homework 5

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97.5% (39/40)

Q1

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MAT334 Problem Set 5 — Due December 7, 2022
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1.

Consider the function g defined by $g(z) = f(ze^{i2\pi/3})f(ze^{i4\pi/3})f(z)$. Since f is analytic on $\overline{B_1(0)}$, then so is g, and thus g must attain maximum and minimum values on $\overline{B_1(0)}$ by the extreme value theorem. However, by the maximum modulus principle, if g is nonconstant, then g should attain said values on $\partial B_1(0)$. Note that since $g(e^{i\theta}) = f(e^{i2\pi/3+\theta})f(e^{i4\pi/3+\theta})f(e^{i\theta}) = 0$ for $0 < \theta < \pi$, then $\pm\mathbb{R}\mathrm{e}\{g\}$ and $|g|$ have attained this maximum and minimum value on $\partial B_1(0)$ since $[0, 2\pi] \subset (0, \pi) \cup (2\pi/3, 5\pi/3) \cup (4\pi/3, 7\pi/3)$. Since there is more than more than one possible θ for which $g(e^{i\theta}) = 0$, it must be that no maximum has occured. Therefore g must be constant on $\overline{B_1(0)}$, and this value is 0. Therefore $g(z) = 0$ for all $z \in \overline{B_1(0)}$.
This implies that for any z in $\overline{B_1(0)}$, one of the values $f(z), f(ze^{i2\pi/3}), f(ze^{i4\pi/3})$ must be zero. Now say, for $z = 0$, $g(0) = [f(0)]^3 = 0$, hence $f(0) = 0$.
If f is nonconstant on $B_1(0)$, then each of the zeroes of f must be isolated. However, for points $z = re^{i\alpha}$ with $r \ll 1$, no matter what value of r, implies that one of $f(re^{i2\pi/3+\alpha}), f(re^{i4\pi/3+\alpha})$, or $f(re^{i\alpha})$ are zero. Furthermore, due to the root at $z = 0$, the zeroes no longer become isolated since r is a continuous variable.
Therefore f cannot be nonconstant, hence $f(z) = 0$ everywhere on $B_1(0)$.

1

Q2a

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2.

(a) The circle $|z-1|^2 = 1$ corresponds to the explicit equation $(x-1)^2+y^2 = 1$, or $x^2+y^2-2x = 0$. For an equation of the form $\alpha(x^2+y^2) + \beta x\gamma y = \delta$, we know that the inversion mapping $w = \frac{1}{z}$ transforms this equation into $\delta(u^2+v^2) - \beta u + \gamma v = \alpha$, hence our circle $|z-1|^2 = 1$ becomes, under the $w = \frac{1}{z}$ transformation, is

$$(x^2+y^2) - 2x = 0 \xrightarrow{w=1/z} (0)(x^2+y^2) - (-2)u + (0)v = 1 \implies 2u = 1,$$

with $w = u + iv$. Therefore $\mathbb{R}\mathrm{e}\{w\} = u = \frac{1}{2}$, which is what I wanted to show.

It would be nice if you showed this fact, or at least stated a theorem from class or from the textbook that lets you conclude this...

2

Q2b

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2.

(b) We wish to define a series of linear fractional transformations such that their composition takes the circle $|z| = 1$ and maps it to the line $\mathbb{R}\mathrm{e}\{w\} = 0$. Let

$$U(z) = z + 1$$
$$V(z) = \frac{1}{z}$$

$$W(z) = z - \frac{1}{2}.$$

Now, $U(z)$ takes the circle $|z| = 1$ and maps it to the new circle $|z-1| = 1$, a shift one unit to the right. This circle correponds to the equation $(x^2+y^2) - 2x = 0$, with $\delta = \gamma = 0$, $\alpha = 1$ and $\beta = -2$. Then, the map $V(z) = \frac{1}{z}$ takes this circle and maps it to the line $(0)(u^2+v^2) + 2u + (0)v = 1$, that is, $u = \frac{1}{2}$. Lastly, the map $W(z) = z - \frac{1}{2}$ takes the line $u = \frac{1}{2}$ and maps it to the line $u = 0$, which is equivalently $\mathbb{R}\text{e}\{w\} = 0$. Therefore composing these indidual linear fractional transformations produces the map which takes the circle $|z| = 1$ to the line $\mathbb{R}\text{e}\{w\} = 0$:

$$\begin{aligned}
W(V(U(z))) &= W(V(z+1)) \\
&= W\left(\frac{1}{z+1}\right) \\
&= \frac{1}{z+1} - \frac{1}{2} \\
&= \frac{2}{2(z+1)} - \frac{z+1}{2(z+1)} \\
&= \frac{2-1-z}{2(z+1)} \\
T(z) &= \frac{1-z}{2(z+1)},
\end{aligned}$$

which is the linear fractional transformation which I wanted to find.

