## Homework 2



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My score **98%** (49/50)

Q1

10 / 10

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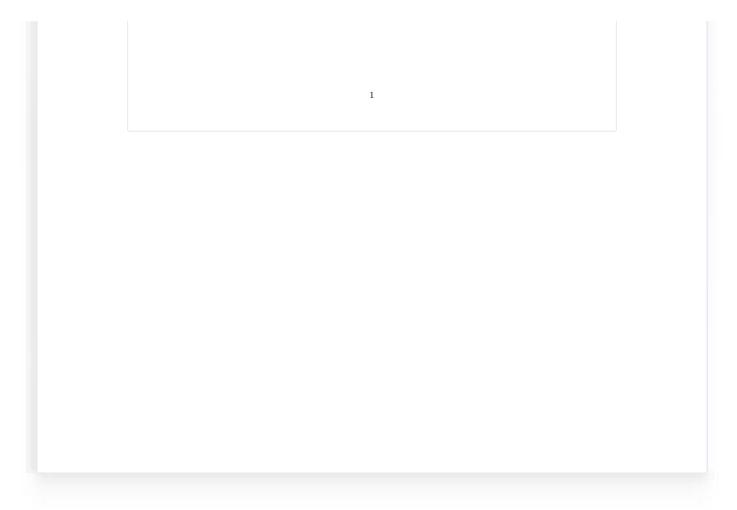
## MAT334 Problem Set 2 — Due Monday, October 24 23:00 ${\scriptstyle 1006940802}$

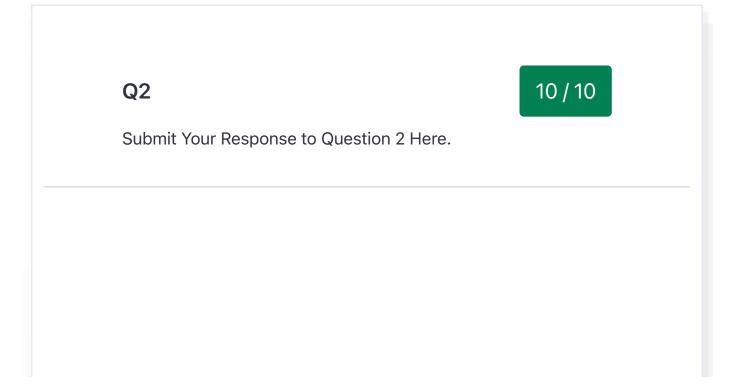
1.

Consider the second-degree continuous differentiable function  $u \in C^2$ , where  $u : \mathbb{R}^2 \to \mathbb{R}$ . Let f(x+iy) = u(x,y) + iu(x,y). To require the function f to be analytic, we require the funtion u to satisfy the Cauchy-Riemann equations:

$$u_x = u_y, \quad u_y = -u_x.$$

The above equations imply that  $u_x=-u_x=u_y$ , so we may proceed by solving the partial differential equations. Firstly, since  $u_x=-u_x$ , then  $2u_x=0$  hence  $u_x=0$ . If  $u_x=0$ , then  $u_y=0$  for all x,y. Thus the only functions u which satisfy the Cauchy-Riemann Equations is any constant function u(x,y)=A for  $A\in\mathbb{C}$ .





2

In this problem, I wish to determine a closed for (or explicit expression of a function) of the power series  $g(z)=\sum_{n=0}^{\infty}n^2z^n$  valid on the disc  $D=\{z\in\mathbb{C}:|z|<1\}$ , the open ball of radius 1 centered at the origin. First, we may begin by factoring a z to see if we have any similarity of other power series:

$$g(z) = \sum_{n=0}^{\infty} n^2 z \cdot z^{n-1} = z \sum_{n=0}^{\infty} n^2 z^{n-1}.$$

Now, notice that  $n^2=n^2-n+n=n(n-1)+n$ , and the first term in that expansion appears as if one had taken a derivative twice. Then

$$\begin{split} g(z) &= z \sum_{n=0}^{\infty} (n(n-1)+n) z^{n-1} \\ &= z \sum_{n=0}^{\infty} n(n-1) z^{n-1} + z \sum_{n=0}^{\infty} n z^{n-1} \\ &= z^2 \sum_{n=0}^{\infty} n(n-1) z^{n-2} + z \sum_{n=0}^{\infty} n z^{n-1}. \end{split}$$

