PHY354 PS2 — 02/11/2022

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1. Find expressions for the angular momentum M_x, M_y, M_z and $\vec{M}^2$ of a single particle in (a) spherical polar (r, θ, ϕ), and (b) cylindrical polar (ρ, ϕ, z) coordinates.

(a) I will begin by introducing the radial and velocity vectors in terms of spherical polar coordinates:

$$\begin{aligned}
\mathbf{r} &= r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}} \\
\mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \\
&= \left(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi\right)\hat{\mathbf{x}} \\
&\quad + \left(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi\right)\hat{\mathbf{y}} + \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{\mathbf{z}}.
\end{aligned}$$

Now,

$$\mathbf{M} \equiv \mathbf{r} \times \mathbf{p} = \det\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ p_x & p_y & p_z \end{pmatrix} = \hat{\mathbf{x}}(r_y p_z - r_z p_y) + \hat{\mathbf{y}}(r_z p_x - r_x p_z) + \hat{\mathbf{z}}(r_x p_y - r_y p_x).$$

Then

$$\begin{aligned}
M_x = r_y p_z - r_z p_y &= m(r\sin\theta\sin\phi)(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) - m(r\cos\theta)(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi) \\
&= m[r\dot{r}\sin\theta\cos\theta\sin\phi - r^2\dot{\theta}\sin\phi(\sin^2\theta + \cos^2\theta) - r\dot{r}\cos\theta\sin\theta\sin\phi - r^2\dot{\phi}\sin\theta\cos\theta\cos\phi] \\
&= -mr^2[\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi].
\end{aligned}$$

$$\begin{aligned}
M_y = r_z p_x - r_x p_z &= m(r\cos\theta)(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi) - m(r\sin\theta\cos\phi)\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right) \\
&= m[r\dot{r}\cos\theta\sin\theta\cos\phi + r^2\dot{\theta}\cos\phi(\cos^2\theta + \sin^2\theta) - r^2\dot{\phi}\sin\theta\cos\theta\sin\phi - r\dot{r}\sin\theta\cos\theta\cos\phi] \\
&= mr^2[\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi]
\end{aligned}$$

$$\begin{aligned}
M_z = (r_x p_y - r_y p_x) &= m(r\sin\theta\cos\phi)\left(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi\right) \\
&\quad - m(r\sin\theta\sin\phi)\left(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi\right) \\
&= m[r\dot{r}\sin^2\theta\sin\phi\cos\phi + r^2\dot{\theta}\cos\theta\cos\phi\sin\theta\sin\phi + r^2\dot{\phi}\sin^2\theta(\cos^2\phi + \sin^2\phi) - r\dot{r}\sin^2\theta\sin\phi\cos\phi \\
&\quad - r^2\dot{\theta}\cos\theta\cos\phi\sin\theta\sin\phi] \\
&= mr^2\dot{\phi}\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
M^2 &= M_x^2 + M_y^2 + M_z^2 \\
&= m^2r^4(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi)^2 + m^2r^4(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi)^2 + m^2r^4\dot{\phi}^2\sin^4\theta \\
&= m^2r^4[\dot{\theta}^2(\sin^2\phi + \cos^2\phi) + \dot{\phi}^2\sin^2\theta\cos^2\theta(\sin^2\phi + \cos^2\phi) + \dot{\phi}^2\sin^4\theta \\
&= m^2r^4[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta].
\end{aligned}$$

(b) Like **(a)**, I will introduce the radial and velocity vectors in terms of cylinderical coordinates (I will use $\mathbf{r}$ instead of ρ):

$$\begin{aligned}
\mathbf{r} &= r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\
\mathbf{v} &= \dot{r}\cos\theta\hat{\mathbf{x}} + \dot{r}\sin\theta\hat{\mathbf{y}} - r\dot{\theta}\sin\theta\hat{\mathbf{x}} + r\dot{\theta}\cos\theta\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}} \\
&= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{\mathbf{x}} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}}
\end{aligned}$$

As before,

$$\mathbf{M} \equiv \mathbf{r}\times\mathbf{p} = \det\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ p_x & p_y & p_z \end{pmatrix} = \hat{\mathbf{x}}(r_y p_z - r_z p_y) + \hat{\mathbf{y}}(r_z p_x - r_x p_z) + \hat{\mathbf{z}}(r_x p_y - r_y p_x).$$

