mt2

My score

70.5% (31/44)

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1. (a) (4 points) Find the power series of $\cos(z)$ centered at $z = 0$.
The identity $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, which you may use without proof, could be useful.

Q1a 4

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right]$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{i^n z^n}{n!} + (-1)^n \frac{i^n z^n}{n!}\right].$$

[illegible] odd integer, this

Note that whenever n is ... expression is $\sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \frac{i^n z^n}{n!} - \frac{i^n z^n}{n!} = 0$, so n is only even. Let $n = 2K$ be such an even integer.

$$= \frac{1}{2} \sum_{K=0}^{\infty} \frac{i^{2K}}{(2K)!} z^{2K} + \frac{i^{2K}}{(2K)!} z^{2K}$$

$$= \frac{2}{2} \sum_{K=0}^{\infty} (i^2)^K \frac{z^{2K}}{(2K)!}$$

and therefore

$$\boxed{\cos(z) = \sum_{K=0}^{\infty} (-1)^K \frac{z^{2K}}{(2K)!}}$$

is the power series of $\cos(z)$ at $z_0 = 0$.

correct / close enough 4

Q1b 4

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(b) (4 points) Derive the Laurent series of the function,

$$f(z) = \frac{z}{z-\mathrm{i}} \cos\left(\frac{1}{(z-\mathrm{i})^2}\right)$$

centered at $z_0 = \mathrm{i}$.

As before, $\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$. This implies that $\cos\left(\frac{1}{(z-i)^2}\right) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{(z-i)^2}\right)^{2k} \frac{1}{(2k)!}$. Furthermore, $\frac{z}{z-i} = \frac{z+i-i}{z-i}$ which is $\frac{i}{z-i} + 1$. Our function expansion is then $f(z) = \left(\frac{i}{z-i} + 1\right) \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} (z-i)^{-4k}$. We can write this expansion this way be

correct / close enough 4

at $z_0 = i$. Thus we have

$$f(z) = i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (z-i)^{-4k-1} + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} (z-i)^{-4k}.$$

Notice that the first term covers odd exponents, while the second even exponents. An index change to the negative regime yields that

$$f(z) = i\sum_{k=-\infty}^{k=0} (-1)^k \frac{1}{(|2k|)!} (z-i)^{4k-1} + \sum_{k=-\infty}^{k=0} (-1)^k \frac{1}{(|2k|)!} (z-i)^{4k}$$

Thus the coefficients of the Laurent series are just

$$f(z) = \sum_{n=-\infty}^{n=0} a_n (z-i)^n$$

where $a_n = \begin{cases} i(-1)^{\frac{n+1}{4}} \frac{1}{(|\frac{n+1}{2}|)!} & \text{for } n = -1, -5, -9, -13, \ldots, -4k-1, \ldots \\ (-1)^{\frac{n}{4}} \frac{1}{(|\frac{n}{2}|)!} & \text{for } n = 0, -4, -8, -12, \ldots, -k \end{cases}$

Here I made the substitutions $n \to 4k-1$, $k = \frac{n+1}{4}$ and $n \to 4k$

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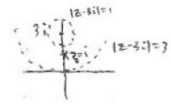
Q1c 4

(c) (4 points) Let $f(z)$ be the function from part (b). Compute both of the integrals

$$\int_{|z-3i|=1} f(z)\mathrm{d}z, \qquad \int_{|z-3i|=3} f(z)\mathrm{d}z$$



using Cauchy's residue theorem or otherwise.

$f(z) = \frac{z}{z-i}\cos\left(\frac{1}{(z-i)^2}\right).$

Since the circle $|z-3i|=1$ does not contain any of the singularities of f, the first integral is zero. $z_0=i$ is the singularity and i is not in $B(3i)$.

