PHY354 PS1 — 01/28/2022

1006940802

---

1. At $t = 0$ a ball is thrown upward from $y = 0$. At a time t_0 later it returns to the same position. Guess a solution for the position $y(t)$ of the ball of the form $y(t) = a_2t^2 + a_1t + a_0$, and by directly minimizing the action between $t = 0$ and $t = t_0$ find $y(t)$. Show that this is equivalent to the result you would obtain from Newton's Second Law. (NOTE: In this question you are to directly minimize the action, not solve the Euler-Lagrange equations.)

To start, we find the kinetic and potential energies of the system:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}(t)^2, \qquad\qquad V = mgy(t).$$

Our Lagrangian is given by $L = T - V = \frac{1}{2}m\dot{y}(t)^2 - mgy(t)$ with the action

$$S = \int_0^{t_0} L[y(t), \dot{y}(t), t]\, dt = \int_0^{t_0} \frac{1}{2}m\dot{y}(t)^2 - mgy(t)\, dt.$$

Guessing $y(t) = a_2t^2 + a_1t + a_0$,

$$\begin{aligned} S &= \int_0^{t_0} \frac{1}{2}m[2a_2t + a_1]^2 - mg[a_2t^2 + a_1t + a_0]\, dt \\ &= \int_0^{t_0} \frac{1}{2}m[4a_2^2t^2 + 4a_1a_2t + a_1^2] - mg[a_2t^2 + a_1t + a_0]\, dt \\ &= \int_0^{t_0} [2ma_2^2 - mga_2]t^2 + [2ma_1a_2 - mga_1]t + \frac{1}{2}ma_1^2 - mga_0\, dt. \end{aligned}$$

Integrating yields

$$S = \frac{1}{3}[2ma_2^2 - mga_2]t_0^3 + \frac{1}{2}[2ma_1a_2 - mga_1]t_0^2 + \frac{1}{2}[ma_1^2 - 2mga_0]t_0.$$

Our boundary conditions require $y(0) = y(t_0) = 0$, which implies that $a_0 = 0$. Furthermore, this implies a constraint that $at_0 + a_1 = 0$ or that $a_1 = -a_2t_0$. Applying this constraint, the action becomes:

$$\begin{aligned} S &= \frac{1}{3}[2ma_2^2 - mga_2]t_0^3 + \frac{1}{2}[-2ma_2^2t_0 + mg[a_2t_0]t_0^2 + \frac{1}{2}ma_2^2t_0^2]t_0 \\ &= \left[\frac{1}{6}a_2^2 + \frac{1}{6}ga_2\right] mt_0^3. \end{aligned}$$

Now, minimizing S with respect to a_2,

$$\begin{aligned} \frac{\partial S}{\partial a_2} &= \left[\frac{1}{3}a_2 + \frac{1}{6}ga_2\right] mt_0^3 = 0 \\ \implies a_2 &= -\frac{1}{2}g \end{aligned}$$

$$\implies a_1 = \frac{1}{2} g t_0.$$

Therefore $\boxed{y(t) = -\frac{1}{2} g t^2 + \frac{1}{2} g t_0 t.}$

How is this equivalent to Newton's laws? First notice that the only force influencing the motion is gravity. This yields $\ddot{y} = -g$. Solving this ODE gives $y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0$, where v_0 is the initial velocity of the object and y_0 is the initial height.

In this problem, since $y(0) = 0$, then $y_0 = 0$. v_0 can be determined by differentiating $y(t)$ and finding when $y'(t) = 0$, which gives the equation of motion $-g\frac{t_0}{2} + v_0 = 0$ if the object reaches a maximum a $t_0/2$. Therefore $v_0 = \frac{1}{2} g t_0$ and $y(t) = -\frac{1}{2} g t^2 + \frac{1}{2} g t_0 t$, which is equivalent to what was shown by minimizing the action.

