PHY256 - Problem Set 1

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1. Let the two matrices be $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

A : The characteristic polynomaial f A is given by $\det(A - \lambda I) = \begin{pmatrix} 0-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (-\lambda)(2-\lambda) - (0)^2 = \lambda(\lambda - 2)$.
Thus $\lambda_1 = 0$ and $\lambda_2 = 2$. A is already a purely diagonal matrix.

For $\lambda = 0$, $\ker(A - 0I) = \begin{pmatrix} 0-0 & 0 \\ 0 & 2-0 \end{pmatrix} = \ker \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. After row reducing, we have the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $x = t$, and thus the solution set is $t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Therefore the eigenvector with eigenvalue $\lambda = 0$ for the matrix A is every vector in the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.
For $\lambda = 2$, we have that $\ker(A - 2I) = \begin{pmatrix} 0-2 & 0 \\ 0 & 2-2 \end{pmatrix} = \ker \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$. Once again after row reducing, we have the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $y = t$, and thus the solution set is $t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Therefore the eigenvector with eigenvalue $\lambda = 2$ for the matrix A is every vector in the basis $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

A^2 : The matrix $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Simply, the eigenvalue for A^2 is 0 with a multiplicity of 2.

The eigenvectors for A^2 are given by $\ker(A^2 - 0I)$. Therefore the eigenvector with eigenvalue $\lambda = 0$ with algebraic multiplicity 2 for the matrix A^2 is any vector in the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ if we let $x = t$ and $y = s$.

B : The characteristic polynomial of B is given by $\det(B - \lambda I) = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(2-\lambda)$.
Thus $\lambda_1 = 0$ and $\lambda_2 = 2$. B will be diagonalized once the eigenvectors are found.

Then $\ker(B-0I) = \begin{pmatrix} 1-0 & 1 \\ 1 & 1-0 \end{pmatrix} = \ker\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. After row reducing, we have $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. If we paratremize $y = t$, then $x = -t$, thus every eigenvalue with eigenvector $\lambda = 0$ for the matrix B is every vector in the basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 2$, we have $\ker(B-2I) = \begin{pmatrix} 1-2 & 1 \\ 1 & 1-2 \end{pmatrix} = \ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. After row reducing, we have $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. If we paratremize $y = t$, then $x = t$, thus every eigenvalue with eigenvector $\lambda = 2$ for the matrix B is every vector in the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Diagonalizing B, we have $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

B^2 : The matrix $B^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The eigenvalues for B^2 are given by $\det(B^2 - \lambda I) = \begin{pmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 4 = \lambda(-4)$, and thus the eigenvalues for B^2 are $\lambda = 0$ and $\lambda = 4$.

The eigenvector associated with $\lambda = 0$ is given by $\ker\begin{pmatrix} 2-0 & 2 \\ 2 & 2-0 \end{pmatrix}$. After row reducing, we have the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Paratremizing, let $y = t$. Then $x = -t$. Thus, the eigenvector with eigenvalue $\lambda = 0$ for the matrix B^2 is any vector in the basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Lastly, for $\lambda = 4$, we have $\ker\begin{pmatrix} 2-4 & 2 \\ 2 & 2-4 \end{pmatrix} = \ker\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$. After row reducing, we have $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Let $y = t$, then $x = t$. Thus, the eigenvector with eigenvalue $\lambda = 4$ for the matrix B^2 is any vector in the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Diagonalizing B^2, we have $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

2. (a) Given: $\lambda = 633\text{nm}, h, c$.

The energy of a single photon emitted from the laser is given by $E = h\nu$. This is equivalent to $E = \dfrac{hc}{\lambda}$, since $c = \lambda\nu \implies \nu = \dfrac{c}{\lambda}$.

Therefore $E = \dfrac{(6.63 \times 10^{-34} J \cdot s)(3.00 \times 10^{8}\, m/s)}{633 \times 10^{-9} m} = 3.14 \times 10^{-19} J$.

(b) Given: $1\text{eV} = 1.6 \times 10^{-19} J$.

Thus, we have that $3.14 \times 10^{-19} J \times \left[\dfrac{1\text{eV}}{1.6 \times 10^{-19} J}\right] = 1.96\text{eV}$.

