PHY483 Problem Set 1

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Problem 1

(a) Consider a Lorentz transformation in the x^1-direction. For two observers, with relative velocity v between them, an event position $(x^{0'}, x^{1'}, 0, 0)$ is related in the unprimed frame $(x^0, x^1, 0, 0)$ as

$$x^1 = vx^0 + x^{1'}\sqrt{1-v^2} \tag{1.1}$$

where vx^0 follows from origin displacement after a time x^0, and $x^{1'}\sqrt{1-v^2}$ follows from length contraction of a difference x^0 of an event in the unprimed frame. With the use of the invariant interval

$$ds^2 = (dx^0)^2 - (dx^1)^2 = (dx^{0'})^2 - (dx^{1'})^2 \tag{1.2}$$

one finds

$$x^{1'} = \gamma(x^1 - vx^0) \tag{1.3}$$
$$x^{0'} = \gamma(x^0 - vx^1) \tag{1.4}$$

We note that $dx^{2'} = dx^2$ and $dx^{3'} = dx^3$ because length is not contracted along an axis which is independent of the boost. Hence

$$dx^0 = \gamma(dx^{0'} + vdx^{1'}) \tag{1.5}$$
$$dx^1 = \gamma(dx^{1'} + vdx^{0'}) \tag{1.6}$$
$$dx^2 = dx^{2'} \tag{1.7}$$
$$dx^3 = dx^{3'} \tag{1.8}$$

I have just taken the inverse Lorentz transformation of (1.3-1.4).
Since coordinates transform as $dx^\mu = \dfrac{\partial x^\mu}{\partial x^{\nu'}}dx^{\nu'}$, we can use (1.5 - 1.8) to obtain the $\dfrac{\partial x^\mu}{\partial x^{\nu'}} \equiv \Lambda^\mu_\nu$ tensor elements. Hence we obtain

$$\Lambda^0_0 = \Lambda^1_1 = \gamma \tag{1.9}$$
$$\Lambda^0_1 = \Lambda^1_0 = \gamma v \tag{1.10}$$
$$\Lambda^2_2 = \Lambda^3_3 = 1 \tag{1.11}$$

while all other elements must be zero. These can be represented in matrix form as

$$\Lambda\nu^\mu = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.12}$$

We now consider the covariant antisymmetric field tensor (in matrix representation)

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{1.13}$$

and examine how each of the components transform under (1.12). I will only be noting the non-zero components of both tensors. Namely, any $\mu = \nu$ component is zero in (1.13). If $\mu, \nu > 1$, we also obtain zero from (1.12) else $\mu = \nu > 1$ in (1.12) and not in (1.13).

$$\begin{aligned}
E_1' = F_{01}' &= \Lambda_0^\alpha \Lambda_1^\beta F_{\alpha\beta} && (1.14)\\
&= \Lambda_0^0 \Lambda_1^1 F_{01} + \Lambda_0^1 \Lambda_1^0 F_{10} && (1.15)\\
&= \gamma^2 E_1 + v^2\gamma^2(-E_1) && (1.16)\\
&= E_1 && (1.17)\\
E_2' = F_{02}' &= \Lambda_0^\alpha \Lambda_2^2 F_{\alpha\beta} && (1.18)\\
&= \Lambda_0^0 \Lambda_2^2 F_{02} + \Lambda_0^1 \Lambda_2^2 F_{12} && (1.19)\\
&= \gamma E_2 + \gamma v(-B_3) && (1.20)\\
&= \gamma(E_2 - vB_3) && (1.21)\\
E_3' = F_{03}' &= \Lambda_0^\alpha \Lambda_3^3 F_{\alpha 3} && (1.22)\\
&= \Lambda_0^0 \Lambda_3^3 F_{03} + \Lambda_0^1 \Lambda_3^3 F_{13} && (1.23)\\
&= \gamma E_3 + \gamma v B_2 && (1.24)\\
&= \gamma(E_3 + vB_2) && (1.25)
\end{aligned}$$

as desired for the $\mathbf{E}'$ fields. We can perform a similar calculation for $\mathbf{B}'$:

$$\begin{aligned}
B_1' = F_{32}' &= \Lambda_3^3 \Lambda_2^2 F_{32} && (1.26)\\
&= F_{32} = B_1 && (1.27)\\
B_2' = F_{13} &= \Lambda_1^\alpha \Lambda_3^3 F_{\alpha 3} && (1.28)\\
&= \Lambda_1^0 \Lambda_3^3 F_{03} + \Lambda_1^1 \Lambda_3^3 F_{13} && (1.29)\\
&= v\gamma E_3 + \gamma B_2 && (1.30)\\
&= \gamma(B_2 + vE_3) && (1.31)\\
B_3' = F_{21}' &= \Lambda_2^2 \Lambda_1^\beta F_{2\beta} && (1.32)\\
&= \Lambda_2^2 \Lambda_1^0 F_{20} + \Lambda_2^2 \Lambda_1^1 F_{21} && (1.33)\\
&= v\gamma(-E_2) + \gamma B_3 && (1.34)\\
&= \gamma(B_3 - vE_2) && (1.35)
\end{aligned}$$

This is the law for field transformations, not for regular 3-vectors. This is because $\mathbf{E}$ and $\mathbf{B}$ are related by (1.13), which are implicity the Maxwell equations (if you minimize the action $\sim F^{\mu\nu}F_{\mu\nu}$). You would get a different result if you were to apply this Lorentz transformation to a different 4-vector. The reason currents / charge distributions are altered under Λ_μ^ν are exactly because the fields satisfy Maxwell equations. For instance, under LT, we would expect a contraction along the boost axis, but the opposite occurs here, where the un-boosted axes are altered.

