PHY350 Problem Set 4 Due December 8, 22

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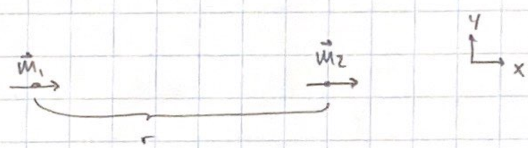
6.3) a) Using $F = 2\pi I R B \cos\theta$.

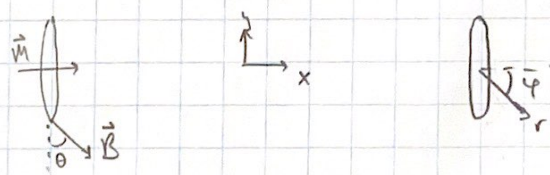
First, recall that the magnetic field produced by a dipole $\vec{m}$ is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m} \right].$$

Suppose $\vec{m}$ is aligned in the $\hat{x}$ direction, as in the diagram,



Assuming that the $\vec{m}_i$'s can be treated as current loops, then $\vec{B}$ varies as a function of θ:



So $B\cos\theta = \vec{B}\cdot\hat{y}$. If $\vec{B}\cdot\hat{y}$, then $\vec{m}_i\cdot\hat{y} = 0$ since $\vec{m}_i$ is orthogonal to $\hat{y}$. Furthermore, $\hat{r}\cdot\hat{y} = \sin\varphi$, which is equivalently $\frac{R}{r}$. This implies that

$$B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_1\cdot\hat{r})(\hat{r}\cdot\hat{y})\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3m_1\cos\theta\sin\varphi\right].$$

However, since $\cos\theta = \frac{\sqrt{r^2-R^2}}{r}$ (since $|r\cos\theta| = \sqrt{r^2-R^2}$ on the circle) then

$$F = 2\pi I R \left(B\cos\theta \right)$$

$$= 2\pi I R \left(\frac{\mu_0}{4\pi r^3} \cdot 3 m_1 \frac{\sqrt{r^2 - R^2}}{r} \cdot \frac{R}{r} \right)$$

$$= \frac{3}{2} \frac{\pi I R^2}{\pi} \mu_0 m_1 \frac{\sqrt{r^2 - R^2}}{r^5}$$

$$= \frac{3\mu_0}{2\pi} m_1 m_2 \frac{\sqrt{r^2 - R^2}}{r^5}$$

since m_2 is defined as $\pi R^2 \cdot I$.

However, since the m_i's are not current loops but rather are infinitessimal, one may take the limit when $r >> R$, when the loop radius is very small.

A binomial expansion on $\sqrt{r^2 - R^2}$ yields that

$$\sqrt{r^2 - R^2} = r\sqrt{1 - R^2/r^2} \approx r\left(\sqrt{1} - \frac{R^2}{r^2} \cdot \frac{1}{2\sqrt{1}} - \mathcal{O}\left(\frac{R^4}{r^4}\right)\right)$$

$$= r$$

if we only keep the first order expansion. Therefore

$$\boxed{\vec{F} = \frac{3}{4} \frac{\mu_0 m_1 m_2}{r^4} \hat{x}}$$

is the force of attraction between the two dipoles.

It can be in the $\pm\hat{x}$ direction depending on which force acts on which dipole, but equivalently so since

$$\vec{F}_{1 \text{ on } 2} = -\vec{F}_{2 \text{ on } 1}.$$

b) Consider now the second method of finding the force,

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

The second formulation of $\vec{B}$ is given by

$$\vec{B} = \frac{\mu_0 m_1}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right].$$

For $\vec{B}_1$, $\theta = 0$ hence $\vec{B}_1 = \frac{\mu_0 m_1}{2\pi r^3}\hat{x}$.

For $\vec{B}_2$, $\theta = \pi$, so $\vec{B}_2 = -\frac{\mu_0 m_2}{2\pi r^3}\hat{x}$.

Then, the force is $\vec{F} = \vec{\nabla}\left(\vec{m}_1 \cdot \frac{-\mu_0 m_2}{2\pi r^3}\hat{x}\right)$

$$= \vec{\nabla}\left(-\frac{\mu_0 m_1 m_2}{2\pi r^3}\right)$$

$$= -\frac{3}{2}\frac{\mu_0 m_1 m_2}{\pi r^3}\hat{x}$$

since $\vec{r} = r\hat{x}$ in the $\hat{x}$ direction. Similarly,

$$\boxed{\vec{F}_{2 on 1} = -\vec{F}_{1 on 2} = \frac{3}{2}\frac{\mu_0 m_1 m_2}{\pi r^3}\hat{x}.}$$

Equvalent to the magnitude determined in part (a).

6.8) For this problem, we consider the bound volume and surface currents produced by the magnetization.

