PHY350 Problem Set 1 — Due Monday October 3, 2022
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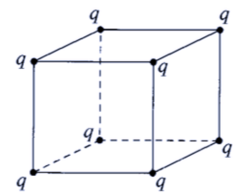
Problem Set 1: Due 3 Oct 2022 (4 problems, submit Online - Weight 2.5%). One of the problems will be marked 1.5%, rest 1% will be for simply completing the homework. Please view these as practice problems for the test/exam.

Problem 3.1 Find the average potential over a spherical surface of radius R due to a point charge q located *inside* (same as above, in other words, only with $z < R$). (In this case, of course, Laplace's equation does not hold within the sphere.) Show that, in general,

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R},$$

where V_{center} is the potential at the center due to all the *external* charges, and Q_{enc} is the total enclosed charge.

Problem 3.2 In one sentence, justify **Earnshaw's Theorem:** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.* As an example, consider the cubical arrangement of fixed charges in Fig. 3.4. It *looks*, off hand, as though a positive charge at the center would be suspended in midair, since it is repelled away from each corner. Where is the leak in this "electrostatic bottle"? [To harness nuclear fusion as a practical energy source it is necessary to heat a plasma (soup of charges particles) to fantastic temperatures—so hot that contact would vaporize any ordinary pot. Earnshaw's theorem says that electrostatic containment is also out of the question. Fortunately, it *is* possible to confine a hot plasma *magnetically*.]



Problem 3.9 A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

(a) Find the potential in the region above the plane.

(b) Find the charge density σ induced on the conducting plane.

Problem 3.10 Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in Fig. 3.15. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

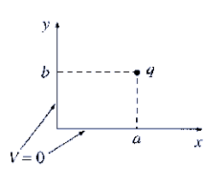


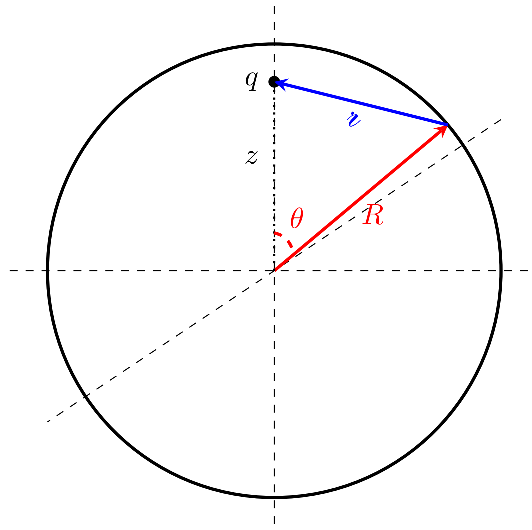
Figure 3.15

(3.1)

The potential of a charge q placed inside a sphere of radius R is given by the definition:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{q}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$

where $\mathbf{r}$ is the vector directed to the spherical surface conductor and $\mathbf{r}'$ is the vector directed to the charge q. The quantity $|\mathbf{r}-\mathbf{r}'|$ can be calculated directly via geometry:



It is given by

$$\begin{aligned}
|\mathscr{r}|^2 &= (z - R\cos\theta)^2 + (R\sin\theta)^2 \\
&= z^2 + R^2\cos^2\theta + R^2\sin^2\theta - 2Rz\cos\theta \\
&= z^2 + R^2 - 2Rz\cos\theta.
\end{aligned}$$

Since R is a constant along the sphere, we must proceed by taking the volume integral

$$\begin{aligned}
V_{\text{avg}} &= \frac{1}{4\pi R^2}\frac{1}{4\pi\varepsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{q}{\sqrt{z^2+R^2-2Rz\cos\theta}}\,R^2\sin\theta\,d\varphi\,d\theta \\
&= \frac{2\pi}{8\pi^2}\frac{q}{2\varepsilon_0}\int_0^{\pi}\frac{d\theta}{\sqrt{z^2+R^2-2Rz\cos\theta}} \\
&= \frac{q}{4\pi\varepsilon_0}\frac{1}{2Rz}\sqrt{z^2+R^2-2Rz\cos\theta}\,\Big|_0^{\pi} \\
&= \frac{q}{4\pi\varepsilon_0}\frac{1}{2Rz}[(R+z)-(R-z)] \\
&= \frac{q}{4\pi\varepsilon_0}\frac{2z}{2Rz} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \equiv \frac{1}{4\pi\varepsilon_0}\frac{Q_{\text{enc}}}{R},
\end{aligned}$$

