Homework 2

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Q1

10 / 10

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MAT334 Problem Set 2 — Due Monday, October 24 23:00
1006940802

1.

Consider the second-degree continuous differentiable function $u \in C^2$, where $u : \mathbb{R}^2 \to \mathbb{R}$. Let $f(x+iy) = u(x,y) + iu(x,y)$. To require the function f to be analytic, we require the funtion u to satisfy the Cauchy-Riemann equations:

$$u_x = u_y, \quad u_y = -u_x.$$

The above equations imply that $u_x = -u_x = u_y$, so we may proceed by solving the partial differential equations. Firstly, since $u_x = -u_x$, then $2u_x = 0$ hence $u_x = 0$. If $u_x = 0$, then $u_y = 0$ for all x, y. Thus the only functions u which satisfy the Cauchy-Riemann Equations is any constant function $u(x,y) = A$ for $A \in \mathbb{C}$.

1

Q2

10 / 10

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2.

In this problem, I wish to determine a closed for (or explicit expression of a function) of the power series $g(z) = \sum_{n=0}^{\infty} n^2 z^n$ valid on the disc $D = \{z \in \mathbb{C} : |z| < 1\}$, the open ball of radius 1 centered at the origin. First, we may begin by factoring a z to see if we have any similarity of other power series:

$$g(z) = \sum_{n=0}^{\infty} n^2 z \cdot z^{n-1} = z \sum_{n=0}^{\infty} n^2 z^{n-1}.$$

Now, notice that $n^2 = n^2 - n + n = n(n-1) + n$, and the first term in that expansion appears as if one had taken a derivative twice. Then

$$\begin{aligned} g(z) &= z \sum_{n=0}^{\infty} (n(n-1) + n) z^{n-1} \\ &= z \sum_{n=0}^{\infty} n(n-1) z^{n-1} + z \sum_{n=0}^{\infty} n z^{n-1} \\ &= z^2 \sum_{n=0}^{\infty} n(n-1) z^{n-2} + z \sum_{n=0}^{\infty} n z^{n-1}. \end{aligned}$$

Now, since $\frac{d^2}{dz^2} z^n = n(n-1) z^{n-2}$, and $\frac{d}{dz} z^n = n z^{n-1}$, we have that

$$g(z) = z^2 \sum_{n=0}^{\infty} \frac{d^2}{dz^2}[z^n] + z \sum_{n=0}^{\infty} \frac{d}{dz}[z^n].$$

Since the sum of polynomials are analytic on $\mathbb{C}$, we may switch the order of differentiation upon summation:

$$g(z) = z^2 \frac{d^2}{dz^2} \sum_{n=0}^{\infty} z^n + z \frac{d}{dz} \sum_{n=0}^{\infty} z^n.$$

However, the terms expressed under the summation are exactly the terms defining the geometric series, since $|z| < 1$: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. Thus we are first required to determine the first and second derivatives of the geometric series:

$$\frac{d^2}{dz^2}\left[\sum_{n=0}^{\infty} z^2\right] = \frac{d^2}{dz^2}\left[\frac{1}{1-z}\right]$$
$$= \frac{d}{dz}\left[\frac{1}{(1-z)^2}\right] \qquad \text{(first derivative)}$$
$$= \frac{2}{(1-z)^3}. \qquad \text{(second derivative)}$$

Therefore our series expansion for $g(z)$ becomes

$$g(z) = z^2 \cdot \frac{2}{(1-z)^3} + z \cdot \frac{1}{(1-z)^2}$$

Good! **10**

2

$$= \frac{2z^2}{(1-z)^3} + \frac{z}{(1-z)^2},$$

which is thus the closed form result of the power series $g(z) = \sum_{n=0}^{\infty} n^2 z^n$, which is what I wanted to determine.

3

Q3

9 / 10

Submit Your Response to Question 3 Here.

3.

To determine the power series on the disc of radius 2 centered at 3, $D = \{z \in \mathbb{C} : |z-3| < 2\}$, one may invoke one of the consequences of Cauchy's Interal Theorem of determining the coefficients of a power series (Fisher, Theorem 2.4.1):

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k \implies a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}}\, d\xi,$$

where γ is the circle of radius $r \in \mathbb{R}^{>0}$ centered at z_0. We wish to determine a power series of the function $f(z) = \frac{1}{1-z}$ valid on D, hence to center it around $z_0 = 3$. Allow me to begin by define a circle of radius 2: $\gamma = \{z \in \mathbb{C} : |z-3| = 2\}$, so $\gamma = \partial D$. Thus (by Fisher, Theorem 2.4.1),

$$a_k = \frac{1}{2\pi i} \int_{|z-3|=2} \frac{f(z)}{(z-3)^{k+1}}\, dz,$$

which is in fact valid of any $z \notin \partial D$. This is the integral we wish to evaluate, and we may do so by applying Cauchy's Generalized Integral Formula as presented to us in lecture: under the same hypothesis as Cauchy's Integral Formula, one has