Now, since  $\frac{d^2}{dz^2}z^n=n(n-1)z^{n-2},$  and  $\frac{d}{dz}z^n=nz^{n-1},$  we have that

$$g(z) = z^2 \sum_{n=0}^{\infty} \frac{d^2}{dz^2} [z^n] + z \sum_{n=0}^{\infty} \frac{d}{dz} [z^n].$$

Since the sum of polynomials are analytic on  $\mathbb{C}$ , we may switch the order of differentiation upon summation:

$$g(z)=z^2\frac{d^2}{dz^2}\sum_{n=0}^{\infty}z^n+z\frac{d}{dz}\sum_{n=0}^{\infty}z^n.$$

However, the terms expressed under the summation are exactly the terms defining the geometric series, since |z|<1:  $\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}$ . Thus we are first required to determine the first and second derivatives of the geometric series:

$$\frac{d^2}{dz^2} \left[ \sum_{n=0}^{\infty} z^2 \right] = \frac{d^2}{dz^2} \left[ \frac{1}{1-z} \right]$$
$$= \frac{d}{dz} \left[ \frac{1}{(1-z)^2} \right]$$
$$= \frac{2}{(1-z)^3}.$$

(first derivative)

 $({\it second \ derivative})$ 

Therefore our series expansion for g(z) becomes

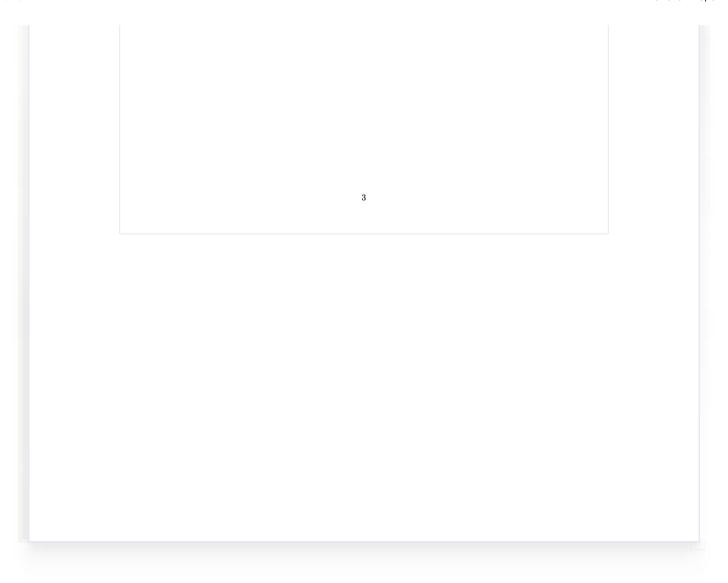
$$g(z) = z^{2} \cdot \frac{2}{(1-z)^{3}} + z \cdot \frac{1}{(1-z)^{2}}$$

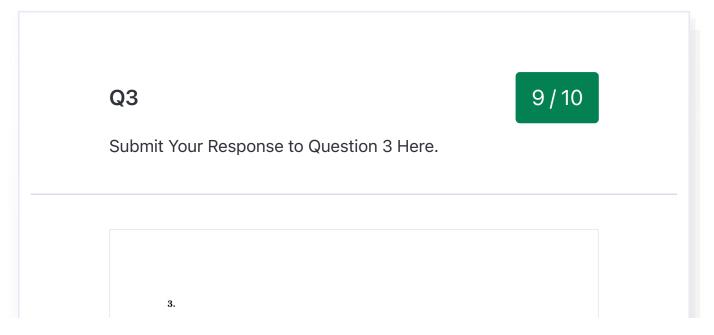
Good! **10** 

9

$$=\frac{2z^2}{(1-z)^3}+\frac{z}{(1-z)^2},$$

which is thus the closed form result of the power series  $g(z)=\sum_{n=0}^{\infty}n^2z^n,$  which is what I wanted to determine.





