(p_1+p_4)^2-m_\gamma^2c^2]^2}[(p_1\cdot p_4)^2-(p_1)^2(p_4)^2+(p_1\cdot p_4)^2]\frac{d^3p_4}{|\mathbf{p}_4|^2c^2}\delta(2p_1c-2p_4c) \tag{4.34}$$

$$= \frac{1}{4(2\pi)^2|\mathbf{p}_1|^2}\frac{g_e^2}{[(p_1+p_4)^2-m_\gamma^2c^2]^2}(p_1\cdot p_4)^2\frac{d^3p_4}{|\mathbf{p}_4|^2c^2}\delta(2p_1c-2p_4c). \tag{4.35}$$

where we have invoked the mass-shell relations $p_1^2 = m_e^2 = 0$, similarly for p_4^2. In (4.34) we take out the 3-momentum space integral d^3p_3, which by the delta-function sends the three-momentum $\mathbf{p}_3 \to -\mathbf{p}_4$. The four-momentum dot product simplifies to

$$\begin{aligned}
p_1\cdot p_4 &= \frac{E_1E_4}{c^2} - \mathbf{p}_1\cdot\mathbf{p}_4 && (4.36)\\
&= |\mathbf{p}_1||\mathbf{p}_4| - |\mathbf{p}_1||\mathbf{p}_4|\cos\theta && (4.37)\\
&= |\mathbf{p}_1||\mathbf{p}_4|(1-\cos\theta) && (4.38)\\
\implies (p_1\cdot p_4)^2 &= |\mathbf{p}_1|^2|\mathbf{p}_4|^2(1-\cos\theta)^2 && (4.39)
\end{aligned}$$

where θ is the scattering angle. We therefore have

$$\begin{aligned}
d\sigma &= \frac{g_e^2}{4(2\pi)^2c^2}\frac{|\mathbf{p}_1|^2|\mathbf{p}_4|^2}{|\mathbf{p}_1|^2|\mathbf{p}_4|^2}\frac{(1-\cos\theta)^2\delta(2p_1c-2p_4c)}{[(p_1+p_4)^2-m_\gamma^2c^2]^2}d^3p_4 && (4.40)\\
&= \frac{g_e^2(1-\cos\theta)^2}{4(2\pi)^2c^2}\frac{\delta(2p_1c-2p_4c)}{[(p_1+p_4)^2-m_\gamma^2c^2]^2}p_4^2dp_4d\Omega && (4.41)
\end{aligned}$$

where I am now including the differential solid angle $d\Omega$ (in which $\mathcal{A}_{fi}$ is independent of, so it's ok to take it out of the integrand). Taking out the radial p_4 component of the remaining differentials, to remove the last delta function, we find that $|\mathbf{p}_1| \to |\mathbf{p}_4|$. We can use this to compute the four-vector sum in the denominator:

$$\begin{aligned}
(p_1+p_4)^2 &= \frac{(E_1+E_4)^2}{c^2} - (\mathbf{p}_1+\mathbf{p}_4)^2 && (4.42)\\
&= \frac{|\mathbf{p}_1|^2c^2+2|\mathbf{p}_1||\mathbf{p}_4|c^2+|\mathbf{p}_4|^2c^2}{c^2} - |\mathbf{p}_1|^2-|\mathbf{p}_4|^2-2|\mathbf{p}_1||\mathbf{p}_4|\cos\theta && (4.43)\\
&= 2|\mathbf{p}_1||\mathbf{p}_4|(1-\cos\theta) && (4.44)\\
\implies &= 2|\mathbf{p}_1|^2(1-\cos\theta) && (4.45)
\end{aligned}$$

where the last line follows from the delta function integral. In terms of the incident electron energy $E_1 = p_1c$, we find the differential cross section to be

$$\frac{d\sigma}{d\Omega} = \frac{g_e^2(1-\cos\theta)^2}{4(2\pi)^2c^4}\frac{E_1^2}{[2E_1^2(1-\cos\theta)/c^2-m_\gamma^2c^2]^2} \tag{4.46}$$

where E_1 is now the incident electron energy. Using the identity $(1-\cos\theta)^2 = 2\sin^2\dfrac{\theta}{2}$ and expanding out the c^4 in the denominator yields

