University of Toronto
Faculty of Arts and Science

MAT237 Multivariable Calculus with Proofs
Term Test 1

T. Janisse, C. Jonker, and A. Zaman

Friday October 29, 2021

Duration: 110 minutes

No aids permitted

Instructions

- **Do not open the exam until you are instructed to do so.** Failure to comply is an academic offence.
- **No aids are permitted on this examination.** Examples of illegal aids include but are not limited to textbooks, notes, calculators, cellphones, or any electronic device.
- **Once the exam begins, check that you have all pages.** This exam contains 12 pages including this cover page, and is printed **double-sided** on 6 sheets of paper. There are 9 problems.
- **Show your work and justify your steps** on every question, unless otherwise indicated.
- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **If you need more space**, use the blank pages at the end of the exam and clearly indicate on the question page when you have done this. **Do not tear any pages off this exam.**

Academic integrity statement

Full Name: Jace Nicklin Alloway

UToronto email: j.alloway@mail.utoronto.ca

Student number: 1006940802

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Question:	1	2	3	4	5	6	7	8	9	Total
Points:	4	3	3	7	5	5	2	5	6	40
Score:										

Do not tear this page off. This page is a formula sheet and will not be graded under any circumstances. It can be used for rough work only.

$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta \qquad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big) \qquad \sin(A+B) = \sin A\cos B + \cos A \sin B$$

$$\cos A \cos B = \frac{1}{2}\big(\cos(A-B) + \cos(A+B)\big) \qquad \cos(A+B) = \cos A\cos B - \sin A \sin B$$

$$\sin A \cos B = \frac{1}{2}\big(\sin(A+B) + \sin(A-B)\big) \qquad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{d}{dx}(x^a) = ax^{a-1} \qquad \frac{d}{dx}(a^x) = a^x \ln a \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \qquad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = 2.718281\ldots \qquad \pi = 3.14159\ldots \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n} \qquad \sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21}\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31}\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

$$a\cdot b = a^T b = a_1 b_1 + \cdots + a_n b_n = ||a||\,||b||\cos\theta \qquad a\times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \qquad e_1 \times e_2 = e_3$$

$$|a\cdot b| \leq ||a||\,||b|| \qquad ||a+b|| \leq ||a|| + ||b|| \qquad (AB)^T = B^T A^T \qquad (AB)^{-1} = B^{-1}A^{-1}$$

$$(x,y,z) = (r\cos\theta, r\sin\theta, z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

1. (4 points) The parts of this question are unrelated. No justification is necessary for any part.
Fill in EXACTLY ONE circle. (unfilled ○ filled ●)

(1a) Belal has a whirlpool and drops a rubber ducky on the edge of the whirlpool. It spins around the centre once every minute and takes 5 minutes to reach its centre. Let $\gamma : [0,5] \to \mathbb{R}^2$ be the path of Belal's ducky (viewed from above) with the origin $(0,0)$ as the centre of the whirlpool.

Which of the following relations could γ realistically obey? Select the best answer.

- ○ $\gamma(t+1) = \gamma(t)$ for $0 \le t \le 5$
- ○ $\gamma(t) = \left(t\cos(\frac{2\pi t}{5}), t\sin(\frac{2\pi t}{5})\right)$ for $0 \le t \le 5$.
- ● The real-valued function $||\gamma(t)||$ is decreasing on $0 \le t \le 5$.
- ○ The derivative $\gamma'(t)$ is constant for all $0 \le t \le 5$.
- ○ The unit tangent $T(t)$ points towards the origin for all $0 \le t \le 5$.

(1b) Let $f(x,y) = xe^y$, $A = \{(x, xe^2) : x \in \mathbb{R}\}$, and $B = \{(x,y) \in \mathbb{R}^2 : xe^y = 3\}$. Which statement is TRUE?