(c) Given: $E_{1photon} = 3.14218... \times 10^{-19} J \cdot s,\ P = 5\text{mW}, \Delta t = 10\text{s}$.

$P = \dfrac{\Delta E}{\Delta t} \implies \Delta E = P\Delta t$. Similarly, $\Delta E = n_{photons} \cdot E_{1photon}$, so $n = \dfrac{P\Delta t}{E_{1photon}} = \dfrac{P\Delta t\lambda}{hc}$.

For energy hitting the paper, we have that $\Delta E = P\Delta t$.

Thus $\Delta E = (0.005\text{mW})(10\text{s}) = 0.05 J$.

We have $n = \dfrac{(0.005\text{W})(10\text{s})}{3.14218... \times 10^{-19} J} = 1.59 \times 10^{17}$ photons.

3. $|\psi_1\rangle = \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle\,, |\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle\,, |\psi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|-\rangle\,.$

(a)

- Find a normalized orthogonal $|\phi_1\rangle$. Let $|\phi_1\rangle = |+\rangle + a|-\rangle\,, a \in \mathbb{C}$.

  $\langle\phi_1|\psi_1\rangle = \frac{1}{\sqrt{3}}\langle +|+\rangle - ai\frac{\sqrt{2}}{\sqrt{3}}\langle -|-\rangle = 0 \implies 1 = ai\sqrt{2}$, or $a = \dfrac{1}{i\sqrt{2}} \cdot \dfrac{i}{i} = \dfrac{-i}{\sqrt{2}}$.

  Now we must normalize $|\phi_1\rangle$:

  $\langle\phi_1|\phi_1\rangle = C^2\left[\langle +|+\rangle + \left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right)\langle -|-\rangle\right] = 1$

  $\implies C^2[1+\frac{1}{2}] = 1 \implies C^2 = \frac{2}{3} \implies C = \frac{\sqrt{2}}{\sqrt{3}}$.

  Therefore $|\phi_1\rangle = \frac{\sqrt{2}}{\sqrt{3}}|+\rangle - \frac{i}{\sqrt{3}}|-\rangle$.

- Find a normalized orthogonal $|\phi_2\rangle$. Let $|\phi_2\rangle = |+\rangle\, ba\,|-\rangle\,, b \in \mathbb{R}$ since $|\psi_2\rangle$ is real.

  $\langle\phi_2|\psi_2\rangle = \frac{1}{\sqrt{5}}\langle +|+\rangle - b\frac{2}{\sqrt{5}}\langle -|-\rangle = 0 \implies \frac{1}{\sqrt{5}} - \frac{2b}{\sqrt{5}}$, or $1 = 2b \implies b = \dfrac{1}{2}$.

  Now we must normalize $|\phi_2\rangle$:

  $\langle\phi_2|\phi_2\rangle = C^2\left[\langle +|+\rangle + \left(\frac{1}{2}\right)^2\langle -|-\rangle\right] = 1$

  $\implies C^2[1+\frac{1}{4}] = 1 \implies C^2\frac{5}{4} = 1 \implies C = \frac{2}{\sqrt{5}}$.

  Therefore $|\phi_2\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle$.

- Find a normalized orthogonal $|\phi_3\rangle$. Let $|\phi_3\rangle = |+\rangle + e^{i\alpha}|-\rangle\,, \alpha \in \mathbb{R}$.

  $\langle\phi_3|\psi_3\rangle = \frac{1}{\sqrt{2}}\langle +|+\rangle + e^{i\pi/4}e^{-i\alpha}\frac{1}{\sqrt{2}}\langle -|-\rangle = 0 \implies e^{i(\pi/4-\alpha)} = -1$, which by Eulers formula we have that $\pi/4 - \alpha = \pi$, and thus $\alpha = -3\pi/4$.

  Now we must normalize $|\phi_3\rangle$:

  $\langle\phi_3|\phi_3\rangle = C^2\left[\langle +|+\rangle + e^{i3\pi/4}e^{-i3\pi/4}\langle -|-\rangle\right] = 1$

  $\implies C^2[1+1] = 1 \implies C^2 = \frac{1}{2} \implies C = \frac{1}{\sqrt{2}}$.

