PHY483 Problem Set 3

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Problem 1

A *black hole* is an object in space for which the curvature of the surrounding spacetime is so great that light cannot exit (which is why it is called 'black', since all light is absorbed). The spacetime curvature is established by the sheer mass of the object, squished into a tiny volume.

Einstein's theory of general relativity explains that the curvature of the surrounding spacetime dictates how much lengths are contracted and time is dilated compared to that of an object in free-fall, which is called following a *geodesic*. For an object in free-fall (following a geodesic), time is maximized, hence every accelerated object not on a geodesic experiences time dilated (lengthened). In the context of a black hole, we observe that light-like geodesics (light paths, travelling at $c = 2.99 \times 10^8$ m/s) cannot overcome the energy required to escape the gravitational well created by the mass, so we only see objects falling in to the black hole and not coming out, except for a small amount of radiation called *Hawking radiation*.

Black holes form in nature by the collapse of heavy stars. This occurs when the inner pressure of the star, established by nuclear fusion, at the end of it's lifetime cannot withstand the outward pressure created by gravity, causing the star to collapse in on itself. A *stellar mass* black hole forms by the collapse of individual stars, around 100 - 900 M_S, where M_S is a solar mass (the mass of our sun, $M_S \approx 1.989 \times 10^30$ kg). *Supermassive* black holes are formed the same way, but with much larger or multiple stars. These are approximately millions or billions of solar masses, and are usually located at the center of galaxies. *Primordial* black holes are black holes which formed at the big bang, and are smaller than the other types of black holes.

The idea of a black hole was first concieved by Schwarzchild, who considered a time-independent, spherically symmetric spacetime with no objects (a vacuum). The curvature of the surrounding spacetime was found to have a critical radius r_s where the equations would diverge to ∞, and this is called the *Schwarzchild radius* or the *event horizon*. The Schwarzchild radius forms at

$$r_s = \frac{2G_N M}{c^2}, \tag{1.1}$$

where M is the mass of the object, c is the speed of light, and $G_N = 6.67 \times 10^{-11}$ Nm^2/kg^2 is Newton's constant. If any mass occupies a radius less than r_s, it is a black hole. The temporal term is of the spacetime proportional to

$$1 - \frac{r_s}{r} \tag{1.2}$$

which means that, as one approaches the event horizon from a point $r_0 > r_s$, time becomes infinitely long and essentially 'stops'. It is for this exact reason that any object which falls into a black hole we observe it stop just as it reaches the event horizon. This is also because the last possible emission of light from that object will be just before it passes the event horizon. The radial term of the spacetime, since black holes are spherically symmetric, is similarly

$$\left(1 - \frac{r_s}{r}\right)^{-1} \tag{1.3}$$

which means that radial lengths are infinitely contracted as one approaches the event horizon, stretching you. There is also a diverging of the equations as r goes to 0, which is called the *singularity*. Any object which falls past the event horizon goes directly to the singularity, a small point of infinite density. It is at the singularity where the laws of physics break down and any object past the event horizon is crushed. As density increases at the singularity, the event horizon becomes larger and the black hole grows. It is assumed that the mass M of a black hole is positive, else the spacetime equations become too complicated to interpret.

There is an extended theoretical solution to the Einstein vacuum equations in which spacetime 'edges' are forbidden (a continuous manifold). This means that, for every black hole, there must also exist a *white hole* for the continuation of the spacetime past the singularity of a Schwarzchild black hole. These points are called *Einstein-Rosen bridges* or *wormholes* and connect two different points in spacetime. With the ideas presented, one may theorically move forward in time and access all parts of space through interlaced connections of wormholes, entering black holes at one point and exiting at another. At each singularity, it is suggested the traveller may access other realities. However, there is yet evidence to be found of ER bridges.