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}}\, d\xi.$$

First, we must note that $f(z) = \frac{1}{1-z}$ is analytic on D, because $f(z)$ and it's derivatives are only discontinuous at $z = 1$, which is not an element in the open disc D but only it's boundary ∂D. Furthermore, $3 = z_0 \in D$ and γ is a simple closed curve. Therefore we have that

$$\begin{aligned}
f^{(k)}(z_0) &= \frac{k!}{2\pi i} \int_{|z-3|=2} \frac{f(z)}{(z-z_0)^{k+1}}\, dz \\
&= f^{(k)}(3) = \frac{k!}{2\pi i} \int_{|z-3|=2} \frac{f(z)}{(z-3)^{k+1}}\, dz = k! \cdot a_k \\
\implies a_k &= \frac{f^{(k)}(3)}{k!},
\end{aligned}$$

which is valid only on the open disc D. Now, I will proceed by determining the k-th derivative of $f(z)$ evaluated at $z = 3$. We have that

$$\begin{aligned}
f^{(0)}(z) &= \frac{1}{1-z}, \quad f(3) \cdot \frac{1}{0!} = -\frac{1}{2}, \\
f^{(1)}(z) &= \frac{1}{(1-z)^2}, \quad f'(3) \cdot \frac{1}{1!} = \frac{1}{4} \\
f^{(2)}(z) &= \frac{2}{(1-z)^3}, \quad f''(3) \cdot \frac{1}{2!} = -\frac{1}{8} \\
f^{(3)}(z) &= \frac{6}{(1-z)^4}, \quad f'''(3) \cdot \frac{1}{3!} = \frac{1}{16}.
\end{aligned}$$

The noticeable pattern is that $a_k = \frac{f^{(k)}(3)}{k!} = (-1)^k \frac{1}{2^k}$. *Proof by induction:* The base case has already been established (above). I now wish to show that if $\frac{f^{(k)}(3)}{k!} = (-1)^k \frac{1}{2^k}$ is true, then $\frac{f^{(k+1)}(3)}{(k+1)!} = (-1)^{(k+1)} \frac{1}{2^{(k+1)}}$ is also true. We have that

$$\frac{f^{(k+1)}(3)}{(k+1)!} = (-1)^{(k+1)} \frac{1}{2^{(k+1)}}$$

$$
\begin{aligned}
&= -(-1)^k \frac{1}{2 \cdot 2^k} \\
&= -\frac{1}{2} \cdot (-1)^k \frac{1}{2^k} \\
&= -\frac{1}{2} \cdot \frac{f^{(k)}(3)}{k!} && \text{(by induction hypothesis)} \\
&= (-1)^1 \frac{1}{2^1} \frac{f^{(k)}(3)}{k!},
\end{aligned}
$$

which is what I wanted to show. Therefore our series of the function $f(z)$ valid on the disc D is

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k} (z-3)^k.$$

You are missing a factor of $(1/2)$ in the final answer **-1**

Q4

10 / 10

Submit Your Response to Question 4 Here.

4.

We wish to determine the integral $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ via complex methods. Consider the complex substitution $z = e^{i\theta}$. Therefore we obtain that

$$z = e^{i\theta} \quad \Longrightarrow \quad dz = ie^{i\theta}\, d\theta = iz\, d\theta, \quad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Invoking this substitution in the integrand yields $\int_{|z|=1} \frac{dz}{iz\left(5 + \frac{4}{2i}\left(z - \frac{1}{z}\right)\right)}$, where the integral is now around the complex unit circle. Expanding the denominator gives:

$$iz\left(5 + \frac{4}{2i}\left(z - \frac{1}{z}\right)\right) = 5iz + 2z\left(z - \frac{1}{z}\right) = 5iz + 2z^2 - 2 = 2\left(z^2 + \frac{5}{2}iz - 1\right).$$

I will proceed by factoring this same polynomial, which will then allow me to apply Cauchy's Integral Formula. By the quadratic formula, the roots of this polynomial are

$$\begin{aligned} z &= \frac{1}{2}\left[-\frac{5}{2}i \pm \sqrt{-\frac{25}{4} - (4)(1)(-1)}\right] \\ &= \frac{1}{4}\left[-5i \pm \sqrt{-25 + 16}\right] \\ &= \frac{1}{4}\left[-5 \pm 3\right] i. \end{aligned}$$

