PS4

My score

71.9% (11.5/16)

Q1

3.5 / 4

1. Nonlinearity

Background: In Lecture 18, I show the nonlinear effect using a simple example: the damped/driven nonlinear pendulum, described as equation (1) in Lecture 18 note. The nonlinear term there is sin(θ). I show in page 4 there that sin(θ) can be expanded by Taylor expansion into θ - $\theta^3/6$ + H.O.T (high order terms). The first term θ is the linear term while all other are the nonlinear terms. If we assume sin(θ) is dominated by the linear term θ, we use sin(θ) ≈ θ and solve equation (1), which is the simple harmonics equation that you are familiar with, and this would cause a solution of θ that is linearly proportional to cos(w_d t), denoted as θ ~ cos(w_d t). Now consider the nonlinear effect: Putting θ ~ cos(w_d t) into the first nonlinear term -$\theta^3/6$ in Taylor expansion above, page 4 shows that this would cause a term of new frequency $3w_d$.

Question: Consider the next nonlinear term (the one higher than -$\theta^3/6$) in H.O.T above, and still use θ ~ cos(w_d t). Shows what new frequency (or frequencies) will this term cause. You need to show the math derivation rigorously. Hint: you may use high school math...

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The Taylor expansion of sin is

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

so we are wanting to expand the nonlinear term $\frac{x^5}{5!}$ with $x = \cos(\omega_d t)$ to find which harmonics the 3rd higher order term will create.

We have that

$$x = \cos(\omega_d t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \implies \frac{x^5}{5!} = \frac{\cos^5(\omega_d t)}{5!} = \frac{(e^{i\omega_d t} + e^{-i\omega_d t})^5}{2^5 \cdot 5!}.$$

By Pascal (high school math), our expansion becomes

$$\begin{aligned}
\frac{(e^{i\omega_d t} + e^{-i\omega_d t})^5}{2 \cdot 5!} &= \frac{e^{5i\omega_d t} + 5e^{4i\omega_d t - i\omega_d t} + 10e^{3i\omega_d t - 2i\omega_d t} + 10e^{2i\omega_d t - 3i\omega_d t} + 5e^{i\omega_d t - 4i\omega_d t} + e^{-5i\omega_d t}}{2^5 \cdot 5!} \\
&= \frac{e^{5i\omega_d t} + 5e^{3i\omega_d t} + 10e^{i\omega_d t} + 10e^{-i\omega_d t} + 5e^{-4i\omega_d t} + e^{-5i\omega_d t}}{2^5 \cdot 5!} \\
&= \frac{10\cos(\omega_d t) + 5\cos(3\omega_d t) + \cos(5\omega_d t)}{4 \cdot 5!}
\end{aligned}$$

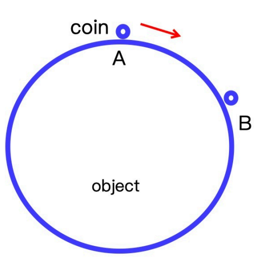
Therefore the 3rd order term in the Taylor expansion of sin x with $x = \cos(\omega_d t)$ yields 3 different harmonics: $\boxed{\omega_d, 3\omega_d, \text{ and } 5\omega_d.}$

Your coefficients are off by a factor of 4.

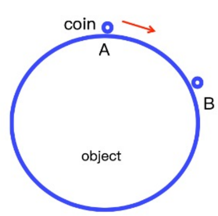
1

Q2

4 / 4



2. Energy
We put a very small coin right at the top of an object. This object is a solid sphere. We assume such a coin is frictionless (no friction with the object) and this object is fixed and has a radius of R. We push this coin only very little bit and then it falls by gravity. At what height below the starting point of this coin will the coin be off the object (without touch with the object)? Say the starting point is A and the detached point is B, the question here asks the height difference between A and B.
Hint 1: Energy conservation.
Hint 2: what is the normal force from the object to the coin?