To determine the power series on the disc of radius 2 centered at 3,  $D=\{z\in\mathbb{C}:|z-3|<2\}$ , one may invoke one of the consequences of Cauchy's Interal Theorem of determining the coefficients of a power series (Fisher, Theorem 2.4.1):

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k \implies a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi,$$

where  $\gamma$  is the circle of radius  $r \in \mathbb{R}^{>0}$  centered at  $z_0$ . We wish to determine a power series of the function  $f(z) = \frac{1}{1-z}$  valid on D, hence to center it around  $z_0 = 3$ . Allow me to begin by define a circle of radius 2:  $\gamma = \{z \in \mathbb{C} : |z-3| = 2\}$ , so  $\gamma = \partial D$ . Thus (by Fisher, Theorem 2.4.1),

$$a_k = \frac{1}{2\pi i} \int_{|z-3|=2} \frac{f(z)}{(z-3)^{k+1}} dz,$$

which is in fact valid of any  $z \notin \partial D$ . This is the integral we wish to evaluate, and we may do so by applying Cauchy's Generalized Integral Formula as presented to us in lecture: under the same hypothesis as Cauchy's Integral Formula, one has

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi.$$

First, we must note that  $f(z)=\frac{1}{1-z}$  is analytic on D, because f(z) and it's derivatives are only discontinuous at z=1, which is not an element in the open disc D but only it's boundary  $\partial D$ . Furthermore,  $3=z_0\in D$  and  $\gamma$  is a simple closed curve. Therefore we have that

$$\begin{split} f^{(k)}(z_0) &= \frac{k!}{2\pi i} \int_{|z-3|=2} \frac{f(z)}{(z-z_0)^{k+1}} \, dz \\ &= f^{(k)}(3) = \frac{k!}{2\pi i} \int_{|z-3|=2} \frac{f(z)}{(z-3)^{k+1}} \, dz = k! \cdot a_k \\ &\implies a_k = \frac{f^{(k)}(3)}{k!}, \end{split}$$

which is valid only on the open disc D. Now, I will proceed by determining the k-th derivative of f(z) evaluated at z=3. We have that

$$\begin{split} f^{(0)}(z) &= \frac{1}{1-z}, \quad f(3) \cdot \frac{1}{0!} = -\frac{1}{2}, \\ f^{(1)}(z) &= \frac{1}{(1-z)^2}, \quad f'(3) \cdot \frac{1}{1!} = \frac{1}{4} \\ f^{(2)}(z) &= \frac{2}{(1-z)^3}, \quad f''(3) \cdot \frac{1}{2!} = -\frac{1}{8} \\ f^{(3)}(z) &= \frac{6}{(1-z)^4}, \quad f'''(3) \cdot \frac{1}{3!} = \frac{1}{16}. \end{split}$$

4

The noticeable pattern is that  $a_k=\frac{f^{(k)}(3)}{k!}=(-1)^k\frac{1}{2^k}$ . Proof by induction: The base case has already been established (above). I now wish to show that if  $\frac{f^{(k)}(3)}{k!}=(-1)^k\frac{1}{2^k}$  is true, then  $\frac{f^{(k+1)}(3)}{(k+1)!}=(-1)^{(k+1)}\frac{1}{2^{(k+1)}}$  is also true. We have that

$$\frac{f^{(k+1)}(3)}{(k+1)!} = (-1)^{(k+1)} \frac{1}{2^{(k+1)}}$$

· · · /

$$= -(-1)^k \frac{1}{2 \cdot 2^k}$$

$$= -\frac{1}{2} \cdot (-1)^k \frac{1}{2^k}$$

$$= -\frac{1}{2} \cdot \frac{f^{(k)}(3)}{k!}$$
 (by induction hypothesis)
$$= (-1)^1 \frac{1}{2^1} \frac{f^{(k)}(3)}{k!},$$

which is what I wanted to show. Therefore our series of the function f(z) valid on the disc D is

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k} (z-3)^k.$$

You are missing a factor of (1/2) in the final answer

-1

5

Q4

10 / 10

Submit Your Response to Question 4 Here.

4.