(**b**) Consider first a point charge. Gauss's law states that the electric field is given as

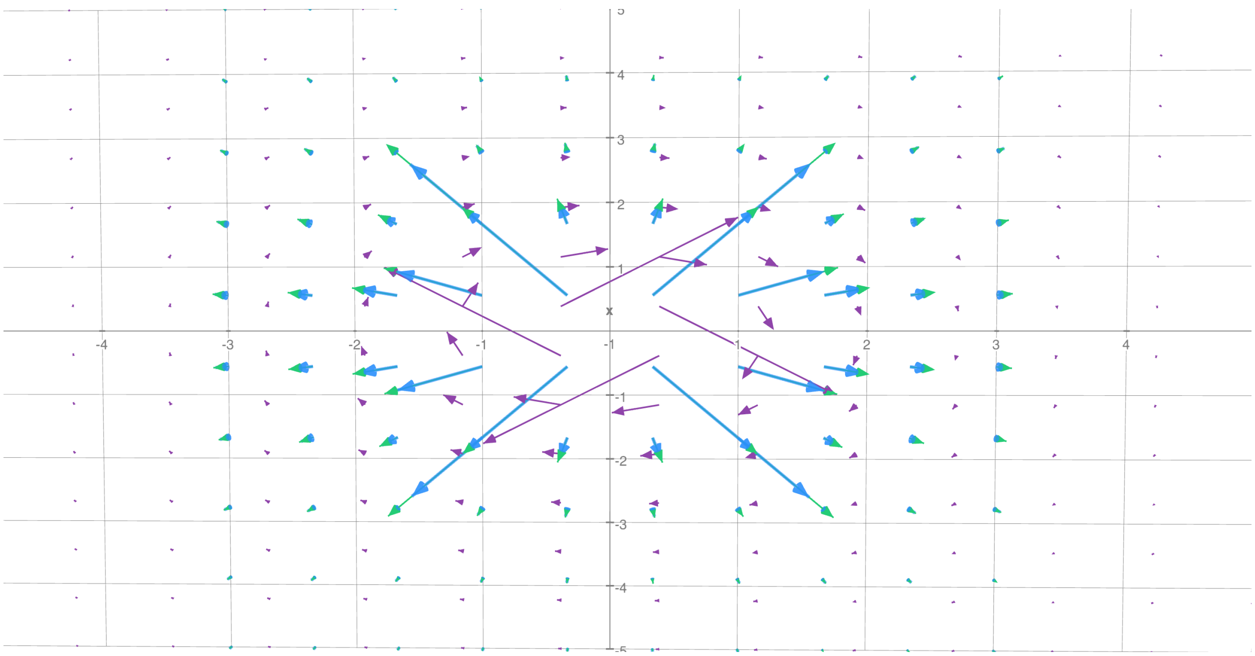
$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}} \tag{1.36}$$

and $\mathbf{B}(x,y,z) = \mathbf{0}$ everywhere, hence the transformed fields would become

$$\mathbf{E}'(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{x}} + \gamma y\hat{\mathbf{y}} + \gamma z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}} \tag{1.37}$$

$$\mathbf{B}'(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{v\gamma z\hat{\mathbf{y}} - v\gamma y\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}} \tag{1.38}$$

Since $\mathbf{E}'$ and $\mathbf{B}'$ don't change along the boost axes, I have plotted the initial and transformed fields in the x^2x^3-plane:



Here the blue field is (1.36), the untransformed field. The green field is (1.37), which is $\mathbf{E}'$, and the purple field is (1.38), the transformed $\mathbf{B}'$. Hence a stationary charge for a relativistically moving observer appears like a current, as expected, since the charge is now moving away from that frame. We further observe that the $\mathbf{E}$ field is magnified along the axes perpendicular to the boost axis. One may also produce similar results for a line current, say, also along the boost axis:

$$\mathbf{B}(r,\varphi,x) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}} = \frac{\mu_0 I}{2\pi r}[-\sin\varphi\hat{\mathbf{y}} + \cos\varphi\hat{\mathbf{z}}] \tag{1.39}$$

with $\hat{\mathbf{J}} \sim \hat{\mathbf{x}}$. Maxwell equations imply

$$\mathbf{E}(r,\varphi,x) = -\frac{\mu_0}{\epsilon_0}\mathbf{J}t \tag{1.40}$$

(easy to check that $\nabla \times \mathbf{E} = \mathbf{0} = -\frac{\partial \mathbf{B}}{\partial t}$). Hence, under an x^1 boost,

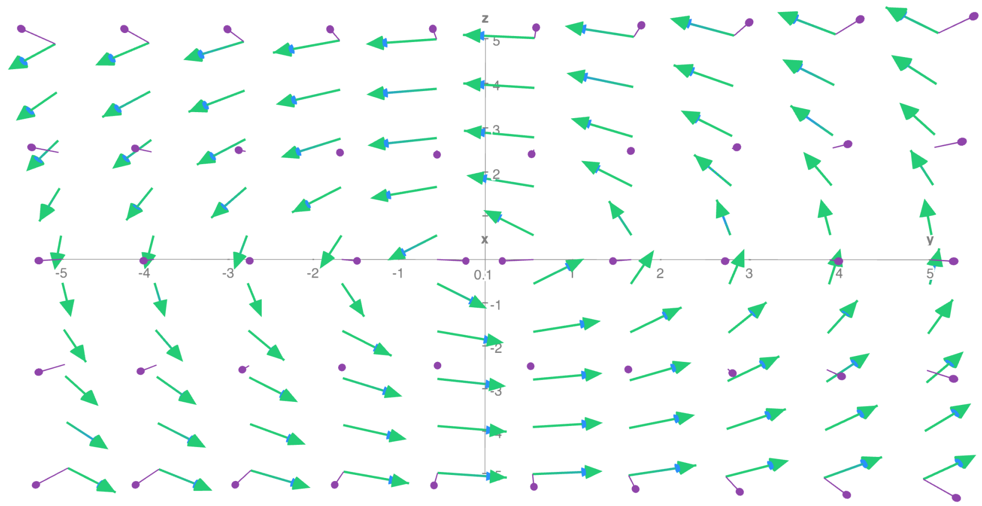
$$\mathbf{B}'(x,y,z) = \gamma\mathbf{B} \tag{1.41}$$