Problem 2

(**a**) Consider the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}. \tag{2.1}$$

Taking the trace of (2.1) with respect to $g^{\mu\nu}$ (raising the index, then contracting to find the scalars), we note that

$$\begin{align} g^{\mu\nu}R_{\mu\nu} &= R \tag{2.2}\\ g^{\mu\nu}g_{\mu\nu} &= 4 \tag{2.3}\\ g^{\mu\nu}T_{\mu\nu} &\equiv T \tag{2.4}\end{align}$$

hence

$$\begin{align} g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} + \Lambda g^{\mu\nu}g_{\mu\nu} &= -8\pi G_N g^{\mu\nu}T_{\mu\nu} \tag{2.5}\\ \implies R - 2R + 4\Lambda &= -8\pi G_N T \tag{2.6}\end{align}$$

Solving for R, we find

$$R = 4\Lambda + 8\pi G_N T. \tag{2.7}$$

Re-substituting (2.7) back into (2.1), we find the alternate form of the Einstein equations:

$$\begin{align} R_{\mu\nu} &= -8\pi G_N T_{\mu\nu} + \frac{1}{2}\left[4\Lambda + 8\pi G_N T\right] g_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2.8}\\ &= -8\pi G_N T_{\mu\nu} + 4\pi G_N T g_{\mu\nu} + (2\Lambda - \Lambda)g_{\mu\nu} \tag{2.9}\\ &= -8\pi G_N\left[T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right] + \Lambda g_{\mu\nu} \tag{2.10}\end{align}$$

as desired.

(**b**) In this problem, I will be referencing equations from notes in *square* brackets $[\ldots]$. We begin with equation(s) [14.15] from the notes, since the metric is the same and we have not imposed any constraints from the field equations yet:

$$\begin{align} R_{tt} &= e^{2(\alpha-\beta)}\left[-(\partial_r^2\alpha) - (\partial_r\alpha)^2 + (\partial_r\alpha)(\partial_r\beta) - \frac{2}{r}(\partial_r\alpha)\right]\\ &\quad + \left\{-(\partial_t^2\beta) - (\partial\beta)^2 + (\partial_t\alpha)(\partial_t\beta)\right\} \tag*{[14.15a]}\\ R_{tr} &= \left\{-\frac{2}{r}(\partial_t\beta)\right\} \tag*{[14.15b]}\\ R_{rr} &= \left[(\partial_r^2\alpha) + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta) - \frac{2}{r}(\partial_r\alpha)\right]\\ &\quad + e^{2(\beta-\alpha)}\left\{-(\partial_t^2\beta) - (\partial\beta)^2 + (\partial_t\alpha)(\partial_t\beta)\right\} \tag*{[14.15c]}\\ R_{\theta\theta} &= -\left[e^{-2\beta}(r(\partial_r\beta) - r(\partial_r\alpha) - 1) + 1\right] \tag*{[14.15d]}\\ R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta}. \tag*{[14.15e]}\end{align}$$

Since the metric is symmetric, [14.15b] is zero, and we again arrive at our conclusion [14.16] that $\beta = \beta(r)$. By [14.17] through [14.24] (the procedure is identical, and we have not imposed any constraints from Λ yet), we also find

$$\beta(r) = C - \alpha(r) \tag*{[14.25]}$$

The difference is imposing the θ equation [14.15d], which is now *nonzero*:

$$R_{\theta\theta} = \Lambda g_{\theta\theta} = -\Lambda r^2 \tag{2.11}$$

$$\implies \Lambda r^2 = \left[e^{2(\alpha-c)}\left(-2r(\partial_r\alpha) - 1\right) + 1\right] \tag{2.12}$$

$$\implies \Lambda r^2 - 1 = -\partial_r\left[re^{2(\alpha-c)}\right] \tag{2.13}$$

which, by a simple integration, implies

$$re^{2\alpha} = r - \frac{1}{3}\Lambda r^3 + D \tag{2.14}$$

$$\implies e^{2\alpha} = 1 - \frac{1}{3}r^2 + \frac{D}{r}, \tag{2.15}$$

where I absorbed e^{-2c} into the time coordinate by a transformation (as per the notes...). Imposing the physical constraint (as $r \to \infty$, g_{tt} goes like $1 - 2G_N M/r$ ($c = 1$) in the Newtonian limit, where $\Lambda = 0$ as in [14.30]),

$$e^{2\alpha} = 1 - \frac{1}{3}\Lambda r^2 - \frac{2G_N M}{r} \tag{2.16}$$

By [14.13], the metric is therefore

$$ds^2 = \left(1 - \frac{1}{3}\Lambda r^2 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{1}{3}\Lambda r^2 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\Omega_2^2. \tag{2.17}$$

where $\mu \equiv G_N M$.