Then

$$\begin{aligned}
M_x &= r_y p_z - r_z p_y = m(r\sin\theta)(\dot{z}) - m(z)(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) \\
&= m[\dot{z}r\sin\theta - z\dot{r}\sin\theta - zr\dot{\theta}\cos\theta] \\
&= m\sin\theta(\dot{z}r - z\dot{r}) - mrz\dot{\theta}\cos\theta
\end{aligned}$$

$$\begin{aligned}
M_y &= r_z p_x - r_x p_z = m(z)(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) - m(r\cos\theta)(\dot{z}) \\
&= m[z\dot{r}\cos\theta - zr\dot{\theta}\sin\theta - \dot{z}r\cos\theta] \\
&= m\cos\theta(z\dot{r} - \dot{z}r) - mzr\dot{\theta}\sin\theta \\
&= -m\cos\theta(\dot{z}r - z\dot{r}) - mzr\dot{\theta}\sin\theta
\end{aligned}$$

$$\begin{aligned}
M_z &= r_x p_y - r_y p_x = m(r\cos\theta)(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) - m(r\sin\theta)(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \\
&= m[r\dot{r}\sin\theta\cos\theta + r^2\dot{\theta}\cos^2\theta - r\dot{r}\sin\theta\cos\theta - r^2\dot{\theta}\sin^2\theta] \\
&= mr^2\dot{\theta}(\cos^2\theta + \sin^2\theta) \\
&= mr^2\dot{\theta}
\end{aligned}$$

$$\begin{aligned}
M^2 = M_x^2 + M_y^2 + M_z^2 &= m^2\sin^2\theta(\dot{z}r - z\dot{r})^2 + m^2r^2z^2\dot{\theta}^2\cos^2\theta - 2m^2rz\dot{\theta}\sin\theta\cos\theta(\dot{z}r - z\dot{r}) \\
&\quad + m^2\cos^2\theta(\dot{z}r - z\dot{r})^2 + m^2r^2z^2\dot{\theta}^2\sin^2\theta + 2m^2rz\dot{\theta}\sin\theta\cos\theta(\dot{z}r - z\dot{r}) \\
&\quad + m^2r^4\dot{\theta}^2 \\
&= m^2(\sin^2\theta + \cos^2\theta)(\dot{z}r - z\dot{r})^2 + m^2r^2z^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) + m^2r^4\dot{\theta}^2 \\
&= m^2(\dot{z}r - z\dot{r})^2 + m^2r^2z^2\dot{\theta}^2 + m^2r^4\dot{\theta}^2 \\
&= m^2(\dot{z}r - z\dot{r})^2 + m^2r^2\dot{\theta}^2(r^2 + z^2).
\end{aligned}$$

2. What components of momentum $\vec{P}$ and angular momentum $\vec{M}$ are conserved in motion in the field (it could be gravitational, electrostatic or something else; it doesn't matter for this problem) produced by the following objects:

   (a) an infinite homogenous plane
   (b) an infinite homogeneous cylinder
   (c) an infinite homogenous prism
   (d) two point particles
   (e) an infinite homogenous half-plane
   (f) a homogenous cone
   (g) a homogeneous circular torus.

   If the angular momentum is only conserved for a particular choice (or class of choices) of the origin, indicate that.

(a) Suppose the infinite plane is located in the xy plane with the normal vector in the $\hat{\mathbf{z}}$ direction. The potential will only contribute along the $\hat{\mathbf{z}}$ direction, which leaves the spacial x and y directions invariant anywhere along the plane for a fixed z. Furthermore, any rotation about a vertical z axis leaves any rotational orientation of the potential invariant. Therefore p_x, p_y, M_z are conserved.

Mathematically, we can examine an infinite plane of electric charge surface density σ.
Then $E_{plane} = \dfrac{\sigma}{2\epsilon_0}$, which by integrating yields $V = -\dfrac{\sigma}{2\epsilon_0}z$ purely along the $\hat{\mathbf{z}}$ direction.
Therefore p_x, p_y, M_z are conserved.

