PHY354 PS4 — 03/18/2022

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Q1.

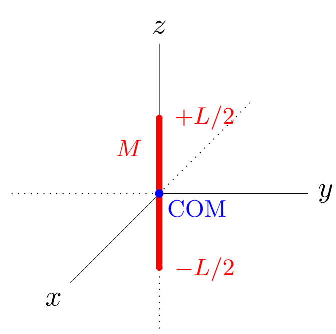
Determine the principal moments of inertia for the following homogeneous bodies: (a) a thin rod of length l, (b) a sphere of radius R, (c) a circular cylinder of radius R and height h, (d) a rectangular parallelepiped of sides a, b, and c, (e) a circular cone of height h and base radius R, (f) an ellipsoid of semiaxes a, b, c.

For continuous bodies, the moment of inertia tensor is given by the integral

$$I = \int d^3\mathbf{r}\rho \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}.$$

For this problem, I will be aligning the bodies along their symmetries, which in these cases give the principle axes of inertia of the object. This is because the centre of mass of the object always will lie on the principle axes of inertia, and the centre of mass can be determined by the symmetrries of the object.

(a) For the rod of length L and mass M, the center of mass is located at the center of the rod, at $L/2$. Orienting the rod along the z axis with the center of the rod being located at the origin $(0, 0, 0)$ gives the principle axes to be the x, y and z axes.



Since we have aligned the rod along its principle axes, $I = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$ which is a diagonal matrix. The moments of inertia are then

$$I_{xx} = \int \rho(y^2+z^2)\,dV, \qquad I_{yy} = \int \rho(x^2+z^2)\,dV, \qquad \text{and} \qquad I_{zz} = \int \rho(x^2+y^2)\,dV.$$

Because the rod is symmetrical about the polar angle (any point in the xy plane), $I_{xx} = I_{yy}$. Furthermore, the rod is assumed to be thin, which means we can neglect the radius. If $r^2 = x^2 + y^2 = 0$, then $I_{zz} = 0$. We proceed by finding $I_{xx} = I_{yy}$.
Along the x axis, y is constrained to be zero while $-L/2 \leq z \leq L/2$. Then

$$I_{xx} = \int_{-L/2}^{L/2} \rho z^2\,dz.$$

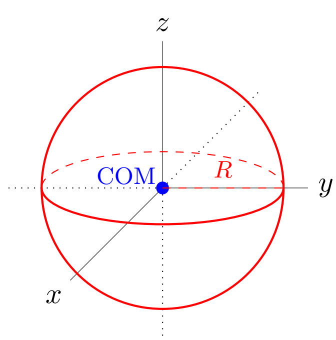
ρ is the density of the rod. Assuming uniform density, $\rho = \dfrac{M}{L}$, and thus

$$\begin{aligned} I_{xx} &= \frac{M}{L}\int_{-L/2}^{L/2} z^2\,dz \\ &= \frac{M}{L}\frac{1}{3}\left(z^3\Big|_{-L/2}^{L/2}\right) \\ &= \frac{M}{L}\frac{1}{3}\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] \\ &= \frac{1}{12}ML^2. \end{aligned}$$

With $I_{xx} = I_{yy}$ and $I_{zz} = 0$, the principle moments of inertia are

$$\boxed{I_{xx} = I_{yy} = \frac{1}{12}ML^2 \qquad \text{and} \qquad I_{zz} = 0.}$$

(1b) For the sphere of radius R, any axis intersecting the center of mass of the sphere will be a principle axis. This is because the sphere is symmetrical from all angles, and conviently assuming uniform density $\rho = \dfrac{M}{V} = \dfrac{M}{\frac{4}{3}\pi R^3}$, the center of the sphere is also the center of mass.



Therefore $I = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$ is diagonal with $I_{xx} = I_{yy} = I_{zz}$. As before, they are

$$I_{xx} = \int \rho(y^2 + z^2)\,dV, \qquad I_{yy} = \int \rho(x^2 + z^2)\,dV, \qquad \text{and} \qquad I_{zz} = \int \rho(x^2 + y^2)\,dV.$$

To simplify the process, we can work in spherical coordinates. To do this, we will compute the sum $I_{xx} + I_{yy} + I_{zz} = 3I$, which we can only do because they are equivalent. Then

$$I_{xx} + I_{yy} + I_{zz} = \int \rho(2x^2 + 2y^2 + 2z^2)\,dV = 2\rho\int (x^2 + y^2 + z^2)\,dV.$$

