PHY224 - Thermal Motion

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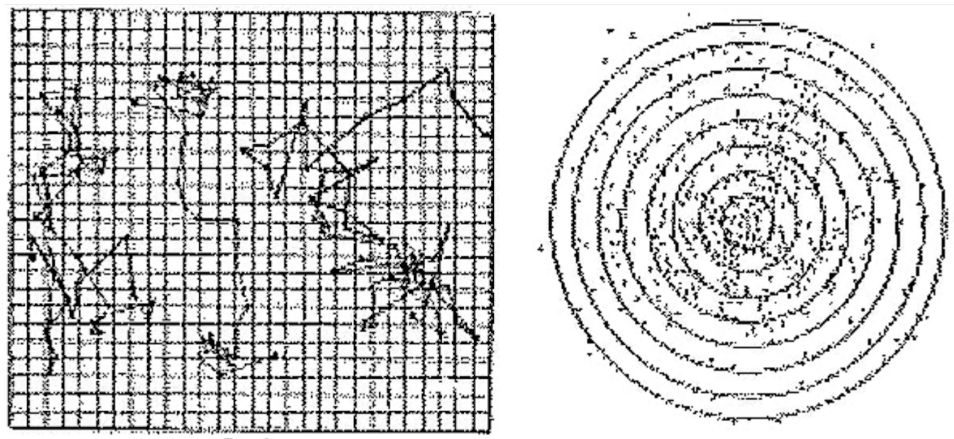


Figure 1: Particles Undergoing Brownian Motion

1 Introduction

The motion of an object due to thermal energy is known as Brownian Motion. First observed on pollen particles by Robert Brown in 1827, Brown later observed the same phenomenon on inorganic particles, leading to the theory of Brownian motion.

1.1 Brief History

Robert Brown made no scientific explanation for his observations, but Albert Einstein managed to explain his findings. Although the existence of atoms wasn't generally accepted in 1905, Einstein published a more detailed paper describing Brown's findings in 1908 while French physicist Jean Baptiste Perrin experimentally confirmed Einstein's theory. This won Einstein the Nobel Prize in Physics in 1926.

1.2 Theory

Applying Newton's law of motion,

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} + X,$$

where $\gamma = 6\pi\eta r$ is the Stokes drag, and η is the viscosity of water, r is the bead radius, and X is the thermal force exhibited on the particle.

Solving the equation yields the Mean-Squared Distance equation for the 1-dimensional path of a particle:

$$< x^2 >= \frac{2kT}{\gamma}t = 2Dt,$$

where $D = \frac{kT}{\gamma}$ is Einstein's Diffusion relation, T is the temperature in Kelvin, and $k = 1.38 \times 10^{-23}\ J/K$ is Boltzmann's constant, and t is the time in seconds.
For an $n-$dimensional case, the Mean-Squared Distance is given by $< r^2 >= n * 2Dt$. Since we are working in two-dimensions, we assume the mean squared distance estimate is given by $< r^2 >= 4Dt = \frac{1}{N}\sum_{i=1}^{N} r_i^2$, thus yielding

$$(2Dt)_{est} = \frac{1}{2N}\sum_{i=1}^{N} r_i^2.$$

The probability density function for the step size of a particle under the influence of thermal motion is given by a $2 - D$ Gaussian,

$$p(x, y, t) = \frac{1}{4\pi Dt} e^{-\frac{x^2+y^2}{4Dt}},$$

yielding the probability density function once integrating

$$P(r, t) = \frac{r}{2Dt} e^{-\frac{r^2}{4Dt}}.$$

2 Materials and Methods

2.1 Materials

- Microscope
- Microscope Cannon Image Capturing Device
- Recording Software, Position Tracker
- Fluorescent Beads Solution (of $1.9 \pm 0.1\mu m$ each) in water at 20^oC
- Vaseline
- Microscope Slides and Seals
- Pipette

2.2 Methods

1. A microscope slide was obtained and prepared by creating a very thin ring of vaseline grease so that water could pool inside of it.
2. Using the pipette, $50 \pm 10L$ of the fluorescent bead solution was placed onto the centre of the slide, creating a vaseline-lined bead pool.

