PS3

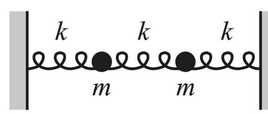
My score

96.9% (15.5/16)

Q1

4 / 4

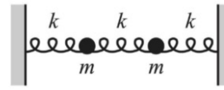
1. Consider two masses m, connected to each other and to two walls by three springs, as shown in the figure. Both masses feel a damping force −bv. The three springs have the same spring constant k. Find the general solution for the positions of the masses as functions of time. Assume underdamping.



PHY254 PS3 — November 21st, 2021
1006940802

Q1.

1. Consider two masses m, connected to each other and to two walls by three springs, as shown in the figure. Both masses feel a damping force −bv. The three springs have the same spring constant k. Find the general solution for the positions of the masses as functions of time. Assume underdamping.



Considering the forces on mass m with position x_1, the left spring produces a restoring force of magnitude $k(\ell - x_1)$ leftward and the middle spring produces a restoring force $-k(\ell - (x_2 - x_1))$ rightward. Our equation of motion becomes

$$\begin{aligned} m\ddot{x}_1 &= k(\ell - x_1) - k(\ell - (x_2 - x_1)) - b\dot{x}_1 \\ &= k(x_2 - 2x_1) - b\dot{x}_1 \\ \implies m\ddot{x}_1 &- k(x_2 - 2x_1) + b\dot{x}_1 = 0 \end{aligned} \tag{1}$$

Similarly, considering the forces on mass m with position x_2, the right spring produces the restoring force $k(\ell - x_2)$ rightward while the middle spring produces the restoring force $-k(\ell - (x_1 - x_2))$ leftward. Our equation of motion becomes

$$\begin{aligned} m\ddot{x}_2 &= k(\ell - x_2) - k(\ell - (x_1 - x_2)) - b\dot{x}_2 \\ &= k(x_1 - 2x_2) - b\dot{x}_2 \\ \implies m\ddot{x}_2 &- k(x_1 - 2x_2) + b\dot{x}_2 = 0 \end{aligned} \tag{2}$$

Since these ordinary differential equations are linear, let us denote the sum $z = x_1 + x_2$ and the difference $q = x_1 - x_2$. Adding $(1) + (2)$,

$$\begin{aligned} 0 &= m(\ddot{x}_1 + \ddot{x}_2) - k((x_1 + x_2) - 2(x_1 + x_2)) + b(\dot{x}_1 + \dot{x}_2) \\ &= m\ddot{z} - k(z - 2z) + b\dot{z} \\ &= m\ddot{z} + b\dot{z} + kz \end{aligned} \tag{3}$$

To solve, let $\gamma = \dfrac{b}{2m}$ and $\omega_0^2 = \dfrac{k}{m}$. Then (3) becomes

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0.$$

This is an equation we know how to solve, taking $z = e^{rt}$ we find the characteristic equation to be $r^2 + 2\gamma r + \omega_z^2 = 0$, thus

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

Since underdamping is assumed, $\gamma^2 < \omega_0^2$, so $r = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$. Our solution is

$$z(t) = e^{-\gamma t}\left[c_1 \cos\left(\sqrt{\omega_0^2 - \gamma^2 t}\right) + c_2 \sin\left(\sqrt{\omega_0^2 - \gamma^2 t}\right)\right]. \tag{4}$$

1

Q2

3.5 / 4

2. Same as in problem 1 above, but let us now assume there are no damping forces anymore. However, the left mass is now driven by a driving force $F_d \cos(2\omega t)$, and the right mass has a driving force $2F_d \cos(2\omega t)$, where ω is the square root of k/m. Find the particular solution for the positions of the masses as functions of time. Explain why your answer makes sense.

Q2.

2. Same as in problem 1 above, but let us now assume there are no damping forces anymore. However, the left mass is now driven by a driving force $F_d \cos(2\omega t)$, and the right mass has a driving force $2F_d \cos(2\omega t)$, where ω is the square root of k/m. Find the particular solution for the positions of the masses as functions of time. Explain why your answer makes sense.

