Problem Set 2: Due 24 Oct 2022 (4 problems, submit Online - Weight 2.5%). One of the problems will be marked 1.5%, rest 1% will be for simply completing the homework. Please view these as practice problems for the test/exam.

Problem 3.19 Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos\theta), \tag{3.88}$$

where

$$C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \tag{3.89}$$

Problem 3.26 A sphere of radius R, centered at the origin, carries charge density

$$\rho(r, \theta) = k \frac{R}{r^2}(R - 2r) \sin\theta,$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

Problem 3.15 A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.

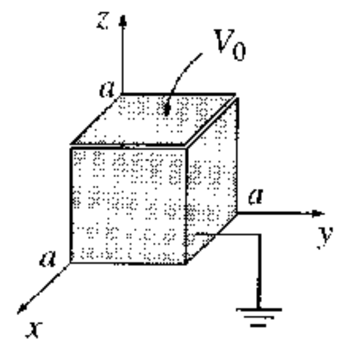

Figure 3.23

Problem 3.49 An ideal electric dipole is situated at the origin, and points in the z direction, as in Fig. 3.36. An electric charge is released from rest at a point in the xy plane. Show that it swings back and forth in a semi-circular arc, as though it were a pendulum supported at the origin. [This charming result is due to R. S. Jones, *Am. J. Phys.* **63**, 1042 (1995).]

PHY350 Problem Set 2 — Due October 24, 23:00

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3.19

To begin, we may consider the spherical expression of the electric potential, then proceed by invoking boundary conditions:

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta).$$

Let R be the radius of the sphere. Since the sphere is hollow, we may consider the potential in two regions - inside and outside the sphere:

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty}(A_l^1 r^l + B_l^1 r^{-l-1})P_l(\cos\theta) & (r \leq R) \\ \sum_{l=0}^{\infty}(A_l^2 r^l + B_l^2 r^{-l-1})P_l(\cos\theta) & (r \geq R). \end{cases}$$

Our first boundary condition is that $V(r=\infty,\theta)=0$, so the potential must vanish at infinity. This implies that $A_l^2 = 0$ for all l. Likewise, we cannot have the potential explode at the origin, and thus $B_l^1 = 0$ for all l. For simplicity, I will now ignore the indices differentiating the outer from the potential from the inner, since there is now only one A and B corresponding to terms r^l and r^{-l-1}, respectively. The potential is now

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & (r \leq R) \\ \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) & (r \geq R). \end{cases}$$

Our third boundary condition is the requirement that the potential must be continuous at the boundary, when $r = R$:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l R^{-l-1} P_l(\cos\theta).$$

From direct observation, if these sums are equal at the boundary, only the Legendre polynomial terms affect the magnitude of the potential, hence the coefficients must be equal:

$$A_l R^l = B_l R^{-l-1},$$

which implies that $B_l = A_l R^{2l+1}$. Now to invoke the last boudnary condition, that the potential at the boundary is specified to be $V_0(\theta)$:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta).$$

We can determine the A_l by applying the orthogonality relations for the Legendre functions, that $\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta = \dfrac{2}{2n+1}\delta_{nm}$. Integrating both sides, we have

$$\begin{aligned}
\sum_{l=0}^{\infty} A_l R^l \int_0^\pi P_l(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta &= \int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,d\theta \\
\implies \sum_{l=0}^{\infty} A_l R^l \cdot \frac{2}{2l+1}\delta_{nm} &= \int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,d\theta \\
\implies A_m &= \frac{2m+1}{2R^m}\int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,d\theta.
\end{aligned}$$

Our boundary conditions are now all invoked, and we may being solving for the surface charge density on the sphere. By definition, the surface charge density is determined by

$$-\frac{\sigma}{\varepsilon_0} = \left(\frac{\partial V_{\text{out}}}{\partial n} - \frac{\partial V_{\text{in}}}{\partial n}\right)\bigg|_{r=R} = \left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\bigg|_{r=R},$$

and $\hat{n} = \hat{r}$ since the normal of the sphere is pointing in the radial direction. Differentiating, we find

$$\begin{aligned}
\sigma &= -\varepsilon_0 \left(\frac{\partial}{\partial r}\sum_{l=0}^{\infty} B_l r^{-l-1}P_l(\cos\theta) - \frac{\partial}{\partial r}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\right)\bigg|_{r=R} \\
&= -\varepsilon_0 \left(\sum_{l=0}^{\infty}(-l-1)(A_l R^{2l+1})r^{-l-2}P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l r^{l-1}P_l(\cos\theta)\right)\bigg|_{r=R} \\
&= \varepsilon_0 \left(\sum_{l=0}^{\infty}(l+1)(A_l R^{2l+1})R^{-l-2}P_l(\cos\theta) + \sum_{l=0}^{\infty} l A_l R^{l-1}P_l(\cos\theta)\right) \\
&= \varepsilon_0 \sum_{l=0}^{\infty} A_l R^{l-1}(l+l+1)P_l(\cos\theta) \\
&= \varepsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta).
\end{aligned}$$