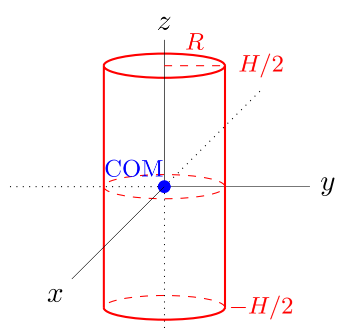
Now we have an expression in which we can apply a spherical coordinate subsitution. Taking $r^2 = x^2 + y^2 + z^2$ and $V = \frac{4}{3}\pi R^3 \implies dV = 4\pi r^2\, dr$, then

$$\begin{aligned} 3I &= 2\rho \int_0^R r^2 \cdot 4\pi r^2\, dr \\ &= 8\pi\rho \cdot \left[\frac{r^5}{5}\right]_0^R \\ &= \frac{8\pi}{5}\frac{M}{\frac{4}{3}\pi R^3} R^5 \\ &= \frac{6}{5}MR^2. \end{aligned}$$

Dividing by 3 gives the three principle moments of inertia:

$$\boxed{I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}MR^2.}$$

(1c) For the cylinder of radius R and height H, we can orient it along the cartesian axes like the rod. Taking the centre axis of the cylinder to be the z axis and placing it an equal height of $H/2$ above and below the xy plane places center of mass directly at the origin. By symmetry, this aligns the principle axes along the x, y and z axes.



As in **(1a)** and **(1b)**, $I = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$ is diagonal. Like the symmetry of the rod, the cylinder oriented along the z axis sets $I_{xx} = I_{yy} \neq I_{zz}$. I will being by computing I_{zz} first, since it will be useful for computing I_{xx} and I_{yy}. The volume of the cylinder is $V = \pi R^2 H$, so a cylindrical z-slice of the volume is $dV = 2\pi r H\, dr$. Then $\rho = \frac{M}{V} = \frac{M}{\pi R^2 H}$. We have

$$I_{zz} = \int \rho(x^2 + y^2)\, dV = \frac{M}{\pi R^2 H}\int_0^R r^2 \cdot 2\pi r H\, dr.$$

This integral can be easily computed to verify that $I_{zz} = \frac{1}{2}MR^2$. Since $I_{xx} = I_{yy}$, we can sum the two to find that

$$I_{xx} + I_{yy} = 2I = \int \rho(x^2 + y^2 + 2z^2)\, dV = \int \rho(x^2 + y^2)\, dV + 2\int \rho z^2\, dV.$$

In this expression, notice that I_{zz} appears as the first term, which we have already found. Then

$$2I = I_{zz} + 2\int \rho z^2 \, dV.$$

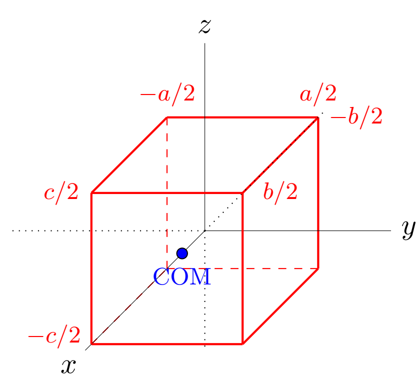
As before, $\rho = \dfrac{M}{\pi R^2 H}$ and a horizontal xy-slice of the volume is $dV \equiv \pi R^2 \, dz$ since we are integrating just over the height of the cylinder. Then

$$\begin{aligned} 2I &= I_{zz} + 2\frac{M}{\pi R^2 H}\int_{-H/2}^{H/2} z^2 \cdot \pi R^2 \, dz \\ &= I_{zz} + 2\frac{M}{H}\frac{1}{3}\left[\frac{H^3}{8} - \left(-\frac{H^3}{8}\right)\right] \\ &= I_{zz} + \frac{1}{6}MH^2. \end{aligned}$$

After substituting in I_{zz} and simplifying, we find that $I_{xx} = I_{yy} = \frac{1}{4}MR^2 + \frac{1}{12}MH^2$. Therefore our principle moments of inertia are

$$\boxed{I_{xx} = I_{yy} = \frac{1}{4}MR^2 + \frac{1}{12}MH^2 \qquad \text{and} \qquad I_{zz} = \frac{1}{2}MR^2.}$$

(1d) The center of mass of the parallelepiped will be determined by the values of a, b, and c. To save ourselves the hassle of finding the center of mass, we can position the parallelepiped on the cartesian axes such that each of the three axes run through the centers of each side, that is $x = \frac{a}{2}$, $y = \frac{b}{2}$, and $z = \frac{c}{2}$.