(c) Let

$$f(r) = 1 - \frac{1}{3}\Lambda r^2 - \frac{2\mu}{r} \tag{2.18}$$

for simplicity. Using (2.17) and our SymPy code, the following non-zero Christoffels were found:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{-\mu + \Lambda r^3/3}{r(2\mu + \Lambda r^3/3 - r)} \tag{2.19}$$

$$= \frac{1}{2}\,\frac{-2\mu/r^2 + 2\Lambda r/3}{2\mu/r + \Lambda r^2/3 - 1} \tag{2.20}$$

$$= \frac{1}{2}\frac{f'(r)}{f(r)} \tag{2.21}$$

$$\Gamma^r_{tt} = \frac{1}{2r^3}\left[2\mu - \frac{2\Lambda r^3}{3}\right]\left[2\mu + r\left(\frac{\Lambda r^3}{3} - 1\right)\right] \tag{2.22}$$

$$= -\frac{1}{2}f'(r)f(r) \tag{2.23}$$

$$\Gamma^r_{rr} = -\Gamma^t_{tr} = -\Gamma^t_{rt} \tag{2.24}$$

$$\Gamma^r_{\theta\theta} = 2\mu + \frac{\Gamma r^3}{3} - r \tag{2.25}$$

$$= \frac{1}{r}f(r) \tag{2.26}$$

$$\Gamma^r_{\varphi\varphi} = \sin^2\theta\,\Gamma^r_{\theta\theta} \tag{2.27}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \tag{2.28}$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta \tag{2.29}$$
$$\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta \tag{2.30}$$

where I have introduced $f(r)$ and $f'(r)$ where necessary. Using the temporal and azimuthal geodesic equations (we want to find the effective potential, which implies finding the energy and angular momentum J),

$$0 = \ddot{t} + \frac{f'(r)}{f(r)}\dot{r}\dot{t} \tag{2.31}$$
$$0 = \ddot{\varphi} + \frac{2}{r}\dot{\varphi}\dot{r} + 2\cot\theta\dot{\varphi}\dot{\theta} \tag{2.32}$$

the first integral of (2.32) is the angular momentum

$$J = r^2\sin^2\theta\dot{\varphi} \tag{2.33}$$

and (2.31) can be integrated to find the energy

$$0 = f(r)\ddot{t} + f'(r)\dot{r}\dot{t} \tag{2.34}$$
$$= \frac{d}{d\lambda}\left[f(r)\dot{t}\right] \tag{2.35}$$
$$\implies E = f(r)\dot{t} \tag{2.36}$$

where (2.35) follows from the chain rule. Lastly, we invoke the tangent norm condition for this metric (2.17)

$$\epsilon = f(r)\dot{t}^2 - [f(r)]^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right) \tag{2.37}$$

which becomes, with the help of our conserved quantities (2.33) and (2.36),

$$\epsilon = [f(r)]^{-1}E^2 - [f(r)]^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - \frac{J^2}{r^2\sin^2\theta} \tag{2.38}$$

By rotational symmetry, we can chose equitorial motion for $\theta = \frac{\pi}{2}$, so $\dot{\theta} = 0$ and $\sin\theta = 1$. The kinetic energy component is $\frac{1}{2}\dot{r}^2$, and the total energy is $\frac{1}{2}E^2$, which implies the potential is

$$\frac{1}{2}E^2 - \frac{1}{2}\dot{r}^2 = \frac{1}{2}f(r)\left[\epsilon + \frac{J^2}{r^2}\right] \tag{2.39}$$
$$= V^{\text{cosmo}}_{\text{eff}}(r) \tag{2.40}$$

Expanding $f(r)$, we find

$$V^{\text{cosmo}}_{\text{eff}}(r) = \frac{1}{2}\left[1 - \frac{1}{3}\Lambda r^2 - \frac{2\mu}{r}\right]\left[\epsilon + \frac{J^2}{r^2}\right] \tag{2.41}$$
$$= \frac{1}{2}\left(1 - \frac{2\mu}{r}\right)\left(\epsilon + \frac{J^2}{r^2}\right) - \frac{\Lambda r^2}{6}\left(\epsilon + \frac{J^2}{r^2}\right) \tag{2.42}$$
$$= V^{\text{Schw}}_{\text{eff}}(r) - \frac{\Lambda}{6}\left(\epsilon r^2 + J^2\right) \tag{2.43}$$

as desired. We note the affects of the addition of Λ to the potential as in (2.43). Generally, Λ is very small ($\mathcal{O}(10^{-52})$), which means that it is only important on a universal scale, and negligible on a solar system scale. For particles (massless or not) with high angular momentum, the addition of the $-\frac{\Lambda}{6}(\epsilon r^2 + J^2)$ just reduces the overall magnitude of the potential. Otherwise, for various values of M and J, orbits (stable or unstable) may still form. For (very) large r or J, the cosmological term will have a greater affect, hence this term is generally only important for large-scale systems. In particular, is has a greater large-scale affect for massive particles, since we include the r^2 term. Otherwise, only massless particle angular momentum is important. The negative sign implies the Λ term produces a repulsion from the attracting mass, more prevalent at large scales / high angular momentum values.