- ● A is an x-slice of the graph of f and B is a contour of f.
- ○ A is a y-slice of the graph of f and B is a contour of f.
- ○ A is a contour of f and B is a x-slice of the graph of f.
- ○ A is a contour of f and B is a y-slice of the graph of f.
- ○ None of the above statements are true.

(1c) Let S be the unit sphere in $\mathbb{R}^3$ centered at the origin. Consider the sets

$$A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\} \qquad B = \{(\rho,\theta,\phi) \in \mathbb{R}^3 : \rho = 1\} \qquad C = \{(r,\theta,z) \in \mathbb{R}^3 : r^2+z^2 = 1\}$$

Which of the following statements is TRUE? Select only one.

- ○ Only A is equal to S.
- ○ Only B is equal to S.
- ○ Only C is equal to S.
- ○ Only A and B are equal to S.
- ● A, B, and C are all equal to S.

(1d) Husky Pup Pet Store sells two competing brands of dog food: Fido Food and Canine Cuisine. Let $F(x,y)$ be the number of bags of Fido Food sold per week, where x is the price (in dollars) of each bag of Fido Food and y is the price (in dollars) of each bag of Canine Cuisine. Please select the statement below that is both correct and well-justified.

- ○ $\frac{\partial F}{\partial y}$ is usually positive, because an increase in the price of Canine Cuisine will result in a decrease in the demand for Canine Cuisine and an increase in the demand for Fido food.
- ● $\frac{\partial F}{\partial y}$ is usually positive, because an increase in the price of Canine Cuisine will result in an increase in the price of Fido Food.
- ○ $\frac{\partial F}{\partial y}$ is usually negative, because if the demand for Canine Cuisine increases, the demand for Fido food will decrease.
- ○ $\frac{\partial F}{\partial y}$ is usually negative, because an increase in the price for Fido Food will result in a decrease in the demand for Fido Food.
- ○ None of the above are both correct and well-justified.

2. (3 points) The parts of this question are unrelated. No justification is necessary for any part.
Fill in EXACTLY ONE circle. (unfilled ○ filled ●)

(2a) Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let a be a limit point of A and let $b \in \mathbb{R}^m$.
Which of the following is EQUIVALENT to $\lim_{x \to a} f(x) = b$?

- ○ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $||x - a|| < \delta \implies ||f(x) - b|| < \varepsilon$
- ○ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < ||x - a|| < \delta \implies ||f(x) - b|| < \varepsilon$
- ○ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$
- ● $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(B_\delta(a) \setminus \{a\}) \subseteq B_\varepsilon(f(a))$
- ○ None of the above are equivalent.

(2b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Which statement must be TRUE?

- ○ f has a maximum and a minimum.
- ○ f has a maximum and a minimum on $(0, 237)^n$.
- ○ If $\lim_{||x|| \to \infty} f(x) = 0$ then f has a maximum.
- ○ If $\lim_{||x|| \to \infty} f(x) = \infty$ then f has a minimum.
- ● None of these are necessarily true.

(2c) Let S be a compact set in $\mathbb{R}^n$. Which of the following statements must be TRUE?

- ○ Every sequence in S converges.
- ○ Every subsequence of a sequence in S converges.
- ○ The set S is path-connected.
- ● The set S is not open.
- ○ None of the above are necessarily true.

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3. (3 points) For each set S, determine whether it satisfies each of the 4 listed properties.
No justification is necessary. **Fill in ALL boxes that apply.** If none apply, leave it blank. (unfilled □ filled ■)

(3a) $\mathbb{R}^{237} \setminus \overline{B_{137}(0)}$

■ open ■ path-connected

□ compact □ closed

(3b) $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \text{ and } x \neq 0\}$