With this, the center of mass lies on each of the principle axes, so again $I = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$ is diagonal. The volume of the parallelepiped is simple given by $V = abc$, so assuming uniform density, $\rho = \dfrac{M}{abc}$. It suffices to show my method and computations for one of the three principle axes, since the others follow via the exact same method. Consider $I_{xx} = \displaystyle\int \rho(y^2 + z^2)\, dV$. This integral can be expanded by linearity:

$$I_{xx} = \int \rho y^2 \, dV + \int \rho z^2 \, dV.$$

The difference in volume along the y axis is a y-slice, so x and z remain fixed lengths a and c respectively, so $dV_y = ac\, dy$. Equivalently with z, $dV_z = ab\, dz$. Since $-b/2 \leq y \leq b/2$ and $-c/2 \leq z \leq c/2$, then

$$I_{xx} = \frac{M}{abc} \int_{-b/2}^{b/2} y^2 \cdot ac\, dy + \frac{M}{abc} \int_{-c/2}^{c/2} z^2 \cdot ab\, dz.$$

The integrands are equivalent to that of the rod examined in **(1a)**, so then

$$I_{xx} = \frac{M}{b} \cdot \frac{1}{3}\frac{b^3}{4} + \frac{M}{c} \cdot \frac{1}{3}\frac{c^3}{4} = \frac{1}{12} M(b^2 + c^2).$$

Following the exact same process for I_{yy} and I_{zz}, we find that the principle moments of inertia are

$$\boxed{I_{xx} = \frac{1}{12} M(b^2 + c^2), \qquad I_{yy} = \frac{1}{12} M(a^2 + c^2), \qquad \text{and} \qquad I_{zz} = \frac{1}{12} M(a^2 + b^2).}$$

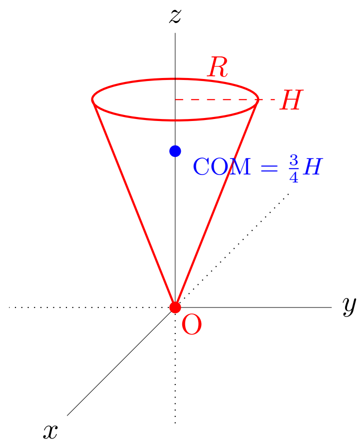
(1e) For the cone, the centre of mass will need to be calculated first to determine the steps necessary to calculate the principle moments of inertia (in case if we have to apply the parallel axis theorem). By symmetry, the center of mass must lie on the axis of the cone, passing through the vertex and the center of the base. The volume of the cone of radius R and height H is $V = \dfrac{1}{3}\pi R^2 H$, and if we assume uniform density that $\rho = \dfrac{M}{V}$, then the mass of the cone is $M = \dfrac{1}{3}\pi R^2 H \rho$. The center of mass is given by the integral

$$COM = \frac{1}{M} \int \rho h \, dV.$$

A volume element of the cone is given by $dV = \pi r^2 \, dh$, since the height and the radius are related by similar trianlges: $r = \dfrac{R}{H} h$. Making this substitution, $dV = \pi \dfrac{R^2}{H^2} h^2 \, dh$. This allows us to integrate over the height of the cone, as required:

$$\begin{aligned} COM &= \frac{1}{M} \int_0^H \rho h \cdot \pi \frac{R^2}{H^2} h^2 \, dh \\ &= \frac{M}{\frac{1}{3}\pi R^2 H} \cdot \frac{1}{M} \cdot \pi \frac{R^2}{H^2} \int_0^H h^3 \, dh \\ &= \frac{3}{4} H. \end{aligned}$$

Therefore the center of mass is located three-quarters up the axis of the cone. To proceed, I will calculated the moments of inertia at the vertex of the cone, centered at the origin, and then apply the parallel axis theorem to 'shift' them up so they pass through the center of mass, hence giving the principle moments of inertia.



I will begin by calculating I_{zz}, since the cone is symmetrical about the z axis. Like the cylinder, finding I_{zz} will also help us find I_{xx} and I_{yy}. Note how any z-slice of the cylinder is a disk, so I will proceed by finding the moment of inertia of a disk, then integrating over every disk to find the net moment of inertia along z.

