MAT224 Linear Algebra II
Assignment 5

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: Jace Alloway

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Full Name:

Student number:

I confirm that:

- I have read and followed the policies described in the document Assignment Policies & FAQ.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway

2)

1. Let F be any field other than $\mathbb{F}_2$, and consider the subspaces $W_1 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in F \right\}$, and $W_2 = \left\{ \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} \mid d \in F \right\}$ of $M_{2\times 2}(F)$.

1(a) Show that $M_{2\times 2}(F) = W_1 \oplus W_2$.

To show that $M_{2\times 2}(F) = W_1 \oplus W_2$, I need to show that $W_1 \cap W_2 = \{\mathbf{0}\}$ and $M_{2\times 2}(F) = W_1 + W_2$.

I will show that $W_1 \cap W_2 = \{\mathbf{0}\}$.

- Let $A_{W_1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $A_{W_2} = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}$ for any $a, b, c, d \in F$. Notice that $A_{W_1} = A_{W_2}$ only when $a = c = 0$ and $b = -d = d$. It is already clear that the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies that $a = c = d = b = 0$, but to show that $b = -d = d = 0$ in any field $F \neq \mathbb{F}_2$, we can show that $d = -d \implies d = 0 \implies b = 0$.

  In a field $F \neq \mathbb{F}_2$, there are either an infinite number of elements in the field or a restricted odd number of elements, where the number of elements in the field is determined by a prime p, which is only if the field is finite.
  In the case where $F = \mathbb{F}_{p\neq 2}$ is finite, then the requirement for $d = -d$ is that $d - pd = -d$ and $d = \frac{p}{2}$, which is not possible if $p \neq 2$ is prime (p must be odd):

  $$\frac{p}{2} - \frac{p^2}{2} = -\frac{p}{2} \implies \implies 1 - p = -1 \implies p = 2, \text{ which contradicts our assumption since } p \neq 2.$$

  Thus the only solution is when $d = 0$ in any finite field $F = \mathbb{F}_{p\neq 2}$.
  In the case where the field is not finite (infinite number of elements), for $d = -d \implies d = 0$, we have

  $$\begin{aligned} -d &= d \\ -d + {\color{red}d} &= d + {\color{red}d} && \text{right-adding 'd' to both sides} \\ {\color{red}0} &= 2d && \text{by axiom \textbf{iv}, existence of additive inverse} \\ &\implies d = 0. \end{aligned}$$

  We have that $d = -d \iff d = 0$ for any field $F \neq \mathbb{F}_2$, and thus $b = 0$ as well.
  Therefore $A_{W_1} = A_{W_2} \implies a = b = c = d = 0$, so $W_1 \cap W_2 = \{\mathbf{0}\}$.

By the addition of the sets $W_1 + W_2$, I need to show that that $M_{2\times 2}(F) = \text{span}\left\{ \begin{bmatrix} a & b+d \\ b-d & c \end{bmatrix} \mid a, b, c, d \in F \right\}$.

- By the properties of matrix addition, a basis for $M_{2\times 2}(F)$ is

  $$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

  It then suffices to show that the scalars $a, b+d, b-d$, and c span the whole space over the field F. We have the set

  $$\left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b+d \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b-d & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \right\},$$

  which then follows that

  $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{r2: r2+r4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{r2: r2 x 1/2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{r4: (-1)r4+r2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

  Therefore each of the basis elements are linearly independent over the field F, so $M_{2\times 2}(F) = W_1 + W_2$.

Therefore $M_{2\times 2}(F) = W_1 \oplus W_2$.

1. Now suppose our field $F = \mathbb{F}_2$, and consider the subspaces $W_1 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in F \right\}$, and $W_2 = \left\{ \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} \mid d \in F \right\}$ of $M_{2\times 2}(F)$.

1(b) Is it still true that $M_{2\times 2}(F) = W_1 \oplus W_2$?

No, it is not true anymore. I will examine the requirement of the direct sum $W_1 \cap W_2 = \{\mathbf{0}\}$. Let $d = 1$. Then $-d = [-1]_2 = 1 = d \neq 0$, since the field is $F = \mathbb{F}_2$. Then $W_1 \cap W_2 \neq \{\mathbf{0}\}$:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} \iff (a = c = 0) \wedge (b = -d = d),$$

however $b = -d = d \iff (b = 0) \vee (b = 1)$. We would have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ in } \mathbb{F}_2.$$

Therefore $W_1 \cap W_2 \neq \{\mathbf{0}\}$, and thus $M_{2\times 2}(F) \neq W_1 \oplus W_2$.

2. Consider the vector space $\mathbb{F}_2^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{F}_2\}$. How many $T \in \mathfrak{L}(\mathbb{F}_2^2)$ are isomorphisms? List them all.

An isomorphism is an invertible linear map. For a linear operator $T \in L(\mathbb{F}_2^2)$ to be invertible, it must be that $\det T \neq 0$.
Let $\mathcal{M} = \left\{ T : \mathbb{F}_2 \to \mathbb{F}_2 : [T_M] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ where } a, b, c, d \in \mathbb{F}_2 \right\}$ denote the set of linear mappings over the finite field $\mathbb{F}_2$.
For T to be an isomorphism, $\det[T_M] \neq 0$, that is, that T is invertible. We have

$$\det[T_M] = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

Then $ad - bc \neq 0 \implies ad \neq bc$.

Since the linear map is over the field $\mathbb{F}_2$, the only entries in the matrix representations of each of the isomorphisms are 0 and 1. Any value outside the set $\{0, 1\}$ would only imply a repeated mapping because we are only examining the vectors whose coordinates are defined in the closed finite field $\mathbb{F}_2$.
Therefore for $ad \neq bc$, it must be that either $ad = 1 \neq 0 = bc$ or $ad = 0 \neq 1 = bc$.

