

Q1. (a)

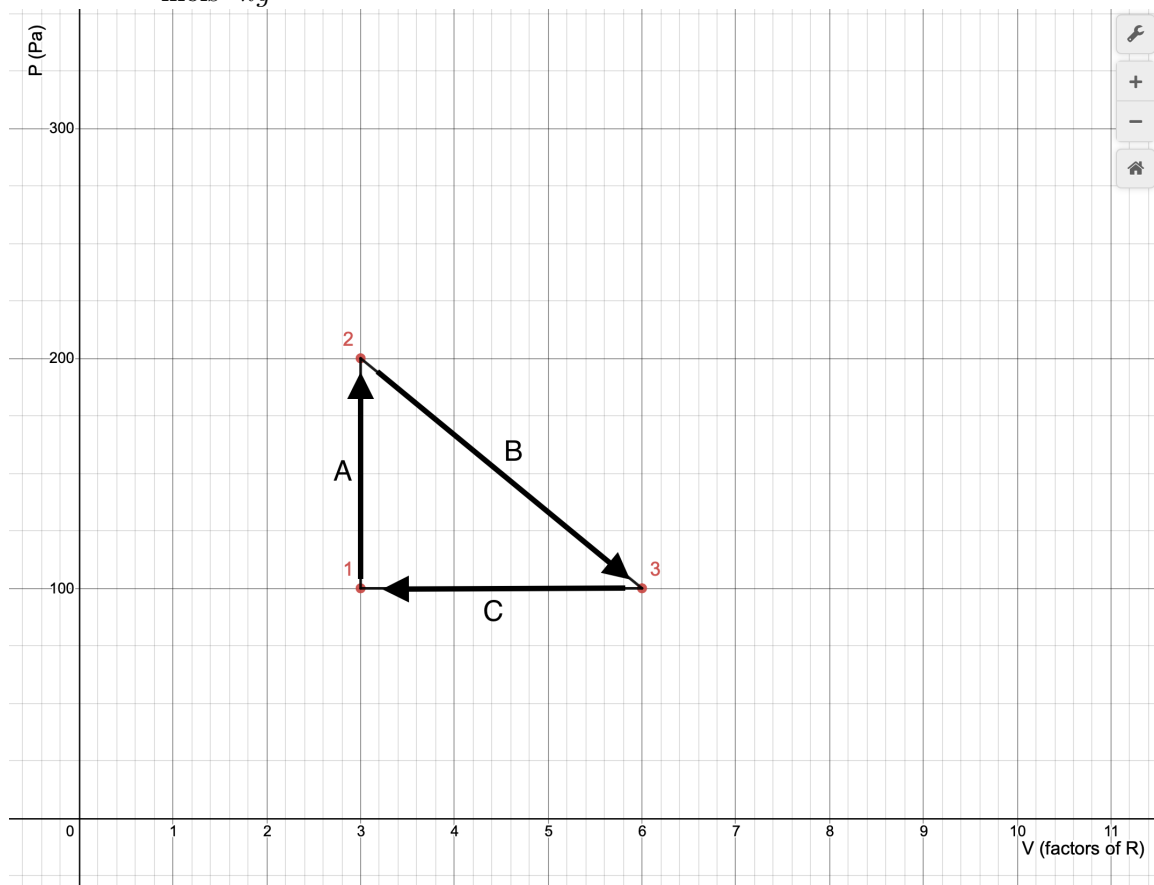
We are given  $PV = nRT$  since we are working with an ideal gas. Here,  $n = 1$  since we are working with 1 mole of a monotonic gas. This rearranges to  $V = \frac{nRT}{P}$  to solve for volume. Then

$$V_1 = R \cdot \frac{300 \text{ K}}{100 \text{ Pa}} = 3R$$

$$V_2 = R \cdot \frac{600 \text{ K}}{200 \text{ Pa}} = 3R$$

$$V_3 = R \cdot \frac{600 \text{ K}}{100 \text{ Pa}} = 6R,$$

where  $R = 8.31 \frac{\text{J}}{\text{mols} \cdot \text{kg}}$ . I left the solution in terms of  $R$  for simplicity. Plotting, we have



(b)

For an ideal gas, the heat capacity at constant pressure is given by  $C_P = Nk \left( \frac{1}{2}f + 1 \right)$  while the heat capacity at constant volume is given by  $C_V = \frac{1}{2}Nfk$ . These are straightforward calculations.

