PHY452 Problem Set 4

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Problem 1

(**a**) Consider a system where helium atoms in a gas can be absorbed onto a metal plate, where it takes a work ϕ to remove the atoms from the plate. To calculate the partition function, we note that we have two systems $\mathcal{Z}_1$ and $\mathcal{Z}_2$ described by the free gas of additional energy per particle ϕ and the bound atoms on the plate. Consider first the Hamiltonian of the gas,

$$\mathcal{H}_1 = \sum_{i=1}^{N_g} \frac{p_i^2}{2m} + \phi N_g \tag{1.1}$$

since the free gas contains an additional energy ϕ per particle to not be bound to the metal plate. Computing the partition function, we have that

$$\mathcal{Z}_1 = \frac{1}{h^{3N_g} N_g!} \int \prod_{i=1}^{N_g} d^3 p_i \, d^3 q_i e^{-\beta \mathcal{H}_1} \tag{1.2}$$

$$= \frac{1}{h^{3N_g} N_g!} \int d^3 p_i \, e^{-\beta \sum_{k=1}^{N_g} p_k^2/2m} \int d^3 q_i \, e^{-\beta \phi N_g} \tag{1.3}$$

$$= \frac{(V e^{-\beta \phi})^{N_g}}{h^{3N_g} N_g!} \left[\int dp \, e^{-\beta p^2/2m} \right]^{3N_g} \tag{1.4}$$

$$= \frac{(V e^{-\beta \phi})^{N_g}}{h^{3N_g} N_g!} \left[\sqrt{2\pi m k T} \right]^{3N_g} . \tag{1.5}$$

We can similarly compute the partition function on the two-dimensional surface of the plate with no additional potential, whose Hamiltonian is simply $\mathcal{H}_2 = \sum_{k=1}^{N_s} \frac{p_k^2}{2m}$,

$$\mathcal{Z}_2 = \frac{1}{h^{2N_s} N_s!} \int \prod_{i=1}^{N_s} d^2 p_i \, d^2 q_i e^{-\beta \mathcal{H}_2} \tag{1.6}$$

$$= \frac{1}{h^{2N_s} N_s!} \int d^2 p_i \, e^{-\beta \sum_{k=1}^{N_s} p_k^2/2m} \int d^2 q_i \tag{1.7}$$

$$= \frac{A^{N_s}}{h^{2N_s} N_s!} \left[\sqrt{2\pi m k T} \right]^{2N_s} \tag{1.8}$$

$$= \frac{A^{N_s}}{N_s!} \left[\frac{2\pi m k T}{h^2} \right]^{N_s} . \tag{1.9}$$

Since the partition function of the combination of multiple systems is the product between each of the functions for the indivial system contributions, we have that

$$\mathcal{Z} = \mathcal{Z}_1 \cdot \mathcal{Z}_2 \tag{1.9}$$

$$= \frac{(Ve^{\beta\phi})^{N_g}}{h^{3N_g} N_g!} \left[\sqrt{2\pi mkT}\right]^{3N_g} \frac{A^{N_s}}{N_s!} \left[\frac{2\pi mkT}{h^2}\right]^{N_s} \tag{1.10}$$

$$= \frac{(Ve^{\beta\phi})^{N_g} A^{N_s}}{h^{3N_g+2N_s} N_g! N_s!} \left[\sqrt{2\pi mkT}\right]^{3N_g+2N_s} \tag{1.11}$$

where the whole system is in equilibrium temperature T, and m is the mass of a helium atom.

(**b**) The first law of thermodynamics states that, for two systems to be in equilibrium at a temperature T, their chemical potentials must be equal. We first compute the Helmholtz free energy $F = -kT \log \mathcal{Z}$, from which we can determine the pressure $P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$ and the chemical potential $\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$:

$$F_1 = -kT \log \mathcal{Z}_1 \tag{1.12}$$

$$= -kT \log \left(\frac{(Ve^{-\beta\phi})^{N_g}}{h^{3N_g} N_g!} [2\pi mkT]^{3N_g/2}\right) \tag{1.13}$$

$$= -kT \left[N_g \log(Ve^{-\beta\phi}) - 3N_g \log h - N_g \log N_g - N_g + \frac{3N_g}{2} \log(2\pi mkT)\right] \tag{1.14}$$

$$\implies P = -\left(-kT \frac{N_g}{Ve^{-\beta\phi}} e^{-\beta p}\right) \tag{1.15}$$

$$= kT \frac{N_g}{V} \tag{1.16}$$

$$\implies \mu_1 = -kT \left[\log(Ve^{-\beta\phi}) - 3\log h - \log N_g - 2 + \frac{3}{2} \log(2\pi mkT)\right] \tag{1.17}$$

and similarly, for the plate system,

$$F_2 = -kT \log \mathcal{Z}_2 \tag{1.18}$$

$$= -kT \log \left(\frac{A^{N_s}}{N_s!} \left[\frac{2\pi mkT}{h^2}\right]^{N_s}\right) \tag{1.19}$$

$$= -kT \left[N_s \log(A) - 2N_s \log h - N_s \log N_s - N_s + N_s \log(2\pi mkT)\right] \tag{1.20}$$

$$\implies \mu_2 = -kT \left[\log(A) - 2\log h - \log N_s - 2 + \log(2\pi mkT)\right] \tag{1.21}$$