Now, invoking the condition that $A_m = \dfrac{2m+1}{2R^m}\int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,d\theta$, we have that

$$\sigma(\theta) = \varepsilon_0 \sum_{l=0}^{\infty}(2l+1)^2\frac{R^{l-1}}{R^l}C_l P_l(\cos\theta) = \frac{\varepsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta),$$

where C_l is just the integral $C_l = \int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,d\theta$. Therefore, this is the expression for the surface charge.

3.26

For points very far away on the z axis, the solid sphere with charge density $\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$ will either appear as if it was a monopole (single charge), dipole, quadrupole, octopole, and so on. Hence we may invoke multipole expansion to determine the approximate potential at points very far away. The potential is given by

$$\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\mathcal{V}} r'^n \rho(r') P_n(\cos\theta)\,d\tau' \\
&= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_{\mathcal{V}}\rho(r')\,d\tau' && \text{(monopole)} \\
&\qquad + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_{\mathcal{V}} r'\rho(r')\cos\theta\,d\tau' && \text{(dipole)} \\
&\qquad\qquad + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_{\mathcal{V}} r'^2\rho(r')\cdot\frac{1}{2}(3\cos^2\theta - 1)\,d\tau' && \text{(quadrupole)} \\
&\qquad\qquad\qquad + \ldots.
\end{aligned}$$

We may proceed by determining each of these integrals until we find the lowest n with a non-zero value. This lowest term (whether it be monopole, dipole, quadrupole, ...) will be how the appoximate potential behaves at far away distances. *(I will be using integral calculator for trigonometric integrals; I do not want to waste time typesetting every step out)* Now:

$$\begin{aligned}
V_{\text{mon}}(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} k\frac{R}{r^2}(R-2r)\sin^2\theta r^2\,dr d\theta d\varphi \\
&= \frac{2\pi}{4\pi\varepsilon_0}\frac{kR}{r}\int_0^{\pi}\int_0^{R}(R-2r)\sin^2\theta\,d\theta dr \\
&= \frac{1}{2\varepsilon_0}\frac{kR}{r}\int_0^{\pi}\sin^2\theta\,d\theta\int_0^{R}(R-2r)\,dr \\
&= \frac{1}{2\varepsilon_0}\frac{kR}{r}\frac{\pi}{2}(R^2 - R^2) && \text{(integral calculator)} \\
&= 0.
\end{aligned}$$

Thus the monopole term is zero. For the dipole term,

$$\begin{aligned}
V_{\text{dip}}(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} k\frac{R}{r^2}(R-2r)\sin^2\theta\cos\theta r^3\,dr d\theta d\varphi \\
&= \frac{2\pi}{4\pi\varepsilon_0}\frac{kR}{r^2}\int_0^{\pi}\int_0^{R}(R-2r)r\sin^2\theta\cos\theta\,dr d\theta \\
&= \frac{1}{2\varepsilon_0}\frac{kR}{r^2}\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta\int_0^{R}(R-2r)r\,dr \\
&= \frac{1}{2\varepsilon_0}\frac{kR}{r^2}(0)\left(-\frac{R^3}{6}\right) && \text{(integral calculator)} \\
&= 0,
\end{aligned}$$

and so the dipole term is zero as well. And the quadrupole term:

$$\begin{aligned}
V_{\text{quad}}(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} k\frac{R}{r^2}(R-2r)\sin^2\theta\frac{1}{2}\left(3\cos^2\theta-1\right)r^4\,drd\theta d\varphi \\
&= \frac{1}{4\varepsilon_0}\frac{kR}{r^3}\int_0^{\pi}\int_0^{R}(R-2r)r^2\sin^2\theta(3\cos^2\theta-1)\,drd\theta \\
&= \frac{1}{4\varepsilon_0}\frac{kR}{r^3}\int_0^{\pi}\sin^2(3\cos^3-1)\int_0^{R}(R-2r)r^2\,dr \\
&= \frac{1}{4\varepsilon_0}\frac{kR}{r^3}\left(-\frac{\pi}{8}\right)\left(-\frac{R^4}{6}\right) && \text{(integral calculator)} \\
&= \frac{\pi}{192\varepsilon_0}\frac{kR^5}{r^3},
\end{aligned}$$