(d) In the Newtonian limit, $\mathbf{g} = -\nabla_r V(r)$, to $\mathcal{O}(r^{-2})$ (any terms smaller we will neglect). The derivative of (2.43) is

$$\mathbf{g} = -\frac{d}{dr}\left[\frac{\epsilon}{2} - \frac{\epsilon\mu}{r} + \frac{J^2}{2r^2} - \frac{\mu J^2}{r^3} - \frac{\Lambda}{6}(\epsilon r^2 + J^2)\right] \tag{2.44}$$

$$= -\left[\frac{\epsilon\mu}{r^2} - \frac{J^2}{r^3} + 3\frac{\mu J^2}{r^4} - \frac{\Lambda}{3}(\epsilon r)\right] \tag{2.45}$$

$$\approx \left(-\frac{\mu}{r^2} + \frac{\Lambda r}{3}\right) + \mathcal{O}(r^{-3}) \tag{2.46}$$

Since we are assuming a massive particle, $\epsilon = 1$. We can lastly re-dimensionalize by multiplying everything by c^2 and re-substituting $\mu = G_N M$:

$$\mathbf{g} = \left(-\frac{G_N M}{r^2} + \frac{c^2\Lambda r}{3}\right) \tag{2.47}$$

which is the weak-field ($r \gg 1$) Newtonian limit gravitational acceleration. As mentioned in part (c), the addition of the Λ term is only crucial for large scale systems, since in reality it is so small. In a solar system context, this term is not very important. But for $r \gg 1$, such as a galaxy-galaxy system, on a universal scale, is more important.

Problem 3

(**a**) Let

$$ds^2 = L^2 \left[dt^2 - \cosh^2 t \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)\right] \tag{3.1}$$

be a 3-dimensional de Sitter metric. The geodesic equations are derived from

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu \dot{x}^\lambda = 0 \tag{3.2}$$

where $\dot{x} \equiv \dfrac{dx}{d\lambda}$ along the affine parameter. Using **[problem set 2, 3b]** and our SymPy code, the only nonzero Christoffels are

$$\Gamma^t_{\theta\theta} = \cosh t \sinh t \qquad \Gamma^t_{\varphi\varphi} = \cosh t \sinh t \sin^2\theta \tag{3.3}$$

$$\Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \tanh t \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \tag{3.4}$$

$$\Gamma^\varphi_{t\varphi} = \Gamma^\varphi_{\varphi t} = \tanh t \qquad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta \tag{3.5}$$

from which the geodesic equations are (by (3.2)),

$$\ddot{t} + \sinh t \cosh t \left[\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right] = 0 \tag{3.6}$$

$$\ddot{\theta} + 2\tanh t\, \dot{t}\dot{\theta} - 2\sin\theta\cos\theta\, \dot{\varphi}^2 = 0 \tag{3.7}$$

$$\ddot{\varphi} + 2\tanh t\, \dot{t}\dot{\varphi} + 2\cot\theta\, \dot{\theta}\dot{\varphi} = 0 \tag{3.8}$$

Note that each term in (3.8) contains derivatives in φ, which implies the existence of a conserved quantity. We have

$$0 = \ddot{\varphi} + 2\dot{\varphi}\left[\tanh t\, \dot{t} + \cot\theta\, \dot{\theta}\right] \tag{3.9}$$

$$= \ddot{\varphi} + 2\dot{\varphi}\frac{d}{d\lambda}\left[\log(\cosh t) + \log(\sin\theta)\right] \tag{3.10}$$

$$= \ddot{\varphi} + \dot{\varphi}\frac{d}{d\lambda}\left[\log(\cosh^2 t \sin^2\theta)\right] \tag{3.11}$$