which is exactly the potential for any charge q inside the sphere, not including any external charge. I must note that the potential is independent of where the charge is placed inside the sphere. The reason I needed to take $[(R+z)-(R-z)]$ instead of $[(z+R)-(z-R)]$ is because of the assumption that $|z| < R$.

Now, if we impose any external charges, in which $\rho_{\text{ext}} = 0$ inside the sphere, then the average potential of those external charges is equal to the potential at the center of the sphere by the mean value theorem in electrostatics. Therefore

$$V_{\text{avg}} = V_{\text{center}} + \frac{1}{4\pi\varepsilon_0}\frac{Q_{\text{enc}}}{R},$$

which is what I wanted to show.

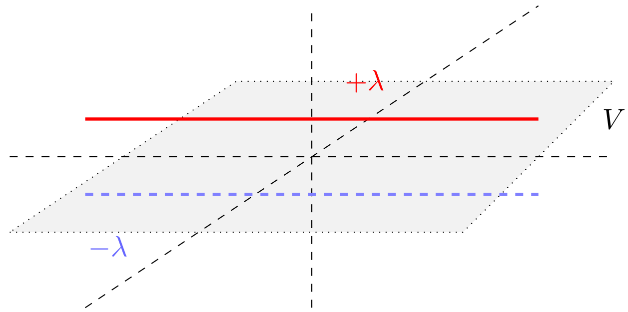
(3.2)

Earnshaw's Theorem: *A charged particle cannot be held in stable equilibrium by electrostatic forces alone.*
The reason as to which is why this theorem is true is because the maximum of any electric potential produced by external charges must attain maximum and minimum values only along the boundary of the domain.

To justify this statement, allow me to consider a one-dimensional example: $\frac{\partial^2}{\partial x^2}V = 0$ which implies that V is linear in x: $V(x) = mx + b$. For any straight line with $m \neq 0$, the function will attain points of extremum only on the endpoints of a closed interval. This exact same property applies to potentials in two and three dimensions. For the cubical arrangement of charges, the centre of the cube, in four-dimensional space, would be a saddle point of unstable equilibrium because the values of extremum of the potential must strictly occur along the boundary of the cube. If all eight charges are the same, the points of maximum value of V occur at the corners of the cube, while the minumum values of V over the cube must occur at the center of each side face.

(3.9)

(a) To find the potential in the region above the plane, we can invoke the method of images and define a new problem with no grounded plane. Place an image infinite line charge a distance $-d$ of charge $-\lambda$ from the z-axis:



Now the potential of a single line charge of length $2L$ is given by $V = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{d\ell}{\sqrt{d^2+\ell^2}}$ in cylindrical coordinates, which then evaluates to

$$V(s,0,0) = \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{L+\sqrt{s^2+L^2}}{-L+\sqrt{s^2+L^2}}\right).$$

Taking the limit of the line charge as $L\to\infty$ produces a divergent limit (ie, the potential explodes), so we shall implement a second reference radius s_0 and compare both potentials from infinity:

$$[V(s,0,0)-V(\infty,0,0)] - [V(s_0,0,0)-V(\infty,0,0)] = V(s)-V(s_0).$$

Because I do not want to type all of this out (and Griffiths has already done this), as we take the limit as $L\to\infty$ and obtain the expression of a potential for a single line charge:

$$V(s,\varphi,x) = \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{s_0}{s}\right).$$