As there are no damping forces and two different driven forces added to (1) and (2), we have

$$F_d\cos(2\omega_0 t) = m\ddot{x}_1 - k(x_2 - 2x_1) \tag{7}$$
$$2F_d\cos(2\omega_0 t) = m\ddot{x}_2 - k(x_1 - 2x_2). \tag{8}$$

We are already familiar with are homogeneous solution if $b = 0$ from $Q1$.
Again, since these are linear ordinary differential equations, we can take the sum and difference of them. As in $Q1$, let $z = x_1 + x_2$ and $q = x_1 - x_2$. Then $(7) + (8)$ is

$$m\ddot{z} + kz = 3F_d\cos(2\omega_0 t) \tag{9}$$

and $(7) - (8)$ becomes

$$m\ddot{q} + 3kq = -F_d\cos(2\omega_0 t). \tag{10}$$

We may proceed to solve for the particular solution of the ODE's via the method of undetermined coefficients. For both equations, guess the solution $F_d\left[A\cos(2\omega_0 t) + B\sin(2\omega_0 t)\right]$. The first two derivatives are given by

$$\begin{aligned} F &= F_d\left[A\cos(2\omega_0 t) + B\sin(2\omega_0 t)\right] \\ F' &= F_d\left[-2\omega_0 A\sin(2\omega_0 t) + 2\omega_0 B\cos(2\omega_0 t)\right] \\ F'' &= F_d\left[-4\omega_0^2 A\cos(2\omega_0 t) - 4\omega_0^2 B\sin(2\omega_0 t)\right]. \end{aligned}$$

Plugging into (9) yields

$$\begin{aligned} 3\frac{F_d}{m}\cos(2\omega_0 t) &= F_d\left[-4\omega_0^2 A\cos(2\omega_0 t) - 4\omega_0^2 B\sin(2\omega_0 t)\right] + \frac{k}{m}F_d\left[A\cos(2\omega_0 t) + B\sin(2\omega_0 t)\right] \\ &= F_d\left[-4\omega_0^2 A\cos(2\omega_0 t) - 4\omega_0^2 B\sin(2\omega_0 t)\right] + \omega_0^2 F_d\left[A\cos(2\omega_0 t) + B\sin(2\omega_0 t)\right]. \end{aligned}$$

We instantly see that $B = 0$. Then

$$\begin{aligned} 3\frac{F_d}{m}\cos(2\omega_0 t) &= -4\omega_0^2 AF_d\cos(2\omega_0 t) + \omega_0^2 AF_d\cos(2\omega_0^2 t) \\ &= -3\omega_0^2 AF_d\cos(2\omega_0 t) \\ \implies A &= -\frac{1}{m\omega_0^2} = -\frac{1}{k}. \end{aligned}$$

$z_p(t)$ is then

$$z_p(t) = -\frac{1}{k}F_d\cos(2\omega_0 t).$$

3

Similarly, plugging into (10) yields

$$
\begin{aligned}
-\frac{F_d}{m}\cos(2\omega_0 t) &= F_d\left[-4\omega_0^2 A\cos(2\omega_0 t) - 4\omega_0^2 B\sin(2\omega_0 t)\right] + \frac{3k}{m}F_d\left[A\cos(2\omega_0 t) + B\sin(2\omega_0 t)\right] \\
&= F_d\left[-4\omega_0^2 A\cos(2\omega_0 t) - 4\omega_0^2 B\sin(2\omega_0 t)\right] + 3\omega_0^2 F_d\left[A\cos(2\omega_0 t) + B\sin(2\omega_0 t)\right].
\end{aligned}
$$

Again, we see that $B = 0$. Then

$$
\begin{aligned}
-\frac{F_d}{m}\cos(2\omega_0 t) &= -4\omega_0^2 AF_d\cos(2\omega_0 t) + 3\omega_0^2 AF_d\cos(2\omega_0^2 t) \\
&= -\omega_0^2 AF_d\cos(2\omega_0 t) \\
\implies A &= \frac{1}{m\omega_0^2} = \frac{1}{k}.
\end{aligned}
$$

$q_p(t)$ is then

$$q_p(t) = \frac{1}{k}F_d\cos(2\omega_0 t).$$

As noted in $Q1$,

$$x_1 = \frac{z+q}{2} \qquad x_2 = \frac{z-q}{2}.$$

The particular solution for x_1 becomes 0 for all t, while the particular solution for x_2 becomes

$$
\begin{aligned}
x_{2_p} &= \frac{-\frac{1}{k}F_d\cos(2\omega_0 t) - \frac{1}{k}F_d\cos(2\omega_0 t)}{2} \\
&= -\frac{1}{k}F_d\cos(2\omega_0 t).
\end{aligned}
$$

Therefore our particular solutions are

$$\boxed{x_{1_p}(t) = 0, \qquad x_{2_p}(t) = -\frac{1}{k}F_d\cos(2\omega_0 t).}$$

These particular solutions represent the long-term (or steady state) behaviour of the masses as $t \to \infty$, since as $t \to \infty$, the homogeneous solutions from $Q1$ approach 0. These solutions make sense while examining the steady state behaviour because the left mass is driven by $F_d\cos(2\omega_0 t)$ from (7) while the right mass is driven by $2F_d\cos(2\omega_0 t)$ from (8).

