Homework 1

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Q1

10 / 10

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MAT334 Problem Set 1 — 09/28/2022
1006940802

1.

$$z^3 + i\bar{z}^2 - 3z - 4 = i \tag{1.1}$$

(a) Assume that $z = q$ and $\mathbb{I}\mathrm{m}\{q\} = 0$, so z is purely a real number. In such a case, (1.1) becomes $q^3 + iq^2 - 3q - 4 = i$, which follows from the fact that $\bar{q} = q$ for real numbers. If a real number were to satisfy this equation, we then require that

$$q^3 - 3q - 4 = 0 \quad \text{and} \quad iq^2 = i. \qquad (1.2,\ 1.3)$$

Now (2.3) is only satisfied if $q = \pm 1$. If $q = 1$, the (2.2) states that $1 - 3 - 4 = -6 \neq 0$, so we obtain a contradiction. Similarly, if $q = -1$, $-1 + 3 - 4 = -2 \neq 0$; a contradiction. This implies that $z = q$ cannot purely be a real number.

(b) To find the modulus, we have

$$\begin{aligned}\left|\frac{1-4i}{(z^3 + i\bar{z}^2 - 3z)^2}\right| &= \frac{|1-4i|}{|(z^3 - i\bar{z}^2 - 3z)^2|} \\ &= \frac{|1-4i|}{|(z^3 - i\bar{z}^2 - 3z - 4 + 4)^2|} \\ &= \frac{|1-4i|}{|(i+4)^2|} \\ &= \frac{1}{|1-4i|} \\ &= \frac{1}{\sqrt{1^2+4^2}} \\ &= \frac{1}{\sqrt{17}}.\end{aligned}$$

(c) Allow me to now calculate $\mathbb{I}\mathrm{m}\{-\bar{z}^3 + iz^2 + 3\bar{z} + i\}$. Notice that

$$\overline{(-\bar{z}^3 + iz^2 + 3\bar{z})} = -z^3 - i\bar{z}^2 + 3z = -4 - i, \tag{1.4}$$

which is given by a slight rearrangemnent of (1.1). However,

$$-\bar{4} - i = -4 + i = -\bar{z}^3 + iz^2 + 3\bar{z}, \tag{1.5}$$

which implies that

$$\mathbb{I}\mathrm{m}\{-\bar{z}^3 + iz^2 + 3\bar{z} + i\} = \mathbb{I}\mathrm{m}\{-4 + i + i\} = 2,$$

as required.

correct / close enough **10**

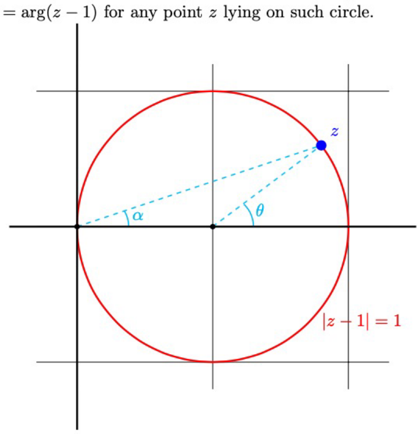
Q2

3 / 5

Submit Your Response to Question 2 Here.

2.

Allow me to provide a geometry interpretation to begin this problem. We have that $|z-1| = 1$ is the circle of radius 1 centered at $x = 1$. The problem is stating to prove that the angle $\alpha = 2\theta$, or equivalently, $2\arg(z) = \arg(z-1)$ for any point z lying on such circle.



For lack of confusion, allow me to utilize the arguments which I previously defined: $\arg(z) = \alpha$ and $\arg(z-1) = \theta$, so it suffices to prove that $2\alpha = \theta$. We begin with the relations

$$|z| \sin\alpha = \sin\theta \tag{2.1}$$
$$|z| \cos\alpha = \cos\theta + 1. \tag{2.2}$$

This implies that $|z| = \dfrac{\sin\theta}{\sin\alpha}$ by (2.1), so $\dfrac{\sin\theta}{\sin\alpha}\cos\alpha = \cos\theta + 1$ by (2.2). Multiplying through by $\sin\alpha$, we have that

$$\sin\theta\cos\alpha - \sin\alpha\cos\theta = \sin\alpha. \tag{2.3}$$

However, we can recognize that the left hand side of (2.3) is equivalent to a difference formula:

$$\sin\theta\cos\alpha - \sin\alpha\cos\theta = \sin(\theta-\alpha) = \sin\alpha, \tag{2.4}$$

which direclty implies that $\theta - \alpha = \alpha$, or that $\theta = 2\alpha$. This is what I wanted to prove. Resubstituting, we then have that $\arg(z-1) = 2\arg(z)$.

correctly showing that $\mathrm{Arg}(z-1) = 2\mathrm{Arg}(z)$. 3

Q3

10 / 10

Submit Your Response to Question 3 Here.

