MAT237 Multivariable Calculus with Proofs
Problem Set 4
Due Friday December 3, 2021 by 13:00 ET

Instructions

This problem set is based on Module D: Inverse and implicit functions (D1 to D7). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 4 sessions are held on Tuesday November 30, 2021 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope.** Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: **Jace Alloway** ________________

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Student number: ________________

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway ________________

2) ________________

Created and revised by C. Jonker, T. Janisse, and A. Zaman. Last updated December 3, 2021 at 15:34:17.

Problems

1. The inverse function theorem is spectacular since it reduces all of your calculations to linear algebra. The computations can be large, but they are tractable and that is a major achievement. Define $F : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$F\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b+c+d-4 \\ ab+ac+ad+bc+bd+cd-6 \\ abc+acd+abd+bcd-4 \\ abcd-1 \end{bmatrix}$$

For all parts of this question, you may use WolframAlpha to do any basic matrix or algebraic computations. If you do, please indicate when you have done so and state the outcome of those calculations.

(1a) Prove that F is a local diffeomorphism at $(a,b,c,d) \in \mathbb{R}^4$ if and only if $a,b,c,d \in \mathbb{R}$ are all distinct.

Proof.

- Assume F is a local diffeomorphism. Then there exists an open subset $U \subseteq \mathbb{R}^4$ containing $(a,b,c,d) \in \mathbb{R}^4$ such that $F|_U : U \to F(U)$ is a diffeomorphism.
- Note that F is C^1 at (a,b,c,d) because it is continuous at (a,b,c,d) and it's derivative is continuous as (a,b,c,d).
- By **Corollary 4.2.4**, since U and $F(U)$ are open subsets, then $DF(a,b,c,d)$ is an invertible $n \times n$ matrix. Computing, we find that

$$DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ b+c+d & a+c+d & a+b+d & a+b+c \\ bc+cd+bd & ac+ad+cd & ab+ad+bd & ac+ab+bc \\ bcd & acd & abd & abc \end{bmatrix}.$$

- Since $DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right)$ is an invertible $n \times n$ matrix, it must be that $\det DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) \neq 0$.
- By the almighty Wolfram Alpha, we find that

$$\det DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

- If $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \neq 0$, we it must be that a, b, c and d are all distinct.
- Conversely, if a, b, c and d are all distinct, we have that $\det DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) \neq 0$. This implies that $DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right)$ is invertible.
- As before, we know that F is a C^1 function and it's Jacobian $DF\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right)$ is invertible at (a,b,c,d). Thus, by the **Inverse Function Theorem (Theorem 4.2.8)**, F is a local diffeomorphism at (a,b,c,d).

$\square$

(1b) Let G be a local C^1 inverse of F at $(1,2,3,4)$, which exists by (1a). Give a linear approximation for $G(w,x,y,z)$ with (w,x,y,z) near $F(1,2,3,4)=(6,29,46,23)$.

- We know that $F(1,2,3,4)=(6,29,46,23)$. Then if G is a local C^1 inverse of F at $(1,2,3,4)$, then $G(6,29,46,23)=(1,2,3,4)$.
- Similarly, by **Theorem 4.2.1**, $DG(6,29,46,23)=[DF(1,2,3,4)]^{-1}$. We find that

$$DF(1,2,3,4)=\begin{bmatrix}1 & 1 & 1 & 1\\ 9 & 8 & 7 & 6\\ 26 & 19 & 14 & 11\\ 24 & 12 & 8 & 6\end{bmatrix}.$$

  Once again, Wolfram Alpha comes in clutch and yields that

$$DG(6,29,46,23)=\begin{bmatrix}49/30 & -13/30 & 1/30 & 1/10\\ -34/5 & 8/5 & -1/5 & -1/10\\ 27/10 & -9/10 & 3/10 & -1/10\\ 52/15 & -4/15 & -2/15 & 1/10\end{bmatrix}$$

  by taking the inverse of $DF(1,2,3,4)$.
- The linear approximation of G near $(6,9,46,23)$ is given by

$$G(w,x,y,z)\approx G(6,29,46,23)+DG(6,29,46,23)((w,x,y,z)-(6,29,46,23)).$$

- We have

$$\begin{aligned}G(w,x,y,z)&\approx\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+\begin{bmatrix}49/30 & -13/30 & 1/30 & 1/10\\ -34/5 & 8/5 & -1/5 & -1/10\\ 27/10 & -9/10 & 3/10 & -1/10\\ 52/15 & -4/15 & -2/15 & 1/10\end{bmatrix}\begin{bmatrix}w-6\\x-29\\y-46\\z-23\end{bmatrix}\\ &=\begin{bmatrix}\frac{49w-13x+y+3z-2}{30}\\ \frac{-68w+16x-2y-z+79}{10}\\ \frac{27w-9x+3y-z+14}{10}\\ \frac{104w-8x-4y+3z-157}{30}\end{bmatrix}\end{aligned}$$