hence the quadrupole term is non-zero. Now, along the z-axis, $\theta = 0$ and $r = z\cos\theta$, hence $r = z$. Therefore the potential far away from the sphere will appear as if it were a quadrupole:

$$V(z) \approx \frac{\pi}{192\varepsilon_0}\frac{kR^5}{z^3}.$$

3.15

In this problem, I will execute a seperation of variables to solve the Laplace equation in 3-dimensional cartesian coordinates. Laplace's equation states

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Guessing the solution $V(x, y, z) = X(x)Y(y)Z(z)$ and proceeding with seperating the variables, Laplace's equation yields that

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

Now, since all of these terms sum to zero, they must all be constant, since each function only depends on a variable independent of the other variables (ie, $X(x)$ is independent of y and z, $Y(y)$ is independent of x and z, and so on). Set $\frac{X''}{X} = c_1$, $\frac{Y''}{Y} = c_2$, and $\frac{Z''}{Z} = c_3$. Thus $c_3 = -c_1 - c_2$. Allow me to invoke some symmetry into the problem. The boundary conditions of the problem state that

$$V(0, y, z) = V(a, y, z) = V(x, 0, z) = V(x, a, z) = V(x, y, 0)$$

while $V(x, y, a) = V_0$, and therefore the solutions given by the X and Y functions must be similar (or identical), since both of their boundary conditions are identical. Let me pick $c_1 = -\lambda^2$ and $c_2 = -\xi^2$. Therefore we must have that $c_3 = \lambda^2 + \xi^2$, which is positive. This will yield us similar solutions for $X(x)$ and $Y(y)$, but a different solution for $Z(z)$, which will be useful for applying the last boundary condition. Therefore we have three individual, non-coupled ordinary differential equations:

$$X'' = -\lambda^2 X, \qquad Y'' = -\xi^2 Y, \qquad Z'' = c_3 Z.$$

The solutions to these ODE's can be easily solved by an appropriate guess. They are

$$\begin{aligned}
X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\
Y(y) &= C\sin(\xi y) + D\cos(\xi y) \\
Z(z) &= Ee^{\sqrt{c_3}z} + Fe^{-\sqrt{c_3}z} = Ee^{\sqrt{\lambda^2+\xi^2}z} + Fe^{-\sqrt{\lambda^2+\xi^2}z}.
\end{aligned}$$

All of the constants (including λ and ξ) are determined by the boundary conditions, as stated above.

1. First off, we require that $X(0) = X(a) = 0$. When $x = 0$, then $X(0) = B = 0$.

2. Likewise, when $x = a$ we obtain $X(a) = A\sin(\lambda a)$. Here, however, A cannot be zero else we arrive at a trivial result for the potential. Instead, $\sin(\lambda a) = 0$, which only occurs if λa is an integer multiple of π, therefore $\lambda = \frac{n\pi}{a}$.

3. Secondly, by a similar argument as applied to the function X, $Y(0) = D = 0$, hence $D = 0$.

4. Furthermore, $Y(a) = C\sin(\xi a) = 0 \implies \xi = \frac{m\pi}{a}$.

5. Third, we require $Z(0) = 0$. Here, $Z(0) = E(1) + F(1) = 0$, which implies that $E = -F$. We now have only one constant in circulation in the potential V, since the product of two constants is indeed a constant...

6. The last boundary condition, $V(x,y,a) = V_0$, will be determined later in the calculation via a Fourier orthogonality relation.

Multiplying each of these terms together, the general expression for our potential is then given by summing over every possible integer n and m which satisfy the boundary conditions:

$$V(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[e^{\frac{\pi}{a}\sqrt{n^2+m^2}\,z} - e^{-\frac{\pi}{a}\sqrt{n^2+m^2}\,z}\right]$$
$$= 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right).$$

I will now proceed by determining the last constant G_{nm}, which is given by the last boundary condition $V(x,y,a) = V_0$:

$$V_0 = V(x,y,a) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\sqrt{n^2+m^2}\right).$$