$$\mathbf{E}'(x,y,z) = -\frac{\mu_0}{\epsilon_0}\mathbf{J}\hat{\mathbf{x}} + \gamma(-v)\frac{\mu_0 I}{2\pi r}\cos\varphi\hat{\mathbf{y}} + \gamma v\frac{\mu_0 I}{2\pi r}(-\sin\varphi)\hat{\mathbf{z}} \tag{1.42}$$

$$= -\mu_0\left[\frac{\hat{\mathbf{J}}}{\epsilon_0} + \frac{\gamma v I}{2\pi r}\hat{\mathbf{r}}\right] \tag{1.43}$$

$$= E_0\hat{\mathbf{x}} - \gamma v|\mathbf{B}|\hat{\mathbf{r}} \tag{1.44}$$

Similarly, we obtain the plots of the fields



where again the blue field represents the untransformed magnetic field, the green represents the transformed magnetic field, and the purple field the resultant electric field from a boost along the x^1 axis (which is out of the page). We observe a reduction in magnitude along the curl of the magnetic field, and an altered electric field with a new inward field component in the radial direction, since we had previously seen the electric fields becoming "squished" inwards along a boost axis from the previous part.

Problem 2

(a) We begin by noting that, for a general acceleration along an axis in the IIRF, that $a^{1'}(\tau) = g(\tau)$, hence

$$a^1(\tau) = \gamma_v^{-3} g(\tau) \implies g(\tau) = \gamma_v^3 a^1(\tau). \tag{2.1}$$

If a general acceleration along the $\hat{\mathbf{1}}$ direction is (in the astronaut's frame)

$$a^1(\tau) = \alpha \sin\left(\frac{2\pi\tau}{T}\right) \tag{2.2}$$

then

$$g(\tau) = \alpha \sin\left(\frac{2\pi\tau}{T}\right) \tag{2.3}$$

(here I have relabelled our constant 'g' to α, for brevity, which will still have units of acceleration) The rapidity ζ along the $\hat{\mathbf{1}}$ direction is given as the integral over the proper time of $g(\tau)$,

$$\zeta(\tau) = \int_0^\tau d\sigma\, g(\sigma) \tag{2.4}$$

which implies that the velocity is

$$u^1(\tau) = \tanh \zeta(\tau). \tag{2.5}$$

(2.4) can be evaluated as

$$\begin{aligned}
\zeta(\tau) &= \alpha \int_0^\tau d\sigma\, \sin\left(\frac{2\pi\sigma}{T}\right) && (2.6)\\
&= \alpha \cdot \frac{T}{2\pi}\left[-\cos\left(\frac{2\pi\sigma}{T}\right)\right]_0^\tau && (2.7)\\
&= \frac{\alpha T}{2\pi}\left[1 - \cos\left(\frac{2\pi\tau}{T}\right)\right] && (2.8)
\end{aligned}$$

Since $u^1(\tau) = \tanh\zeta(\tau)$ (velocity along the axis direction), then we both have

$$\begin{aligned}
dt &= \cosh\zeta(\tau)\, d\tau \qquad \text{(time dilation formula)} && (2.8)\\
\implies dx^1 &= \sinh\zeta(\tau)\, d\tau. && (2.9)
\end{aligned}$$

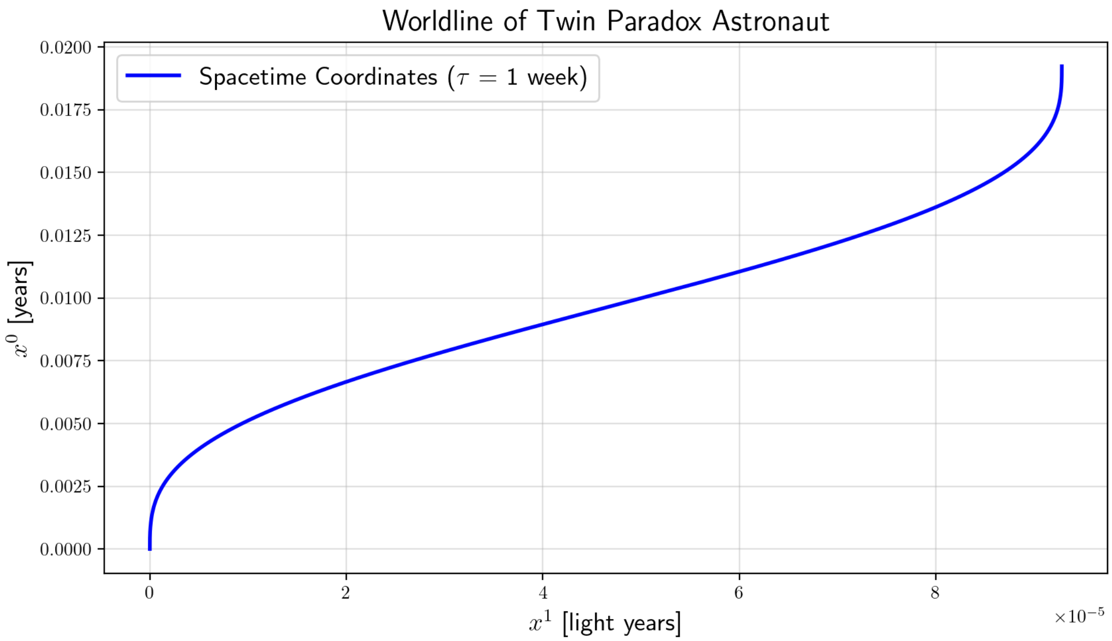
These can then be integrated to obtain the quadrature reduction of the explicit motion of the astronaut in spacetime coordinates, as a function of their proper time:

$$\begin{aligned}
t(\tau) &= \int_0^\tau d\sigma\, \cosh\left[\frac{\alpha T}{2\pi}\left[1 - \cos\left(\frac{2\pi\sigma}{T}\right)\right]\right] && (2.10)\\
x(\tau) &= \int_0^\tau d\sigma\, \sinh\left[\frac{\alpha T}{2\pi}\left[1 - \cos\left(\frac{2\pi\sigma}{T}\right)\right]\right]. && (2.11)
\end{aligned}$$

