University of Toronto
Faculty of Arts and Science

MAT237 Multivariable Calculus with Proofs
Term Test 3

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Friday March 4, 2022
Duration: 110 minutes
No aids permitted

Instructions

- **Do not open the exam until you are instructed to do so.** Failure to comply is an academic offence.
- **No aids are permitted on this examination.** Examples of illegal aids include but are not limited to textbooks, notes, calculators, cellphones, or any electronic device.
- **Once the exam begins, check that you have all pages.** This exam contains 12 pages including this cover page, and is printed **double-sided** on 6 sheets of paper. There are 7 problems.
- **Show your work and justify your steps** on every question, unless otherwise indicated.
- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **If you need more space**, use the blank pages at the end of the exam and clearly indicate on the question page when you have done this. **Do not tear any pages off this exam.**

Academic integrity statement

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I confirm that:

- I have not used or been in possession of an unauthorized aid while writing this exam.
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- I immediately stopped writing when an invigilator announced the exam was over.
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Question:	1	2	3	4	5	6	7	Total
Points:	5	5	5	8	6	6	5	40
Score:								

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Trigonometry

$$\sin\frac{\pi}{6}=\cos\frac{\pi}{3}=\frac{1}{2}\qquad \sin\frac{\pi}{3}=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}\qquad \sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$\sin(-\theta)=-\sin\theta\qquad \cos(-\theta)=\cos\theta\qquad \tan(-\theta)=-\tan\theta$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}\qquad \sec\theta=\frac{1}{\cos\theta}\qquad \cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{1}{\tan\theta}\qquad \csc\theta=\frac{1}{\sin\theta}$$

$$\sin^2\theta+\cos^2\theta=1\qquad 1+\tan^2\theta=\sec^2\theta\qquad 1+\cot^2\theta=\csc^2\theta$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta\qquad \sin 2\theta=2\sin\theta\cos\theta\qquad \tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos^2\theta=\frac{1+\cos 2\theta}{2}\qquad \sin^2\theta=\frac{1-\cos 2\theta}{2}\qquad \tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Antiderivatives

$$\int x^n dx=\frac{x^{n+1}}{n+1}+C\qquad \int \frac{1}{x}dx=\ln|x|+C\qquad \int a^x dx=\frac{a^x}{\ln a}+C$$

$$\int \sin x dx=-\cos x+C\qquad \int \cos x dx=\sin x+C\qquad \int \tan x dx=-\ln|\cos x|+C$$

$$\int \sec x dx=\ln|\sec x+\tan x|+C\qquad \int \frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C\qquad \int \frac{1}{1+x^2}dx=\arctan x+C$$

$$\int u dv=uv+\int v du\qquad \int f(g(x))g'(x)dx=\int f(u)du$$

Linear algebra

$$\det\begin{bmatrix} a & b\\ c & d\end{bmatrix}=ad-bc\qquad \det\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=a_{11}\det\begin{bmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix}-a_{21}\begin{bmatrix} a_{12} & a_{13}\\ a_{32} & a_{33}\end{bmatrix}+a_{31}\begin{bmatrix} a_{12} & a_{13}\\ a_{22} & a_{23}\end{bmatrix}$$

$$a\cdot b=a^Tb=a_1b_1+\cdots+a_nb_n=||a||||b||\cos\theta\qquad a\times b=(a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1)\qquad e_1\times e_2=e_3$$

$$|a\cdot b|\leq ||a||||b||\qquad ||a+b||\leq ||a||+||b||\qquad (AB)^T=B^TA^T\qquad (AB)^{-1}=B^{-1}A^{-1}$$

Coordinate systems

$$(x,y)=(r\cos\theta, r\sin\theta)$$
$$dA=dxdy=rdrd\theta$$

$$(x,y,z)=(r\cos\theta, r\sin\theta, z)=(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$
$$r^2=x^2+y^2,\qquad \rho^2=x^2+y^2+z^2$$
$$dV=dxdydz=rdrd\theta dz=\rho^2\sin\phi d\rho\theta d\phi$$

$$\int_{g(\Omega)} f dV=\int_{\Omega}(f\circ g)|\det Dg|dV$$

1. (5 points) The parts of this question are unrelated. No justification is necessary. When necessary, **fill in EXACTLY ONE circle.** (unfilled ○ filled ●)

(1a) Let R be a rectangle in $\mathbb{R}^n$. Let P, P', P'' be partitions of the rectangle R.
Which statement is necessarily TRUE?