□ open ■ path-connected

□ compact □ closed

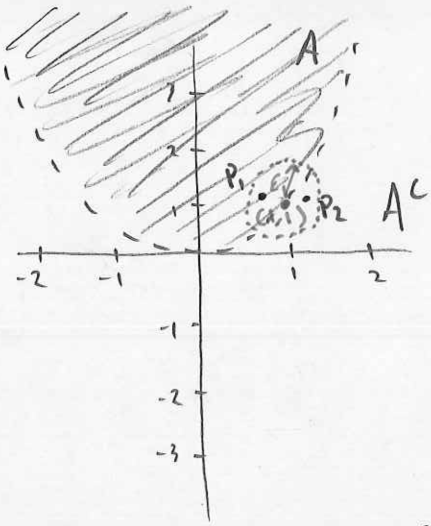
(3c) $S = \{(xy^3, e^{xy}, x^3, \sin y) : x, y \in [0, 1]\}$

□ open ■ path-connected

■ compact ■ closed

4. The parts of this question are unrelated.

(4a) (3 points) Let $A = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$. Draw a "picture proof" that $(1, 1)$ is a boundary point of A. Label your picture with quantities according to how a proof would be written. **Do not write a proof.**



For a point $(x,y) \in \mathbb{R}^2$,
$\varepsilon = \|(x,y) - (1,1)\|$

$B_\varepsilon((1,1)) \cap A \neq \emptyset$ and $B_\varepsilon((1,1)) \cap A^c \neq \emptyset$.

(4b) (4 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$ does not exist.

Choose two sequences $S_K = \frac{1}{K}$, and $t_K = \frac{1}{K^3}$.

By definition, $\lim_{K\to\infty} S_K = 0$ and $\lim_{K\to\infty} t_K = 0$.

Consider the limit along the line $(0,1)$. Then $\lim_{K\to\infty} (0, S_K) = (0,0)$.

- $$\lim_{K\to\infty} \frac{(0)^3\left(\frac{1}{K}\right)}{(0)^6 + \left(\frac{1}{K}\right)^2} = \frac{0}{\frac{1}{K^2}} = 0.$$

Now consider the limit along the line $(1,1)$. Then $\lim_{K\to\infty} (t_K, S_K) = (0,0)$.

- $$\lim_{K\to\infty} \frac{\left(\frac{1}{K}\right)^3\left(\frac{1}{K^3}\right)}{\left(\frac{1}{K}\right)^6 + \left(\frac{1}{K^3}\right)^2} = \lim_{K\to\infty} \frac{\frac{1}{K^6}}{\frac{1}{K^6} + \frac{1}{K^6}} = \lim_{K\to\infty} \frac{\cancel{\frac{1}{K^6}}}{\frac{2}{\cancel{K^6}}}$$

$$= \lim_{K\to\infty} \frac{1}{2} = \frac{1}{2}.$$

The limits along the line $(0,1)$ is different from the limit along the $(1,1)$ line, since $0 \neq \frac{1}{2}$. $\lim_{K\to\infty} (0, S_K) = \lim_{K\to\infty} (t_K, S_K) = (0,0)$

- Therefore since the limit of sequences along each line computes two different values as the sequences approach $(0,0)$, the limit does not exist.

5. (5 points) Prove that $f(x,y) = 3x + 2y$ is continuous at $(1,5)$ using the ε-δ definition.

Rough Work

$$\lim_{(x,y)\to(1,5)} f(x,y) = f(1,5) = 13$$

$0 < \left\| \binom{x}{y} - \binom{1}{5} \right\| < \delta \Rightarrow \| f(x,y) - 13 \| < \varepsilon$

$\sqrt{(x-1)^2 + (y-5)^2}$ $\quad \| 3x + 2y - 13 \| < \varepsilon$

$x - 1 \leq \| \cdots \| < \delta$

$y - 5 \leq \| \cdots \| < \delta$

$y < \delta + 5 \quad x < \delta + 1$

choose $\delta = 1$ $\quad y < 6 \quad x < 2$

$2y < 12 \quad 3x < 6$

$3x + 2y - 13 < 18 - 13$

$3x + 2y - \cancel{13} < \cancel{18}\ 5 \quad 3x + 2y - 13 < 5$

$\delta = \frac{\varepsilon}{5}$

$2y < 2\delta + 10 \quad 3x + 2y < 5\delta + 13$

$3x < 3\delta + 3 \quad 3x + 2y - 13 < 5\delta.$

$\leq \varepsilon$

I want to prove that $\forall \varepsilon > 0, \exists \delta > 0$ s.t $\forall (x,y) \in \mathbb{R}^2$, $0 < \left\| \binom{x}{y} - \binom{1}{5} \right\| < \delta$ $\Rightarrow \| f(x,y) - f(1,5) \| = \| f(x,y) - 13 \| < \varepsilon$.

proof:

- Let $\varepsilon > 0$ be given.
- Choose $\delta = \frac{\varepsilon}{5}$.
- Fix $(x,y) \in \mathbb{R}^2$. Assume $0 < \left\| \binom{x}{y} - \binom{1}{5} \right\| < \delta$.

  This implies that $0 < \sqrt{(x-1)^2 + (y-5)^2} < \delta$.

  Furthermore, $x - 1 \leq \sqrt{(x-1)^2 + (y-5)^2} < \delta$

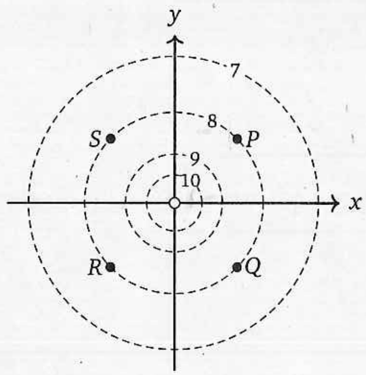
  and $y - 5 \leq \sqrt{(x-1)^2 + (y-5)^2} < \delta$,

  and thus $x - 1 < \delta$, $y - 5 < \delta$.
- It follows that $x < \delta + 1 \Rightarrow 3x < 3\delta + 3$ and $y < \delta + 5$ $\Rightarrow 2y < 2\delta + 10$. Thus $3x + 2y < 3\delta + 2\delta + 10 + 3$, so

$$\begin{aligned} 3x + 2y - 13 &< 5\delta \\ &= 5\left(\frac{\varepsilon}{5}\right) \\ &= \varepsilon, \end{aligned}$$

as required. Therefore $f(x,y) = 3x + 2y$ is continuous at $(1,5)$ $\square$

6. (5 points) A contour plot of a differentiable function $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is provided below. Four points P, Q, R, and S are labelled. Use this diagram for every part below. No justification is necessary.



Fill in ALL boxes that apply. If none apply, leave it blank. (unfilled ☐ filled ■)

(6a) At which of these 4 points is g_x negative?

■ P ■ Q ☐ R ☐ S

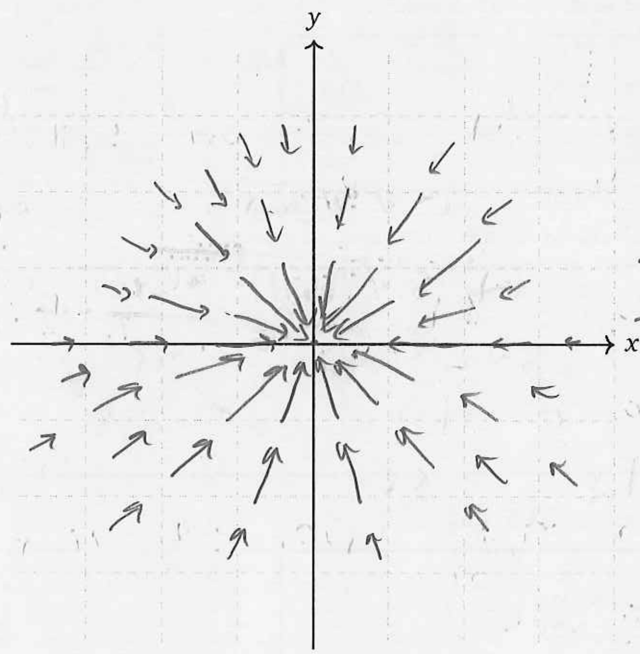
(6b) At which of these 4 points is $D_{(1,1)}g$ approximately zero?