  Therefore $|\phi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + e^{-i3\pi/4}\frac{1}{\sqrt{2}}|-\rangle$.

(b) Recall that for two states $|\phi_i\rangle$ and $|\psi_i\rangle$, $\langle\psi_i|\phi_i\rangle = \langle\phi_i|\psi_i\rangle^*$.

- $\langle\psi_1|\psi_2\rangle = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{5}}\langle +|+\rangle + \left(-i\frac{\sqrt{2}}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{5}}\right)\langle -|-\rangle$

  which is $\langle\psi_1|\psi_2\rangle = \frac{1}{\sqrt{15}} - i\frac{2\sqrt{2}}{\sqrt{15}}$.

- Similarly, then $\langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle^* = \frac{1}{\sqrt{15}} + i\frac{2\sqrt{2}}{\sqrt{15}}$.

- $\langle\psi_1|\psi_3\rangle = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} \langle+|+\rangle + \left(-i\frac{\sqrt{2}}{\sqrt{3}}\right)\left(e^{i\pi/4}\frac{1}{\sqrt{2}}\right)\langle-|-\rangle$. From Eulers formula, $-i = e^{i3\pi/2}$.
  Thus $\langle\psi_1|\psi_3\rangle = \frac{1}{\sqrt{6}} + e^{i3\pi/2+i\pi/4}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{6}} + e^{i7\pi/4}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{6}} + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.
  Therefore $\langle\psi_1|\psi_3\rangle = \frac{\sqrt{6}+3\sqrt{2}}{6} - i\frac{\sqrt{2}}{2}$.

- Similarly, then $\langle\psi_3|\psi_1\rangle = \langle\psi_1|\psi_3\rangle^* = \frac{\sqrt{6}+3\sqrt{2}}{6} + i\frac{\sqrt{2}}{2}$.

- $\langle\psi_2|\psi_3\rangle = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{2}} \langle+|+\rangle + e^{i\pi/4}\frac{2}{\sqrt{5}\cdot\sqrt{2}}\langle-|-\rangle$. From Eulers formula, $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = e^{i\pi/4}$.
  This implies that $\langle\psi_2|\psi_3\rangle = \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] \implies \frac{1+\sqrt{5}}{\sqrt{10}} + i\frac{\sqrt{2}}{\sqrt{10}}$.
  Therefore $\langle\psi_2|\psi_3\rangle = \frac{1+\sqrt{2}}{\sqrt{10}} + i\frac{\sqrt{5}}{5}$.

- Similarly, then $\langle\psi_3|\psi_2\rangle = \langle\psi_2|\psi_3\rangle^* = \frac{1+\sqrt{2}}{\sqrt{10}} - i\frac{\sqrt{5}}{5}$.

- Lastly, $\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = \langle\psi_3|\psi_3\rangle = 1$.

4. **(a)** On one hand, we have

$$
\begin{aligned}
|\langle R|\psi\rangle|^2 &= \left| \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} \right|^2 \\
&= \left(\frac{a}{\sqrt{2}} - \frac{ibe^{i\phi}}{\sqrt{2}} \right) \left(\frac{a}{\sqrt{2}} + \frac{ibe^{-i\phi}}{\sqrt{2}} \right) \\
&= \frac{a^2}{2} + \frac{b^2}{2} + \frac{abie^{-i\phi}}{2} - \frac{abie^{i\phi}}{2} \\
&= \frac{1}{4} \left[a^2 + b^2 + abi[e^{-i\phi} - e^{i\phi}] \right]
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
|\langle L|\psi\rangle|^2 &= \left| \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} \right|^2 \\
&= \left(\frac{a}{\sqrt{2}} + \frac{ibe^{i\phi}}{\sqrt{2}} \right) \left(\frac{a}{\sqrt{2}} - \frac{ibe^{-i\phi}}{\sqrt{2}} \right) \\
&= \frac{a^2}{2} + \frac{b^2}{2} + \frac{abie^{i\phi}}{2} - \frac{abie^{-i\phi}}{2} \\
&= \frac{1}{4} \left[a^2 + b^2 + abi[e^{i\phi} - e^{-i\phi}] \right]
\end{aligned}
$$