(1c) Can you use your approximation to estimate $G(6.1,28.9,46.02,23)$? Explain in at most 2 full sentences.

Yes, you can use the approximation to estimate $G(6.1,28.9,46.02,23)$ (this is why it is called an approximation). This is because you are plugging in a point close to $G(6,29,46,23)$, and so the linear approximation should yield a point close to $(1,2,3,4)$. Computing,

$$G(6.1,28.9,46.02,23)\approx\begin{bmatrix}\frac{49(6.1)-13(28.9)+(46.02)+3(23)-2}{30}\\ \frac{-68(6.1)+16(28.9)-2(46.02)-(23)+79}{10}\\ \frac{27(6.1)-9(28.9)+3(46.02)-(23)+14}{10}\\ \frac{104(6.1)-8(28.9)-4(46.02)+3(23)-157}{30}\end{bmatrix}=\begin{bmatrix}1.207\\1.156\\3.366\\4.371\end{bmatrix}.$$

2. The implicit function theorem (IFT) allows you to exhibit local solutions, but it cannot prove that none exist. You shall see this with the figure eight curve.

   (2a) Use the IFT to prove that $x^4 = x^2 - y^2$ locally defines x as a C^1 function of y near $(1,0)$.

Proof.

- Define the set $U = (\sqrt{2}, \infty) \times (-0.5, 0.5) \subseteq \mathbb{R} \times \mathbb{R}$.
- Define the function $F : U \to \mathbb{R}$ by $F(x,y) = x^4 - x^2 + y^2$. Since F is a linear combination of polynomials, by **Lemma 2.7.24**, F is continuous. Likewise, all of the partials of F exist and are again polynomials (and hence continuous), and thus F is a real-valued C^1 function on U.
- We know $(1,0) \in U$ since $1 \in (\sqrt{2}, \infty)$ and $0 \in (-0.5, 0.5)$.
- We find that $F(1,0) = (1)^4 - (1)^2 + (0)^2 = 0$ and $\dfrac{\partial F}{\partial x}(1,0) = 4 - 2 = 2 \neq 0$.
- Therefore, by the **Implicit Function Theorem (Theorem 4.4.6)**, the equation $F(x,y) = x^4 - x^2 + y^2 = 0 \implies x^4 = x^2 - y^2$ locally defines x as a C^1 function of y near $(1,0)$.

$\square$

   (2b) Here is a flawed proof that $x^4 = x^2 - y^2$ does not locally define y as a C^1 function of x near $(0,0)$.

> 1. *Define* $F : \mathbb{R}^2 \to \mathbb{R}$ *by* $F(x,y) = x^4 - x^2 + y^2$.
> 2. *Suppose, for a contradiction, that* $F(x,y) = 0$ *locally defines* y *as a* C^1 *function of* x *near* $(0,0)$.
> 3. *There exists* $V, W \subseteq \mathbb{R}$ *open sets containing* 0 *and a* C^1 *function* $\phi : V \to W$ *such that*
> $$\forall (x,y) \in V \times W, \quad F(x,y) = 0 \iff y = \phi(x). \tag{1}$$
> 4. *Since* V *is open and* $0 \in V$, *there exists* $0 < \varepsilon < 1$ *such that* $(-2\varepsilon, 2\varepsilon) \subseteq V$.
> 5. *Notice that* $F(\varepsilon, \sqrt{\varepsilon^2 - \varepsilon^4}) = 0$ *and* $F(\varepsilon, -\sqrt{\varepsilon^2 - \varepsilon^4}) = 0$.
> 6. *From* (1), *this implies that* $\phi(\varepsilon) = \sqrt{\varepsilon^2 - \varepsilon^4}$ *and* $\phi(\varepsilon) = -\sqrt{\varepsilon^2 - \varepsilon^4}$.
> 7. *This is a contradiction since* ϕ *is a function.*

Identify and briefly explain the flaw in at most two full sentences. Do not explain how to fix it.