$$= \ddot{\varphi} + \dot{\varphi}\left[\cosh^2 t\sin^2\theta\right]^{-1}\frac{d}{d\lambda}\left[\cosh^2 t\sin^2\theta\right] \tag{3.12}$$

$$= \ddot{\varphi}(\cosh^2 t\sin^2\theta) + \dot{\varphi}\frac{d}{d\lambda}\left[\cosh^2 t\sin^2\theta\right] \tag{3.13}$$

$$= \frac{d}{d\lambda}\left[\dot{\varphi}\cosh^2 t\sin^2\theta\right] \tag{3.14}$$

hence

$$J \equiv \dot{\varphi}\cosh^2 t\sin^2\theta \tag{3.15}$$

is constant (conserved quantity). Lastly, returning to (3.7), we note that if $\theta = \dfrac{\pi}{2}$ (equitorial motion) is constant, then $\dot{\theta} = \ddot{\theta} = 0$ and $\cos\dfrac{\pi}{2} = 0$, which solves the equation in θ for all λ.

(**b**) A massless particle following a lightlike geodesic has the tangent vector norm condition

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{3.16}$$

For an arbitrary $\theta = \dfrac{\pi}{2}$ (we can choose this by the rotational symmetry of the metric), this condition expands to

$$0 = L^2\dot{t}^2 - \cosh^2 t\dot{\varphi}^2 \tag{3.17}$$

where we have invoked $\dot{\theta} = 0$ and $\sin\frac{\pi}{2} = 1$. We re-arrange this, and invoke (3.15) to write

$$L^2\dot{t}^2 = \cosh^2 t\dot{\varphi}^2 \tag{3.18}$$

$$L^2\dot{t}^2 = \frac{J^2}{\cosh^2 t} \tag{3.19}$$

$$\implies \frac{dt}{d\lambda} = +\frac{J}{L\cosh t} \tag{3.20}$$

where we have taken the '+' from the square root since time propagates forwards from $\lambda = 0$. Seperating variables and integrating,

$$dt\cosh t = d\lambda\frac{J}{L} \tag{3.21}$$

$$\implies \sinh t = \frac{J}{L}\lambda + C \tag{3.22}$$

$$\implies t(\lambda) = \operatorname{arcsinh}\left[\frac{J}{L}\lambda + C\right] \tag{3.23}$$

where C is a constant imposed by the initial condition of time. Since we can always perform a translation in t, we can choose C to be zero, so the origin of time corresponds to $\lambda = 0$. Note that we deal with the inverse hyperbolic functions via

$$\cosh^2 x - \sinh^2 x = 1 \implies \text{, for} \qquad x = \operatorname{arcsinh}(y),$$

$$\cosh^2(\operatorname{arcsinh}(y)) - y^2 = 1 \implies \cosh(\operatorname{arcsinh}(y)) = \sqrt{1+y^2} \tag{3.24}$$

Imposing the function for $t(\lambda)$ (3.23) on our condition for J (3.15) to solve for $\varphi(\lambda)$,