3.

$$\text{Define the set} \quad \Gamma = \{|z-p| = cx \,:\, z \in \mathbb{C},\ c, p \in \mathbb{R}\}.$$

(a) Suppose that $c \in (0,1)$. I will show that Γ is the set of numbers which compose an ellipse. For each of the following cases, we have that

$$|z-p|^2 = |x+iy-p|^2$$

$$
\begin{aligned}
&= (x-p)^2 + y^2 \\
\implies c^2x^2 &= (x-p)^2 + y^2. \qquad (3.1)
\end{aligned}
$$

Now I will define a new term $d^2 \equiv 1 - c^2 \neq 0$. An expansion of the $(x-p)^2$ term promotes a completion of the square of the polynomial:

$$
\begin{aligned}
0 &= x^2(1-c^2) - 2px + p^2 + y^2 = d^2x^2 - 2px + p^2 + y^2 \\
&= d^2\left[x^2 - \frac{2p}{d^2} + \frac{p^2}{d^2}\right] + y^2 \\
&= d^2\left[x^2 - \frac{2p}{d^2} + \frac{p^2}{d^2} + \frac{p^2}{d^4} - \frac{p^2}{d^4}\right] + y^2 \\
&= d^2\left[\left(x - \frac{p}{d^2}\right)^2 + \frac{p^2}{d^2} - \frac{p^2}{d^4}\right] + y^2 \\
&= d^2\left(x - \frac{p}{d^2}\right)^2 + p^2 - \frac{p^2}{d^2} + y^2. \qquad (3.2)
\end{aligned}
$$

After rearranging, we find that

$$
-p^2 + \frac{p^2}{d^2} = d^2\left[x - \frac{p}{d^2}\right]^2 + y^2. \qquad (3.3)
$$

which is only valid for $d \neq 0$, or equivalently $c \neq 0$. However, since $0 < c < 1$, then $0 < d^2 < 1$, which is positive. We see that equation (3.3) takes the form of an ellipse:

$$
C = \frac{(x-a)^2}{A} + \frac{(y-b)^2}{B}, \qquad (3.4)
$$

with $C = -p^2 + \dfrac{p^2}{1-c^2}$, $a = -\dfrac{p}{1-c^2}$, $A = \dfrac{1}{1-c^2}$ and $b = 0$, $B = 1$. For an ellipse, we require $A, B > 0$, which is true since $c \in (0,1)$ and $B = 1$, which is what I wanted to show.

(b) Allow me to now consider the case if $c = 1$. Allow me to return to equation (3.1):

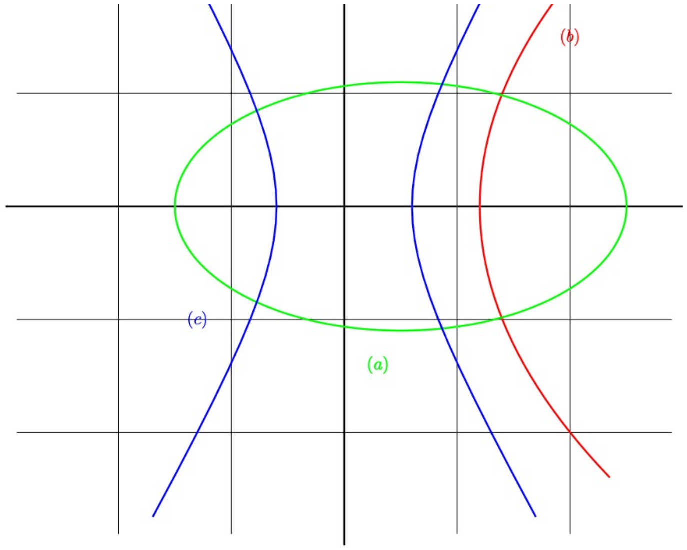
$$
x^2 = x^2 - 2px + p^2 + y^2 \implies 0 = -2px + p^2 + y^2. \qquad (3.5)
$$

(3.5) here is an implicit equation equivalent to that of a horizontal parabola, since we can solve for y:

$$
y^2 = 2px - p^2 \implies y = \pm\sqrt{2px - p^2}. \qquad (3.6)
$$

The domain of this equation is given by the real part, hence $2px - p^2 \geq 0$ else $y(x)$ becomes complex. We find that $x \geq \dfrac{p}{2}$ is the values of x for which y is defined. Therefore when $c = 1$, (3.1) defines a horizontal parabola.