We wish to determine the integral  $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$  via complex methods. Consider the complex substitution  $z=e^{i\theta}$ . Therefore we obtain that

$$z=e^{i\theta}\quad \implies\quad dz=ie^{i\theta}\,d\theta=iz\,d\theta,\quad \sin(\theta)=\frac{1}{2i}\left(z-\frac{1}{z}\right).$$

Invoking this substitution in the integrand yields  $\int_{|z|=1} \frac{dz}{iz\left(5+\frac{4}{2i}\left(z-\frac{1}{z}\right)\right)}$ , where the integral is now around the complex unit circle. Expanding the denominator gives:

$$iz\left(5+\frac{4}{2i}\left(z-\frac{1}{z}\right)\right)=5iz+2z\left(z-\frac{1}{z}\right)=5iz+2z^2-2=2\left(z^2+\frac{5}{2}iz-1\right).$$

I will proceed by factoring this same polynomial, which will then allow me to apply Cauchy's Integral Formula. By the quadratic formula, the roots of this polynomial are

$$\begin{split} z &= \frac{1}{2} \left[ -\frac{5}{2} i \pm \sqrt{-\frac{25}{4}} - (4)(1)(-1) \right] \\ &= \frac{1}{4} \left[ -5 i \pm \sqrt{-25 + 16} \right] \\ &= \frac{1}{4} \left[ -5 \pm 3 \right] i. \end{split}$$

Then, we obtain that

$$2\left(z^2+\frac{5}{2}iz-1\right)=2\left(z+\frac{5-3}{4}i\right)\left(z+\frac{5+3}{4}i\right)=2\left(z+\frac{i}{2}\right)(z+2i)\,.$$

With our denominator factored, we can proceed by applying Cauchy's Integral Formula to  $\int_{|z|=1} \frac{dz}{2\left(z+i/2\right)\left(z+2i\right)}.$  To begin, Cauchy's formula states that for an analytic complex function

 $f: D \to \mathbb{C}$  on a simply open connected set D and a simple smooth closed piecewise curve  $\gamma$  with  $\operatorname{In}(\gamma) \subset D$ , then for  $z_0 \in \operatorname{In}(\gamma)$ ,

$$2\pi i \, f(z_0) = \int_{\gamma} \frac{f(\xi)}{\xi - z_0} \, d\xi.$$

Allow me to choose the open set  $D=\{z\in\mathbb{C}:|z|<\frac{3}{2}\}$ . It follows that D is open and connected, with  $\frac{i}{2}\in D$ , and  $2i\notin D$ . The unit circle lies inside of D because it's radius only extends to 1 and not beyond 1.5. Since  $2i\notin D$ , it follows that the function  $f:D\to\mathbb{C}$  given by  $f(z)=\frac{1}{2(z+2i)}$  is analytic on D, since it's derivative  $f'(z)=-\frac{1}{2(z+2i)^2}$  is continuous everywhere except  $z=-2i\notin D$ . Therefore by Cauchy's Integral Formula, we obtain

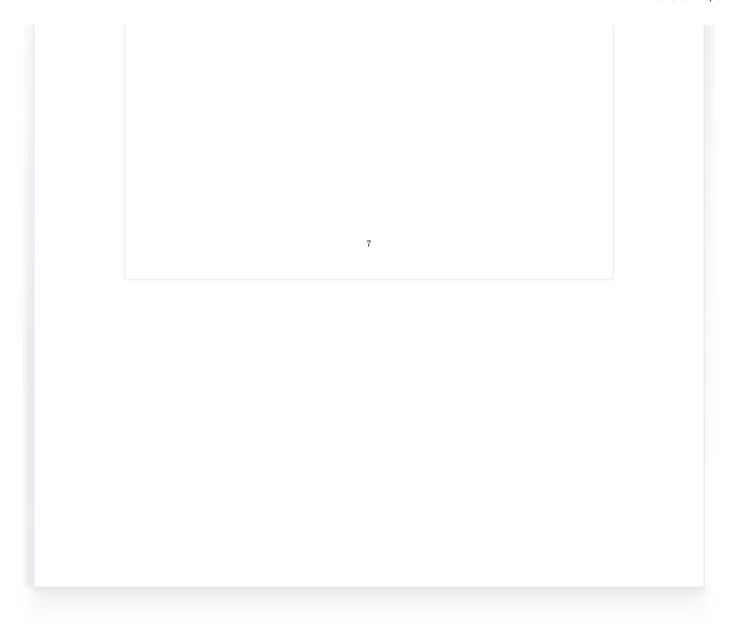
$$2\pi i f(-i/2) = \int_{|z|=1} \frac{f(z)}{z + i/2} dz$$