Q3

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3.

Consider the function defined by $f(z) = i\dfrac{1+z}{1-z}$. Note that this linear fractional transformation is a composition of other linear fractional transformations. Let

$$V(z) = \frac{1}{z}$$
$$T(z) = \frac{1}{2}z - \frac{1}{2}$$
$$W(z) = iz + i.$$

Then,

$$\begin{aligned} f(z) &= W(V(T(V(z)))) \\ &= W(V(T(1/z))) \\ &= W\left(V\left(\frac{1}{2z}-\frac{1}{2}\right)\right) \\ &= W\left(\frac{1}{1/2z}-1/2\right) \\ &= i\left(\frac{1}{1/2z}-z/2z\right)+i \\ &= i\left(\frac{2z}{1-z}+1\right) \\ &= i\left(\frac{2z+(1-z)}{1-z}\right) \\ &= i\frac{1+z}{1-z}. \end{aligned}$$

Now consider the unit disc $|z|\leq 1$. We wish to find the mapping of $|z|\leq 1$ under f.

- First, note that the mapping of $|z|\leq 1$ under V is still $|z|\leq 1$, since V acts to invert the unit circle. We have that $(x^2+y^2)\leq 1$, which is equivalent to the equation $\alpha(x^2+y^2)+\beta x+\gamma y=\delta$ with $\alpha=\delta=1$ and $\beta=\gamma=0$. Therefore under V, we obtain the inverted circle $\delta(x^2+y^2)-\beta x+\gamma y=\alpha$, hence we still have that $x^2+y^2\leq 1$. Thus
$$|z|\leq 1 \xrightarrow{V} |z|\leq 1.$$
- Second, consider the mapping of $|z|\leq 1$ under T. For a mapping of the form $g(z)=az+b$, then g takes a circle $|z-z_0|\leq r$ to the new translated disc $|z-(az_0+b)|\leq |a|r$. Therefore $|z|\leq 1$ is translated to $|z-\frac{1}{2}(0)+\frac{1}{2}|\leq\frac{1}{2}$:
$$|z|\leq 1 \xrightarrow{T} |z+\frac{1}{2}|\leq\frac{1}{2}.$$
- Third, now V again acts on the circle $|z+\frac{1}{2}|\leq\frac{1}{2}$ and converts it to another equation. In explicit form as we did before, we have that $\left(x+\frac{1}{2}\right)^2+y^2=x^2+y^2+x+\frac{1}{4}\leq\frac{1}{4}\implies$

$x^2+y^2+x\leq 0$, hence V takes this circle and converts it to the line $(0)(x^2+y^2)-x+(0)y\leq 1$, which is the line $x\geq -1$. Therefore
$$|z+\frac{1}{2}|\leq\frac{1}{2}\xrightarrow{V} x\geq -1.$$

- Lastly, consider the image of $\{z:\mathbb{R}\mathrm{e}\{z\}\geq -1\}$ under W. First note that for lines $\mathbb{R}\mathrm{e}\{Az+B\}=0$ under the linear fractional transformation $g(z)=az+b$ is taken to the new line $\mathbb{R}\mathrm{e}\{(A/a)z+B-b(A/a)\}=0$. Hence, for our line $\mathbb{R}\mathrm{e}\{z+1\}\geq 0$, $A=B=1$, and our map

W $a = b = i$. Thus, we obtain the new line $\mathbb{R}\mathrm{e}\{-iz + 1 - i(-i)\} = \mathbb{R}\mathrm{e}\{-iz\} \geq 0$ which is equivalently $\mathbb{R}\mathrm{e}\{-i(x+iy)\} = \mathbb{R}\mathrm{e}\{-ix + y)\} = y \geq 0$, which is identically the upper half plane. Thus

$$\mathbb{R}\mathrm{e}\{z\} \geq -1 \xrightarrow{W} \mathbb{I}\mathrm{m}\{z\} \geq 0,$$

which is what I wanted to show. I will next find the inverse. For a linear fractional transformation $T(z) = \dfrac{az+b}{cz+d}$, we have that the inverse is given by $T^{-1}(w) = \dfrac{-dw+b}{cw-a}$. For f, we have that $a = b = i$ and $c = -1$, $d = 1$. Then $f^{-1}(z) = \dfrac{-z+i}{-z-i}$. *Proof:*