Thus, the linear operators defined by the matrices

$$[\mathrm{T}_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [\mathrm{T}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad [\mathrm{T}_3] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[\mathrm{T}_4] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad [\mathrm{T}_5] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad [\mathrm{T}_6] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

are the only linear transformations $T \in \mathfrak{L}(\mathbb{F}_2^2)$ which are isomorphisms.

3. Let V be a finite dimensional vector space over a field F, and let $T \in \mathfrak{L}(V)$.

3(a) Suppose that W is a T-invariant subspace, and that $V = \ker T \oplus W$. Prove that $W = \operatorname{im} T$.

I want to prove that for a T-invariant subspace W and finite-dimensional vector space V, if $V = \ker T \oplus W$ then $W = \operatorname{im} T$.

Proof.
Assume $V = \ker T \oplus W$. Then

$$\ker T \cap W = \{\mathbf{0}\} \quad \text{and} \quad V = \ker T + W.$$

Assume W is a T-invariant subspace, that is that $T(W) \subseteq W$, or equivilantly $\forall \mathbf{x} \in W,\ T(\mathbf{x}) \in W$.

Suppose $\dim V = n$.
Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a basis for W, and extend to a basis V which is $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ where $\{\mathbf{x}_{k+1}, \mathbf{x}_{k+2} \ldots, \mathbf{x}_n\}$ is a basis for $\ker T$.

Since the addition of two subspaces is the union of their spans, we have that

$$V = \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\} = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \cup \operatorname{span}\{\mathbf{x}_{k+1}, \mathbf{x}_{k+2} \ldots, \mathbf{x}_n\} = W + \ker T.$$

By applying T, we have

$$\operatorname{span}\{T\mathbf{x}_1, \ldots, T\mathbf{x}_n\} = \operatorname{span}\{T\mathbf{x}_1, T\mathbf{x}_2, \ldots, T\mathbf{x}_k\} \cup \operatorname{span}\{T\mathbf{x}_{k+1}, T\mathbf{x}_{k+2} \ldots, T\mathbf{x}_n\} = \operatorname{span}\{T\mathbf{x}_1, T\mathbf{x}_2, \ldots, T\mathbf{x}_k\},$$

since the vectors in the basis of the kernel go to zero under $T:$

$$\operatorname{span}\{T\mathbf{x}_{k+1}, T\mathbf{x}_{k+2}, \ldots, T\mathbf{x}_n\} = \operatorname{span}\{\mathbf{0}\} = \{\mathbf{0}\}.$$

W is T-invariant and $\{\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \ldots, \mathbf{x}_n\}$ is a basis for W, so $\{T\mathbf{x}_{k+1}, T\mathbf{x}_{k+2}, \ldots, T\mathbf{x}_n\}$ is also a basis for W. This is because the restriction of T to W, denoted by $T|_W : W \to W$, is injective because $\ker(T|_W) = \{\mathbf{0}\}$ by **Proposition 2.4.2**, and thus since $T|_W$ is an operator, then by **Proposition 2.4.10**, $T|_W$ is also surjective. Therefore $T(W) = W$.

It then follows that

$$T(V) = \operatorname{span}\{T\mathbf{x}_1, \ldots, T\mathbf{x}_k, T\mathbf{x}_{k+1}, \ldots, T\mathbf{x}_n\} = \operatorname{span}\{T\mathbf{x}_1, \ldots, T\mathbf{x}_k\} = T(W) = W,$$

and thus $\operatorname{im} T = W$, which is what I needed to prove.

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3. Let V be a finite dimensional vector space over a field F, and let $T \in \mathfrak{L}(V)$.

3(b) Suppose that W is the span of all eigenvectors of T corresponding to non-zero eigenvalues. Prove that if T is diagonalizable then $W = \operatorname{im} T$.

I want to prove that if T is diagonalizable, then $W = \operatorname{im} T$ where W is the span of all eigenvectors of T corresponding to a non-zero eigenvalue.

Proof.
Let $\dim V = n$, and assume that T is diagonalizable. If T is diagonalizable, then by **Corollary 4.2.8**, there are n distinct eigenvectors with corresponding non-zero eigenvalues for T. Assume that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are eigenvectors of $T \in \mathfrak{L}(V)$ with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Let $W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, the span of all eigenvectors of T. Let $E_{\lambda_1}(T), E_{\lambda_2}(T), \ldots, E_{\lambda_n}(T)$ be the corresponding eigenspaces of each eigenvector $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively.

Since T is diagonalizable, then by **Quiri Li Diagonalization Slides Definition 5**, the direct sum of the span of all distinct eigenspaces is V, that is,

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_n}.$$

However since $W = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_n}$, the span of all eigenvectors with non-zero eigenvalue, then it must be that $W = V$ and $\dim W = \dim V$.

If W is the span of all eigenvectors with corresponding non-zero eigenvalues, then $\ker T = \{\mathbf{0}\}$. This is because $\ker T$ would be an eigenvector with eigenvalue $\lambda = 0$, which would contradict our assumption since all eigenvalues are assumed to be non-zero.

By the Rank-Nullity Theorem, we have

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T) = 0 + \dim(\operatorname{im} T) = \dim(\operatorname{im} T).$$

By **Proposition 2.4.7**, since $\dim V = \dim \operatorname{im} T$, then T must be surjective. Since T is surjective, it then follows that $\operatorname{im} T = V = W$, which is what I needed to prove.

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