The velocity v is given as $\dfrac{dx^1}{dt} = \tanh\zeta(\tau)$, which is obtained by integrating (2.2):

$$v(\tau) = -\frac{\alpha T}{2\pi}\cos\left(\frac{2\pi\tau}{T}\right) \tag{2.12}$$

(**b**) We will numerically integrate using Simpson's rule (using quadratics to approximate the curve, then to integrate), since my computer usually integrates different values if I were to use Gaussian quadrature instead. I have attached the code in addition to my solutions. Here, I just took $g = 15\,\dfrac{\mathrm{m}^2}{\mathrm{s}} = 10 * (3600 * 24 * 365)^2 * 1.057 \times 10^{-16}\,\dfrac{\text{light years}}{\text{years}^2}$ (if it was around 9.81 $\mathrm{m/s}^2$, there wouldn't be that much of a difference, since gravity on earth is approximately 9.81 $\mathrm{m/s}^2$), and $T = 1/52$ years. This was plotted in `matplotlib.pyplot`, using τ as a 1-week parametrization.



The resulting time dilation was taken as two times the maximum value of x^0 in the homebody frame, minus the $1/52$ year time (1 week) that the astronaut experienced. The two is because we are assuming a round trip. The resulting output was about 21.19 s, that is, how much older the homebody frame would be.

Problem 3

The Twin's "Paradox":

As you have previously seen, an observer who travels at relativistic speeds close to the speed of light ($c \approx 2.99 \times 10^8$ m/s) experiences reality differently than someone who isn't moving. These differences are a result of Einstein's special theory of relativity (SR), a consequence being the fact that time is *not* absolute. Instead, the speed of light *is* absolute, meaning that it is the same speed regardless of which reference frame you are in. The coordinates of one frame of reference S are related to another, moving frame, S', by a set of rules called *Lorentz transformations*.

One way to quickly derive a Lorentz transformation, say, a 'boost' (which is a change of reference along an axis, not a rotation) along a certain axis (say, the $\hat{\mathbf{x}}$ axis) is to consider the frame S' moving at a speed $\mathbf{v} = v\hat{\mathbf{x}}$ relative to frame S, and to consider an event at a spacetime point $E' = (t', x', 0, 0)$ in S'. We wish to determine the representation of E' in S. The general *Galilean transformation* implies that

$$\begin{aligned} t &= t' \\ x &= x' + vt \\ y &= y' \\ z &= z' \qquad \text{(Galilean)} \end{aligned} \tag{2.1}$$

however, the Galilean transformation does not consider the maximum propagation of information in spacetime, which is c, and the fact that time is not absolute. A spherical wavefront of light, emitted at $t = 0$, travelling at speed c at a time t, must obey the equation of a sphere in $\mathbb{R}^3$ because it will travel in all directions equally after a time t with radius ct:

$$(ct)^2 - x^2 - y^2 - z^2 = 0 \tag{2.2}$$

which must be true in every frame, hence

$$(ct')^2 - x'^2 - y'^2 - z'^2 = 0 \tag{2.3}$$

Since the velocity of the boost is only in the $\hat{\mathbf{x}}$ direction, then following from (2.1) $y = y'$ and $z = z'$. Concatenating the wavefront relations yields an 'interval term' which is the same in all reference frames, and we call this the *invariant interval* s^2:

$$s^2 = (ct')^2 - x'^2 = (ct)^2 - x^2. \tag{2.4}$$

The interesting thing about s^2 is the relations of whether or not it is positive or negative, and how it determine the kind of seperation between various events, but we will not look at this here. Let's go back to our events E' in S' and in S. Thus, after a time t, S' will move a distance vt, plus a term accounting for the length contraction between the event E' and the origin of S, which is $x'\sqrt{1 - v^2/c^2}$. Hence, the event x' in S can be written as

$$x = vt + x'\sqrt{1 - v^2/c^2}. \tag{2.5}$$

Plugging this into the invariant interval (2.4) implies the relation between t and t':

$$(ct)^2 - (vt + x'\sqrt{1 - v^2/c^2})^2 = (ct')^2 - (x')^2 \implies t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \tag{2.6}$$

furthermore the relation for x,

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{2.7}$$

which give us the Lorentz transformations (this should just be review),

$$\begin{aligned} t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \\ x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \qquad \text{(Lorentz)} \end{aligned} \tag{2.8}$$

I now want to introduce you to a specific thought experiment, with the use of time dilation, called the *Twin's Paradox*. We first return to time dilation: the concept that time is slowed for moving observers. For a time-like seperated event ($t_2 - t_1 \equiv \Delta t$) in a frame S, which is stationary ($x_2 - x_1 = \Delta x = 0$), time is transformed according to (2.8) in a moving frame S' at speed v as