☐ P ■ Q ☐ R ■ S

(6c) Choose the minimum of these 4 real numbers. If several choices apply, select them all.

■ $D_{(1,1)}g(P)$ ☐ $D_{(1,1)}g(Q)$ ☐ $D_{(1,1)}g(R)$ ☐ $D_{(1,1)}g(S)$

(6d) Sketch the gradient vector field of g.



- undefined at (0,0)
- increasing in magnitude as you approach the origin

7. (2 points) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = xy^2$. **No justification is necessary for any part below.**

(7a) Calculate $\nabla g(-1,2)$.

$\nabla g(-1,2) = (4,-4)$

(7b) For which **unit** vector $u \in \mathbb{R}^2$ is the quantity $D_u g(-1,2)$ maximized?

$u = \frac{1}{\sqrt{32}}(4,-4)$

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8. (5 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable and fix $p = (1,2) \in \mathbb{R}^2$. Suppose

$$f(p) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \quad \text{and} \quad Df(p) = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}.$$

No justification is necessary for any part below.

(8a) Compute $\partial_2 f(p)$.

$\partial_2 f(p) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(8b) Compute $D_v f(p)$ where $v = (0.5, -0.5)$.

$D_v f(p) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(8c) Give an explicit formula for the differential of f at p.

$df_p(h) = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} h, \quad h \in \mathbb{R}^2.$

(8d) Linearly approximate $f(1.5, 1.5)$.

$f(1.5, 1.5) \approx \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

9. (6 points) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$.
Using the limit definition of differentiability, prove that the function $f+g$ is differentiable at a and

$$d(f+g)_a = df_a + dg_a.$$

I ~~want to prove that~~ $\lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a) - d(f+g)_a(h)}{h} = 0.$

PLEASE SEE PAGE 12 :) Thanks

~~Proof~~

- ~~Assume~~ $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable. Then, their differentials df_a and dg_a exist and are linear maps approximating the derivatives of f and g near a.
- By definition, we have

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - df_a(h)}{\|h\|} = 0 \quad \text{and}$$

$$\lim_{h\to 0} \frac{g(a+h) - g(a) - dg_a(h)}{\|h\|} = 0.$$

- By the linearity of limits, [insert textbook theorem here],

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - df_a(h)}{\|h\|} + \lim_{h\to 0} \frac{g(a+h) - g(a) - dg_a(h)}{\|h\|}$$

$$= \lim_{h\to 0} \left[\frac{f(a+h) - f(a) - df_a(h)}{\|h\|} + \frac{g(a+h) - g(a) - dg_a(h)}{\|h\|} \right] = 0.$$

$$= \lim_{h\to 0} \left[\frac{f(a+h) + g(a+h) - f(a) - g(a) - df_a(h) - dg_a(h)}{\|h\|} \right]$$

$$= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a) - d(f+g)_a(h)}{\|h\|} = 0.$$

- Therefore the function $f+g$ is differentiable at a, and the differentials (linear maps) are $df_a(h) + dg_a(h) = d(f+g)_a(h)$, which implies that $d(f+g)_a = df_a + dg_a$, which is what I wanted to show. $\square$

Do not tear this page off. This page is for additional work and will not be graded, unless you clearly indicate it on the original question page.