The surface area of a disk is $A = \pi r^2$, so a difference in the area is $dA = 2\pi r\,dr$. The mass density is then $\rho = \dfrac{M}{\pi r^2}$. In polar coordinates, $x^2 + y^2 = r^2$, so we have that

$$\begin{aligned}
I_{zz,\,disk} &= \int \rho(x^2 + y^2)\,dA \\
&= \frac{M}{\pi r^2}\int_0^r r'^2 \cdot 2\pi r'\,dr \\
&= \frac{M}{r^2}\frac{2}{4}r^4 \\
&= \frac{1}{2}Mr^2.
\end{aligned}$$

Back to the cone, as before, the volume is $V = \dfrac{1}{3}\pi R^2 H$. The difference in the volume along z, using similar triangles, is $dV = \pi r^2\,dr = \pi\dfrac{R^2}{H^2}h^2\,dh$, with $r = \dfrac{R}{H}h$. We wish to integrate over all disks with radius $0 \le r \le R$ along z, so our bounds of integration actually become $0 \le h \le H$. This gives

$$\begin{aligned}
I_{zz} &= \frac{1}{2}r^2\,dM = \frac{1}{2}r^2\rho dV \\
\implies I_{zz} &= \frac{1}{2}\int \rho r^2\,dV \\
&= \frac{1}{2}\int_0^H \frac{M}{\frac{1}{3}\pi R^2 H}\pi\left(\frac{R}{H}h\right)^4\,dh \\
&= \frac{3}{2}\frac{MR^2}{H^5}\cdot\frac{H^5}{5}.
\end{aligned}$$

Simplifying further, we find that $I_{zz} = \dfrac{3}{10}MR^2$. Next is to find the moments of inertia about the vertex of the cone. By symmetry, like the cylinder, $I_{xx} = I_{yy}$. Summing the two gives

$$I_{xx} + I_{yy} = 2I = \int \rho(x^2 + y^2 + 2z^2)\,dV = I_{zz} + 2\int \rho z^2\,dV.$$

Since we already have an expression from I_{zz}, it suffices to just integrate the second term. Following the same process with a small volume element being $dV = \pi \dfrac{R^2}{H^2} h^2\, dh$ and $z \to h$ with $0 \leq h \leq H$, integrating over height, we have

$$\begin{aligned}
2I &= I_{zz} + 2\int_0^H \frac{M}{\frac{1}{3}\pi R^2 H} h^2 \cdot \pi \frac{R^2}{H^2} h^2\, dh \\
&= I_{zz} + 6\frac{M}{H^3}\int_0^H h^4\, dh \\
&= \frac{3}{10}MR^2 + \frac{6}{5}MH^2.
\end{aligned}$$

Thus, $I_{xx} = I_{yy} = \dfrac{3}{20}MR^2 + \dfrac{3}{5}MH^2$, so our moments of inertia at the vertex of the cone are

$$I_{xx} = I_{yy} = \frac{3}{20}MR^2 + \frac{3}{5}MH^2 \qquad \text{and} \qquad I_{zz} = \frac{3}{10}MR^2.$$

To find the principle moments of inertia intersecting the center of mass, we can apply the parallel axis theorem. For shifting the moments of inertia along a vector $\mathbf{a}$, $L\,\&\,L$ gives

$$I'_{ik} = I_{ik} - (a^2\delta_{ik} - a_i a_k).$$

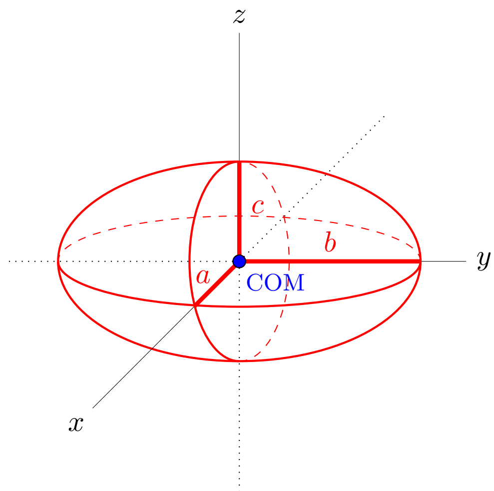
In this case, the vector from the origin to the center of mass is $\mathbf{a} = \left(0, 0, \dfrac{3}{4}H\right)$. Any index $i \neq k$ in the formula then produces the same result, so we have that

$$\begin{aligned}
I'_{xx} &= I_{xx} - M\left(\frac{9}{16}H^2\delta_{11} - a_1^2\right) = I_{xx} - M\left(\frac{9}{16}H^2 - 0\right) = I_{xx} - \frac{9}{16}MH^2 \\
I'_{yy} &= I_{yy} - M\left(\frac{9}{16}H^2\delta_{22} - a_2^2\right) = I_{yy} - M\left(\frac{9}{16}H^2 - 0\right) = I_{yy} - \frac{9}{16}MH^2 \\
I'_{zz} &= I_{zz} - M\left(\frac{9}{16}H^2\delta_{33} - a_3^2\right) = I_{xx} - M\left(\frac{9}{16}H^2 - \frac{9}{16}H^2\right) = I_{zz}.
\end{aligned}$$