- ○ If R_1 and R_2 are distinct subrectangles of P, then $R_1 \cap R_2 = \varnothing$.
- ● The set $P' \cup P''$ is a partition of R.
- ○ If P' is a refinement of P, then P' contains P.
- ○ If P is the common refinement of P' and P'', then P contains P' and P''.
- ○ None of the above statements are true.

(1b) Let P be the partition of $R = [0,2] \times [-1,3]$ defined by the partitions $\{0,1,2\}$ of $[0,2]$ and partition $\{-1,3\}$ of $[-1,3]$. Let $f(x,y) = x^2 + y$. Compute the lower sum $L_P(f)$. Write your final answer only.

$L_P(f) =$ -1

(1c) Let R be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be bounded.
Which statement is necessarily TRUE?

- ○ For every partition P of R, $\underline{I_R} f \leq L_P(f)$.
- ○ There exists a partition P of R such that $\underline{I_R} f = L_P(f)$.
- ○ If f is continuous on R, then there exists a partition P of R such that $\int_R f\, dV = L_P(f)$.
- ● There exists a partition P of R such that $\underline{I_R} f - 0.01 \leq L_P(f)$.
- ○ None of the above statements are true.

(1d) Let R be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be bounded. Here is an attempted proof that $\underline{I_R} f \leq \overline{I_R} f$.

> 1. *For any partition P of R, we have that $L_P(f) \leq U_P(f)$.*
> 2. *Taking the supremum and infimum over all partitions P, $\sup_P L_P(f) \leq \inf_P U_P(f)$.*
> 3. *Thus, $\underline{I_R} f \leq \overline{I_R} f$.*

Select the most accurate assessment of this argument.

- ○ The proof is essentially correct, but missing some minor details and justifications.
- ○ Line 1 is flawed since it is possible that $L_P(f) \geq U_P(f)$.
- ○ Line 1 is flawed since $L_P(f)$ or $U_P(f)$ are not necessarily defined.
- ● Line 2 is flawed since it assumes an invalid property of suprema and infima.
- ○ Line 3 is flawed since $\underline{I_R} f = \inf_P L_P(f)$ and $\overline{I_R} f = \sup_P U_P(f)$.

(1e) Let $D \subseteq \mathbb{R}^n$ be a set and let $f : D \to \mathbb{R}^m$ be a function.
Which statement is EQUIVALENT to "f is uniformly continuous on D"?

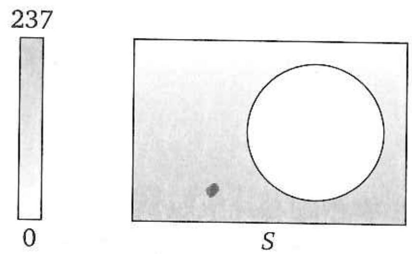
- ○ $\forall a \in D, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in D, ||x - a|| < \delta \implies ||f(x) - f(a)|| < \varepsilon$
- ○ $\forall \varepsilon > 0, \forall a \in D, \exists \delta > 0$ s.t. $\forall x \in D, ||x - a|| < \delta \implies ||f(x) - f(a)|| < \varepsilon$
- ● $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in D, \forall a \in D, ||x - a|| < \delta \implies ||f(x) - f(a)|| < \varepsilon$
- ○ $\forall \varepsilon > 0, \forall a \in D, \exists \delta > 0$ s.t. $f(B_\delta(a) \cap D) \subseteq B_\varepsilon(f(a))$

2. (5 points) The parts of this question are unrelated. No justification is necessary.
Fill in EXACTLY ONE circle. (unfilled ○ filled ●)

(2a) Let $S \subseteq \mathbb{R}^n$ be a set. Which statement is necessarily TRUE?

