MAT237 Multivariable Calculus with Proofs
Problem Set 1
Due Friday October 1, 2021 by 13:00 ET

Instructions

This problem set is based on Module A: Maps (A1 to A5) and Module B: Topology (B1 to B2). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 1 sessions are held on Tuesday September 28, 2021 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope.** Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: **Jace Alloway**

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Full Name: ______

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I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway

2) ______

Created and revised by B. Abuelnasr, T. Janisse, and A. Zaman. Last updated September 30, 2021 at 00:09:36.

Problems

1. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a parametric curve with components f, g, h so $\gamma(t) = (f(t), g(t), h(t))$. Assume its component functions are infinitely differentiable. Determine which of the following statements are true or false. If false, give a counterexample. If true, explain why.

   (1a) For all $t \in \mathbb{R}$, the unit tangent vector $T(t)$ is defined.

   ◯ TRUE √ **FALSE**

Let $f(t) = g(t) = h(t) = 0$, the zero function. These functions are certainly infinitely differentiable, since $\dfrac{d}{dt}[0] = 0$.

If we consider the norm of $\gamma(t)$, we have $||\gamma'(t)|| = \sqrt{(0,0,0)\cdot(0,0,0)} = 0$. Since the unit tangent vector is defined as $T(t) = \dfrac{\gamma'(t)}{||\gamma'(t)||}$, we have that the tangent vector is undefined for all $t \in \mathbb{R}$, since we have $T(t) = \dfrac{(0,0,0)}{0}$.

   (1b) For all $t \in \mathbb{R}$, the unit normal vector $N(t)$, if it exists, satisfies $N(t) = \dfrac{\gamma''(t)}{||\gamma''(t)||}$.

   ◯ TRUE √ **FALSE**

Let $\gamma(t) = \left(e^t, e^t, t^2\right)$. Then $\gamma'(t) = (e^t, e^t, 2t)$ and $||\gamma'(t)|| = \sqrt{2e^{2t} + 4t^2}$.

$T(t)$ then is $\dfrac{(e^t, e^t, 2t)}{\sqrt{2e^{2t} + 4t^2}}$. Differentiating using the quotient rule, we have

$$T'(t) = \left(\frac{e^t(2e^{2t} + 4t^2) - e^t(4e^{2t} + 8t)}{2(2e^{2t} + 4t^2)^{3/2}}, \frac{e^t(2e^{2t} + 4t^2) - e^t(4e^{2t} + 8t)}{2(2e^{2t} + 4t^2)^{3/2}}, \frac{2(2e^{2t} + 4t^2) - (4e^{2t} + 8t)}{(2e^{2t} + 4t^2)^{3/2}} \right).$$

Then

$$||T'(t)|| = \sqrt{2\left(\frac{e^t(2e^{2t} + 4t^2) - e^t(4e^{2t} + 8t)}{2(2e^{2t} + 4t^2)^{3/2}}\right)^2 + \left(\frac{2(2e^{2t} + 4t^2) - (4e^{2t} + 8t)}{(2e^{2t} + 4t^2)^{3/2}}\right)^2}.$$

Since $N(t) = \dfrac{T'(t)}{||T'(t)||}$, then if this statement was true, $T'(t) = \gamma''(t)$ and $||T'(t)|| = ||\gamma''(t)||$. However, this is clearly not the case since $\gamma''(t) = (e^t, e^t, 2)$ and $||\gamma''(t)|| = \sqrt{2e^{2t} + 4}$.

Therefore this statement is false.

   (1c) For all $t \in \mathbb{R}$, the unit binormal vector $B(t)$ is uniquely defined provided $T(t)$ and $N(t)$ are both defined.