Therefore by the parallel axis theorem, our principle moments of inertia for the cone about its center of mass are

$$\boxed{I_{xx} = I_{yy} = \frac{3}{20}MR^2 + \frac{3}{80}MH^2 \qquad \text{and} \qquad I_{zz} = \frac{3}{10}MR^2.}$$

(1f) By symmetry, the center of mass of the ellipsoid is located at the midpoint of each axis. Orienting the ellipsoid such that each semiaxis is parallel with each cartesian axis places the center of mass at the origin.



The equation of an ellipsoid is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

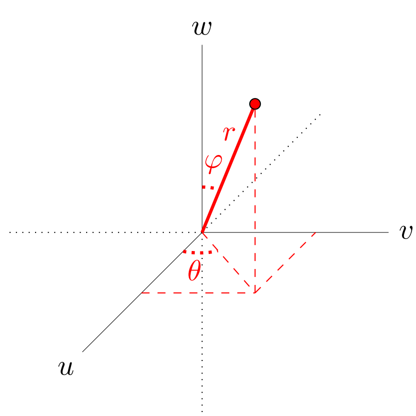
To proceed, we can perform a change of variables to transform the ellipsoid into a sphere of radius 1. Making the substitutions

$$u = \frac{x}{a}, \qquad v = \frac{y}{b}, \qquad \text{and} \qquad w = \frac{z}{c},$$

our equation for the ellipsoid becomes $u^2 + v^2 + w^2 = 1$. Define the function $g : \mathbb{R}^3 \to \mathbb{R}^3$ by $g(u, v, w) = (au, bv, cw)$. This is the inverse mapping between bases, in which the stretch factor is the determinant of the Jacobian of g:

$$\det(Dg) = \det \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = abc.$$

With these substitutions, the ellipsoid in this basis is a sphere of radius 1. Since each principle axis is aligned along the semi axes of the ellipsoid, then the inertia tensor $I = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$ is diagonal. By symmetry, it suffices to compute one of the three principle moments of inertia, since the other two follow by the exact same process. Since the change of variables has transformed the ellipsoid into a sphere of radius 1, we can proceed by integrating in spherical coordinates.



The component of r along u is $r\cos\theta\sin\varphi$, the compnent of r along v is $r\sin\theta\sin\varphi$, and the component of r along w is $r\cos\varphi$. In polar coordinates, $dV' \to r^2\sin\varphi\, dr d\varphi d\theta$. Let us consider I_{xx}:

$$\begin{aligned}
I_{xx} &= \int \rho(y^2+z^2)\,dV && \text{by definition}\\
&= abc\rho\int (b^2v^2+c^2w^2)\,dV' && \text{by change of variables}\\
&= abc\rho\int_0^{2\pi}\int_0^{\pi}\int_0^1 (b^2r^2\sin^2\theta\sin^2\varphi + c^2r^2\cos^2\varphi)\cdot r^2\sin\varphi\, dr d\varphi d\theta && \text{spherical coordinate subsitution}\\
&= abc\rho\bigg[\, b^2\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^4\sin^2\theta\sin^3\varphi\, dr d\varphi d\theta\\
&\qquad\qquad + c^2\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^4\cos^2\varphi\sin\varphi\, dr d\varphi d\theta\,\bigg] && \text{by linearity}\\
&= abc\rho\left[\, b^2\left(\frac{1}{5}\right)(\pi)\left(\frac{4}{3}\right) + c^2\left(\frac{1}{5}\right)(2\pi)\left(\frac{2}{3}\right)\right] && \text{by Wolfram Alpha}\\
&= \frac{4\pi}{15}abc\rho(b^2+c^2).
\end{aligned}$$

The volume of an ellipsoid is given by $V = \dfrac{4}{3}\pi abc$, so the mass density is $\rho = \dfrac{M}{\frac{4}{3}\pi abc}$. Substituting ρ into the expression form I_{xx}, we have that

$$I_{xx} = \frac{4\pi}{15}abc\frac{M}{\frac{4}{3}\pi abc}(b^2+c^2) = \frac{1}{5}M(b^2+c^2).$$