(b) This follows from a similar argument as **(a)**. Let the cylinders axis run along the $\hat{\mathbf{z}}$ axis. Notice that the potential field is not invariant with respect to any radius r away, since the strength may differ. Since r is dependent of x and y, then the potential is not invariant under any x or y direction spatial translations. However, for a fixed radius r, the electric field is invariant of any height along z, implying that there is spatial invariance over z. The electric field changes for any rotation about the x or y axes, but for a fixed r and z, the rotation about the $\mathbf{z}$ direction is invariant. Therefore p_z, M_z are conserved.

(c) Let the prism be oriented such that its axis is in the $\hat{\mathbf{z}}$ direction. For any prism, it is easy to see that any spatial translations along any x or y axes are not invariant. Moving along x or y, the potentials strength differs. However, since the prism is infinitely long, there is an invariance in strength along the $\hat{\mathbf{z}}$ direction. The same argument is applied for any isotropy about the x, y or z axes. There is certainly no invariance under rotation about the $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ directions, however since we are examining a prism (with I should say, 'discrete' walls), there are points in rotation about $\hat{\mathbf{z}}$ that will differ from other points. Thus there is no rotational invariance.
Therefore p_z is the only conserved quantity.

(d) Let the z axis be the line which joins the two point particles. For any translation along the x, y or z axes, there is no invariance. This is proven by knowing that $V \to 0$ as $r \to \infty$, so the potential's strength differs for any spatial translation. Likewise, any rotation about the x or y axes

encounter regions where the potential varies, however any rotational along the $\hat{\mathbf{z}}$ direction, for a fixed radius r and height z, the potential field will be left unchanged for any angle, hence invariance about isotropy about the z axis exists.
Therefore just M_z is conserved.

(e) Let the plane be located in the xy plane (normal vector along $\hat{\mathbf{z}}$) with the x axis 'cutting' the plane in half. This is trivial, since the only noticeable invariance would be along the x direction for a fixed height z and distance into/away from the plane y.
Therefore just p_x is conserved.

(f) Let the z axis run down the centre of the cone. The similar arguments I have been applying to the other problems hold. The field magnitude will change for any spatial translation about any of the 3 axes, and rotation about the x (and hence y) axes are not invariant. For a fixed radius r and height z, the system is rotationally invariant about the $\hat{\mathbf{z}}$ axis.
Therefore M_z is conserved.

(g) This is applying the similar argument to **(f)**. Let z be the axis which runs down the centre of the torus. Then for a fixed radius r and height z, any rotation about the z axis remains invariant.
Therefore M_z is conserved.

3. Consider a particle moving in the external potential field of an infinite homogenous cylindrical helix (see figure). The system has neither rotational or translational symmetry, but there is still a linear linear combination of components of $\vec{P}$ and $\vec{M}$ which is conserved - find it.

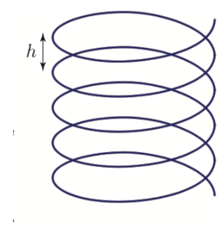

Figure 1: Problem 3

I will parametrize the helix defined by the function $f : \mathbb{R} \to \mathbb{R}^3$ by

$$f(t) = \left(R\cos t,\ R\sin t,\ \frac{ht}{2\pi} \right)$$

for an arbitrary radius R. The z component is just given by the fact that for every $t = 2\pi$, $z = h$. Notice that any rotation about the x or y axes is not invariant (the field differs in magnitude). Similarly, suppose $r \to \infty$. The field will differ in magnitude greatly, therefore the spatial translations about x and y are not invariant. Suppose a particle traverses the curve $g : \mathbb{R} \to \mathbb{R}^3$ defined by

$$g(t) = \left(r\cos t,\ r\sin t,\ \frac{ht}{2\pi} \right)$$

for $r > R$. Then the electric field magnitude will be invariant if and only if the particle traverses this path. It directly follows that the linear combination $M_z + \dfrac{h}{2\pi} p_z$ is the conserved quantity.

4. Let's return to the problem of a bead on a rotating hoop, Problem 5 from Problem Set #1.

   (a) Find the generalized momentum p_ϕ. Is it conserved?

   (b) Find the Hamiltonian of the system. Is it conserved? Check your result explicitly by using the equations of motion. Compare the Hamiltonian with the energy of the bead. Is the energy of the bead conserved? Explain why H and E differ. How does this argument change when $\omega = 0$?