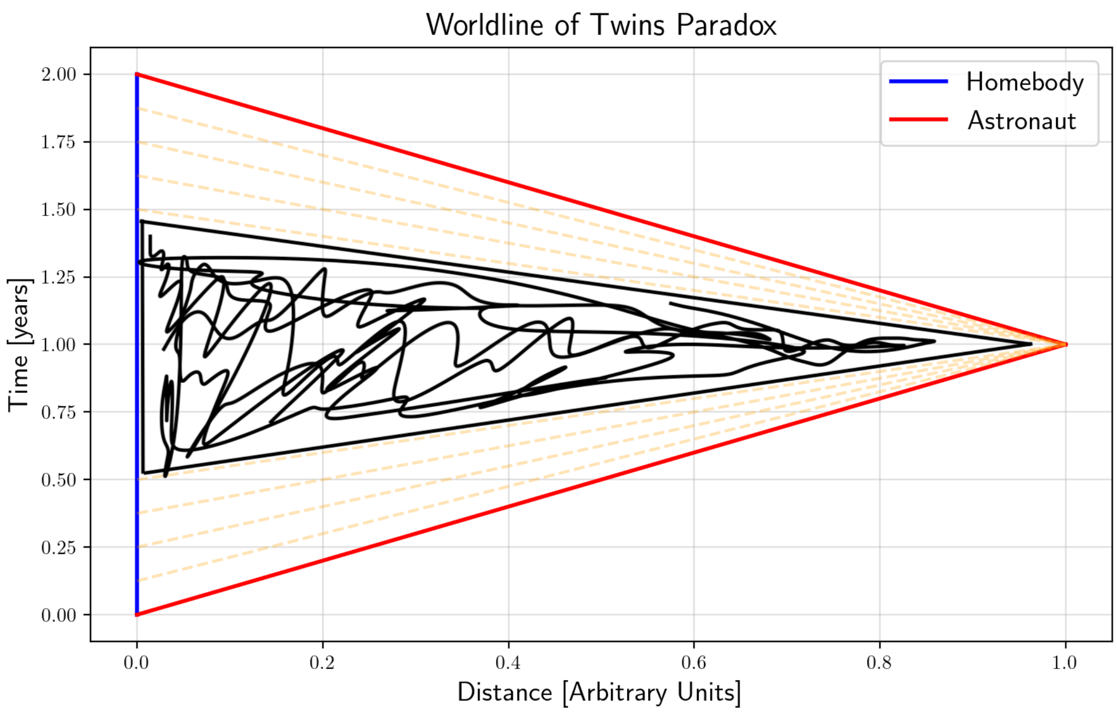
$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} \tag{2.9}$$

which is the relation for time dilation. That is, as v increases, $\gamma_v \equiv \dfrac{1}{\sqrt{1 - v^2/c^2}} \to 0$ in the $v \to c$ limit, so time actually *increases* for the observer in the moving frame S'!

Now consider two observers. Both observers, say, are twins born identically at the same time here on earth at the same location. Let me call them Sam and Jess. Now, let's suppose that Sam goes off and gets an advanced degree in physics and engineering and is recruited to pilot a spacecraft to a distant galaxy during the course of a two-year round trip. Since the planet is so far away, Sam is required to travel at a fraction of the speed of light to reach his destination on time. For generality, let's assume that Sam doesn't spend any time accelerating to the required speed, but decelerates temporarily to change direction for a couple days to collect samples from the distant planet before returning at the same speed. Sam's *worldline* over the two-year period may appear as

However, what is important to note that Sam is travelling a greater distance in the same amount of time at a higher speed, which implies that time must *slow down* for him in the frame of the homebody! Not only does this occur on his trip towards the planet, but also on his return trip. The paradox is that, from the perspective of Sam on the stationary spacecraft, the homebody is the frame which is moving, implying that from Sam's perspective, time slows down for Jess! However, once Sam arrives home again, he is actually younger than Jess would be.

The twins paradox in and of itself is not really a paradox, but appears as a paradoxical result of SR. The key difference between Sam and Jess is Sam must produce a negative acceleration to turn around and return home once he has reached his destination. In doing this, Sam experiences a time dilation forward in time, then backwards in time, skipping over a whole "region" of spacetime!



From the diagram above, for instance (I am greatly overexaggerate here), Sam skips an entire year $(1.5 - 0.5)$ of Jess's life, simply because his acceleration had changed directions. In reality this difference is a lot smaller. Altogether, the Twin's "paradox" is just a consequence of SR, result being due to time dilation as noted by the Lorentz transformations as derived above. There are more advanced mathematical explanations for this, involving rapidity measurements of acceleration integrals and general proper time transformations, these of which I will not mention here. (Note: I got the some of the help for this explanation from https://www.youtube.com/watch?v=0iJZ_QGMLD0, which is not my direct thinking).

Problem 4

(a) Consider the following Poisson bracket relations:

$$\{P_\mu, P_\nu\} = 0 \tag{4.1}$$
$$\{J_{\mu\nu}, P_\alpha\} = \eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu \tag{4.2}$$
$$\{J_{\mu\nu}, J_{\alpha\beta}\} = \eta_{\mu\alpha} J_{\nu\beta} - \eta_{\mu\beta} J_{\nu\alpha} + \eta_{\nu\beta} J_{\mu\alpha} - \eta_{\nu\alpha} J_{\mu\beta} \tag{4.3}$$

and define

$$W^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma} P_\delta. \tag{4.4}$$

Recall that the linearity of Poisson brackets, along with the product rule, implies (generally)

$$\{f + ag, h\} = \{f, h\} + a\{g, h\} \tag{4.5}$$
$$\{fg, h\} = \{f, h\}g + f\{g, h\} \tag{4.6}$$

