PHY356 PS4 — Due November 15 11pm

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1. Consider a harmonic oscillator of mass m and angular frequency ω. At time $t = 0$, the state of this oscillator is given by:

$$|\psi(0)\rangle = \sum_n c_n |\varphi_n\rangle$$

where the states $|\varphi_n\rangle$ are stationary states with energies $(n + 1/2)\hbar\omega$.

a. What is the probability $\mathcal{P}$ that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$, will yield a result greater than $2\hbar\omega$? When $\mathcal{P} = 0$, what are the non-zero coefficients c_n?

b. From now on, assume that only c_0 and c_1 are different from zero. Write the normalization condition for $|\psi(0)\rangle$ and the mean value $\langle H \rangle$ of the energy in terms of c_0 and c_1. With the additional requirement $\langle H \rangle = \hbar\omega$, calculate $|c_0|^2$ and $|c_1|^2$.

c. As the normalized state vector $|\psi(0)\rangle$ is defined only to within a global phase factor, we fix this factor by choosing c_0 real and positive. We set: $c_1 = |c_1|\mathrm{e}^{i\theta_1}$. We assume that $\langle H \rangle = \hbar\omega$ and that:

$$\langle X \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

Calculate θ_1.

d. With $|\psi(0)\rangle$ so determined, write $|\psi(t)\rangle$ for $t > 0$ and calculate the value of θ_1 at t. Deduce the mean value $\langle X \rangle(t)$ of the position at t.

(a) To begin, I will invoke one of the consequences of Schrödinger's equation which governs the time evolution of a state, that is

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\varphi_n\rangle = \sum_n c_n e^{-i\omega\left(n+\frac{1}{2}\right)t} |\varphi_n\rangle .$$

Notice that regardless of time evolution, we must have the normalization condition that the total probability outcome must be equal to one: $1 = \sum_n |c_n|^2$. If we wish to determine the probability that an energy in a one-dimensional quantum harmonic oscillator greater than $2\hbar\omega$ is measured, I will first note that the energies are quantized in the oscillator by $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$, which implies that $n + \frac{1}{2} > 2 \implies n > \frac{3}{2}$ for this to be true. Since n must be integral, an appropriate selection would be $n \geq 2$.

Now, for measuring a state with energy greather than $2\hbar\omega$, the probability is then given by summing every square modulus coefficient beginning from $n = 2$ to infinity, since $n \geq 2$ as determined. Thus

$\mathbb{P}_{>2\hbar\omega} = \sum_{n=2}^{\infty} |c_n|^2$. However, we may also determine a simpler closed-form expression for this probability. Note that

$$\begin{aligned}
1 &= \sum_{n=0}^{\infty} |c_n|^2 \\
&= |c_0|^2 + |c_1|^2 + \sum_{n=2}^{\infty} |c_n|^2 \\
&= |c_0|^2 + |c_1|^2 + \mathbb{P}_{>2\hbar\omega} \\
\implies \mathbb{P}_{>2\hbar\omega} &= 1 - |c_0|^2 - |c_1|^2,
\end{aligned}$$

as desired. When $\mathbb{P}_{>2\hbar\omega} = 0$, we simply just get that $1 = |c_0|^2 + |c_1|^2$ while the rest of the coefficients are zero, since $|c_n|^2$ is strictly greater than or equal to zero, but cannot be negative. Thus $|c_n|^2$ for $n \geq 2$ must be zero if we never measure an energy greater than $2\hbar\omega$. Therefore the only nonzero coefficients are c_0 and c_1.

(b) Assuming the only nonzero coefficients are c_0 and c_1, then the normalization condition, as stated before, just becomes $1 = |c_0|^2 + |c_1|^2$. For the expectation value of the Hamiltonian,

$$\begin{aligned}
\hbar\omega = \langle H \rangle &= \langle \psi(0) | H | \psi(0) \rangle \\
&= \langle \psi(0) | \left[c_0 E_0 \left| \varphi_0 \right\rangle + c_1 E_1 \left| \psi_1 \right\rangle \right] \\
&= \left[\left\langle \varphi_0 \right| c_0^* + \left\langle \varphi_1 \right| c_1^* \right] \left[c_0 E_0 \left| \varphi_0 \right\rangle + c_1 E_1 \left| \psi_1 \right\rangle \right] \\
&= |c_0|^2 E_0 + |c_1|^2 E_1.
\end{aligned}$$

