Motion in a Central Field

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§1. Introduction

Many instances of physics deal with the problem of a particle located in a potential whose magnitude depends only on the distance $\mathbf{r}$ away from a fixed point. This is termed a *central field*. In the same fashion, the magnitude of the central force acting on the particle is also dependent only on the distance $\mathbf{r}$:

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}} = -\frac{\partial U}{\partial r}\hat{\mathbf{r}}. \tag{1.1}$$

§2. Angular Momentum

One property of the central field is the isotropy of space, which is the invariance of the motion under rotations about the fixed point. By Noether's theorem, this symmetry allows us to arrive at the conservation of angular momentum about the axis. Let me prove this. Allow us to consider an infinitesimal rotation, represented by the vector $\delta\boldsymbol{\varphi}$, of the radius vector $\mathbf{r}$ which is an angle θ from the rotation axis (**Fig. 1**). The small change in the radius can be determined by applying the arc length relation $a = r\theta$, so $|\delta\mathbf{r}| = |\mathbf{r}|\sin\theta|\delta\boldsymbol{\varphi}|$. Equivalent to the cross product definition, then

$$\delta\mathbf{r} = \delta\boldsymbol{\varphi} \times \mathbf{r}. \tag{2.1}$$

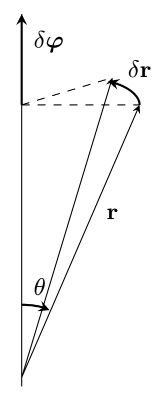

Fig. 1

In the same manner, all velocity vectors are also transformed by

$$\delta\mathbf{v} = \delta\boldsymbol{\varphi} \times \mathbf{v}. \tag{3.2}$$

Now the Lagrangian of the system is $\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$, so a small change in rotation of the Lagrangian yields

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\mathbf{r}}_i \cdot \delta\mathbf{r}_i + \frac{\partial\mathcal{L}}{\partial\mathbf{v}_i} \cdot \delta\mathbf{v}_i && (2.3)\\
&= \dot{\mathbf{p}}_i \cdot \delta\boldsymbol{\varphi} \times \mathbf{r}_i + \mathbf{p}_i \cdot \delta\boldsymbol{\varphi} \times \mathbf{v}_i && (2.4)\\
&= \delta\boldsymbol{\varphi}\left[\mathbf{r}_i \times \dot{\mathbf{p}}_i + \mathbf{v}_i \times \mathbf{p}_i\right] && (2.5)
\end{aligned}$$

$$= \delta\varphi \frac{d}{dt}\left[\mathbf{r}_i \times \mathbf{p}_i\right], \tag{2.6}$$

where Einstein summation is invoked over i. The relations $\mathbf{p}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{r}_i}$ and $\dot{\mathbf{p}}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{v}_a}$ come from the Euler-Lagrange equations. Noether's theorem states that the condition for invariance is that the Lagrangian is left unchanged, or $\delta\mathcal{L} = 0$. Since $\delta\varphi$ is arbitrary, then the angular momentum vector

$$\mathbf{M} \equiv \mathbf{r}_i \times \mathbf{p}_i \tag{2.7}$$

is conserved in the central field. In terms of the integral of motion,

$$p_\varphi = \mathbf{M} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\dot{\varphi} = \text{constant.} \tag{2.8}$$

§3. The Effective Potential

We now proceed by reducing a problem of two degrees of freedom to a one-dimensional problem. Since M is constant, then we can re-write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} - U(r). \tag{3.1}$$

