# PHY250 PS1 — 01/31/2022

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## 1.1

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $C^2$ function. Since $F$ is $C^2$, the gradient of $F$ is

$$\nabla F = \partial_x F \hat{\mathbf{x}} + \partial_y F \hat{\mathbf{y}} + \partial_z F \hat{\mathbf{z}}.$$

Taking the curl of $F$ yields

$$\begin{aligned}
\nabla \times (\nabla F) &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ \partial_x F & \partial_y F & \partial_z F \end{pmatrix} \\
&= \hat{\mathbf{x}} \left( \partial_y \partial_z F - \partial_z \partial_y F \right) - \hat{\mathbf{y}} \left( \partial_x \partial_z F - \partial_z \partial_z F \right) + \hat{\mathbf{z}} \left( \partial_x \partial_y F - \partial_y \partial_x F \right).
\end{aligned}$$

Since $F$ is $C^2$, by Clairaut's Theorem, then each $\partial_i \partial_j F = \partial_j \partial_i F$ for any $i, j \in \{x, y, z\}$. Therefore

$$\nabla \times (\nabla F) = \hat{\mathbf{x}}(0) - \hat{\mathbf{y}}(0) + \hat{\mathbf{z}}(0) = 0,$$

and therefore the curl of a gradient is always zero.

## 1.2

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $F(x, y, z) = x^2 y^3 z^4$. $F$ is certainly $C^2$ on $\mathbb{R}^3$ since it is a monomial. The gradient of $F$ is

$$\nabla F = 2xy^3 z^4 \hat{\mathbf{x}} + 3x^2 y^2 z^4 \hat{\mathbf{y}} + 4x^2 y^3 z^3 \hat{\mathbf{z}}.$$

Then

$$\begin{aligned}
\nabla \times (\nabla F) &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 2xy^3 z^4 & 3x^2 y^2 z^4 & 4x^2 y^3 z^3 \end{pmatrix} \\
&= \hat{\mathbf{x}}(12x^2 y^2 z^3 - 12x^2 y^2 z^3) - \hat{\mathbf{y}}(8xy^3 z^4 - 8xy^2 z^3) + \hat{\mathbf{z}}(6xy^2 z^4 - 6xy^2 z^4) \\
&= \hat{\mathbf{x}}(0) - \hat{\mathbf{y}}(0) + \hat{\mathbf{z}}(0) \\
&= 0.
\end{aligned}$$

## 2

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}$ be a $C^2$ function. Then

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{F}) &= \nabla \times \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{pmatrix} \\
&= \nabla \times [\hat{\mathbf{x}}(\partial_y F_z - \partial_z F_y) + \hat{\mathbf{y}}(\partial_z F_x - \partial_x F_z) + \hat{\mathbf{z}}(\partial_x F_y - \partial_y F_x)] \\
&= \hat{\mathbf{x}}[\partial_y \partial_x F_y - \partial_y^2 F_x - \partial_z^2 F_x + \partial_z \partial_x F_z] + \hat{\mathbf{y}}[\partial_z \partial_y F_z - \partial_z^2 F_y - \partial_x^2 F_y + \partial_x \partial_y F_x] \\
&\qquad + \hat{\mathbf{z}}[\partial_x \partial_z F_x - \partial_x^2 F_z - \partial_y^2 F_z + \partial_y \partial_z F_y] \\
&= \hat{\mathbf{x}}[\partial_y \partial_x F_y - \partial_y^2 F_x - \partial_z^2 F_x + \partial_z \partial_x F_z + \partial_x^2 F_x - \partial_x^2 F_x] \\
&\qquad + \hat{\mathbf{y}}[\partial_z \partial_y F_z - \partial_z^2 F_y - \partial_x^2 F_y + \partial_x \partial_y F_x + \partial_y^2 F_y - \partial_y^2 F_y] \\
&\qquad + \hat{\mathbf{z}}[\partial_x \partial_z F_x - \partial_x^2 F_z - \partial_y^2 F_z + \partial_y \partial_z F_y + \partial_z^2 F_z - \partial_z^2 F_z] \\
&= \hat{\mathbf{x}}[\partial_x^2 F_x + \partial_x \partial_y F_y + \partial_x \partial_z F_z] + \hat{\mathbf{y}}[\partial_y \partial_x F_x + \partial_y^2 F_y + \partial_y \partial_z F_z] + \hat{\mathbf{z}}[\partial_z \partial_x F_x + \partial_z \partial_y F_y + \partial_z^2 F_z] \\
&\qquad - \left[\hat{\mathbf{x}}[\partial_x^2 + \partial_y^2 + \partial_z^2] F_x + \hat{\mathbf{y}}[\partial_x^2 + \partial_y^2 + \partial_z^2] F_y + \hat{\mathbf{z}}[\partial_x^2 + \partial_y^2 + \partial_z^2] F_z\right] \\
&= \nabla[\partial_x F_x + \partial_y F_y + \partial_z F_z] - \left[\nabla^2 F_x \hat{\mathbf{x}} + \nabla^2 F_y \hat{\mathbf{y}} + \nabla^2 F_z \hat{\mathbf{z}}\right] \\
&= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}
\end{aligned}$$