Setting the chemical potentials (1.17) and (1.21) equal, we find

$$\log(Ve^{-\beta\phi}) - 3\log h - \log N_g - 2 + \frac{3}{2}\log(2\pi mkT) = \log(A) - 2\log h - \log N_s - 2 + \log(2\pi mkT) \tag{1.22}$$

$$\implies \log(A) - \log N_s = \log(Ve^{-\beta\phi}) - \log h - \log N_g + \frac{1}{2}\log(2\pi mkT) \tag{1.23}$$

$$\implies \log\left(\frac{A}{N_s}\right) = \log\left(\frac{Ve^{-\beta\phi}}{N_g h}\sqrt{2\pi mkT}\right) \tag{1.24}$$

$$\implies \frac{N_s}{A} = \frac{N_g}{V} \frac{h}{e^{-\beta\phi}\sqrt{2\pi mkT}} \tag{1.45}$$

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$$= \frac{Ph}{kT} \left(\frac{1}{\sqrt{2\pi mkT}} \right) e^{\beta\phi} \tag{1.26}$$

which is the mean number of atoms absored onto the plate, as desired. Note that the last line followed since our pressure is given by (1.16), $\frac{N_g}{V} = \frac{P}{kT}$.

Problem 2

(a) The energy of relativistic fermions with two spin states is $\varepsilon_p = \sqrt{p^2 c^2 + m^2 c^4}$, and spin multiplicity $g = 2$. **Zero-Temperature Conditions:** Suppose that the gas is contained within some arbitrary volume in three dimensions, V, so that the number of particles occupying the it at zero temperature $T = 0$ is

$$N = g \int d^3p \, \theta(p - p_F) \tag{2.1}$$

$$= 4\pi g \frac{V}{(2\pi)^3} \int p^2 dp \, \theta(p - p_F) \tag{2.2}$$

$$= \frac{4\pi g V}{3(2\pi)^3} p_F^3 \tag{2.3}$$

$$\implies p_F = \left(\frac{6\pi^2 N}{gV} \right)^{1/3} \tag{2.4}$$

as specfied by the Fermi surface. The Fermi energy is therefore

$$\varepsilon_F = \left[\left(\frac{6\pi^2 nc^3}{g} \right)^{2/3} + m^2 c^4 \right]^{1/2} \tag{2.5}$$

We can also calculate the total energy at zero temperature inside the volume by using ε_p, which is

$$E = g \frac{V}{(2\pi)^3} \int d^3p \, \sqrt{p^2 c^2 + m^2 c^4} \, \theta(p - p_F) \tag{2.6}$$

$$= g \frac{V}{(2\pi)^3} \int_0^{p_F} dp \, p^2 \sqrt{p^2 c^2 + m^2 c^4} \tag{2.7}$$

$$= \frac{gV}{4(2\pi)^3} m^4 c^5 \left[x(x^2 + 1)^{3/2} - \frac{1}{2} x\sqrt{x^2 + 1} - \frac{1}{2} \operatorname{arcsinh} x \right] \tag{2.8}$$

Integral evaluation of (2.7): Let $p = mc \tan \theta$ so $dp = mc \sec^2 \theta d\theta$:

$$mc^2 \int_0^{p_F} dp \, p^2 \sqrt{p^2/(mc)^2 + 1} = mc^2 \int_0^{\arctan(p_F/mc)} mc \sec^2 \theta d\theta \, m^2 c^2 \tan^2 \theta \sqrt{\tan^2 \theta + 1} \tag{2.7.1}$$

$$= m^4 c^5 \int_0^{\arctan(p_F/mc)} d\theta \, \sec^3 \theta (\sec^2 \theta - 1) \tag{2.7.2}$$

Let $x \equiv p_F/mc$. Note that we have sec integral type $\int d\theta \sec^n \theta = \frac{n-2}{n-1} \int d\theta \sec^{n-2} \theta + \frac{\sec^{n-2} \theta \tan \theta}{n-1}$ and hence

$$\int_0^{\arctan x} d\theta \, \sec^5 \theta - \int_0^{\arctan x} d\theta \, \sec^3 \theta = \frac{\sec^3 \theta \tan \theta}{4} \Big|_0^{\arctan x} + \left(\frac{3}{4} - 1 \right) \int_0^{\arctan x} d\theta \, \sec^3 \theta \tag{2.7.3}$$

$$= \frac{\sec^3 \theta \tan \theta}{4} \Big|_0^{\arctan x} - \frac{1}{4} \left[\frac{\sec \theta \tan \theta}{2} \Big|_0^{\arctan x} + \frac{1}{2} \int_0^{\arctan x} d\theta \, \sec \theta \right] \tag{2.7.4}$$

$$= \left[\frac{\sec^3 \theta \tan \theta}{4} - \frac{1}{4} \frac{\sec \theta \tan \theta}{2} - \frac{1}{8} \operatorname{arcsinh}(\tan \theta) \right] \Big|_0^{\arctan x} \tag{2.7.5}$$