By symmetry, this potential should only be dependent on the distance away from the line charge - which it is. Now, if we impose a second line charge a distance $2d$ apart from the first of radius s', we may superpose the two potentials from each distribution:

$$\begin{aligned} V_{+\lambda}+V_{-\lambda} &= \frac{\lambda}{2\pi\varepsilon_0}\left[\log\left(\frac{s_0}{s}\right)-\log\left(\frac{s_0}{s'}\right)\right] \\ &= \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{s'}{s}\right). \end{aligned}$$

Shifting the point of origin down d so that it is located in between the line charges, the potential then becomes

$$\begin{aligned} V &= \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{\sqrt{(z+d)^2+x^2}}{\sqrt{(z-d)^2+x^2}}\right) \\ &= \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{(z+d)^2+x^2}{(z-d)^2+x^2}\right) \end{aligned}$$

in cartesian coordinates. Restricting z to be positive yields the potential in the region above the plane by the method of images:

$$V(x, y, z > 0) = \frac{\lambda}{4\pi\varepsilon_0} \log\left(\frac{(z+d)^2 + x^2}{(z-d)^2 + x^2}\right).$$

(b) Now the induced surface charge on the plane is given by $\dfrac{\sigma}{\varepsilon_0} = -\dfrac{\partial V}{\partial n} = -\dfrac{\partial V}{\partial z}$ in this case, since the normal vector of the plane points upwards in the z-direction. I will proceed by differentiating:

$$\begin{aligned}
\sigma(x, y, z) &= -\varepsilon_0 \frac{\partial V}{\partial z} \\
&= -\frac{\lambda}{4\pi}\frac{\partial}{\partial z}[\log((z+d)^2 + x^2) - \log((z-d)^2 + x^2)] \\
&= -\frac{\lambda}{4\pi}\left[\frac{2(z+d)}{(z+d)^2 + x^2} - \frac{2(z-d)}{(z-d)^2 + x^2}\right].
\end{aligned}$$

Now, at the surface of the plane, we have that $z = 0$, thus it follows that the surface charge on the surface of the plane is

$$\begin{aligned}
\sigma(x, y, z = 0) &= -\frac{2\lambda}{4\pi\varepsilon_0}\left[\frac{d}{d^2 + x^2} + \frac{d}{d^2 + x^2}\right] \\
&= \frac{4\lambda}{4\pi\varepsilon_0}\left[\frac{d}{d^2 + x^2}\right] \\
&= -\frac{\lambda d}{\pi(d^2 + x^2)}.
\end{aligned}$$

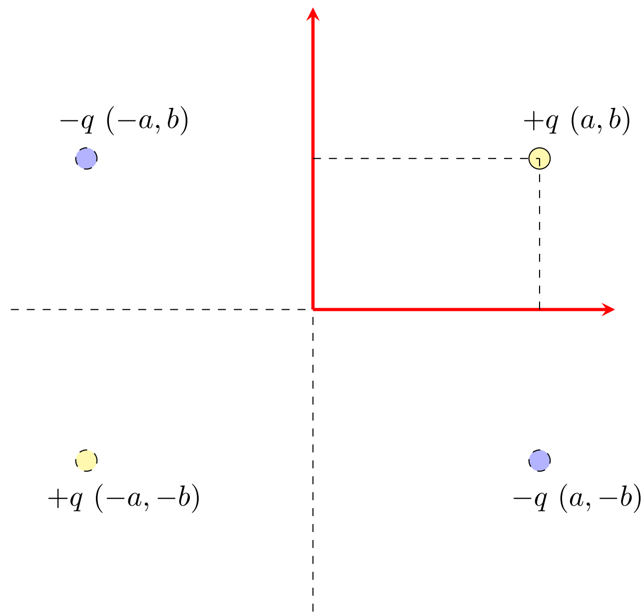
Intuitively, this makes sense because the surface charge is only dependent on x while restricted to the plane, and the surface charge is a maximum in magnitude right underneath the wire when $x = 0$. I wish to conduct a sanity check by determining the amount of induced charge per unit length on the plane. Consider a segment of wire of length L:

$$\begin{aligned}
Q_{\text{tot}} = L \int_{\mathbb{R}} dx\, \sigma(x, y, z = 0) &= -\frac{L\lambda d}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x^2 + d^2} \\
&= -\frac{L\lambda d}{\pi}\frac{1}{d} \arctan\left(\frac{x}{d}\right)\Bigg|_{-\infty}^{\infty} \\
&= -\frac{L\lambda}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= -L\lambda,
\end{aligned}$$

and therefore for the whole plane the induced charge is $-\lambda$, as required.