3. The slide was sealed with a microscope slide seal by placing the seal onto the vaseline ring and not pressing down. This was to allow the beads to move under the seal.

4. The microscope was turned on and the illuminating intensity was turned up. The phase-ring was adjusted to 1 (Ph1) and a piece of white paper was placed underneath the microscope and the focus of the lens was adjusted.

5. The microscope camera acquisition device was turned on and was set to 'fluorescent illumination' (the X-Cite box). The GFP fluorescence cube was then put in place with the dial on the upper right hand side of the microscope. The camera shutter was then opened.

6. The prepared slide was set underneath the objective lens and the objective lens was adjusted using the fine up/down adjustment knob on the microscope and the phase ring 2 (Ph2) microscopy such that the image was focused.

7. Using the x and y position adjustment knobs on the microscope, a moving bead was located. The fine focus adjustment knob (up/down) was then used to focus the bead in the camera.

8. Using microscope camera acquisition software, the brightness and opacity of the image capture was adjusted so that only the bead could be observed with no background information.

9. Again using the microscope camera recording acquisition, 200 frames were recorded in photo-format at 0.1s / frame. The photos of the bead location were saved in a folder.

10. Accessing the folder for the captured bead, the location of the bead was tracked over the 200 frames using position tracking software. The x and y positions of the bead for each frame was tracked in pixels and exported into a **.txt** file.

11. Steps 7-10 were carefully repeated 9 more times for a total of 10 beads, hence 10 data sets.

12. Each data set was imported into Python to be analysed. This was done by converting the pixel coordinates into microns then using Pythagoras' Theorem to measure the distance the bead travelled during 1 frame:

$$\Delta d = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

for each $i = 0, \ldots, 199$. The distances for each bead were plotted into a Histogram using **numpy.hist** and **matplotlib.pyplot.hist**. The uncertainty in the pixel conversion is given by 1 pixel $= 0.12048 \pm 0.003\mu m$ while the uncertainty in the distance is given by

$$u_z = \frac{(u_{x_i} + u_{x_{i+1}})|x_{i+1} - x_i| + (u_{y_i} + u_{y_{i+1}})|y_{i+1} - y_i|}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}.$$

Each histogram was fitted using **scipy.optimize**'s **curve-fit** function to the model function

$$P(r,t) = \frac{r}{2Dt} e^{-\frac{r^2}{4Dt}}.$$

The value of D was calculated using the model function parameters, hence yielding k by applying Einstein's Diffusion Relation $k = \frac{D\gamma}{T}$. The uncertainty in approximating was calculated using **pcov**.

The data was further analysed by calculating the Mean-Squared Distance estimate for D using the formula

$$(2Dt)_{est} = \frac{1}{2N}\sum_{i=1}^{N} r_i^2.$$

The uncertainty value of D_{est} was found by the formula

$$z = cx^2 \implies u_z = 2\left(\frac{u_x}{x}\right) z.$$

Again, k could be estimated from this value of D_{est} by applying Einstein's Diffusion Relation. This uncertainty value of k was also calculated using the uncertainty propogation from Einstein's Relation:

$$u_k = 6\pi \cdot \left[\left(\frac{u_\eta}{\eta}\right)^2 + \left(\frac{u_r}{r}\right)^2 + \left(\frac{u_D}{D}\right)^2 + \left(\frac{u_T}{T}\right)^2\right]^{1/2} \cdot k.$$

The error and percentage error for each trial was calculated and compared. This was repeated for all 10 data sets.

3 Results

4 data sets were lost due to poor tracking. Because of this, the 6 best trials were analyzed.