○ If $S^c = \mathbb{R}^n \setminus S$ does not have zero Jordan measure, then S has zero Jordan measure.
○ If S° is empty, then S has zero Jordan measure.
○ If ∂S has zero Jordan measure, then S is bounded.
○ If $\overline{S}$ has zero Jordan measure, then S has zero Jordan measure.
● None of the above statements are true.

(2b) For a cheese slice $S \subseteq \mathbb{R}^2$ with mass density δ, approximate the position of its centre of mass.



(2c) Let $(\Omega, \Sigma, \mathbb{P})$ be a continuous probability space in $\mathbb{R}^n$ with probability density function ϕ. Which statement is necessarily TRUE?

● Σ is the set of all subsets of Ω.
○ $0 \leq \phi(x) \leq 1$ for all $x \in \Omega$.
○ $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ for any collection of disjoint events $\{A_n\}_{n=1}^{\infty}$ in Σ.
○ If $A \in \Sigma$, then $\overline{A} \in \Sigma$ and $\mathbb{P}(A) = \mathbb{P}(\overline{A})$.
○ None of the above statements are true.

(2d) Let $f : [0,1] \times [2,3] \to \mathbb{R}$ be bounded. Define the three quantities

$$A = \int_0^1 \int_2^3 f(x,y)\,dy\,dx, \qquad B = \int_2^3 \int_0^1 f(x,y)\,dx\,dy, \qquad C = \iint_{[0,1]\times[2,3]} f\,dA.$$

Which statement is necessarily TRUE?

○ If A and B exist, then C exists.
● If A and B exist and $A = B$, then C exists.
○ If C exists, then both A and B exist.
○ If C exists, then at least one of A or B exists.
○ None of the above statements are true.

(2e) Evaluate $I = \displaystyle\int_0^1 \int_0^x (x + 2y)\,dy\,dx$. Write your final numerical answer only.

$$\boxed{I = \frac{2}{3}}$$

$\int_0^1 xy + y^2 \Big|_0^x \, dx$
$= \int_0^1 x^2 + x^2$
$= 2\int_0^1 x^2 \, dx$
$= \frac{2}{3} x^3 \Big|_0^1$

3. (5 points) The parts of this question are unrelated. No justification is necessary.
Fill in ALL boxes that apply. If none apply, leave it blank. (unfilled ☐ filled ■)

(3a) Which of the following sets have zero Jordan measure?

- ☐ $B_1(0)$
- ☐ $\{(x,y)\in\mathbb{R}^2 : y=2x\}$
- ■ $\{1,2\}\times([0,1]\cap\mathbb{Q})$
- ■ $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=4, 2\le z\le 37\}$

(3b) Which of the following sets are Jordan measurable?

- ☐ $\mathbb{R}^n$
- ■ $\partial B_1(0)$
- ☐ $[3,4]^n\cap\mathbb{Q}^n$
- ■ $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2<4, 2\le z<37\}$

(3c) Which of the following functions f are integrable on the sets S?

- ■ $S=\{(x,y)\in\mathbb{R}^2 : 0<x^2+y^2\le 1\}$ and $f(x,y)=x^2+y^2$.
- ■ $S=\{(x,y)\in\mathbb{R}^2 : 0<x^2+y^2\le 1\}$ and $f(x,y)=1/(x^2+y^2)$.
- ☐ $S=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$ and $f(x,y)=237$ if $(x,y)\in\mathbb{Q}^2$ and $f(x,y)=0$ otherwise.
- ☐ $S=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$ and $f(x,y)=237$ if $(x,y)\in\mathbb{Q}^2$ and $f(x,y)=0$ otherwise.

(3d) Let $(\Omega,\Sigma,\mathbb{P})$ be a continuous probability space in $\mathbb{R}^2$ for selecting a vector uniformly inside the square $\Omega=[-8,8]^2$. Which statements are events occurring with probability zero?