2. Consider the most general Lagrangian for a system of n dynamical degrees of freedom q_i, $i = 1 \dots n$, with only kinetic, and no potential, energy:

$$L = \frac{1}{2} \sum_{a,b} g_{ab}(q_i) \dot{q}_a \dot{q}_b \tag{1}$$

where the functions $g_{ab}(q_i) = g_{ba}(q_i)$ are the components of a symmetric $n \times n$ matrix, and depend on all the generalized coordinates.

(a) Show that the Euler-Lagrange equations for this system are

$$\ddot{q}_a + \sum_{b,c} \Gamma^a_{bc} \dot{q}_b \dot{q}_c = 0 \tag{2}$$

where

$$\Gamma^a_{bc} = \sum_d \frac{1}{2} (g^{-1})_{ad} \left(\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d} \right). \tag{3}$$

The matrix g^{-1} is the inverse of g: $\sum_b (g^{-1})_{ab} g_{bc} = \delta_{ac}$, where δ_{ac} is the Kronecker delta (or equivalently, the unit matrix)

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

We begin with the Euler-Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] = \frac{\partial L}{\partial q_i}, \qquad i = 1, \dots n$$

for $L = \frac{1}{2} \sum_{a,b} g_{ab}(q_i) \dot{q}_a \dot{q}_b$:

$$\frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i} \sum_{a,b} g_{ab}(q_i) \dot{q}_a \dot{q}_b\right] = \frac{1}{2}\frac{\partial}{\partial q_i} \sum_{a,b} g_{ab}(q_i) \dot{q}_a \dot{q}_b.$$

Now the $\frac{1}{2}$'s cancel and the right hand side simplifies to $\sum_{a,b} \frac{\partial g_{ab}}{\partial q_i} \dot{q}_a \dot{q}_b$, since $\dot{q}_a$ and $\dot{q}_b$ do not depend on each q_i.

The left hand side, by the produce rule, yields

$$\frac{d}{dt}\left[\sum_{a,b} g_{ab} \left(\dot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \dot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right)\right].$$

Taking the time derivative,

$$\frac{d}{dt} \sum_{a,b} \left[g_{ab} \left(\dot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \dot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right)\right] = \sum_{a,b} \left[\frac{d g_{ab}}{dt} \left(\dot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \dot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right) + g_{ab} \left(\ddot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \ddot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right)\right]$$

$$= \sum_{a,b} \frac{dg_{ab}}{dt} \left(\dot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \dot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right) + \sum_{a,b} g_{ab} \left(\ddot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \ddot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right).$$

The second term yields delta functions which can then be re-indexed with the sum:

$$\begin{aligned}
\sum_{a,b} g_{ab} \left(\ddot{q}_b \frac{\partial \dot{q}_a}{\partial \dot{q}_i} + \ddot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_i} \right) &= \sum_{a,b} g_{ab} \left(\ddot{q}_b \delta_{ai} + \ddot{q}_a \delta_{bi} \right) \\
&= \sum_b g_{ib} \ddot{q}_b + \sum_a g_{ai} \ddot{q}_a \\
&= 2 \sum_a g_{ia} \ddot{q}_a.
\end{aligned}$$

Our Lagrangian is then

$$\begin{aligned}
& 2 \sum_a g_{ia} \ddot{q}_a + \sum_{b,c} \frac{dg_{bc}}{dt} \left(\dot{q}_c \frac{\partial \dot{q}_b}{\partial \dot{q}_i} + \dot{q}_b \frac{\partial \dot{q}_c}{\partial \dot{q}_i} \right) = \sum_{b,c} \frac{\partial g_{bc}}{\partial q_i} \dot{q}_b \dot{q}_c \\
\Longrightarrow\ & 2 \sum_a g_{ia} \ddot{q}_a + \sum_{b,c} \left[\frac{dg_{bc}}{dt} \left(\dot{q}_c \frac{\partial \dot{q}_b}{\partial \dot{q}_i} + \dot{q}_b \frac{\partial \dot{q}_c}{\partial \dot{q}_i} \right) - \frac{\partial g_{bc}}{\partial q_i} \dot{q}_b \dot{q}_c \right] = 0,
\end{aligned}$$

which appears by re-indexing the sum. We now examine the partial derivatives in the second term in the equation above.

The time derivative $\frac{dg_{bc}}{dt}$, by the chain rule, becomes $\sum_d \frac{\partial g_{bc}}{\partial q_d} \frac{dq_d}{dt} = \sum_d \frac{\partial g_{bc}}{\partial q_d} \dot{q}_d$. Furthermore, as before, the partial derivatives $\frac{\partial \dot{q}_b}{\partial \dot{q}_i}$ and $\frac{\partial \dot{q}_c}{\partial \dot{q}_i}$ give delta functions δ_{ib} and δ_{ic}, respectively. The second term is then

$$\sum_{b,c,d} \left[\frac{\partial g_{bc}}{\partial q_d} \dot{q}_d \left(\dot{q}_c \delta_{ib} + \dot{q}_b \delta_{ic} \right) - \frac{\partial g_{bc}}{\partial q_i} \dot{q}_b \dot{q}_c \right] = \sum_{b,c,d} \frac{\partial g_{bc}}{\partial q_d} \dot{q}_d \dot{q}_c \delta_{ib} + \sum_{b,c,d} \frac{\partial g_{bc}}{\partial q_d} \dot{q}_d \dot{q}_b \delta_{ic} - \sum_{b,c,d} \frac{\partial g_{bc}}{\partial q_i} \dot{q}_b \dot{q}_c$$

and re-indexing yields

$$\sum_{b,c} \frac{\partial g_{ib}}{\partial q_c} \dot{q}_c \dot{q}_b + \sum_{b,c} \frac{\partial g_{ci}}{\partial q_b} \dot{q}_b \dot{q}_c - \sum_{b,c} \frac{\partial g_{bc}}{\partial q_i} \dot{q}_b \dot{q}_c.$$

Factoring the like terms, we have

$$\sum_{b,c} \dot{q}_b \dot{q}_c \left(\frac{\partial g_{ib}}{\partial q_c} + \frac{\partial g_{ci}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_i} \right).$$

Our Lagrangian is then

$$2 \sum_a g_{ia} \ddot{q}_a + \sum_{b,c} \left(\frac{\partial g_{ib}}{\partial q_c} + \frac{\partial g_{ci}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_i} \right) \dot{q}_b \dot{q}_c = 0.$$

By multiplying the equation by $\frac{1}{2} g_{ia}^{-1}$, we have

$$\sum_a g_{ia}^{-1} g_{ai} \ddot{q}_a + \frac{1}{2} \sum_{b,c} g_{ia}^{-1} \left(\frac{\partial g_{ib}}{\partial q_c} + \frac{\partial g_{ci}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_i} \right) \dot{q}_b \dot{q}_c = 0.$$

Now, we have that $\sum_a g^{-1}_{ia} g_{ai} \ddot{q}_a = \delta_{ii} \ddot{q}_a = \ddot{q}_a$. Since $i = 1, \ldots, n$ was fixed, we have obtained a solution for a single $i - th$ coordinate. To account for all solutions, our Lagrangian must sum over all i. Let each i be replaced by the summation index d. This yields

$$\ddot{q}_a + \frac{1}{2} \sum_{b,c} \sum_{d} g^{-1}_{ad} \left(\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d} \right) \dot{q}_b \dot{q}_c = 0.$$

Therefore

$$\ddot{q}_a + \frac{1}{2} \sum_{b,c} \left(\sum_{d} g^{-1}_{ad} \left[\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d} \right] \right) \dot{q}_b \dot{q}_c = 0$$

$$\boxed{\ddot{q}_a + \frac{1}{2} \sum_{b,c} \Gamma^a_{bc} \dot{q}_b \dot{q}_c = 0,}$$

as required.

(b) For polar coordinates, $q_1 = r$ and $q_2 = \theta$, show that

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{4}$$

and the Γ^a's are

$$\begin{aligned} \Gamma^1 &= \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix}, \\ \Gamma^2 &= \begin{pmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{pmatrix}. \end{aligned} \tag{5}$$

Use these to find the equations of motion of a free particle in two dimensions in polar coordinates from Equation (2), and show that these are identical to those obtained in class (the ball on a spring, in the special case $g = k = 0$).