   √ **TRUE** ◯ FALSE

If $T(t)$ and $N(t)$ are both defined, then their magnitudes are both non-zero. Since the vector $B(t)$ is defined by the cross product

$$B(t) = T(t) \times N(t) = ||T(t)||||N(t)|| \sin(\pi/2)\hat{k} = ||T(t)||||N(t)||\hat{k}.$$

Since $T(t)$ and $N(t)$ by definition are orthogonal to each other, then $B(t)$ is uniquely defined.

2. Factory wind tunnels are used to analyze the aerodynamic properties of fast moving objects like bikes, cars, trucks, planes, or spaceships. Large wind turbines combined with pulses of smoke create flow lines (see 0:20 to 0:40 of this video) that can be studied to perfect a design.

   Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the time-dependent velocity vector field of the air flow in a wind tunnel. At time $t \in \mathbb{R}$, the vector $F(x,t)$ is the velocity of a particle in the air at point $x \in \mathbb{R}^3$.

   (2a) A **pathline** is the trajectory that individual fluid particles follow. The velocity of the particle will be determined by the velocity vector field at each moment in time. Write a formal statement that describes when the trace of $\gamma : \mathbb{R} \to \mathbb{R}^3$ is a pathline of F.

Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be a time-dependent vector field. A function $\gamma : \mathbb{R} \to \mathbb{R}^3$ describing a particle's motion in F is a **pathline** of F if for all $t \in \mathbb{R}$, $\gamma(t) = F(x,t)$.

   (2b) A **streamline** is the trajectory that individual fluid particles would follow if the flow were stable from a fixed moment in time and onwards. The velocity of the particle would be determined by the velocity vector field at that fixed moment in time. Write a formal statement that describes when the trace of $\gamma : \mathbb{R} \to \mathbb{R}^3$ is a streamline of F.

Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be a time-dependent vector field. A function $\gamma : \mathbb{R} \to \mathbb{R}^3$ describing a particle's motion in F is a **streamline** of F if there exists a t_0 such that for all $t \geq t_0$, $\gamma(t) = F(x, t_0)$.

   (2c) Streamlines and pathlines are sometimes the same and sometimes not.

   - Do you expect them to be nearly the same in factory wind tunnels?
   - Do you expect them to be nearly the same close to a moving tornado?

   In a single full sentence each, explain why or why not.

- I do expect streamlines and pathlines to be the same in factory wind tunnels in certain conditions because more often than not, the flow is laminar, which means that there is no turbulence disturbing the direction of the flow vectors in terms of time, hence pathlines would be the same as streamlines (which are the vectors held at a constant time).
- I do not expect streamlines and pathlines to be the same close to a moving tornado simply because there is too much turbulence changing the direction of the flow vectors through time, which would be constantly changing the pathline (since the pathline is time-dependent), thus the pathlines and streamlines (which is are vectors held at a constant time) would rarely be the same.

3. A grayscale picture can be thought of as a function $f : A \to [0,1]$ where $A \subseteq \mathbb{R}^2$ is a rectangular region and $[0,1]$ is the **intensity**. Higher values are brighter.

(3a) In a single full sentence, describe what the 1-level set of f represents. Use plain language.

The 1-level set of f would represent every section of the picture that is completely white.

(3b) You want to invert the grayscale intensity of your picture f. For example, black switches to white and white switches to black. Define a function $g : A \to [0,1]$ in terms of f that would be the resulting picture after this inversion. No justification is required.

Assume that the domain of f takes (x,y) inputs where $(x,y) \in A \subseteq \mathbb{R}^2$.

Define $g : A \to [0,1]$ by the function $g(x,y) = 1 - f(x,y)$. This will flip the grayscale image, so white appears black and black appears white.

(3c) A colour version of your picture f can be described using the RGB colour model (see *Numeric representations*). How can you think of your picture as a function h using RGB? State the domain and codomain of h and briefly explain what a value of your function h represents.

Let $R : \mathbb{R}^2 \to [0,1]$ described intensity of the colour red on the photo. '0' refers to no red, '1' refers to completely red (or 100%). Define the functions $G : \mathbb{R}^2 \to [0,1]$, $B : \mathbb{R}^2 \to [0,1]$ to describe the intensity of the colours green and blue, respectively, in terms of x and y.