The fatal flaw of the proof is located in line 4.

- The problem with the proof is that the author has defined a new subset $Q = (-2\varepsilon, 2\varepsilon) \subseteq V$ with $0 < \varepsilon < 1$, which is defined *after* defining V and W.
- Although $\phi : V \to W$, we do not have any information to show that $\phi(Q)$ is a subset of W, which is the actual function the author is using, and hence failing to defining y as a C^1 function of x near (0,0).

3. Let $F : \mathbb{R}^3 \to \mathbb{R}$ and $G : \mathbb{R}^3 \to \mathbb{R}$ be C^1 functions. Consider the implicit curve given by

$$S = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0, G(x, y, z) = 0\}$$

Fix a point $p = (a, b, c) \in S$.

(3a) Prove that if $\{\nabla F(p), \nabla G(p)\}$ is linearly independent, then S is a regular 1-dimensional surface at p.

Proof.

- Define a new function $H : \mathbb{R}^3 \to \mathbb{R}^2$ by $H(x, y, z) = (F(x, y, z), G(x, y, z))$. Note that since F and G are both C^1 on $\mathbb{R}^3$, H is a vector-valued C^1 function on $\mathbb{R}^3$.
- The set S becomes
$$S = \{(x, y, z) \in \mathbb{R}^3 : H(x, y, z) = 0\}.$$
Note that S is non-empty since $p \in S$.
- Assume that $\{\nabla F(p), \nabla G(p)\}$ is linearly independent. Taking the Jacobian of H, we have that
$$DH(p) = \begin{bmatrix} \partial_1 F(p) & \partial_2 F(p) & \partial_3 F(p) \\ \partial_1 G(p) & \partial_2 G(p) & \partial_3 G(p) \end{bmatrix}.$$
- Since the rows $(\partial_1 F(p) \quad \partial_2 F(p) \quad \partial_3 F(p)) = \nabla F(p)^T$ and $(\partial_1 G(p) \quad \partial_2 G(p) \quad \partial_3 G(p)) = \nabla G(p)^T$ are linearly independent, it follows that the Jacobian of H at p has full rank, which is 2.
- Therefore, by **Theorem 4.5.7**, S is a 3-2=1 dimensional regular surface at p.

$\square$

(3b) For (3b) and (3c), assume that

$$\partial_2 F(p)\partial_3 G(p) - \partial_3 F(p)\partial_2 G(p) \neq 0.$$

Show that S is a regular curve at p and, also, S locally defines (y, z) as a C^1 function ϕ of x near p.

- As in **3a**, define a function $H : \mathbb{R}^3 \to \mathbb{R}^2$ by $H(x, y, z) = (F(x, y, z), G(x, y, z))$. H is C^1 because both F and G are C^1 on $\mathbb{R}^3$.
- Again, S becomes
$$S = \{(x, y, z) \in \mathbb{R}^3 : H(x, y, z) = 0\}.$$
Note that S is non-empty since $p \in S$.
- Taking the Jacobian of H at p yields
$$DH(p) = \begin{bmatrix} \partial_1 F(p) & \partial_2 F(p) & \partial_3 F(p) \\ \partial_1 G(p) & \partial_2 G(p) & \partial_3 G(p) \end{bmatrix}.$$
- Assume that
$$\partial_2 F(p)\partial_3 G(p) - \partial_3 F(p)\partial_2 G(p) = \det\begin{pmatrix} \partial_2 F(p) & \partial_3 F(p) \\ \partial_2 G(p) & \partial_3 G(p) \end{pmatrix} \neq 0,$$
which implies that the pair
$$\left\{ \begin{pmatrix} \partial_2 F(p) \\ \partial_2 G(p) \end{pmatrix}, \begin{pmatrix} \partial_3 F(p) \\ \partial_3 G(p) \end{pmatrix} \right\}$$
is linearly independent, which directly implies that $DH(p)$ has full rank, which is 2.
- From **3a** and **Theorem 4.5.7**, S is a 1-dimensional regular curve at p.
- Furthermore, let $U = \mathbb{R} \times \mathbb{R}^2$ such that $H : U \to \mathbb{R}^2$. As before, H is a C^1 map. Since $p \in S$, then $H(p) = 0$.
- As assumed, thus the 2×2 matrix
$$\frac{\partial(F, G)}{\partial(y, z)} = \begin{pmatrix} \partial_2 F(p) & \partial_3 F(p) \\ \partial_2 G(p) & \partial_3 G(p) \end{pmatrix}$$
is invertible since
$$\det\begin{pmatrix} \partial_2 F(p) & \partial_3 F(p) \\ \partial_2 G(p) & \partial_3 G(p) \end{pmatrix} \neq 0.$$
- By the **Implicit Function Theorem (Theorem 4.4.13)**, $H(x, y, z) = 0$ locally defines (y, z) as an $\mathbb{R}^2$ valued C^1 function ϕ of x near p.