$N$  is the number of molecules ( $1 \text{ mol} \implies N = 6.02 \times 10^{23}$ ) and  $f$  for a monotonic molecule is 3 ( $x$ ,  $y$ , and  $z$  directions):

$$\begin{aligned} C_P &= \frac{5}{2} \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \\ &= 20.8 \frac{J}{K}. \\ C_V &= \frac{3}{2} \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \\ &= 12.5 \frac{J}{K}. \end{aligned}$$

(c)

As  $C_P = \frac{Q}{\Delta T}$ , then the heat flow is given by  $Q = C_P \Delta T$ :

$$\begin{aligned} A : \quad \Delta T_A &= 300 \text{ K}, Q_A = \frac{5}{2} \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 300 \text{ K} \\ &= 6.23 \times 10^3 \text{ J} \\ B : \quad \Delta T_B &= 0 \text{ K}, Q_B = 0 \text{ J}. \\ C : \quad \Delta T_C &= -300 \text{ K}, Q_C = -\frac{5}{2} \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 300 \text{ K} \\ &= -6.23 \times 10^3 \text{ J}. \end{aligned}$$

The work done on a quasistatic, ideal gas system is given by

$$W = P_2 V_2 - P_1 V_1.$$

As calculated in part (a),

$$\begin{aligned} A : \quad W_A &= (200 \text{ Pa} \cdot 3R) - (100 \text{ Pa} \cdot 3R) \\ &= 100 \text{ Pa} \cdot 3R \approx 2.49 \times 10^3 \text{ J} \\ B : \quad W_B &= (100 \text{ Pa} \cdot 6R) - (200 \text{ Pa} \cdot 3R) \\ &= 0 \text{ J} \\ C : \quad W_C &= (100 \text{ Pa} \cdot 3R) - (100 \text{ Pa} \cdot 6R) \\ &= -100 \text{ Pa} \cdot 3R \approx -2.49 \times 10^3 \text{ J}. \end{aligned}$$

Then the total change in energy  $\Delta U$  is given by the first law of thermodynamics,  $\Delta U = Q + W$ :

$$\begin{aligned} A : \quad \Delta U_A &= Q_A + W_A = 6.23 \times 10^3 \text{ J} + 2.49 \times 10^3 \text{ J} \\ &= 8.72 \times 10^3 \text{ J} \\ B : \quad \Delta U_B &= Q_B + W_B = 0 \text{ J} \\ C : \quad \Delta U_C &= Q_C + W_C = -6.23 \times 10^3 \text{ J} - 2.49 \times 10^3 \text{ J} \\ &= -8.72 \times 10^3 \text{ J}. \end{aligned}$$

**Q2. (a)**

The energy of an ideal gas is given by  $U = \frac{1}{2}NfkT$ , and so the energy of container  $A$  is

$$\begin{aligned} U_A &= \frac{1}{2}NfkT_A \\ &= \frac{1}{2} \cdot 5 \cdot 6.02 \times 10^{23} \cdot 5 \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 293 K \\ &= 30.4 kJ. \end{aligned}$$

Since we required  $U_A + U_B = 50 kJ$ , then  $U_B$  must be  $50 kJ - 30.4 kJ = 19.6 kJ$ . Then we can solve for the temperature  $T_B$ ,  $T_B = \frac{2U_B}{Nfk}$ :

$$\begin{aligned} T_B &= \frac{2U_B}{Nfk} \\ &= \frac{2 \cdot 19.573415 \times 10^3 J}{7 \cdot 6.02 \times 10^{23} \cdot 5 \cdot 1.38 \times 10^{-23} \frac{J}{K}} \\ &= 134 K. \end{aligned}$$