Since $E_0 = \frac{1}{2}\hbar\omega$ and $E_1 = \frac{3}{2}\hbar\omega$, then the above relation becomes

$$\begin{aligned}
1 &= \frac{1}{2}|c_0|^2 + \frac{3}{2}|c_1|^2 \\
&= \frac{1}{2}\left(1 - |c_1|^2\right) + \frac{3}{2}|c_1|^2 \\
&= \frac{1}{2} + |c_1|^2 \\
\implies |c_1|^2 &= \frac{1}{2} \quad \text{and} \quad |c_0|^2 = \frac{1}{2}.
\end{aligned}$$

(c) Under the new condition that c_0 must be real and positive, and $c_1 \to |c_1|e^{i\theta_1}$, we have that $c_0 = \frac{1}{\sqrt{2}}$ and $c_1 = \frac{e^{i\theta_1}}{\sqrt{2}}$, which are determined by the normalization condition and that $\langle H \rangle = \hbar\omega$, as found in the previous part of this question. To determine θ_1, we invoke the relation

$$\begin{aligned}
\langle X \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} &= \langle \psi(0) | \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) | \psi(0) \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{2} \left\langle \psi(0) \right\rangle (a^\dagger + a) \left[\left| \varphi_0 \right\rangle + e^{i\theta_1} \left| \varphi_1 \right\rangle \right] \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} \left(\frac{1}{\sqrt{2}} \left\langle \varphi_0 \right| + \frac{e^{-i\theta_1}}{\sqrt{2}} \left\langle \varphi_1 \right| \right) \left(\sqrt{1} \left| \varphi_1 \right\rangle + e^{i\theta_1}\sqrt{2} \left| \varphi_2 \right\rangle + 0 + e^{i\theta_1}\sqrt{1} \left| \varphi_0 \right\rangle \right)
\end{aligned}$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2}}\left(e^{i\theta_1} + e^{-i\theta_1}\right).$$

We then must require that $e^{i\theta_1} + e^{-i\theta_1} = 2\cos\theta_1 = \sqrt{2}$ for this condition on $\langle X \rangle$ to be true. This implies that $\theta_1 = \arccos\frac{\sqrt{2}}{2}$, and therefore $\theta_1 = \frac{\pi}{4}$.

(d) As stated in part (a), the time evolution of the state $|\psi(t)\rangle$ is governed by the Schrödinger equation:

$$\begin{aligned}
|\psi(t)\rangle &= c_0 e^{-i\omega\left(0+\frac{1}{2}\right)t}\,|\varphi_0\rangle + c_1 e^{-i\omega\left(1+\frac{1}{2}\right)t}\,|\varphi_1\rangle \\
&= \frac{1}{\sqrt{2}}e^{-\frac{1}{2}i\omega t}\,|\varphi_0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}e^{-\frac{3}{2}i\omega t}\,|\varphi_1\rangle\,,
\end{aligned}$$

in which I have used the previous parts of this question to determine the coefficients c_0, c_1 and θ_1. The mean value of the position $\langle X \rangle(t)$ can be determined by a similar method as applied in the previous part:

$$\begin{aligned}
\langle X \rangle(t) &= \langle\psi(t)|\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|\psi(t)\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\,\langle\psi(t)\rangle\,(a^\dagger + a)\left[e^{-\frac{1}{2}i\omega t}\,|\varphi_0\rangle + e^{i\theta_1}e^{-\frac{3}{2}i\omega t}\,|\varphi_1\rangle\right] \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{1}{\sqrt{2}}e^{\frac{1}{2}i\omega t}\,\langle\varphi_0| + \frac{e^{-i\theta_1}}{\sqrt{2}}e^{\frac{3}{2}i\omega t}\,\langle\varphi_1|\right) \\
&\qquad\qquad \cdot\left(\sqrt{1}e^{-\frac{1}{2}i\omega t}\,|\varphi_1\rangle + e^{i\theta_1}e^{-\frac{3}{2}i\omega t}\sqrt{2}\,|\varphi_2\rangle + 0 + e^{-\frac{3}{2}i\omega t}e^{i\theta_1}\sqrt{1}\,|\varphi_0\rangle\right) \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2}}\left(e^{-i\theta_1+\omega t} + e^{i\theta_1-\omega t}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\theta_1 - \omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos\left(\frac{\pi}{4} - \omega t\right),
\end{aligned}$$

hence $\theta(t) = \frac{\pi}{4} - \omega t$ yields the value of θ_1 at time t — although this part of the problem is ambiguous, since θ_1 is an initial phase condition (phase difference) and should not time-dependent. Therefore $\langle X \rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\cos\left(\frac{\pi}{4} - \omega t\right)$, which is what I wanted to determined.

