MAT244 PS6

Q1) a)

$$x' = 2x - y + e^t \qquad y' = 3x - 2y - e^t \longrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t.$$

Solve the homogeneous eq$\underline{n}$:

$$\det\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) + 3$$

$$= -4 + 3 + \lambda^2$$

$$0 = -1 + \lambda^2 \implies \lambda = \pm 1.$$

$$\ker\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \ker\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The homogeneous equation is then

$$X_h(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Solve the particular solution, guess $v = ate^t + be^t$.

$$v' = ae^t(t+1) + be^t = Aate^t + Abe^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

Collecting like terms, we find

$$Aa = a \qquad \text{and}$$

$$Ab = a + b + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since a is an eigenvector of A with $\lambda = 1$, a must be of the form $a = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$ with $\alpha \in \mathbb{R}$.

$$\begin{pmatrix} b_1 - b_2 \\ 3b_1 - 3b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 1 \\ \alpha + 1 \end{pmatrix} \implies \text{let } \alpha = 2. \text{ Then } a = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

satisfies the matrix equation. Our full general solution is then

$$\boxed{X(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t}$$

b) $x' = 2x - 5y + 1$

$y' = x - 2y - \frac{1}{t}$ $\longrightarrow$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1/t \end{bmatrix}$

Find the homogeneous solution:

$$\det \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) + 5$$

$$0 = -4 + \lambda^2 + 5 = 1 + \lambda^2. \implies \lambda = \pm i.$$

$$\operatorname{Ker}\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \to \operatorname{Ker}\begin{pmatrix} 5 & 5(-2-i) \\ 1 & -2i \end{pmatrix} \to \operatorname{Ker}\begin{pmatrix} 1 & -2-i \\ 0 & 0 \end{pmatrix} \to v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}, \lambda = i.$$

Then $v_{1,2} = \begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$ with $\lambda_{1,2} = \pm i$. $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The homogeneous solution is then

$$X_h(t) = C_1\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t\right] + C_2\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t\right].$$

The fundamental matrix is given by

$$\Psi(t) = \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}.$$

We have, by variation of parameters,

$$\Psi(t) \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 1 \\ -1/t \end{bmatrix}.$$

After row reducing (a little), we have that

$$\begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 1 + 2/t \\ -1/t \end{bmatrix}.$$

We find that $u_2' = \dfrac{1 + 2/t + u_1' \sin t}{\cos t}$.

Then $u_1' = \dfrac{-1/t - \tan t - \frac{2\tan t}{t}}{\sin t \tan t + \cos t}$, which implies

$$u_2' = \frac{1 + \frac{2}{t} + \sin t \left(\dfrac{-1/t - \tan t - \frac{2\tan t}{t}}{\sin t \tan t + \cos t} \right)}{\cos t}$$

Now we must integrate:

$$u_1 = -\int_0^t ds \frac{1/s + \tan s + \frac{2\tan s}{s}}{\sin s \tan s + \cos s}$$

$$= -\int_0^t ds \frac{1 + s\frac{\sin s}{\cos s} + 2\frac{\sin s}{\cos s}}{s\left(\sin s \frac{\sin s}{\cos s} + \cos s\right)}$$

$$= -\int_0^t ds \frac{\frac{\cos s + s\cdot\sin s + 2\sin s}{\cancel{\cos s}}}{s\left(\frac{\cancel{\sin^2 s + \cos^2 s}^{\,1}}{\cancel{\cos s}}\right)}$$

$$= -\int_0^t ds \frac{\cos s + 2\sin s}{s} + \sin s$$

$$= -\left[Ci(t) + 2Si(t) - \cos t\right] = \cos t - Ci(t) - 2Si(t).$$

$$u_2 = \int_0^t ds \frac{1 + \frac{2}{s} - \sin s\left(\frac{\cos s + 2\sin s + \sin s\cdot s}{s}\right)}{\cos s}$$

$$= \int_0^t ds \frac{s + 2 - \sin s\cos s - 2\sin^2 s - s\sin^2 s}{s\cos s}$$

$$= \int_0^t ds \frac{s + 2 - \sin s\cos s - 2(1-\cos^2 s) - s(1-\cos^2 s)}{s\cos s}$$

$$= \int_0^t ds \frac{\cancel{s + 2 - 2 - s}^{\,0} - \sin s\cos s + 2\cos^2 s + s\cos^2 s}{s\cos s}$$

$$= \int_0^t ds \cos s + \frac{2\cos s}{s} - \frac{\sin s}{s}$$

$$= \sin t + 2Ci(t) - Si(t).$$

Therefore $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos t - Ci(t) - 2Si(t) \\ \sin t + 2Ci(t) - Si(t) \end{bmatrix}$.