**(b)**

The pressure in each container can be calculated by the ideal gas law:  $PV = nkT$ :

$$\begin{aligned} P_A &= \frac{NkT_A}{V} \\ &= \frac{5 \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 293 K}{3 m^3} \\ &= 4.06 \times 10^3 Pa \\ P_B &= \frac{NkT_B}{V} \\ &= \frac{7 \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 134.6334 K}{3 m^3} \\ &= 2.61 \times 10^3 Pa. \end{aligned}$$

**(c)**

This is possible to determine using energy conservation. Since no work is done on the system,  $U = Q = C_{V,A}T_A + C_{V,B}T_B = (C_{V,A} + C_{V,B})T_{eq}$ . Like before,  $C_V = \frac{1}{2}NfkT$ . Then

$$\begin{aligned} C_{V,A} &= \frac{1}{2} \cdot 5 \cdot 6.02 \times 10^{23} \cdot 5 \cdot 1.38 \times 10^{-23} \frac{J}{K} \\ &= 104 \frac{J}{K} \\ C_{V,B} &= \frac{1}{2} \cdot 7 \cdot 6.02 \times 10^{23} \cdot 5 \cdot 1.38 \times 10^{-23} \frac{J}{K} \end{aligned}$$

$$= 145 \frac{J}{K}.$$

Then

$$T_{eq} = \frac{103.845 \frac{J}{K} \cdot 296 K + 145.383 \frac{J}{K} \cdot 134.6334 K}{103.845 \frac{J}{K} + 145.383 \frac{J}{K}} = 202 K.$$

The volume of the containers resolves to  $6m^3$  once the barrier is removed. The pressure in the containers will resolve to

$$P = \frac{NkT_{eq}}{V} = \frac{12 \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 201.8695 K}{6m^3} = 3.35 \times 10^3 Pa.$$

(d)

If the wall is thin enough to allow heat to permeate through, then the temperature will still resolve to  $202 K$ , however the relaxation time will increase. The pressure of each container will now vary, which can be calculated by the ideal gas law:

$$\begin{aligned} P_A &= \frac{NkT_{eq}}{V} \\ &= \frac{5 \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 201.8695 K}{3m^3} \\ &= 2.80 \times 10^3 Pa \\ P_B &= \frac{NkT_{eq}}{V} \\ &= \frac{7 \cdot 6.02 \times 10^{23} \cdot 1.38 \times 10^{-23} \frac{J}{K} \cdot 201.8695 K}{3m^3} \\ &= 3.91 \times 10^3 Pa. \end{aligned}$$

**Q3. (a)**

Since the gas is able to freely exchange heat with its surrounding environment, this is not an adiabatic process. Since the gas rapidly expands, this is not an isothermal process either. By Boyle's law,  $P \propto \frac{1}{V}$  which implies  $P_i V_i = P_f V_f$ . This allows us to calculate the final volume of the cylinder:  $P_1 V_1 = P_2 V_2 \implies V_2 = V_1 \cdot \frac{P_1}{P_2}$ .

Work done on the piston is  $W = F \Delta h = P(V_f - V_i)$ . All at once, the pressure exerted on the piston by the gas is  $1 \text{ atm} = 101325 \text{ Pa}$ . Then

$$\begin{aligned}
 W &= P(V_f - V_i) \\
 &= P \left( V_1 \cdot \frac{P_1}{P_2} - V_1 \right) \\
 &= P V_1 \left( \frac{P_1}{P_2} - 1 \right) \\
 &= (101325 \text{ Pa}) \cdot 0.002 \text{ m}^3 \left( \frac{5 \text{ Pa}}{1 \text{ Pa}} - 1 \right) \\
 &= 8.10 \times 10^2 \text{ J}.
 \end{aligned}$$

**(b)**

When the system is allowed to re-equilibrate after each weight is removed, this becomes an isothermal process as the gas exchanges heat through the piston. The total work done on the system is then the sum of the work done when each individual weight is taken off.