- ☐ The vector lies inside the first quadrant. 25%
- ■ The vector has magnitude 1.
- ☐ The vector points downward but not directly down. 50%
- ■ The vector has two rational components.

(3e) Let $R=[a,b]\times[c,d]\times[e,f]$ be a rectangle in $\mathbb{R}^3$ and let $\varphi : R\to\mathbb{R}$ be bounded. According to Fubini's theorem, the identity

$$\int_R \varphi\, dV = \int_a^b \int_e^f \int_c^d \varphi(x,y,z)\,dy\,dz\,dx$$

holds and both quantities exist provided which assumption(s) hold? Select as few as possible.

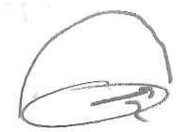
nah

- ■ φ is integrable on $[a,b]\times[c,d]\times[e,f]$.
- ■ For every $x\in[a,b]$, the x-slice φ^x is integrable on $[c,d]\times[e,f]$
- ■ For every $y\in[c,d]$, the y-slice φ^y is integrable on $[a,b]\times[e,f]$
- ■ For every $z\in[e,f]$, the z-slice φ^z is integrable on $[a,b]\times[c,d]$
- ■ For every $x\in[a,b], y\in[c,d]$, the (x,y)-slice $\varphi^{x,y}$ is integrable on $[e,f]$
- ■ For every $x\in[a,b], z\in[e,f]$, the (x,z)-slice $\varphi^{x,z}$ is integrable on $[c,d]$
- ■ For every $y\in[c,d], z\in[e,f]$, the (y,z)-slice $\varphi^{y,z}$ is integrable on $[a,b]$.

4. (8 points) The parts below are unrelated. No justification is necessary. **Do not evaluate any integral(s)**

(4a) Let $W = \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$ be a plate with continuous density $f : \mathbb{R}^2 \to \mathbb{R}$. Express the mass of W as an iterated double integral using polar coordinates with order $dr d\theta$.

$$\int_0^{2\pi} \int_1^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



(4b) Let $H = \{x^2 + y^2 + z^2 \le 4, z \ge 0\}$ be the solid upper hemisphere of radius 2. Express vol(H) as an iterated **double** integral with order $dydx$.

$0 \le z \le \sqrt{4-x^2-y^2}$

$f(x,y) = \sqrt{4-x^2-y^2}$

$y^2 \le 4-x^2$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\, dy\, dx$$

Express vol(H) as an iterated triple integral using cylindrical coordinates with order $dr d\theta dz$.

$$\int_0^2 \int_0^{2\pi} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} r\, dr\, d\theta\, dz$$

Express vol(H) as an iterated triple integral using spherical coordinates with order $d\rho d\theta d\phi$.

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$$

5. (6 points) Cheese is sliced in the shape of a parallelogram $P = \{(x,y) \in \mathbb{R}^2 : 0 \leq x - y \leq 3, 1 \leq x + y \leq 2\}$ with mass density $\delta(x,y) = (x+y)^2$. Use a change variables to transform P into a rectangle and find the total mass of the cheese. Briefly verify the assumptions of the change of variables theorem.

Let $u = x - y$ $\Rightarrow$ $0 \leq u \leq 3$
$v = x + y$ $\quad 1 \leq v \leq 2$. $\quad \delta(x,y) \to \delta(u,v) = v^2$.

$h(x,y) = (x-y, x+y) = (u,v)$.

Find inverse: $x = u + y$, $v = u + y + y \Rightarrow \boxed{y = \frac{v-u}{2}, \quad x = \frac{u+v}{2}.}$

$g(u,v) = h^{-1}(x,y) = \left(\frac{v-u}{2}, \frac{u+v}{2}\right) = (x,y)$.

note g is bijective, C^1, and $\det(Dg) \neq 0$:

$|Dg| = \frac{1}{2}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = \frac{1}{2}(-1-1) = -1$. Then $P = g(\Omega)$ for $\Omega = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq 3, 1 \leq v \leq 2\}$.