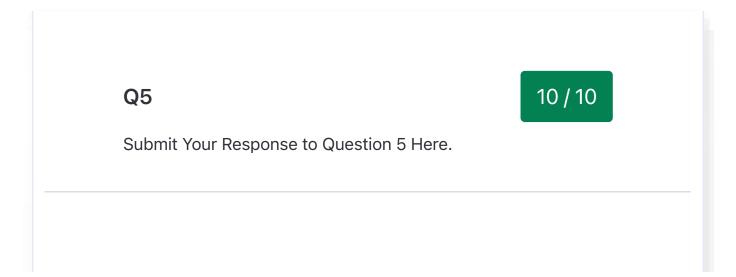
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$$=2\pi i \left(\frac{1}{2\left(-i/2+2i\right)}\right)$$
 
$$=\frac{\pi i}{3/2i}$$
 
$$=\frac{2\pi}{3},$$

which is the value of the original integral, what I initially desired to determine:

Correct.  $10^{\frac{2\pi}{5+4\sin\theta}} = \frac{2\pi}{3}.$ 





5.

We now wish to determine the integral  $\int_{\gamma} \frac{e^z}{z(z-i)(z-3)} \, dz$ , where  $\gamma$  is the parametrization of the circle of radius 2 centered at the origin. To proceed, I will apply Cauchy's Integral Formula, which I restated before in problem (4). Choose my open set to be  $D=\{z\in\mathbb{C}:|z|<5/2\}$  and let  $f:D\to\mathbb{C}$  be the analytic function on D defined by  $f(z)=\frac{e^z}{z-3}$ . f is analytic on D because  $3\notin D$ , hence the derivative is defined and continuous everywhere on D. Furthermore, the curve  $\gamma$ defining the circle of radius 2 centered at the origin is contained in D because it's radius  $2 < \frac{5}{2}$ , and therefore  $In(\gamma) \subset D$ . The integral then becomes

$$\int_{|z|=2} \frac{f(z)}{z(z-i)} \, dz.$$

We may not begin to apply Cauchy's Integral Formula because the denominator exposes two separate discontinuities of the integrand, that at z = 0 and z = i. To resolve this issue, I will proceed by apply partial fraction decomposition to the fraction. Thus we desire to determine coefficients Aand B such that

$$\frac{A}{z} + \frac{B}{z - i} = \frac{1}{z(z - i)}.$$

We have that

$$\begin{aligned} A(z-i) + B(z) &= 1 \\ \Longrightarrow (A+B)z - Ai &= 1 \\ \Longrightarrow A+B &= 0 \text{ and } A = -\frac{1}{i}. \end{aligned}$$

This implies that A=i, so B=-i, hence our integral is then decomposed into fractions:

$$\int_{|z|=2} f(z) \left( \frac{i}{z} - \frac{i}{z-i} \right).$$

 $\int_{|z|=2} f(z) \left(\frac{i}{z} - \frac{i}{z-i}\right).$  Now, I can apply Cauchy's integral formula to each of the two individual integrals by linearity:

$$\begin{split} \int_{|z|=2} f(z) \left(\frac{i}{z} - \frac{i}{z-i}\right) &= i \int_{|z|=2} \frac{f(z)}{z} - i \int_{|z|=2} \frac{f(z)}{z-i} \\ &= 2\pi f(0) - 2\pi f(i) \\ &= 2\pi \left(\frac{e^0}{0-3} - \frac{e^i}{i-3}\right) \\ &= 2\pi \left(-\frac{1}{3} - \frac{e^i}{i-3}\right) \\ &= 2\pi \left(\frac{e^i}{3-i} - \frac{1}{3}\right). \end{split}$$

By Cauchy's Integral Formula, this is the value of the integral we were trying to determine:

$$\int_{\gamma} \frac{e^z}{z(z-i)(z-3)} \, dz = 2\pi \left(\frac{e^i}{3-i} - \frac{1}{3}\right).$$