(c) Now allow me to consider the case when $c > 1$. I will return to (3.2) as solved in (a), and in this case $d^2 = 1 - c^2 < 0$ because $c > 1$. Similarly to (3.4), we obtain an equation where $A < 0$ and $B > 0$, which defines a hyperbola. Here, $C = -p^2 + \dfrac{p^2}{1-c^2}$, $a = -\dfrac{p}{1-c^2}$, $A = \dfrac{1}{1-c^2} < 0$ and $b = 0$, $B = 1$. The following plot exemplifies each of the three equations defining the ellipse, parabola, and hyperbola:


(b)
(c)
(a)

Q4

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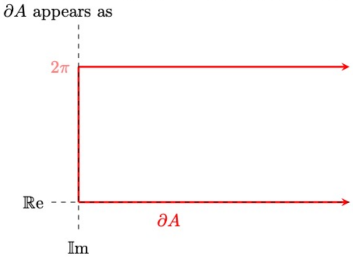
Submit Your Response to Question 4 Here.

4.

Define the set $A = \{z \in \mathbb{C} : 0 < \mathbb{R}\text{e}(z),\ 0 \leq \mathbb{I}\text{m}(z) < 2\pi\}$ and the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = i + e^{-z}$.

(a) To find the boundary ∂A, it may help to draw what A looks like. A appears to be a rectangle in the complex plane, whose real part extends to infinity and whose imaginary components are in between 0 and 2π. Then ∂A appears as



Allow me to find each of the mappings of ∂A to their respective image $f(\partial A)$. Recall that

$$\begin{aligned} f(z) &= i + e^{-z} \\ &= i + e^{-z} e^{-iy} \\ &= i + e^{-x}(\cos(y) - i\sin(y)) \end{aligned} \tag{4.1}$$

Equivalently,

$$f(\mathbb{R}\text{e}(z), \mathbb{I}\text{m}(z)) = e^{-\mathbb{R}\text{e}(z)} \cos(\mathbb{I}\text{m}(z)) + i(1 - e^{-\mathbb{R}\text{e}(z)} \sin(\mathbb{I}\text{m}(z))). \tag{4.2}$$

Notice that for $x \in [0, \infty)$, $e^{-x} \in [1, 0)$ and thus when $y = 0$ or $y = 2\pi$, we have that

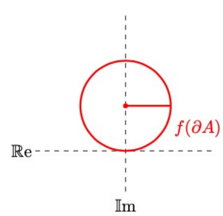
$$f(\mathbb{R}\text{e}(z), 0) = f(\mathbb{R}\text{e}(z), 2\pi) = i + e^{-\mathbb{R}\text{e}(z)}$$

so the horizontal lines of the boundary correspond to a line in the complex plane with points $(1 + i) \to (i)$ because $\mathbb{R}\text{e}(z) \to \infty$.

Similarly, we find that when $\mathbb{R}\mathrm{e}(z) = 0$ and $0 \leq \mathbb{I}\mathrm{m}(z) << \pi$, (4.2) becomes

$$f(0, \mathbb{I}\mathrm{m}(z)) = \cos(y) + i(1 - \sin(y)) = \cos(y) - i\sin(y) + i = e^{-iy} + i,$$

so as $\mathbb{I}\mathrm{m}(z)$ varies from 0 to 2π, $f(0, \mathbb{I}\mathrm{m}(z))$ parametrizes a circle centered at i. The corresponding graphical transformation appears as



Note that the full transformation $f(A)$ leaves a hole in the center of the circle at $z = i$, while the outside of the boundary $\partial(f(A))$ is still open. This is due to the fact that as $\mathbb{R}\mathrm{e}(z) \in (0, \infty)$, $e^{-\mathbb{R}\mathrm{e}(z)} \in (0, 1)$ is open on both ends of the interval.