$$\frac{d\varphi}{d\lambda} = \frac{J}{\cosh^2(t(\lambda))} \tag{3.25}$$

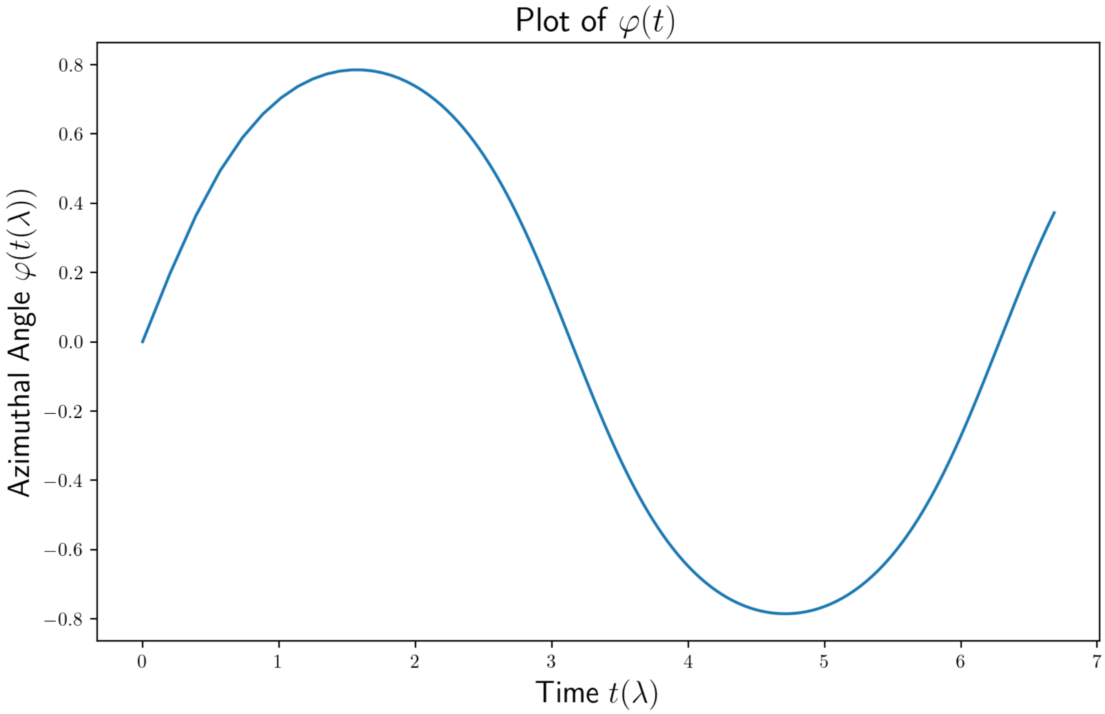
$$= \frac{J}{1 + J^2\lambda^2/L^2} \tag{3.26}$$

$$\implies d\varphi = d\lambda\frac{J}{1 + J^2\lambda^2/L^2}. \tag{3.27}$$

Using $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ and making the substitution $u = \frac{J}{L}\lambda$, we can integrate to find

$$\varphi(\lambda) = L\arctan\left(\frac{J}{L}\lambda\right) + D, \tag{3.28}$$

where D is another constant of integration, determining the initial condition on $\varphi(\lambda)$, which we may also just take to be zero since it is arbitrary. We plot $\varphi(t)$ by noting $\varphi(\lambda) = \varphi((L/J)\sin t(\lambda))$:



This occurs because time runs on a hyperbolic scale. If one were to plot the cartesian projections from the polar coordinates, $(\cos\varphi(t(\lambda)), \sin\varphi(t(\lambda)))$, one would find that the solution converges to a single point as $t \to \infty$. This is due to the hyperbolic nature of three-dimensional de Sitter spacetime.

Problem 4

(**a**) Let

$$\epsilon = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{4.1}$$

be the tangent vector norm condition, where $\dot{x} \equiv \frac{dx}{d\lambda}$, λ being the affine parameter. We can derive the geodesic equations from (4.1) by showing that ϵ is a conserved quantity along the path.

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\left[g_{\mu\nu}(x^\sigma)\dot{x}^\mu \dot{x}^\nu\right] &(4.2)\\
&= \frac{dg_{\mu\nu}(x^\sigma)}{d\lambda}\dot{x}^\mu \dot{x}^\nu + g_{\mu\nu}(x^\sigma)\ddot{x}^\mu \dot{x}^\nu + g_{\mu\nu}(x^\sigma)\dot{x}^\mu \ddot{x}^\nu &(4.3)\\
&= \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\dot{x}^\sigma \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu}(x^\sigma)\ddot{x}^\mu \dot{x}^\nu + g_{\mu\nu}(x^\sigma)\dot{x}^\mu \ddot{x}^\nu. &(4.4)
\end{aligned}$$

Since $g_{\mu\nu}(x^\sigma)$ is symmetric, then the last two terms are identical under exchange of indices. Hence

$$0 = \partial_\sigma g_{\mu\nu}\dot{x}^\sigma \dot{x}^\mu \dot{x}^\nu + 2g_{\mu\nu}\dot{x}^\nu \ddot{x}^\mu. \tag{4.5}$$

In a similar fashion, since the first term contracts all indices, and $g_{\mu\nu}$ is symmetric, we can add and subtract a derivative term,

$$0 = \left[\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\mu\sigma}\right]\dot{x}^\sigma \dot{x}^\mu \dot{x}^\nu + 2g_{\mu\nu}\dot{x}^\nu \ddot{x}^\mu \tag{4.6}$$

which is equivalent to the downstairs, covariant Christoffel symbol

$$2\Gamma_{\nu,\mu\sigma} = \partial_\sigma g_{\mu\nu} + \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\mu\sigma} \tag{4.7}$$

