Midterm 1

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1. Wave Equation

Q1 14

[15 marks] Find the solution to the wave equation:
$u_{tt} = c^2 u_{xx}$ on $0 < x < l$,
with $u(0,t) = 0$, $u(l,t) = 0$, and:
$u(x,0) = \phi(x) = 3\sin\left(\frac{2\pi x}{l}\right)$
$u_t(x,0) = \psi(x) = \sin\left(\frac{5\pi x}{l}\right)$

$X\ddot{T} = c^2 X''T \rightarrow \frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$ $\quad X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$
$T(t) = C\cos(\sqrt{\lambda}ct) + D\sin(\sqrt{\lambda}ct)$

Border Conditions: $X(0) = 0 = A\cancel{\cos(0)}^{1} \Rightarrow A = 0.$
$X(l) = 0 = B\sin(\sqrt{\lambda}l) = 0, \quad \sqrt{\lambda}l = n\pi, \quad \lambda = \frac{n^2\pi^2}{l^2}.$

$\rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[C_n\cos\left(\frac{n\pi t}{l}\right) + D_n\sin\left(\frac{n\pi t}{l}\right)\right]$

Time conditions: $u(x,0) = 3\sin\left(\frac{2\pi x}{l}\right) = \sum^{\infty} \sin\left(\frac{n\pi x}{l}\right) C$

$$3\underbrace{\int_0^{\ell} \sin\frac{2\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx}_{\frac{\ell}{2}\delta_{2m}} = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^{\ell} \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx}_{\frac{\ell}{2}\delta_{mn}}$$

Thus $n=2=m$: $3\frac{\cancel{\ell}}{\cancel{2}} = C_2\frac{\cancel{\ell}}{\cancel{2}}$, all other C_n $(n\neq 2)$ $=0$.

$$\dot{u}(x,t) = \frac{d}{dt}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}\left[C_n\cos\frac{n\pi t}{\ell} + D_n\sin\frac{n\pi t}{\ell}\right] = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}\left[-C_n\sin\frac{n\pi t}{\ell} + D_n\cos\frac{n\pi t}{\ell}\right]\times\frac{n\pi}{\ell}$$

$\dot{u}(x,0) = \sin\frac{5\pi x}{\ell} = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}\left(\frac{n\pi}{\ell}\right)D_n$ By a similar argument, $D_5 = \frac{\ell}{5\pi}$,

all other D_n $(n\neq 5)=0$, Thus our solution is

$$\boxed{u(x,t) = 3\sin\frac{2\pi x}{\ell}\cos\frac{2\pi t}{\ell} + \frac{\ell}{5\pi}\sin\frac{5\pi x}{\ell}\sin\frac{5\pi t}{\ell}}$$

small mistake -1

correct solution 15

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(extra space)

Justification

The solution to the equation is

$X(x)\,T(t)$, so $X(0)=0$ implies $A\cos(0)+B\sin(0) = A(1)=0$,

so $A=1$.

As $X(\ell)=0$, we then require $B\sin\sqrt{\lambda}\ell=0$, but $B=0$ yields a trivial solution and thus $\sin\sqrt{\lambda}\ell=0$. However, this only occurs when $\sqrt{\lambda}\ell = n\pi$ for an integer n, Then $\lambda = \frac{\pi^2 n^2}{\ell^2}$.

At time $t=0$, the fourier coefficient condition yields that

$$3\int_0^\ell \sin\frac{2\pi x}{\ell}\sin\frac{m\pi x}{\ell} = 3\cdot\delta_{2m}\frac{\ell}{2} = \sum_{n=1}^{\infty} C_n\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}$$

$$= \sum_{n=1}^{\infty}\frac{\ell}{2}\delta_{mn}C_n,$$ yet δ_{2m} implies $n=m=2$. Thus

$C_2=3$, and all other C_n $(n\neq 2)=0$.

The derivative yields that (in time),

$$\psi(x) = \sin\frac{5\pi x}{\ell} = \sum_{\substack{n=1,\\ n\neq 2}}^{\infty}\sin\frac{n\pi x}{\ell}\cos\frac{n\pi x}{\ell}\cdot\frac{\pi n}{\ell} + \cancel{C_2\frac{2\pi}{\ell}\sin\frac{2\pi x}{\ell}\sin\frac{2\pi t}{\ell}}^{0},$$

and the second term is 0 since $\sin(0)=0$. The first term is

$\sum_{\substack{n=1,\\ n\neq 2}}^{\infty}\sin\frac{n\pi x}{\ell}\cdot\frac{\pi n}{\ell}$, which must be $5=n$ by the orthogonality relation. All other D_n must be zero.

$$u(x,t) = 3\sin\frac{2\pi x}{\ell}\cos\frac{2\pi t}{\ell} + \frac{\ell}{5\pi}\sin\frac{5\pi x}{\ell}\sin\frac{5\pi t}{\ell}$$ is the solution.