Define $h : \mathbb{R}^2 \to [0,1]^3$ by $h = R(x,y)\mathbf{r} + G(x,y)\mathbf{g} + B(x,y)\mathbf{b}$, where $\mathbf{r}$, $\mathbf{g}$, and $\mathbf{b}$ define 'unit color vectors' (ie. '$\mathbf{r}$' is red, '$\mathbf{g}$' is green, and '$\mathbf{b}$' is blue).

The domain of h is $\mathbb{R}^2$, since a location on the photo can be described by an x and y coordinate. The codomain of h is $[0,1]^3$, since h is describing 3 different colors' intensity: red, green, and blue, each ranging from $[0,1]$. 0 implying that none of that certain color, 1 implying that all of that color is shown.

(3d) The grayscale intensity of a pixel can be obtained by averaging the three RGB values of that pixel. This gives a relationship between f and h. State that relationship. No justification is required.

We can define f in terms of h by taking the sums of R, G, and B and dividing them by 3. Assuming $\mathbf{r}$, $\mathbf{g}$, and $\mathbf{b}$ are orthogonal, we can take the dot product:

$$f(x,y) = \frac{1}{3}\left(R(x,y)\ G(x,y)\ B(x,y)\right)\begin{pmatrix}\mathbf{r}\\ \mathbf{g}\\ \mathbf{b}\end{pmatrix} = \frac{1}{3}h(x,y)\cdot\begin{pmatrix}\mathbf{r}\\ \mathbf{g}\\ \mathbf{b}\end{pmatrix} = \frac{1}{3}\left(R(x,y) + G(x,y) + B(x,y)\right).$$

f can still be defined as $f : \mathbb{R}^2 \to [0,1]$. The photo is white when $R = G = B = 1$ and black and $R = G = B = 0$.

4. Humans across history have produced so many different maps of the Earth, because projecting the image of a sphere in $\mathbb{R}^3$ onto a flat 2-dimensional surface poses a lot of challenges. Here you will construct such a map based on a few geometrically described steps.

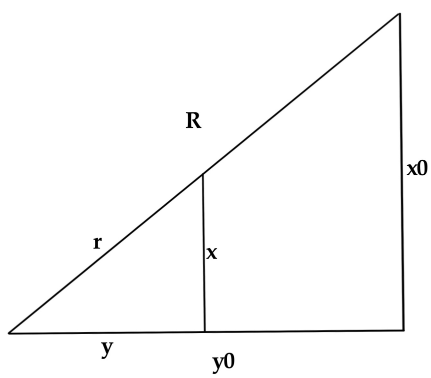
   (4a) Let S_R be a sphere of radius R centred at the origin. Place the sphere inside a (infinite) cylinder C_R of the same radius. The z-axis should pass through the center of the cylinder and the north and south pole of the sphere. Each z-slice between $-R$ and R cuts through the z-axis, the sphere, and the cylinder. Let

$$f : S_R \setminus \{(0,0,\pm R)\} \to C_R$$

   be the map that sends each z-slice of the sphere radially outward from the z-axis to the corresponding z-slice of the cylinder. Formally define f and include a well-labelled sketch that illustrates f.
   Optional: View this video from 0:00 to 1:10 for a quick visual of how f works.

We are wanting to map a point (x,y,z) on a sphere to another point (x_0,y_0,z) on a cylinder by creating $z-$slices and projecting each point outwards in a straight line through the $z-$axis and that point.