Let $C_1 = P^\alpha P_\alpha$ be a contraction of momentum 4-vectors. If C_1 is an invariant of the Poincaré group, it has zero Poisson bracket with both P_μ and $J_{\mu\nu}$. We have, by (4.6),

$$\begin{aligned} \{P^\alpha P_\alpha, P_\mu\} &= \{\eta^{\alpha\beta} P_\beta P_\alpha, P_\mu\} && (4.7)\\ &= \eta^{\alpha\beta} \left[\{P_\beta, P_\mu\} P_\alpha + P_\beta \{P_\alpha, P_\mu\}\right] && (4.8)\\ &= 0 \end{aligned}$$

which follows from (4.1). Similarly,

$$\begin{aligned} \{P^\alpha P_\alpha, J_{\mu\nu}\} &= \eta^{\alpha\beta} [\{P_\beta, J_{\mu\nu}\} P_\alpha + P_\beta \{P_\alpha, J_{\mu\nu}\}] && (4.9)\\ &= \eta^{\alpha\beta} \left[(\eta_{\nu\beta} P_\mu - \eta_{\mu\beta} P_\nu) P_\alpha + P_\beta (\eta_{\nu\alpha} P_\mu - \eta_{\mu\alpha} P_\nu)\right] && (4.10)\\ &= \eta^{\alpha\beta} \eta_{\nu\beta} P_\mu P_\alpha - \eta^{\alpha\beta} \eta_{\mu\beta} P_\nu P_\alpha + \eta^{\alpha\beta} \eta_{\nu\alpha} P_\beta P_\mu - \eta^{\alpha\beta} \eta_{\mu\alpha} P_\beta P_\nu && (4.11)\\ &= P_\nu P_\mu - P_\nu P_\mu + P_\nu P_\mu - P_\mu P_\nu && (4.12)\\ &= 0. \end{aligned}$$

Hence C_1 is an invariant of the Poincaré group.

(b) Now let $C_2 = W^\alpha W_\alpha$. This contraction is equivalently

$$\begin{aligned} W^\alpha W_\alpha &= \frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma} P_\delta \epsilon_{\alpha\mu\nu\lambda} J^{\mu\nu} P^\lambda && (4.13)\\ &= -\frac{1}{4} \delta^{[\beta\gamma\delta]}_{\mu\nu\lambda} J_{\beta\gamma} P_\delta J^{\mu\nu} P^\lambda && (4.14) \end{aligned}$$

We find that (see part (c))

$$W^\alpha W_\alpha = -\frac{1}{2} J_{\mu\nu} P_\lambda J^{\mu\nu} P^\lambda + J_{\lambda\mu} J^{\mu\nu} P_\nu P^\lambda \tag{4.15}$$

Thus, the Poisson bracket is zero only if the bracket between the first and second terms are both zero, or equal but opposite in sign. For $\{C_2, P_\nu\}$, by lowering the indices a factor of $\eta^{\alpha\mu} \eta^{\beta\nu} \eta^{\lambda\gamma}$, we have

$$\{J_{\mu\nu} J_{\alpha\beta} P_\lambda P_\gamma, P_\sigma\} = \{J_{\mu\nu}, P_\sigma\} J_{\alpha\beta} P_\lambda P_\gamma + J_{\mu\nu} \{J_{\alpha\beta} P_\lambda P_\gamma, P_\sigma\} \tag{4.16}$$

$$
\begin{aligned}
&= \{J_{\mu\nu}, P_\sigma\}J_{\alpha\beta}P_\lambda P_\gamma + J_{\mu\nu}[\{J_{\alpha\beta}, P_\sigma\}P_\lambda P_\gamma + J_{\alpha\beta}\cancelto{0}{\{P_\lambda P_\gamma, P_\sigma\}}] && (4.17)\\
&= (\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu)J_{\alpha\beta}P_\lambda P_\gamma + J_{\mu\nu}P_\lambda P_\gamma(\eta_{\alpha\sigma}P_\beta - \eta_{\beta\sigma}P_\alpha) && (4.18)\\
&= \eta_{\mu\sigma}J_{\alpha\beta}P_\lambda P_\gamma P_\nu - \eta_{\nu\sigma}J_{\alpha\beta}P_\mu P_\lambda P_\gamma \\
&\qquad + \eta_{\alpha\sigma}J_{\mu\nu}P_\lambda P_\gamma P_\beta - \eta_{\beta\sigma}J_{\mu\nu}P_\lambda P_\gamma P_\alpha && (4.19)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\lambda\gamma}\{J_{\mu\nu}J_{\alpha\beta}P_\lambda P_\gamma, P_\sigma\} &= J_{\sigma\beta}P^\beta P^\gamma P_\gamma - J_{\alpha\sigma}P^\alpha P^\gamma P_\gamma + J_{\sigma\nu}P^\nu P^\gamma P_\gamma - J_{\mu\sigma}P^\mu P^\gamma P_\gamma && (4.20)\\
&= 4P_\lambda P^\lambda J_{\sigma\nu}P^\nu && (4.21)
\end{aligned}
$$

so the first term gives a factor of $-2J_{\sigma\nu}P^\nu P_\lambda P^\lambda$. Lowering the second term indices by $\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\lambda\gamma}$,

$$
\begin{aligned}
\{J_{\lambda\mu}J_{\alpha\beta}P_\nu P_\gamma, P_\sigma\} &= \eta_{\mu\sigma}J_{\alpha\beta}P_\lambda P_\nu P_\gamma - \eta_{\lambda\sigma}J_{\alpha\beta}P_\nu P_\gamma P_\mu \\
&\qquad + \eta_{\beta\sigma}J_{\lambda\mu}P_\nu P_\gamma P_\alpha - \eta_{\alpha\sigma}J_{\lambda\mu}P_\nu P_\gamma P_\beta && (4.22)
\end{aligned}
$$