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From which $\sec(\arctan x) = \sqrt{x^2 + 1}$, hence we obtain

$$mc^2 \int_0^{p_F} dp\, p^2 \sqrt{p^2/(mc)^2 + 1} = \frac{1}{4} m^4 c^5 \left[x(x^2 + 1)^{3/2} - \frac{1}{2} x \sqrt{x^2 + 1} - \frac{1}{2} \operatorname{arcsinh} x \right] \qquad (2.7.6)$$

and (2.7) becomes (2.8):

$$\frac{gV}{(2\pi)^3} \times \int_0^{p_F} dp\, p^2 \sqrt{p^2 c^2 + m^2 c^4} = \frac{gV}{4(2\pi)^3} m^4 c^5 \left[x(x^2 + 1)^{3/2} - \frac{1}{2} x \sqrt{x^2 + 1} - \frac{1}{2} \operatorname{arcsinh} x \right]$$
$$(2.7.7)$$

by letting $x = p_F/(mc)$. **Finite-Temperature Conditions:** Now accounting for the temperature of the system, the number of particles (and energy) is given by relations with the Fermi-Dirac function $n_F(\varepsilon)$,

$$N = g \frac{V}{(2\pi)^3} \int d^3 p\, n_F(\varepsilon_p) \qquad (2.9)$$

where $n_F(\varepsilon_p) = \dfrac{1}{e^{\beta(\varepsilon_p - \mu)} + 1}$ is the probability that a state of energy ε_p is occupied. The integrand in (2.9) is spherically symmetric ($d^3 p \to 4\pi p^2\, dp$), and the energy relation implies that $p = \dfrac{1}{c} \sqrt{\varepsilon_p^2 - m^2 c^4}$, so a change of variables yields that

$$N = g \frac{V}{(2\pi)^3} (4\pi) \frac{1}{c} \int_{mc^2}^{\infty} d\varepsilon\, \frac{\varepsilon}{\sqrt{\varepsilon^2 - m^2 c^4}(e^{\beta(\varepsilon - \mu)} + 1)} \frac{1}{c^2} (\varepsilon^2 - m^2 c^4) \qquad (2.10)$$

$$= \frac{4\pi gV}{(2\pi c)^3} \int_{mc^2}^{\infty} d\varepsilon\, \frac{\varepsilon \sqrt{\varepsilon^2 - m^2 c^4}}{e^{\beta(\varepsilon - \mu)} + 1}. \qquad (2.11)$$

Recall that the Sommerfeld expansion defines a class of integrals $I[f] = \displaystyle\int_{mc^2}^{\infty} d\varepsilon\, f(\varepsilon) n_F(\varepsilon)$ and $F(\varepsilon) = \displaystyle\int_{mc^2}^{\varepsilon} d\varepsilon'\, f(\varepsilon')$ (in this problem, our lowest energy is mc^2, not zero) so that $I[f]$ can be expanded about the chemical potential μ as

$$I[f] \approx F(\mu) + \frac{\pi^2}{6} (kT)^2 F''(\mu). \qquad (2.12)$$

Applying $I[f]$ to our relativistic integral in (2.11) using (2.12), we find a first order correction to the number of particles in terms of the chemical potential μ:

$$f(\varepsilon) = \varepsilon \sqrt{\varepsilon^2 - m^2 c^4} \qquad (2.13)$$

$$F(\varepsilon) = \frac{1}{3} (\varepsilon^2 - m^2 c^4)^{3/2} \qquad (2.14)$$

$$F''(\varepsilon) = \sqrt{\varepsilon^2 - m^2 c^4} + \frac{\varepsilon^2}{\sqrt{\varepsilon^2 - m^2 c^4}} \qquad (2.15)$$

from which (2.11) approximates into

$$N \approx \frac{4\pi gV}{(2\pi c)^3} \left[\frac{1}{3} (\mu^2 - m^2 c^4)^{3/2} + \frac{\pi^2}{6} (kT)^2 \left(\sqrt{\mu^2 - m^2 c^4} + \frac{\mu^2}{\sqrt{\mu^2 - m^2 c^4}} \right) \right]. \qquad (2.16)$$

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Keeping N constant, we use p_F^3 from (2.4) to re-write this as

$$N = \frac{N}{c^3} \frac{1}{p_F^3} \left[(\mu^2 - m^2c^4)^{3/2} + \frac{\pi^2}{2}(kT)^2 \left(\sqrt{\mu^2 - m^2c^4} + \frac{\mu^2}{\sqrt{\mu^2 - m^2c^4}} \right) \right] \tag{2.17}$$

$$\implies (\varepsilon_F^2 - m^2c^4)^{3/2} = (\mu^2 - m^2c^4)^{3/2} + \frac{\pi^2}{2}(kT)^2 \left(\sqrt{\mu^2 - m^2c^4} + \frac{\mu^2}{\sqrt{\mu^2 - m^2c^4}} \right) \tag{2.18}$$

which follows from $p_F = \frac{1}{c}\sqrt{\varepsilon_F^2 - m^2c^4}$. Since $\mu(T = 0) = \varepsilon_F$, and the above represents the chemical potential for $0 < T << 1$, we let $\mu = \varepsilon_F + \delta$ be a perturbation to the chemical potential, for which we solve for δ after expansions. By the binomial expansions, we have that

$$\mu^n = \varepsilon_F^n \left(1 + \frac{\delta}{\varepsilon_F} \right)^n \approx \varepsilon_F^n + n\delta\varepsilon_F \tag{2.19}$$

for real numbers n, thus

$$(\mu^2 - m^2c^4)^n \approx (\varepsilon_F^2 + 2\delta\varepsilon_F - m^2c^4)^n \tag{2.20}$$

$$= (\varepsilon_F^2 - m^2c^4)^n \left(1 + \frac{2\delta\varepsilon_F}{\varepsilon_F^2 - m^2c^4} \right)^n \tag{2.21}$$