We raise the index by multiplying through an upstairs $g^{\nu\kappa}$ metric:

$$\begin{aligned}
0 &= 2g^{\nu\kappa}\Gamma_{\nu,\mu\sigma}\dot{x}^\sigma \dot{x}^\mu \dot{x}^\nu + 2g^{\nu\kappa}g_{\mu\nu}\dot{x}^\nu \ddot{x}^\mu &(4.8)\\
&= 2\Gamma^\kappa_{\mu\sigma}\dot{x}^\sigma \dot{x}^\mu \dot{x}^\nu + 2\delta^\kappa_\mu \dot{x}^\nu \ddot{x}^\mu &(4.9)\\
&= 2\dot{x}^\nu\left[\Gamma^\kappa_{\mu\sigma}\dot{x}^\sigma \dot{x}^\mu + \ddot{x}^\kappa\right] &(4.10)
\end{aligned}$$

After dividing both sides by $2\dot{x}^\nu$, we arrive at the geodesic equation (I will explicitly write the derivatives here)

$$\frac{d^2x^\kappa}{d\lambda^2} + \Gamma^\kappa_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{4.11}$$

and therefore (4.1) is a conserved quantity, a first integral of the geodesic equation.

(**b**) We consider purely radial motion for a massive object to fall into the Schwarzchild field. The geodesic equations are

$$\begin{aligned}
&\ddot{t} + \frac{r_s}{r(r-r_s)}\dot{t}\dot{r} = 0 &(4.12)\\
&\ddot{r} + \frac{r_s(r-r_s)}{2r^3}\dot{t}^2 - \frac{r_s}{2r(r-r_s)}\dot{r}^2 - (r-r_s)\left[\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right] = 0 &(4.13)\\
&\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta\cos\theta\dot{\varphi}^2 = 0 &(4.14)\\
&\ddot{\varphi} + \frac{2}{r}\dot{\varphi}\dot{r} + 2\cot\theta\dot{\theta}\dot{\varphi} = 0 &(4.15)
\end{aligned}$$

where $\dot{x} \equiv \dfrac{dx}{d\lambda}$ along the affine parameter. Furthermore, we consider the conservation of the geodesic four-velocity squared magnitude,

$$\epsilon = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right] \tag{4.16}$$

where $\epsilon = 0$ for massless particles and $\epsilon = 1$ for massive particles. The initial conditions of the problem impose $\epsilon = 1$ for a massive buoy, and $r_0 > r_s$. Purely radial motion further implies that, by the rotational symmetry of the Schwarzchild field, $\dot{\varphi} = \dot{\theta} = 0$. For a buoy released from rest, $\dot{r}(\lambda = 0) = 0$ as well, if we let $r(\lambda = 0) = r_0$ at some time $t(\lambda = 0) = t_0$. Since we may always perform a translation in time to the event when the buoy is released, we can just pick $t_0 = 0$. Then the geodesic equations and norm condition reduce to (we will only need the temporal equation and the norm constraint, not the radial geodesic equation)

$$\ddot{t} + \frac{r_s}{r(r - r_s)}\dot{t}\dot{r} = 0 \tag{4.17}$$

$$\left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 = 1 \tag{4.18}$$

The temporal equation implies that energy is conserved with a first integral,

$$E = p_t = \left(1 - \frac{r_s}{r}\right)\dot{t} = \text{constant} \tag{4.19}$$

hence the norm condition becomes

$$1 = \left(1 - \frac{r_s}{r}\right)^{-1}\left[E^2 - \dot{r}^2\right] \tag{4.20}$$

which implies

$$\dot{r}^2 = E^2 - \left(1 - \frac{r_s}{r}\right). \tag{4.21}$$

Using this, and noting that at $\dot{r}(\lambda = 0) = 0$ and the potential is $\left(1 - \dfrac{r_s}{r}\right)$, then we can invoke the total energy as a constraint:

$$E^2 = 1 - \frac{r_s}{r_0}. \tag{4.22}$$

Let the time t be a function of r, so we can use r as a parametrization from the moment the buoy was released at r_0 to find the total amount of time elapsed at a radius r. By the chain rule,

$$E = \left(1 - \frac{r_s}{r}\right)\frac{dt}{dr}\dot{r} \tag{4.23}$$

$$= \left(1 - \frac{r_s}{r}\right)\frac{dt}{dr}\left[E^2 - \left(1 - \frac{r_s}{r}\right)\right]^{1/2} \tag{4.24}$$

so, by some simple re-arranging,

$$dt = dr\, E\left(1 - \frac{r_s}{r}\right)^{-1}\left[E^2 - \left(1 - \frac{r_s}{r}\right)\right]^{-1/2} \tag{4.25}$$

which implies

$$t - t_0 = E\int_r^{r_0} dr'\,\left(1 - \frac{r_s}{r'}\right)^{-1}\left[E^2 - \left(1 - \frac{r_s}{r'}\right)\right]^{-1/2} \tag{4.26}$$