2. Anisotropic three-dimensional harmonic oscillator

In a three-dimensional problem, consider a particle of mass m and of potential energy:

$$V(X,Y,Z) = \frac{m\omega^2}{2}\left[\left(1+\frac{2\lambda}{3}\right)(X^2+Y^2)+\left(1-\frac{4\lambda}{3}\right)Z^2\right]$$

where ω and λ are constants which satisfy:

$$\omega \geq 0 \quad , \qquad 0 \leq \lambda < \frac{3}{4}$$

a. What are the eigenstates of the Hamiltonian and the corresponding energies?

b. Calculate and discuss, as functions of λ, the variation of the energy, the parity and the degree of degeneracy of the ground state and the first two excited states.

(a) For this problem, I will first introduce a useful change of variables to avoid any kind of messy algebra. I will let $q^2 = 1+\dfrac{2\lambda}{3}$ and $k^2 = 1-\dfrac{4\lambda}{3}$ (note that these quantities are dimensionless, hence they may be easily manipulated as numbers). The potential then becomes

$$V(X,Y,Z) = \frac{m\omega^2}{2}\left[q^2(X^2+Y^2)+k^2Z^2\right].$$

Under representation of states in 3-dimensional space, we obtain a seperable tensor product of the states in x, y and z coordinate spaces. That is, $\mathcal{E}_{\boldsymbol{r}} = \mathcal{E}_x \otimes \mathcal{E}_y \otimes \mathcal{E}_z$. Thus our Hamiltonian can be written as three-seperate components, with $H = H_x + H_y + H_z$:

$$H_x = \frac{P_x^2}{2m} + \frac{m\omega^2 q^2}{2}X^2$$
$$H_y = \frac{P_y^2}{2m} + \frac{m\omega^2 q^2}{2}Y^2$$
$$H_z = \frac{P_z^2}{2m} + \frac{m\omega^2 k^2}{2}Z^2.$$

The first important observation is that the Hamiltonians in x and y coordinate space are identical, hence their eigenvalues and stationary states will also be identical. The change of variables I invoked, as introduced earlier, will then allow me to only solve one of the three eigenvalue equations $H_i\left|\varphi_{n_i}\right\rangle = E_{n_i}\left|\varphi_{n_i}\right\rangle$ for $i = x, y, z$, since the Hamiltonian for z-coordinate space is just given by swapping q with k then re-substituting.

To solve the eigenvalue equation for x, $H_x\left|\varphi_{n_x}\right\rangle = E_{n_x}\left|\varphi_{n_x}\right\rangle$, I will first introduce the non-dimensional operators $\hat{X} = q\sqrt{\dfrac{m\omega}{\hbar}}X$ and $\hat{P}_x = \dfrac{1}{\sqrt{m\omega\hbar}}P_x$. The dimensionless Hamiltonian $\hat{H} = \hbar\omega H$can then be put in the form $\hat{H}_x = \dfrac{1}{2}\left(\hat{P}_x^2+\hat{X}^2\right)$, which has the exact same structure as a one-dimensional harmonic oscillator! The corresponding lowering and raising operators are equivalently defined

$$a_x = \frac{1}{\sqrt{2}}(\hat{X}+i\hat{P}_x) \qquad a_x^\dagger = \frac{1}{\sqrt{2}}\left(\hat{X}-i\hat{P}_x\right).$$

As before, the energy levels and stationary states are determined by the Hamiltonian eigenvalue equation. As in the one-dimensional isotropic harmonic oscillator, it is useful to determine such energies and states by first calculating the commutation relations between the raising and lower operators, as well as the non-dimensional position and momentum operators:

$$\begin{aligned}
[\hat{X}, \hat{P}_x] &= q\sqrt{\frac{m\omega}{\hbar}}\frac{1}{\sqrt{m\omega\hbar}}[X, P_x] \\
&= i\hbar q \cdot \frac{1}{\hbar} \\
&= iq
\end{aligned}$$

$$\begin{aligned}
\implies [a_x, a_x^\dagger] &= \frac{i}{2}[\hat{P}_x, \hat{X}] - \frac{i}{2}[\hat{X}, \hat{P}_x] \\
&= -\frac{i^2}{2}q - \frac{i^2}{2}q \\
&= q.
\end{aligned}$$