This is not true here, and is only true when there is damping which kills the homogeneous solution long term

Because the force driving the right mass is twice the magnitude of the force driving the left mass, the long term behaviour of the two masses being driven will end up cancelling the motion of the left mass entirely because of the reflected driven force from the right mass on the left wall (it first gets dissipated by the driving force from the left mass, then reflected, cancelling the long-term motion of the left mass to 0). This too occurs with the left mass on the right mass, however it does not cancel out the long-term motion of the right mass.

This isn't quite right; the driving force on mass 1 is cancelled by the spring force of the middle spring

Examining x_{2_p}, we have

$$x_{2_p} = -\frac{1}{k}F_d\cos(2\omega_0 t).$$

If $k \to \infty$, we find $x_{p_2} \to 0$. Physically, this means that the springs are very stiff, so [illegible] enough time for the right mass to oscillate significantly and will result in low displa[illegible] mass has no affect on the steady state behaviour because the driving forces do not de[illegible] masses.

Fair enough point

Q3

4 / 4

3. Fourier Series:

(a) Consider the function $f(t)$ which is periodic with period $T = 2\pi$ and looks as follows from 0 to 2π:

$$f(t) = \begin{cases} t & \text{for} \quad 0 \le t \le \pi \\ 2\pi - t & \text{for} \quad \pi < t \le 2\pi \end{cases}$$

Is this function odd, even or asymmetric? Based on your answer, do you anticipate any of the function's Fourier coefficients to be 0? Then find the Fourier coefficients A0, An, and Bn for this function by rigorous math derivation.

(b) Consider the driving force given by

$$F_d(t) = F_0 \cos(\omega_d t - \phi_0)$$

where $F_0 > 0$ is the amplitude of the driving force, ω_d is the driving frequency and ϕ_0 is the phase.

- Find the Fourier series coefficients for this function $F_d(t)$.
  (Hint: there is trivial method to do this)
- If this driving force is applied to a pendulum bob of mass m, length l and with no damping, what is the total (particular plus homogeneous) solution for the angle θ if $\theta(0) = 0$ and $\dot{\theta}(0) = 0$?

Q3a.

3. Fourier Series:

(a) Consider the function $f(t)$ which is periodic with period $T = 2\pi$ and looks as follows from 0 to 2π:

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Is this function odd, even or asymmetric? Based on your answer, do you anticipate any of the function's Fourier coefficients to be 0? Then find the Fourier coefficients A0, An, and Bn for this function by rigorous math derivation.

We begin by calculating the Fourier Coefficients given by

$$A_0 = \frac{1}{T}\int_0^T f(t)\,dt, \qquad A_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)\,dt, \qquad B_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)\,dt.$$

Since $f(t)$ is an even function, we can expect $B_n = 0$ for all $n \in \mathbb{N}^+$ because $\sin(t)$ is an odd function. Our period is $T = 2\pi$.

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} f(t)\,dt + \int_{\pi}^{2\pi} f(t)\,dt\right] \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} t\,dt + \int_{\pi}^{2\pi}(2\pi - t)\,dt\right] \\ &= \frac{1}{2\pi}\left[\frac{t^2}{2}\Big|_0^{\pi} + 2\pi t\Big|_{\pi}^{2\pi} - \frac{t^2}{2}\Big|_{\pi}^{2\pi}\right] \\ &= \frac{\pi}{4} + \pi - \frac{3\pi}{4} = \frac{\pi}{2} \end{aligned}$$

Similarly,

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt \\ &= \frac{1}{\pi}\left[\int_0^{\pi} t\cos(t)\,dt + \int_{\pi}^{2\pi}(2\pi - t)\,dt\right] \\ &= \frac{1}{\pi}\left[\int_0^{\pi} t\cos(t)\,dt + 2\pi\int_{\pi}^{2\pi} dt - \int_{\pi}^{2\pi} t\,dt\right] \end{aligned}$$