(a) The Lagrangian of the system is again given by

$$L = \frac{1}{2}mR^2\dot{\phi}^2 + mR^2\omega^2(1+\cos\phi).$$

The generalized momentum of the system is given by $\frac{\partial L}{\partial \dot{\phi}}$, and thus

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}.$$

Notice that

$$\frac{\partial L}{\partial \phi} = -mR^2\omega^2\sin\phi \neq 0$$

which implies that p_ϕ is not conserved (if ϕ was cyclic, then p_ϕ would be conserved).

(b) The Hamiltonian is given by

$$H \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L,$$

so

$$\begin{aligned} H &= \dot{\phi}\frac{\partial L}{\partial \dot{\phi}_i} - \left[\frac{1}{2}mR^2\dot{\phi}^2 + mR^2\omega^2(1+\cos\phi)\right] \\ &= mR^2\dot{\phi}^2 - \frac{1}{2}mR^2\dot{\phi}^2 - mR^2\omega^2(1+\cos\phi) \\ &= \frac{1}{2}mR^2\dot{\phi}^2 - mR^2\omega^2(1+\cos\phi). \end{aligned}$$

Since the Lagrangian has no explicit time dependence, H is conserved. Recall that the equation of motion of the bead is

$$\ddot{\phi} = -\omega^2\sin\phi.$$

This is also reflected in the equations of motion, because they too are time independent. Since the kinetic energy is quadratic in the generalized coordinates, H is the total energy of the system. The energy of the bead is equivalent to the Lagrangian,

$$E = \frac{1}{2}mR^2\dot{\phi}^2 + mR^2\omega^2(1+\cos\phi).$$

Notice that $E \neq H$, so E is not the total energy of the system. It follows that the energy E is not conserved. The Hamiltonian and the energy differ because the system is subject to a time-dependent constrained: ω, so energy is being added through ωt. This argument changes when $\omega = 0$ because there is now no energy being driven into the system. In this case,

$$H = E = mR^2\dot{\phi}\ddot{\phi},$$

which implies that the Hamiltonian and E are both conserved. When $\omega = 0$, H and E do not differ.

5. * Consider a particle of mass m moving in a central force, $\vec{F}(\vec{r}) = f(r)\hat{r}$.

   (a) Show that

$$\frac{d}{dt}\left(\vec{p}\times\vec{M}\right) = mf(r)\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right)$$

   where $r \equiv |\vec{r}|$.

Note that

$$\begin{aligned}
\vec{r} &= r\hat{r} \\
\dot{\vec{r}} &= \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
\ddot{\vec{r}} &= \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta},
\end{aligned}$$

where $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ and $\ddot{\hat{r}} = -\dot{\theta}^2\hat{r}$. In a central potential, angular momentum is always conserved by definition. so $\dfrac{dM}{dt} = 0$. Furthermore, the force acting on the particle must move the particle directly into or away from it's origin. This implies there is no $\hat{\theta}$ components in the motion:

$$\begin{aligned}
\vec{r} &= r\hat{r} \\
\dot{\vec{r}} &= \vec{v} = \dot{r}\hat{r} \\
\ddot{\vec{r}} &= \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r},
\end{aligned}$$

Since $\vec{F}(\vec{r}) = -\vec{\nabla}U(r)$, our Lagrangian becomes

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r).$$

Applying the Euler-Lagrange equations yield

$$m\ddot{r} = mr\dot{\theta}^2 + f(r) \implies f(r) = m(\ddot{r} - r\dot{\theta}^2).$$

By the product rule,

$$\begin{aligned}
\frac{d}{dt}[\vec{p}\times\vec{M}] &= \left(\frac{d\vec{p}}{dt}\times\vec{M}\right) + \left(\vec{p}\times\frac{d\vec{M}}{dt}\right) \\
&= \dot{\vec{p}}\times\vec{M} + 0.
\end{aligned}$$