The Lagrangian for a single particle in polar coordinates, with no potential, is given by $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. The generalized Lagrangian in question (2a) was given by

$$L = \frac{1}{2}m\sum_{a,b} g_{ab}\dot{q}_a\dot{q}_b = \frac{1}{2}m\begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

in matrix notation. The equating and expansion of these Lagrangian's give

$$\dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 g_{11} + 2\dot{r}\dot{\theta}g_{12} + \dot{\theta}^2 g_{22}.$$

This implies that $g_{12} = g_{21} = 0$, $g_{11} = 1$ and $g_{22} = r^2$. Therefore $\boxed{g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}$. Furthermore, the inverse of g can be easily calculated as $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$.

The following $\Gamma^{a's}$ can be calculated by the explicit formula from (2a),

$$\begin{aligned} \Gamma^1_{bc} &= \frac{1}{2}g^{-1}_{11}\left(\frac{\partial g_{b1}}{\partial q_c} + \frac{\partial g_{c1}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_1}\right) + \frac{1}{2}g^{-1}_{12}\left(\frac{\partial g_{b2}}{\partial q_c} + \frac{\partial g_{c2}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_2}\right) \\ \Gamma^2_{bc} &= \frac{1}{2}g^{-1}_{21}\left(\frac{\partial g_{b1}}{\partial q_c} + \frac{\partial g_{c1}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_1}\right) + \frac{1}{2}g^{-1}_{22}\left(\frac{\partial g_{b2}}{\partial q_c} + \frac{\partial g_{c2}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_2}\right). \end{aligned}$$

Since $g^{-1}_{12} = g^{-1}_{21} = 0$, we can simplify our expressions to

$$\begin{aligned} \Gamma^1_{bc} &= \frac{1}{2}g^{-1}_{11}\left(\frac{\partial g_{b1}}{\partial q_c} + \frac{\partial g_{c1}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_1}\right) \\ \Gamma^2_{bc} &= \frac{1}{2}g^{-1}_{22}\left(\frac{\partial g_{b2}}{\partial q_c} + \frac{\partial g_{c2}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_2}\right). \end{aligned}$$

Each entry of each Γ will need to be calculated explicitly. For Γ^1,

$$\Gamma^1_{11} = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial q_1} + \frac{\partial g_{11}}{\partial q_1} - \frac{\partial g_{11}}{\partial q_1}\right) = \frac{1}{2}(0 + 0 - 0) = 0$$

$$
\begin{aligned}
\Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2}\left(\frac{\partial g_{11}}{\partial q_2} + \frac{\partial g_{21}}{\partial q_1} - \frac{\partial g_{12}}{\partial q_1}\right) = \frac{1}{2}(0+0-0) = 0 \\
\Gamma^1_{22} &= \frac{1}{2}g_{11}^{-1}\left(\frac{\partial g_{21}}{\partial q_2} + \frac{\partial g_{21}}{\partial q_2} - \frac{\partial g_{22}}{\partial q_1}\right) = \frac{1}{2}(0+0-2r) = -r.
\end{aligned}
$$

Therefore $\boxed{\Gamma^1 = \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix}}$. For Γ^2,

$$
\begin{aligned}
\Gamma^2_{11} &= \frac{1}{2}g_{22}^{-1}\left(\frac{\partial g_{12}}{\partial q_1} + \frac{\partial g_{12}}{\partial q_1} - \frac{\partial g_{11}}{\partial q_2}\right) = \frac{1}{2}r^{-2}(0+0-0) = 0 \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2}g_{22}^{-1}\left(\frac{\partial g_{12}}{\partial q_2} + \frac{\partial g_{22}}{\partial q_1} - \frac{\partial g_{12}}{\partial q_2}\right) = \frac{1}{2}r^{-2}(0+2r-0) = r^{-1} \\
\Gamma^2_{22} &= \frac{1}{2}g_{22}^{-1}\left(\frac{\partial g_{22}}{\partial q_2} + \frac{\partial g_{22}}{\partial q_2} - \frac{\partial g_{22}}{\partial q_2}\right) = \frac{1}{2}r^{-2}(0+0-0) = 0,
\end{aligned}
$$

and thus $\boxed{\Gamma^2 = \begin{pmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{pmatrix}}$.