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To solve this problem, we consider the energy conservation of the coin and the normal force the sphere exerts on the coin.

We are able to think of the position of the coin in term of it's polar angle with respect to the vertical. For energy, we have that

$$E_0 = E_f \implies mg(\Delta h) = \frac{1}{2}mv^2, \tag{1}$$

where $\Delta h = R - R\cos\theta = R(1-\cos\theta)$ is the change in height of the coin.

For the normal force, we know the radial components are given by $\frac{mv^2}{R}$ and $mg\cos\theta$, and thus

$$\begin{aligned} \frac{mv^2}{R} &= mg\cos\theta - N \\ \implies mv^2 &= R(mg\cos\theta - N). \end{aligned} \tag{2}$$

Subbing (2) into (1), we have that

$$\begin{aligned} mgR(1-\cos\theta) &= \frac{1}{2}mgR\cos\theta - RN \\ \implies N &= \frac{3}{2}mg\cos\theta - mg \\ \implies N &= mg(\frac{3}{2}\cos\theta - 1) \\ \implies N &= mg(3\cos\theta - 2). \end{aligned}$$

Now, for the coin to remain on the surface of the sphere, it must be that $N > 0$. Thus, the coin falls off of the surface when $N \le 0$. From direct calculation we have that $\cos\theta \le \frac{3}{2}$ for when the coin falls off the surface.

Thus $\Delta h = R(1-\cos\theta) < R(1 - \frac{2}{-} = \frac{R}{-}$.

Therefore $\boxed{h = \frac{R}{3}}$ is when the coin falls off of the sphere.

2

Q3

2 / 4

3. Energy.

On a horizontal surface, we have a massless spring that is connected to an object with mass m. The spring constant is k. The friction of this object with the surface is the product of a constant b with the velocity of this object. Consider 1-dimensional motion of this object in x direction: when the object is located at x_2, the spring has no force on the object. Say initially the object's location is x_1, with a velocity of v_1. There is no other external forces like driving force or so. Assume no static friction.

Question (a): After very long time, how much heat will be generated due to the dissipation caused by the friction of this object? Consider from the energy perspective.

Question (b): Prove your answer in (a) above, using the fact that the work done by a force is equal to the integral of this force over distance. Hint: Consider the equation of motion; Also, (dv/dt)•dx =dv•v = vdv

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(a)

Thermal energy is defined as $\Delta E_{th} = W = \int_{x_0}^{x} dx' F_f(x')$, where F_f is the force of friction acting on the object.
This is quite easy to explain. All of the initial energy, since it is conserved, must all contribute to the thermal energy once the object has stopped moving. The initial and final positions of the masses are well known, and thus we can calculate the thermal energy:

$$\begin{aligned}\Delta E_{th} = W &= \int_{x_1}^{x_2} dx\, F(x) = -b \int_{x_1}^{x_2} dt\, \dot{x}(t) \\ &= -b \int_{x_1}^{x_2} dt\, \frac{dx}{dt} = -b \int_{x_1}^{x_2} dx = b(x_1 - x_2) \\ &= b\Delta x.\end{aligned}$$

Therefore the amount of thermal energy generated due to dissipation caused by friction of the mass on the spring is $\boxed{\Delta E_{th} = b(x_2 - x_1) = b\Delta x.}$

Note that we cannot actually calculate temperature because we do not have any information about the specific heat capacity of the mass against the surface.