### 3.1

$\mathbf{F}(x,y,z) = (2xy^2 + z^3)\hat{\mathbf{x}} + 2x^2y\hat{\mathbf{y}} + 3xz^2\hat{\mathbf{z}}$. Then

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}[2xy^2 + z^3] + \frac{\partial}{\partial y}[2x^2y] + \frac{\partial}{\partial z}[3xz^2] \\
&= 2y^2 + 2x^2 + 6xz. \\
\nabla \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 2xy^2 + z^3 & 2x^2y & 3xz^2 \end{pmatrix} \\
&= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y}[3xz^2] - \frac{\partial}{\partial z}[2x^2y] \right) - \hat{\mathbf{y}} \left( \frac{\partial}{\partial x}[3xz^2] - \frac{\partial}{\partial z}[2xy^2 + z^3] \right) \\
&\qquad + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x}[2x^2y] - \frac{\partial}{\partial y}[2xy^2 + z^3] \right) \\
&= \hat{\mathbf{x}}(0 - 0) + \hat{\mathbf{y}}(3z^2 - 3z^2) + \hat{\mathbf{z}}(4xy - 4xy) \\
&= 0.
\end{aligned}$$

### 3.2

$\mathbf{F}(x,y,z) = (x^2 - z^2)\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 2xz\hat{\mathbf{z}}$.

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}[x^2 - z^2] + \frac{\partial}{\partial y}[2] + \frac{\partial}{\partial z}[2xz] \\
&= 2x + 2x = 4x. \\
\nabla \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x^2 - z^2 & 2 & 2xz \end{pmatrix} \\
&= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y}[2xz] - \frac{\partial}{\partial z}[2] \right) - \hat{\mathbf{y}} \left( \frac{\partial}{\partial x}[2xz] - \frac{\partial}{\partial z}[x^2 - z^2] \right) \\
&\qquad + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x}[2] - \frac{\partial}{\partial y}[x^2 - z^2] \right) \\
&= \hat{\mathbf{x}}(0 - 0) - \hat{\mathbf{y}}(4z) + \hat{\mathbf{z}}(0 - 0) \\
&= -4z\hat{\mathbf{y}}.
\end{aligned}$$

### 3.3

$\mathbf{F}(x,y,z) = e^{yz}\hat{\mathbf{x}} + e^{xz}\hat{\mathbf{y}} + e^{xy}\hat{\mathbf{z}}$.

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}[e^{yz}] + \frac{\partial}{\partial y}[e^{xz}] + \frac{\partial}{\partial z}[e^{xy}] \\
&= 0 + 0 + 0 = 0.
\end{aligned}$$

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ e^{yz} & e^{xz} & e^{xy} \end{pmatrix} \\
&= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y}[e^{xy}] - \frac{\partial}{\partial z}[e^{xz}] \right) - \hat{\mathbf{y}} \left( \frac{\partial}{\partial x}[e^{xy}] - \frac{\partial}{\partial z}[e^{yz}] \right) \\
&\qquad\qquad\qquad\qquad + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x}[e^{xz}] - \frac{\partial}{\partial y}[e^{yz}] \right) \\
&= \hat{\mathbf{x}}(xe^{xy} - xe^{xz}) + \hat{\mathbf{y}}(ye^{yz} - ye^{xy}) + \hat{\mathbf{z}}(ze^{xz} - ze^{yz}).
\end{aligned}
$$