Using tabular integration, we have

sign	D	I
$+$	t	$\cos nt$
$-$	1	$\frac{1}{n}\sin nt$
$+$	0	$-\frac{1}{n^2}\cos nt$

which yields

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\left(\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right)\Big|_0^{\pi} + \left(\frac{2\pi}{n}\right)\Big|_{\pi}^{2\pi} - \left(\frac{t}{n}\sin nt - \frac{1}{n^2}\cos nt\right)\Big|_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos(\pi n) - 1) - \frac{1}{n^2}(\cos(2\pi n) - \cos(\pi n))\right] \\
&= \frac{1}{\pi n^2}\left(2\cos(\pi n) - \cos(2\pi n) - 1\right).
\end{aligned}
$$

Therefore our Fourier Coefficients are

$$\boxed{A_0 = \frac{\pi}{2} \qquad A_n = \frac{1}{\pi n^2}\left(2\cos(\pi n) - \cos(2\pi n) - 1\right) \qquad B_n = 0}$$

Q3b.

(b) Consider the driving force given by

$$F_d(t) = F_0 \cos(\omega_d t - \phi_0)$$

where $F_0 > 0$ is the amplitude of the driving force, ω_d is the driving frequency and ϕ_0 is the phase.

- Find the Fourier series coefficients for this function $F_d(t)$.
  (Hint: there is trivial method to do this)
- If this driving force is applied to a pendulum bob of mass m, length l and with no damping, what is the total (particular plus homogeneous) solution for the angle θ if $\theta(0) = 0$ and $\dot{\theta}(0) = 0$?

Our function is given by

$$F_d(t) = F_0 \cos(\omega_d t - \phi_0)$$

which is equivalent to

$$F_d(t) = F_0 \left[\cos(\phi_0)\cos(\omega_d t) + \sin(\phi_0)\sin(\omega_d t)\right].$$

The Fourier Series expansion is given by the formula

$$F_d(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n t}{T}\right) + B_n \sin\left(\frac{2\pi n t}{T}\right).$$

The period T of $F_d(t)$ is given by $T = \dfrac{2\pi}{\omega_d}$. Our Fourier expansion then becomes

$$F_d(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_d n t) + B_n \sin(\omega_d n t).$$

Calculating A_n, B_n, we find

$$\begin{aligned}
A_n &= \frac{F_0\omega_d}{\pi}\left[\cos(\phi_0)\int_0^T \cos(\omega_d t)\cos(\omega_d n t)\,dt + \sin(\phi_0)\int_0^T \sin(\omega_d t)\cos(\omega_d n t)\,dt\right] \\
&= \frac{F_0\omega_d}{\pi}\frac{\pi}{\omega_d}\cos(\phi_0)\delta_{n\omega_d,\omega_d} + 0 \\
&= F_0\cos(\phi_0)\delta_{n\omega_d,\omega_d} \\
B_n &= \frac{F_0\omega_d}{\pi}\left[\cos(\phi_0)\int_0^T \cos(\omega_d t)\sin(\omega_d n t)\,dt + \sin(\phi_0)\int_0^T \sin(\omega_d t)\sin(\omega_d n t)\,dt\right] \\
&= \frac{F_0\omega_d}{\pi}\frac{\pi}{\omega_d}\sin(\phi_0)\delta_{n\omega_d,\omega_d} + 0 \\
&= F_0\sin(\phi_0)\delta_{n\omega_d,\omega_d}
\end{aligned}$$

The Delta Functions imply that the only nonzero terms in the Fourier Expansion is when $n = 1$ and will be zero for any other n since the frequencies are different, by the orthogonality properties of sin and cos.

The term $A_0 = 0$ because the integral over the period of any periodic function will be zero:

$$\begin{aligned} A_0 &= \frac{F_0\omega_d}{2\pi}\left[\cos(\phi_0)\int_0^T \cos(\omega_d t)\,dt + \sin(\phi_0)\int_0^T \sin(\omega_d t)\,dt\right] \\ &= \frac{F_0\omega_d}{2\pi}\left[\cos(\phi_0)(1-1) + \sin(\phi_0)(1-1)\right] \\ &= 0 \end{aligned}$$

Therefore our Fourier Expansion of $F_d(t)$ is

$$F_d(t) = F_0\cos(\omega_d t - \phi_0)$$

with

$$\boxed{A_0 = 0 \qquad A_n = F_0\cos(\phi_0)\delta_{n\omega_d,\omega_d} \qquad B_n = F_0\sin(\phi_0)\delta_{n\omega_d,\omega_d}}$$