(b) Now fix an arbitrary $z \in A$. Note that

$$|f(z) - i| = |i + e^{-z} - i| = |e^{-z}| = e^{-x}|e^{-iy}| = e^{-\cancel{\mathbb{R}}\mathrm{e}(z)} \tag{4.3}$$

Now since $z \in A$ consitutes that $\mathbb{R}\mathrm{e}(z) > 0$, then $e^{-\mathbb{R}\mathrm{e}(z)} \in (0, 1)$ which is less than 1, hence $|f(z) - i| < 1$.

(c) Now consider the set $D = \{z \in \mathbb{C} : 0 < |z - i| < 1\}$. I want to prove that for $\forall w \in D$, $\exists z \in A$ such that $f(z) = w$ and hence $\mathrm{Img}(f(A)) = D$.

Proof.

Why is $i \not \in f(A)$? -1

- Let $w \in D$ be arbitrary and suppose that we express w as $w = a + ib$. I will follow by considering the following two cases: values of w for which $a \neq 0$ and values of w for which $a = 0$.
- *($a \neq 0$)*: Choose

$$z = -\log\sqrt{(1 - b^2) + a^2} + i\arctan\left(\frac{b - 1}{a}\right) + i\pi. \tag{4.4}$$

z is certainly in A because $0 < -\log\sqrt{(1 - b^2) + a^2} \iff \sqrt{(1 - b^2) + a^2} \in (0, 1)$, however $\sqrt{(1 - b^2) + a^2} = |w - i| \in (0, 1)$, as required by the set D. Similarly,

$0 < \frac{\pi}{2} < \arctan\left(\frac{b-1}{a}\right) + \pi < \frac{3\pi}{2} < 2\pi$ so $\mathbb{R}\mathrm{e}(z) \in A$ and $\mathbb{I}\mathrm{m}(z) \in A$, thus $z \in A$.

- We find that

$$\begin{aligned} f(z) = e^{-z} + i &= \exp\left[\log\sqrt{(1-b^2)+a^2} - i\arctan\left(\frac{b-1}{a}\right) - i\pi\right] + i \\ &= \exp\left[\log\sqrt{(1-b^2)+a^2}\right] \cdot \left[\cos\left(\arctan\left(\frac{b-1}{a}\right)\right) - i\sin\left(\arctan\left(\frac{b-1}{a}\right)\right)\right] \cdot \exp[-i\pi] + i \\ &= \sqrt{(1-b^2)+a^2} \cdot (-1) \cdot \left[\frac{1}{\sqrt{(b-1)^2/a^2+1}} - i\frac{(b-1)/a}{\sqrt{(b-1)^2/a^2+1}}\right] + i \\ &= -a\sqrt{(1-b)^2/a^2+1} \cdot \frac{1 - i\cdot(b-1)/a}{\sqrt{(b-1)^2/a^2+1}} + i \end{aligned}$$

6

$$\begin{aligned} &= -a\left(1 - i\cdot\frac{b-1}{a}\right) + i \\ &= -a + ia\frac{1}{a}(b-1) + i \\ &= -a - i + ib + i \\ &= -a + ib. \end{aligned}$$

very ugly solution, but it's satisfying in its own way to just write down an equation and plug it in.

Regardless of the $-a$ term, which just constitutes the left/right $\mathbb{R}\mathrm{e}(z)$ for z in the set D, we still have that $w \in D$ where the $-a$ indicates a reflection about the $\mathbb{I}\mathrm{m}$-axis. We still have $|w - i| < 1$.

- Allow me to now consider the case when $a = 0$. This implies that $b \neq 0, 1, 2$ because $0, i, 2i \notin D$. When $b \in (0,1)$, choose $z = -\log(b) + \frac{\pi}{2}i$. Then

$$\begin{aligned} f(z) = e^{-z} + i = e^{\log(b)}\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] + i \\ = b\,[(0) - i(1)] + i \\ = i(1-b) \in D, \end{aligned}$$

which constitutes a line along the imaginary axis from $(0,1)$, open. To map the other line of the set, I will still set $b \in (0,1)$ however choose $z = -\log(b) + \frac{3\pi}{2}i$. We have that

$$\begin{aligned} f(z) = e^{-z} + i = e^{\log(b)}\left[\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}\right] + i \\ = b\,[(0) - i(-1)] + i \\ = i(1+b) \in D. \end{aligned}$$

- We have found that for any $w \in D$, there exists a $z \in A$ such that $f(z) = w$. By definition, it follows that $\mathrm{Img}(f(A)) = D$, which is what I wanted to prove.

□

It follows that $D \subset (f(A))$.