Again, by Boyle's law,

$$\begin{aligned}
 V_1 &= V_0 \cdot \frac{P_0}{P_1} = 0.002 \text{ m}^3 \cdot \frac{5 \text{ atm}}{4 \text{ atm}} \\
 &= 0.0025 \text{ m}^3 \\
 V_2 &= V_1 \cdot \frac{P_1}{P_2} = 0.0025 \text{ m}^3 \cdot \frac{4 \text{ atm}}{3 \text{ atm}} \\
 &= 0.0033 \dots \text{ m}^3 \\
 V_3 &= V_2 \cdot \frac{P_2}{P_3} = 0.0033 \text{ m}^3 \cdot \frac{3 \text{ atm}}{2 \text{ atm}} \\
 &= 0.0050 \text{ m}^3 \\
 V_4 &= V_3 \cdot \frac{P_3}{P_4} = 0.005 \text{ m}^3 \cdot \frac{2 \text{ atm}}{1 \text{ atm}} \\
 &= 0.01 \text{ m}^3.
 \end{aligned}$$

Then

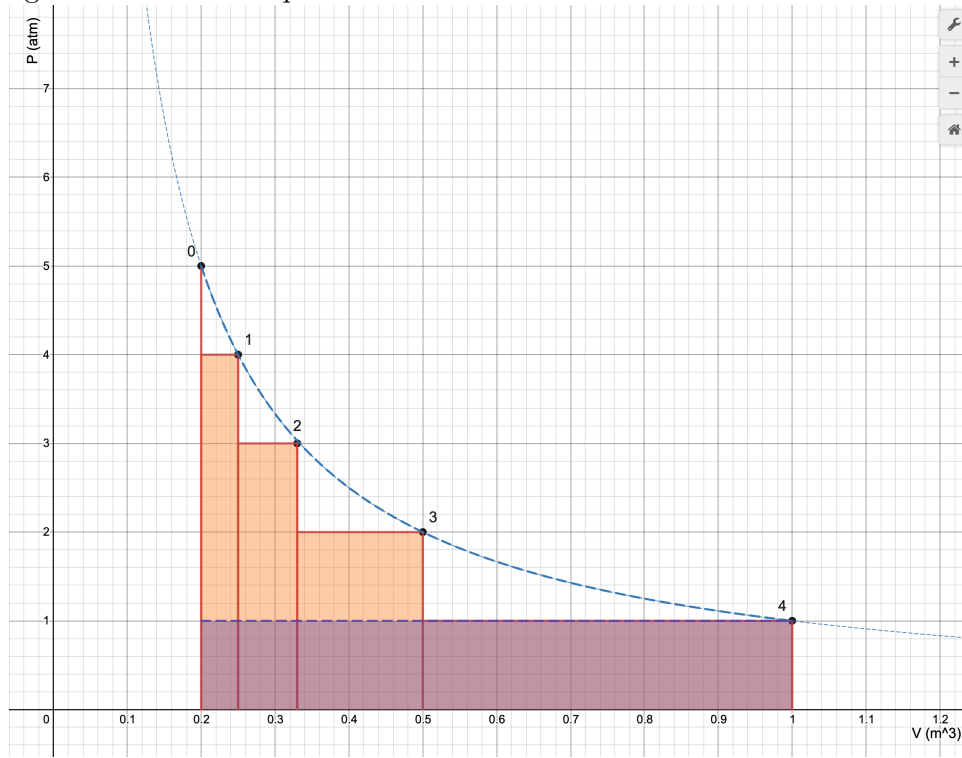
$$\begin{aligned}
 W &= P \Delta V \\
 &= (4 \cdot 101325 \text{ Pa}) \cdot (0.0025 - 0.002) \text{ m}^3 + (3 \cdot 101325 \text{ Pa}) \cdot (0.0033 - 0.0025) \text{ m}^3 + \\
 &\quad (2 \cdot 101325 \text{ Pa}) \cdot (0.005 - 0.0033) \text{ m}^3 + (1 \cdot 101325 \text{ Pa}) \cdot (0.01 - 0.005) \text{ m}^3
 \end{aligned}$$

$$= 1.3 \times 10^3 J.$$

(c)

To extract the maximum amount of work done by the system, the process must be done quasistatically. This is allowing for a reversible process. The work done by the gas during expansion (calculated in part (b)) is then the maximum amount of work the piston can do when restricted to removing one weight at a time. The process of removing all the weights at once creates entropy, since the rapid expansion of the gas can only contribute so much work due to heat being converted into mechanical work. Since the system creates entropy, the process cannot be reversible.