Then $X = \Psi(t)c + \Psi(t)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, which yields

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}\begin{bmatrix} \cos t - Ci(t) - 2Si(t) \\ \sin t + 2Ci(t) - Si(t) \end{bmatrix}}$$

c) You certainly could use the method of undetermined coefficients, however guessing a particular solution may be too difficult to guess.

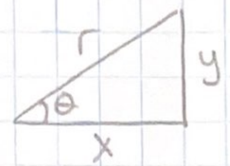
You would need to guess a solution

$$v = a\cos t + b\sin t + c\,Si(t) + d\,Ci(t)$$

then determine the vectors a, b, c, and d.

MAT244 PS6

Q2) a) We can visualize the polar coordinate transformation by a right triangle:



· By pythagoras' theorem,

$$\boxed{r^2 = x^2 + y^2.}$$

· Similarly, since $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$,

then $\boxed{\theta = \arctan\left(\frac{y}{x}\right).}$

Note: The problem with this interpretation is that θ is not defined for when $x=0$, since $\arctan\left(\frac{y}{x}\right)$ is undefined.

b) $\frac{d}{dt}[r^2] = \frac{d}{dt}[x^2+y^2]$

$2rr' = 2xx' + 2yy'$

$\Longrightarrow \quad rr' = xx' + yy'.$

$\frac{d}{dt}[\theta] = \frac{d}{dt}\left[\arctan\left(\frac{y}{x}\right)\right]$

$= \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{xy' - yx'}{x^2}$

$\theta' = \frac{xy'-yx'}{x^2+y^2} = \frac{xy'-yx'}{r^2}$

$\Longrightarrow \quad \boxed{\theta' r^2 = xy' - yx'}$

$\frac{d}{dt}[x] = \frac{d}{dt}[r\cos\theta] \Longrightarrow \boxed{x' = r'\cos\theta - r\theta'\sin\theta}$ by chain rule

$\frac{d}{dt}[y] = \frac{d}{dt}[r\sin\theta] \longrightarrow \boxed{y' = r'\sin\theta + r\theta'\cos\theta.}$

c) $x' = (x^2+y^2)\,y \qquad \longrightarrow \qquad r'\cos\theta - r\theta'\sin\theta = r^3\sin\theta$

$y' = -(x^2+y^2)\,x \qquad\qquad\qquad r'\sin\theta + r\theta'\cos\theta = -r^3\cos\theta$

Which is equivalent to

$$\cancel{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} r' \\ r\theta' \end{bmatrix} = \cancel{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} 0 \\ -r^3 \end{bmatrix}$$

So

$$r' = 0$$
$$r\theta' = -r^3 \quad \rightarrow \quad \theta' = -r^2.$$

Since $r' = 0$, then $r = c_1$ ($c_1 \in \mathbb{R}$ is a constant).

Then $\theta' = -c_1^2 \longrightarrow \theta = -c_1^2 t + c_2$ (and $c_2 \in \mathbb{R}$ too).

Subbing r and θ back into x and y, we find

$$\boxed{\begin{aligned} x &= c_1 \cos(c_2 - c_1^2 t) \\ y &= c_1 \sin(c_2 - c_1^2 t) \end{aligned}}$$

MAT244 PS6

Q3) A:
$$x' = y + x(x^2+y^2)$$
$$y' = -x + y(x^2+y^2)$$

B:
$$x' = y - x(x^2+y^2)$$
$$y' = -x - y(x^2+y^2)$$

a) If $(0,0)$ is a critical point of A and B, then $x'=0$, $y'=0$ for both A and B.

A: $x' = (0) + (0)(0^2+0^2) = 0$

$y' = -(0) + (0)(0^2+0^2) = 0$

B: $x' = (0) - (0)(0^2+0^2) = 0$

$y' = -(0) - (0)(0^2+0^2) = 0$

And thus $(0,0)$ is a critical point. To prove that it is an isolated point, it suffices to prove that $(0,0)$ is a centre of both systems. We can examine the local linearity of each system.