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Q2 30

2. Separation of Variables

[30 marks] The **Laplace Equation** is given by $u_{xx} + u_{yy} = 0$, for $u(x, y)$.
Solve the laplace equation on the infinite rectangle:

$$S = \{0 \leq x \leq l, 0 \leq y < \infty\}$$

with the boundary conditions:

$$u(0, y) = 0 \quad u(l, y) = 0$$

$$\lim_{y \to \infty} u(x, y) = 0$$

$$u(x, 0) = \phi(x)$$

(Intuitively, we can view the variable y as the variable t.)

1. Using separation of variables $u(x, y) = X(x)Y(y)$, define the separation constant with

$$\frac{X''}{X} = -\lambda$$

2. Recall for $f(x)$, the solutions to the ODE $f''(x) = kf(x)$ is given by $f(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$.
3. Apply the boundary conditions $u(0, y) = 0$ and $u(l, y) = 0$, to find λ.
4. Apply the boundary condition $\lim_{y \to \infty} u(x, y) = 0$.
5. Apply the boundary condition $u(x, 0) = \phi(x)$, and write the solution $u(x, y)$.

1) $X''y + y''X = 0 \rightarrow \frac{X''}{X} = -\frac{y''}{y} = -\lambda$.

2) $X'' = -\lambda X \Rightarrow X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$ by solving the ODE.
$y'' = \lambda y \Rightarrow y(y) = Ce^{\sqrt{\lambda}y} + De^{-\sqrt{\lambda}y}$

3) $X(0) = B(1) = 0 \Rightarrow B = 0$. $X(l) = 0 = A\sin(\sqrt{\lambda}l)$, $\boxed{\lambda = \frac{n^2\pi^2}{l^2}}$.

4) We have that $u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(C_n e^{\frac{n\pi y}{l}} + D_n e^{-\frac{n\pi y}{l}}\right)$.
As $y \to \infty$, $e^{\frac{n\pi y}{l}} \to \infty$ thus $C_n = 0$, $D_n \neq 0$, Thus
(for all n)
$u(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l} e^{-\frac{n\pi y}{l}}$. The Solution must remain bounded as $y \to \infty$.

5) At $y = 0$, $u(x,y) = \varphi(x)$, $u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} D_n$. Determine D_n:
$\sum_n D_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \varphi(x)\sin\frac{m\pi x}{l}\,dx$, $= D_n \cdot \frac{l}{2}\delta_{mn}$. By Fourier orthogonality.

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(extra space)

Therefore our Solution is

$$\boxed{\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{n\pi}{\ell} y} \\ \text{with} \quad D_n &= \frac{2}{\ell}\int_0^{\ell} \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right). \end{aligned}}$$

Is there really any justification to make here?

?

correct general solution and constant using orthogonality of sin functions. **30**

(extra space)

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Q3 18

3. Schrödinger Equation

[20 marks] Consider a quantum particle in a box of length $\frac{l}{2}$, obeying the Schrödinger Equation:

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} \qquad 0 < x < \frac{l}{2}$$

Initially, its probability density is **uniformly distributed** in the box (of size $\frac{l}{2}$).
At time $t = 0$, the box suddenly doubled in size to length l.
Thus the domain has increased to $0 < x < l$, while the initial condition is:

$$\Psi(x,0) = \phi(x) = \begin{cases} N & 0 < x < \frac{l}{2} \\ 0 & \frac{l}{2} < x < l \end{cases}$$

1. Find the normalization constant N.
2. Find the solution $\Psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} e^{\frac{-iE_n t}{\hbar}}$
3. Using Parseval's Equality, evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}(1-\cos\frac{n\pi}{2})^2$

   Explain how this series is different to the standard p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

   i.e. How is the extra term $(1-\cos\frac{n\pi}{2})^2$ affecting the series?

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1) We require $\int_0^{l/2} |N|^2\,dx = 1$ before the box doubles in size.