A bird's eye view may look like:



where the initial point is (x,y) and we want to project this point outwards to a point (x_0,y_0).
By the laws of similar triangles, since the angle between the hypotenuse and base isn't changing, we have that $\frac{r}{y} = Ry_0$ and $\frac{r}{x} = Rx_0$. Since the small radius r is given by $r = \sqrt{x^2+y^2}$ and the radius of the sphere itself is given by $R = \sqrt{x^2+y^2+z^2}$, then

$$x_0 = \frac{R}{r}x = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}x$$
$$y_0 = \frac{R}{r}x = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}y$$

Therefore our transformation from the sphere to the cylinder is given by

$$f(x,y,z) = \left(\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}x, \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}y, z\right)$$

(4b) Define a projection $g : S_R \setminus \{(0,0,\pm R)\} \to [-\pi,\pi]\times[-R,R]$ using the map f. The range of g should contain $(-\pi,\pi)\times(-R,R)$. *Hint:* Use a parametrization for the cylinder.

We can parametrize the cylinder by the function $\gamma : [-\pi,\pi]\times[-R,R]$ defined by $\gamma(\theta,t) = (R\cos\theta, R\sin\theta, t)$, where R is the fixed radius of the cylinder. In order to define the map $g : S_R \setminus \{(0,0,\pm R)\} \to [-\pi,\pi]\times[-R,R]$, we need to find the inverse of γ.

Because 'θ' is an angle in the parametrization, we can only isolate θ by using the inverse trig functions. The problem with this is that each inverse trig function has a restricted domain and codomain, and thus we will need to define more than one projection: one for the left hand side of the sphere ($[-\pi,0)$), and one for the right ($[0,\pi)$).

Let (x_0,y_0,z_0) be the translated point on the cylinder. We want to move (x_0,y_0,z_0) to a point in $[-\pi,\pi]\times[-R,R]$. Thus on the parametrization, $x_0 = R\cos\theta$, $y_0 = y\cos\theta$, and $z_0 = t$. Since $R = \sqrt{x^2+y^2+z^2}$, then $\cos\theta = \dfrac{x_0}{\sqrt{x^2+y^2+z^2}} = \dfrac{x}{\sqrt{x^2+y^2}}$.
We can also write $\sin\theta$ in terms of $\cos\theta$ from Pythagoras' identity, $\sin\theta = \sqrt{1-\cos^2\theta}$. Then

$$y_0 = R\sqrt{1-\cos^2\theta} \implies \frac{y_0^2}{R^2} = 1-\cos^2\theta \implies \sqrt{1-\frac{y_0^2}{R^2}} = \cos\theta \implies \sqrt{1-\frac{y^2}{x^2+y^2}} = \cos\theta.$$

Now $\theta = \arccos\dfrac{x}{\sqrt{x^2+y^2}}$, however this only contains points from in the interval from $[0,\pi]$ because that is the range of arccos. To cover the other half of the rectangle, $\theta' = -\arccos\dfrac{x}{\sqrt{x^2+y^2}}$ where $\theta' \in (-\pi,0)$ (*eg. the function would map two different points to the same location*).

Define the following subsets:

$$\begin{aligned}
S_{RL} \subset S \text{ where } S_{RL} &= \{(x,y,z)\in S_R : y<0\}\\
S_{RR} \subset S \text{ where } S_{RR} &= \{(x,y,z)\in S_R : y\ge 0\}\\
R_L \subset [-\pi,\pi]\times[-R,R] \text{ where } R_L &= \{(x,y)\in[-\pi,\pi]\times[-R,R] : -\pi<x<0\} = (-\pi,0)\times[-R,R]\\
R_R \subset [-\pi,\pi]\times[-R,R] \text{ where } R_R &= \{(x,y)\in[-\pi,\pi]\times[-R,R] : x\ge 0\} = [0,\pi]\times[-R,R]
\end{aligned}$$

Then $S_{RL}\cup R_{RR} = S_R$ and $R_L \cup R_R = [-\pi,\pi]\times[R,R]$. Now we are able to project S_{RL} to R_L and S_{RR} to R_R without any points being doubled or misprojected.