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{s}\left(\frac{\partial M_z}{\partial \varphi} - s\frac{\partial M_\varphi}{\partial z}\right)\hat{s} + \left(\frac{\partial M_s}{\partial z} - \frac{\partial M_z}{\partial s}\right)\hat{\varphi}$$
$$+ \frac{1}{s}\left(\frac{\partial}{\partial s}(sM_\varphi) - \frac{\partial M_s}{\partial \varphi}\right)\hat{z}$$
$$= \frac{1}{s}\frac{\partial}{\partial s}(sKs^2)\hat{z}$$
$$= 3Ks\,\hat{z}$$

$$\vec{K}_b = \vec{M} \times \hat{n} = KR^2(\hat{\varphi} \times \hat{s}) = -KR^2\hat{z}.$$

We may now use Ampere's Law for a loop of enclosed current running around the cylinder (the current runs through the loop)

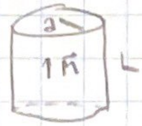
For points outside,

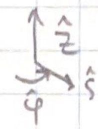
$$I_{enc} = \int_0^{2\pi}\int_0^R 3Ks \cdot s\,ds\,d\varphi + \int_0^{2\pi}(-KR^2)R\,d\varphi$$
$$= 2\pi KR^3 - 2\pi KR^3$$
$$= 0 \implies \boxed{B_{out} = 0}$$

and inside, $\mu_0 I_{enc} = \mu_0\int_0^{2\pi}\int_0^R 3Ks^2\,ds\,d\varphi = \mu_0 2\pi Ks^3 = 2\pi s B$

which implies $\boxed{\vec{B}_{in} = \mu_0 Ks^2\hat{z}}$ since $\vec{B}$ must be parallel to $\vec{M}$.

6.4)





Since $\vec{M}$ is uniform, $\vec{M} = M\hat{z}$.

We instantly observe that $\vec{\nabla} \times \vec{M} = \vec{J}_b = 0$.

Furthermore, on the top and bottom faces

$\hat{n} \parallel \vec{M}$, so $\vec{M} \times \hat{n}_{T,B} = 0$.

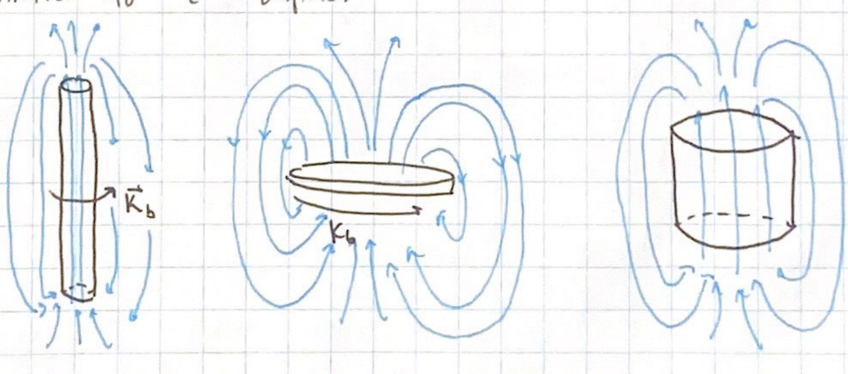
$\hat{n} = \hat{s}$ along the sides, so $\vec{K}_b = M(\hat{z} \times \hat{s}) = M\hat{\varphi}$,

so this is the only bound current.

$$\boxed{\vec{K}_b = M\hat{\varphi}}.$$

Note that this current wraps around the cylinder.

Thus for $L \gg a$, $\vec{K}_b$ makes the cylinder appear like a solenoid. For $L \ll a$, we observe a current distribution similar to a dipole.



"Solenoid" $L \gg a$

"dipole" $L \ll a$

$L \simeq a$.

6.12) a) Consider the infinitely polarized cylinder with

$$\vec{M} = ks\,\hat{z}.$$

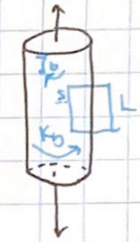
There is no free current in the system.

Now, the bound currents are given by

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \ldots = -k\,\hat{\varphi},$$

$$\vec{K}_b = \vec{M}\times\hat{n} = \ldots = kR\,\hat{\varphi} \quad \text{at the boundary } s=R.$$

Now, we may proceed by finding the enclosed current of a loop with the current running through it.



First note that, outside the cylinder,

$$I_{enc} = L\int_0^{2\pi}\int_0^R (-k)\,R\,ds\,d\varphi + L\int_0^{2\pi}(kR^2)\,d\varphi$$

$$= -2\pi kR^2L + 2\pi kR^2L = 0.$$

Inside the cylinder, however,

$$I_{encl} = L\cdot kR + L\cdot(-k)\int_s^R ds'$$

$$= L\cdot kR - kL(R-s)$$

$$= kLs,$$

so, by Ampere's Law,

$$\boxed{B_{out} = 0,} \quad \text{and} \quad BL = kLs\mu_0, \quad \boxed{\vec{B}_{in} = \mu_0 ks\,\hat{z}}$$

b) Since $J_f = 0$, $\oint \vec{H}\cdot d\vec{l} = I_{f,enc} = 0$

hence $\vec{H} = 0$. Thus $0 = \frac{1}{\mu_0}\vec{B} - \vec{M}$, or

$$\boxed{\vec{B}_{in} = \mu_0\vec{M} = \mu_0 ks\,\hat{z}}$$ while $\vec{B}_{out} = 0$, as desired.