(I just copied the formula from (4.19)) Hence

$$
\begin{aligned}
\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\lambda\gamma}\{J_{\lambda\mu}J_{\alpha\beta}P_\nu P_\gamma, P_\sigma\} &= J_{\sigma\beta}P^\beta P^\lambda P_\lambda - J^{\mu\nu}P_\nu P_\mu P_\sigma + J_{\mu\lambda}P^\mu P^\lambda P_\sigma - J_{\lambda\sigma}P^\lambda P^\nu P_\nu && (4.23)\\
&= (J^{\mu\nu}P_\mu P_\nu P_\sigma - J^{\mu\nu}P_\mu P_\nu P_\sigma) + 2J_{\sigma\beta}P^\beta P^\lambda P_\lambda && (4.24)\\
&= 0 + 2J_{\sigma\nu}P^\nu P_\lambda P^\lambda && (4.25)
\end{aligned}
$$

therefore

$$
\begin{aligned}
\{C_2, P_\sigma\} &= -2J_{\sigma\nu}P^\nu P_\lambda P^\lambda + 2J_{\sigma\nu}P^\nu P_\lambda P^\lambda && (4.26)\\
&= 0. && (4.27)
\end{aligned}
$$

Performing a similar calculation for the J Poisson bracket, it expands as

$$
\begin{aligned}
&\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\gamma\lambda}\{J_{\mu\nu}J_{\alpha\beta}P_\gamma P_\lambda, J_{\varphi\psi}\} \\
\implies \{J_{\mu\nu}J_{\alpha\beta}P_\gamma P_\lambda, J_{\varphi\psi}\} &= \{J_{\mu\nu}, J_{\varphi\psi}\}J_{\alpha\beta}P_\lambda P_\gamma + J_{\mu\nu}\Big[\{J_{\alpha\beta}, J_{\varphi\psi}\}P_\gamma P_\lambda \\
&\qquad + J_{\alpha\beta}[\{P_\gamma, J_{\varphi\psi}\}P_\gamma + P_\gamma\{P_\lambda, J_{\varphi\psi}\}]\Big] && (4.28)\\
&= \{J_{\mu\nu}, J_{\varphi\psi}\}J_{\alpha\beta}P_\gamma P_\lambda + \{J_{\alpha\beta}, J_{\varphi\psi}\}J_{\mu\nu}P_\gamma P_\lambda \\
&\qquad + \{P_\gamma, P_{\varphi\psi}\}J_{\mu\nu}J_{\alpha\beta}P_\gamma + \{P_\lambda, J_{\varphi\psi}\}J_{\mu\nu}J_{\alpha\beta}P_\gamma && (4.29)
\end{aligned}
$$

One can note that, upon re-indexing and grouping with the metric tensors, that the symmetry involves the terms cancelling out the like previous parts (I'm not doing this algebra. It's a waste of time). Thus

$$
\{C_2, J_{\varphi\psi}\} = 0. \tag{4.30}
$$

(c) When expanding the delta function, we find that

$$
\begin{aligned}
W^\alpha W_\alpha &= -\frac{1}{4}J_{\beta\gamma}P_\delta J^{\mu\nu}P^\lambda\left[\delta^\beta_\mu\delta^\gamma_\nu\delta^\delta_\lambda + \delta^\gamma_\mu\delta^\delta_\nu\delta^\beta_\lambda + \delta^\delta_\mu\delta^\beta_\nu\delta^\gamma_\lambda - \delta^\beta_\lambda\delta^\gamma_\nu\delta^\delta_\mu - \delta^\gamma_\lambda\delta^\delta_\nu\delta^\beta_\mu - \delta^\delta_\lambda\delta^\beta_\nu\delta^\gamma_\mu\right] && (4.31)\\
&= -\frac{1}{4}\Big[J_{\mu\nu}P_\lambda J^{\mu\nu}P^\lambda + J_{\lambda\mu}P_\nu J^{\mu\nu}P^\lambda + J_{\nu\lambda}P_\mu J^{\mu\nu}P^\lambda
\end{aligned}
$$

$$- J_{\lambda\nu}P_\mu J^{\mu\nu}P^\lambda - J_{\mu\lambda}P_\nu J^{\mu\nu}P^\lambda - J_{\nu\mu}P_\lambda J^{\mu\nu}P^\lambda\Big] \tag{4.32}$$

$$= -\frac{1}{4}\left[2J_{\mu\nu}P_\lambda J^{\mu\nu}P^\lambda + 2J_{\lambda\mu}P_\nu J^{\mu\nu}P^\lambda + 2J_{\nu\lambda}P_\mu J^{\mu\nu}P^\lambda\right] \tag{4.33}$$

$$= -\frac{1}{2}J_{\mu\nu}P_\lambda J^{\mu\nu}P^\lambda + J_{\lambda\mu}J^{\mu\nu}P_\nu P^\lambda \tag{4.34}$$

where (4.) follows from re-indexing. Defining $J_{\beta\gamma} = X_\beta P_\gamma - X_\gamma P_\beta + S_{\beta\gamma}$, we write W^α as

$$W^\alpha = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}J_{\beta\gamma}P_\delta \tag{4.35}$$

$$= -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}[X_\beta P_\gamma - X_\gamma P_\beta + S_{\beta\gamma}]P_\delta \tag{4.36}$$

$$= -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}S_{\beta\gamma}P_\delta + \frac{1}{2}\left[\epsilon^{\alpha\beta\gamma\delta}X_\gamma P_\beta - \epsilon^{\alpha\beta\gamma\delta}X_\beta P_\gamma\right]P_\delta \tag{4.37}$$

$$= -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}S_{\beta\gamma}P_\delta + \frac{1}{2}\left[-\epsilon^{\alpha\gamma\beta\delta}X_\gamma P_\beta - \epsilon^{\alpha\beta\gamma\delta}X_\beta P_\gamma\right]P_\delta \tag{4.38}$$

$$= -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}S_{\beta\gamma}P_\delta - \epsilon^{\alpha\beta\gamma\delta}X_\beta P_\gamma P_\delta \tag{4.39}$$