We now solve the equation of motion for the pendulum of mass m, length l, and no damping. Our equation of motion is given by

$$\begin{aligned} I\alpha &= F_G\ell\sin\theta + F_d\cos(\omega_d t - \phi_0) \\ m\ell^2\ddot{\theta}(t) &= -mg\ell\sin\theta + F_d\cos(\omega_d t - \phi_0)\ell\sin\theta \\ \ddot{\theta}(t) &= -\frac{g}{\ell}\sin\theta + \frac{F_d}{m\ell}\cos(\omega_d t - \phi_0)(1) \\ &\approx -\frac{g}{\ell}\theta + \frac{F_d}{m\ell}\cos(\omega_d t - \phi_0) \\ \implies \ddot{\theta}(t) + \frac{g}{\ell}\theta(t) &= \frac{F_d}{m\ell}\cos(\omega_d t - \phi_0) \end{aligned} \tag{11}$$

for small angles.

Let $F_0 = \dfrac{F_d}{m\ell}$ and let $\omega_0 = \dfrac{g}{\ell}$. Our homogeneous equation is

$$\begin{aligned} 0 &= \ddot{\theta}(t) + \frac{g}{\ell}\theta(t) \\ \implies r^2 &= -\omega_0^2 \\ \implies r &= \pm i\,\omega. \end{aligned}$$

The homogeneous solution is then

$$\theta_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t).$$

Like $Q2$, we will solve for the particular solution via the method of undetermined coefficients. Guess the solution

$$\begin{aligned} F &= AF_0\cos(\omega_d t - \phi_0) + BF_0\sin(\omega_d t - \phi_0) \\ F' &= -A\omega_d F_0\sin(\omega_d t - \phi_0) + B\omega_d F_0\cos(\omega_d t - \phi_0) \\ F'' &= -\omega_d^2\left[AF_0\cos(\omega_d t - \phi_0) + BF_0\sin(\omega_d t - \phi_0)\right]. \end{aligned}$$

Subbing into (11), we determine that $B = 0$ and

$$\begin{aligned} F_0 \cos(\omega_d^2 t - \phi_0) &= -\omega_d^2 A F_0 \cos(\omega_d t - \phi_0) + \omega_0^2 A F_0 \cos(\omega_d t - \phi_0) \\ \implies 1 &= -\omega_d^2 A + \omega_0^2 A \\ \implies A &= \frac{1}{\omega_0^2 - \omega_d^2}. \end{aligned}$$

Therefore the total solution to (11) is

$$\boxed{\theta(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{1}{m\ell(\omega_0^2 - \omega_d^2)} F_d \cos(\omega_d t - \phi_0)}$$

Solving the initial conditions,

$$\begin{aligned} \theta(0) &= c_1 + c_2 + \frac{1}{\omega_0^2 - \omega_d^2} F_0 \cos(\phi_0) = 0 \\ \dot{\theta}(0) &= -\omega_0 c_1 + \omega_0 c_2 - \frac{\omega_d}{\omega_0^2 - \omega_d^2} F_0 \cos(\phi_0) = 0 \end{aligned}$$

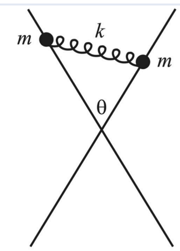
and after some mathematical magic, we find

$$\boxed{c_1 = \frac{1}{2}\left[\frac{F_d \cos(\phi_0)}{m\ell(\omega_0^2 - \omega_d^2)}\left(\frac{\omega_d}{\omega_0} - 3\right)\right] \qquad c_2 = \frac{1}{2}\left[\frac{F_d \cos(\phi_0)}{m\ell(\omega_0^2 - \omega_d^2)}\left(1 - \frac{\omega_d}{\omega_0}\right)\right]}$$

Q4

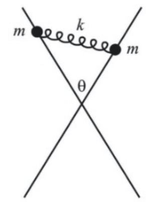
4 / 4

4. Two horizontal frictionless rails make an angle θ with each other, as shown in the figure. Each rail has a bead of mass m on it, and the beads are connected by a spring with spring constant k and relaxed length zero. Assume that one of the rails is positioned a tiny distance above the other, so that the beads can pass freely through the crossing. Find the general solution for the positions of each mass as functions of time.

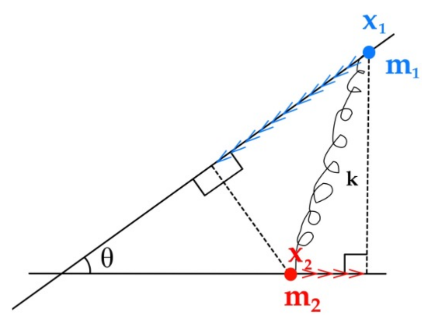


Q4.