Then $\int_P \delta(x,y)\, dA = \int_\Omega \delta \circ g \, |\cancel{-}1| \, du\,dv$

$= \int_1^2 \int_0^3 v^2 \, du\,dv$

$= \int_1^2 v^2 u \Big|_0^3 \, dv = 3 \int_1^2 v^2 \, dv$

$= 3 \left(\frac{1}{3} v^3 \Big|_1^2\right)$

$= 8 - 1$

$\boxed{\text{mass}_\delta(P) = 7.}$

6. (6 points) Let $\Omega \subseteq \mathbb{R}^n$ be a set with an exhaustion by compact Jordan measurable sets. Let f and g be real-valued locally integrable functions on Ω. Assume that $0 \le f \le g$ on Ω and the improper integral $\int_\Omega g\,dV$ converges. Prove that the improper integral $\int_\Omega f\,dV$ converges.

$\{\Omega_K\}_{K=1}^{\infty}$ exhaustion, assume $0 \le f \le g$, $\lim_{K\to\infty} \int_{\Omega_K} g\,dV = \int_\Omega g\,dV$.

$\int_\Omega f\,dV$ converges if the limit $\lim_{K\to\infty} \int_{\Omega_K} f\,dV$ exists.

Since f, g locally integrable on Ω, they are locally integrable on each Ω_K.

~~proof:~~

~~• Assume $0 \le f \le g$ on Ω. By assumption, $\int_\Omega g\,dV$ converges.~~

~~• Note that since f and g are locally integrable on Ω,~~

pls ignore thx

proof:

- Let $\{\Omega_K\}_{K=1}^{\infty}$ be the exhaustion of Ω by compact Jordan measurable sets.
- Since each Ω_K is Jordan measurable and f and g are locally integrable, then both f and g are locally integrable on Ω_K.
- By assumption, $0 \le f \le g$ and $\int_\Omega g\,dV$ converges. This implies that the limit $\lim_{K\to\infty} \int_{\Omega_K} g\,dV$ exists, and let its value be M.
- We have that $0 \le f \le g$, which implies that for each Ω_K,

$0 \le \int_{\Omega_K} f\,dV \le \int_{\Omega_K} g\,dV$. Taking the limit of the expression as $K \to \infty$,

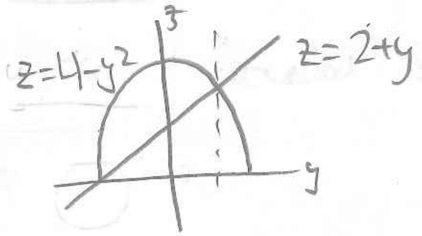
$0 \le \lim_{K\to\infty} \int_{\Omega_K} f\,dV \le \lim_{K\to\infty} \int_{\Omega_K} g\,dV = M$. Thus $0 \le \lim_{K\to\infty} \int_{\Omega_K} f\,dV = \int_\Omega f\,dV \le M$

or $0 \le \int_\Omega f\,dV \le M$. This implies that $\int_\Omega f\,dV$ is bounded above and below, so $\int_\Omega f\,dV$ converges (this is also a result from a comparison test). $\square$

7. (5 points) Consider the solid S above $z = 0$, below $z = y + 2$, and below the paraboloid $z = 4 - x^2 - y^2$. Use **slices** to set up the volume of S as a sum of iterated integral(s) with the order $dx\ dy\ dz$. Explain the geometry of your process, including the area of each slice and a labelled sketch of typical slice(s).

Perspectives of the solid are on the next page. Do not evaluate any integral(s).

Let us examine a x-slice at $x=0$:



The two points of intersection are given by $4-y^2 = 2+y \iff y = -2, 1$.

We wish to integrate first the x-slices from $-2 \le y \le 1$ and the integrate again from $1 \le y \le 2$, using $z = 2+y$ and $z = 4-y^2$ as the upper bounds, respectfully.