$$\text{Let } g_1 : S_{RL} \to R_L \text{ be the map defined by } g_1(x,y,z) = \left(-\arccos\frac{x}{\sqrt{x^2+y^2}}, z\right)$$

$$\text{Let } g_2 : S_{RR} \to R_R \text{ be the map defined by } g_2(x,y,z) = \left(\arccos\frac{x}{\sqrt{x^2+y^2}}, z\right)$$

Then define $g : S_R \setminus \{(0,0,\pm R)\} \to [-\pi,\pi]\times[-R,R]$ by

$$g(x,y,z) = \begin{cases} \left(\arccos\frac{x}{\sqrt{x^2+y^2}}, z\right) & \text{if } \quad y\ge 0\\ \left(-\arccos\frac{x}{\sqrt{x^2+y^2}}, z\right) & \text{if } \quad y<0\end{cases}$$

(4c) You've constructed a projection from a sphere in $\mathbb{R}^3$ to a rectangle in $\mathbb{R}^2$. Now, use it to define a projection that models the creation of a world map. You may need to make choices or assumptions, look up basic information about the Earth, or introduce constants. Regardless, you must explain your rationale. Do not simply write down your answer with little to no explanation.

To define a projection that would model the world map, we need to use latitude and longitude coordinates. Latitude refers to the radial angle of the "sphere" of the earth (the earth could be flat), and longitude refers to the azimuthal angle with respect to the axis going through the earths poles. We need to convert the coordinates (x, y, z) into polar coordinates describing latitude and longitude, which is not difficult.

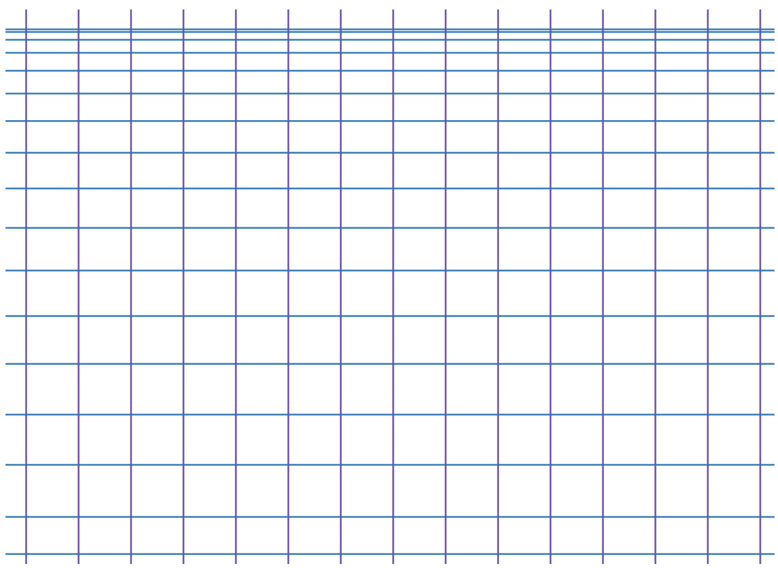
Now that we are using polar coordinates, the latitude radial angle corresponds to single points since it ranges from $-\pi$ to π. Thus our 'x' component of our projection is just given by θ, or latitude.

Our z component is just given by $R\cos\phi$, where ϕ is the azimuthal angle or longitude. The radius of the earth in meters is approximately $6.37 \times 10^6 m$, and thus our z component of our transformation becomes $6.37 \times 10^6 \cos\phi$.

Then we define $T : [-\pi, \pi] \times [0, \pi] \to [-\pi, \pi] \times [-R, R]$ by

$$T(\theta, \phi) = \left(\theta, 6.37 \times 10^6 \cos\phi\right).$$

For instance, the top-half of the map may look like this:



where the vertical lines represent the projection of the latitude coordinates and the horizontal lines represent the projection through the z-slice onto the cylinder then onto the plane.