$$\approx (\varepsilon_F^2 - m^2c^4)^n \left(1 + \frac{2n\delta\varepsilon_F}{\varepsilon_F^2 - m^2c^4} \right) \tag{2.22}$$

$$= (\varepsilon_F^2 - m^2c^4)^n + 2n\delta\varepsilon_F(\varepsilon_F^2 - m^2c^4)^{n-1} \tag{2.23}$$

which implies that

$$(\varepsilon_F^2 - m^2c^4)^{3/2} = (\varepsilon_F^2 - m^2c^4)^{3/2} + 3\delta\varepsilon_F\sqrt{\varepsilon_F^2 - m^2c^4}$$
$$+ \frac{(\pi kT)^2}{2} \left(\sqrt{\varepsilon_F^2 - m^2c^4} + \frac{\varepsilon_F^2}{\sqrt{\varepsilon_F^2 - m^2c^4}} \right) \mathcal{O}((kT)^2\delta; \delta^2) \tag{2.24}$$

$$\implies 0 = 3\delta\varepsilon_F\sqrt{\varepsilon_F^2 - m^2c^4} + \frac{(\pi kT)^2}{2}\sqrt{\varepsilon_F^2 - m^2c^4} \left(1 + \frac{\varepsilon_F^2}{\varepsilon_F^2 - m^2c^4} \right) \tag{2.25}$$

$$\implies -3\delta\varepsilon_F = \frac{(\pi kT)^2}{2} \left(1 + \frac{\varepsilon_F^2}{\varepsilon_F^2 - m^2c^4} \right) \tag{2.26}$$

$$\implies \delta = -\frac{1}{6}(\pi kT)^2 \frac{1}{\varepsilon_F} \left(1 + \frac{\varepsilon_F^2}{\varepsilon_F^2 - m^2c^4} \right) \tag{2.27}$$

and therefore the first order correction to the chemical potential near zero temperature is

$$\mu(T) = \varepsilon_F - \frac{1}{6}(\pi kT)^2 \frac{1}{\varepsilon_F} \left(1 + \frac{\varepsilon_F^2}{\varepsilon_F^2 - m^2c^4} \right) \tag{2.28}$$

As before, in determining N at low T, we can perform a similar procedure to determine the internal energy at finite temperatures,

$$E = \frac{gV}{(2\pi)^3} \int_0^\infty 4\pi p^2 \, dp \frac{\varepsilon_p}{e^{\beta(\varepsilon_p - \mu)} + 1} \tag{2.29}$$

$$= \frac{4\pi g V}{(2\pi c)^3} \int_{mc^2}^{\infty} d\varepsilon \, \frac{\varepsilon^2 \sqrt{\varepsilon^2 - m^2 c^4}}{e^{\beta(\varepsilon - \mu)} + 1} \tag{2.30}$$

from which

$$f(\varepsilon) = \varepsilon^2 \sqrt{\varepsilon^2 - m^2 c^4} \tag{2.31}$$

$$F(\varepsilon) = \frac{1}{4} m^4 c^8 \left[\sec^3 \theta \tan \theta - \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \operatorname{arcsinh}(\tan \theta) \right]_{\operatorname{arcsec} 1}^{\operatorname{arcsec}(\varepsilon/mc^2)} \tag{2.32}$$

$$= \frac{1}{4} m^4 c^8 \left[y^3 \sqrt{y^2 - 1} - \frac{1}{2} y \sqrt{y^2 - 1} - \frac{1}{2} \operatorname{arcsinh}(\sqrt{y^2 - 1}) - \mathcal{C} \right] \tag{2.33}$$

$$= \frac{1}{4} \left[\varepsilon^3 \sqrt{\varepsilon^2 - m^2 c^4} - \frac{(mc^2)^2}{2} \varepsilon \sqrt{\varepsilon^2 - m^2 c^4} - \frac{(mc^2)^4}{2} \operatorname{arcsinh}\left(\sqrt{\left(\frac{\varepsilon}{mc^2}\right)^2 - 1} \right) - (mc^2)^4 \mathcal{C} \right] \tag{2.34}$$

$$F''(\varepsilon) = 2\varepsilon \sqrt{\varepsilon^2 - m^2 c^4} + \frac{\varepsilon^3}{\sqrt{\varepsilon^2 - m^2 c^4}} \tag{2.35}$$

where $y \equiv \frac{\varepsilon}{mc^2}$ and $\mathcal{C} = \sqrt{2} - \frac{\sqrt{2}}{2} - \frac{1}{2} \operatorname{arcsinh}(\sqrt{2}) \approx 1.3399\ldots$ is a constant (this integral was computed similarly to the previous energy integral in the zero-temperature régime, but with the substitution $\varepsilon' = mc^2 \sec \theta$. The same secant reduction formula was used). Then, invoking the Sommerfeld expansion,

$$E \approx \frac{gV}{2\pi^2 c^3} \left[\frac{1}{4} \left(\mu^3 \sqrt{\mu^2 - m^2 c^4} - \frac{m^2 c^4}{2} \mu \sqrt{\mu^2 - m^2 c^4} - \frac{(mc^2)^4}{2} \operatorname{arcsinh}\left(\frac{1}{mc^2} \sqrt{\mu^2 - m^2 c^4} \right) \right) \right.$$
$$\left. + \frac{(\pi k T)^2}{6} \left(2\mu \sqrt{\mu^2 - m^2 c^4} + \frac{\mu^3}{\sqrt{\mu^2 - m^2 c^4}} \right) \right] + C \tag{2.36}$$

$$\frac{E}{N} = \frac{3}{p_F^3 c^3} \left[\frac{1}{4} \left(\mu^3 \sqrt{\mu^2 - m^2 c^4} - \frac{m^2 c^4}{2} \mu \sqrt{\mu^2 - m^2 c^4} - \frac{(mc^2)^4}{2} \operatorname{arcsinh}\left(\frac{1}{mc^2} \sqrt{\mu^2 - m^2 c^4} \right) \right) \right.$$
$$\left. + \frac{(\pi k T)^2}{6} \left(2\mu \sqrt{\mu^2 - m^2 c^4} + \frac{\mu^3}{\sqrt{\mu^2 - m^2 c^4}} \right) \right] + C \tag{2.37}$$