After cancelling some terms, we have that

$$
\begin{aligned}
& e^{-i\phi} - e^{i\phi} = e^{i\phi} - e^{-i\phi} \\
\implies & e^{i\phi} = e^{-i\phi} \\
\implies & i \sin\phi = -i \sin\phi \implies \sin\phi = 0 \\
\implies & \phi = k\pi \text{ for some } k \in \mathbb{Z}.
\end{aligned}
$$

We cannot conclude anything else about a or b just from this information because those terms cancel out in the calculation, however we can conclude that ϕ must be an integer multiple of π.

(b) Since we know $\phi = k\pi$ for some $k \in \mathbb{Z}$, then $e^{i\phi} = e^{-i\phi} = -1$ by Eulers formula. On one hand, we have

$$
\begin{aligned}
|\langle 45^o|\psi\rangle|^2 &= \left| \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix} \right|^2 \\
&= \left[\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} \right]^2
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
|\langle -45^o|\psi\rangle|^2 &= \left| \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix} \right|^2 \\
&= \left[\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \right]^2
\end{aligned}
$$

After taking the square root of both sides and cancelling some terms, we have that

$$\frac{b}{\sqrt{2}} = -\frac{b}{\sqrt{2}}$$
$$\implies b = -b$$
$$\implies b = 0.$$

From this, we can conclude that $a = 1$ in order for $|\psi\rangle$ to be normalized. Therefore our polarization state of the photon is given by

$$|\psi\rangle = |H\rangle .$$

(c) We can conclude that the photon is horizontally polarized, since our polarization state is $|\psi\rangle = |H\rangle$.

5. **(a)** The $\mathcal{S}_x$ basis in terms of the $\mathcal{S}_z$ basis is

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right), \quad |-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right).$$

To find the probability along the $\mathcal{S}_{z+}$ axis, we take the magnitude of the inner product squared with $\langle +|$:

$$\begin{aligned}
\langle +|\psi\rangle &= \langle +| \left[\frac{2}{\sqrt{13}}\,|+\rangle_x + i\frac{3}{\sqrt{13}}\,|-\rangle_x\right] \\
&= \frac{2}{\sqrt{13}}\,\langle +|+\rangle_x + i\frac{3}{\sqrt{13}}\,\langle +|-\rangle_x \\
&= \frac{2}{\sqrt{13}}\,\langle +| \left[\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right] + i\frac{3}{\sqrt{13}}\,\langle +| \left[\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right] \\
&= \frac{2}{\sqrt{26}}\left[\langle +|+\rangle + \langle +|-\rangle\right] + i\frac{3}{\sqrt{26}}\left[\langle +|+\rangle - \langle +|-\rangle\right] \\
&= \frac{2}{\sqrt{26}} + i\frac{3}{\sqrt{26}} \\
|\langle +|\psi\rangle|^2 &= \left|\frac{2}{\sqrt{26}} + i\frac{3}{\sqrt{26}}\right|^2 = \left(\frac{2}{\sqrt{26}} + i\frac{3}{\sqrt{26}}\right)\cdot\left(\frac{2}{\sqrt{26}} - i\frac{3}{\sqrt{26}}\right) = \frac{4}{26} + \frac{9}{26} = \frac{1}{2}
\end{aligned}$$

Similarly, for the $\mathcal{S}_{z-}$ axis, we take the magnitude of the inner product squared with $\langle -|$:

$$\begin{aligned}
\langle -|\psi\rangle &= \langle -| \left[\frac{2}{\sqrt{13}}\,|+\rangle_x + i\frac{3}{\sqrt{13}}\,|-\rangle_x\right] \\
&= \frac{2}{\sqrt{13}}\,\langle -|+\rangle_x + i\frac{3}{\sqrt{13}}\,\langle -|-\rangle_x \\
&= \frac{2}{\sqrt{13}}\,\langle -| \left[\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right] + i\frac{3}{\sqrt{13}}\,\langle -| \left[\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right] \\
&= \frac{2}{\sqrt{26}}\left[\langle -|+\rangle + \langle -|-\rangle\right] + i\frac{3}{\sqrt{26}}\left[\langle -|+\rangle - \langle -|-\rangle\right] \\
&= \frac{2}{\sqrt{26}} - i\frac{3}{\sqrt{26}} \\
|\langle -|\psi\rangle|^2 &= \left|\frac{2}{\sqrt{26}} - i\frac{3}{\sqrt{26}}\right|^2 = \left(\frac{2}{\sqrt{26}} - i\frac{3}{\sqrt{26}}\right)\cdot\left(\frac{2}{\sqrt{26}} + i\frac{3}{\sqrt{26}}\right) = \frac{4}{26} + \frac{9}{26} = \frac{1}{2}
\end{aligned}$$

Therefore the possible measurements for the spin components along the $\mathcal{S}_z$ axis are either $|+\rangle$ (spin up z) or $|-\rangle$ (spin down z), each with a probability of 50%.

(b) To find the probability along the $\mathcal{S}_{x+}$ axis, we take the magnitude of the inner product squared with ${}_x\langle +|$:

$$\begin{aligned}
{}_x\langle + | \psi \rangle &= {}_x\langle + | \left[\frac{2}{\sqrt{13}} \left| + \right\rangle_x + i \frac{3}{\sqrt{13}} \left| - \right\rangle_x \right] \\
&= \frac{2}{\sqrt{13}} {}_x\langle + | + \rangle_x + \frac{3}{\sqrt{13}} {}_x\langle + | - \rangle_x \\
&= \frac{2}{\sqrt{13}} \\
|{}_x\langle + | \psi \rangle|^2 &= \frac{4}{13}
\end{aligned}$$

Similarly, to find the probability along the $\mathcal{S}_{x-}$ axis, we take the magnitude of the inner product squared with ${}_x\langle - |$:

$$\begin{aligned}
{}_x\langle - | \psi \rangle &= {}_x\langle - | \left[\frac{2}{\sqrt{13}} \left| + \right\rangle_x + i \frac{3}{\sqrt{13}} \left| - \right\rangle_x \right] \\
&= \frac{2}{\sqrt{13}} {}_x\langle - | + \rangle_x + i \frac{3}{\sqrt{13}} {}_x\langle - | - \rangle_x \\
&= i \frac{3}{\sqrt{13}} \\
|{}_x\langle + | \psi \rangle|^2 &= \frac{9}{13}
\end{aligned}$$

Therefore the possible measurements for the spin components along the $\mathcal{S}_x$ axis are either $\left| + \right\rangle_x$ (spin up x) with a probability of $\frac{4}{13} \approx 31\%$ or $\left| - \right\rangle_x$ (spin down x) with a probability of $\frac{9}{13} \approx 69\%$.

6. (a) We have that $|\langle +|\psi\rangle|^2 = 64\% = \frac{16}{25}$ and $|\langle -|\psi\rangle|^2 = 36\% = \frac{9}{25}$.

$$\begin{aligned}
|\langle +|\psi\rangle|^2 &= |\langle +|\, a\, |+\rangle + \langle +|\, b\, |-\rangle|^2 \\
&= |a\langle +|+\rangle + b\langle +|-\rangle|^2 \\
&= |a|^2 = \frac{16}{25} \\
\implies a &= \frac{4}{5} \text{ or } -\frac{4}{5} \text{ or } i\frac{4}{5} \text{ or } -i\frac{4}{5}.
\end{aligned}$$

For simplicity assume that $a = \frac{4}{5}$. Similarly, for spin down,

$$\begin{aligned}
|\langle -|\psi\rangle|^2 &= \left|\langle -|\, \frac{4}{5}\, |+\rangle + \langle -|\, b\, |-\rangle\right|^2 \\
&= \left|\frac{4}{5}\langle -|+\rangle + b\langle -|-\rangle\right|^2 \\
&= |b|^2 = \frac{9}{25} \\
\implies b &= \frac{3}{5} \text{ or } -\frac{3}{5} \text{ or } i\frac{3}{5} \text{ or } -i\frac{3}{5}.
\end{aligned}$$