(4d) Congratulations on becoming an amateur cartographer! You've made a world map. Cartographers usually describe the surface of the Earth using latitude and longitude as a coordinate system. Your world map should correspond to this standard coordinate system in some way. Explain where Toronto is located on your world map by converting its latitude and longitude to your coordinate system.

The coordinates for Toronto, in degrees, are $43.6532N$, $79.3832W$. Converting this into radians $\times\left(\frac{\pi}{180^o}\right)$, the coordinates are $0.761892069N$, $1.385498211W = -1.385498211$ in the $[-\pi, \pi]$ interval.

The coordinates of Toronto on my world map are then

$$\begin{aligned} T(-1.385498211, 0.761892069) &= (-1.385498211 \text{ rad}, 4'609'170.929 \text{ m}) \\ &= (-1.385498211 \text{ rad}, +4'609.171 \text{ km}). \end{aligned}$$

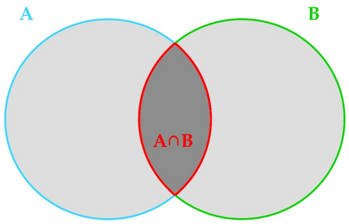
Logically these new coordinates make sense, because Toronto is to the left of the Prime Meridian (0 rad latitude) and far above the equator.

5. Determine if each statement is true or false. If true, prove it. If false, give a counterexample.

(5a) Let $A, B \subseteq \mathbb{R}^n$ be sets. Then $\partial(A \cup B) = \partial A \cup \partial B$.

◯ TRUE √ **FALSE**

Counterexample: Consider the following diagram of the two sets A and B.



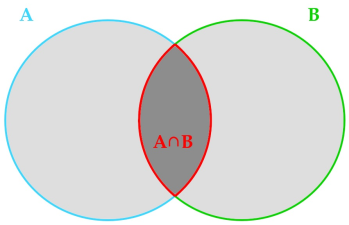
The set $\partial(A \cup B)$ is given by the two sets blue and green together, which is the outer boundary. Notice that this is not equivalent to the set $\partial A \cup \partial B$, which is the set containing every colored line (every boundary) in the diagram. The set containing the red boundaries is $\partial(A \cap B)$.

Thus we have that $\partial(A \cup B) = (\partial A \cup \partial B) \setminus \partial(A \cap B)$.
Geometrically this makes sense, since $\partial A \cup \partial B = (\partial(A \cup B)) \cup (\partial(A \cap B))$, as *'all of the lines = reds + blue + green'*.

(5b) Let $A, B \subseteq \mathbb{R}^n$ be sets. Then $\partial(A \cap B) = \partial A \cap \partial B$.

◯ TRUE √ **FALSE**

Counterexample: Again, consider the following diagram of the two sets A and B.



The set $\partial(A \cap B)$ is given by the red boundary lines in the diagram. This is not equivalent to the set $\partial A \cap \partial B$, which is just the set containing two points, just where the outer boundary circles of A and B overlap. The blue and green lines together is the set $\partial(A \cup B)$.

Instead, we have $\partial(A \cap B) = (\partial A \cup \partial B) \setminus \partial(A \cup B)$. Again geometrically this makes sense, since $\partial A \cup \partial B = (\partial(A \cup B)) \cup (\partial(A \cap B))$, as *'all of the lines = reds + blue + green'*.

6. Like limit points, interior points have an equivalent sequential definition.

   Lemma. *Let $S \subseteq \mathbb{R}^n$ be a set. Let $x \in S$. The point x is an interior point of S if and only if for any sequence $\{x_k\}_k$ in $\mathbb{R}^n$ converging to x, there exists $K \in \mathbb{N}$ such that $\{x_k\}_{k=K}^{\infty} \subseteq S$.*

   You will study the proof the this lemma.

   (6a) Here is a flawed proof of the "only if" direction.