$N^2 \cdot \frac{l}{2} = 1$, $\boxed{N = \sqrt{\frac{2}{l}}}$

a) normalization constant $N = \sqrt{\frac{2}{l}}$ 2.5

2) We need to have that $A_n = \frac{2}{l}\int_0^{l} \varphi(x) \sin\frac{n\pi x}{l}\,dx$ by the Fourier Sine series definition.

This is just: $A_n = \frac{2}{l}\int_0^{l/2} N \sin\frac{\pi n x}{l}\,dx$ ($\varphi(x)$ not defined for $\frac{l}{2} < x < l$)

$= \frac{2}{l} N \int_0^{l/2} \sin\frac{n\pi x}{l} = -\frac{2}{l} N \frac{l}{n\pi}\cos\left(\frac{n\pi}{l}x\right)\Big|_0^{l/2} = -\frac{2}{n\pi}N\left(\cos\left(\frac{n\pi}{2}\right)-1\right)$

Thus $\boxed{\Psi(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sqrt{\frac{2}{l}}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{l}\right) e^{-iE_n t/\hbar}}$

8

b) correct value of A_n and solution 7.5

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3) Parseval's Inequality we require normalization

$$1 = \int_0^\ell |\Psi(x,t)|^2 dx = \int_0^\ell \left| \sum_{n=1}^{\infty} \frac{2}{n\pi} \sqrt{\frac{2}{\ell}} \left(1-\cos\frac{n\pi}{2}\right) \sin\frac{n\pi x}{\ell} e^{-iE_n t/\hbar} \right|^2 dx$$

$$= \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2\ell} \cdot \left(1-\cos\frac{n\pi}{2}\right)^2 \underbrace{\int_0^\ell \sin^2\frac{n\pi x}{\ell} dx}_{=\ell/2}$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{8}\,4}{n^2\pi^2\cancel{\ell}} \cdot \frac{\cancel{\ell}}{\cancel{2}} \left(1-\cos\frac{n\pi}{2}\right)^2.$$

Thus $1 = \sum_{n=0}^{\infty} \frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)^2$, so the series is

$$\boxed{\sum_{n=1}^{\infty} \frac{\left(1-\cos\frac{n\pi}{2}\right)^2}{n^2} = \frac{\pi^2}{4}}.$$

c) series is $\frac{\pi^2}{4}$ 7.5

As $n \to \infty$, $\cos\frac{n\pi}{2}$ alternates between ±1 and 0, the odd terms yield zero, thus $\sum_{n\,odd} \frac{\left(1-\cos\frac{n\pi}{2}\right)^2}{n^2} = \sum_{n\,odd} \frac{1}{n^2}$.

Thus $1-\cos\frac{n\pi}{2}$ alternates between 0, 1, and 4, 1 Different from the p-series $\sum^{\infty} \frac{1}{n^2}$ this series removes some terms and doubles some terms:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(1-\cos(\frac{n\pi}{2}))^2 = \frac{1}{1^2} + \frac{2}{2^2} + \frac{1}{\ldots}$$

which is ~~not~~ equivalent to $\frac{1}{1} \ldots$

mistake -1

c) values of $(1-\cos{\frac{n \pi}{2}})^2$ are $1, 4, 1, 0$ and it makes the series bigger. 2.5

how does it compare to the p-series $\sum \frac{1}{n^2}$? -1

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4. Fourier Series

[20 marks] Let $f(x) = x$ with $x \in (0, l)$.
Consider its Fourier Sine Series $F(x)$, and its Fourier Cosine Series $G(x)$, on $x \in (0, l)$,

$$x = F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) \qquad x = G(x) = \frac{l}{2} + \sum_{\text{n odd}}^{\infty} \frac{-4l}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

For each of the above 2 series, explain whether we are able to take derivatives on both sides of the equation to attain the Fourier Series of $f'(x)$.

If so, take derivatives on both sides, and find $f'(x)$ with its Fourier Series on the appropriate domain.
If not, explain why not.