$$\begin{aligned} f(f^{-1}(z)) &= \frac{i\left(1 + \left(\frac{-z+i}{-z-i}\right)\right)}{1 - \left(\frac{-z+i}{-z-i}\right)} \\ &= \frac{i[(-z-i) + (-z+i)]}{-z-i-(-z+i)} \\ &= \frac{-2i}{-2i} z \\ &= z, \end{aligned}$$

hence the inverse is $f^{-1}(z) = \dfrac{-z+i}{-z-i}$. Now, since f is a linear fractional transformation, f is already injective (see section 3.3, pg. 196 Fisher). It suffices to show that it is bijective. Fix $w \in \{w : \mathbb{I}\mathrm{m}\{w\} \geq 0\}$ and choose $z \in \overline{B_1(0)}$ as $z = \dfrac{w-i}{w+i}$. Clearly, $z \in \overline{B_1(0)}$ since $|z| = \underline{\dfrac{|w-i|}{|w+i|}} \leq \underline{\dfrac{|w|+1}{|w|+1}} = 1$, and

$$\begin{aligned} f(z) &= \frac{i\left(1 + \left(\frac{w-i}{w+i}\right)\right)}{1 - \left(\frac{w-i}{w+i}\right)} \\ &= \frac{i[(w+i) + (w-i)]}{w+i-(w-i)} \\ &= \frac{-2i}{-2i} w \\ &= w, \end{aligned}$$

as desired. Therefore $f(z)$ is bijective, which is what I wanted to show.

Q4

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4.

So far, I have not had time to study yet because I have had 5 assignments all due on Wednesday, December 7, 2022. However, in preparation for the final exam, I will be eventually reviewing the concept of Laurant series. Let f be an analytic function on an open disc of radius R centered at a point z_0, $B_R(z_0)$. By definition, we are able to expand f in terms of a power series of radius of convergence R centered ar z_0. Allow us to now consider a punctured disc of inner radius $0 \leq r$ and outer radius R such that $r < |z - z_0| < R$, or $B_R(z_0) \setminus \overline{B_r(0)}$. Now, we are able to expand an analytic function f on this punctured disc by a means of a sum of two power series f_1 and f_2, where f_1 is analytic on $B_R(z_0)$ and f_2 is analytic on $B_r(z_0) \cup \{\infty\}$. Here, f is expressed as the sum of the two power series terms

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$$f(z) = f_1(z) + f_2(z) = \sum_{k=0} a_k(z - z_0)^k + \sum_{k=1} b_k(z - z_0)^{-k}.$$

Note that f_2 is often referred to as the 'principle part' of the expansion of f. Similarly to the power series of f, the coefficients $a_{-k} = b_k$ are given by $a_k = \dfrac{1}{2\pi i}\displaystyle\int_{|w-z_0|=s} \frac{f(w)}{(w - z_0)^{k+1}}\, dw$, $k = 0, \pm 1, \pm 2, \ldots$ for a circle of radius $r < s < R$ centered at z_0. By this method of expansion, we are able to express functions in terms of their Laurant series expansions which will allow us to quickly determine residues (the value of the coefficient a_{-1}) and determine principle components of functions.

Consider the following example of finding the principle part and residue of the function $f(z) = \dfrac{z^3 + z^2}{(z-1)^2}$ centered at $z_0 = 1$. We will first proceed by the expanding $z^3 + z^2$ in terms of $(z - 1)$:

$$\begin{aligned} a(z-1)^3 + b(z-1)^2 + c(z-1) + d &= a(z^3 - 3z^2 + 3z - 1) + b(z^2 - z + 1) + c(z - 1) + d \\ &= az^3 + (-3a + b)z^2 + (2a - b + c)z + (-a + b - c + d) \\ &\implies a = 1, b = 4, c = 5, d = 2 \\ &\implies z^3 + z^2 = (z-1)^3 + 4(z-1)^2 + 5(z-1) + 2 \end{aligned}$$

Therefore f becomes

$$f(z) = \frac{(z-1)^3 + 4(z-1)^2 + 5(z-1) + 2}{(z-1)^2} = (z-1) + 4 + \frac{5}{z-1} + \frac{2}{(z-1)^2},$$

which is equivalently the Laurant series expasion of f. The residue at z_0 is then 5, and the principle part is $\dfrac{5}{z-1} + \dfrac{2}{(z-1)^2}$.