Similarly, $i \in B_3(3i)$, hence by Cauchy's Residue theorem,

$$\int_{|z-3i|=3} \frac{z}{z-i}\cos\left(\frac{1}{(z-i)^2}\right)dz = 2\pi i \cdot \mathrm{Res}(f, i).$$

correct / close enough 4

Now $\mathrm{Res}(f,i)$ is given by the a_{-1} term in the Laurent series expansion of f, which as previously determined is when $k=0$,

$$i \cdot (-1)^0 \frac{1}{(2(0))!}(z-i)^{0-1} \text{ which is just } i.$$

Therefore $\displaystyle\int_{|z-3i|=3} \frac{z}{z-i}\cos\left(\frac{1}{(z-i)^2}\right)dz = i \cdot 2\pi i = -2\pi.$

The first integral $\displaystyle\int_{|z-3i|=1} f(z)dz = 0$, since $z_0=i$ doesn't lie inside that curve, by Cauchy's Residue theorem.

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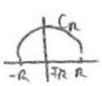
Q2 7

2. (8 points) Calculate
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x+1)^2+4}dx.$$

Hint: consider the function $f(z) := \frac{e^{iz}}{(z+1)^2+4}$, and apply Cauchy's residue theorem to the line integral along the closed semicircle in the upper half-plane of radius R, centered at the origin. Let $R \to \infty$. Make sure to show that the contribution of the integral from the semicircular arc $\{|z| = R, \text{Im}[z] > 0\}$ tends to 0 as $R \to \infty$.



$$\int_{\gamma_R} \frac{e^{iz}}{(z+1)^2+4} dz = \int_{I_R} \frac{e^{iz}}{(z+1)^2+4} dz + \int_{C_R} \frac{e^{iz}}{(z+1)^2+4} dz.$$

Note $\text{Re}\left\{\int_{I_R} \frac{e^{iz}}{(z+1)^2+4} dz\right\} = \int_{-R}^{R} \frac{\cos x}{(x+1)^2+4} dx$, then take $R \to \infty$.

We have that $(z+1)^2+4 = z^2+2z+1+4 = z^2+2z+5$, which has roots $z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 4i$. In the upper half plane, $z_0 = -1+4i$ is the only singularity. Thus

$$\lim_{R\to\infty} \int_{I_R} \frac{e^{iz}}{(z+1-4i)(z+1+4i)} dz = 2\pi i \cdot \text{Res}\left(f(z), -1+4i\right) = 2\pi i \lim_{z\to -1+4i} \ldots$$
$$= 2\pi i \frac{e^{-i-4}}{8i} = \frac{\pi}{4}\left(e^{-i-4}\right).$$

Poles are not correct.

Likewise, $\left|\int_{C_R} \frac{e^{iz}}{(z+1)^2+4} dz\right| \le \max_{z\in C_R} \{|f(z)|\} \cdot \pi R$ by M-L estimation,
$$\le \left|\frac{1}{(|R|+1)^2+4}\right| \cdot \pi R$$

which thus $\lim_{R\to\infty} \frac{\pi R}{R^2+2R+5} = 0$, thus

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{(z+1)^2+4} dz = \lim_{R\to\infty} \int_{I_R} \frac{e^{iz}}{(z+1)^2+4} dz + 0 = \frac{\pi}{4}\left(e^{-i-4}\right).$$

The initial value of the integral is therefore $\text{Re}\left\{\frac{\pi}{4} e^{-i-4}\right\} = \frac{\pi}{4} e^{-4} c$

Hence $$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x+1)^2+4} dx = \frac{\pi}{4} e^{-4}\cos(1), \text{ as desired.}$$

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3. (a) (2 points) State Liouville's theorem. You do not need to provide a proof.

Q3a 2

Let $f: \mathbb{C} \to \mathbb{C}$ be entire, If $\exists M > 0$ such that $|f(z)| \leq M$, then $f(z)$ is identically constant.

(b) (6 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function satisfying $|f(z)| \geq 1$ $\forall z \in \mathbb{C}$. Show that f is constant.