Q4 9

For the F. sine series $F(x)$, we may not take a derivative.
If we were to, we would obtain

$$F'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \cdot \cos\left(\frac{n\pi x}{l}\right) \cdot \frac{n\pi}{l} = \sum_{n=1}^{\infty}$$

This series does not conv[erge, and instead will]
alternate values in sign,

For the Cosine series, we [may take a derivative]
will converge after differentiation:

$$\sum^{\infty} \frac{-4l}{} \cdot \left(\frac{n\pi x}{l}\right) \frac{n\pi}{} = \sum^{\infty} \frac{4}{} \sin\left(\frac{n\pi x}{}\right)$$

need to look at extensions of $f(x)$ onto $(-l,l)$ and think about if the series becomes a full Fourier series

a) arguing that derivative of series diverges 5

$$G'(x) = \sum_{n \text{ odd}} \frac{\cdots}{n^2\pi^2} \cdot \left(-\sin\left(\frac{n\pi x}{\ell}\right)\right) \cdot \frac{\cdots}{\ell} = \sum_{n \text{ odd}} \frac{\cdots}{n\pi} \cdots \left(\frac{\cdots}{\ell}\right).$$

We may check this: for $f(x)=1$ on $(0,\ell)$: its F. Sine series is

$$A_n = \frac{2}{\ell}\int_0^\ell (1)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\cdot\left[-\cos\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell = \frac{2}{n\pi}\left(-\cos(n\pi)+1\right)$$

But for odd n, we have that $\cos(n\pi) = -1$, thus $A_n = \frac{4}{n\pi}$.

This is as obtained above. Therefore

$1 = G'(x) = \sum_{n \text{ odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right)$ is the derivative of $G(x)$ and it

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b) derivative of $G(x)$ **4**

(extra space)

exists due to the series converging.

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(extra space)

5. Fourier Series

[15 marks] Consider real valued functions defined on $[-l, l]$.
Define the **inner product** (or "dot product") between 2 functions f, g to be

$$\langle f, g\rangle = \int_{-l}^{l} f(x)g(x)\,dx \qquad \text{(when this integral exists)}$$

This in turn, gives, $\|f\|$, the **norm** (or length) of a function f, as

$$\|f\|^2 = \langle f, f\rangle = \int_{-l}^{l} f(x)^2\,dx \qquad \text{(when this integral exists)}$$

This in turn, gives $d(f, g)$, the **"distance"** between 2 functions f, g, as

$$d(f,g)^2 = \|f - g\|^2 = \int_{-l}^{l} \big(f(x) - g(x)\big)^2\,dx \qquad \text{(when this integral exists)}$$

This space of functions is typically called L^2, with the above distance (metric).
(Each function f, is a "point" in this space of functions L^2.)

Take a function f in this space. We would like to investigate how the Fourier Series approximate the function f, in this distance (metric).

Consider a Fourier Series $g(x)$ written in general form with coefficients a_k:

$$g(x) = \sum_{k=1}^{\infty} a_k X_k(x)$$

where $X_k(x)$ are orthogonal functions: $\langle X_k, X_m\rangle = 0$ if $k \neq m$.

1. Consider the first n terms of the series:

$$g_n(x) = a_1X_1(x) + a_2X_2(x) + ... + a_nX_n(x) = \sum_{k=1}^{n} a_k X_k(x)$$

Consider the **(squared) distance** between f and g_n:

$$D_n = d(f, g_n)^2 = \|f - g_n\|^2 = \langle f - g_n, f - g_n\rangle = \langle f - \sum_{k=1}^{n} a_k X_k, f - \sum_{m=1}^{n} a_m X_m\rangle$$

Expand and simplify this expression using the orthogonality relation.

2. Choose the coefficients a_k to be the Fourier coefficients: $a_k = \dfrac{\langle f, X_k\rangle}{\langle X_k, X_k\rangle}$.
Using the fact that $0 \leq D_n$, show **Bessel's Inequality**:

$$\sum_{k=1}^{n} |a_k|^2 \int_{-l}^{l} |X_k(x)|^2 \leq \int_{-l}^{l} |f(x)|^2$$

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3. Recall the definition of a series, given $\{s_k\}$:

$$\sum_{k=1}^{\infty} s_k = \lim_{n\to\infty} \sum_{k=1}^{n} s_k$$

We say the Fourier Series $g(x)$ **converges** to $f(x)$ in the L^2 distance when:

$$\lim_{n\to\infty} D_n = 0$$

Show that if this is the case, then we attain **Parseval's Equality**:

$$\sum_{k=1}^{\infty} |a_k|^2 \int_{-l}^{l} |X_k(x)|^2 = \int_{-l}^{l} |f(x)|^2$$