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Q5

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Submit Your Response to Question 5 Here.

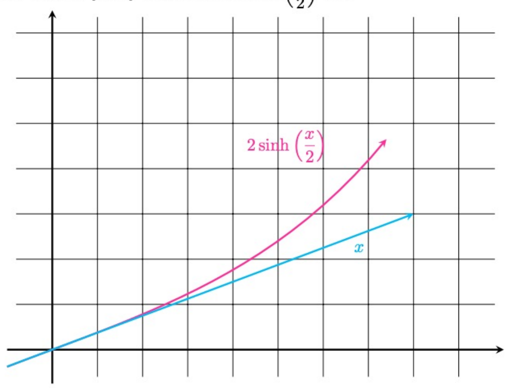
5.

(a) Consider the series $\sum_{n=1}^{\infty} \frac{ne^{i\pi/4}}{e^n - 1}$. This series is convergent, and I will show why.

By the absolute convergence test, if the series $\sum_{n=1}^{\infty} \left|\frac{ne^{i\pi/4}}{e^n - 1}\right|$ converges, then our original series converges absolutely. Since $e^{i\pi/4}$ is just a rotation about the unit circle in the complex plane, we find that $|e^{i\pi/4}| = 1$ so

$$\sum_{n=1}^{\infty} \left|\frac{ne^{i\pi/4}}{e^n - 1}\right| = \sum_{n=1}^{\infty} \left|\frac{n}{e^n - 1}\right|.$$

I must note that $\left|\frac{n}{e^n - 1}\right| = \frac{n}{e^n - 1} > 0$, so to find whether $\sum_{n=1}^{\infty} \left|\frac{n}{e^n - 1}\right|$ converges we just need to check if $\sum_{n=1}^{\infty} \frac{n}{e^n - 1}$ is convergent. Allow me to follow with an integral test, that is, consider the function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{x}{e^x - 1}$. The theorem states that $\sum_{n=1}^{\infty} \frac{n}{e^n - 1}$ is convergent if and only if the improper integral $\int_1^{\infty} \frac{x}{e^x - 1}\, dx$ is convergent. This integral may be too complicated for me to solve in a short amount of space, so I will apply a comparison test. Notice that on the domain $[1, \infty]$ we find that $2\sinh\left(\frac{x}{2}\right) \geq x$:



This implies that:

$$x \leq 2\sinh\left(\frac{x}{2}\right) = 2\left[\frac{e^{x/2} - e^{-x/2}}{2}\right]$$

$$= e^{x/2} - e^{-x/2}$$
$$= (e^x - 1)e^{-x/2}$$
$$\implies \frac{x}{e^x - 1} \leq e^{-x/2},$$

thus if $\int_1^\infty e^{-x/2}\,dx$ is convergent, then so is $\int_1^\infty \frac{x}{e^x-1}\,dx$ by the basic comparison test. Integrating, we find that

$$\begin{aligned} \int_1^\infty e^{-x/2}\,dx &= -2e^{-x/2}\Big|_1^\infty \\ &= -2\left[\lim_{a\to\infty} e^{-a/2} - e^{-1/2}\right] \\ &= -2(0 - e^{-1/2}) \\ &= 2e^{-1/2} \end{aligned}$$

and therefore $\int_1^\infty e^{-x/2}\,dx$ converges to $\frac{2}{\sqrt{e}}$. To follow the line of logic back to our original problem,

- $\int_1^\infty e^{-x/2}\,dx$ converges, thus $\int_1^\infty \frac{x}{e^x-1}\,dx$ converges since $\frac{x}{e^x-1} < e^{-x/2}$ on $[1,\infty)$ by the basic comparion test.
- Since $\int_1^\infty \frac{x}{e^x-1}\,dx$ converges, so $\sum_{n=1}^\infty \frac{n}{e^n-1}$ must converge by the integral test.
- Since $\frac{n}{e^n-1} > 0$ for $n \geq 1$, then $\frac{n}{e^n-1} = \left|\frac{n}{e^n-1}\right|$. Therefore if $\sum_{n=1}^\infty \frac{n}{e^n-1}$ converges, so does $\sum_{n=1}^\infty \left|\frac{n}{e^n-1}\right|$.
- Because $\sum_{n=1}^\infty \left|\frac{n}{e^n-1}\right|$ converges, then $\sum_{n=1}^\infty \left|\frac{ne^{i\pi/4}}{e^n-1}\right|$ converges, and therefore $\sum_{n=1}^\infty \frac{ne^{i\pi/4}}{e^n-1}$ converges (absolutely), which is what I wanted to justify.