(3c) Use ϕ from (3b) to formally prove the identity

$$\frac{\partial y}{\partial x} = \frac{\partial_3 F \partial_1 G - \partial_1 F \partial_3 G}{\partial_2 F \partial_3 G - \partial_3 F \partial_2 G}.$$

Be sure to clearly state at which points this identity holds.

Proof.

- As in **3a** and **3b**, define the function $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by $H(x, y, z) = (F(x, y, z), G(x, y, z))$. Since both F and G are C^1 functions on $\mathbb{R} \times \mathbb{R}^2$, H is a C^1 function on $\mathbb{R} \times \mathbb{R}^2$. Note that $\mathbb{R} \times \mathbb{R}^2$ is certainly open.
- From **3b**, the equation $H(x, y, z) = 0$ locally defines (y, z) as an $\mathbb{R}^2$ valued C^1 function $\phi : V \to W$ of x near $p = (a, b, c) \in S$, where $V \subseteq \mathbb{R}$ and $W \subseteq \mathbb{R}^2$.
  Thus, we have that $a \in V$ and $(b, c) \in W$
- Define a new function $\Sigma : V \to V \times W$ by $\Sigma(x) = (x, \phi(x))$. For $x \in V$, we then have that $H \circ \Sigma(x) = 0$.
- Fix $v \in V$ and $w = \phi(v)$. It must be that the Jacobian of $H \circ \Sigma : V \to \mathbb{R}^2$ at v must be the 2×1 zero matrix, which is
$$D(H \circ \Sigma)(v) = 0_{2\times 1}.$$
- Since $\Sigma(v) = (v, \phi(v)) = (v, w)$, by the chain rule, $D(H \circ \Sigma)(v) = DH(v, w)D\Sigma(v)$.
  By direct calculation, we verify that
$$0 = \frac{\partial H}{\partial x}(v, w) + \frac{\partial H}{\partial y}(v, w)D\phi(v).$$
- From **3b**, since the 2×2 matrix $\frac{\partial H}{\partial y}(v, w) = \begin{pmatrix} \partial_2 F(v, w) & \partial_3 F(v, w) \\ \partial_2 G(v, w) & \partial_3 G(v, w) \end{pmatrix}$ is invertible at p, then the Jacobian $D\phi(p)$ is a 2×1 matrix satisfying
$$\begin{pmatrix} \partial_1 \phi_1(v) \\ \partial_1 \phi_2(v) \end{pmatrix} = -\left[\begin{pmatrix} \partial_2 F(v, w) & \partial_3 F(v, w) \\ \partial_2 G(v, w) & \partial_3 G(v, w) \end{pmatrix}\right]^{-1} \begin{pmatrix} \partial_1 F(v, w) \\ \partial_1 G(v, w) \end{pmatrix}.$$
- It is easily verifiable that
$$\begin{pmatrix} \partial_1 \phi_1 \\ \partial_1 \phi_2 \end{pmatrix}(v) = -\left[\begin{pmatrix} \partial_2 F & \partial_3 F \\ \partial_2 G & \partial_3 G \end{pmatrix}\right]^{-1}(v, w) \begin{pmatrix} \partial_1 F \\ \partial_1 G \end{pmatrix}(v, w) = \frac{1}{\partial_2 F(v, w)\partial_3 G(v, w) - \partial_3 F(v, w)\partial_2 G(v, w)} \begin{pmatrix} \partial_1 G \partial_3 F - \partial_1 F \partial_3 G \\ \partial_1 F \partial_2 G - \partial_1 G \partial_2 F \end{pmatrix}(v, w).$$
- Since $\phi(x) = (\phi_1(x), \phi_2(x))$ is a C^1 function of x defining (y, z) near p, then $\phi_1(x) = y$ and $\phi_2(x) = z$ near p. Then
$$\partial_1 \phi(v) = \frac{\partial_3 F(v, w)\partial_1 G(v, w) - \partial_1 F(v, w)\partial_3 G(v, w)}{\partial_2 F(v, w)\partial_3 G(v, w) - \partial_3 F(v, w)\partial_2 G(v, w)} \implies \frac{\partial y}{\partial x} = \frac{\partial_3 F \partial_1 G - \partial_1 F \partial_3 G}{\partial_2 F \partial_3 G - \partial_3 F \partial_2 G},$$
  which holds for any $v \in V$ in the neighborhood of p for $\phi : V \to W$.