To extract the maximum amount of work done on the system, the weights need not be removed one at a time (which is discrete), but rather linearly. I have drawn the plot representing the work done by the gas in these various processes:



The purple region is the work calculated in (a) while the orange region is the work calculated in the discrete steps in (b). Then

$$W = \int_{V_i}^{V_f} \frac{dV}{V} = \log \frac{V_f}{V_i}.$$

The work is missing a factor of  $10^{-2}$  because of the scale of the volume. Then the maximum amount of work done by the system is

$$W = \frac{1}{100} \cdot 101325 \cdot \log \frac{0.01}{0.002} = 1.63 \times 10^3 J.$$

**Q4. (a)**

By the equipartition theorem, each degree of freedom contains energy  $E = \frac{1}{2}kT$ . Since we are only considering the  $\hat{\mathbf{x}}$  component of the velocity of the particle (assuming we are working in just 2D), then

$$E = \frac{1}{2}mv_x^2 = \frac{1}{2}kT \implies v_{rms} = \sqrt{\frac{kT}{m}}.$$

Now, if the container is located in 3D with the wall of the container along the  $\hat{\mathbf{z}}$  component facing upwards (in the wall direction), then we must consider the  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  components of the particles velocity. In this case (following from the previous calculation),

$$E = \frac{1}{2}mv_{xy}^2 = 2 \cdot \frac{1}{2}kT \implies v_{rms} = \sqrt{\frac{2kT}{m}}$$

since the particles velocity along the  $\hat{\mathbf{z}}$  axis is 0.

**(b)**

The number of particles in the container can be given by  $N(t) = N_0 - N'(t)$ , where  $N_0$  is the initial number of particles in the container, and  $N'(t)$  is the number of particles escaping through the hole. This implies that  $N'(t) = N_0 - N(t)$ . Then  $\frac{dN'}{dt} = -\frac{dN}{dt}$ .

By the ideal gas law,  $P = \frac{NkT}{V}$ , and  $P_A = \frac{2(N_0 - N(t))m \langle v_x \rangle}{A\Delta t} \rightarrow -\frac{2 \langle v_x \rangle}{A} \frac{dN}{dt}$ . Equating,

$$\frac{NkT}{V} = -\frac{2m \langle v_x \rangle}{A} \dot{N}.$$

Making the approximation  $\langle v_x \rangle = v_{rms} = \sqrt{\frac{kT}{m}}$ , we have

$$\dot{N} = -N \frac{A}{2V} \sqrt{\frac{kT}{m}} = -N \frac{A}{2V} v_{rms}.$$

**(c)**

Since this is a separable first order DE,

$$\int \frac{dN}{N} = -\frac{A v_{rms}}{2V} \int dt \implies \log N = -\frac{A v_{rms}}{2V} t + C.$$

Then  $N = C' \exp\left(\frac{-t A v_{rms}}{2V}\right)$ , which becomes

$$N(t) = N(0)e^{-t/\tau}$$

where  $\tau = \frac{2V}{A \langle v_x \rangle}$  is the characteristic time and  $C' = N(0)$  is the initial number of particles in the container, as stated previously.

(d)

The volume of the tire is given by

$$V = \left[ \pi \left( \frac{33''}{2} \right)^2 - \pi \left( \frac{15''}{2} \right)^2 \right] \cdot 12.5'' = 2700\pi \text{ in}^3.$$

Now since 1 cubic inch is  $1.63871 \times 10^{-5} m^3$ , then the volume of the tire becomes  $V = 1.39 \times 10^{-1} m^3$ .