Now for $q_1 = r$,

$$
\ddot{r} = -\begin{pmatrix} \dot{r} & \dot{\theta} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = r\dot{\theta}^2
$$

and for $q_2 = \theta$

$$
\ddot{\theta} = -\begin{pmatrix} \dot{r} & \dot{\theta} \end{pmatrix} \begin{pmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = -2\dot{r}\dot{\theta}r^{-1}.
$$

As in class, let us solve the Lagrangian with no potential energy given by $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$:

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] &= m\ddot{r} \\
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 \\
\implies \ddot{r} &= r\dot{\theta}^2 \qquad\qquad \text{and} \\
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] &= \frac{d}{dt}\left[mr^2\dot{\theta}\right] = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= 0 \\
\implies \ddot{\theta} &= -2m\dot{r}\dot{\theta}r^{-1}.
\end{aligned}
$$

Therefore $\boxed{\ddot{r} = r\dot{\theta}^2 \text{ and } \ddot{\theta} = -2m\dot{r}\dot{\theta}r^{-1},}$ so the two forms are identical.

3. The pivot end of a simple pendulum of mass m and length L is attached to the edge of a disk of radius R, rotating about its centre with angular frequency ω, as shown in the figure. Write down the Lagrangian and derive the equations of motion for the angle θ between the pendulum and the vertical axis. (HINT: you may find it most straightforward to determine the velocity of the mass by writing down an expression for its Cartesian coordinates at time t in terms of the generalized coordinates, and then differentiating.)

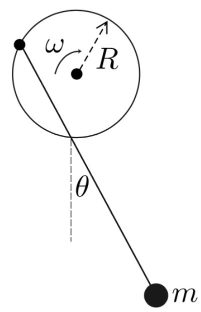

Figure 1: Problem 3.

Let ϕ be the angle of the fixed point on the disk with respect to the vertical, so $\dot{\phi} = \omega$. Then $\phi(t) = \omega t + \phi_0$.
The x and y cartesian coordinates of the mass are then given by

$$\begin{aligned} x &= L\sin\theta + R\sin\phi \\ y &= L\cos\theta + R\cos\phi. \end{aligned}$$

Then

$$\begin{aligned} \dot{x} &= L\cos\theta\dot{\theta} + R\cos\phi\dot{\phi} \\ \dot{y} &= -L\sin\theta\dot{\theta} - R\sin\phi\dot{\phi}. \end{aligned}$$

Our Lagrangian is given by

$$\begin{aligned} L = T - V &= \frac{1}{2}m[\dot{x}^2 + \dot{y}^2] + mgy \\ &= \frac{1}{2}m[(L\cos\theta\dot{\theta} + R\cos\phi\dot{\phi})^2 + (-L\sin\theta\dot{\theta} - R\sin\phi\dot{\phi})^2] + mg[L\cos\theta + R\cos\phi] \\ &= \frac{1}{2}m[L^2\cos^2\theta\dot{\theta}^2 + 2LR\cos\theta\cos\phi\dot{\theta}\dot{\phi} + R^2\cos^2\phi\dot{\phi}^2 + \\ &\quad L^2\sin^2\theta\dot{\theta}^2 + 2LR\sin\theta\sin\phi\dot{\theta}\dot{\phi} + R^2\sin^2\phi\dot{\phi}^2] + mg[L\cos\theta + R\cos\phi] \\ &= \frac{1}{2}m[L^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2LR\dot{\theta}\dot{\phi}(\cos\theta\cos\phi + \sin\theta\sin\phi)] + mg[L\cos\theta + R\cos\phi] \\ &= \frac{1}{2}m[L^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2LR\dot{\theta}\dot{\phi}\cos(\theta - \phi)] + mg[L\cos\theta + R\cos\phi]. \end{aligned}$$