Q3b 3

Since f is entire, $e^{-f(z)}$ is also entire. If $|f(z)| \geq 1$,

then $|e^{-f(z)}|$ is b

$|e^{-f(z)}| = e^{-|f(z)|}$,

Since $e^{-f(z)}$ is bo

$e^{-f(z)}$ is constant;

$e^{-f(z)}(-f'(z)) = 0$, However, since $e^{-f(z)}$ is never zero,

it must be that $f'(z)$ is zero,

and thus $f(z)$ is constant too.

Wrong choice of function, it is not true that $g(z) = e^{-f(z)}$ is bounded under the assumption |f(z)| \ge 1. Try instead g(z)=1/f(z). -3

Q4 2

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4. (8 points) Let $f : \{0 < |z| < 1\} \to \mathbb{C}$ be an analytic function on the punctured unit disc $\{0 < |z| < 1\}$. Suppose that for some $A > 0$ and $1 > \epsilon > 0$ we have,

$$|f(z)| \leq A|z|^{-1+\epsilon}$$

for all $0 < |z| < 1$. Prove that f has a removeable singularity at $z = 0$.

Hint: consider the function $g(z) = zf(z)$ defined on the punctured disc $\{0 < |z| < 1\}$.

If f has a removable singularity at $z=0$, it suffices to prove that there are no terms a_k in the Laurent series of f such that $a_k = 0$ for $k < 0$.

Define the function $g(z) = z f(z)$ on the punctured disk $\{0 < |z| < 1\}$. Since g is analytic, its power series is given by $g(z) = \sum_{k=0}^{\infty} a_k z^k$, and $a_k = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{z^{k-1}} dz$, which follows since $z f(z)$ is analytic.

This is equivalent to $|a_k| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{z f(z)}{z^{k+1}} dz \right| \leq \frac{1}{2\pi} \max_{z \in \{|z|=1\}} \left| \frac{f(z)}{z^k} \right| \cdot 2\pi$

$\leq \max_{z \in \{|z|=1\}} \frac{A|z|^{-1+\epsilon}}{|z|^k}$ (by assumption)

$\leq \max_{z \in \{|z|=1\}} \frac{A}{|z|^{k+1-\epsilon}}$ which is not [...] any $k < 1-\epsilon$, so $k > 1-\epsilon$, so [...] a_k for $k < 0$ that are nonzero, $a_k > 0$ for all $k \geq 0$, no i

Therefore $z=0$ is a removable singularity, as desired. □

You fixed $|z|=1$ so k is irrelevant in the equation.

You have started to compute the coefficients of the power series of f, but you did not seem to think of varying the radius of the integration contour. 2

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Q5 5

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5. (8 points) Let $k \in \mathbb{Z}$ be an integer. Calculate,

$$\int_0^{2\pi} \frac{e^{ik\theta}}{2 - e^{i\theta}} d\theta.$$

Hint: substitute $z = e^{i\theta}$.

We have $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow d\theta$ = ...

$\int_0^{2\pi} \to \int_{|z|=1}$, here $\int_{|z|=1} \frac{z^k}{2-z} \cdot \frac{1}{iz} dz$. The ... becomes $iz(2-z)$, so $\frac{z^k}{iz(2-z)}$ has a removable singularity at $z_0 = 0$. Note that the function $\frac{z^k}{i(2-z)}$ is holomorphic on $|z|=1$, so by Cauchy's Integral formula,

$$\int_{|z|=1} \frac{z^k}{iz(2-z)} dz = 2\pi i \left. \frac{z^k}{i(2-z)} \right|_{z=0},$$

... ...lly 0 if $k \geq 1$. It is π if $k = 0$ and is undefined, not existing when $k < 0$.

Correct line integral. 2

Partial mark. 1

Case $k\geq1$ 2

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Extra work for Question ____
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Extra work for Question ____

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