Computing the Jacobian matrix for both systems yield

$$DA(x,y) = \begin{pmatrix} 3x^2+y^2 & 1+2xy \\ -1+2xy & 3y^2+x^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$DB(x,y) = \begin{pmatrix} -3x^2-y^2 & 1-2xy \\ -1-2xy & -3y^2-x^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues of $DA(0,0) = DB(0,0)$ are $\lambda = \pm i$, which implies that $\mathrm{Re}(\lambda_1) = \mathrm{R}(\lambda_2) = 0$, thus the point $(0,0)$ is a centre and furthermore an isolated critical point, for both systems.

b) We can prove this two different ways.

From (a), we know that the Jacobian matrix at $(0,0)$ is given by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and therefore the system is locally linear.

The second method is to show that $\lim_{(x,y)\to(0,0)} \frac{\|g(x,y)\|}{\|x,y\|} = 0$.

In both systems A and B, we can rewrite them as matrices.

$$A: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} y(x^2+y^2) \\ x(x^2+y^2) \end{bmatrix}$$

$$B: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -y(x^2+y^2) \\ -x(x^2+y^2) \end{bmatrix}.$$

We require that $\lim_{(x,y)\to(0,0)} \frac{\|g(x,y)\|}{\|(x,y)\|} = 0$. Since $g_A(x,y) = -g_B(x,y)$, and the matrix $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the same in both A and B, it just suffices to show that the limit above is 0. We have:

$$\lim_{(x,y)\to(0,0)} \frac{\|g(x,y)\|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2(x^2+y^2)^2 + y^2(x^2+y^2)^2}}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}}$$

$$= \lim_{(x,y)\to(0,0)} x^2+y^2 = 0,$$

and thus the system is locally linear around $(0,0)$ given the matrix $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, for both systems A and B.

c) We have, from 2b,

A: $x' = y + x(x^2+y^2)$

$y' = -x + y(x^2+y^2)$

B: $x' = y - x(x^2+y^2)$

$y' = -x - y(x^2+y^2)$

$\longrightarrow$ A: $r'\cos\theta - r\theta'\sin\theta = r\sin\theta + r^3\cos\theta$

$r'\sin\theta + r\theta'\cos\theta = -r\cos\theta + r^3\sin\theta$

$\longrightarrow$ B: $r'\cos\theta - r\theta'\sin\theta = r\sin\theta - r^3\cos\theta$

$r'\sin\theta + r\theta'\cos\theta = -r\cos\theta - r^3\sin\theta$

Which implies

A: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r' \\ r\theta' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r^3 \\ -r \end{bmatrix}$

$\Longrightarrow \begin{matrix} r' = r^3 \\ \theta' = -1 \end{matrix} \Longrightarrow \boxed{\begin{matrix} r = \frac{\pm 1}{\sqrt{2(c_2 - t)}} \\ \theta = c_1 - t \end{matrix}}$ $\leftarrow$ (we can obtain x and y from this)

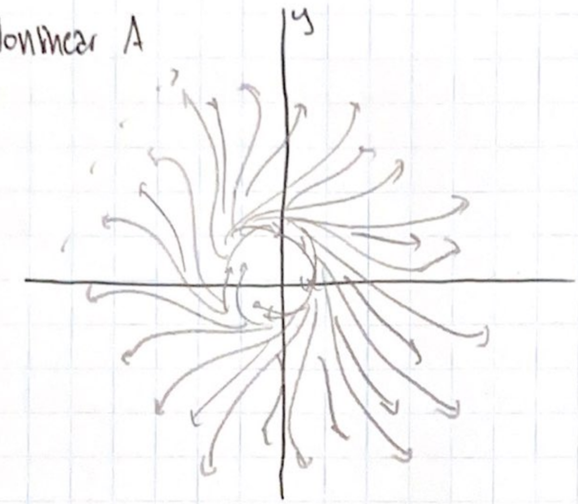
B: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r' \\ r\theta' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -r^3 \\ -r \end{bmatrix}$

$\Longrightarrow \begin{matrix} r' = -r^3 \\ \theta' = -1 \end{matrix} \Longrightarrow \boxed{\begin{matrix} r = \frac{\pm 1}{\sqrt{2(c_2 + t)}} \\ \theta = c_1 - t \end{matrix}}$

d) Notice for A, $r' = r^3$, which implies that the radius r will be rapidly increasing. Thus (0,0) for A is unstable, diverging to ∞.

For B, since $r' = -r^3$, the radius r will be rapidly converging to some circle for some t, hence (0,0) being a stable centre.