Again, expanding μ as a Taylor series and only keeping terms $\mathcal{O}((kT)^2 \delta; \delta^2)$, term by term,

$$\mu^3 \sqrt{\mu^2 - m^2 c^4} \approx \varepsilon_F^3 p_F c + \delta \left[3\varepsilon_F^2 p_F c + \frac{\varepsilon_F^4}{p_F c} \right] \tag{2.38}$$

$$= \varepsilon_F^3 p_F c - \frac{(\pi k T)^2}{6} \frac{1}{\varepsilon_F} \left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2} \right) \left[3\varepsilon_F^2 p_F c + \frac{\varepsilon_F^4}{p_F c} \right] \tag{2.39}$$

$$\mu \sqrt{\mu^2 - m^2 c^4} \approx \varepsilon_F p_F c + \delta \left[p_F c + \frac{\varepsilon_F^2}{p_F c} \right] \tag{2.40}$$

$$= \varepsilon_F p_F c - \frac{(\pi k T)^2}{6} \frac{1}{\varepsilon_F} \left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2} \right) \left[p_F c + \frac{\varepsilon_F^2}{p_F c} \right] \tag{2.41}$$

$$\operatorname{arcsinh}\left(\frac{1}{mc^2} \sqrt{\mu^2 - m^2 c^4} \right) \approx \operatorname{arcsinh}\left(\frac{p_F c}{mc^2} \right) + \delta \frac{1}{\sqrt{x^2 + 1}} \tag{2.42}$$

$$= \operatorname{arcsinh}\left(\frac{p_F}{mc} \right) - \frac{(\pi k T)^2}{6} \frac{1}{\varepsilon_F} \left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2} \right) \frac{1}{\sqrt{x^2 + 1}} \frac{1}{mc^2} \frac{\varepsilon_F}{p_F c} \tag{2.43}$$

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$$= \operatorname{arcsinh}\left(\frac{p_F}{mc}\right) - \frac{(\pi kT)^2}{6}\frac{1}{\varepsilon_F}\left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2}\right)\frac{1}{p_F c} \tag{2.44}$$

where $x = \dfrac{p_F}{mc}$ is defined as before. Whence, just taking leading order in the latter half, and only considering the $\mathcal{O}(T^2)$ terms (since we're differentation constants away anyway, I don't want to keep doing pointless algebra), we obtain

$$\frac{E}{N} \approx \frac{3}{p_F^3 c^3}\frac{(\pi kT)^2}{6}\frac{1}{\varepsilon_F}\left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2}\right)\left[\frac{1}{4}\left(-3\varepsilon_F^2 p_F c - \frac{\varepsilon_F^4}{p_F c} + \frac{m^2 c^4}{2}\left(p_F c + \frac{\varepsilon_F^2}{p_F c}\right) + \frac{m^4 c^8}{2}\frac{1}{p_F c}\right)\right]$$
$$+ \frac{3}{p_F^3 c^3}\frac{(\pi kT)^2}{6}\left[2\varepsilon_F p_F c + \frac{\varepsilon_F^3}{p_F c}\right] \tag{2.45}$$

$$= \frac{(\pi kT)^2}{2p_F^3 c^3}\frac{1}{\varepsilon_F}\left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2}\right)\left[\frac{1}{4}\left(-3\varepsilon_F^2 p_F c - \frac{\varepsilon_F^4}{p_F c} + \frac{1}{2}\varepsilon_F^2 p_F c + \frac{\varepsilon_F^4}{2p_F c} - \frac{1}{2}\varepsilon_F^2 p_F c - \frac{1}{2}p_F^3 c^3\right.\right.$$
$$\left.\left. + \frac{1}{2}\frac{\varepsilon_F^4}{p_F c} + \frac{1}{2}p_F^3 c^3 - \varepsilon_F^2 p_F c\right)\right] + \frac{(\pi kT)^2}{2p_F^3 c^3}\left[2\varepsilon_F p_F c + \frac{\varepsilon_F^3}{p_F c}\right] \tag{2.46}$$

$$= \frac{(\pi kT)^2}{2p_F^3 c^3}\left[\left(1 + \frac{\varepsilon_F^2}{p_F^2 c^2}\right)\left(-\varepsilon_F p_F c\right) + 2\varepsilon_F p_F c + \frac{\varepsilon_F^3}{p_F c}\right] \tag{2.47}$$

$$= \frac{(\pi kT)^2}{2p_F^3 c^3}\left[\varepsilon_F p_F c - \frac{\varepsilon_F^3}{p_F c} + \frac{\varepsilon_F^3}{p_F c}\right] \tag{2.48}$$