Taking the product $a_x^\dagger a_x$ yields the Hamiltonian in x-coordinate space:

$$\begin{aligned}
a_x^\dagger a_x &= \frac{1}{2}(\hat{X} - i\hat{P}_x)(\hat{X} + i\hat{P}_x) \\
&= \frac{1}{2}(\hat{X}^2 + \hat{P}_x^2 + i[\hat{X}, \hat{P}_x]) \\
&= \frac{1}{2}(\hat{X}^2 + \hat{P}_x^2 - q) \\
&= H - \frac{q}{2}. \implies \hat{H}_x = a_x^\dagger a_x + \frac{q}{2} = \hat{N}_x + \frac{q}{2}.
\end{aligned}$$

Since the raising and lowering operators act on the states in x-coodinate space as they would in a one-dimensional problem, the number operator $\hat{N}_x = a_x^\dagger a_x$ also acts on the state in the same way by drawing out an eigenvalue. Therefore the eigenvalue equation in x is

$$\begin{aligned}
\hat{H}_x \left|\varphi_{n_x}\right\rangle &= \left(\hat{N}_x + \frac{q}{2}\right)\left|\varphi_{n_x}\right\rangle \\
&= (n_x + \frac{q}{2})\left|\varphi_{n_x}\right\rangle.
\end{aligned}$$

Therefore, re-dimensionalizing the Hamiltonian by $\hbar\omega$ yields the energy eigenvalues for the oscillator states in $\mathcal{E}_x$ space:

$$E_{n_x} = \hbar\omega\left(n_x + \frac{q}{2}\right) = \hbar\omega\left(n_x + \frac{1}{2}\sqrt{1 + \frac{2\lambda}{3}}\right).$$

The equivalent calculator for k determines the energy eigenvalues for y and z coordinate spaces. By taking the tensor product of all three state spaces, the energies for each of the three oscillators superpose, hence obtaining the full energy spectrum for the anisotropic harmonic oscillator:

$$\boxed{E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \sqrt{1 + \frac{2\lambda}{3}} + \frac{1}{2}\sqrt{1 - \frac{4\lambda}{3}}\right).}$$

The process of determining eigenstates is once again the same as determining them in a one-dimensional case. The general spectrum of states are hence given by the tensor product between

each of the three coordinate spaces, $|\varphi_{n_x,n_y,n_z}\rangle = |\varphi_{n_x}\rangle\,|\varphi_{n_y}\rangle\,|\varphi_{n_z}\rangle$. Then, the Hamiltonian H_x (for instance) acting on the ket $|\varphi_{n_x,n_y,n_z}\rangle$ then only acts on $|\varphi_{n_x}\rangle$, drawing out the energy E_{n_x}. The y and z Hamiltonians act on $|\varphi_{n_x,n_y,n_z}\rangle$ the same way. This implies that the full 3-dimensional Hamiltonian H acts on $|\varphi_{n_x,n_y,n_z}\rangle$ and draws out the energy E_{n_x,n_y,n_z} as found above. I will proceed by solving for the state in the x-coordinate space representation, still using the change of variables I initially introduced, then generalizing for the y and z coordinate spaces.
As in the one-dimensional problem, the lowering operator a_x acting the ground state in the x-coordinate representation yields zero. Determining the ground state henceforth determines the rest of the eigenstates, because the raising operator acting on the ground state yields the first excited state $a_x^\dagger\,|\varphi_{n_x}\rangle = |\varphi_{n+1_x}\rangle$. Therefore to determine the ground state, we have the relation

$$\begin{aligned}
&\langle x|a_x|\varphi_{0_x}\rangle = 0\\
&\implies 0 = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}qx + \frac{i}{\sqrt{m\hbar\omega}}P_x\right)\varphi_{0_x}(x)\\
&\implies 0 = \left(\frac{m\omega q}{\hbar}x + \frac{\partial}{\partial x}\right)\varphi_{0_x}(x).
\end{aligned}$$