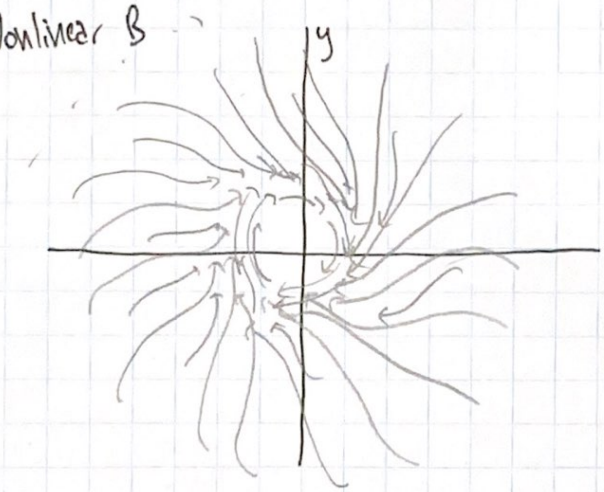
e) Nonlinear A



- Centre at (0,0)
- Diverging solutions as $t \to \infty$
- Clockwise rotation

(0,0) → diverging centre

Nonlinear B



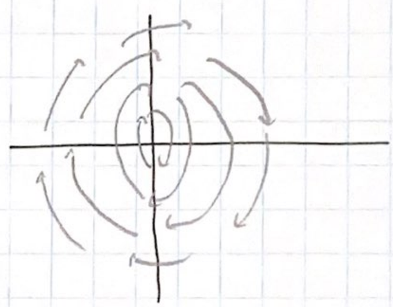
- Converging solutions to stable centre, "circle" at (0,0).

(0,0) → converging centre

Linear A

(0,0) → Stable Centre

Linear B



$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 0.$

f) Since the behaviour of each linear system is equivalent, and the stability of both the nonlinear systems A and B are different (but their linear systems are the same), and thus you cannot deduce nonlinear behaviour from linear behaviour.

MAT244 PS6

Q4) $x' = y + 2x(1 - x^2 - y^2)$

$y' = -x$

a) We have

x': $\cos(t) = \cos(t) + 2\sin(t)\left[1 - (\cancel{\sin^2(t) + \cos^2(t)})\right]$

$0 = 2\sin(t)(1-1) = 0$

y': $-\sin(t) = -(\sin(t))$

and thus $\phi_0(t) = (\sin t, \cos t)$ is a solution to the system.

We find that as $t \to \infty$, $\phi_0(t)$ oscillates around $(0,0)$.

Likewise, $x' = (0) + 2(0)(1 - 0^2 - 0^2) = 0$

$y' = -(0) = 0$,

and therefore $(0,0)$ is a critical point of $\phi_0(t)$.

For uniqueness, we find that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

and thus $(0,0)$ is the only critical point to the system, and therefore it is unique.

b) We have that $\psi(t) = -(x(t), y(t))$.

If $\phi(t) = (x(t), y(t))$, we have that $y'(t) = -x(t)$ must satisfy the system. Then

$-y''(t) = y + 2x(1 - x^2 - y^2)$.

Now, suppose for a contradiction that $\psi(t) = (-x(t), -y(t))$ is not a solution to the system

Since $-y'(t) = -(-x(t)) \Rightarrow y'(t) = -x'(t)$,

it must be that $x'(t) \neq y + 2x(1-x^2-y^2)$.

However, since $x'(t) = -y'(t)$, then

$$-y''(t) \neq y + 2x(1-x^2-y^2),$$

and thus it must be that $\Psi(t)$ is a solution to the system since these are the same equations reached when plugging $\phi(t)$ into the system, and so we have a contradiction.

Therefore $\Psi(t) = -\phi(t)$ is also a solution to the system.

We have that the solutions on one side of the phase plane are just negative solutions on another side, regardless of orientation.

Since $(0,0)$ is the only critical point of the system, it must be that the solutions are symmetric with respect to the y-axis, centred at $(0,0)$.

More generally, solutions on the left are negative of the solutions on the right, hence symmetry.