   1. *Let $S \subseteq \mathbb{R}^n$ and assume $x \in S$ is an interior point.*
   2. *Let $\{x_k\}_k$ in $\mathbb{R}^n$ be a sequence converging to x.*
   3. *Since x is an interior point of S, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.*
   4. *Since $x_k \to x$, there exists $K \in \mathbb{N}$ such that $\|x_K - x\| < \varepsilon$.*
   5. *Thus, $\{x_k\}_{k=K}^{\infty} \subseteq S$ as required.*

   One line has a critical flaw. Identify this line and briefly explain the issue.

- Line 4 contains a critical flaw: the author of the proof is assuming for all $i \geq K, \|x_i - x\| < \varepsilon$. This is clearly false, since the sequence may hop 'in and out' of the epsilon ball centred at x.

  Furthermore, line 4 fails to mention that we are trying to prove '$\forall k \in \mathbb{N}$ s.t $k \geq K$', not just one value of the sequence K.

  For instance, if the open boundary of the epsilon ball was located at the boundary of S, and suppose that $x_{K+1} \notin S$, then $\{x_K\}_{k=K}^{\infty} \not\subseteq S$ because there are still points in the sequence not in the set S. Then $\|x_{K+1} - x\| < \varepsilon$ fails.

   (6b) Suggest how to fix the above proof by replacing/adding one or more lines.

- Replace line 4:

  4. *Since $x_k \to x$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$ where $k \in \mathbb{N}, \|x_k - x\| < \varepsilon$.*

- Replace line 5:

  5. *Since $\|x_k - x\| \in B_\varepsilon(x)$ for all $k \geq K$ and $B_\varepsilon(x) \subseteq S$, then $\{x_k\}_{k=K}^{\infty} \subseteq S$, as required.*

(6c) Prove the "if" direction of the lemma. Include a sketch illustrating your proof.

I need to prove that if for any sequence $\{x_k\}_k$ in $\mathbb{R}^n$ converging to a point $p \in \mathbb{R}^n$ such that there exists a $K \in \mathbb{N}$ where $\{x_k\}_{k=K}^{\infty} \subseteq S$, then $p \in S$ is an interior point of S.

To prove the "if" direction, it suffices to show that $\exists \varepsilon > 0$ such that $B_\varepsilon(p) \subseteq S$.

Proof:

- Let $p \in S$. Assume $\{x_k\}_k$ in $\mathbb{R}^n$ is a sequence of points converging to p:

$$\forall \varepsilon > 0, \exists K \in \mathbb{N} \text{ s.t } \forall k \in \mathbb{N}, k \geq K \implies \|x_k - p\| < \varepsilon.$$

- Let $z \in \left(\overline{S} \setminus S^o\right)$ be the point satisfying $\min\|z - p\|$.
- Choose $\varepsilon = \min\|z - p\|$. Choose a $K \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}, k \geq K \implies \|x_k - p\| < \min\|z - p\|.$$

- Notice that $\min\|z - p\| = 0$ if and only if $p \in \left(\overline{S} \setminus S^o\right)$, however this would not satisfy our assumption since it is assumed that $\varepsilon > 0$. This directly implies that p must be an interior point since $z \in \left(\overline{S} \setminus A^o\right)$.
- Then $B_\varepsilon(p) = \{x \in \mathbb{R}^n : \|x - p\| < \min\|z - p\|\}$, and for all $k \geq K$, $\{x_k\}_{k=K}^{\infty} \subseteq B_\varepsilon(p)$.
- Now let $x \in B_\varepsilon(p)$. To show $B_\varepsilon(p) \subseteq S$, I will show that any $x \in B_\varepsilon(p)$ is also in S.
  For any $x \in B_\varepsilon(p)$, $\max\|x - p\| < \min\|z - p\|$ because the ε-ball is open, and thus x must be contained in S because the maximum distance away from p must be less than ε, which is contained in S already.
- Therefore $B_\varepsilon(p) \subseteq S$, which is what I needed to prove.

$\square$

Picture proof: