JPE395 Problem Set 2 March 16, 2023

Problem 1:

a) To show that the hypothesis is true, we must consider why subducted plate does not sink into the mantle. This can only happen if the amount of time it takes to cool the oceanic plate is longer than the amount of time it takes for the crust material to subduct, therefore $\tau_{conduct} > \tau_{sink}$.

b) To determine $\tau_{conduct}$, we can examine the dimensional timescales of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

Now, $\frac{\partial T}{\partial t} \propto \frac{T}{\tau}$ and $\nabla^2 T \propto \frac{T}{D^2}$,

which implies $\frac{1}{\tau_{conduct}} \propto \kappa \cdot \frac{1}{D^2_{conduct}}$,

hence

$$\tau_{conduct} \propto \frac{D^2_{conduct}}{\kappa}.$$

From lecture 2, the oceanic crust depth is approximately

$$D_{conduct} \sim 100'000 \text{ m},$$

with $\kappa = 10^{-6}$ m^2/s, then

$$\tau_{conduct} \sim \frac{(1 \times 10^5)^2 \text{ m}^2}{10^{-6} \frac{\text{m}^2}{\text{s}}} = 10^{16} \text{ s}.$$

Then, for the sink timescale, we can examine the subducting velocity, given by the range.

$$V_{subduct} \sim 1 - 10 \ cm/yr = 3.17 \times 10^{-10} - 31.7 \times 10^{-10} \ m/s$$

The distance which plates must subduct into the mantle can be given by the distances in lecture 2,

$$D_{sink} = \underbrace{108'000 \ m}_{\text{oceanic dist.}} + \underbrace{2'900'000 m}_{\text{mantle}}$$

$$= 3'008'000 \ m.$$

Therefore since $V = \frac{d}{\tau}$, $\tau_{sink} = \frac{D_{sink}}{V_{sink}}$

$$\tau_{sink} \sim \frac{3'008'000}{31.7 \times 10^{-10}} \ s - \frac{3'008'000}{3.17 \times 10^{-10} \ s^{-1}}$$

$$\tau_{sink} \sim 9.48 \times 10^{14} s - 9.48 \times 10^{15} s.$$

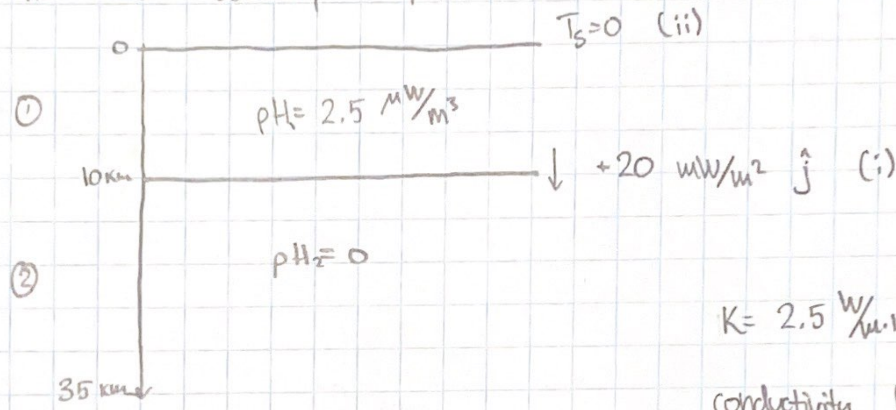
We find that

$$9.48 \times 10^{15} \ s < 10^{16} \ s,$$

therefore $\tau_{conduct} > \tau_{sink}$, which proves the theory.

Problem 2:

We have the heat relation $\overset{\text{Steady state}}{\cancel{\frac{\partial T}{\partial t}}^{0}} = \nabla^2 T + \rho H$ with the two boundary layers



$K = 2.5\ W/m \cdot K$ thermal conductivity for both régimes.

In ①, $\quad -K \frac{\partial^2 T_1}{\partial y^2} = \rho H \implies \frac{\partial T_1}{\partial y} = -\frac{\rho H}{K} y + \frac{C_1}{K}$.

Apply boundary condition (i),

$$\left.\frac{\partial T_1}{\partial y}\right|_{10 \cdot 10^3} = \frac{20}{K}\ mW/m^2 = -\frac{\rho H_1}{K}(10'000) + C_1$$

$$\implies C_1 = \frac{20}{K}\ mW/m^2 + \frac{\rho H_1}{K}(10'000)$$

Integrating a second time, $\quad T_1(y) = -\frac{\rho H}{2K} y^2 + C_1 y + C_2$,

where by (ii) implies $C_2 = 273$ K

In ②, $\quad \frac{\partial^2 T_2}{\partial y^2} = 0, \quad$ for $10 < y < 35$,

$\frac{\partial T_2}{\partial y} = C_1' \quad$ which, evaluated at $y = 10,000$ km,

$C_1' = 20\ mW/m^2$.

This implies that

$$T_2(y) = C_1' y + C_2',$$

where C_2' may be determined by the continuity condition

$$T_1(10) = T_2(10).$$

Therefore we obtain that

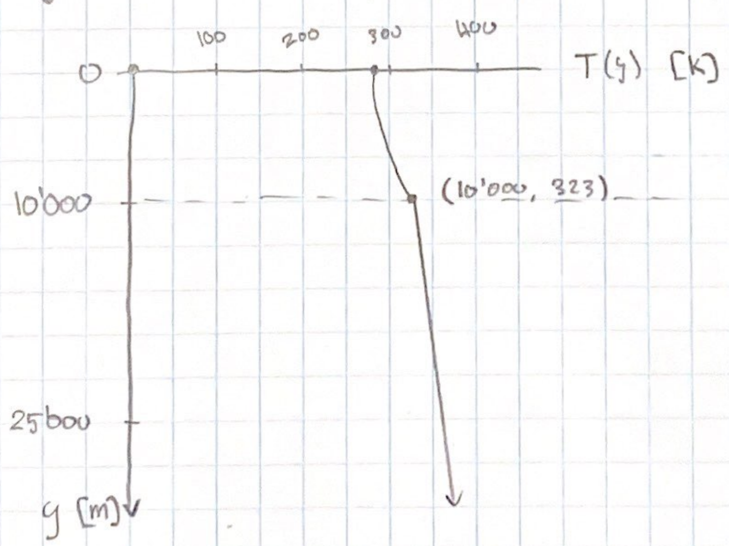
$$\frac{20 \frac{mW}{m^2}}{K} \cdot 10'000\,m + C_2' = -\frac{\rho H_1}{2K}(10'000\,m)^2 + \left(\frac{20}{K} \,mW/m^2 + \frac{\rho H_1}{K}(10'000)\right)(10'000) + 273$$

$$C_2' = \frac{\rho H_1}{2K}(10'000)^2 + 273$$

$$T(y) = \begin{cases} -\frac{\rho H_1}{2K} y^2 + \left(\frac{20 \,mW/m^2}{K} + \frac{\rho H_1}{K} \cdot 10'000\right) y + 273 & 0 \le y \le 10 \text{ km} \\ \frac{20 \,mW/m^2}{K} y + \frac{\rho H_1}{2K}(10'000)^2 + 273 & 10 \le y \le 35 \text{ km}. \end{cases}$$

$$T(y) = \begin{cases} 5 \times 10^{-7} \frac{K}{m^2} y^2 + 0.013 \frac{K}{m} y + 273 \text{ K} & 0 \le y \le 10'000 \text{ m} \\ 0.008 \frac{K}{m} y + 323 \text{ K} & 10'000 \text{ m} \le y \le 35'000 \text{ m} \end{cases}$$

Plotting this function,



Problem 3:

a) Annual Surface Temp. $T_s = -50°c = 223$ K

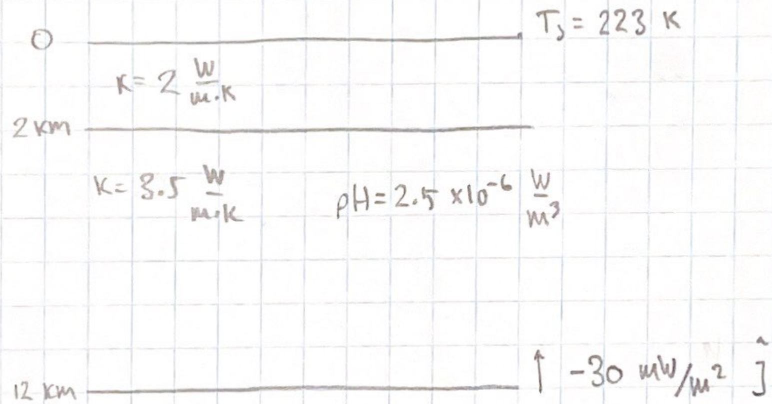
$K_{bedrock} = 3.5$ W/m·K

$K_{ice} = 2$ W/m·K

$\rho H = 2.5 \times 10^{-6}$ W/m³

Crust is 10'000 m thick.

Assume $q_b = 3 \times 10^{-2}$ W/m². We have the distribution



① $\nabla^2 T = 0$

② $k \nabla^2 T = -\rho H$

In ①, $\frac{\partial T}{\partial y} = \frac{C_1}{K}$ (unknown), then $T(y) = \frac{C_1}{K} y + C_2$.

At $T_s = 223$, $C_2 = 223$ K.

In region ②, $\frac{\partial T}{\partial y} = -\frac{\rho H}{k} y + C' = \frac{30}{k}$ mW/m² (flux at base).

$$\Rightarrow C_1' = -3 \times 10^{-2} \frac{W}{m^2} \cdot \frac{1}{3.5} \frac{m \cdot K}{W} - \frac{2.5 \times 10^{-6}}{3.5} \frac{W}{m^3} \frac{m \cdot K}{W} \cdot (12'000 \text{ m})$$

$$= -0.0171428 \frac{K}{m}$$

Then $T(y) = -\frac{\rho H}{2k} y^2 + C_1' y + C_2'$.

We must have $T(2 \text{ km})$ continuous and $\left.\frac{\partial T}{\partial y}\right|_{2 \text{ km}}$ continuous:

$$\frac{C_1}{K} = -\frac{\rho H}{\kappa} \cdot 2000 + C_1'$$

$$= -\frac{2.5 \times 10^{-6}}{3.5} \cdot 2000 + \frac{0.006}{3.5} \qquad \frac{K}{m}$$

$$C_1 = 0.015714 \ \frac{W}{m^2} \cdot 2 \ \frac{m \cdot K}{W} \quad \text{(continuous derivative)}$$

And $\quad C_1 \cdot 2'000 + 223 = -\frac{\rho H}{2K}(2'000)^2 + C_1' \cdot 2'000 + C_2'$

$$\Rightarrow \quad C_2' = 223 + 2 \cdot 0.015714 \cdot 2'000 + \frac{\rho H}{2K}(2'000)^2 - \frac{0.006}{3.5} \cdot 2'000$$

$$C_2' = 221.15278 \ K$$

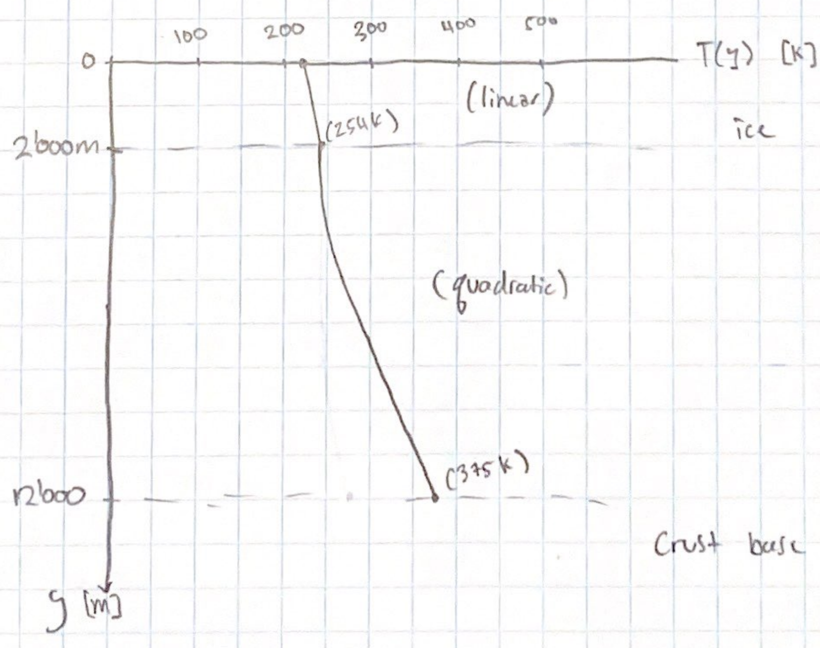
The temperature of the crust surface is then 254.42 K

The temperature of the crust base is therefore

$$T(12'000) = -\frac{\rho H}{2K}(12'000)^2 + \frac{0.06}{3.5} \cdot 12000 + 221.15278$$

$$= 375.86 \ K$$

Plotting the geotherm:



b) To determine the percentage of radiogenic heating contribution, we must compare heat fluxes $\frac{q_s - q_0}{q_s} = \%$.

Here, q_0 is the heat contribution as if $pH=0$, which may be determined by

$$\frac{q_0}{K_1} = \left(\frac{q_s}{K_1} = \frac{\partial T}{\partial y}\bigg|_{2000\,m}\right) \quad \text{(heat flux into ice)}$$

$$-\left(\frac{q_b}{K_2} = \frac{\partial T}{\partial y}\bigg|_{12000\,m}\right) \quad \text{(heat flux entering)}$$

with $pH=0$. $\frac{q_s}{K_1} = C_1 \Rightarrow q_s = C_1K_1 = 0.0157\ldots \cdot 2$

Hence $q_0 = 0 - 3\times10^{-3} \cdot \frac{2}{3.5} = -\frac{3}{175}$

$q_b = -3\times10^{-2}\ \frac{W}{m^2}$ (given)

and $q_s = \frac{\partial T}{\partial y}\bigg|_{2000m} = 0.015714\ \frac{W}{m^2} \cdot 2$ $(K_1=2)$

$q_0 = -\frac{3}{175}$.

thus $\frac{q_s - q_0}{q_s} \times 100\% = \frac{0.031428 - 3/175}{0.031428} \times 100\%$

$$= \underline{45.45\%}$$

c) To melt water at the surface, we require

$T_s = 0^{\circ}C$. Thus, using $T_1(y) = C_1 y + 223$ $(C_1 \approx 0.006)$.

We have

$$T_1(0) = 273\ K = \frac{C_1}{K} y + 223$$

Hence $\frac{50}{0.0157} = y$ in meters

Thickness $= y = 3'181.87$ m.

Q4) Let $H = 7.4 \times 10^{-12}$ W/kg

a) With a steady-state temperature, we have the ODE

$$-\rho H = \frac{K}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right).$$

This equation may be solved with a series solution

$$T(r) = \sum_n a_n r^n + C$$

where C is a boundary condition. We have that

$$T(r=a) = 0$$

as our boundary condition. Then,

$$-\rho H = \frac{K}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\left[\sum_n a_n r^n\right]\right)$$

$$= \frac{K}{r^2}\frac{d}{dr}\left(\sum_n a_n n r^{n+1}\right)$$

$$= \frac{K}{r^2}\sum_n a_n n(n+1) r^n$$

$$= K \sum_n a_n n(n+1) r^{n-2}$$

Hence $n=2$ to match LHS = RHS. This implies

$$a_2 = -\frac{\rho H}{6K}.$$

We obtain $T(r) = -\frac{\rho H}{6K} r^2 + C$, by which $T(a) = 0$

$$\Rightarrow \frac{\rho H a^2}{6K} = C \quad \Rightarrow \quad \boxed{T(r) = \frac{\rho H}{6K}(a^2 - r^2)}$$

At the bottom of the moon's mantle,

$r = 330'000 \text{ m} = R_{in}$

$K = 3.3 \text{ W/m·K}$, $\rho = 3'440 \text{ Kg/m}^3$

$a = 1'740'000 \text{ m}$.

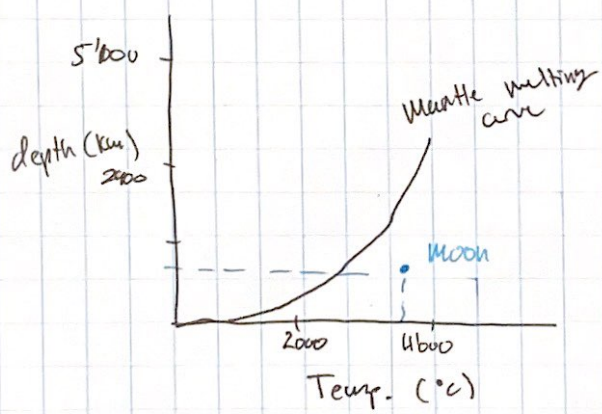
Then $$T(R_{in}) = \frac{(3'440) \cdot (7.4 \times 10^{-12} \text{ W/Kg})}{6 \cdot 3.3 \text{ W/m·K}} \left(1'740'000^2 - 330'000^2\right)$$

$$= 3'752.44 \text{ °C}$$

b) From lecture 8, the depth in the mantle is given by

$$1'740'000 - 330'000 = 1'410'000 \text{ m}.$$

According to the plot



which shows that the moon's mantle is liquid. This disagrees with seismic measurements, primarily because we have assumed the moons surface temperature is 0°c instead of ~ -200°c, which is the real measurement. In this case, we will end up on the LHS of the solidus curve shown above.

Problem 5:

a) According to half-space cooling model relationship

$$\frac{T-T_0}{T_s-T_0} = \theta(\eta) = \mathrm{erfc}\left(\frac{y}{2\sqrt{\kappa t}}\right).$$

If $\theta(\eta) \simeq 0.15$, the corresponding erfc value is ~ 1.0,

hence $$y/_{2\sqrt{\kappa t}} = 1.0$$

$$\Rightarrow \quad y = 2\sqrt{\kappa t}.$$

b) If $T_s \to T_0 + 500K$ initially and $T \to T_0 + 10$ finally,

then

$$\frac{T_0 + 10 - T_0}{T_0 + 500 - T_0} = \frac{1}{50} = \mathrm{erfc}\left(\frac{10}{2\sqrt{10^{-6} \cdot t}}\right)$$

If $\mathrm{erfc}(x) = \frac{1}{50}$, $x \simeq 1.6$ by the erfc table (given in lecture), thus

$$1.6 \simeq \frac{5}{\sqrt{10^{-6}\,t}} \Rightarrow t \simeq \left(\frac{5}{1.6}\right)^2 \cdot \frac{1}{10^{-6}}$$

$$\underline{t \simeq 9'765'625\ s}$$

Problem 6

$$\nabla T \simeq \frac{\partial}{\partial y} T\Big|_{y=0} = 35\,°C/km.$$

Mean thermal conductivity $\bar{k} = 1.7 \frac{W}{m \cdot K}$.

We have the relationship

$$q_s = \bar{K} \frac{\partial T}{\partial y}\Big|_{y=0}$$

$$= 1.7 \frac{W}{m \cdot K} \cdot 35\,°C/km \cdot \frac{1\,km}{1000\,m}$$

$$= 0.0595 \frac{W}{m^2}. \qquad \left(q_s = 59.5 \frac{mW}{m^2}\right)$$

According to fig 2.16, $59.5 \frac{mW}{m^2}$ is approximately 50 myr,

thus with the $\frac{1}{2}$-spreadrate velocity $v = 3\ cm/yr = 3 \times 10^{-2}\ m/yr$,

and $v = \frac{d}{t} \Rightarrow d = vt$

$$d = 3 \times 10^{-2}\ m/yr \cdot 5 \times 10^{7}\ yr$$

$$= 15 \times 10^{5}\ m$$

$$= 1500\ km$$

which is the approximate distance away from the ridge.