This equivalently occurs with solutions above and below the x-axis, and therefore all solutions are symmetrical around $(0,0)$

(ie, if you flip y and x, you will obtain a 'mirrored' version of the solution).

c) We have that

$$r' \cos\theta - r\theta' \sin\theta = r\sin\theta + 2r\cos\theta\,(1-r^2)$$

$$r' \sin\theta + r\theta' \cos\theta = -r\cos\theta.$$

This implies that

$$\theta' = -1 - \frac{r'}{r}\tan\theta \quad \text{and}$$

$$r' = -r'\tan^2\theta + 2r(1-r^2)$$

Thus

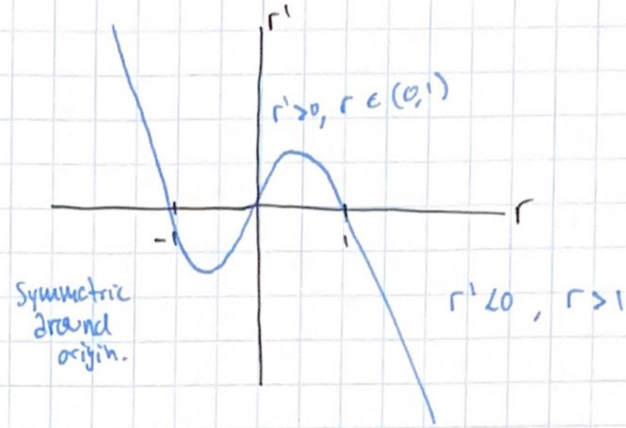
$$\boxed{r' = 2r(1-r^2)\cos^2\theta \text{ and } \theta' = -1-2\sin\theta\cos\theta.}$$

d) We see that $-2 \leq -1-2\sin\theta\cos\theta \leq 0$, which implies that $\theta' \leq 0$. Because of this, we can conclude that the rate of change of θ with respect to time is negative, and thus we will have counterclockwise motion.

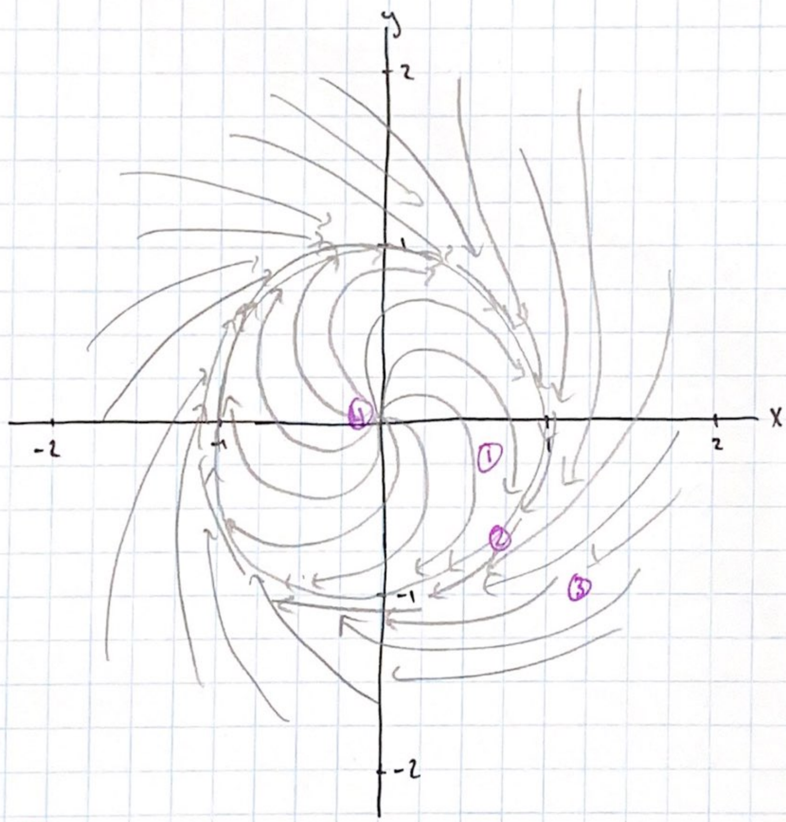
For r', we see that r is a constant, or $r'=0$, if and only if $r = \pm 1$.

Furthermore, Studying the r' vs r graph, we find that $r'>0$ if $0<r<1$ (hence outwards convergence to the unit circle from the origin) and that $r'<0$ if $r>1$ (hence inwards convergence to the unit circle from anywhere $\|r\|>1$). This equivalently happens if $r<-1$ and $-1<r<0$ ($r'>0$, $r'<0$, respectively, both converging towards unit circle).

Graph for d):



e)



probably wrong but you get the idea.

Notes:

① Solutions with $0<r<1$ converge outwards from $(0,0)$ towards unit circle

② Counterclockwise rotation around unit circle

③ Solutions with $r>1$ converges towards unit circle from where $r>0$.

④ $(0,0)$ is a unique centre + only critical point of the system.