□

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4. Mega-gaming company SO-KNEE is now planning production of the PayStation 3. They want to sell each PS3 for \$550. To produce N million PayStation consoles, SO-KNEE must spend L millions of dollars on their employees, K millions of dollars on capital, and M millions of dollars on materials. This will generate P millions of dollars in profit. A consultant's model estimates that N is given by the Cobb-Douglas function
$$N = \frac{1}{25}K^{1/4}L^{1/3}M^{1/6}.$$
PayStations are a hot commodity, so every console they produce will sell. Their total budget is 2 billion dollars, but they do not necessarily want to spend all of it. SO-KNEE wants to maximize their profit.

   For both (4a) and (4b), you may use WolframAlpha to solve systems of equations. Indicate when you do so.
   Do not write any part of your solution to this question on this page.

(4a) Show that SO-KNEE must spend all of their budget to maximize their profit. Do not find the maximum profit when they spend their entire budget; that will be computed in (4b).

- The budget constraint is given by $K+L+M \leq 2000$, in millions of dollars (their budget is 2 billion dollars).
- Define the set $U = \{(K,L,M) \in \mathbb{R}^3 : K,L,M \geq 0,\ K+L+M \leq 2000\}$. We are wanting to maximize profit on U. I will show that the only way profit can be maximized is if SO-KNEE spends all of their budget.
- Define the profit function $P : U \to \mathbb{R}$ by
$P(K,L,M) = 550N(K,L,M) - (K+L+M) = 22K^{1/4}L^{1/3}M^{1/6} - (K+L+M)$.
- It is easily verifiable that U is compact and non-empty. Since P is continuous on U then by the **Global Extreme Value Theorem**, P attains maximum and minimum values at points in U.
- Define the following subsets of U:

$$\begin{aligned}
&A = \{(K,L,M) \in \mathbb{R}^3 : K,L,M > 0,\ K+L+M < 2000\} \qquad \text{(interior)}\\
&B = \{(K,L,M) \in \mathbb{R}^3 : L = 0, 0 < K+M \leq 2000\} \cup \{(K,L,M) \in \mathbb{R}^3 : K = 0, 0 < L+M \leq 2000\} \cup\\
&\qquad \{(K,L,M) \in \mathbb{R}^3 : M = 0, 0 < K+L \leq 2000\} \qquad \text{(side planes)}\\
&C = \{\{(K,L,M) \in \mathbb{R}^3 : K,L,M > 0, K+L+M = 2000\}\} \qquad \text{(outer plane)}\\
&D = \{(2000,0,0),(0,2000,0),(0,0,2000)\} \qquad \text{(axes)}
\end{aligned}$$

- Therefore if P attains a maximum on U, P either attains its maximum on A, B, C or D because $U = A \cup B \cup C \cup D$.
- We begin by checking A for critical points:

$$\nabla P(K,L,M) = \left(\frac{22L^{1/3}M^{1/6}}{4K^{3/4}} - 1, \frac{22K^{1/4}M^{1/6}}{3L^{2/3}} - 1, \frac{22K^{1/4}L^{1/3}}{6M^{5/6}} - 1\right) = (0,0,0).$$

Again, Wolfram Alpha saves the day and yields that the solution to this system of 3 equations is

$$K = \frac{14641}{9(2)^{2/3}}, \qquad L = \frac{29282(2)^{1/3}}{27}, \qquad M = \frac{14641(2)^{1/3}}{27}.$$

The issue with this critical point is that it is actually over budget, yielding $K+L+M \approx 3'074 > 2000$. Therefore there are actually no points of extrema of P on A.
- Next, we can determine that there are no critical points in B: since for any subset of B, either K, L, or $M = 0$, which would imply that the profit function becomes negative. Therefore there are no critical points of P on B.
- Likewise with the subset boundary D, there are can be no critical points in D because in any case, the profit function P would be negative, therefore not maximizing profit on U.
- Thus, by the process of elimination, it must be that the maximum of P on U is found in the boundary subset C, when SO-KNEE spends all of their budget.