Again for simplicity assume that $b = \frac{3}{5}$. Since $\frac{16}{25} + \frac{9}{25} = 1$, $|\psi\rangle$ is already normalized. Thus our state is:

$$|\psi\rangle = \frac{4}{5}|+\rangle + \frac{3}{5}|-\rangle\,.$$

(b) By measuring one state (spin up or spin down along the $\mathbf{x}$ component), we will receive the probability of both states since $|_x\langle +|\psi\rangle|^2 + |_x\langle -|\psi\rangle|^2 = 1$.
The $\mathcal{S}_x$ basis in terms of the $\mathcal{S}_z$ basis is given by

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right), \quad |-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right).$$

We have:

$$
\begin{aligned}
{}_x\langle +|\psi\rangle &= {}_x\langle +|\, a\,|+\rangle + {}_x\langle +|\, b\,|-\rangle \\
&= a\,{}_x\langle +|+\rangle + b\,{}_x\langle +|-\rangle \\
&= a\left[\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right]|+\rangle + b\left[\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|\right]|-\rangle \\
&= \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \\
\implies |{}_x\langle +|\psi\rangle|^2 &= \left|\frac{a+b}{\sqrt{2}}\right|^2 \\
&= \frac{1}{2}|a+b|^2 \,\%\text{ spin up } \mathbf{x} \\
\implies |{}_x\langle -|\psi\rangle|^2 &= 1 - \frac{1}{2}|a+b|^2 \,\%\text{ spin down } \mathbf{x}
\end{aligned}
$$

When $a = \frac{4}{5}$ and $b = \frac{3}{5}$, we have that $|{}_x\langle +|\psi\rangle|^2 = \left|\frac{4/5+3/5}{\sqrt{2}}\right|^2 = \frac{49}{50} = 98\%$.

Similarly for spin down $\mathbf{x}$, we have $|{}_x\langle +|\psi\rangle|^2 = 1 - \left|\frac{4/5+3/5}{\sqrt{2}}\right|^2 = 1 - \frac{49}{50} = \frac{1}{50} = 2\%$.

The maximum value for spin up $\mathbf{x}$ is 1, when $a = b$. Then the minimum value for spin down $\mathbf{x}$ is 0. The minimum value for spin up $\mathbf{x}$ is 0, when $a = -b$. Then the maximum value for spin down $\mathbf{x}$ is 1:

$$
\begin{aligned}
&\max\{|{}_x\langle +|\psi\rangle|^2\} = 1, \text{ when } a = b \\
&\min\{|{}_x\langle +|\psi\rangle|^2\} = 0, \text{ when } a = -b \\
&\max\{|{}_x\langle -|\psi\rangle|^2\} = 1, \text{ when } a = -b \\
&\min\{|{}_x\langle -|\psi\rangle|^2\} = 0, \text{ when } a = b
\end{aligned}
$$

(c) To measure the spin through the $\mathbf{x}$ analyzer of the 64% that went up, we take the inner product of the pre-measured state $|\psi\rangle_{z+} = \frac{4}{5}|+\rangle$ and ${}_x\langle +|$, which is just

$$
\begin{aligned}
|{}_x\langle +|\psi\rangle_{z+}|^2 &= \left|\frac{4}{5\sqrt{2}}\right|^2 \\
&= \frac{8}{25} = 32\% \quad |+\rangle \to |+\rangle_x .
\end{aligned}
$$

It is important to notice that 32% is half of the amount of atoms that went in, which was 64%. Thus $P(\pm x) = 50\%$ each.

(d) To measure the spin through the $\mathbf{x}$ analyzer of the 36% that went down, we take the inner product of the pre-measured state $|\psi\rangle_{z-} = \frac{3}{5}|-\rangle$ and ${}_x\langle+|$, which is just

$$\begin{aligned} |{}_x\langle+|\psi\rangle_{z-}|^2 &= \left|\frac{3}{5\sqrt{2}}\right|^2 \\ &= \frac{6}{50} = 18\% \quad |-\rangle \to |+\rangle_x \,. \end{aligned}$$

It is important to notice that 18% is half of the amount of atoms that went in, which was 26%. Thus $P(\pm x) = 50\%$ each.