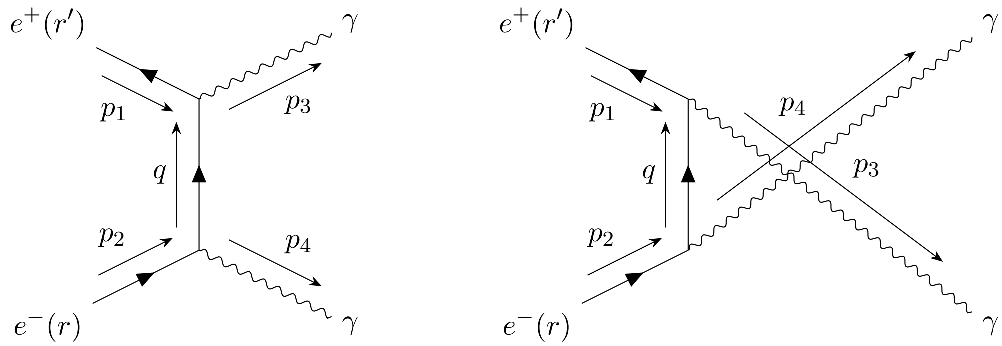
$$\frac{d\sigma}{d\Omega} = \frac{g_e^2\sin^4\frac{\theta}{2}}{4\pi^2}\frac{E_1^2}{[4E_1^2\sin^2\frac{\theta}{2}-m_\gamma^2c^4]^2} \tag{4.47}$$

as desired, and θ is the scattering angle. Backscattering ($\theta = \pi$) then implies a maximum in the differential cross-section.

Problem 5

Next, we consider pair annihilation into gamma rays $e^+e^- \to \gamma\gamma$ using the same spin-0 massive scalar photon model from the previous problem.

(**a**) There are now two leading order $(O)(g_e^2)$ Feynman diagrams due to the degeneracy of the final state (photons are indistinguishable), so we must also include the S factor of $\frac{1}{2}$ for identical final-state particles. We have



where the propagator is now the fermion propagator $\dfrac{i(\not{p} + mc)}{q^2 - m_e^2c^2}$ and we still include the vertex factors of $ig_e\mathbf{1}$. Let's begin constructing the amplitude factors. For the first diagram, we have an incoming fermion and antifermion, and two outgoing "mesons". Working backwards along each fermion line, according to the Feynman rules, we include a Dirac adjoint antispinor $\overline{v}_{p_3}^{(r')}$ for the incoming e^+ and a regular spinor $u_{p_2}^{(r)}$ for the incoming e^-. We have, for the first diagram,

$$i\mathcal{A}_{fi} = \int \frac{d^4q}{(2\pi)^4}\overline{v}_{p_1}^{(r')}(ig_e\mathbf{1})\left[\frac{i(\not{q} + m_ec)}{q^2 - m_e^2c^2}\right](ig_e\mathbf{1})u_{p_2}^{(r)}(2\pi)^4\delta^{(4)}(p_1 + q - p_3)(2\pi)^4\delta^{(4)}(p_2 - q - p_4) \tag{5.1}$$

$$= i^3g_e^2\overline{v}_{p_1}^{(r')}\left[\frac{(\not{p}_2 - \not{p}_4 + m_ec)}{(p_2 - p_4)^2 - m_e^2c^2}\right]u_{p_2}^{(r)}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \tag{5.2}$$

$$= -ig_e^2\overline{v}_{p_1}^{(r')}\left[\frac{(\not{p}_2 - \not{p}_4 + m_ec)}{(p_2 - p_4)^2 - m_e^2c^2}\right]u_{p_2}^{(r)}. \tag{5.3}$$

Since both photons are spinless and indistinguishable, the amplitude for the second diagram is equivalent to that of the first, so we may just omit the statistical factor of $\frac{1}{2}$ and only work with the amplitude determined above. In the case that the photons were (at least) polarizable, there would be a second term in the amplitude arising from the exchange of polarization vectors in the second diagram.

Also notice that the 4×4 identity matrices have been multiplied into the expression (they do not change anything; the matrix element is still a number).

(**b**) Similar to problem 4, we now square the amplitude and take the spin sum-averages of the spinors. Let's take the high energy limit such that the electron and photon masses can be set to zero. Take $m_e = 0$ (m_γ should not be in the above expression anyway), we have that

$$\mathcal{A}_{fi} = -g_e^2\overline{v}_{p_1}^{(r')}\left[\frac{\not{p}_2 - \not{p}_4}{(p_2 - p_4)^2}\right]u_{p_2}^{(r)} \tag{5.4}$$

by which can be expanded under the expansion of the slashed contractions:

$$\mathcal{A}_{fi} = \frac{-g_e^2}{(p_2 - p_4)^2}\left[p_{2\mu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)} - p_{4\nu}\overline{v}_{p_1}^{(r')}\gamma^\nu u_{p_2}^{(r)}\right]. \tag{5.5}$$

Squaring gives

$$|\mathcal{A}_{fi}|^2 = \frac{g_e^4}{(p_2 - p_4)^4}\left[(p_{2\mu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}) - (p_{4\mu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)})\right]\left[(p_{2\nu}\overline{v}_{p_1}^{(r')}\gamma^\nu u_{p_2}^{(r)})^\dagger - (p_{4\nu}\overline{v}_{p_1}^{(r')}\gamma^\nu u_{p_2}^{(r)})^\dagger\right] \tag{5.6}$$

$$= \frac{g_e^4}{(p_2 - p_4)^4}\left[(p_{2\mu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}) - (p_{4\mu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)})\right]\left[(p_{2\nu}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')}) - (p_{4\nu}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')})\right] \tag{5.7}$$

$$= \frac{g_e^4}{(p_2 - p_4)^4}\Big[p_{2\mu}p_{2\nu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')} - p_{2\mu}p_{4\nu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')}$$

$$- p_{2\mu}p_{4\nu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')} + p_{4\mu}p_{4\nu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')}\Big] \tag{5.8}$$

so we must deal with the various spinors and adjoints in product with the gamma matrices. This again can be taken care of by taking the trace of the expression (it is just a number). Again using the fact that the trace of a product of matrices is invariant under cyclic permutations of factors, we can calculate one term and apply it to all four (since the only differing terms are the four-momenta involved). We have (since electrons are "massless" in this limit)

$$\mathrm{tr}(p_{i\mu}p_{j\nu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')}) = p_{i\mu}p_{j\nu}\mathrm{tr}(\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')}) \tag{5.9}$$

$$= p_{i\mu}p_{j\nu}\mathrm{tr}(v_{p_1}^{(r')}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu) \tag{5.10}$$

$$\implies \sum_{r,r'=1,2}\mathrm{tr}(p_{i\mu}p_{j\nu}\overline{v}_{p_1}^{(r')}\gamma^\mu u_{p_2}^{(r)}\overline{u}_{p_2}^{(r)}\gamma^\nu v_{p_1}^{(r')}) = p_{i\mu}p_{j\nu}\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu) \tag{5.11}$$

$$= p_{i\mu}p_{j\nu}p_{1\lambda}p_{2\sigma}\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu) \tag{5.12}$$

$$= 4p_{i\mu}p_{j\nu}p_{1\lambda}p_{2\sigma}(g^{\lambda\mu}g^{\sigma\nu} - g^{\lambda\sigma}g^{\mu\nu} + g^{\lambda\nu}g^{\mu\sigma}) \tag{5.13}$$

$$= 4\left[(p_i\cdot p_1)(p_j\cdot p_2) - (p_1\cdot p_2)(p_i\cdot p_j) + (p_j\cdot p_1)(p_i\cdot p_2)\right] \tag{5.14}$$

thus, averaging over all initial spins yields

$$\frac{1}{2}\sum_{r,r'=1,2}|\mathcal{A}_{fi}|^2 = \frac{2g_e^4}{(p_2 - p_4)^4}\Big[(p_2\cdot p_1)(p_2\cdot p_2) - (p_1\cdot p_2)(p_2\cdot p_2) + (p_2\cdot p_1)(p_2\cdot p_2)$$

$$+ (p_2\cdot p_1)(p_4\cdot p_2) - (p_1\cdot p_2)(p_2\cdot p_4) + (p_4\cdot p_1)(p_2\cdot p_2)$$

$$+ (p_2\cdot p_1)(p_4\cdot p_2) - (p_1\cdot p_2)(p_2\cdot p_4) + (p_4\cdot p_1)(p_2\cdot p_2)$$

$$+ (p_4\cdot p_1)(p_4\cdot p_2) - (p_1\cdot p_2)(p_4\cdot p_4) + (p_4\cdot p_1)(p_4\cdot p_2)\Big]. \tag{5.15}$$

Since all particles are massless, we can compute the inner product for any two four-momenta, then substitute appropriately to simplify (5.15):

$$p_i\cdot p_j = \frac{E_iE_j}{c^2} - \mathbf{p}_i\cdot\mathbf{p}_j \tag{5.16}$$

$$= \frac{|\mathbf{p}_i||\mathbf{p}_j|c^2}{c^2} - |\mathbf{p}_i||\mathbf{p}_j|\cos\alpha_{ij} \tag{5.17}$$

$$= |\mathbf{p}_i||\mathbf{p}_j|(1-\cos\alpha_{ij}) \tag{5.18}$$

where α_{ij} is now the angle between the particles' three-momenta. This implies that for $i=j$, their four-momenta inner product is zero. Furthermore, since the four-momenta inner product is commutative, more terms cancel in (5.15), simplifying the spin-averaged squared Feynman amplitude to be

$$\frac{1}{2}\sum_{r,r'=1,2}|\mathcal{A}_{fi}|^2 = \frac{2g_e^4}{(p_2-p_4)^4}\left[2(p_4\cdot p_1)(p_4\cdot p_2)\right] \tag{5.19}$$

$$= \frac{4g_e^4}{(p_2-p_4)^4}(p_1\cdot p_4)(p_2\cdot p_4) \tag{5.20}$$

which is the spin-averaged amplitude.