You drastically over-complicated (a) by using this test in particular

(b) Now let me consider the series $\sum_{n=1}^\infty \frac{1}{z+n}$ for $z \in \mathbb{C}\setminus\mathbb{R}$. This series is divergent. Allow me to justify why:
I will continue by applying an integral test. Fix $z \in \mathbb{C}\setminus\mathbb{R}$, so z is a constant. Therefore $\frac{d}{dn}[z] = 0$. Define the function $f : [1,\infty) \to \mathbb{C}$ by $f(n) = \frac{1}{z+n} = \frac{1}{x+iy+n}$, where I am expressing $z = x+iy$. Now, if $\mathbb{I}\mathrm{m}(z) = y = 0$, then $\mathbb{R}\mathrm{e}(z) \notin \mathbb{Z}$. Due to this fact, the function f will always be defined, hence we can compute $\int_1^\infty f(n)\,dn$. By the integral test, if $\int_1^\infty f(n)\,dn$ is divergent, then so is $\sum_{n=1}^\infty \frac{1}{z+n}$.

$$\int_1^\infty \frac{dn}{z+n} \xrightarrow{u=z+n} \int_{z+1}^\infty \frac{du}{u}$$

The integral test applies to real series. if you want to use it here you need to split the terms in the series into their real and imaginary parts. Make sure you don't assume tools from $\mathbb{R}$ automatically generalize to $\mathbb{C}$ like this -2

$$
\begin{aligned}
&= \log|u|\Big|_{z+1}^{\infty} \\
&= \log|z+n|\Big|_{1}^{\infty} \\
&= \lim_{b\to\infty} \log|z+b| - \log|z+1| \\
&= \infty,
\end{aligned}
$$

and so the integral is divergent. Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z+n}$ diverges to $+\infty$, which is what I wanted to show.

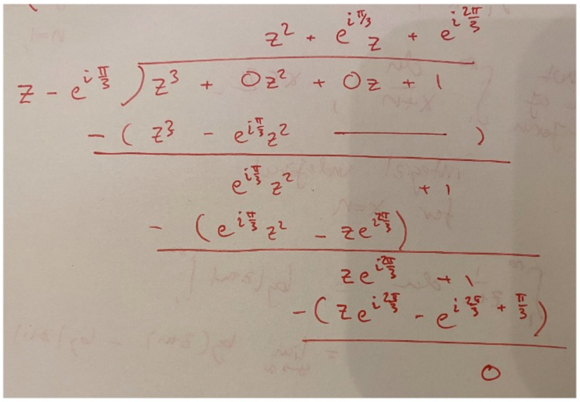
Q6

5 / 5

Submit Your Response to Question 6 Here.

6.

(a) Find $\lim_{z \to e^{i\pi/3}} \left(z - e^{i\pi/3}\right) \frac{z}{z^3+1}$. To solve this, we may utilize L'Hôpital's Rule, or we can divide the polynomial z^3+1 by $z - e^{i\pi/3}$. I will do the latter, because it is more interesting. We want to divide z^3+1 by $z - e^{i\pi/3}$, which will allow us to factor out that term to cancel it from the numerator. We have:



(I didn't know how to type this out). Which then implies that

$$z^3 + 1 = (z - e^{i\pi/3})(z^2 + ze^{i\pi/3} + e^{i2\pi/3}). \tag{6.1}$$

The limit then becomes:

$$\lim_{z \to e^{i\pi/3}} \left(z - e^{i\pi/3}\right) \frac{z}{z^3+1} = \lim_{z \to e^{i\pi/3}} \frac{(z - e^{i\pi/3})z}{(z - e^{i\pi/3})(z^2 + ze^{i\pi/3} + e^{i2\pi/3})}$$

$$
\begin{aligned}
&= \lim_{z\to e^{i\pi/3}} \frac{z}{z^2 + ze^{i\pi/3} + e^{i2\pi/3}} \\
&= \frac{e^{i\pi/3}}{e^{i2\pi/3} + e^{i2\pi/3} + e^{i2\pi/3}} \\
&= \frac{e^{i\pi/3}}{3e^{i2\pi/3}} \\
&= \frac{1}{3e^{i\pi/3}},
\end{aligned}
$$

which is the exact same result obtained if one were to proceed with L'Hôpital's rule from the original limit.