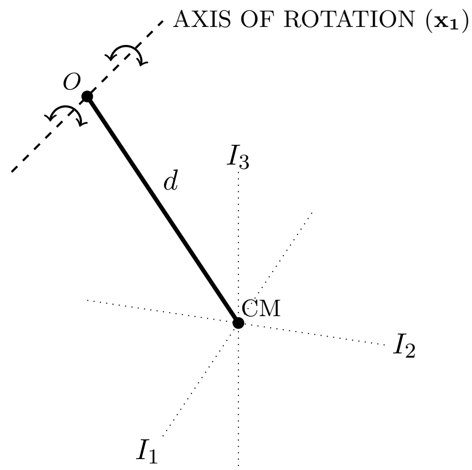
The expressions for the moments of inertia I_{yy} and I_{zz} follow by the exact same process, so by linearity, the principle moments of inertia of an ellipsoid of semiaxes a, b, and c are

$$\boxed{I_{xx} = \frac{1}{5}M(b^2+c^2), \qquad I_{yy} = \frac{1}{5}M(a^2+c^2) \qquad \text{and} \qquad I_{zz} = \frac{1}{5}M(a^2+b^2).}$$

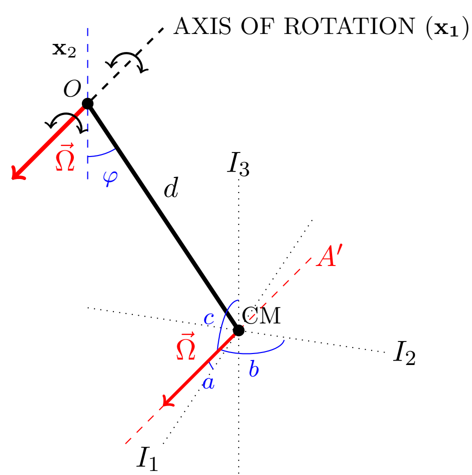
Q2.

Determine the frequency of small oscillations of a compound pendulum (a rigid body swinging about a fixed horizontal axis in a gravitational field).

We begin by analyzing the kinetic energy of the body. Let its mass be M. In three dimensions, every rigid body has 3 principle moments of inertia. Suppose the pivot point O of the pendulum is located a distance d from the center of mass, which I may note (O) may not lie on one of the three principle axes. With this, the orientation of the principle axes may be completely arbitrary from the axis of rotation, which I will call x_1.



Let the angle between d and the verticle be φ, so the angular velocity of the mass rotating about the axis of rotation is $\vec{\Omega} = \dot{\varphi}\,\hat{\mathbf{x}}_\mathbf{1}$. $\vec{\Omega}$ points in the parallel direction of the axis of rotation and is the same vector for any point on the body - including the center of mass. Let A' be the axis which intersects the center of mass and is parallel to the axis of rotation. Note that the principle axes form angles a, b and c with A', and hence with $\vec{\Omega}$. These angles are constant because O does not change with respect to the center of mass, and hence the principle axes.



The kinetic energy of the body is given by the addition of the translational energy of the center of mass with the rotational energy about the principle axes. The speed of the center of mass is given by $V_{CM} = d\dot{\varphi}$, so the kinetic energy of translation is

$$T_{trans} = \frac{1}{2}MV_{CM}^2 = \frac{1}{2}Md^2\dot{\varphi}^2.$$

The rotational kinetic energy can be found by decomposing $\vec{\Omega}$ such that $\vec{\Omega}$ has three components along each of the principle axes. We can do this because $\vec{\Omega}$ is the same for every point on the body. Projecting, the I_1 component of $\vec{\Omega}$ is $\Omega_1 = \varphi \cos a$. Taking the other projections in the same way, the components of $\vec{\Omega}$ along I_2 and I_3 are $\dot{\varphi}\cos b$ and $\dot{\varphi}\cos c$, respectively. The kinetic energy of rotation is then

$$T_{rot} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) = \frac{1}{2}\dot{\varphi}^2(I_1\cos^2 a + I_2\cos^2 b + I_3\cos^2 c).$$

The potential energy is purely gravitational. Since any force acting on a rigid body acts as if it was applied at the center of mass, then the potential energy is given by

$$U = Mgd - Mgd\cos\varphi = Mgd(1 - \cos\varphi).$$

Since φ is small, we can proceed by applying a Taylor expansion. We have that

$$\cos\varphi \approx 1 - \frac{1}{2}\varphi^2 + \ldots,$$

And thus

$$\begin{aligned} U &\approx Mgd\left(1 - 1 + \frac{1}{2}\varphi^2\right) \\ &= \frac{1}{2}Mgd\varphi^2. \end{aligned}$$