Solving the ordinary differential equation via the method of integrating factors then yields the general solution $\varphi_{0_x}(x) = Ae^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2}$ for some constant A, determined by the normalization. We have that

$$\begin{aligned}
1 &= |A|^2\int_{-\infty}^{\infty} dx\, e^{-\frac{m\omega q}{\hbar}x^2}\\
&= |A|^2\sqrt{\frac{\pi}{\frac{m\omega}{\hbar}q}}\\
\implies A &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4}.
\end{aligned}$$

Therefore the ground state is given by $\varphi_{0_x}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4}e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2}$, and the general ground state in 3-dimensional space for the 3-dimensional anisotropic Hamiltonian is given by the product of $\varphi_{0_x}(x)$, $\varphi_{0_y}(y)$ and $\varphi_{0_z}(z)$:

$$\boxed{\psi_0(x,y,z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} q^{1/2}\,k^{1/4}\,e^{-\frac{1}{2}\frac{m\omega}{\hbar}(qx^2+qy^2+kz^2)}.}$$

Allow me to return to our ground state wavefunction in x-coordinate space. The determination of higher-level excited states is followed by the process of applying the raising operator n_x number of times on the ground state $|\varphi_{0_x}\rangle$. Thus, every state is proportional with is neighbouring state: $|\varphi_{n_x}\rangle = c_{n_x}a_x^\dagger\,|\varphi_{n-1_x}\rangle$, generally. First off, for the ground state, since it is normalized we obtain that $\langle\varphi_{0_x}|\varphi_{0_x}\rangle = 1$. For the first exicted state in x,

$$\begin{aligned}
\langle\varphi_{1_x}|\varphi_{1_x}\rangle &= |c_{0_x}|^2\,\langle\varphi_{0_x}|\,(a_x a_x^\dagger)\,|\varphi_{0_x}\rangle\\
&= |c_{0_x}|^2\,\langle\varphi_{0_x}|\left(\hat{N}_x + q\right)|\varphi_{0_x}\rangle\\
&= |c_{0_x}|^2 q = 1\\
\implies c_{0_x} &= \frac{1}{\sqrt{q}}.
\end{aligned}$$

Applying the exact same calculation to the next excited state,

$$\begin{aligned}
\langle \varphi_{2_x} | \varphi_{2_x} \rangle &= |c_{1_x}|^2 \langle \varphi_{1_x} | (a_x a_x^\dagger) | \varphi_{1_x} \rangle \\
&= |c_{1_x}|^2 \langle \varphi_{1_x} | \left(\hat{N}_x + q \right) | \varphi_{1_x} \rangle \\
&= |c_{1_x}|^2 (1+q) = 1 \\
\implies c_{1_x} &= \frac{1}{\sqrt{1+q}}
\end{aligned}$$

The third excited state,

$$\begin{aligned}
\langle \varphi_{3_x} | \varphi_{3_x} \rangle &= |c_{2_x}|^2 \langle \varphi_{2_x} | (a_x a_x^\dagger) | \varphi_{2_x} \rangle \\
&= |c_{2_x}|^2 \langle \varphi_{2_x} | \left(\hat{N}_x + q \right) | \varphi_{2_x} \rangle \\
&= |c_{2_x}|^2 (2+q) = 1 \\
\implies c_{2_x} &= \frac{1}{\sqrt{2+q}}
\end{aligned}$$

and so on. Thus:

$$\begin{aligned}
|\varphi_{3_x}\rangle &= \frac{1}{\sqrt{2+q}} a_x^\dagger |\varphi_{2_x}\rangle \\
&= \frac{1}{\sqrt{2+q}} \frac{1}{\sqrt{1+q}} (a_x^\dagger)^2 |\varphi_{1_x}\rangle \\
&= \frac{1}{\sqrt{2+q}} \frac{1}{\sqrt{1+q}} \frac{1}{\sqrt{q}} (a_x^\dagger)^3 |\varphi_{0_x}\rangle .
\end{aligned}$$

For a general state $|\varphi_{n_x}\rangle$, we then have the product

$$|\varphi_{n_x}\rangle = \left[\prod_{\ell=0}^{n_x-1} \frac{1}{\sqrt{q+\ell}} \right] (a_x^\dagger)^{n_x} |\varphi_{0_x}\rangle$$

valid for $n_x > 0$.