Yet, we can still simplify this expression. Using the constraint on E in terms of r_0, and taking $t_0 = 0$, this simplifies to

$$t = \sqrt{1 - \frac{r_s}{r_0}} \int_r^{r_0} dr' \left[\left(1 - \frac{r_s}{r'}\right) \sqrt{\frac{r_s}{r'} - \frac{r_s}{r_0}} \right]^{-1} \tag{4.27}$$

is the amount of time the buoy will take to fall into the black hole (the 'time' here is referring to us, on the spaceship - since the proper time for the buoy will remain linear as it's maximized along the geodesic, according to the choice of affine parameter $\lambda = a\tau + c$). One key observation is that the integral diverges as $r \to r_s$ (just plot it in Desmos). Thus, from the perspective of the stewards on the ship, we observe the buoy make it's way into the black hole until we do not see it moving anymore: time has 'stopped' for it once it reaches the event horizon. The buoy may physically pass through the event horizon to the other side, however since no information will leave the black hole, we will see the last stationary position of the buoy prior to crossing r_s.

The proper time τ of the buoy should be finite as it goes to the singularity. The interval is

$$\Delta\tau = \tau - \tau_0 = \int_S ds \tag{4.28}$$

where ds is the line element formed by the metric

$$ds = \left[\left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 \right]^{1/2} \tag{4.29}$$

$$= \left[\left(1 - \frac{r_s}{r}\right) \left(\frac{dt}{dr}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \right]^{1/2} dr \tag{4.30}$$

and S is the path from r_0 to $r = 0$. Using the fundamental theorem of calculus on (4.27), taking the derivative with respect to r, we find

$$\frac{dt}{dr} = -\sqrt{1 - \frac{r_s}{r_0}} \left[\left(1 - \frac{r_s}{r}\right) \sqrt{\frac{r_s}{r} - \frac{r_s}{r_0}} \right]^{-1} \tag{4.31}$$

$$\implies \left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{r_s}{r_0}\right) \left(1 - \frac{r_s}{r}\right)^{-2} \left(\frac{r_s}{r} - \frac{r_s}{r_0}\right)^{-1} \tag{4.32}$$

hence, from (4.30), we have the line element

$$ds = \left[\left(1 - \frac{r_s}{r_0}\right) \left(1 - \frac{r_s}{r}\right)^{-1} \left(\frac{r_s}{r} - \frac{r_s}{r_0}\right)^{-1} - \left(1 - \frac{r_s}{r}\right)^{-1} \right]^{1/2} dr \tag{4.33}$$

$$= \left[\left(\frac{r(r_0 - r_s)}{r_s(r_0 - r)} - 1\right) \left(1 - \frac{r_s}{r}\right)^{-1} \right]^{1/2} dr \tag{4.34}$$

$$= \left[\frac{r^2(r_0 - r_s) - r r_s(r_0 - r)}{r_s(r_0 - r)(r - r_s)} \right]^{1/2} dr \tag{4.35}$$

$$= \left[\frac{r_0 r}{r_s r_0 - r_s r} \right]^{1/2} dr \tag{4.36}$$

$$= \left[\frac{r_s}{r} - \frac{r_s}{r_0} \right]^{-1/2} dr \tag{4.37}$$

Hence, using (4.28), and integrating from r_0 inwards to the singularity $r = 0$, we have

$$\tau = \int_{r_0}^{0} dr \left[\frac{r_s}{r} - \frac{r_s}{r_0} \right]^{-1/2} \tag{4.38}$$

which is the total proper time for the buoy to reach the singularity (I have taken $\tau_0 = 0$ to coincide with the release of the buoy at $t = 0$). If one were to plot (4.38) as a function of r (the upper integral bound is r instead of zero), one would observe a convergence over the event horizon r_s.