The following table is the collective data of the values of the diffusion coefficient estimate, D, taken from the analysis of each data set:

Table 1: Values of Diffusion Coefficient for a Bead Undergoing Brownian Motion:

Data Set	Maximum Likelihood Estimate (MSD) for D (m^2/s)	Uncertainty in D_{est} (m^2/s)	Curve-Fit Estimate for D (m^2/s)	Uncertainty in Curve-Fit Value of D (m^2/s)
4	2.563e-13	1.142e-15	1.748e-13	7.622e-40
5	2.123e-13	1.623e-15	1.500e-13	4.349e-40
6	3.841e-13	2.215e-15	1.638e-13	1.031e-39
7	2.996e-13	1.514e-15	1.388e-13	5.995e-40
8	2.172e-13	9.935e-16	1.420e-13	4.516e-40
9	4.151e-13	1.673e-15	2.057e-13	1.800e-39
10	3.229e-13	1.660e-15	1.659e-13	9.071e-40

The following table is the collective data of the values of the calculated Boltzmann Constant, k, taken from the analysis of each data set via Mean-Squared Distance and compared to the actual value:

Table 2: Values of Calculated Boltzmann Constant for a Bead Undergoing Brownian Motion via the Method of Mean-Squared-Distance:

Data Set	Value of k_{est} from D_{est} (J/K)	Uncertainty in k_{est} (J/K)	Expected Value of k (J/K)	Overlap	Percentage Error (%)
4	1.55e-23	2.12e-23	1.38e-23	Yes	12.17
5	1.28e-23	1.76e-23	1.38e-23	Yes	7.10
6	2.32e-23	3.19e-23	1.38e-23	Yes	68.09
7	1.81e-23	2.48e-23	1.38e-23	Yes	39.24
8	1.31e-23	1.80e-23	1.38e-23	Yes	4.93
9	2.51e-23	3.44e-23	1.38e-23	Yes	81.65
10	1.95e-23	2.68e-23	1.38e-23	Yes	41.33

The following table is the collective data of the values of the calculated Boltzmann Constant, k, taken from the analysis of each data set via Curve-Fitting to a Histogram Distribution:
Table 3: Values of Calculated Boltzmann Constant for a Bead Undergoing Brownian Motion via the Method of Curve-Fitting to a Histogram:

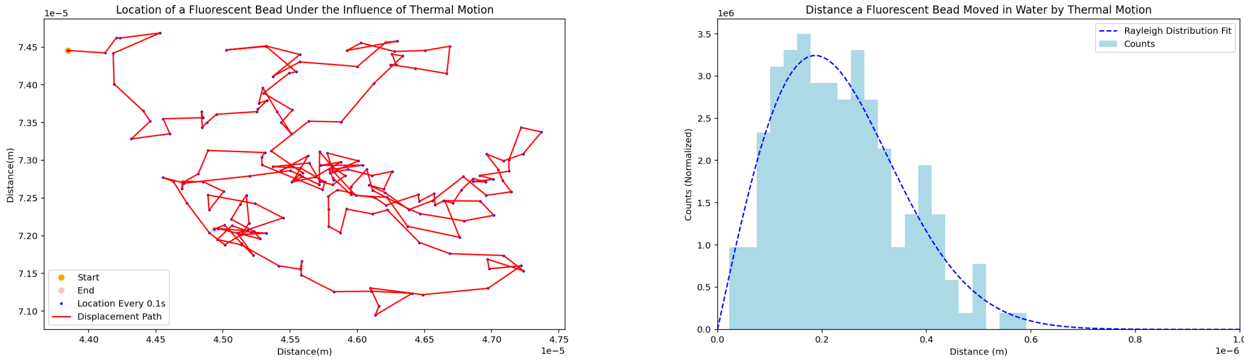
Data Set	Value of k_{popt} from **popt** (J/K)	Uncertainty in k_{popt} (J/K)	Expected Value of k (J/K)	Overlap	Percentage Error (%)
4	1.06e-23	1.44e-23	1.38e-23	Yes	23.52
5	9.1e-24	1.24e-23	1.38e-23	Yes	34.34
6	9.9e-24	1.35e-23	1.38e-23	Yes	28.31
7	8.4e-24	1.15e-23	1.38e-23	Yes	39.24
8	8.6e-24	1.17e-23	1.38e-23	Yes	37.84
9	1.24e-24	1.70e-23	1.38e-23	Yes	10.00
10	1.00e-23	1.37e-23	1.38e-23	Yes	27.38

3.1 Observations

- Throughout experimentation, it was noted that smaller beads were subject to more noticeable movement and larger beads were subject to less noticeable movement.
- Bead motion is completely random.
- Pressing down on a slide created less movement within the slide.

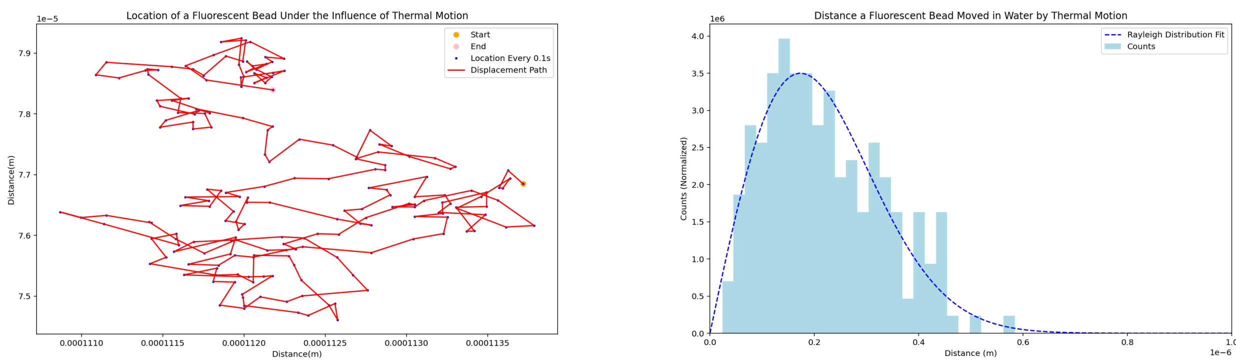
3.2 Plot Data

The following plots are the tracked positions of the 4th bead and the histogram of the distances the bead moved every frame, respectively:
Figure 2: Dataset 4



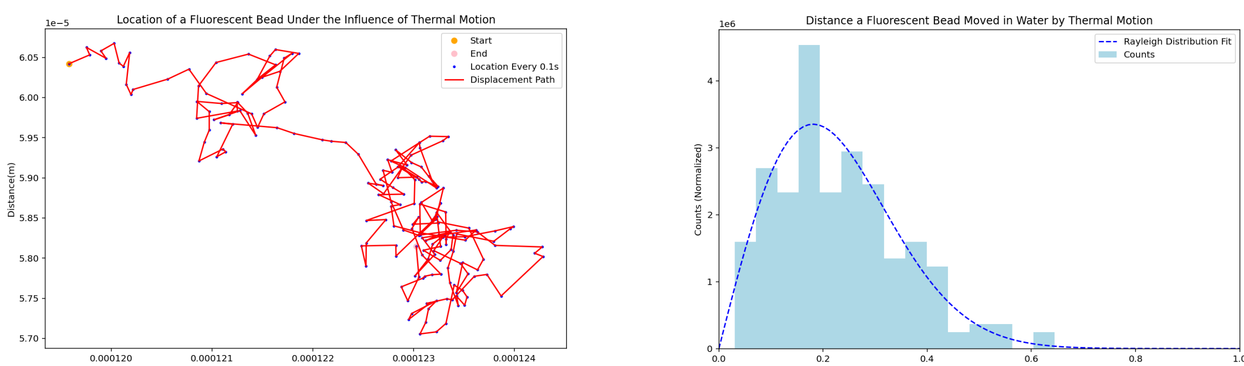
The following plots are the tracked positions of the 5th bead and the histogram of the distances the bead moved every frame, respectively:

Figure 3: Dataset 5



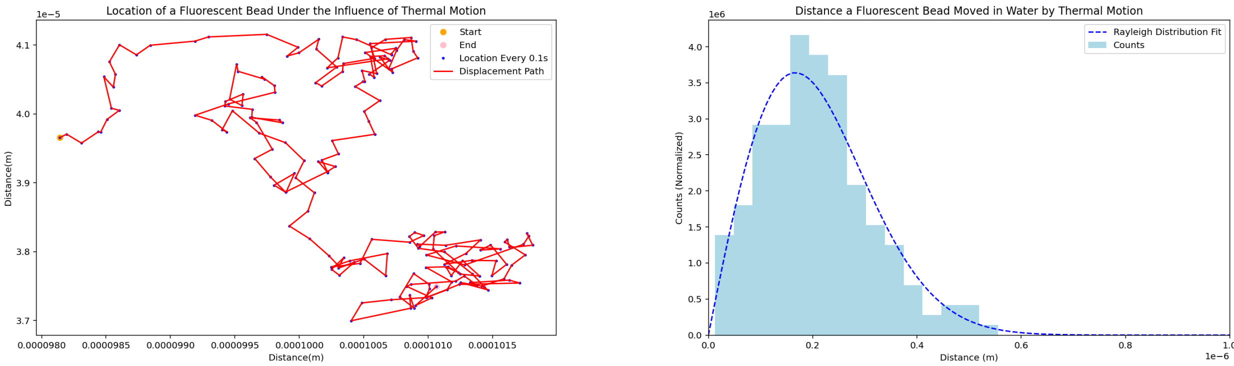
The following plots are the tracked positions of the 6th bead and the histogram of the distances the bead moved every frame, respectively:

Figure 4: Dataset 6



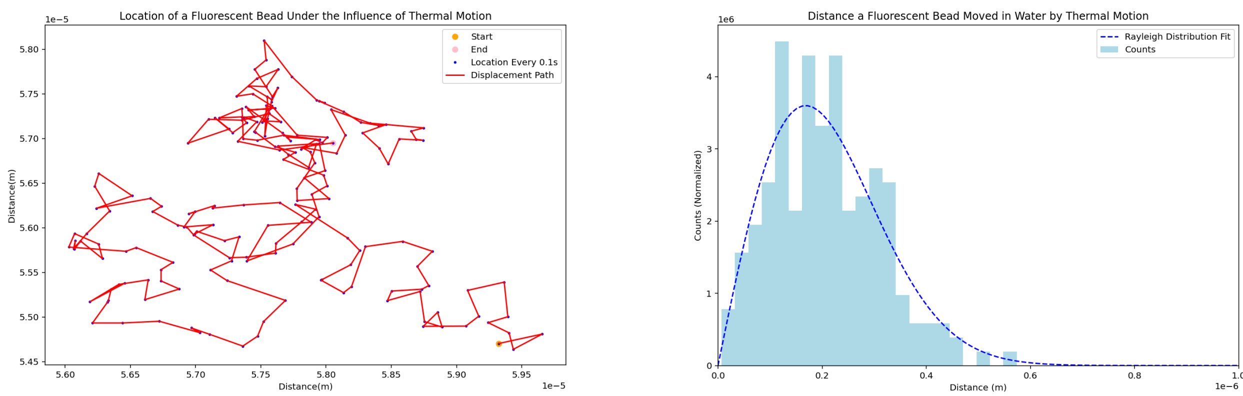
The following plots are the tracked positions of the 7th bead and the histogram of the distances the bead moved every frame, respectively:

Figure 5: Dataset 7



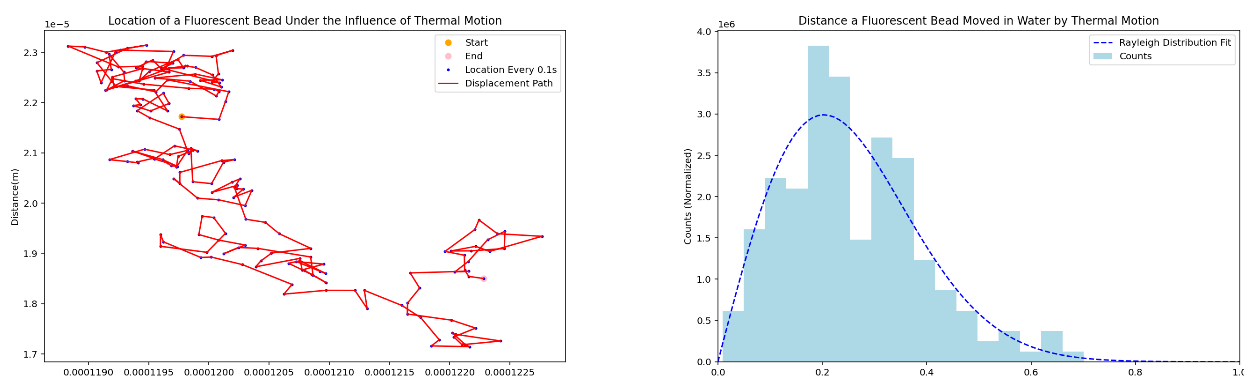
The following plots are the tracked positions of the 8th bead and the histogram of the distances the bead moved every frame, respectively:

Figure 6: Dataset 8



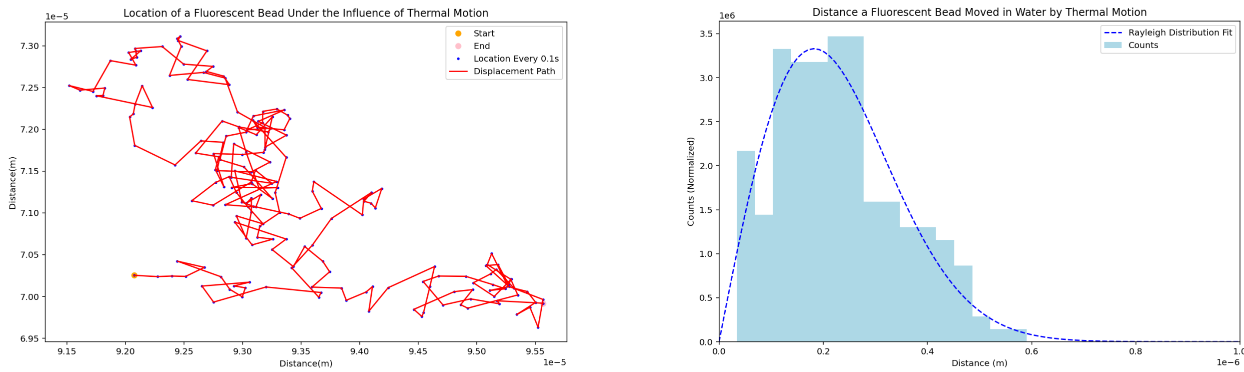
The following plots are the tracked positions of the 9th bead and the histogram of the distances the bead moved every frame, respectively:

Figure 7: Dataset 9



The following plots are the tracked positions of the 10th bead and the histogram of the distances the bead moved every frame, respectively:

Figure 8: Dataset 10



4 Comparison of Results, Analysis, Discussion

Comparison to Rayleigh Distribution:

A Rayleigh Distribution function is given by

$$f(x,\sigma) = \frac{x}{\sigma^2} e^{(x^2/2\sigma^2)},$$

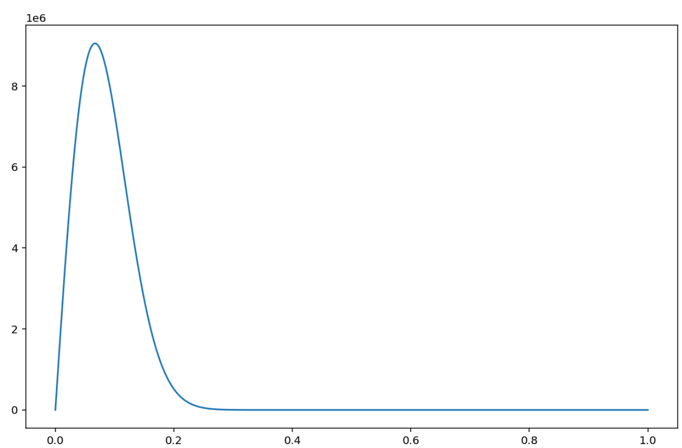
while the function we are curve-fitting to is give by

$$P(r,t) = \frac{r}{2Dt} e^{-\frac{r^2}{4Dt}},$$

and thus we can take $\sigma = \sqrt{2Dt}$.