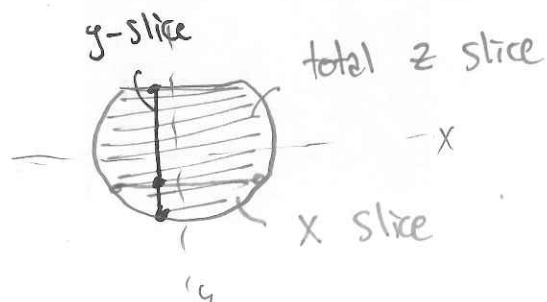
An x-slice is given by $-\sqrt{4-y^2-z} \le x \le \sqrt{4-y^2-z}$.

The y-slice will be, first, $z-2 \le y \le 1$ and second $1 \le y \le \sqrt{4-z}$.

We will then integrate over all of the z slices, going from 0 to 3, which is the "upper bound" of the volume:

$$\boxed{\mathrm{Vol}(S) = \int_0^3 \left[\int_{z-2}^{1} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx\,dy + \int_1^{\sqrt{4-z}} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx\,dy \right] dz}.$$

Typical slice:



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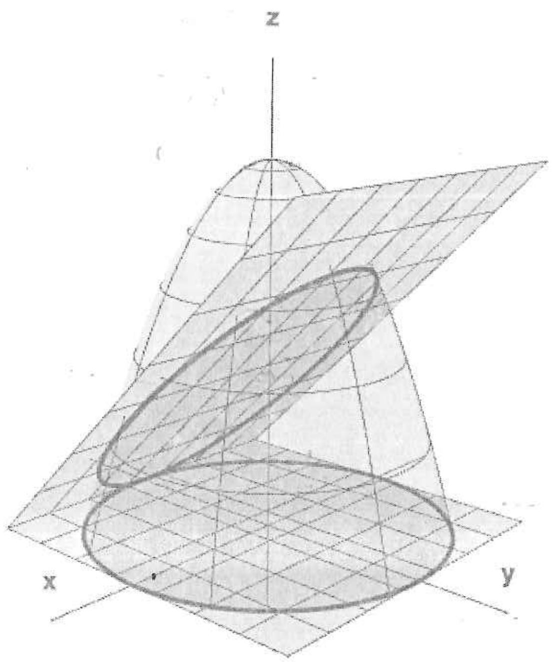