(b) Consider now the limit $\lim_{z\to 0} \frac{|z|}{z}$. This limit does not exist because $z \to 0$ from more than one point. Allow me to consider just the $\mathbb{R}\mathrm{e}(z)$ for now.
We have that as $z \to 0^+$,the numerator and denominator are both positive, hence the limit is 1. However, as $z \to 0^-$, the numerator is positive and the denominator is negative, so the limit is -1:

$$
\begin{aligned}
\lim_{z\to 0^+} \frac{|z|}{z} &= \lim_{z\to 0^+} \frac{z}{z} = 1 \\
\lim_{z\to 0^-} \frac{|z|}{z} &= \lim_{z\to 0^-} -\frac{z}{z} = -1.
\end{aligned}
$$

Now if I were to consider just the $\mathbb{I}\mathrm{m}(z)$, as $z \to 0^{+\mathbb{I}\mathrm{m}}$, then the numerator is $|i| = 1$, and the denominator is i, which yields a limit of $-i$. Likewise, as $z \to 0^{-\mathbb{I}\mathrm{m}}$, the limit approaches $+i$:

$$
\begin{aligned}
\lim_{z\to 0^{+\mathbb{I}\mathrm{m}}} \frac{|z|}{z} &= \lim_{y\to 0^+} \frac{|iy|}{iy} = \lim_{y\to 0^+} \frac{y}{iy} = \lim_{y\to 0^+} \frac{1}{i} = -i \\
\lim_{z\to 0^{-\mathbb{I}\mathrm{m}}} \frac{|z|}{z} &= \lim_{y\to 0^-} \frac{|iy|}{iy} = \lim_{y\to 0^-} -\frac{y}{iy} = \lim_{y\to 0^+} -\frac{1}{i} = i.
\end{aligned}
$$

Therefore this limit cannot exist.

12

Q7

7 / 10

Submit Your Response to Question 7 Here.

7.

Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be distinct. Define the function $f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ defined by

$$f(z) = \sum_{k=1}^{n} \frac{1}{a_k - z}. \tag{7.1}$$

(a) Define the set $\mathbb{H}_+ = \{z = x + iy \,:\, x \in \mathbb{R}, y > 0\}$ which corresponds to the upper-half plane. Let $z \in \mathbb{H}_+$, so $\mathbb{Im}(z) > 0$. Now we have that

$$\begin{aligned} f(z) &= \sum_{k=1}^{n} \frac{1}{a_k - z} = \sum_{k=1}^{n} \frac{1}{a_k - x - iy} \\ &= \sum_{k=1}^{n} \frac{1}{a_k - x - iy} \cdot \frac{a_k - x + iy}{a_k - x + iy} \\ &= \sum_{k=1}^{n} \frac{a_k - x + iy}{(a_k - x)^2 + y^2}. \end{aligned} \tag{7.2}$$

Here, (7.2) shows that

$$\mathbb{Im}(f(z)) = \mathbb{Im}\left(\sum_{k=1}^{n} \frac{a_k - x + iy}{(a_k - x)^2 + y^2}\right) = \left(\sum_{k=1}^{n} \frac{y}{(a_k - x)^2 + y^2}\right), \tag{7.3}$$

and $\displaystyle\sum_{k=1}^{n} \frac{y}{(a_k - x)^2 + y^2} > 0$ because $y > 0$ and the denominator is always positive by the definition of a norm in each term summed. Therefore $\mathbb{Im}(f(z)) > 0$.

(b) Now let me find $\lim_{y \to 0} f(x + iy)$. As calculated in (7.2), we have found that

$$f(x + iy) = \sum_{k=1}^{n} \frac{a_k - x + iy}{(a_k - x)^2 + y^2}. \tag{7.4}$$

Now, allow us to suppose that $\exists j \in (1, \ldots, n)$ such that $x = a_j$ in the summation. By the linearity of the limit, it follows that

$$\begin{aligned} \lim_{y \to 0} f(x + iy) &= \lim_{y \to 0} \sum_{k=1}^{n} \frac{a_k - x + iy}{(a_k - x)^2 + y^2} \\ &= \sum_{k=1,\, k \neq j}^{n} \frac{a_k - x}{(a_k - x)^2} + \frac{a_j - x}{(a_j - x)^2} \\ &= \sum_{k=1,\, k \neq j}^{n} \frac{1}{a_k - x} + \frac{1}{a_j - x} \end{aligned} \tag{7.5}$$