(4b) What is SO-KNEE's projected maximum profit and how should they spend their money to achieve it? Use (4a) and the method of Lagrange multipliers. Round to the nearest million dollars and summarize your final answer in a full sentence.

- As in **4a**, define the subset of U $C = \{\{(K,L,M) \in \mathbb{R}^3 : K,L,M > 0, K+L+M = 2000\}\}$.
- Similarly, recall the profit function define in **4a** $P : U \to \mathbb{R}$ by $P(K,L,M) = 550N(K,L,M) - (K+L+M) = 22K^{1/4}L^{1/3}M^{1/6} - (K+L+M)$.
- Define the budget function $B : \mathbb{R}^3 \to R$ by $B(K,L,M) = K+L+M$. We are wanting to maximize P on C subject to the constraint $B(K,L,M) = 2000$.
- By the method of Lagrange Multipliers, the maximum of P on C must satisfy the system of equations given by

$$\begin{aligned} &\nabla P(K,L,M) = \lambda \nabla B(K,L,M); \qquad B(K,L,M) = 2000 \\ \iff &\frac{22L^{1/3}M^{1/6}}{4K^{3/4}} = \lambda + 1, \qquad \frac{22K^{1/4}M^{1/6}}{3L^{2/3}} = \lambda + 1, \qquad \frac{22K^{1/4}L^{1/3}}{6M^{5/6}} = \lambda + 1, \qquad K+L+M = 2000. \end{aligned}$$

- After working incredibly hard, Wolfram Alpha solves the system and gives:

$$K = \frac{2000}{3},\ L = \frac{8000}{9},\ M = \frac{4000}{9},\ \lambda = \frac{11}{2(2)^{1/6}(3)^{1/4}(5)^{3/4}} - 1.$$

- Thus, the solution is $(K,L,M) = \left(\frac{2000}{3}, \frac{8000}{9}, \frac{4000}{9}\right)$. From **4a**, this is the only critical point of P on U. Evaluating,

$$P\left(\frac{2000}{3}, \frac{8000}{9}, \frac{4000}{9}\right) = 969.285,$$

  in millions of dollars.
- Overall, from **4a**, the maximum of P on U occurs only when SO-KNEE spends all of their budget. This maximum value of profit is \$969 million dollars, and to achieve this, SO-KNEE should spend $\frac{2000}{3} \approx 667$ million dollars on capital, $\frac{8000}{9} \approx 889$ million dollars on their employees, and $\frac{4000}{9} \approx 444$ million dollars on materials.

5. Here you will establish the raison d'être for diffeomorphisms.

A diffeomorphism between surfaces yields an isomorphism between tangent spaces.

Note an isomorphism is an invertible linear transformation between subspaces of $\mathbb{R}^n$.

Let $S \subseteq \mathbb{R}^n$ be a set and let $p \in S$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism.

(5a) Let v be a tangent vector of S at p. Show that $dF_p(v) \in \mathbb{R}^n$ is a tangent vector of $F(S)$ at $F(p)$.

- Assume v is a tangent vector of S at p. Then, there exists an open interval $I \subseteq \mathbb{R}$ containing 0 and a differentiable function $\gamma : I \to \mathbb{R}^n$ such that $\gamma(I) \subseteq S, \gamma(0) = p$, and $\gamma'(0) = v$.
- We have that if $\gamma(0) = p$, then $F(p) = F(\gamma(0))$. Furthermore, since $\gamma(I) \subseteq S$, then $F(\gamma(I)) \subseteq F(S)$.
- Suppose $t \in I$. We have $F \circ \gamma : I \to \mathbb{R}^n$, and thus by differentiating using the chain rule,

$$D(F \circ \gamma)(t) = DF(\gamma(t))D\gamma(t).$$

- Evaluating at $t = 0$ ($\gamma(0) = p$), we have that

$$DF(\gamma(0))D\gamma(0) = DF(p)\gamma'(0) = DF(p)v = dF_p(v)$$

  by **Theorem 3.5.17**, and therefore $dF_p(v) \in \mathbb{R}^n$ is a tangent vector of $F(S)$ at $F(p)$.

(5b) Let w be a tangent vector of $F(S)$ at $F(p)$. Use (5a) to show $(dF_p)^{-1}(w)$ is a tangent vector of S at p.