Since $\vec{M} \equiv \vec{r}\times\vec{p}$,

$$\begin{aligned}
\frac{d}{dt}[\vec{p}\times\vec{M}] = \dot{\vec{p}}\times\vec{M} &= \dot{\vec{p}}\times(\vec{r}\times\vec{p}) \\
&= (\dot{\vec{p}}\cdot\vec{p})\vec{r} - (\dot{\vec{p}}\cdot\vec{r})\vec{p} \\
&= m^2\left[\dot{r}\left(\ddot{r} - r\dot{\theta}^2\right)\right]\vec{r} + m^2\left[r(\ddot{r} - r\dot{\theta}^2)\right]\dot{\vec{r}} \\
&= m\left[m(\ddot{r} - r\dot{\theta}^2)\right](\dot{r}\vec{r} + r\dot{\vec{r}}) \\
&= mf(r)(\dot{r}\vec{r} + r\dot{\vec{r}}).
\end{aligned}$$

(b) Show that

$$\frac{d}{dt}\hat{r} = -\frac{1}{r^2}\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right).$$

From problem **(5a)**,

$$\begin{aligned}
\vec{r} &= r\hat{r} \\
\dot{\vec{r}} &= \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
\ddot{\vec{r}} &= \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta},
\end{aligned}$$

where $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ and $\ddot{\hat{r}} = -\dot{\theta}^2\hat{r}$.
Now

$$\begin{aligned}
\dot{\hat{r}} = \dot{\hat{r}} \implies r^2\dot{\hat{r}} &= r^2\dot{\hat{r}} \\
&= -r\dot{r}\hat{r} + r\dot{r}\hat{r} + r^2\dot{\hat{r}} \\
&= -\left(r\dot{r}\hat{r} - r\dot{r}\hat{r} - r^2\dot{\hat{r}}\right) \\
&= -\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right) \\
\implies \dot{\hat{r}} &= -\frac{1}{r^2}\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right).
\end{aligned}$$

(c) From the previous two results, show that in the case of an attractive gravitational (or Coulomb) force, $f(r) = -K/r^2$, there is an additional conserved quantity, the *Laplace-Runge-Lenz Vector*,

$$\vec{A} = \vec{p} \times \vec{M} - mK\hat{r}$$

which is not conserved for a general central potential. Show that $\vec{A} \cdot \vec{M} = 0$ and $|\vec{A}|^2$ is not independent of the other integrals of motion to argue that the three components of $\vec{A}$ only give one independent integral of the motion. In class, we will show how conservation of $\vec{A}$ allows us to solve the Kepler problem without solving the equations of motion or doing any integrals.

From **(5a), (5b)**,

$$\frac{d}{dt}[\vec{p} \times \vec{M}] = mf(r)(\dot{r}\vec{r} + r\dot{\vec{r}}) \qquad \text{and} \qquad \dot{\hat{r}} = -\frac{1}{r^2}\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right).$$

If $\vec{A}$ is a conserved quantity, then $\frac{d}{dt}\vec{A} = 0$:

$$\begin{aligned}
\frac{d}{dt}\vec{A} &= \frac{d}{dt}[\vec{p} \times \vec{M}] - mK\frac{d}{dt}[\hat{r}] \\
&= mf(r)(\dot{r}\vec{r} - r\dot{\vec{r}}) + mK\frac{1}{r^2}(\dot{r}\vec{r} - r\dot{\vec{r}}) \\
&= mf(r)(\dot{r}\vec{r} - r\dot{\vec{r}}) - mf(r)(\dot{r}\vec{r} - r\dot{\vec{r}}) \\
&= 0.
\end{aligned}$$

Note that $\vec{M} \times \vec{M} = 0$ and $\hat{r} \times \vec{r} = r(\hat{r} \times \hat{r}) = 0$. Then

$$\begin{aligned}
\vec{A} \cdot \vec{M} &= \left(\vec{p} \times \vec{M}\right) \cdot \vec{M} - mK\hat{r} \cdot \vec{M} \\
&= \vec{p} \cdot \left(\vec{M} \times \vec{M}\right) - mK\hat{r} \cdot (\vec{r} \times \vec{p}) \\
&= \vec{p} \cdot (0) - mK\vec{p} \cdot (\hat{r} \times \vec{r}) \\
&= 0 - mK\vec{p} \cdot (0) \\
&= 0.
\end{aligned}$$

To find $|\vec{A}|^2$, we must find $|\vec{p} \times \vec{M}|^2$ and $\hat{r} \cdot \left(\vec{p} \times \vec{M}\right)$.