(e) We have that $|\langle+|\psi\rangle|^2 = 50\% = \frac{1}{2}$ and $|\langle-|\psi\rangle|^2 = 50\% = \frac{1}{2}$.

$$\begin{aligned} |\langle+|\psi\rangle|^2 &= |\langle+|\,a\,|+\rangle + \langle+|\,b\,|-\rangle|^2 \\ &= |a\langle+|+\rangle + b\langle+|-\rangle|^2 \\ &= |a|^2 = \frac{1}{2} \\ \implies a &= \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}} \text{ or } \frac{i}{\sqrt{2}} \text{ or } -\frac{i}{\sqrt{2}}. \end{aligned}$$

For simplicity assume that $a = \frac{1}{\sqrt{2}}$. Similarly, for spin down,

$$\begin{aligned} |\langle-|\psi\rangle|^2 &= \left|\langle-|\,\frac{1}{\sqrt{2}}\,|+\rangle + \langle-|\,b\,|-\rangle\right|^2 \\ &= \left|\frac{1}{\sqrt{2}}\langle-|+\rangle + b\langle-|-\rangle\right|^2 \\ &= |b|^2 = \frac{9}{25} \\ \implies b &= \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}} \text{ or } \frac{i}{\sqrt{2}} \text{ or } -\frac{i}{\sqrt{2}}. \end{aligned}$$

Again for simplicity assume that $b = \frac{1}{\sqrt{2}}$. Since $\frac{1}{2} + \frac{1}{2} = 1$, $|\psi\rangle$ is already normalized. Thus our state is:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\,.$$

By measuring one state (spin up or spin down along the $\mathbf{x}$ component), we will receive the probability of both states since $|{}_x\langle+|\psi\rangle|^2 + |{}_x\langle-|\psi\rangle|^2 = 1$.
The $\mathcal{S}_x$ basis in terms of the $\mathcal{S}_z$ basis is given by

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \quad |-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\,.$$

We have:

$$\begin{aligned}
{}_x\langle +|\psi\rangle &= {}_x\langle +|\, a\,|+\rangle + {}_x\langle +|\, b\,|-\rangle \\
&= a\,{}_x\langle +|+\rangle + b\,{}_x\langle +|-\rangle \\
&= a\left[\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right]|+\rangle + b\left[\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|\right]|-\rangle \\
&= \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \\
\implies |{}_x\langle +|\psi\rangle|^2 &= \left|\frac{a+b}{\sqrt{2}}\right|^2 \\
&= \frac{1}{2}|a+b|^2 \,\%\ \text{spin up } \mathbf{x} \\
\implies |{}_x\langle -|\psi\rangle|^2 &= 1 - \frac{1}{2}|a+b|^2 \,\%\ \text{spin down } \mathbf{x}
\end{aligned}$$

When $a = b = \frac{1}{\sqrt{2}}$, we have that $|{}_x\langle +|\psi\rangle|^2 = \left|\frac{1+1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$.

Similarly for spin down $\mathbf{x}$, we have $|{}_x\langle +|\psi\rangle|^2 = 1 - \left|\frac{1+1}{\sqrt{2}}\right|^2 = 1 - \frac{1}{2} = \frac{1}{2} = 50\%$.

The maximum value for spin up $\mathbf{x}$ is 1, when $a = b$. Then the minimum value for spin down $\mathbf{x}$ is 0. The minimum value for spin up $\mathbf{x}$ is 0, when $a = -b$. Then the maximum value for spin down $\mathbf{x}$ is 1:

$$\begin{aligned}
&\max\{|{}_x\langle +|\psi\rangle|^2\} = 1, \ \text{when } a = b \\
&\min\{|{}_x\langle +|\psi\rangle|^2\} = 0, \ \text{when } a = -b \\
&\max\{|{}_x\langle -|\psi\rangle|^2\} = 1, \ \text{when } a = -b \\
&\min\{|{}_x\langle -|\psi\rangle|^2\} = 0, \ \text{when } a = b
\end{aligned}$$