- If w is a tangent vector of $F(S)$ at $F(p)$, then there exists an open interval $I \subseteq \mathbb{R}$ containing 0 and a differentiable function $\gamma : I \to \mathbb{R}^n$ such that $\gamma(I) \subseteq F(S)$, $\gamma(0) = F(p)$, and $\gamma'(0) = w$.
- Since F is a diffeomorphism, by **Definition 4.1.4**, F has a C^1 inverse function $G : \mathbb{R}^n \to \mathbb{R}^n$.
- We have that if $\gamma(0) = F(p)$, then $G(F(p)) = G(\gamma(0)) = p$. Furthermore, since $\gamma(I) \subseteq F(S)$, then $G(\gamma(I)) \subseteq G(F(S)) = S$.
- Suppose $t \in I$. We have $G \circ \gamma : I \to \mathbb{R}^n$, and thus by differentiating using the chain rule,

$$D(G \circ \gamma)(t) = DG(\gamma(t))D\gamma(t).$$

- Evaluating at $t = 0(\gamma(0) = F(p))$, we have that

$$DG(\gamma(0))D\gamma(0) = DG(F(p))\gamma'(0) = DG(F(p))w = [DF(p)]^{-1}(w) = (dF_p)^{-1}(w)$$

  by **Theorem 3.5.17**, and therefore $(dF_p)^{-1}(w) \in \mathbb{R}^n$ is a tangent vector of S at p.

(5c) Show that if T_pS is a k-dimensional subspace of $\mathbb{R}^n$, then $T_{F(p)}F(S)$ is a k-dimensional subspace of $\mathbb{R}^n$.
Hint: Use both (5a) and (5b) and some linear algebra facts.

- Assume T_pS is a k-dimensional subspace of $\mathbb{R}^n$ and let $\{v_1, v_2, \dots, v_k\}$ be a basis for T_pS.
- Since F is a diffeomorphism, then F is a diffeomorphism at $p \in S$. Since $\mathbb{R}^n$ is open, by **Theorem 4.2.1**, the Jacobian of F at p is an invertible $n \times n$ matrix and the Jacobian of the inverse $G = F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ both satisfy

$$DG(y) = [DF(x)]^{-1} \qquad \text{for every } x \in U \text{ and } y = F(x).$$

- Define the linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ by $L(x) = DF(x)$. L is an isomorphism because it is invertible (since $DF(x)$ is invertible, as before).
- It now suffices to show that $L(T_pS) = T_{F(P)}F(S)$. Notice that each basis element $v_i \in T_pS \subseteq \mathbb{R}^n$ for $i = 1, 2, \dots, k$ are tangent vectors to S at p. By **3a**, $L(v_i) = dF_p(v_i)$ for $i = 1, 2, \dots, k$, and thus $\{dF_p(v_1), dF_p(v_2), \dots, dF_p(v_k)\}$ is a basis (with k elements) for $T_{F(p)}F(S)$ since each $dF_p(v_i)$ is a tangent vector to $F(S)$ at $F(p)$. Thus $L(T_pS) = T_{F(P)}F(S)$ and $\dim T_{F(p)}F(S) = k$.
- Conversely, it now suffices to show that $L^{-1}T_{F(p)}F(S) = T_pS$. Let $\{w_1, w_2, \dots, w_k\}$ be a basis for $T_{F(p)}F(S)$. Then, each $w_i \in T_{F(S)}F(p) \subseteq \mathbb{R}^n$ for $i = 1, 2, \dots, k$ are tangent vectors to $F(S)$ at $F(p)$. By **3b**, $L^{-1}(w_i) = (dF_p)^{-1}(w_i)$ for $i = 1, 2, \dots, k$, and thus $\{(dF_p)^{-1}(w_1), (dF_p)^{-1}(w_2), \dots, (dF_p)^{-1}(w_k)\}$ is a basis (with k elements) for T_pS since each $(dF_p)^{-1}(w_i)$ is a tangent vector to S at p. Thus $L^{-1}(T_{F(S)}F(p)) = T_pS$ and $\dim T_pS = k$.
- Therefore since $\dim T_pS = \dim T_{F(p)}F(S) = k$, then if T_pS is a k-dimensional subspace of $\mathbb{R}^n$, then $T_F(p)F(S)$ is a k-dimensional subspace of $\mathbb{R}^n$.