We're asking for the limits of the imaginary parts. -3

whose second term is undefined, so the limit does not necessarily exist. Now if there does not exists a j such that $a_j = x$, and the $(a_1, a_2, \ldots, a_n, x)$ are all distinct, we find that

$$\lim_{y \to 0} f(x + iy) = \lim_{y \to 0} \sum_{k=1}^{n} \frac{a_k - x + iy}{(a_k - x)^2 + y^2} = \sum_{k=1}^{n} \frac{1}{a_k - x}. \tag{7.6}$$

(c) Now consider $\lim_{y \to 0} \mathbb{Re}(f(x + iy))$. (7.4) tells us that

$$\lim_{y\to 0} \mathbb{R}\mathrm{e}(f(x+iy)) = \lim_{y\to 0} \mathbb{R}\mathrm{e}\left(\sum_{k=1} \frac{a_k - x + iy}{(a_k-x)^2+y^2}\right) = \lim_{y\to 0} \sum_{k=1} \frac{a_k - x}{(a_k-x)^2+y^2}. \tag{7.7}$$

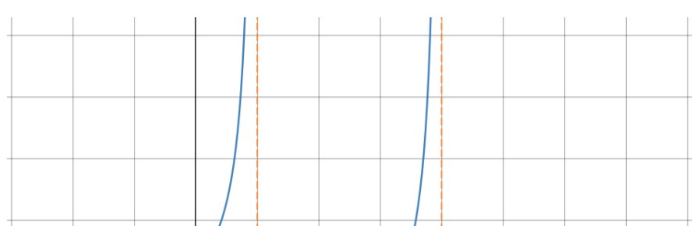
In the case where $\exists j \in (1, \ldots, n)$ such that $a_j = x$, we can first separate the terms then take the limit, since the function manipulation is first:

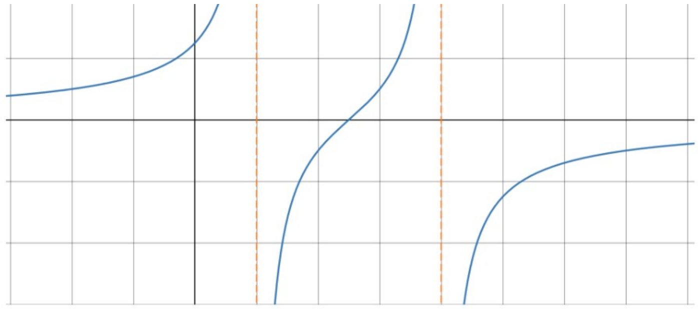
$$\begin{aligned}
\lim_{y\to 0} \mathbb{R}\mathrm{e}(f(x+iy)) &= \lim_{y\to 0} \sum_{k=1,\, k\neq j}^{n} \frac{a_k - x}{(a_k-x)^2+y^2} + \frac{a_j - x}{(a_j-x)^2+y^2} \\
&= \lim_{y\to 0} \sum_{k=1,\, k\neq j}^{n} \frac{a_k - x}{(a_k-x)^2+y^2} + \frac{0}{0^2+y^2} \\
&= \sum_{k=1,\, k\neq j}^{n} \frac{1}{a_k - x},
\end{aligned} \tag{7.8}$$

which is always real and defined for terms $a_k \neq x$. Lastly, if there exists no j such that $a_j = x$, the limit of the real part of the function is just defined as

$$\lim_{y\to 0} \mathbb{R}\mathrm{e}(f(x+iy)) = \lim_{y\to 0} \sum_{k=1}^{n} \frac{a_k - x}{(a_k-x)^2+y^2} = \sum_{k=1}^{n} \frac{1}{a_k - x}$$

which is always defined. Therefore the limit $g(x) = \lim_{y\to 0} \mathbb{R}\mathrm{e}(f(x+iy))$ always exists for every real number $x \in \mathbb{R}$. The function $g(x)$ itself is not continuous, because the function constitutes a finite sum of reciprocal terms, which establish vertical asymptotes of discontinuities whenever $x = a_j$, since x is continuous while a_j is discrete. If we were to take a limit of x say, towards a_j, then the limit would not exist. However, as $y \to 0$, the limit exists for every $x \in \mathbb{R}$, but the function $g(x)$ itself is not continuous.
For instance, just consider $g(x) = \dfrac{1}{1-x} + \dfrac{1}{4-x}$:





which is discontinuous at $x = 1$ and $x = 4$.