The Lagranian of the body is then $L = T - V$,

$$L = \frac{1}{2}Md^2\dot{\varphi}^2 + \frac{1}{2}\dot{\varphi}^2(I_1\cos^2 a + I_2\cos^2 b + I_3\cos^2 c) - \frac{1}{2}Mgd\varphi^2,$$

with φ as the generalized coordinate. We can proceed by applying the Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}}\right] &= Md^2\ddot{\varphi} + \ddot{\varphi}(I_1\cos^2 a + I_2\cos^2 b + I_3\cos^2 c) \\ = \frac{\partial L}{\partial \varphi} &= -Mgd\varphi. \end{aligned}$$

Factoring out the $\ddot{\varphi}$ gives the expression for frequency (like a simple pendulum),

$$\begin{aligned} Mgd &= \ddot{\varphi}\left(Md^2 + (I_1\cos^2 a + I_2\cos^2 b + I_3\cos^2 c)\right) \\ \implies \omega^2 &= \frac{Mgd}{Md^2 + (I_1\cos^2 a + I_2\cos^2 b + I_3\cos^2 c)}. \end{aligned}$$

Therefore the frequency of small oscillations of a compound pendulum is given by

$$\boxed{\omega = \sqrt{\frac{Mgd}{Md^2 + I_1\cos^2 a + I_2\cos^2 b + I_3\cos^2 c}}.}$$

Q3.

Find the kinetic energy of a homogeneous cylinder of radius a rolling inside a cylindrical surface of radius R (Fig. 41).

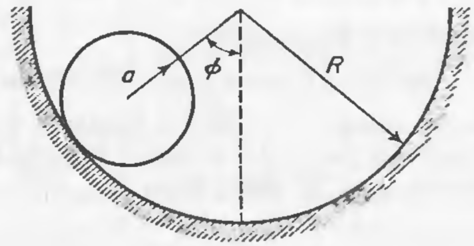


FIG. 41

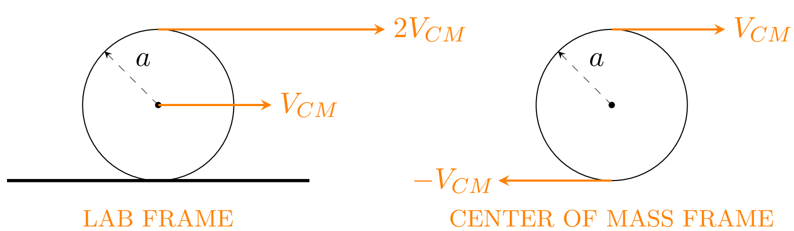
The total kinetic energy of the cylinder is given by the kinetic energy of the translational motion of the center of mass, superposed with the rotational kinetic energy of the cylinder along is center axis, since that is the only principle axis by which the cylinder rotates. It is

$$T = T_{CM} + T_{rot} = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_3\Omega_3^2.$$

Since the tangential speed of a rotating circular body rotating at angular velocity ω is $v = \omega r$, then the speed of the center of mass, a distance $(R-a)$ away is given by

$$V_{CM} = (R-a)\dot{\varphi}.$$

This argument can also be applied to find the rotational kinetic energy of the cylinder. Allow us to consider the rotation of the cylinder in the center of mass frame:



The tangential speed in the center of mass frame is $V_{CM} = (R-a)\dot{\varphi}$. This implies that the angular velocity about the center of mass is $\Omega_3 = \dfrac{V_{CM}}{a} = \dfrac{(R-a)\dot{\varphi}}{a}$. The kinetic energy of the cylinder is then

$$T = \frac{1}{2}M(R-a)^2\dot{\varphi}^2 + \frac{1}{2}I_3\frac{(R-a)^2\dot{\varphi}^2}{a^2}.$$

From question **(1c)**, the principle moment of inertia along the center axis of rotation of the cylinder is given by $I_3 = \frac{1}{2}Ma^2$. Substituting this value into the kinetic energy and simplifying, we find that

$$\boxed{T = \frac{3}{4}M(R-a)^2\dot{\varphi}^2.}$$

Q4.

The same as Problem 9, but for the case where the axis AB is not perpendicular to CD and is an axis of symmetry of the ellipsoid (Fig. 45).