3

(b)
This is the rigorous derivation for **(a)**. The equation of motion for the mass on the spring is given by

$$m\ddot{x} = -b\dot{x} - k(x - x_2) \implies m\ddot{x} + b\dot{x} + k(x - x_2) = 0.$$

As always, let $\gamma = \dfrac{b}{2m}$ and $\omega_0^2 = \dfrac{k}{m}$. I will assume underdamping, and define $\Omega^2 = \omega_0^2 - \gamma^2$. Solving, we find our general solution to the equation of motion is

$$x(t) = e^{-\gamma t}\left[c_1\cos(\Omega t) + c_2\sin(\Omega t)\right].$$

We can now solve the IVP, with $x(t=0) = x_1$ and $\dot{x}(t=0) = v_0$:

$$x_1 = c_1 + x_2 \implies c_1 = x_1 - x_2$$
$$v_0 = -\gamma(x_1 - x_2) + c_2\Omega \implies c_2 = \frac{v_0 + \gamma(x_1 - x_2)}{\Omega}$$

This yields the complete solution

$$x(t) = e^{-\gamma t}\left[(x_1 - x_2)\cos(\Omega t) + \frac{v_0 + \gamma(x_1 - x_2)}{\Omega}\sin(\Omega t)\right] + x_1.$$

Now to find the thermal energy as $t \to \infty$, we evaluate the improper integral
$\Delta E_{th} = W = \displaystyle\int_0^\infty dt\, F(\dot{x}(t)) = -b\int_0^\infty dt\, \dot{x}(t)$. We have

$$\begin{aligned}
-b\int_0^\infty dt\, \dot{x}(t) &= -b\int_0^\infty dt\, \left[-\gamma e^{-\gamma t}\left(c_1\cos(\Omega t) + c_2\sin(\Omega t)\right) + e^{-\gamma t}\left(c_2\Omega\cos(\Omega t) - c_1\Omega\sin(\Omega t)\right)\right] \\
&= -b\int_0^\infty dt\, \left[e^{-\gamma t}\cos(\Omega t)(c_2\Omega - \gamma c_1) + e^{-\gamma t}\sin(\Omega t)(-c_1\Omega - \gamma c_2)\right] \\
&= -b(c_2\Omega - \gamma c_1)\int_0^\infty dt\, e^{-\gamma t}\cos(\Omega t) + b(c_1\Omega + \gamma c_2)\int_0^\infty dt\, e^{-\gamma t}\sin(\Omega t). \qquad (3)
\end{aligned}$$

Briefly aside, we know the following integrals:

$$\int_0^\infty dt\, e^{-\gamma t}\cos(\Omega t) = \left[\frac{\Omega e^{-\gamma t}\sin(\Omega t)}{\gamma^2 + \Omega^2} - \frac{\gamma e^{-\gamma t}\cos(\Omega t)}{\gamma^2 + \Omega^2}\right]_0^\infty = \frac{\gamma}{\omega_0^2} \qquad (4)$$
$$\int_0^\infty dt\, e^{-\gamma t}\sin(\Omega t) = \left[\frac{-\Omega e^{-\gamma t}\cos(\Omega t)}{\gamma^2 + \Omega^2} - \frac{\gamma e^{-\gamma t}\sin(\Omega t)}{\gamma^2 + \Omega^2}\right]_0^\infty = \frac{\Omega}{\omega_0^2} \qquad (5)$$

Subbing (4) and (5) into (3) yields

$$\begin{aligned}
-b\int_0^\infty dt\, \dot{x}(t) &= -b(c_2\Omega - \gamma c_1)\left[\frac{\gamma}{\omega_0^2}\right] + b(c_1\Omega + \gamma c_2)\left[\frac{\Omega}{\omega_0^2}\right] \\
&= -b\left(\left(\frac{v_0 + \gamma(x_1 - x_2)}{\Omega}\right)\Omega - \gamma(x_1 - x_2)\right)\left[\frac{\gamma}{\omega_0^2}\right] + b\left((x_1 - x_2)\Omega + \gamma\left(\frac{v_0 + \gamma(x_1 - x_2)}{\Omega}\right)\right)\left[\frac{\Omega}{\omega_0^2}\right] \\
&= -bv_0\frac{\gamma}{\omega_0^2} + \frac{b}{\omega_0^2}\left((x_1 - x_2)\Omega^2 + \gamma v_0 + \gamma^2(x_1 - x_2)\right) \\
&= -bv_0\frac{\gamma}{\omega_0^2} + \frac{b}{\omega_0^2}((x_1 - x_2)(\omega_0^2 - \gamma^2) + \gamma v_0 + \gamma^2(x_1 - x_2))
\end{aligned}$$