Then, we obtain that

$$2\left(z^2 + \frac{5}{2}iz - 1\right) = 2\left(z + \frac{5-3}{4}i\right)\left(z + \frac{5+3}{4}i\right) = 2\left(z + \frac{i}{2}\right)(z + 2i).$$

With our denominator factored, we can proceed by applying Cauchy's Integral Formula to $\int_{|z|=1} \frac{dz}{2\,(z + i/2)\,(z + 2i)}$. To begin, Cauchy's formula states that for an analytic complex function

$f : D \to \mathbb{C}$ on a simply open connected set D and a simple smooth closed piecewise curve γ with $\text{In}(\gamma) \subset D$, then for $z_0 \in \text{In}(\gamma)$,

$$2\pi i\, f(z_0) = \int_\gamma \frac{f(\xi)}{\xi - z_0}\, d\xi.$$

Allow me to choose the open set $D = \{z \in \mathbb{C} : |z| < \frac{3}{2}\}$. It follows that D is open and connected, with $\frac{i}{2} \in D$, and $2i \notin D$. The unit circle lies inside of D because it's radius only extends to 1 and not beyond 1.5. Since $2i \notin D$, it follows that the function $f : D \to \mathbb{C}$ given by $f(z) = \frac{1}{2(z+2i)}$ is analytic on D, since it's derivative $f'(z) = -\frac{1}{2(z+2i)^2}$ is continuous everywhere except $z = -2i \notin D$. Therefore by Cauchy's Integral Formula, we obtain

$$2\pi i\, f(-i/2) = \int_{|z|=1} \frac{f(z)}{z + i/2}\, dz$$

$$\begin{aligned} &= 2\pi i \left(\frac{1}{2\,(-i/2 + 2i)} \right) \\ &= \frac{\pi i}{3/2i} \\ &= \frac{2\pi}{3}, \end{aligned}$$

which is the value of the original integral, what I initially desired to determine:

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}.$$

Correct. 10

Q5

10 / 10

Submit Your Response to Question 5 Here.

5.

We now wish to determine the integral $\int_\gamma \frac{e^z}{z(z-i)(z-3)}\,dz$, where γ is the parametrization of the circle of radius 2 centered at the origin. To proceed, I will apply Cauchy's Integral Formula, which I restated before in problem (4). Choose my open set to be $D = \{z \in \mathbb{C} : |z| < 5/2\}$ and let $f : D \to \mathbb{C}$ be the analytic function on D defined by $f(z) = \frac{e^z}{z-3}$. f is analytic on D because $3 \notin D$, hence the derivative is defined and continuous everywhere on D. Furthermore, the curve γ defining the circle of radius 2 centered at the origin is contained in D because it's radius $2 < \frac{5}{2}$, and therefore $\text{In}(\gamma) \subset D$. The integral then becomes

$$\int_{|z|=2} \frac{f(z)}{z(z-i)}\,dz.$$

We may not begin to apply Cauchy's Integral Formula because the denominator exposes two separate discontinuities of the integrand, that at $z = 0$ and $z = i$. To resolve this issue, I will proceed by apply partial fraction decomposition to the fraction. Thus we desire to determine coefficients A and B such that

$$\frac{A}{z} + \frac{B}{z-i} = \frac{1}{z(z-i)}.$$

We have that

$$\begin{aligned} A(z-i) + B(z) &= 1 \\ \implies (A+B)z - Ai &= 1 \\ &\implies A + B = 0 \text{ and } A = -\frac{1}{i}. \end{aligned}$$

This implies that $A = i$, so $B = -i$, hence our integral is then decomposed into fractions:

$$\int_{|z|=2} f(z)\left(\frac{i}{z} - \frac{i}{z-i}\right).$$

Now, I can apply Cauchy's integral formula to each of the two individual integrals by linearity:

$$\begin{aligned} \int_{|z|=2} f(z)\left(\frac{i}{z} - \frac{i}{z-i}\right) &= i\int_{|z|=2} \frac{f(z)}{z} - i\int_{|z|=2} \frac{f(z)}{z-i} \\ &= 2\pi f(0) - 2\pi f(i) \\ &= 2\pi\left(\frac{e^0}{0-3} - \frac{e^i}{i-3}\right) \\ &= 2\pi\left(-\frac{1}{3} - \frac{e^i}{i-3}\right) \\ &= 2\pi\left(\frac{e^i}{3-i} - \frac{1}{3}\right). \end{aligned}$$

By Cauchy's Integral Formula, this is the value of the integral we were trying to determine:

$$\int_\gamma \frac{e^z}{z(z-i)(z-3)}\,dz = 2\pi\left(\frac{e^i}{3-i} - \frac{1}{3}\right).$$