Taking $D = 2.28 \times 10^{-14}$ (the expected value) and $t = 0.1$ (the frame per second time used), we find $\sigma \approx 6.7 \times 10^{-8}\, m$. Comparing this value to the plot of a Rayleigh Distribution with this value of σ,

Figure 9: Sample Rayleigh Distribution with $\sigma = 6.7 \times 10^{-8} m$



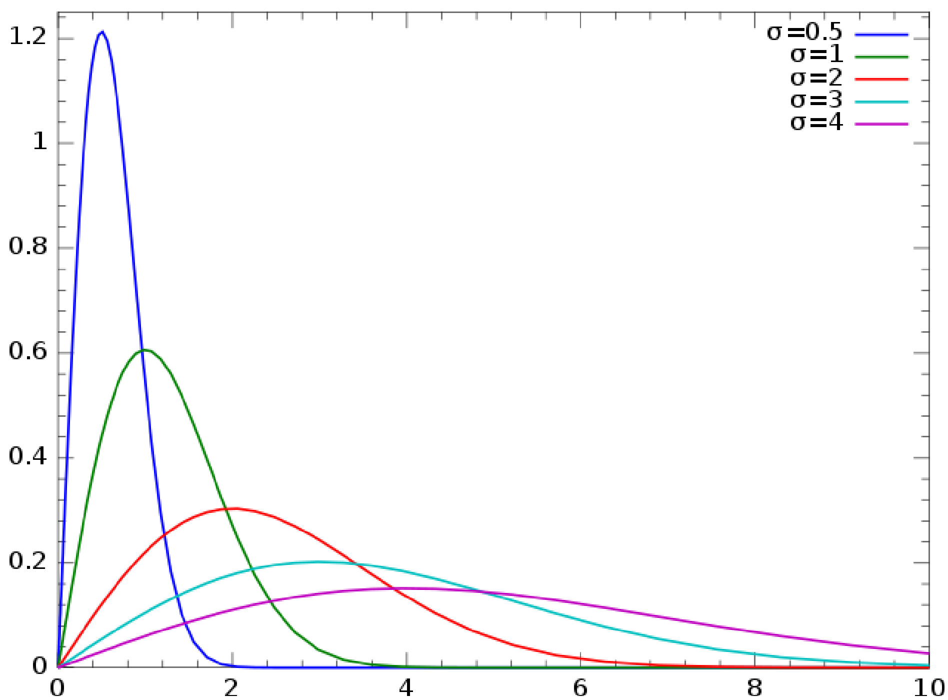
we find that the distribution plots for each of the 6 beads coincide with the sample distribution above.

Of course, different Rayleigh distributions will correspond to different Mean-Squared Distance dimensions, given by

$$< r^2 >= n * 2Dt,$$

where n is the dimension. This is because the value of σ changes:

Figure 10: Rayleigh Distribution Plot with Various Values of σ



Review of Data in Comparison to the Expected Value of k:
It was observed that as though the Mean-Squared Distance calculations given by $(2Dt)_{est} = \frac{1}{2N}\sum_{i=1}^{N} r_i^2$ were more accurate in some trials than the values given by the Rayleigh Distribution Fit, the values of k given from the distribution curve-fit were more consistent. This is expected to be because of how Python curve-fits the Rayleigh distribution to the histogram and because of the uncertainty in the Histogram.

Each calculated value of k given by the Mean Squared Distance analysis was within range of the expected value of the Boltzmann Constant, with the percentage error values given in the rightmost column of tables 2 and 3.

It is believed that the reason the uncertainty propagation values for both k_{est} and k_{popt} are so large is because of the error sum of the squared distance terms given by the MSD calculation and the small values of the curve-fit uncertainty **pcov** in the rightmost column of table 1, respectively.