$f(1.5, 1.5) \approx$

$f((1,2)) = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$(-2x^{2/3}$ $)$

$f(1.5, 1.5)$

$f(1,2) = (-2, 5)$

$v = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$

$g(x,y) = xy^2$

$\nabla g(x,y) = (y^2, 2xy)$

$\nabla g(-1,2) = (4, 2(-1)(2)) = (4,-4)$

$u = \dfrac{\nabla g(-1,2)}{\|\nabla g(-1,2)\|} = \dfrac{(4,-4)}{\sqrt{4^2+4^2}}$

$\sqrt{4^2+4^2} = \sqrt{16+16}$

$\sqrt{32}$

$f(p) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \quad Df(p) = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} \partial F_1 & \partial_2 F_1 \\ \partial_1 F_2 & \ldots \end{bmatrix}$

$\nabla g = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Do not tear this page off. This page is for additional work and will not be graded, unless you clearly indicate it on the original question page.

Differentiable $\Rightarrow$ $\exists$ linear map df_a s.t

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - df_a(h)}{h} = 0.$$

$$\lim_{h \to 0} \frac{(f+g)(a+h) - (f+g)(a) - (df_a + dg_a)(h)}{h} = 0.$$

$\Rightarrow$ $f+g$ differentiable at a.

$\frac{1}{k^3} \cdot \frac{1}{k^2}$

$\frac{1}{k^3} \to \frac{1}{k^6}$

$\underbrace{\underbrace{2 \times 2}_{4} \times 2}_{8} \times 2 \times 2 \times 2 \times 2$

$8 \quad 16 \quad 32 \quad 64$

$-(x^2+y^2) + C$

QUESTION 9

Do not tear this page off. This page is for additional work and will not be graded, unless you clearly indicate it on the original question page.

To prove $f+g$ is differentiable, it suffices to prove that for one component $f_i : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, for $i \in (1, \cdots, m)$, if f_i and g_i are differentiable, then $f_i + g_i$ is differentiable and the differential linear map $df_{i,a} + dg_{i,a}$ is a $m \times 1$ matrix.

proof

- Assume f and g are differentiable at a. Then each of their components f^i, g^i for $i \in (1, \cdots, m)$ are differentiable at a.
- By definition,

$$\lim_{h_i \to 0} \frac{f^i(a_i+h_i) - f^i(a_i) - df^i_a(h_i)}{h_i} = 0 \quad \text{and}$$

$$\lim_{h_i \to 0} \frac{g^i(a_i+h_i) - f^i(a_i) - dg^i_a(h_i)}{h_i} = 0.$$

- By the linearity addition of limits,

$$\lim_{h_i \to 0} \frac{f^i(a_i+h_i) - f^i(a_i) - df^i_a(h_i)}{h_i} + \lim_{h_i \to 0} \frac{g^i(a_i+h_i) - g^i(a_i) - dg^i_a(h_i)}{h_i} =$$

$$\lim_{h_i \to 0} \frac{(f+g)^i(a_i+h_i) - (f+g)^i(a_i) - d(f+g)^i_a(h_i)}{h_i} = 0,$$

and thus $(f+g)^i(a)$ is differentiable for all $i \in (1, \cdots, m)$.

- Furthermore, $df^i_a(h_i) + dg^i_a(h_i) = d(f+g)^i_a(h_i)$, which is a row vector which implies that $df^i_a + dg^i_a = d(f+g)^i_a$.
- The matrix $\begin{bmatrix} d(f+g)^1_a \\ \vdots \\ d(f+g)^m_a \end{bmatrix}$ is the differential of $f+g$, hence

Since $df^i_a + dg^i_a = d(f+g)^i_a$, for all $i \in (1, \cdots, m)$, then

$$df_a + dg_a = \begin{bmatrix} df^1_a \\ \vdots \\ df^m_a \end{bmatrix} + \begin{bmatrix} dg^1_a \\ \vdots \\ dg^m_a \end{bmatrix} = \begin{bmatrix} d(f+g)^1_a \\ \vdots \\ d(f+g)^m_a \end{bmatrix} = d(f+g)_a.$$

- Therefore $f+g$ is differentiable and $df_a + dg_a = d(f+g)_a$, as desired. $\square$