## 4

The surface of the hemispherical bowl can be described by the set

$$S = \{(R, \theta, \phi) \in \mathbb{R}^3 : 0 \leq \theta \leq \pi/2,\ 0 \leq \phi < 2\pi\} \cup \{(r, \pi, \phi) \in \mathbb{R}^3 : 0 \leq r \leq R, 0 \leq \phi \leq 2\pi\}.$$

Similarly, the volume of the bowl is described by the set

$$V = \{(r, \theta, \phi) \in \mathbb{R}^3 : 0 \leq r \leq R, 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq 2\pi\}.$$

Notice that $\mathbf{A}(r, \theta, \phi) = r \sin\theta \cos\phi \hat{\mathbf{x}} + r \sin\theta \sin\phi \hat{\mathbf{y}} + r \cos\theta \hat{\mathbf{z}}$, which simplifies to $\mathbf{A}(r, \theta, \phi) = r\hat{\mathbf{r}}$ in spherical coordinates. The divergence theorem states

$$\int_V (\nabla \cdot \mathbf{A}) d\tau = \oint_S \mathbf{A} \cdot d\mathbf{a}.$$

I will work in spherical coordinates and show that the left hand side and right hand sides of the theorem yield the same result.

To begin, notice that $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$ in spherical coordinates. Also,

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r}[r^2 A_r] + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}[\sin\theta A_\theta] + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}[A_\phi] \\
&= \frac{1}{r^2} \frac{\partial}{\partial r}[r^3] + 0 + 0 \\
&= 3.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_V (\nabla \cdot \mathbf{A}) d\tau &= \iiint_V 3r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= 3 \int_0^R r^2 \, dr \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} \, d\phi \\
&= 3 \left[\frac{1}{3} r^3\right]_0^R [-\cos\theta]_0^{\pi/2} [\phi]_0^{2\pi} \\
&= 3 \cdot \frac{1}{3} R^3 \cdot 1 \cdot 2\pi \\
&= 2\pi R^3.
\end{aligned}$$

For the right hand side, we will divide $S$ into two surfaces $S_1$ and $S_2$ with different area elements $d\mathbf{a}_1$ and $d\mathbf{a}_2$, representing the spherical surface and the bottom disc, respectively. At $d\mathbf{a}_1$, $R$ is held constant while $\theta$ and $\phi$ change, so our surface element is given by $d\mathbf{a}_1 = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$. Likewise with the bottom disc, $\theta$ is held constant at $\pi/2$, so $d\mathbf{a}_2 = r \, dr \, d\phi \, \hat{\boldsymbol{\theta}}$. *See Griffiths page 40 for a more formal derivation.*

Since $\mathbf{A} = A_r \hat{\mathbf{r}}$ and $\mathbf{A} \cdot d\mathbf{a}_1 = R^3 \sin\theta \, d\theta \, d\phi \hat{\mathbf{r}}$, then $\mathbf{A} \cdot d\mathbf{a}_2 = 0$. At the curved surface of the bowl $r = R$, so $\mathbf{A} = R\hat{\mathbf{r}}$. Then

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_{S_1} \mathbf{A} \cdot d\mathbf{a}_1 + \int_{S_2} \mathbf{A} \cdot d\mathbf{a}_2$$

$$
\begin{aligned}
&= \iint_{S_1} R^3 \sin\theta \, d\theta \, d\phi + 0 \\
&= R^3 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} \, d\phi \\
&= R^3 \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{2\pi} \\
&= R^3 \cdot 1 \cdot 2\pi \\
&= 2\pi R^3.
\end{aligned}
$$

Therefore the divergence theorem holds.

### 5.1

$\mathbf{F}(x,y) = x\hat{\mathbf{x}} + (x-y)\hat{\mathbf{y}}$ along $y = x^2$ for $0 \le x \le 1$. Let

$$C = \{(x, x^2) \in \mathbb{R}^2 : 0 \le x \le 1\}$$

be that curve. We wish to evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{l}$. Here, $d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \int_C [x\,dx + (x-y)\,dy].$$