The Hamiltonian can also be determined, which is defined by $H = \sum_i \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \mathcal{L}$. Since the kinetic energy of the Lagrangian is quadratic in $\dot{r}$, then by Euler's theorem for homogeneous functions, the Hamilton becomes

$$\begin{aligned}
H &= \sum_i \dot{r}\frac{\partial T}{\partial \dot{r}} - \mathcal{L} && (3.2)\\
&= 2T - (T - V) && (3.3)\\
&= T + V \equiv E && (3.4)\\
&= \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} + U(r) && (3.5)\\
&\equiv \frac{1}{2}m\dot{r}^2 + \mathcal{U}_{\text{eff}}(r), && (3.6)
\end{aligned}$$

where $\mathcal{U}_{\text{eff}}(r) = \frac{M^2}{2mr^2} + U(r)$ is the *effective potential* of the particle in the central field. Invoking the effective potential re-writes the problem in terms of only one degree of freedom, and the expressions for the motion can be determined by reducing the Hamiltonian to quadrature. Depending on the form of $U(r)$, $\mathcal{U}_{\text{eff}}(r)$ may or may not have an extremum. It can be easily plotted for different functions of $U(r)$, which is something I will discuss later.

§4. Quadratures

By solving the differential equation in (3.6), we find that

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}[E - \mathcal{U}_{\text{eff}}(r)]} \tag{4.1}$$

or, by integrating,

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - \mathcal{U}_{\text{eff}}(r)]}}. \tag{4.2}$$

This is called *reducing the problem to quadrature*, and is useful for determining explicit solutions for the motion. Equation (4.2) gives the solution for orbit time, and evaluating the integral across two different radii gives the expression for the half-period:

$$\frac{T}{2} = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{\frac{2}{m}[E - \mathcal{U}_{\text{eff}}(r)]}}. \tag{4.3}$$

This integral can be solved numerically. Now by re-writing the angular momentum in terms of the differential $d\varphi = \dfrac{M}{mr^2}\,dt$, then we can write $\dot{r} = \dfrac{dr}{d\varphi}\dfrac{M}{mr^2}$. Following the same process, Equation (4.1) becomes

$$\varphi = \int \frac{M}{r^2} \frac{dr}{\sqrt{\frac{2}{m}[E - \mathcal{U}_{\text{eff}}(r)]}}. \tag{4.4}$$

Here, Equation (4.4) gives the relation between r and φ, the equation of the path of the particle. This can also be solved numerically. Again, evaluating the integral in (4.4) between two different radii gives the change in φ over one half period. Thus

$$\Delta\varphi = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{M}{r^2} \frac{dr}{\sqrt{\frac{2}{m}[E - \mathcal{U}_{\text{eff}}(r)]}}. \tag{4.5}$$

I should note that these expressions are only valid for energy values greater than the minimum of $\mathcal{U}_{\text{eff}}(r)$, else the integrals would be undefined. A *closed path* occurs when the initial and final positions of the partical are identicle after a certain amount of time. This is easy to determine, since the condition is that $\Delta\varphi$ is a fraction of 2π: $\Delta\varphi = 2\pi\dfrac{n}{m}$, $n, m \in \mathbb{N}^+$. The most common visualization is that of an *isochrone orbit*[1] (**Fig. 2**):