$$= \frac{(\pi kT)^2}{2}\frac{\varepsilon_F}{p_F^2 c^2} \tag{2.49}$$

$$= \frac{(\pi kT)^2}{2}\frac{\sqrt{p_F^2 c^2 + m^2 c^4}}{p_F^2 c^2} \tag{2.50}$$

$$= \frac{(\pi kT)^2}{2}\frac{mc^2\sqrt{x^2 + 1}}{p_F^2 c^2} \times \frac{mc^2}{mc^2} \tag{2.51}$$

$$= \frac{(\pi kT)^2}{2mc^2}\frac{\sqrt{x^2 + 1}}{x^2} \tag{2.52}$$

where $x = \dfrac{p_F}{mc^2}$ again. The heat capacity at constant volume is just given by one derivative in T, which is

$$\frac{C_V}{N} = \pi^2\frac{\sqrt{x^2 + 1}}{x^2}\frac{k^2 T}{mc^2} \implies \frac{C_V}{Nk} = \pi^2\frac{\sqrt{x^2 + 1}}{x^2}\frac{kT}{mc^2} \tag{2.53}$$

as desired.

(**b**) In the non-relativistic limit, we have that $c \to \infty$ and $\varepsilon_F = \dfrac{p_F^2}{2m}$. Then, piggybacking off of our solution from (a), we have that

$$\frac{C_V}{Nk} = \frac{\pi^2 kT}{m}\lim_{c\to\infty}\frac{1}{c^2}\frac{\sqrt{p_F^2/(m^2 c^2) + 1}}{p_F^2/(m^2 c^2)} \tag{2.54}$$

$$= \frac{\pi^2 kT}{m}\lim_{c\to\infty}\frac{\sqrt{p_F^2/(m^2 c^2) + 1}}{p_F^2/m^2} \tag{2.55}$$

$$= \frac{\pi^2 kT}{m}\frac{1}{p_F^2/m^2} \tag{2.56}$$

$$= \frac{\pi^2 kT}{2\varepsilon_F} \tag{2.57}$$

as found with the regular, quantum-mechanical Fermi gas. We can also take the ultra-relativistic limit, $p_F c \gg mc^2 \implies \dfrac{mc^2}{p_F c} \ll 1$, hence $\varepsilon_F \to p_F c$. We have that

$$\frac{C_V}{Nk} = \pi^2 kT \lim_{mc/p_F \to 0} \frac{p_F c \sqrt{1 + m^2 c^4/(p_F^2 c^2)}}{p_F^2 c^2} \tag{2.58}$$

$$\approx \pi^2 kT \lim_{mc/p_F \to 0} \frac{1}{p_F c} \left[1 + \frac{1}{2} \frac{m^2 c^4}{p_F^2 c^2} + \dots \right] \tag{2.59}$$

$$= \frac{\pi^2 kT}{p_F c} \tag{2.60}$$

$$= \frac{\pi^2 kT}{\varepsilon_F} \tag{2.61}$$

in the ultra-relativistic limit.

Problem 3

(a) Consider a box V of spinless ($g = 1$) Fermions with energy $\varepsilon = |p/\hbar|^s$ in d dimensions. Let $k_n = p_n/\hbar$ be the wavevector of the n-th fermion. We first begin by calculating the grand partition function $\mathcal{Q}$,

$$\mathcal{Q} = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{n_k\}} e^{-\beta \sum_k \varepsilon_k n_k} \tag{3.1}$$

$$= \prod_k \sum_{n_k=0,1} e^{\beta(\mu-\varepsilon_k)n_k} \tag{3.2}$$

$$= \prod_k \left[1 + e^{\beta(\mu-\varepsilon_k)}\right] \tag{3.3}$$

which follows by the Pauli exclusion principle, fermionic states are either unoccupied ($n_k = 0$) or occupied with capacity 1 ($n_k = 1$). Since we are aiming to find the grand potential G, we take the logarithm of (3.3):

$$\log \mathcal{Q} = \log \prod_k (1 + e^{\beta(\mu-\varepsilon_k)}) \tag{3.4}$$

$$= \sum_k \log(1 + e^{\beta(\mu-\varepsilon_k)}) \tag{3.5}$$

for which k is continuous (we integrate) in d dimensions,

$$\log \mathcal{Q} = \frac{V}{(2\pi)^d} \int d^d k \, \log(1 + e^{\beta(\mu-\varepsilon_k)}) \tag{3.6}$$

$$= \frac{V}{(2\pi)^d} \int d^d k \, \log(1 + ze^{-\beta|k|^s}) \tag{3.7}$$

$$= \frac{V}{(2\pi)^d} \int_0^{\infty} k^{d-1} dk \, \log(1 + ze^{-\beta|k|^s}) \int d\Omega_d \tag{3.8}$$

The integrand in (3.8) is spherically symmetric, and $d\Omega_d$ is the solid angle in d dimensions. If we recall the formulas for d-dimensional sphere volumes and surface areas to be

$$V_d(R) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} R^d \tag{3.9}$$

$$S_d(R) = \frac{d}{dR} V_d(R) = \frac{d\pi^{d/2}}{\Gamma(d/2+1)} R^{d-1} \tag{3.10}$$

we note that the solid angle integral is just

$$\int d\Omega_d = \frac{d\pi^{d/2}}{\Gamma(d/2+1)} \tag{3.10}$$

since radial integration provides $\left.\frac{1}{d} r^d\right|_0^R$. Thus (3.8) becomes

$$\log \mathcal{Q} = \frac{V}{(2\pi)^d} \frac{d\pi^{d/2}}{\Gamma(d/2+1)} \int_0^{\infty} dk \, k^{d-1} \log(1 + ze^{-\beta k^s}) \tag{3.11}$$