(c) We lastly calculate the differential cross section of the pair annihilation in the center of mass frame (as before, $\mathbf{p}_1=-\mathbf{p_2}$ so $p_1^\mu = p_2^\mu$) using (5.19) as the spin-averaged Feynman amplitude. As I mentioned before, the statistical factor would be $\frac{1}{2}$ for photon degeneracy, however both Feynman amplitudes are equivalent and do not differ by a minus sign (the photon is scalar-valued, so there's no polarization to account for), so the $\frac{1}{2}$ factor cancels when amplitudes are summed. We have

$$d\sigma = \frac{1}{2|\mathbf{p_1}|^2}\int_{\text{FINAL STATES}}|\mathcal{A}_{fi}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4} \tag{5.21}$$

$$= \frac{4g_e^4}{8(2\pi)^2|\mathbf{p_1}|^2}\int\frac{(p_1\cdot p_4)^2}{(p_1-p_4)^4}\frac{d^3p_3d^3p_4}{|\mathbf{p}_3||\mathbf{p}_4|c^2}\delta(2p_1c-p_3c-p_4c)\delta^{(3)}(\mathbf{p}_3+\mathbf{p}_4). \tag{5.22}$$

Taking out the d^3p_3 integral removes the 3-delta function and sends $\mathbf{p_3}\to-\mathbf{p}_4$, hence

$$d\sigma = \frac{g_e^4}{2(2\pi)^2|\mathbf{p_1}|^2}\frac{(p_1\cdot p_4)^2}{(p_1-p_4)^4}\frac{d^3p_4}{|\mathbf{p}_4|^2c^2}\delta(2p_1c-2p_4c) \tag{5.23}$$

$$= \frac{g_e^4}{2(2\pi)^2|\mathbf{p_1}|^2}\frac{|\mathbf{p}_1|^2|\mathbf{p}_4|^2(1-\cos\theta)^2}{(p_1-p_4)^4}\frac{d^3p_4}{|\mathbf{p}_4|^2c^2}\delta(2p_1c-2p_4c) \tag{5.24}$$

$$= \frac{g_e^4}{2(2\pi)^2|\mathbf{p_1}|^2c^2}\frac{|\mathbf{p}_1|^2(1-\cos\theta)^2}{(p_1-p_4)^4}\delta(2p_1c-2p_4c)d^3p_4 \tag{5.25}$$

where the last line follows from (5.18) and θ is now the scattering angle. As in the previous problem, we can expand $(p_1-p_4)^4$ and then take out the radial momentum dp_4 integral. Again, notice that

$$(p_1-p_4)^2 = \frac{(|\mathbf{p}_1|c-|\mathbf{p}_4|c)^2}{c^2}-(\mathbf{p}_1-\mathbf{p}_4)^2 \tag{5.26}$$

$$= |\mathbf{p}_1|^2+|\mathbf{p}_4|^2-2|\mathbf{p}_1||\mathbf{p}_4|-|\mathbf{p}_1|^2-|\mathbf{p}_4|^2-2|\mathbf{p}_1||\mathbf{p}_4|\cos\theta \tag{5.27}$$

$$= -2|\mathbf{p}_1||\mathbf{p}_4|(1-\cos\theta) \tag{5.28}$$

$$\implies (p_1-p_4)^4 = 4|\mathbf{p}_1|^2|\mathbf{p}_4|^2(1-\cos\theta)^2. \tag{5.29}$$

Therefore, with $d^3p_4 = p_4^2dp_4d\Omega$, we have the differential cross section

$$\frac{d\sigma}{d\Omega} = \int dp_4\frac{g_e^4}{2(2\pi)^2|\mathbf{p_1}|^2c^2}\frac{|\mathbf{p}_1|^2(1-\cos\theta)^2}{4|\mathbf{p}_1|^2|\mathbf{p}_4|^2(1-\cos\theta)^2}|\mathbf{p}_4|^2\delta(2p_1c-2p_4c) \tag{5.30}$$

$$= \frac{g_e^4}{8(2\pi)^2 E_1^2} \tag{5.31}$$

where $E_1 = |\mathbf{p_1}|c$ is now the incident electron energy. Observe that the differential cross section is independent of the scattering angle, which arises from the high-energy limit taking a massless fermion propagator $m_e = 0$.