Q5 15

We have that

$$D_p = \langle f - \sum_{k=1}^{p} a_k X_k(x),\ f - \sum_{m=1}^{p} a_m X_m(x) \rangle = \langle f, f\rangle - \sum_{k=1}^{p} a_k \langle f, X_k(x)\rangle$$

$$- \sum_{m=1}^{p} a_m \langle f, X_m\rangle + \sum_{k=1}^{p}\sum_{m=1}^{p} a_k a_m \langle X_k, X_m\rangle$$

$$= \|f\|^2 - \sum_{k=1}^{p} a_k \langle f, X_k\rangle - \sum_{m=1}^{p} a_m \langle f, X_m\rangle + \sum_{n=1}^{p} |a_n|^2 \|X_n\|^2$$

$$= \|f\|^2 - 2\sum_{m=1}^{p} a_m \langle f, X_m\rangle + \sum_{n=1}^{p} |a_n|^2 \|X_n\|^2,$$

The last step follows by a change in index since they are the same series, and the last term follows since $\sum_{k=1}^{p}\sum_{m=1}^{p} a_k a_m \delta_{km} \|X_n\|^2 = \sum_{n=1}^{p} |a_n|^2 \|X_n\|^2$, the delta function removes all cross terms

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Now, with the Fourier coefficient being imposed

$a_k = \frac{\langle f, X_k \rangle}{\|X_k\|^2}$, we have that

$$D_p = \|f\|^2 - 2\sum_{m=1}^{p} \frac{\langle f, X_m\rangle}{\|X_m\|^2}\langle f, X_m\rangle + \sum_{k=1}^{p} \frac{(\langle f, X_k\rangle)^2}{(\|X_k\|^2)^2}\|X_k\|^2$$

$$= \|f\|^2 - 2\sum_{m=1}^{p} \frac{(\langle f, X_m\rangle)^2}{\|X_m\|^2} + \sum_{m=1}^{p} \frac{(\langle f, X_m\rangle)^2}{(\|X_m\|^2)}$$

$$\boxed{D_p = \|f\|^2 - \sum_{m=1}^{p} \frac{(\langle f, X_m\rangle)^2}{\|X_m\|^2}}$$

Since they are the same series, an index change was invoked and they were added.

2) Since $D_p \geq 0$, we require that $\|f\|^2 - \sum_{m=1}^{\infty} \frac{(\langle f, X_m\rangle)^2}{\|X_m\|^2} \geq 0$.

This implies that $\|f\|^2 \geq \sum_{m=1}^{p} \frac{(\langle f, X_m\rangle)^2}{\|X_m\|^2}$.

Yet, since our Fourier coefficients are given by $a_k = \frac{\langle f, X_k\rangle}{\|X_k\|^2}$, we have on the right hand side that

$$\|f\|^2 \geq \sum_{m=1}^{p} \left[\frac{\langle f, X_m\rangle}{\|X_m\|^2}\right]^2 \|X_m\|^2 = \sum_{m=1}^{p} |a_m|^2 \|X_m\|^2.$$

By definition of the inner product, $\|f\|^2 = \int_{-\ell}^{\ell} [f(x)]^2\,dx$ and thus $\boxed{\int_{-\ell}^{\ell} |f(x)|^2 dx \geq \sum_{m=1}^{p} |a_k|^2 \int_{-\ell}^{\ell} |X_m(x)|^2\,dx}$, so we arrive at Bessel's Inequality.

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If we now consider a the series as a limit,

$\lim_{p\to\infty} \sum_{n=1}^{p} |a_n|^2 \|X_n\|^2$, and take the limit of boths sides,

$D_p = \|f\|^2 - \sum_{n=1}^{p} |a_n|^2 \|X_n\|^2$

$\Longrightarrow \lim_{p\to\infty} D_p = 0 = \|f\|^2 - \lim_{p\to\infty} \sum_{n=1}^{p} |a_n|^2 \|X_n\|^2$

Which, by definition of the series, is just

5 5

$0 = \|f\|^2 - \sum_{n=1}^{\infty} |a_n|^2 \|X_n\|^2$

5 5 5 5

Here invoking the definition of the inner product $\|f\|^2 = \int_{-\ell}^{\ell} |f(x)|^2\,dx$,

we arrive at Parseval's Inequality

$$\boxed{\int_{-\ell}^{\ell} |f(x)|^2\,dx = \sum_{n=1}^{\infty} |a_n|^2 \int_{-\ell}^{\ell} |X_n(x)|^2\,dx.}$$

As desired.

* I apologize about my work being located on the page margins; I hope this is not an inconvienence. I didn't exactly 'remember' this rule throughout the ~~thes.~~ test. Thx