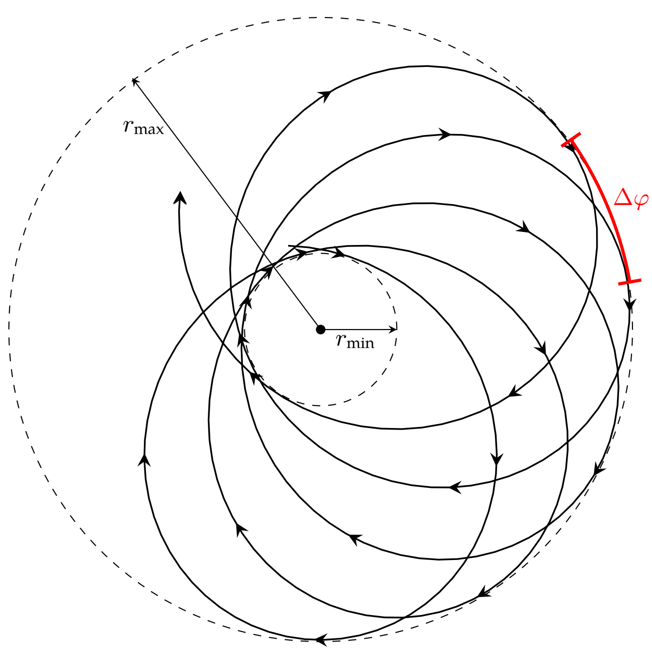

Fig. 2

[1]Isochrone functions not only apply to orbits, but other oscillatory motion, such as a pendulum. For more information, I shall direct you to Ramond, P., Perez, J. (2021). *The Geometry of Isochrone Orbits.* Arxivlabs Cornell University.

§5. A Special Class of Potentials

I want to take some time to draw the attention to the very common set of potentials in most physics: $U(r) = \dfrac{k}{r^n}$ for $k, n \in \mathbb{R}$. We are interested in discussing the behaviour of $\mathcal{U}_{\text{eff}}(r)$ for this set of potentials - which values of k, n does $\mathcal{U}_{\text{eff}}$ give extrema, bound the particle, give stable and unstable orbits, and where the particle can diverge to.

I will not bother examining the case when $n = 0$ or $k = 0$, since this is just the act of slapping a constant onto the angular energy term $\dfrac{M^2}{2mr^2}$. Now, taking the derivative of the effective potential gives

$$\mathcal{U}'_{\text{eff}}(r) = \frac{d}{dr}\left[\frac{M^2}{2mr^2} + \frac{k}{r^n}\right] \tag{5.1}$$

$$= -\frac{M^2}{mr^3} - \frac{kn}{r^{n+1}}. \tag{5.2}$$

Setting this equal to zero gives the relation $\dfrac{M^2}{m} = -\dfrac{kn}{r^{n-2}}$, or that

$$r_0 = \left(-\frac{knm}{M^2}\right)^{1/(n-2)}, \tag{5.3}$$

which is the radial location of the maximum or minimum of the potential. We instantly find that for any $k > 0$ or $n = 2$, the effective potential has no minimum, and will act as a monotonically decreasing function coming from $r = 0$. In this case, the particle will never be able to spiral into the origin since the effective potential goes to ∞ as $r \to 0$.

Now for values $k < 0$, we find that there is an interesting occurence for values $0 < n < 2$ and $n > 2$. An extremum of the effective potential exists, which means that there are values of the energy for which the particle can attain a circular orbit. I shall apply the second derivative test:

$$\mathcal{U}''_{\text{eff}}(r) = \frac{3M^2}{mr^4} + \frac{kn(n+1)}{r^{n+2}}. \tag{5.4}$$

Substituting in the value of r_0 in Equation 5.4,

$$\mathcal{U}''_{\text{eff}}(r_0) = \frac{3M^2}{m}\left(-\frac{knm}{M^2}\right)^{4/(2-n)} + kn(n+1)\left(-\frac{knm}{M^2}\right)^{(n+2)/(2-n)}. \tag{5.5}$$

For values $0 < n < 2$, the exponents over r_0 are raised to be greater than zero, and thus the second derivative is positive since the leading order term is of degree $4/(2-n)$. This implies that a minimum is given for $0 < n < 2$. For $n > 2$, the leading term is of degree $n + 2/(2-n)$, with the negative leading coefficient k. Therefore the second derivative for $n > 2$ is negative, hence a maximum is given.

Therefore $0 < n < 2$ gives a stable circular orbit at r_0, while $n > 2$ gives an unstable orbit. Notice that when $r \to \infty$, $\mathcal{U}_{\text{eff}}(r) \to 0$ for both cases. For the $0 < n < 2$ case, $\mathcal{U}_{\text{eff}}(r) \to \infty$ as $r \to 0$, since the leading term is $\dfrac{1}{r^2}$. For $n > 2$, since $k < 0$, the effective potential has a negative leading coefficient k on the second term, hence $\mathcal{U}_{\text{eff}}(r) \to -\infty$ as $r \to 0$. This is the behaviour of the potentials, however it might suffice to visualize these.