To isolate G_{nm}, we may multiply and integrate by orthogonal terms. Recall that for integers k and l, the Fourier orthogonality relation dictates that $\int_0^a \sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi x}{a}\right)\,dx = \frac{a}{2}\delta_{kl}$. Therefore we have that

$$\int_0^a\int_0^a V_0 \sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi y}{a}\right)\,dydx = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{nm}\sinh\left(\pi\sqrt{n^2+m^2}\right).$$
$$\cdot\int_0^a \sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\,dx \int_0^a \sin\left(\frac{l\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\,dy$$
$$= 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{nm}\sinh\left(\pi\sqrt{n^2+m^2}\right).\frac{a^2}{4}\delta_{kn}\delta_{lm}$$
$$= \frac{a^2}{2}G_{kl}\sinh\left(\pi\sqrt{k^2+l^2}\right)$$
$$\implies G_{kl} = \frac{2}{a^2\sinh\left(\pi\sqrt{k^2+l^2}\right)}\int_0^a\int_0^a V_0\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi y}{a}\right)\,dydx.$$

Since the double integral on the right hand side of the above relation is just two seperate integrals of seperate variables (x and y), we may evaluate it. It is

$$G_{kl} = \frac{2}{a^2\sinh\left(\pi\sqrt{k^2+l^2}\right)}\int_0^a\int_0^a V_0\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi y}{a}\right)\,dydx$$
$$= \frac{2V_0}{a^2\sinh\left(\pi\sqrt{k^2+l^2}\right)}\cdot\frac{a}{k\pi}\cdot\frac{a}{l\pi}(-1)^2\cos\left(\frac{k\pi x}{a}\right)\Bigg|_{x=0}^{x=a}\cos\left(\frac{l\pi y}{a}\right)\Bigg|_{y=0}^{y=a}$$
$$= \frac{2V_0}{\pi^2 kl\sinh\left(\pi\sqrt{k^2+l^2}\right)}(\cos(k\pi)-1)(\cos(l\pi)-1).$$

It is now important to notice that whenever $\cos(k\pi)$ or $\cos(l\pi)$ are 1, then $G_{kl} = 0$. This only occurs when k and l are even multiples of π $(0, 2\pi, 4\pi, \ldots)$. G_{kl} does not vanish at all if both k

and l are even integers. Therefore

$$G_{kl} = \begin{cases} 0 & \text{if } m \text{ or } n \text{ even} \\ \dfrac{8V_0}{\pi^2 kl \sinh\left(\pi\sqrt{k^2+l^2}\right)} & \text{if } m \text{ and } n \text{ odd.} \end{cases}$$

Under a brief index change, we just have G_{nm}, which we may not include in our initial expression for the potential above. Summing over odd values of m and n only, we finally obtain the potential inside the cube:

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\, z\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}.$$

3.49

For this problem, we may consider the expression of the potential of a dipole in spherical coordinates:

$$V_{\mathrm{dip}}(r,\theta,\varphi) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}.$$

It suffices to first determine the electric field of this potential. Once the electric field is found, we may apply the proportionality relation $\mathbf{F} = q\mathbf{E} = -q\boldsymbol{\nabla}V$. We have, in spherical coordinates,

$$\begin{aligned}\mathbf{F} = -q\boldsymbol{\nabla}V &= -q\left[\frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\varphi}}\right] \\ &= -\frac{q}{4\pi\varepsilon_0}\left[-\frac{2p\cos\theta}{r^3}\hat{\mathbf{r}} - \frac{1}{r}\frac{p\sin\theta}{r^2}\hat{\boldsymbol{\theta}}\right] \\ &= \frac{q}{4\pi\varepsilon_0}\frac{p}{r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right],\end{aligned}$$

which gives the expression for the force on the charge in the presence of the dipole. Now, for a pendulum in the classical regime, the Euler-Lagrange equations yield the equations of motion for the Lagrangian in polar coordinates,

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta \implies m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta, \quad mr^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = mgr\sin\theta.$$

I will note here that θ represents the angle apart from the vertical z axis along unstable equilibrium, and not the angle with respect to equilbrium. For constant r, we obtain the single equation of motion $ml\ddot{\theta} = mg\sin\theta = F_\theta$, we notice the drastic similarity with the above expression for the radial force of the charge q being a distance l away from the dipole:

$$F_\theta = \frac{q}{4\pi\varepsilon_0}\frac{p}{l^3}\sin\theta \longleftrightarrow F_\theta = mg\sin\theta.$$

With the radial component of the force being independent of the motion of the angle, the charge thus behaves as if it were a pendulum mounted at the origin.