Given $y = x^2$ is the constraint, $dy = 2xdx$, and so

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{l} &= \int_0^1 [x + (x - x^2)2x]\,dx \\
&= \int_0^1 x + 2x^2 - 2x^3\,dx \\
&= \int_0^1 x\,dx + 2\int_0^1 x^2\,dx - 2\int_0^1 x^3\,dx \\
&= \frac{1}{2} + \frac{2}{3} - \frac{2}{4} \\
&= \frac{2}{3}.
\end{aligned}$$

### 5.2

$\mathbf{F}(x,y) = (x^2 + 2y)\hat{\mathbf{x}} - y^2\hat{\mathbf{y}}$ along the ellipse $x^2 + 9y^2 = 9$ from $(0,-1) \to (0,1)$. If we consider a parameterization of the ellipse, we can define $x$ as a function of $y$ with the bounds of integration being $y = -1$ and $y = 1$. Then

$$x = -3\sqrt{1-y^2},$$

which follows from integrating over the left hand side of the ellipse (would be positive if right hand). Let

$$C = \{y, -3\sqrt{1-y^2} \in \mathbb{R}^2 : -1 \le y \le 1\}$$

be the curve we want to integrate over. It follows that since $x = -3\sqrt{1-y^2}$, then $dx = \dfrac{3y}{\sqrt{1-y^2}}\,dy$. Therefore

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{l} &= \int_C [(x^2 - 2y)\,dx - y^2\,dy] \\
&= \int_{-1}^1 \left[\frac{27(y - y^3) - 6y^2}{\sqrt{1-y^2}} - y^2\right] dy \\
&= 27\int_{-1}^1 \frac{y}{\sqrt{1-y^2}}\,dy - 27\int_{-1}^1 \frac{y^3}{\sqrt{1-y^2}}\,dy - 6\int_{-1}^1 \frac{y^2}{\sqrt{1-y^2}}\,dy - \int_{-1}^1 y^2\,dy
\end{aligned}$$

I will now integrate each of these separately, moving from left to right (the first two are zero since the integrand is an odd function, but I will continue to prove this anyway).

The first is given by letting $u = \sqrt{1-y^2}$, so $du = \dfrac{-y}{\sqrt{1-y^2}}\,dy$ which implies that $dy = \frac{-\sqrt{1-y^2}}{y}\,du$, so

$$\begin{aligned}\int_{-1}^{1} \frac{y}{\sqrt{1-y^2}}\,dy &= -\int_{u(-1)}^{u(1)} du = u\Big|_{u(-1)}^{u(1)} \\ &= -\sqrt{1-(1)^2} - \sqrt{1-(-1)^2} \\ &= 0.\end{aligned}$$

The second can be evaluated by making another $u$-substitution, $u = 1-y^2$, so $y^2 = 1-u$ and $du = -2y\,dy \implies dy = -\dfrac{1}{2y}\,du$. Then

$$\begin{aligned}\int_{-1}^{1} \frac{y^3}{\sqrt{1-y^2}}\,dy &= -\frac{1}{2}\int_{-1}^{1} \frac{(y^2)(-2y)}{\sqrt{1-y^2}}\,dy \\ &= -\frac{1}{2}\int_{u(-1)}^{u(1)} \frac{1-u}{\sqrt{u}}\,du \\ &= -\frac{1}{2}\int_{u(-1)}^{u(1)} u^{-1/2}\,du + \frac{1}{2}\int_{u(-1)}^{u(1)} u^{1/2}\,du \\ &= -\frac{1}{2}[2\sqrt{u}]_{u(-1)}^{u(1)} + \frac{1}{2}\frac{2}{3}[u^{3/2}]_{u(-1)}^{u(1)} \\ &= -\sqrt{1-y^2} + \frac{1}{3}(1-y^2)^{3/2}\Big|_{-1}^{1} \\ &= 0.\end{aligned}$$

The third integral I will evaluate using a trigonometric substitution. Let $y = \sin u$ (so $u = \arcsin(y)$), and $dy = \cos u\,du$. Therefore