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and k is always positive at this point. Letting $x = \beta k^s$, then $k = \left(\dfrac{x}{\beta}\right)^{1/s}$ and $dk = \dfrac{1}{s\beta}\left(\dfrac{x}{\beta}\right)^{1/s-1} dx$,

$$\log \mathcal{Q} = \frac{V}{(2\pi)^d} \frac{d\pi^{d/2}}{\Gamma(d/2+1)} \int_0^\infty \frac{1}{s\beta}\left(\frac{x}{\beta}\right)^{1/s-1} dx \left(\frac{x}{\beta}\right)^{(d-1)/s} \log(1 + ze^{-x}) \tag{3.12}$$

$$= \frac{V}{(2\pi)^d} \frac{d\pi^{d/2}}{\Gamma(d/2+1)} \frac{1}{s\beta} \int_0^\infty dx \left(\frac{x}{\beta}\right)^{d/s-1} \log(1 + ze^{-x}) \tag{3.13}$$

$$= \frac{V}{(2\pi)^d} \frac{d\pi^{d/2}}{\Gamma(d/2+1)} \frac{1}{s\beta^{d/s}} \int_0^\infty dx\, x^{d/s-1} \log(1 + ze^{-x}) \tag{3.14}$$

Now, integrating by parts, with $u = \log(1 + ze^{-x})$, $dv = x^{d/s-1}\, dx$, we find

$$\log \mathcal{Q} = \frac{V}{(2\pi)^d} \frac{d\pi^{d/2}}{\Gamma(d/2+1)} \frac{1}{s\beta^{d/s}} \left[\frac{s}{d} \cancel{\log(1+ze^{-x})x^{d/s}}\Big|_0^\infty + \frac{s}{d}\int_0^\infty dx\, x^{d/s} \frac{ze^{-x}}{1 + ze^{-x}} \right] \tag{3.15}$$

$$= \frac{V}{(2\sqrt{\pi})^d} \frac{1}{\Gamma(d/2+1)} \frac{1}{\beta^{d/s}} \int_0^\infty dx\, x^{d/s} \frac{ze^{-x}}{1 + ze^{-x}} \tag{3.16}$$

$$= \frac{V}{(2\sqrt{\pi})^d} \frac{1}{\Gamma(d/2+1)} \frac{1}{\beta^{d/s}} \int_0^\infty dx\, \frac{x^{d/s}}{1 + z^{-1}e^x} \tag{3.17}$$

Since we define $f_n(z) = \dfrac{1}{\Gamma(n)} \displaystyle\int_0^\infty dx\, \dfrac{x^{n-1}}{z^{-1}e^x + 1}$ for fermions, (3.17) becomes

$$\log \mathcal{Q} = \frac{V}{(2\sqrt{\pi})^d} \frac{\Gamma(d/s+1)}{\Gamma(d/2+1)} \frac{1}{\beta^{d/s}} f_{d/s+1}(z). \tag{3.18}$$

Since the grand potential is defined as $G = -kT \log \mathcal{Q}$, by expanding out $\beta = \dfrac{1}{kT}$ we have that

$$G = -\frac{V}{(2\sqrt{\pi})^d} \frac{\Gamma(d/s+1)}{\Gamma(d/2+1)} (kT)^{d/s+1} f_{d/s+1}(z). \tag{3.19}$$

Furthermore, we can calculate the density of particles $\dfrac{N}{V} \equiv n$ at a chemical potential μ, by letting $a = \beta\mu$,

$$N = \frac{\partial(\log \mathcal{Q})}{\partial a} \tag{3.20}$$

$$= \frac{V}{(2\sqrt{\pi})^d} \frac{\Gamma(d/s+1)}{\Gamma(d/2+1)} \frac{1}{\beta^{d/s}} \int_0^\infty dx\, x^{d/s} \frac{\partial}{\partial a}\left[\frac{1}{e^{-a}e^x + 1}\right] \tag{3.21}$$

$$= \frac{V}{(2\sqrt{\pi})^d} \frac{\Gamma(d/s+1)}{\Gamma(d/2+1)} \frac{1}{\beta^{d/s}} \int_0^\infty dx\, x^{d/s} \frac{e^{-a}e^x}{(e^{-a}e^x + 1)^2} \tag{3.22}$$

$$= \frac{V}{(2\sqrt{\pi})^d} \frac{\Gamma(d/s+1)}{\Gamma(d/2+1)} \frac{1}{\beta^{d/s}} \int_0^\infty dx\, x^{d/s} \frac{z^{-1}e^x}{(z^{-1}e^x + 1)^2} \tag{3.23}$$

Taking a moment to simplify the integral, we notice that

$$\frac{\partial}{\partial a}\left[\frac{1}{z^{-1}e^x + 1}\right] = \frac{z^{-1}e^x}{(z^{-1}e^x + 1)^2} = -\frac{\partial}{\partial x}\left[\frac{1}{z^{-1}e^x + 1}\right] \tag{3.24}$$