Now, notice that this relation is not resolved if $n_x = 0$, because we obtain the product $\displaystyle\prod_{\ell=0}^{-1} \frac{1}{\sqrt{q+\ell}}$ which does not make sense, so every state in the 3-dimensional oscillator with any $n_i = 0$ must be expressed differently due to the normalization factor difference between ground states and excited states. This is only due to the fact that the number operator pulls out a 0 when acting on the ground state, but non-zero when acting on any other excited state.

To determine the explicit forms of the position representations of the excited states, we may expand the raising operator:

$$\begin{aligned}
\langle x | \varphi_{n_x} \rangle &= \left[\prod_{\ell=0}^{n_x-1} \frac{1}{\sqrt{q+\ell}} \right] \left[\frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}} q x - \frac{\hbar}{\sqrt{m\omega\hbar}} \frac{\partial}{\partial x} \right) \right]^{n_x} \varphi_{0_x}(x) \\
\varphi_{n_x}(x) &= \left[\prod_{\ell=0}^{n_x-1} \frac{1}{\sqrt{q+\ell}} \right] \left(\frac{\hbar}{2m\omega} \right)^{n_x/2} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} q^{1/4} \left[\frac{m\omega}{\hbar} q x - \frac{\partial}{\partial x} \right]^{n_x} e^{-\frac{1}{2} \frac{m\omega q}{\hbar} x^2} .
\end{aligned}$$

Once again, for $n_x > 0$. The coordinate space states for y and z are identical (let $q \to k$ for z). Therefore the ground and excited states for the oscillator are

$$\varphi_{0_x}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4} e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \qquad (n_x = 0)$$

$$\varphi_{n_x}(x) = \left[\prod_{\ell=0}^{n_x-1} \frac{1}{\sqrt{q+\ell}}\right] \left(\frac{\hbar}{2m\omega}\right)^{n_x/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4} \left[\frac{m\omega}{\hbar}qx - \frac{\partial}{\partial x}\right]^{n_x} e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \qquad (n_x \geq 1)$$

$$\varphi_{0_y}(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4} e^{-\frac{1}{2}\frac{m\omega q}{\hbar}y^2} \qquad (n_y = 0)$$

$$\varphi_{n_y}(y) = \left[\prod_{j=0}^{n_y-1} \frac{1}{\sqrt{q+j}}\right] \left(\frac{\hbar}{2m\omega}\right)^{n_y/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4} \left[\frac{m\omega}{\hbar}qy - \frac{\partial}{\partial y}\right]^{n_y} e^{-\frac{1}{2}\frac{m\omega q}{\hbar}y^2} \qquad (n_y \geq 1)$$

$$\varphi_{0_z}(z) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} k^{1/4} e^{-\frac{1}{2}\frac{m\omega k}{\hbar}z^2} \qquad (n_z = 0)$$

$$\varphi_{n_z}(z) = \left[\prod_{m=0}^{n_z-1} \frac{1}{\sqrt{k+m}}\right] \left(\frac{\hbar}{2m\omega}\right)^{n_z/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} k^{1/4} \left[\frac{m\omega}{\hbar}kz - \frac{\partial}{\partial z}\right]^{n_z} e^{-\frac{1}{2}\frac{m\omega k}{\hbar}z^2} \qquad (n_z \geq 1)$$

with $\psi_{n_x,n_y,n_z}(x,y,z) = \varphi_{n_x}(x)\,\varphi_{n_y}(y)\,\varphi_{n_z}(z)$ which is piecewise, for general n_x, n_y, n_z.

This is what I wanted to determine. It should not be difficult to resubstitute $q = \sqrt{1 + \frac{2\lambda}{3}}$ and $k = \sqrt{1 - \frac{4\lambda}{3}}$. It is important to note that when $\lambda = 0$, each of the three states return to their general form as if the state was in a spherically symmetric isotropic potential, but more on this in part (b).

(b-i) For the ground state, I will invoke the energy relation which I have found in the previous part of this question. Letting $n_x = n_y = n_z = 0$ for the lowest energy level, we obtain

$$E_{0,0,0} = \hbar\omega\left(\sqrt{1 + \frac{2\lambda}{3}} + \frac{1}{2}\sqrt{1 - \frac{4\lambda}{3}}\right).$$