$$
\begin{aligned}
&= -bv_0\frac{\gamma}{\omega_0^2} + \frac{b}{\omega_0^2}((x_1 - x_2)\omega_0^2 + \gamma v_0) \\
&= b(x_1 - x_2) + \frac{bv_0\gamma}{\omega_0^2} - \frac{bv_0\gamma}{\omega_0^2} \\
&= b(x_1 - x_2) \\
&\implies \boxed{\Delta E_{th} = b\Delta x}
\end{aligned}
$$

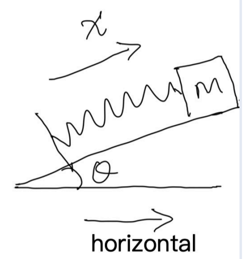
which is equivalent to the answer derived in **3a**.

Q4

2 / 4

4. Energy

We have the exact same situation as question 3 above; the only difference is that we are now on an inclined plane with an angle of θ. See the figure here. Answer question 3(a) again in this situation.



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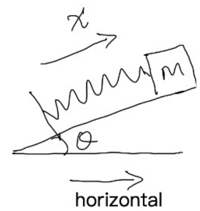

In order to find ΔE, we must find $\Delta x = x_f - x_1$, assuming x_f is the final resting position of the mass and x_2 is the equilibrium position from **Question 3** where the spring will not exert any force on the mass.

Analyzing the forces on the mass, we find $mg\sin\theta = -k(x_f - x_2) \implies (x_f - x_2) = -\dfrac{mg\sin\theta}{k}$.
By energy conservation,

$$\begin{aligned}
E_0 &= \frac{1}{2}mv_0^2 + mgx_1\sin\theta + \frac{1}{2}k(x_1 - x_2)^2 \\
E_f &= mgx_f\sin\theta + \frac{1}{2}k\left(x_f - x_2\right)^2 \\
&= mgx_f\sin\theta + \frac{1}{2}k\left(\frac{mg\sin\theta}{k}\right)^2.
\end{aligned}$$

$E_i = E_f + \text{heat}$

No

Equating, since $E_0 = E_f$, we have

$$\frac{1}{2}mv_0^2 + mgx_0\sin\theta + \frac{1}{2}k(x_1 - x_2)^2 - \frac{m^2g^2\sin^2\theta}{k^2} = mgx_f\sin\theta.$$

$\text{heat} = E_i - E_f$

Solving for x_f, we have

$$x_f = \frac{v_0^2}{2g\sin\theta} + \frac{k(x_1 - x_2)^2}{2mg\sin\theta} - \frac{mg\sin\theta}{2k} + x_1.$$

Now, evaluating ΔE_{th}, we have

$$\begin{aligned}
\Delta E_{th} = W &= \int_{x_1}^{x_f} dx\, F(x) = -b\int_{x_1}^{x_f} dt\, \dot{x}(t) \\
&= -b\int_{x_1}^{x_f} dt\, \frac{dx}{dt} = -b\int_{x_1}^{x_f} dx\, = b(x_1 - x_f) \\
&= -b\left(\frac{v_0^2}{2g\sin\theta} + \frac{k(x_1 - x_2)^2}{2mg\sin\theta} - \frac{mg\sin\theta}{2k} + x_1 - x_1\right)
\end{aligned}$$

And thus

$$\boxed{\Delta E_{th} = b\left(\frac{mg\sin\theta}{2k} - \frac{v_0^2}{2g\sin\theta} - \frac{k(x_1 - x_2)^2}{2mg\sin\theta}\right)}$$