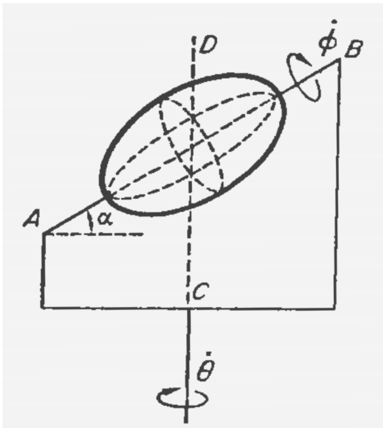
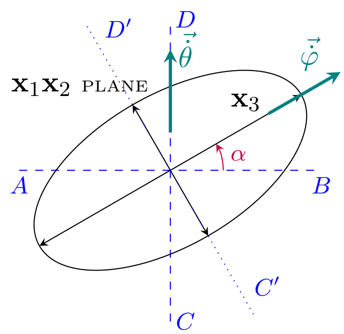
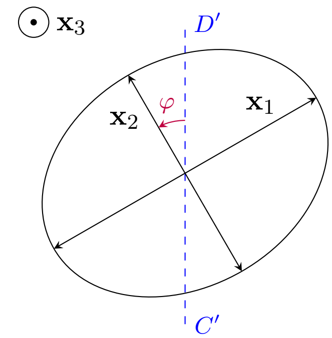


FIG. 45

The kinetic energy of the ellipsoid is strictly rotational, since the center of mass is stationary throughout rotation. The principle axes are oriented the same as the ellipsoid in **(1f)**. The line CD makes an angle α with the x_1x_2 plane, and an angle $\alpha+\pi/2$ with the x_3 axis. We proceed by finding the components of $\dot\theta$ and $\dot\varphi$ along the three principle axes.





Since the $\mathbf{x}_1\mathbf{x}_2$ plane is orthogonal to the $\mathbf{x}_3$ principle axis, then the component of $\dot\theta$ along $\mathbf{x}_3$ is simply $\dot\theta\sin\alpha$. Since $\mathbf{x}_3$ rotates with angular velocity $\dot\varphi$, which is parallel to the principle axis, then the angular velocity component along $\mathbf{x}_3$ is $\Omega_3 = \dot\varphi + \dot\theta\cos\alpha$.

The component of $\dot\theta$ along the $\mathbf{x}_1\mathbf{x}_2$ plane is given by $\dot\theta\cos\alpha$, since they are an angle α apart. Let the new line $C'D'$ be the projection of the CD line onto this plane. Now let the angle between the $\mathbf{x}_2$ principle axis and the $C'D'$ axis be φ (it rotates with angular velocity $\dot\varphi$). Since the projection of $\vec{\dot\theta}$ onto the $\mathbf{x}_1\mathbf{x}_2$ plane is $\dot\theta\cos\alpha$, then the components of $\dot\theta\cos\alpha$ along the $\mathbf{x}_2$ principle axis is $\dot\theta\cos\alpha\cos\varphi$. Again, since the principle axes $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal, then the component along the $\mathbf{x}_1$ axis is $\dot\theta\cos\alpha\sin\varphi$. Therefore our angular velocity compnents alone each other the three principle axes are

$$\Omega_1 = \dot\theta\cos\alpha\sin\varphi \quad \Omega_2 = \dot\theta\cos\alpha\cos\varphi \quad \text{and} \quad \Omega_3 = \dot\varphi + \dot\theta\sin\alpha.$$

The kinetic energy of the ellisoid is then

$$\begin{aligned} T_{rot} &= \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right). \\ &= \frac{1}{2}\left[I_1\dot\theta^2\cos^2\alpha\sin^2\varphi + I_2\dot\theta^2\cos^2\alpha\cos^2\varphi + I_3(\dot\varphi + \dot\theta\cos\alpha)^2\right] \\ &= \frac{1}{2}\dot\theta^2\cos^2\alpha(I_1\sin^2\varphi + I_2\cos^2\varphi) + \frac{1}{2}I_3(\dot\varphi + \dot\theta\sin\alpha)^2. \end{aligned}$$

Since AB is an axis of symmetry of the ellipsoid, then $I_1 = I_2$. After applying the pythagorean identity, the kinetic energy of the ellipsoid is

$$\boxed{T = \frac{1}{2}I_1\dot\theta^2\cos^2\alpha + \frac{1}{2}I_3(\dot\varphi + \dot\theta\sin\alpha)^2.}$$