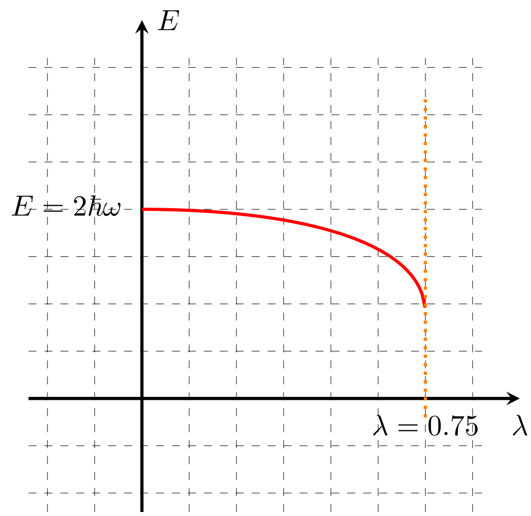
The corresponding ground state, which I have also previously determined, is just

$$\psi_0(x,y,z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} q^{1/2}\, k^{1/4}\, e^{-\frac{1}{2}\frac{m\omega}{\hbar}(qx^2+qy^2+kz^2)}.$$

Degeneracy arises in the above energy relation for the grouns state if $E(\lambda)$ gives the same value for two different values of λ, that is $E(\lambda_1) = E(\lambda_2)$ for $\lambda_1 \neq \lambda 2$. To show that this energy value is non-degenerate, we just need to examine the derivative:

$$\frac{dE}{d\lambda} = \hbar\omega\left(\frac{1}{3\sqrt{1+2\lambda/3}} - \frac{1}{3\sqrt{1-4\lambda/3}}\right).$$

It is easy to see that for $\lambda \in [0, 3/4)$, this derivative is always negative, hence the corresponding energy function is monotonic, and thus non-degenerate because it is injective. We may visualize this via a plot of the energy:



As for the ground eigenstate, it is simply a 3-dimensional gaussian function, which has even parity due to it's spherical symmetry and since q and k are always positive.

(b-ii) As I stated before, when $\lambda = 0$, the anisotropic harhomic oscillator now becomes isotropic, and the energy levels are easily given by

$$E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right).$$

The degeneracy of the energies are determined by how many combinations of n_x, n_y, n_z yield the same value of E. This is no more than a stars and bars problem (combinatorics). The degree of degeneracy is given by

$$\begin{aligned}\binom{k+n-1}{n} &= \frac{(k+n-1)!}{n!(k+n-1-n)!} \\ &= \frac{(k+n-1)!}{n!(k-1)!},\end{aligned}$$

where n is the level of the excited state and k is the number of different inputs (here, $k = 3$ always because we have 3 inputs n_x, n_y, n_z). Therefore the degree of degeneracy of the $n-th$ excited state is

$$D(n) = \frac{(2+n)!}{2n!}.$$

For $n = 0$, $D = 1$, which implies no degeneracy. There is only 1 combination of inputs which produces the ground state. When $n = 1$, $D(1) = 3$, so there are 3 different combinations which produce the first excited state (that is, either n_x, n_y,or $n_z = 1$). When $n = 2$, $D(2) = 6$, so there are 6 ways to create the second-highest excited state. For $n = 6$, for instance, the degree of degeneracy is 28, so there are 28 combinations of n_x, n_y and n_z which produce the 6-th level exicted state.

As for the eigenstates, it suffices to determine the first two exicted states of just one of the three eigenvectors, since the other two follow a similar way. I will leave q and k nonzero to determine generality for now, then I will proceed with $\lambda = 0$. For $n_x = 1$, we have

$$\varphi_{1_x}(x) = \frac{1}{\sqrt{q}}\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4}\left[\frac{m\omega q}{\hbar}x - \frac{\partial}{\partial x}\right] e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2}$$

$$
\begin{aligned}
&= q^{-1/4}\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\frac{m\omega q}{\hbar}xe^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2}+e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2}\left(\frac{m\omega q}{\hbar}x\right)\right] \\
&= q^{-1/4}\sqrt{\frac{2\hbar}{m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{m\omega q}{\hbar}xe^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \\
&= \sqrt{2}\left(\frac{m\omega q}{\hbar\pi^{1/3}}\right)^{3/4} xe^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2},
\end{aligned}
$$

which is an odd parity state. In the product with the other states $\varphi_{0_y}(y)$ and $\varphi_{0_z}(z)$, the state is still odd parity. Likewise with the other first-excited states, they too are odd. The collection of first-exicted odd parity states are just