$$\begin{aligned}
|\vec{p} \times \vec{M}|^2 &= \left(\vec{p} \times \vec{M}\right) \cdot \left(\vec{p} \times \vec{M}\right) \\
&= \det\begin{pmatrix} \vec{p} \cdot \vec{p} & \vec{p} \cdot \vec{M} \\ \vec{M} \cdot \vec{p} & \vec{M} \cdot \vec{M} \end{pmatrix} \\
&= (\vec{p} \cdot \vec{p})(\vec{M} \cdot \vec{M}) - (\vec{M} \cdot \vec{p})^2 \\
&= (m^2(\dot{r}^2))(m^2r^2\dot{\theta}^4) - 0 \\
&= m^4r^2\dot{r}^2\dot{\theta}^4
\end{aligned}$$

$$\hat{r} \cdot \left(\vec{p} \times \vec{M}\right) = \frac{1}{r}\vec{r} \cdot \left(\vec{p} \times \vec{M}\right)$$

$$
\begin{aligned}
&= \frac{1}{r}\ (\vec{r} \times \vec{p}) \cdot \vec{M} \\
&= \frac{1}{r}\ |\vec{M}|^2 \\
&= \frac{1}{r} m^2 r^2 \dot{\theta}^4 \\
&= m^2 r \dot{\theta}^4.
\end{aligned}
$$

Now $|\vec{A}|^2 \equiv \vec{A} \cdot \vec{A}$, so

$$
\begin{aligned}
\vec{A} \cdot \vec{A} &= \left(\vec{p} \times \vec{M} - mK\hat{r}\right) \cdot \left(\vec{p} \times \vec{M} - mK\hat{r}\right) \\
&= |\vec{p} \times \vec{M}|^2 + m^2K^2(1) - 2mK\left[\hat{r} \cdot \left(\vec{p} \times \vec{M}\right)\right] \\
&= m^4 r^2 r^2 \dot{\theta}^4 + m^2 K^2 - 2m^3 r K \dot{\theta}^4 \\
&= m^2 r^2 \dot{\theta}^4 \left(m^2 \dot{r}^2 - 2m\frac{K}{r}\right) - (mK)^2 \\
&= 2m\left(\frac{1}{2} m\dot{r}^2 - \frac{K}{r}\right) m^2 r^2 \dot{\theta}^4 + (mK)^2 \\
&= 2mE|\vec{M}|^2 + (mK)^2.
\end{aligned}
$$

Since $E = H = \dfrac{1}{2}m\dot{r}^2 - \dfrac{K}{r}$ is an integral of motion and $M = mr\dot{\theta}^2$ is an integral of motion, then $|\vec{A}|^2$ is dependent on the other integrals of motion. This argues that $\vec{A}$ is another integral of motion, since it is orthogonal to $\vec{M}$ and the magnitude is dependent on other conserved quantities: E and $\vec{M}$. Therefore the three components of $\vec{A}$ also give another integral of motion independent of the others.

6. * Consider the Atwood's machine shown in the figure, consisting of two massless pulleys supporting three masses m_1, m_2 and m_3 via massless strings. Write the Lagrangian for the system in terms of the generalized coordinates x_1 and x_2. Identify a symmetry of the corresponding Lagrangian which is not simple time or space translation and use Noether's theorem to find the corresponding conserved quantity. Verify directly from the equations of motion that this quantity is conserved.

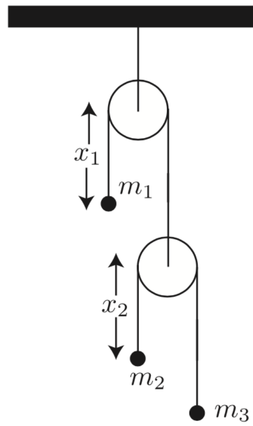

Figure 2: Problem 6

The two symmetries of this system are given by the conservation of the string length: as m_1 moves up x_1, m_2, m_3 move down $-x_1$. Furthermore, as m_2 moves up x_2, m_3 moves down $-x_2$. The Lagrangian for this system is

$$L = \frac{1}{2}m_1\dot{x}_1 + \frac{1}{2}m_2[\dot{x}_1 - \dot{x}_2]^2 + \frac{1}{2}m_3[\dot{x}_1 + \dot{x}_2]^2 - m_1 g x_1 - m_2 g[x_2 - x_1] - m_3 g[x_1 + x_2],$$

which simplifies to

$$L = \frac{1}{2}\dot{x}_1^2(m_1 + m_2 + m_3) - (m_2 - m_3)\dot{x}_1\dot{x}_2 + \frac{1}{2}\dot{x}_2^2(m_2 + m_3) - x_1 g(m_1 - m_2 + m_3) - x_2 g(m_2 + m_3).$$

