PHY489 Problem Set 3

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Problem 1

(**a**) To convert values in natural units ($\hbar = c = 1$) to SI units, we just need to re-introduce factors of c (units of m/s) and $\hbar$ (units of eV s). Lifetimes are given in units of seconds, thus it is simple to convert units by just multiplying through by $\hbar$:

$$\begin{aligned}\tau &= 0.5\,\frac{1}{\text{GeV}}\\ &= 0.5\hbar\,\frac{\text{GeV s}}{\text{GeV}}\\ &= 0.5(6.58211957\times 10^{-25})\text{ s}\\ &= 3.3\times 10^{-25}\text{ s.}\end{aligned} \tag{1.1}$$

(**b**) Similarly, we can convert units of barns (that of cross-sections) into natural units of energy:

$$\begin{aligned}1\,\text{pb} &= \left[\frac{10^{-40}\text{ m}^2}{1\text{ pb}}\right]\times\left[\frac{1\text{ s}}{c\,\text{m}}\right]^2\times\left[\frac{1}{\hbar\,\text{GeV s}}\right]^2\\ &= 10^{-40}\frac{1}{c^2\hbar^2}\,\frac{1}{\text{GeV}^2}\\ &= 10^{-40}\frac{1}{(3.0\times 10^8)^2(6.58211957\times 10^{-25})^2}\,\frac{1}{\text{GeV}^2}\\ &= 3.9\times 10^{-8}\,\frac{1}{\text{GeV}^2}.\end{aligned} \tag{1.2}$$

Problem 2

(a), (b) To determine the masses of resonant particles produced in high energy interactions, we can turn to the center of mass energy. The (square of the) center of mass energy, which is a Lorentz invariant quantity (the invariant mass) and is conserved throughout the process of the interaction. For a beam of π^- directed at stationary protons, the invariant mass is

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ &= \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2 \\ &= (E_\pi + m_p)^2 - \mathbf{p}_\pi^2 \\ &= m_p^2 + m_\pi^2 + 2E_\pi m_p. \end{aligned} \tag{2.1}$$

Taking the proton mass to be 938 MeV and the pion mass to be 139 MeV, we find that, for $|\mathbf{p}_\pi| = 300$ MeV,

$$\begin{aligned} s &= (938)^2 + (139)^2 + 2\sqrt{(139)^2 + (300)^2}(938) \\ &= 1'519'440 \text{ MeV}^2 \end{aligned} \tag{2.2}$$

which corresponds to the mass $\sqrt{s} = 1232$ MeV - the Δ baryon.

Problem 3

Consider the weak decay of the tau lepton $\tau^- \to \pi^- \nu_\tau$. In decay, the angle between the emitted π^- and the axis defined by the spin of the τ^- is θ^*. Furthermore, the angular distribution of the pion emission $\dfrac{dN_\pi}{d(\cos\theta^*)} \propto 1 + \cos\theta^*$. To determine the energy distribution in the lab frame, $\dfrac{dN_\pi}{dE_\pi}$, we must consider the boost of the pion four-momentum along the spin axis of the τ^-.

Let $\hat{\mathbf{z}}$ be the axis defined by the spin of the tau particle. In the τ^- rest frame, we can consider the pion momentum vector to be in spherical coordinates $\mathbf{p} = (p_r, p_\theta, p_\varphi)$, which can be represented in the cartesian coordinate basis of the τ^- allowing to compute the Lorentz boost:

$$\begin{aligned} p_\pi'^\mu &= (E_\pi', p_r, p_\theta, p_\varphi) \\ &= (E_\pi', p\sin\theta^*\sin\varphi^*, p\sin\theta^*\cos\varphi^*, p\cos\theta^*). \end{aligned} \tag{3.1}$$

Since we are only interested in the energy distribution in the lab frame, we only need to apply the boost to the temporal component of p_π^μ, with

$$\Lambda_\mu^\nu = \begin{pmatrix} \gamma(v) & 0 & 0 & -v\gamma(v) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma(v) & 0 & 0 & \gamma(v) \end{pmatrix} \tag{3.2}$$

being the Lorentz boost along the $\hat{\mathbf{z}}$ axis of the τ^-, where v is the velocity of the τ^- in the lab frame. This means that, for flight direction begin aligned with the spin of the τ^-, $v \to v$. Similarly, for flight direction being anti-aligned, $v \to -v$. The z-component of the pion momentum is unaffected, because the spin direction is left unchanged (we're only changing the boost direction). To not lose generality, let me transform $-v\gamma(v) = \Lambda_3^0 = \Lambda_0^3 \to \mp v\gamma(v)$, where the $-$ indicates boost alignment, the $+$ boost anti-alignment with the spin axis. Transforming the temporal component of $p_\pi'^\mu$ into the lab frame,

$$\begin{aligned} p_\pi^0 &= \Lambda_\mu^0 p_\pi'^\mu \\ E_\pi &= \gamma(v)E_\pi' \mp v\gamma(v)p\cos\theta^* \end{aligned} \tag{3.3}$$

which gives the energy of the pion in the lab frame. Since v is constant, the variation of θ^* in $\cos\theta^*$ implies that maxima and minima exist for E_π. The derivative is

$$\begin{aligned} \frac{dE_\pi}{d\theta^*} &= \mp v\gamma(v)p\frac{d}{d\theta^*}\cos\theta^* \\ &= \mp v\gamma(v)p(-\sin\theta^*) \\ 0 &= \pm v\gamma(v)p\sin\theta^* \end{aligned} \tag{3.4}$$

which has critical values when $\theta^* = 0, \pi$. When $\theta^* = 0$, $\cos 0 = 1$ so $E_\pi = \gamma(v)E_\pi' \mp v\gamma(v)p$, a minimum for alignment and maximum for anti-alignment. Furthermore, $\cos\pi = -1$ so $E_\pi = \gamma(v)E_\pi' \pm v\gamma(v)p$, a maximum for alignment and a minimum for anti-alignment. Let me label the maximum and minimum values $E_\pm, E_\mp$ respectively, for each of the given cases.

Observe that

$$E_\pm + E_\mp = 2\gamma(v)E_\pi' \tag{3.5}$$

$$E_\pm - E_\mp = \pm 2v\gamma(v)p \tag{3.6}$$

which implies that the lab frame energy can be written in the compact form

$$E_\pi = \frac{1}{2}(E_\pm + E_\mp) - \frac{1}{2}(E_\pm - E_\mp)\cos\theta^*. \tag{3.7}$$

An expansion therefore allows us to isolate the $(1+\cos\theta^*)$ required to determine the energy distribution:

$$\begin{aligned} E_\pi &= \frac{1}{2}E_\pm + \frac{1}{2}E_\mp - \frac{1}{2}E_\pm\cos\theta^* - \frac{-1}{2}E_\mp\cos\theta^* \\ &= \frac{1}{2}E_\pm(1-\cos\theta^*) + \frac{1}{2}E_\mp(1+\cos\theta^*). \end{aligned} \tag{3.8}$$

Note that, by factoring out a minus sign and adding/subtracting 2 inside the bracket allows us to isolate the $(1+\cos\theta^*)$ term:

$$\begin{aligned} E_\pi &= -\frac{1}{2}E_\pm(1+\cos\theta-2) + \frac{1}{2}E_\mp(1+\cos\theta^*) \\ &= E_\pm + \frac{1}{2}(E_\mp - E_\pm)(1+\cos\theta^*). \end{aligned} \tag{3.9}$$

Solving for $1+\cos\theta^*$ is simple now:

$$1+\cos\theta^* = \frac{2(E_\pi - E_\pm)}{E_\mp - E_\pm}. \tag{3.10}$$

Calculating the lab frame energy distribution from this step is a straightforward application of the chain rule:

$$\begin{aligned} \frac{dN_\pi}{dE_\pi} &= \frac{dN_\pi}{d(\cos\theta^*)}\frac{d(\cos\theta^*)}{dE_\pi} \\ &= A(1+\cos\theta^*)\cdot\frac{2}{E_\mp - E_\pm} \end{aligned} \tag{3.11}$$

where again, E_- represents the minimum of E_π when $\theta^* = 0$ and E_+ represents the maximum when $\theta^* = \pi$. A is a constant of proportionality, which I assumed exists because $\frac{dN_\pi}{d(\cos\theta^*)} \propto 1+\cos\theta^*$ is given for the angular distribution.
Therefore, when the spin is aligned with boost direction, we find the energy distribution to be

$$\frac{dN_\pi}{dE_\pi} = 2A\frac{(1+\cos\theta^*)}{E_- - E_+}$$

and

$$\frac{dN_\pi}{dE_\pi} = 2A\frac{(1+\cos\theta^*)}{E_+ - E_-}$$

when the spin is anti-aligned with boost direction.

Problem 4

The energy factor for scattering processes of incident particles is given by $\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}$ (I am not going to use natural units here). Expanding this term out yields various expressions for the energy factor, which is frame-dependent. First note that for two four-vectors p_1^μ and p_2^μ,

$$\begin{aligned}
(p_1 \cdot p_2)^2 &= (p_1^\mu p_{2\,\mu})^2 \\
&= ((E_1/c)(E_2/c) - \mathbf{p}_1 \cdot \mathbf{p}_2)^2 \\
&= \frac{E_1^2 E_2^2}{c^4} + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{2E_1E_2}{c^2}\mathbf{p}_1 \cdot \mathbf{p}_2 \\
&= \frac{(m_1^2c^4 + p_1^2c^2)(m_2^2c^4 + p_2^2c^2)}{c^4} + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{2E_1E_2}{c^2}\mathbf{p}_1 \cdot \mathbf{p}_2 \\
&= \frac{m_1^2m_2^2c^8}{c^4} + \frac{p_1^2p_2^2c^4}{c^4} + \frac{m_1^2p_2^2c^6}{c^4} + \frac{m_2^2p_1^2c^6}{c^4} + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{2E_1E_2}{c^2}\mathbf{p}_1 \cdot \mathbf{p}_2 \\
&= (m_1m_2c^2)^2 + p_1^2p_2^2 + m_1^2p_2^2c^2 + m_2^2p_1^2c^2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{2E_1E_2}{c^2}\mathbf{p}_1 \cdot \mathbf{p}_2. \qquad (4.1)
\end{aligned}$$

Substituting this expression into the energy factor for two-incident particle scattering, the first term in (4.1) is subtracted and the square root of the remaining expression can be taken out last.

(a) For an observer in the center of mass frame, $|\mathbf{p}_1| = |\mathbf{p}_2|$ and $\mathbf{p}_1 = -\mathbf{p}_2$, taking expression (4.1) into

$$(p_1 \cdot p_2)^2 - (m_1m_2c^2)^2 = p_1^2p_2^2 + m_1^2p_1^2c^2 + m_2^2c^2p_1^2 + p_1^4 + \frac{2E_1E_2}{c^2}p_1^2. \qquad (4.2)$$

Here I have chosen to explicity not convert one $p_2^2 \to p_1^2$ so that the energy E_2 can be recovered. Factoring out a $\frac{p_1^2}{c^2}$ gives

$$\begin{aligned}
(p_1 \cdot p_2)^2 - (m_1m_2c^2)^2 &= \left(p_2^2c^2 + m_1^2c^4 + m_2^2c^4 + p_1^2c^2 + 2E_1E_2\right)\frac{p_1^2}{c^2} \\
&= (E_2^2 + E_1^2 + 2E_1E_2)\frac{p_1^2}{c^2} \\
&= (E_1 + E_2)^2\frac{p_1^2}{c^2}, \qquad (4.3)
\end{aligned}$$

where we can now take out the square root sign and obtain the center of mass energy factor:

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1m_2c^2)^2} = (E_1 + E_2)\frac{|\mathbf{p}_1|}{c} \qquad (4.4)$$

which is what I wanted to show.

(b) A similar calculation can be done for an observer in the lab frame, where one of the initial particles (I'll say particle 2) is at rest: $|\mathbf{p}_2| = 0$. In this case, (4.1) goes into

$$\begin{aligned}
(p_1 \cdot p_2)^2 - (m_1m_2c^2)^2 &= p_1^2(0)^2 + m_1^2(0)^2c^2 + m_2^2p_1^2c^2 + (\mathbf{p}_1 \cdot \mathbf{0})^2 - \frac{2E_1E_2}{c^2}(\mathbf{p}_1 \cdot \mathbf{0}) \\
&= m_2^2p_1^2c^2. \qquad (4.5)
\end{aligned}$$

Taking the square root yields the lab frame factor:

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1m_2c^2)^2} = m_2c|\mathbf{p}_1| \qquad (4.6)$$

as desired.

Problem 5

In this problem, we consider two-body elastic scattering in the lab frame, where particle 2 is initially at rest. Let me therefore label the interaction $A + B \to 1 + 2$, where B is at rest. Masses and momenta will be labelled respectfully.

(**a**) We wish to determine the differential cross-section $\frac{d\sigma}{d\Omega}$, which is given by the golden rule:

$$d\sigma = \frac{S}{4\sqrt{(p_A \cdot p_B)^2 - (m_A m_B)^2}} \int_{\text{Final States}} \prod_{k=1}^{2} \frac{d^3 p_k}{(2\pi)^3 2E_k} (2\pi)^4 \delta^{(4)}(P_I - P_F) |\mathcal{A}_{fi}|^2 \tag{5.1}$$

The '$d\sigma$' here indicates that not all integrals will be taken out, since we are not wanting to find the total cross-section. The energy factor in the first term for lab frame rest particles is given by (4.6), and the initial and final four-momenta delta function $\delta^{(4)}(p_A + p_B - p_1 - p_2)$. The statistical factor S is left undetermined, since the types of particles associated in the scatter aren't defined ($S = 1/2$ for $AA \to AA$, but $S = 1$ for $AB \to AB$ processes). We therefore have

$$d\sigma = \frac{S}{4 m_B p_A} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2) |\mathcal{A}_{fi}|^2 \tag{5.2}$$

with the Feynman amplitude $i\mathcal{A}_{fi}$ being left variable as the scattering process is arbitrary. Taking out all the factors of 2π and all the constants in front of the integral, and re-writing the four-dimensional delta function as the product between the energy and vector three-momenta (so that energy and three-momenta can be treated individually), we have that

$$\begin{aligned} d\sigma &= \frac{S}{16(2\pi)^2 m_B p_A} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(3)}(\mathbf{p}_A + \mathbf{p}_B - \mathbf{p}_1 - \mathbf{p}_2) \delta(E_A + E_B - E_1 - E_2) |\mathcal{A}_{fi}|^2 && (5.3) \\ &= \frac{S}{16(2\pi)^2 m_B p_A} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(3)}(\mathbf{p}_A - \mathbf{p}_1 - \mathbf{p}_2) \delta(E_A + m_B - E_1 - E_2) |\mathcal{A}_{fi}|^2 && (5.4) \end{aligned}$$

where I have evaluated the particle B momentum to be zero in the last line. To simplify matters even more, we can take out the p_1 volume integral, where the $\delta^{(3)}$ function just sends $\mathbf{p_1} \to \mathbf{p}_A - \mathbf{p}_2$:

$$d\sigma = \frac{S}{16(2\pi)^2 m_B p_A} \int \frac{d^3 p_2}{E_A(\mathbf{p}_A - \mathbf{p}_2) E_B(\mathbf{p}_2)} \delta(E_A(\mathbf{p}_A) + m_B - E_A(\mathbf{p}_A - \mathbf{p}_2) - E_B(\mathbf{p_2})) |\mathcal{A}_{fi}|^2 \tag{5.5}$$

and I have introduced the notation $E_i(\mathbf{p}) = \sqrt{m_i^2 + p^2}$ for the momentum conversion, letting $m_1 = m_A$ and $m_2 = m_B$ for elastic scattering. The Feynman amplitude should also go over into the new momentum value, but I will leave this implicit as we are just looking for a general expression for the differential cross section. Assuming that the amplitude is simple enough to be independent of the direction of $\mathbf{p}_2$, we can expand the differential $d^3 p_2$ into the spherical components, including the solid angle: $d^3 p_2 = p_2^2 dp_2 d\Omega_2$. We can therefore take the solid angle out of the integral, as we are not looking to evaluate for the total cross section, only the differential. We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{S}{16(2\pi)^2 m_B p_A} \int p_2^2 \, dp_2 \, |\mathcal{A}_{fi}|^2 \, \frac{\delta\left(E_A(\mathbf{p}_A) + m_B - \sqrt{m_A^2 + (\mathbf{p}_A - \mathbf{p}_2)^2} - \sqrt{m_B^2 + p_2^2}\right)}{\sqrt{m_A^2 + (\mathbf{p}_A - \mathbf{p}_2)^2}\,\sqrt{m_B^2 + p_2^2}}. \tag{5.6}$$

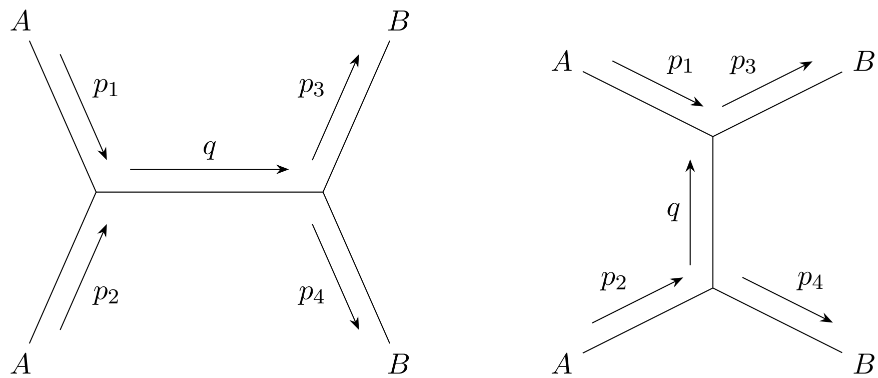
(**b**) In the case that one of the incident particles is massless, say particle A, then $m_A = 0$ before and after the scatter. This simplifies matters significantly in equation (5.6), and we find that

$$\frac{d\sigma}{d\Omega} = \frac{S}{16(2\pi)^2 m_B p_A} \int p_2^2 \, dp_2 \, |\mathcal{A}_{fi}|^2 \, \frac{\delta\left(p_A + m_B - |\mathbf{p}_A - \mathbf{p}_2| - \sqrt{m_B^2 + p_2^2}\right)}{|\mathbf{p}_A - \mathbf{p}_2| \sqrt{m_B^2 + p_2^2}}. \tag{5.7}$$

where the $E_A(\mathbf{p})$ energies have gone over into $|\mathbf{p}| = p$. Note that the integrand is just the energy conservation delta function upon the product of the final state energies, which is as expected, and the length of the momentum (the magnitude) p_2 is being integrated over.

Problem 6

(**a**) In this problem, we consider the scattering process $AA \to BB$ in the lab frame (one of the initial state particles A is at rest), where the particle B is massless, and the virtual particle C whose propagator is the scalar propagator is also massless. I will generalize for clarity, labelling the particles $1 + 2 \to 3 + 4$ so that no arbitrariness arises from defining various energies and momenta. We begin by calculating the amplitude of the process, given diagrammatically by two tree-level Feynman diagrams of $\mathcal{O}(g^2)$:



The Feynman amplitude $i\mathcal{A}_{fi}$ is therefore the sum of the amplitudes given by both diagrams.

For the first diagram, we see that there are two vertices, and therefore two factors of $(-ig)$ and two energy-momentum conserving delta-functions, $(2\pi)^4\delta^{(4)}(p_1 + p_2 - q)$, $(2\pi)^4\delta^{(4)}(q - p_3 - p_4)$. There is one scalar propagator $\dfrac{i}{q^2 - m^2}$ (I have omitted isospin indices, as this is only a toy model), and one internal momentum line $\displaystyle\int \frac{d^4q}{(2\pi)^4}$. Altogether, we have

$$\begin{aligned} i\mathcal{A}_1 &= (-ig)^2(2\pi)^4 \int d^4q \frac{i}{q^2 - m^2}\delta^{(4)}(p_1 + p_2 - q)\delta^{(4)}(q - p_3 - p_4) \\ &= -ig^2(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\frac{1}{(p_1 + p_2)^2} \end{aligned} \tag{6.1}$$

where I have taken out the integral, and let $m \to 0$ for a massless scalar propagator. We can perform the same amplitude construction for the second diagram, and find that

$$\begin{aligned} i\mathcal{A}_2 &= (-ig)^2(2\pi)^4 \int d^4q \frac{i}{q^2 - m^2}\delta^{(4)}(p_2 - p_4 - q)\delta^{(4)}(p_1 + q - p_3) \\ &= -ig^2(2\pi^4)\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\frac{1}{(p_4 - p_2)^2}. \end{aligned} \tag{6.2}$$

By enforcing momentum conservation the delta functions are satisfied, as this is already going to be included in differential cross-section expression, and therefore

$$i\mathcal{A}_{fi} = i\mathcal{A}_1 + i\mathcal{A}_2 \tag{6.3}$$

implies that

$$|\mathcal{A}_{fi}|^2 = \left|\frac{g^2}{(p_1 + p_2)^2} + \frac{g^2}{(p_2 - p_4)^2}\right|^2. \tag{6.4}$$

We can now proceed by evaluating the differential cross section, whose formula is given in (5.1):

$$d\sigma = \frac{S}{4\sqrt{(p_A \cdot p_B)^2 - (m_A m_B)^2}} \int_{\text{Final States}} \prod_{k=1}^{2} \frac{d^3 p_k}{(2\pi)^3 2E_k} (2\pi)^4 \delta^{(4)}(P_I - P_F) |\mathcal{A}_{fi}|^2 \tag{6.5}$$

First note some modifications to this expression before moving forward. First, in the lab frame (particle 2 is at rest), we have that $\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} \to m_2 p_1$ by (4.6). On the products side, two identical B particles are scattered, giving $S = \frac{1}{2}$ as a statistical factor. Particle B is also massless, thus the energies $E_i \to p_i$, but this will be taken out once the energy-momentum conserving delta function is seperated into energy and three-momentum components. Thus far, we have

$$\begin{aligned} d\sigma &= \frac{1}{8(2\pi)^2 m_2 p_1} \int \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(E_1 + E_2 - E_3 - E_4) |\mathcal{A}_{fi}|^2 \\ &= \frac{1}{32(2\pi)^2 m_2 p_1} \int \frac{d^3 p_3}{p_3} \frac{d^3 p_4}{p_4} \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_4) \delta(E_1 + m_2 - p_3 - p_4) \\ &\qquad \times \left| \frac{g^2}{m_1^2 + m_2^2 + 2E_1 m_2} + \frac{g^2}{m_2^2 - 2m_2 p_4} \right|^2 \end{aligned} \tag{6.6}$$

where the zero momentum and massless substitutions have been made in the second line and the Feynman amplitude (6.4) has been introduced. If we strive for the differential cross section, we need not take out both integrals, but only the p_3 integral, which is only over the three-momentum and sends $p_3 \to |\mathbf{p}_1 - \mathbf{p}_4|$ by the delta function. Note that the amplitude is unaffected by this since it is independent of p_3^μ.

$$\begin{aligned} d\sigma &= \frac{1}{32(2\pi)^2 m_2 p_1} \int \frac{d^3 p_4}{p_4} \frac{1}{|\mathbf{p}_1 - \mathbf{p}_4|} \delta(E_1 + m_2 - |\mathbf{p}_1 - \mathbf{p}_4| - p_4) \\ &\qquad \times \left| \frac{g^2}{m_1^2 + m_2^2 + 2E_1 m_2} + \frac{g^2}{m_2^2 - 2m_2 p_4} \right|^2 \end{aligned} \tag{6.7}$$

Lastly, since the amplitude is independent of the orientation of p_4, we can seperate the differential $d^3 p_4$ into radial and angular components, introducing the solid angle $d\Omega = \sin\theta d\theta d\varphi$, hence $d^3 p_4 = p_4^2 dp_4 d\Omega$ (the '4' subscript is omitted on the solid angle, as the choice of solid angle for 3 or 4 is arbitrary, and can be taken out of the integral). Thus the differential cross section for the $AA \to BB$ process is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{32(2\pi)^2 m_2 p_1} \int \frac{p_4 \, dp_4}{|\mathbf{p}_1 - \mathbf{p}_4|} \delta(E_1 + m_2 - |\mathbf{p}_1 - \mathbf{p}_4| - p_4) \\ &\qquad \times \left| \frac{g^2}{m_1^2 + m_2^2 + 2E_1 m_2} + \frac{g^2}{m_2^2 - 2m_2 p_4} \right|^2 \end{aligned} \tag{6.8}$$

(**b**) The second component of this problem is considering both the low- and high-energy limiting cases for the incident particle 1 beam momentum. In particular, we want to evaluate the differential cross section for when $\frac{p_1}{m_1} \ll 1$ (that is, the low-energy limit $p_1 \ll m_1 c$) and when $p_1 \to m_1$ (that is, the high-energy limit $p_1 \to m_1 c$) and $c = 1$ is still invoked here.

Beginning with the low-energy limit, first observe the approximation of relativistic energy

$$E_1 = \sqrt{m_1^2 + p_1^2}$$

$$
\begin{aligned}
&= m_1\sqrt{1 + p_1^2/m_1^2} \\
&\approx m_1\left(1 + \frac{1}{2}\frac{p_1^2}{m_1^2} + \mathcal{O}(p_1^4/m_1^4)\right) \\
&\approx m_1\left(1 + \mathcal{O}(p_1^2/m_1^2)\right) \\
&= m_1,
\end{aligned} \tag{6.9}
$$

which is sensible, as the rest mass energy component is much larger than that of the kinetic component. Furthermore, we can evaluate the approximation of the momentum difference $|\mathbf{p}_1 - \mathbf{p}_4|$:

$$
\begin{aligned}
|\mathbf{p}_1 - \mathbf{p}_4| &= \sqrt{(p_1 - p_4)^2} \\
&= m_1\sqrt{p_1^2/m_1^2 + p_4^2/m_1^2 - 2p_1p_4/m_1^2} \\
&\approx m_1\left[\sqrt{p_4^2/m_1^2 - 2p_1p_4/m_1^2} + \frac{1}{2}p_1^2/m_1^2\left(p_4^2/m_1^2 - 2p_1p_4/m_1^2\right)^{-1/2} + \mathcal{O}(p_1^4/m_1^4)\right] \\
&\approx m_1\left[\sqrt{p_4^2/m_1^2 - 2p_1p_4/m_1^2} + \mathcal{O}(p_1^2/m_1^2)\right] \\
&= \sqrt{p_4^2 - 2p_1p_4}.
\end{aligned} \tag{6.10}
$$

With these approximations, we have that

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} = \frac{1}{32(2\pi)^2 m_2 p_1}\int \frac{p_4\, dp_4}{\sqrt{p_4^2 - 2p_1p_4}}&\delta(m_1 + m_2 - \sqrt{p_4^2 - 2p_1p_4} - p_4) \\
&\times\left|\frac{g^2}{(m_1 + m_2)^2} + \frac{g^2}{m_2^2 - 2m_2p_4}\right|^2
\end{aligned} \tag{6.11}
$$

whose integral in p_4 may be able to be taken out, assuming the energy-conserving delta function can be gotten into a form which isolates p_4 (possibly a u-substitution).

Next, we can consider the high energy limit as the velocity of the particle approaches c, or in natural units, $p_1 \to m_1$. This subsitution can directly be made, as $E_1 \to \sqrt{2}m_1$ and $|\mathbf{p}_1 - \mathbf{p}_4| \to |m_1 - p_4|$:

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} = \frac{1}{32(2\pi)^2 m_1 m_2}\int \frac{p_4\, dp_4}{|m_1 - p_4|}&\delta(\sqrt{2}m_1 + m_2 - |m_1 - p_4| - p_4) \\
&\times\left|\frac{g^2}{m_1^2 + m_2^2 + 2\sqrt{2}m_1m_2} + \frac{g^2}{m_2^2 - 2m_2p_4}\right|^2
\end{aligned} \tag{6.12}
$$

Similarly, this integral may be taken out via a substitution.