Now our Euler-Lagrange equations on the generalized coordinate θ gives

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}}\right] = \frac{d}{dt}\left[\frac{1}{2}m[2L^2\dot{\theta} + 2LR\dot{\phi}\cos(\theta - \phi)]\right]$$

$$
\begin{aligned}
&= \frac{d}{dt}\left[m[L^2\dot{\theta} + LR\dot{\phi}\cos(\theta-\phi)]\right] \\
&= mL^2\ddot{\theta} + mLR\ddot{\phi}\cos(\theta-\phi) - LR\dot{\phi}\sin(\theta-\phi)(\dot{\theta}-\dot{\phi})
\end{aligned}
$$

$$
= \frac{\partial L}{\partial \theta} = \frac{1}{2}m[-2LR\dot{\theta}\dot{\phi}\sin(\theta-\phi)] - mgL\sin\theta.
$$

Notice that since $\dot{\phi} = \omega$, then $\ddot{\phi} = 0$. This yields

$$
mL^2\ddot{\theta} - mLR\omega\sin(\theta-\phi)(\dot{\theta}-\omega) = -mLR\dot{\theta}\omega\sin(\theta-\phi) - mgL\sin\theta.
$$

Furthermore, since $\dot{\phi} = \omega$ then $\omega t + \phi_0 = \phi(t)$, where ϕ_0 is the initial value of the pivot point on the disk. Rearranging the Lagrangian and factoring,

$$
\begin{aligned}
0 &= mL^2\ddot{\theta} - mLR\omega\sin(\theta - \omega t + \phi_0)[\dot{\theta} - (\dot{\theta}-\omega)] + mgL\sin\theta \\
&= mL^2\ddot{\theta} + mLR\omega^2\sin(\theta - \omega t + \phi_0) + mgL\sin\theta.
\end{aligned}
$$

Cancelling m and dividing by L^2 on each side, our equations of motion becomes

$$
\boxed{\ddot{\theta} = -\frac{R}{L}\omega^2\sin(\theta - \omega t + \phi_0) - \frac{g}{L}\sin\theta.}
$$

4. * A coffee cup of mass M is connected to a mass m by a string. The coffee cup hangs over a frictionless pulley of negligible size, and the mass m is initially held with the string horizontal, as shown in part (a) of the figure below. The mass m is then released. Find the equations of motion for r, the length of string between m and the pulley, and θ, the angle that the string to m makes with the horizontal. Assume that m somehow doesn't run into the string holding the cup up.

In this problem I will work with polar coordinates, with the pulley at the origin. Let r be the distance from the origin and let θ be the angle of the swinging string with respect to the horizontal. Our kinetic and potential energy, respectively, is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2, \qquad V = mgr\cos\theta - Mgr.$$

Notice the plus sign in the potential energy term. Assuming $m > M$, this is a result of the mass m moving upwards while the mass M moves downwards. Our Lagrangian is then

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2 - mgr\cos\theta + Mgr.$$

Applying the Euler-Lagrange equation yields

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] &= \frac{d}{dt}\left[m\dot{r} + M\dot{r}\right] = (m+M)\ddot{r} \\ &= \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta + Mg. \end{aligned}$$

and for θ

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] &= \frac{d}{dt}[mr^2\dot{\theta}] = m[2r\dot{r}\dot{\theta} + r^2\ddot{\theta}] \\ &= \frac{\partial L}{\partial \theta} = mgr\sin\theta. \end{aligned}$$

Therefore our equations of motion are

$$\boxed{\ddot{r}(m+M) = mr\dot{\theta}^2 - mg\cos\theta + Mg \qquad \text{and} \qquad r\ddot{\theta} = g\sin\theta - 2\dot{r}\dot{\theta}.}$$

5. * A circular wire hoop rotates in the horizontal plane at constant angular velocity ω about a vertical axis through the point A in part (b) of the figure below (the figure is shown viewed from above). A bead of mass m is threaded on the hoop and free to move around it, with its position specified by the angle ϕ shown in the figure. Find the Lagrangian for this system using ϕ as your generalized coordinate. Show that the bead oscillates about the point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small?

Let R be the radius of the hoop and let r be the distance from the mass to point A. The kinetic energy of this system is rotational: $T = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}mr^2\omega^2$. There is no $\dot{r}$ term since the bead is fixed to the ring, the kinetic energy would be accounted for in the $\frac{1}{2}mR^2\dot{\phi}^2$ term. There is no potential energy since the hoop is located in the horizontal plane.
The distance r from A to m can be given by cosine law:

$$\begin{aligned} r^2 &= R^2 + R^2 - 2R\cos(\pi - \phi) \\ &= 2R^2(1 - \cos(\pi - \phi)) \\ &= 2R^2(1 + \cos\phi) \\ \implies r &= \sqrt{2}R\sqrt{1 + \cos\phi}. \end{aligned}$$

Then $\boxed{L = \frac{1}{2}mR^2\dot{\phi}^2 + mR^2\omega^2(1 + \cos\phi)}$. The Euler-Lagrange equations give

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}}\right] &= mR^2\ddot{\phi} \\ \frac{\partial L}{\partial \phi} &= -mR^2\omega^2\sin\phi \\ &\boxed{\implies \ddot{\phi} = -\omega^2\sin\phi,} \end{aligned}$$

which is the equation for a simple pendulum oscillating about $\phi = 0$, which is B. For small amplitudes, $\sin\phi \approx \phi$, which gives the frequency ω.

6. * Consider a bead of mass m sliding on a wire that is bent in the shape of a parabola and is being spun with constant angular velocity ω about its vertical axis, as shown in part (c) of the figure. Take the equation of the parabola to be $z = k\rho^2$ for some constant k. Find the Lagrangian in terms of the generalized coordinate ρ, and find the equation of motion of the bead. Are there any points of equilibrium (values of ρ at which the bead can remain fixed, without sliding up or down the spinning wire)? Discuss the stability of any equilibrium positions you find, as a function of the frequency ω. (Recall: an equilibrium position is stable if there is a restoring force for small perturbations about the equilibrium point, otherwise it is unstable).

The kinetic energy of the mass, in cylindrical coordinates, will be $T = \dfrac{1}{2}m(\dot\rho^2 + \rho^2\omega^2 + \dot z^2)$, while the potential energy is given by $V = -mgk\rho^2$. The Lagrangian is then

$$L = \frac{1}{2}m(\dot\rho^2 + \rho^2\omega^2 + 4k^2\rho^2\dot\rho^2) + mgk\rho^2.$$

It follows by the Euler-Lagrange equations

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot\rho}\right] &= \frac{d}{dt}\left[m\dot\rho + 4mk^2\rho^2\dot\rho\right] = m\ddot\rho + 4mk^2\rho^2\ddot\rho + 8mk^2\rho\dot\rho^2 \\
&= \frac{\partial L}{\partial \rho} = m\rho\omega^2 + 4mk^2\rho\dot\rho^2 + 2mgk\rho \\
&\boxed{\implies \ddot\rho\left(1 + 4k^2\rho^2\right) = -4k^2\rho\dot\rho^2 + \rho\omega^2 + 2gk\rho.}
\end{aligned}$$

Now, if ρ is constant at equilibrium points, then $\dot\rho = 0$ and $\ddot\rho = 0$. Thus

$$0 = \rho(\omega^2 + 2gk).$$

This can only be true if $\rho = 0$ or $\omega^2 = -2gk$. This implies that $\dfrac{-\omega^2}{2gk} = -1$, so ρ is constant. The equilibrium positions of the mass are then

$$\begin{aligned}
\text{[unstable]} \qquad & \rho = 0,\ \omega > 0 \\
\text{[stable, converges to each } \rho\text{]} \qquad & \rho(\omega) = \frac{\omega^2}{2gk}.
\end{aligned}$$