where (4.38) follows since the Levi-Civita tensor (for Minkowski metric, at least, where $|\det(\eta_{\alpha\beta})| = 1$) is antisymmetric under the exchange of any two indices. However, upon expanding the second term, since there are two indistinct momentum 4-vectors, will evaluate to zero. For instance, consider the permutation when $\alpha = 0$ (it is the same to look at 1, 2, 3, as well):

$$\begin{aligned}
&(0123) : (+1) \qquad X_1P_2P_3 \\
&(0132) : (-1) \qquad X_1P_3P_2 \\
&(0312) : (+1) \qquad X_3P_1P_2 \\
&(0321) : (-1) \qquad X_3P_2P_1 \\
&(0231) : (+1) \qquad X_2P_3P_1 \\
&(0213) : (-1) \qquad X_2P_1P_3
\end{aligned}$$

which will all sum to 0 upon index contraction. Thus

$$W^\alpha = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}S_{\beta\gamma}P_\delta. \tag{4.40}$$

Therefore, if we contract (4.40),

$$W^\alpha W_\alpha = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\theta\sigma\varphi}S_{\beta\gamma}S^{\theta\sigma}P_\delta P^\varphi \tag{4.41}$$

$$= \frac{1}{4}\delta^{[\beta\gamma\delta]}_{\theta\sigma\varphi}S_{\beta\gamma}S^{\theta\sigma}P_\delta P^\varphi \tag{4.42}$$

$$= \frac{1}{4}\begin{pmatrix}\delta^\beta_\theta & \delta^\gamma_\theta & \delta^\delta_\theta \\ \delta^\beta_\sigma & \delta^\gamma_\sigma & \delta^\delta_\sigma \\ \delta^\beta_\varphi & \delta^\gamma_\varphi & \delta^\delta_\varphi\end{pmatrix} S_{\beta\gamma}S^{\theta\sigma}P_\delta P^\varphi \tag{4.43}$$

$$= \frac{1}{4}\left[\delta^\beta_\theta\delta^\gamma_\sigma\delta^\delta_\varphi + \delta^\gamma_\theta\delta^\delta_\sigma\delta^\beta_\varphi + \delta^\delta_\theta\delta^\beta_\sigma\delta^\gamma_\varphi - \delta^\beta_\varphi\delta^\gamma_\sigma\delta^\delta_\theta - \delta^\gamma_\varphi\delta^\delta_\sigma\delta^\beta_\theta - \delta^\delta_\varphi\delta^\beta_\sigma\delta^\gamma_\theta\right] S_{\beta\gamma}S^{\theta\sigma}P_\delta P^\varphi \tag{4.44}$$

$$= \frac{1}{4}[S_{\theta\sigma}S^{\theta\sigma}P^\delta P_\delta + S_{\varphi\theta}S^{\theta\sigma}P_\sigma P^\varphi + S_{\sigma\gamma}S^{\theta\sigma}P_\theta P^\gamma$$

$$- S_{\varphi\sigma} S^{\theta\sigma} P_\theta P^\varphi - S_{\theta\varphi} S^{\theta\sigma} P_\sigma P^\varphi - S_{\sigma\theta} S^{\theta\sigma} P_\varphi P^\varphi] \tag{4.45}$$
$$= \frac{1}{4}[S_{\theta\sigma} S^{\theta\sigma} P_\varphi P^\varphi - S_{\sigma\theta} S^{\theta\sigma} P_\varphi P^\varphi + (0)]$$

where the middle terms go to zero because under cross-contraction of P's and S's. Since S is completely antisymmetric, repeated indices to validate the P contractions (the only non-zero term of P is P^0, which is the mass in the rest frame) and an index cannot be repeated. Furthermore, $P^\varphi P_\varphi = E^2 - |p|^2 = M^2$ (in rest frame), therefore

$$W^\alpha W_\alpha = \frac{M^2}{4}[S_{\theta\sigma} S^{\theta\sigma} - S_{\sigma\theta} S^{\theta\sigma}] \tag{4.46}$$

Note that

$$H^{\mu\nu} H_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\gamma\delta} S_{\alpha\beta} S^{\gamma\delta} \tag{4.47}$$
$$= -2(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma) S_{\alpha\beta} S^{\gamma\delta} \tag{4.48}$$
$$= 2(S_{\delta\gamma} S^{\gamma\delta} - S_{\gamma\delta} S^{\gamma\delta}) \tag{4.49}$$

which is the equivalent expression in (4.46). Hence

$$W^\alpha W_\alpha = \frac{M^2}{2} \cdot (S_{\delta\gamma} S^{\gamma\delta} - S_{\gamma\delta} S^{\gamma\delta}) \tag{4.50}$$

where I have interchanged indices and added the values, just for clarity. If we recall from classical mechanics, angular momentum is defined as $L_i = \epsilon_{ijk} x_j p_k$, for indices in (1,2,3). Summing over the '0' indices then implies the remaining indices to be $1, 2, 3$, which brings out another factor of 2 in (4.47) from the contraction of the Levi-Civita 4-tensor, just in two indices (see lecture notes). Therefore defining $S^c = \epsilon^{cab} S_{ab}$, where $\epsilon^{cab} \epsilon_{cde}$ is the reduced contraction of $\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\gamma\delta}$ (times $-2!$), we find

$$W^\alpha W_\alpha = \frac{M^2}{4} \cdot -2(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma) S_{\alpha\beta} S^{\gamma\delta} \tag{4.51}$$
$$= \frac{M^2}{4} \cdot -2(-2!)(\epsilon^{cab} \epsilon_{cde}) S_{ab} S^{de} \tag{4.52}$$
$$= M^2 S^c S_c. \tag{4.53}$$