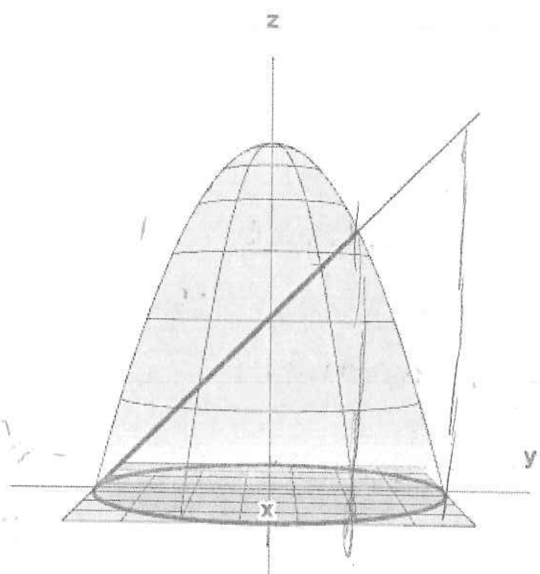

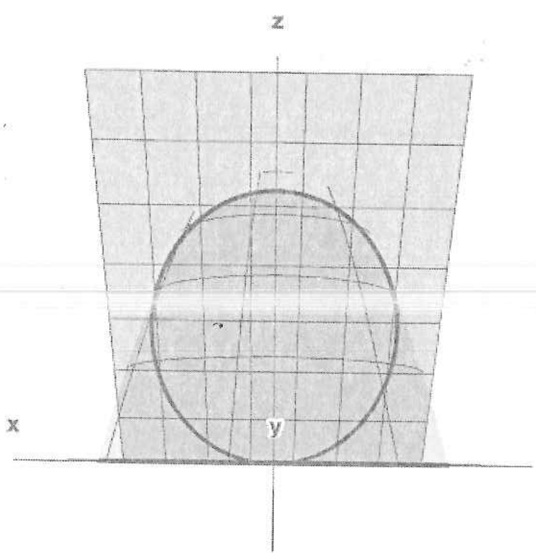

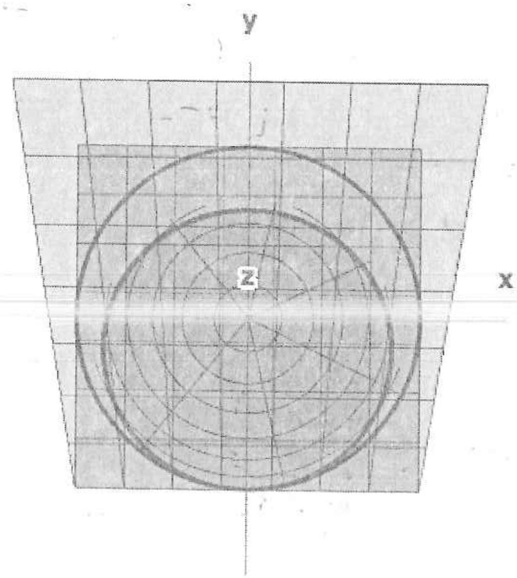

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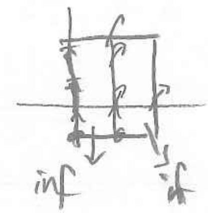
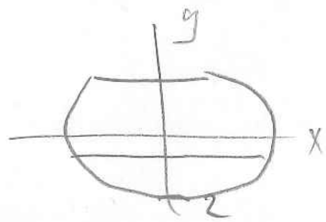


$f = x^2 + y$

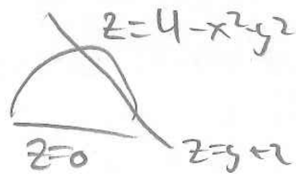
$\nabla f = (2x, 1)$

$f(0,-1) = -1$

$f(1,-1) = 0$



Break along x axis



Let c be fixed of z.

$x^2 = 4 - c - y^2$

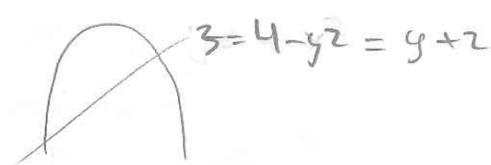
$-\sqrt{4-y^2-c} \leq x \leq \sqrt{4-y^2-c}$

now $y \geq 0$, $y \leq 0$.

$\int_0^2 \quad \int_{z-2}^0$

$$\int_0^4 \int_{z-2}^2 \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx\, dy\, dz.$$

$z = y + 2$



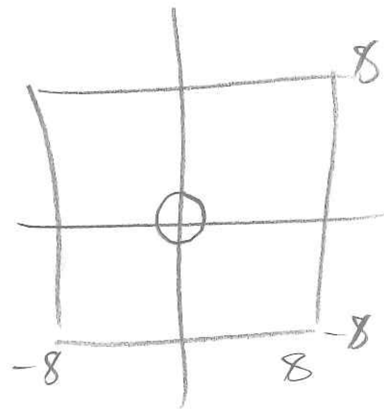
$2 = y + y^2$

$y^2 + y - 2 = 0$

$$y = \frac{-1 \pm \sqrt{1^2 + 4(1)(2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}$$

$\boxed{y_1 = 1}$, $y_2 = -2$

Do not tear this page off. This page is for additional work and will not be graded, unless you clearly indicate it on the original question page.



prob of vector being on unit circle

$16 \cdot 16 = (10+6)\cdot(10+6) = 100 + 36 + 20\cdot 6 = 100 + 120 + 36 = 256.$

2π