4. Two horizontal frictionless rails make an angle θ with each other, as shown in the figure. Each rail has a bead of mass m on it, and the beads are connected by a spring with spring constant k and relaxed length zero. Assume that one of the rails is positioned a tiny distance above the other, so that the beads can pass freely through the crossing. Find the general solution for the positions of each mass as functions of time.



Simplifying the orientation of the system yields



The length of the line from the distance x_1 along axis 1 to the orthogonal projection of the distance x_2 onto axis 1 (the line with the blue arrows) is given by

$$\Delta x_1 = x_1 - x_2 \cos\theta$$

while the length of the line from the distance x_2 along axis 2 to the orthogonal projection of the distance x_1 onto axis 2 (the line with the red arrows) is given by

$$\Delta x_2 = x_1 \cos\theta - x_2.$$

Now, the spring force acting on x_1 will want to move towards x_2 but is restricted along the x_1 axis, and vice versa for x_2. This yields the equations of motion

$$m\ddot{x}_1 = -k(x_1 - x_2\cos\theta) \tag{12}$$
$$m\ddot{x}_2 = -k(x_2 - x_1\cos\theta) \tag{13}$$

As in $Q1$ and $Q2$, let $z = x_1 + x_2$ and $q = x_1 - x_2$. Adding (12) + (13) yields

$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= -k((x_1 + x_2) - \cos\theta(x_1 + x_2)) \\ m\ddot{z} &= -kz(1 - \cos\theta) \qquad (14) \end{aligned}$$

and

$$\begin{aligned} m(\ddot{x}_1 - \ddot{x}_2) &= -kx_1 + kx_2\cos\theta - [-kx_2 + kx_1\cos\theta] \\ &= -kx_1 + kx_2\cos\theta + kx_2 - kx_1\cos\theta \\ &= -k(x_1 - x_2) - k\cos\theta(x_1 - x_2) \\ m\ddot{q} &= -kq(1 + \cos\theta) \qquad (15) \end{aligned}$$

Solving (15), let $\omega_0^2 = \frac{k}{m}$. Then

$$\begin{aligned} \ddot{z} &= -\frac{k}{m}(1 - \cos\theta)z \\ \implies r^2 &= -\omega_0^2(1 - \cos\theta) \\ \implies r &= \pm i\,\omega_0\sqrt{1 - \cos\theta}. \end{aligned}$$

The solution for z is then

$$z(t) = c_1\cos\left(\omega_0\sqrt{1 - \cos\theta}\,t\right) + c_2\sin\left(\omega_0\sqrt{1 - \cos\theta}\,t\right). \qquad (16)$$

Similarly, solving (15) yields

$$\begin{aligned} \ddot{q} &= -\frac{k}{m}(1 + \cos\theta)q \\ \implies r^2 &= -\omega_0^2(1 + \cos\theta) \\ \implies r &= \pm i\,\omega_0\sqrt{1 + \cos\theta}. \end{aligned}$$

The solution for q is then

$$q(t) = c_3\cos\left(\omega_0\sqrt{1 + \cos\theta}\,t\right) + c_4\sin\left(\omega_0\sqrt{1 + \cos\theta}\,t\right). \qquad (17)$$

As in $Q1$, we know that $x_1 = \dfrac{z + q}{2}$ and $x_2 = \dfrac{z - q}{2}$. Using (16) and (17), we have that

$$\boxed{x_1 = \frac{1}{2}\left[c_1\cos\left(\omega_0\sqrt{1 - \cos\theta}\,t\right) + c_2\sin\left(\omega_0\sqrt{1 - \cos\theta}\,t\right) + c_3\cos\left(\omega_0\sqrt{1 + \cos\theta}\,t\right) + c_4\sin\left(\omega_0\sqrt{1 + \cos\theta}\,t\right)\right]}$$

and

$$\boxed{x_2 = \frac{1}{2}\left[c_1\cos\left(\omega_0\sqrt{1 - \cos\theta}\,t\right) + c_2\sin\left(\omega_0\sqrt{1 - \cos\theta}\,t\right) - c_3\cos\left(\omega_0\sqrt{1 + \cos\theta}\,t\right) - c_4\sin\left(\omega_0\sqrt{1 + \cos\theta}\,t\right)\right]}$$