(3.10)

Consider the configuration of planes and charge as stated in the problem. Notice that to make the potential zero on both planes, we reflect an image charge of negative value on either side of each of the planes equidistant of that of the first charge. However, doing so does not make the potential at the origin zero! This is because the potential created by the negative charges will overwhelm the potential created by the single positive charge. The solution to this is to place a third mirror charge, which is positive this time, reflected in the line $y = -x$ from the first charge. The mirror charge configuration is then



which, by symmetry, allows us to easily calculate the potential in the region of the planes by the method of superposition. The distances between the charges are simply given in terms of x and y by $\sqrt{(x \pm a)^2 + (y \pm b)^2}$, and so from $V(x, y) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{|\mathscr{r}|}$, we obtain

$$\begin{aligned}
V(x, y) &= \sum_i V_i(x, y) \\
&= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \right. \\
&\qquad \left. - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} \right].
\end{aligned}$$

The first two terms are taken by the positive charges, while the following two terms the negative charges. By the uniqueness theorem, this is the potential in the $(+x, +y)$ region, which is what I needed to find. Now, the electrostatic force on the charge q can be found a similar way by invoking Coulombs law:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_i q_j}{r_{ij}^2} \hat{\mathbf{r}}_{ij}.$$

Expanding this out:

$$\mathbf{F} = \frac{q}{4\pi\varepsilon_0} \left[\frac{+q}{(2a)^2 + (2b)^2} \hat{\mathbf{r}}_{++} + \frac{-q}{(2a)^2} \hat{\mathbf{r}}_{-+} + \frac{-q}{(2b)^2} \hat{\mathbf{r}}_{+-} \right]$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{4a^2+4b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\hat{\mathbf{x}} + \frac{b}{\sqrt{a^2+b^2}}\hat{\mathbf{y}}\right] - \frac{1}{4a^2}\hat{\mathbf{x}} - \frac{1}{4b^2}\hat{\mathbf{y}}\right]$$
$$= \frac{q^2}{16\pi\varepsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{\mathbf{y}}\right],$$

which gives the total force on the charge in the region of the two planes. To calculate the amount of work it takes to bring the charge in from infinity, we may utilize the expression $W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$ for each of the three image charges acting on the initial charge. The initial charge is located at (a, b) and hence, by imposing the expression we have found for electric potential before, we have that

$$W = \frac{1}{2}\frac{q \cdot q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(a+a)^2+(b+b)^2}} - \frac{1}{\sqrt{(a+a)^2+(b-b)^2}} - \frac{1}{\sqrt{(a-a)^2+(b+b)^2}}\right]$$
$$= \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{2\sqrt{a^2+b^2}} - \frac{1}{2a} - \frac{1}{2b}\right]$$
$$= \frac{q^2}{16\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right],$$

which is the amount of energy required to bring the particle in from infinity towards the semi-infinite plane conductors. Note that the method of images does work for specific other angle other than 90°, and these specific angles are determined by the symmetry of the system. When the angle is 180°, the image charge placed is negatve of opposite distance from that of the plane. When $\theta = 60^{\circ}$, a symmetric charge configuration can be established. Similarly for 45°, 30°, and so on. Notice that each of these values are then integer divisors of 180°, because by this method a completely symmetric system of image charges can be placed which equally reflect the potential across the planes. We can think of this in terms of polygons: with the centre of the polygon placed at the origin, image charges are placed on the vertices of the polygon: triangles, rectangles, pentagons, hexagons, heptagons, and so on, are all representations of the angles which add symmetrically to 180°.