$$\begin{aligned}\int_{-1}^{1} \frac{y^2}{\sqrt{1-y^2}}\,dy &= \int_{u(-1)}^{u(1)} \frac{\cos u \sin^2 u}{\sqrt{1-\sin^2 u}}\,du \\ &= \int_{u(-1)}^{u(1)} \frac{\cos u \sin^2 u}{\cos(u)}\,du = \int_{u(-1)}^{u(1)} \sin^2 u\,du \\ &= \int_{u(-1)}^{u(1)} \frac{1-\cos(2u)}{2}\,du \\ &= \frac{1}{2}\int_{u(-1)}^{u(1)} du - \frac{1}{2}\int_{u(-1)}^{u(1)} \cos(2u)\,du \\ &= \frac{1}{2}u\Big|_{u(-1)}^{u(1)} - \frac{1}{4}\sin(2u)\Big|_{u(-1)}^{u(1)} \\ &= \frac{1}{2}\arcsin(y)\Big|_{-1}^{1} - \frac{1}{4}\sin(2\arcsin(y))\Big|_{-1}^{1} \\ &= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] - \frac{1}{4}[0-0] \\ &= \frac{\pi}{2}.\end{aligned}$$

The last integral is simply

$$\begin{aligned}\int_{-1}^{1} y^2\,dy &= \frac{1}{3}y^3\Big|_{-1}^{1} \\ &= \frac{2}{3}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{l} &= 27(0) - 27(0) - 6\cdot\frac{\pi}{2} - \frac{2}{3} \\ &= -3\pi - \frac{2}{3} \approx -10.09.\end{aligned}$$

## 6

$\mathbf{F}_1(x,y,z) = (x+y)\hat{\mathbf{x}} + (-x+y)\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}$ and $\mathbf{F}_2(x,y,z) = 2y\hat{\mathbf{x}} + (2x+3z)\hat{\mathbf{y}} + 3y\hat{\mathbf{z}}$.
We have that

$$\begin{aligned}
\nabla \cdot \mathbf{F}_1 &= \frac{\partial}{\partial x}[x+y] + \frac{\partial}{\partial y}[-x+y] + \frac{\partial}{\partial z}[-2z] \\
&= 1 + 1 - 2 = 0 \\
\nabla \times \mathbf{F}_1 &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x+y & -x+y & -2z \end{pmatrix} \\
&= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y}[-2z] - \frac{\partial}{\partial z}[-x+y] \right) - \hat{\mathbf{y}} \left( \frac{\partial}{\partial x}[-2z] - \frac{\partial}{\partial z}[x+y] \right) \\
&\qquad + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x}[-x+y] - \frac{\partial}{\partial y}[x+y] \right) \\
&= \hat{\mathbf{z}}(-1-1) = -2\hat{\mathbf{z}} \neq 0,
\end{aligned}$$

and for $\mathbf{F}_2$,

$$\begin{aligned}
\nabla \cdot \mathbf{F}_2 &= \frac{\partial}{\partial x}[2y] + \frac{\partial}{\partial y}[2x+3z] + \frac{\partial}{\partial z}[3y] \\
&= 0 + 0 + 0 = 0 \\
\nabla \times \mathbf{F}_2 &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 2y & 2x+3z & 3y \end{pmatrix} \\
&= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y}[3y] - \frac{\partial}{\partial z}[2x+3z] \right) - \hat{\mathbf{y}} \left( \frac{\partial}{\partial x}[3y] - \frac{\partial}{\partial z}[2y] \right) \\
&\qquad + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x}[2x+3y] - \frac{\partial}{\partial y}[2y] \right) \\
&= \hat{\mathbf{x}}[3-3] - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}[2-2] \\
&= 0.
\end{aligned}$$

Since $\nabla \times \mathbf{F}_2 = 0$, there exists some scalar potential $V$ such that $\mathbf{F}_2 = -\nabla V$. We must have

$$\frac{\partial}{\partial x}[V(x,y,z)] = -2y, \qquad \frac{\partial}{\partial y}[V(x,y,z)] = -2x - 3z, \qquad \frac{\partial}{\partial z}[V(x,y,z)] = -3y.$$

Integrating the partial derivatives yields the relations

$$\begin{aligned}
V(x,y,z) &= -2yx + F(y,z) \\
V(x,y,z) &= -2xy - 3yz + G(x,z) \\
V(x,y,z) &= -3yz + H(x,y),
\end{aligned}$$

where the functions $F, G$ and $H$ are terms of higher order variables which may have vanished when taking the partial derivatives. It can easily be seen that $F = G = H = c$ for some $c \in \mathbb{R}$, so $V(x,y,z) = -2xy - 3yz + c$. Therefore $\mathbf{F}_2(x,y,z) = -\nabla V(x,y,z)$.