§6. Visualizing the Effective Potential

The following plots are cases of the potential $U(r) = \dfrac{k}{r^n}$ as discussed in the previous part. In each case I have set $k < 0$ to magnify how each value of n changes the function. I will begin with the sample case $n = 1$, $k = -3.5$:

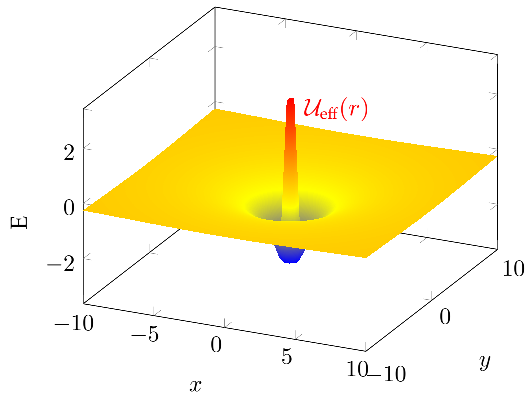

Fig. 3

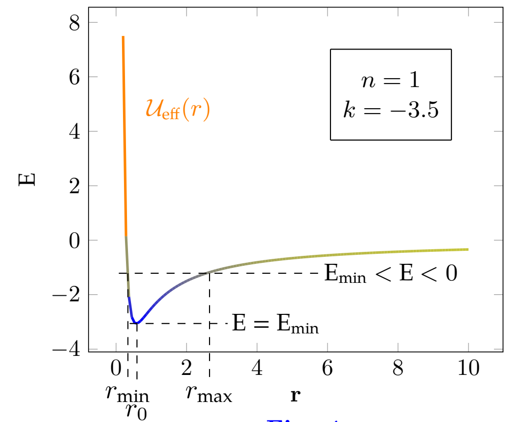

Fig. 4

We find that the minimum of $\mathcal{U}_{\text{eff}}(r)$ in (**Fig. 4**) gives a circular orbit, stable under small perturbations. For an energy $E_{\text{min}} < E < 0$, the particle is bounded in between two different radii $\mathbf{r}_1$ and $\mathbf{r}_2$. As described previously, this gives an isochrone orbit. For $E > 0$, the particle is unbound to ∞, however it will never spiral into the origin. Now for the case when $n = 2$, $k = -1$:

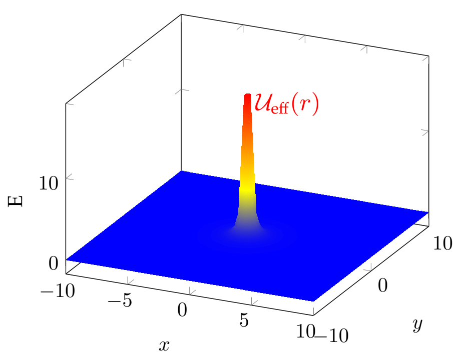

Fig. 5

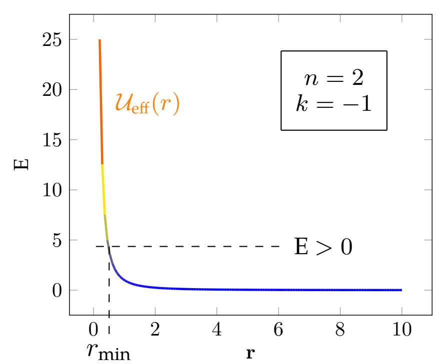

Fig. 6

or for smaller values of k, say $k = -4$,

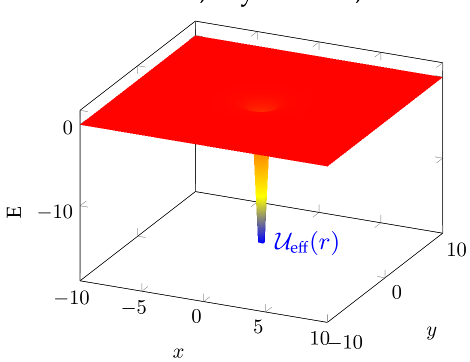

Fig. 7

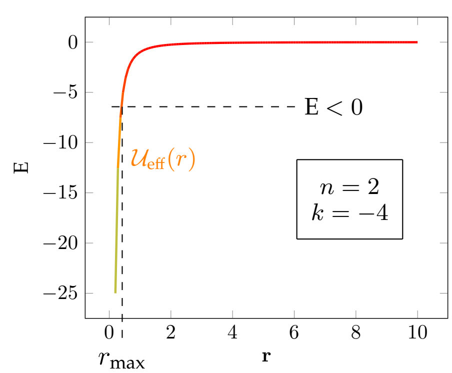

Fig. 8

and so the $n = 2$ case is dependent on the value of k. For larger k, say $k = -1$ as plotted in (**Fig. 5**) and (**Fig. 6**), any allowed energy $E > 0$ causes the particle to always diverge to $+\infty$ and never reach the origin. Once again, due to the spike in the angular energy. For smaller k, like $k = -4$ as plotted in (**Fig. 7**) and (**Fig. 8**), any energy $E > 0$ causes the particle to again, be unbound. However, any $E < 0$ causes the particle to be bound into a region close to the origin, and eventually causing the particle to spiral into the origin. The time for this to occur can be deduced from the path expressions Equations 4.3 and 4.5. The last case, of course, being $n > 2$ and $k < 0$. For $n = 3$ and $k = -1$,

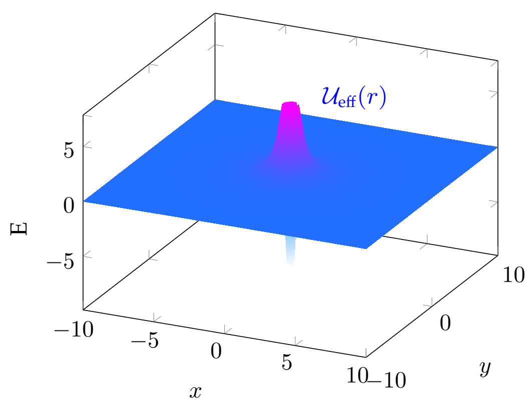

Fig. 9

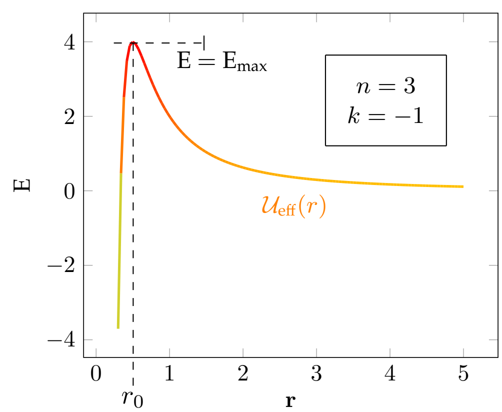

Fig. 10

For $E > E_{\text{max}}$, the particle again is unbound. For $E = E_{\text{max}}$, the particle attains a circular orbit at the maximum of the effective potential, however small perturbations leave the orbit unstable - either sending the particle to the origin, or to ∞. For $E_{\text{max}} > E > 0$, the particle is unbound to ∞ or spirals into the origin, depending on its initial conditions. Likewise, for $E < 0$, the particle spirals into the origin.

§7. Bibliography

1. Landau, L.D., Lifshitz, E.M. (1976). *Mechanics, Third Edition: Course of Theoretical Physics*. (J.B. Sykes, J.S. Bell). Elsevier Butterworth Heinemann.

2. Ramond, P., Perez, J. (2021). *The Geometry of Isochrone Orbits*. Arxivlabs Cornell University.