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so we can integrate (3.23) by parts, letting $u = x^{d/s}$ and $dv = -\dfrac{\partial}{\partial x}\left[\dfrac{z^{-1}e^x}{(z^{-1}e^x + 1)^2}\right]dx$, we find

$$\int_0^\infty dx\, x^{d/s}\frac{z^{-1}e^x}{(z^{-1}e^x + 1)^2} = \frac{x^{d/s}}{z^{-1}e^x + 1}\Big|_0^\infty + \int_0^\infty dx\,\frac{1}{(d/s + 1)}\frac{x^{d/s+1}}{z^{-1}e^x + 1} \tag{3.25}$$

$$= \frac{1}{(d/s + 1)}\Gamma(d/s + 2)f_{d/s+2}(z) \tag{3.26}$$

$$= \Gamma(d/s + 1)f_{d/s+2}(z) \tag{3.27}$$

since $\dfrac{1}{a+1}\Gamma(a + 2) = \dfrac{1}{a+1}(a + 1)a! = \Gamma(a + 1)$. Substituting our simplified integral in terms of $f_n(z)$ back into (3.23) and dividing both sides by V, we obtain the density of particles to be

$$n = \frac{(kT)^{d/s}}{(2\sqrt{\pi})^d}\frac{[\Gamma(d/s + 1)]^2}{\Gamma(d/2 + 1)}f_{d/s+2}(z). \tag{3.28}$$

(b) We now aim to determine the ratio between pressure and volume with energy, $\dfrac{PV}{E}$. Since we already have found $\log \mathcal{Q}$ and that

$$\beta PV = \log \mathcal{Q}, \tag{3.29}$$

we just need to find the energy

$$E = \left(\frac{\partial(\log \mathcal{Q})}{\partial \beta}\right)_{\beta\mu} \tag{3.30}$$

at constant $a = \beta\mu$ (from (a)). Taking from (3.18), we notice that $\log \mathcal{Q}$ is a polynomial in β, since $f_{d/s+1}(z)$ is just a number (and $z = e^{-a}$ is held constant), hence we quickly find that

$$E = \frac{V}{(2\sqrt{\pi})^d}\frac{\Gamma(d/s + 1)}{\Gamma(d/2 + 1)}f_{d/s+1}(z)\frac{\partial}{\partial \beta}\left[\frac{1}{\beta^{d/s}}\right] \tag{3.31}$$

$$= -\frac{d}{s}\frac{V}{(2\sqrt{\pi})^d}\frac{\Gamma(d/s + 1)}{\Gamma(d/2 + 1)}\frac{1}{\beta^{d/s+1}}f_{d/s+1}(z) \tag{3.32}$$

which immediatley implies that

$$\frac{PV}{E} = -\frac{s}{d}. \tag{3.33}$$

For $s = 2$ in 3-dimensions, we regain our relation $E = -\dfrac{3}{2}PV$.

(c) We lastly consider the low-temperature limit as $T \to 0$, so $z \to \infty$ in the function $f_n(z)$. In general, we can approximate this as

$$f_n(z) \xrightarrow{z\to\infty} \frac{(\log z)^n}{n!} = \frac{(\beta\mu)^n}{n!}. \tag{3.35}$$

I will begin by noting that the number of particles, at zero temperature, is the number of states within the d-dimensional sphere of radius k_F:

$$N = \frac{V}{(2\pi)^d}\int d^d k\, \theta(k - k_F) \tag{3.36}$$

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$$= \frac{V}{(2\pi)^d} \frac{\pi^{d/2}}{(d/2)!} k_F^d \tag{3.37}$$

$$\implies n = \frac{1}{(2\sqrt{\pi})^d} \frac{1}{(d/2)!} k_F^d \tag{3.37}$$

$$\implies k_F = 2\sqrt{\pi}\left[(d/2)!\, n\right]^{1/d} \tag{3.38}$$

The energy is given by (3.32), where diving both sides by N give

$$\frac{E}{N} = -\frac{d}{s}\frac{1}{n}\frac{1}{(2\sqrt{\pi})^d}\frac{(d/s)!}{(d/2)!}\frac{1}{\beta^{d/s+1}}\frac{(\beta\mu)^{d/s+1}}{(d/s+1)(d/s)!} \tag{3.39}$$

$$= -\frac{1}{n}\frac{1}{(2\sqrt{\pi})^d}\frac{1}{(d/2)!}\frac{\mu^{d/s+1}}{(1+s/d)} \tag{3.40}$$

And since $\mu(T=0) = \varepsilon_F = k_F^s$, we further find, using (3.38), that

$$\frac{E}{N} = -\frac{1}{n}\frac{1}{(2\sqrt{\pi})^d}\frac{1}{(d/2)!}(2\sqrt{\pi})^{d+s}((d/2)!\, n)^{1+d/s} \tag{3.41}$$

$$= -[2\sqrt{\pi}]^s[(d/2)!]^{d/s}\, n^{d/s} \tag{3.42}$$

which implies that the energy per particle (at $T=0$) decreases as density increases.

$$\frac{E}{N} \propto -n^{d/s}. \tag{3.43}$$

We can similarly determine the dependence of the pressure P from (3.33) on the gas. We find that

$$P = -\frac{s}{d}\frac{E}{V} = -\frac{s}{d}\frac{E}{N}n \tag{3.44}$$

$$\implies P = \frac{s}{d}[2\sqrt{\pi}]^s[(d/2)!]^{d/s}\, n^{d/s+1} \tag{3.45}$$

hence

$$P \propto n^{d/s+1} \tag{3.44}$$

which means that pressure (at $T=0$) increases as the density increases.