Notice, for the potential energy term, that

$$-x_1 g(m_1 - m_2 + m_3) - x_2 g(m_2 + m_3) = -g\left[x_1(m_1 - m_2 + m_3) + x_2(m_2 + m_3)\right],$$

so L is invariant under the transformation

$$\begin{aligned} x_1 &\to x_1 - \epsilon(m_2 + m_3) \\ x_2 &\to x_2 + \epsilon(m_1 - m_2 + m_3), \end{aligned}$$

where I will let $K_{x_1} = -m_2 - m_3$ and $K_{x_2} = m_1 - m_2 + m_3$. The conserved momentum is then given by

$$\begin{aligned} P &= \frac{\partial L}{\partial \dot{x}_1}K_{x_1} + \frac{\partial L}{\partial \dot{x}_2}K_{x_2} \\ &= (\dot{x}_1(m_1 + m_2 + m_3) - \dot{x}_2(m_2 - m_3))\left[-m_2 - m_3\right] \end{aligned}$$

$$
\begin{aligned}
&+ (\dot{x}_2(m_2+m_3) - \dot{x}_1(m_2-m_3))\,[m_1 - m_2 + m_3] \\
&= \dot{x}_1\,[(m_2-m_3)(m_1-m_2+m_3) - (m_2+m_3)(m_1+m_2+m_3)] - \dot{x}_2 m_1(m_2+m_3).
\end{aligned}
$$

This p is constant and hence conserved. To verify this with the equations of motion, applying the E-L equations yields:

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= \ddot{x}_1(m_1+m_2+m_3) - \ddot{x}_2(m_2-m_3) \\
= \frac{\partial L}{\partial x_1} &= -g(m_1-m_2+m_3) \qquad \text{and} \\
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_2}\right] &= \ddot{x}_2(m_2+m_3) - \ddot{x}_1(m_2-m_3) \\
\frac{dL}{dx_2} &= -g(m_2+m_3)
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \ddot{x}_1(m_1+m_2+m_3) - \ddot{x}_2(m_2-m_3) = -g(m_1-m_2+m_3) \\
&\Longrightarrow \ddot{x}_2(m_2+m_3) - \ddot{x}_1(m_2-m_3) = -g(m_2+m_3).
\end{aligned}
$$

Multiplying the first equation of motion by $-(m_2+m_3)$ and the second by $(m_1-m_2+m_3)$,

$$
\begin{aligned}
-\ddot{x}_1(m_1+m_2+m_3)(m_2+m_3) + \ddot{x}_2(m_2-m_3)(m_2+m_3) &= -g(m_1-m_2+m_3)(m_2+m_3) \\
-\ddot{x}_1(m_2-m_3)(m_1-m_2+m_3) + \ddot{x}_2(m_2+m_3)(m_1-m_2+m_3) &= -g(m_2+m_3)(m_1-m_2+m_3).
\end{aligned}
$$

Subtracting the first equation of motion from the second yields

$$
\begin{aligned}
&\ddot{x}_1\,[(m_2-m_3)(m_1-m_2+m_3) - (m_2+m_3)(m_1+m_2+m_3)] \\
&\qquad\qquad + \ddot{x}_2(m_2+m_3)\,[(m_2-m_3) + (m_1-m_2+m_3))] = 0 \\
&\ddot{x}_1\,[(m_2-m_3)(m_1-m_2+m_3) - (m_2+m_3)(m_1+m_2+m_3)] + \ddot{x}_2 m_1(m_2+m_3) = 0
\end{aligned}
$$

$$
\Longrightarrow \frac{d}{dt}\left[-2\dot{x}_1(m_2^2+m_1m_3+m_3^2) - \dot{x}_2 m_1(m_2+m_3)\right] = 0,
$$

and thus

$$
p = -2\dot{x}_1(m_2^2+m_1m_3+m_3^2) - \dot{x}_2\,m_1(m_2+m_3)
$$

is the conserved quantity.