From (c), the characteristic time is given by  $\tau = \frac{2V}{A \langle v_x \rangle}$ .

Since  $\langle v_x \rangle \approx v_{rms} = \sqrt{\frac{kT}{m}}$ , if the tire is kept at room temperature then

$$v_{rms} = \sqrt{\frac{1.38 \times 10^{-23} \frac{J}{K} \cdot 293 K}{4.81 \times 10^{-26} kg}} = 289.935 m s^{-1}.$$

Therefore,

$$A = \frac{2V}{\tau v_{rms}} = \frac{2 \cdot 1.39 \times 10^{-1} m^3}{3600 s \cdot 289.9351135 m s^{-1}} \approx 2.66 \times 10^{-14} m^2 = 2.66 \times 10^{-8} mm^2.$$



**Q5. (a)**

The Fourier heat conduction law is given by

$$\frac{Q}{\Delta t} = -k_t A \frac{dT}{dx}.$$

Now, the heat capacity is  $C \equiv \frac{Q}{\Delta T}$ . Furthermore, since we are assuming the gradient of the temperature across the rod to be uniform,  $\frac{dT}{dx} \rightarrow \frac{\Delta T}{\Delta x} \rightarrow \frac{T - T_E}{\ell}$ . Then

$$\frac{C \Delta T}{\Delta t} = -k_t A \frac{T - T_E}{\ell} \implies \frac{dT}{dt} = -k_t A \frac{T - T_E}{C \ell}.$$

Solving the equation,

$$\begin{aligned} \frac{dT}{dt} &= -k_t A \frac{T - T_E}{C \ell} \\ \implies \frac{dT}{T - T_E} &= -\frac{k_t A}{C \ell} dt \\ \implies \int \frac{dT}{T - T_E} &= -\int \frac{k_t A}{C \ell} dt \\ \implies \log(T - T_E) &= -\frac{k_t A t}{C \ell} + P, \quad P \in \mathbb{R} \\ \implies T - T_E &= \exp\left(-\frac{k_t A t}{C \ell} + P\right) = P' \exp\left(-\frac{k_t A t}{C \ell}\right) \\ \implies T &= T_E + P' \exp\left(-\frac{k_t A t}{C \ell}\right). \end{aligned}$$

The initial temperature of the block will be dictated by  $t = 0$ , or the constant  $P'$ :

$$\begin{aligned} T(t = 0) &= T_0 = T_E + P' \\ \implies P' &= T_0 - T_E. \end{aligned}$$

Then

$$T = T_E + (T_0 - T_E) \exp\left(-\frac{k_t A t}{C \ell}\right).$$

**(b)**

We are given that

$$\begin{aligned} c &\equiv \frac{C}{m} = 600 \frac{J}{kg \, K} \\ k_t &= 250 \frac{W}{m \, K} \\ \ell &= 10 \, mm = 0.01 \, m \\ A &= 3 \, mm^2 = 3 \times 10^{-6} \, m^2 \\ T_0 &= 30^\circ C = 303.15 \, K \end{aligned}$$

$$\begin{aligned}
T_F &= -19^\circ C = 254.15\ K \\
T_E &= -20^\circ C = 253.15\ K \\
t &= 12\ h = 43'200\ s.
\end{aligned}$$

Then

$$254.15\ K = 253.15\ K + (303.15\ K - 253.15\ K) \exp \left( -\frac{250\ W/m\ K \cdot 3 \times 10^{-6}\ m^2 \cdot 43200\ s}{m \cdot 600\ J/kg\ K \cdot 0.01\ m} \right).$$

Simplifying,

$$1\ K = 50\ K \exp \left( -\frac{5.4}{m} \right).$$

This implies

$$\begin{aligned}
\frac{1}{50} &= \exp \left( -\frac{5.4}{m} \right) \\
\Rightarrow\ m &= -\frac{5.4}{\log(1/50)} = 1.38\ kg.
\end{aligned}$$