Discussion of Experimental Observations

As noted before, it was observed that larger beads in the microscope capture view were subject to less noticeable movement in the fluid. This is believed to be so because of Newton's Law of Motion for a particle undergoing a thermal force in a fluid, given by the differential equation

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} + X$$

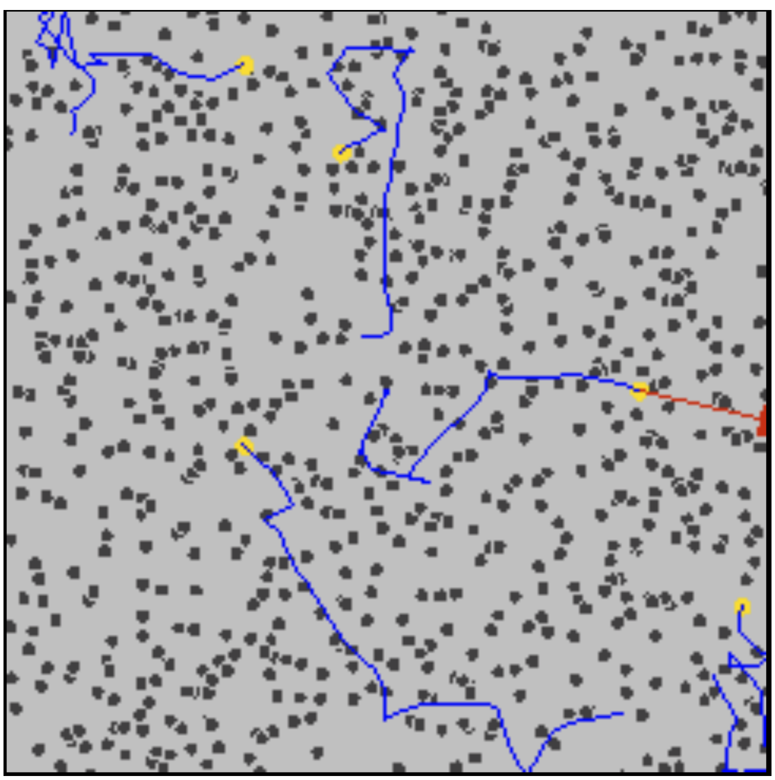
(see **1.1**).

The thermal force exerted on the particle X, with the damping term, is proportional to the mass of the object with it's acceleration. As m increases, the thermal force required to move the bead becomes greater, which coincides with the conservation of momentum principle.

The thermal force X exerted on the particle is a result of molecules undergoing brief collisions with the fluorescent bead, causing the bead to undergo motion.

On Thermal Energy: As given by Einstein's Relation $D = \dfrac{kT}{\gamma}$, it was concluded that different medium temperatures affect the particle motion. Logically, if more thermal energy (heat) is put into the fluorescent bead - water system, the resulting motion of the bead may be more noticeable depending on it's size. This is because of the vast larger number of collisions the bead has with surrounding particles since the rapid particle motion is fueled by the conversion of thermal energy into kinetic energy:

Figure 11: Brownian Motion Simulation Freezeframe



Observations Made by Analyzing Data Sets:

The most probabilistic distances a fluorescent bead subject to thermal motion moves is given by the Rayleigh Distributions. For each data set, the average distance the bead moved is $0.2 \times 10^{-6}\, m$, which can be compared to other particles of similar mass. Given this, information may be extracted about the differential equation of motion given in **1.1**.

Furthermore, temperature, viscosity, and bead size are proportional to the movement of the bead: as temperature increases, thermal energy increases, which implies that the thermal force exerted on the bead is greater. Similarly, as viscosity increases, the bead is subject to a greater drag force, implying the amount of movement on the bead will decrease. Lastly, as before, as bead size increases, a greater force is required to displace it a certain distance. This is a direct result of Newton's Third Law.

5 Conclusion

The information gathered in this experiment aligns with the theory of Brownian motion as well as Einstein's conclusions. As predicted, the motion of the beads was random and the distribution of the distances was Rayleigh. Furthermore, the motion of the fluorescent beads coincided with Newton's Third Law of Motion. The experimental values of the Boltzmann Constant k overlapped with the expected value of $k = 1.38 \times 10^{-23} J/K$, hence supporting the theory of thermal motion.

6 References

- `https://en.wikipedia.org/wiki/Brownian_motion`
- `https://en.wikipedia.org/wiki/Rayleigh_distribution`
- `https://www.feynmanlectures.caltech.edu/I_41.html`
- `https://www.isobudgets.com/propagation-of-uncertainty-rules/`
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- Lab Manual: `https://www.physics.utoronto.ca/~phy224_324/experiments/thermal-motion/Thermal%20Motion.pdf`