$$
\begin{aligned}
\varphi_{1_x}(x) &= \left[\frac{4}{\pi}\left(\frac{m\omega q}{\hbar}\right)^3\right]^{1/4} xe^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \\
\varphi_{1_y}(y) &= \left[\frac{4}{\pi}\left(\frac{m\omega q}{\hbar}\right)^3\right]^{1/4} ye^{-\frac{1}{2}\frac{m\omega q}{\hbar}y^2} \\
\varphi_{1_z}(z) &= \left[\frac{4}{\pi}\left(\frac{m\omega k}{\hbar}\right)^3\right]^{1/4} ze^{-\frac{1}{2}\frac{m\omega k}{\hbar}z^2},
\end{aligned}
$$

and hence the first excited state of the oscillator is given when any of $n_x, n_y, n_z = 1$ and the other are ground. For the second-excited state,

$$
\begin{aligned}
\varphi_{2_x}(x) &= \frac{1}{\sqrt{q(q+1)}}\left(\frac{\hbar}{2m\omega}\right)^{2/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} q^{1/4}\left[\frac{m\omega q}{\hbar}x-\frac{\partial}{\partial x}\right]^2 e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \\
&= \frac{q^{1/4}}{\sqrt{q(q+1)}}\left(\frac{\hbar}{2m\omega}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\frac{m\omega q}{\hbar}x-\frac{\partial}{\partial x}\right]2\frac{m\omega q}{\hbar}xe^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \\
&= \frac{q^{1/4}}{\sqrt{q(q+1)}}\left(\frac{\hbar}{2m\omega}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2\frac{m\omega q}{\hbar}\left[\frac{m\omega q}{\hbar}x^2-1+\frac{m\omega q}{\hbar}x^2\right] e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2} \\
&= \frac{q^{3/4}}{\sqrt{(q+1)}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[2\frac{m\omega q}{\hbar}x^2-1\right] e^{-\frac{1}{2}\frac{m\omega q}{\hbar}x^2}.
\end{aligned}
$$

This is an even parity state. I shall now invoke the fact that $\lambda = 0$, which thus implies that $q = k = 1$. Therefore our first and second excited states in the x-representation coordinate space are identical to the ones as if the oscillator was isotropic:

$$
\begin{aligned}
\varphi_{1_x}(x) &= \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} xe^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \\
\varphi_{2_x}(x) &= \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[2\frac{m\omega}{\hbar}x^2-1\right] e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2},
\end{aligned}
$$

which also holds identically for y and z (they are all the same function). Now, due to degeneracies, our wavefunction with $n = 2$ may also be a combination of two single-excited states, such as $\psi_{1,1,0}(x, y, z) = \varphi_{1_x}(x)\varphi_{1_y}(y)\varphi_{0_z}(z)$. In this case, believe it or not, we actually an even parity state:

$$
\psi_{1,1,0}(x, y, z) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/2} xy\, e^{-\frac{1}{2}\frac{m\omega}{\hbar}(x^2+y^2+z^2)}.
$$

When x or y are negative, they are opposite each other, however when x and y are both negative or both positive, the wavefunction is positive, hence resulting in an even symetric state. Equivalently, this occurs for any combination of the $n_i's$ making $n = 2$.

(b-iii) *For this part of the question, I am assuming that it is asking how the stationary states evolve over time, since the energy levels themselves stay constant for a fixed* λ*.* For $\lambda \ll 1$, I will invoke the binomial expansion on q and k. Firstly, Schrödinger's equation depicts the time evolution of the oscillatot state by

$$\psi_{n_x,n_y,n_z}(x,y,z,t) = \varphi_{n_x}(x)\varphi_{n_y}(y)\varphi_{n_z}(z)e^{-i\omega(n_x+n_y+n_z+\sqrt{1+2\lambda/3}+0.5\sqrt{1-4\lambda/3})t}.$$

(Allow me to briefly draw your attention to the fact that the time evolution of the state is the product of the time evolution of the states $\varphi_{n_i}(i,t)$ for $i = x, y, z$) Invoking the binomial expansion $f(1+\varepsilon) \simeq f(1) + \varepsilon f'(1) + \mathcal{O}(\varepsilon^2)$ to the function $\sqrt{1+\varepsilon}$ with $\varepsilon = \dfrac{2\lambda}{3}, \dfrac{-4\lambda}{3}$, we obtain

$$\sqrt{1+\frac{2\lambda}{3}} \simeq 1 + \frac{1}{2}\frac{2\lambda}{3} = 1 + \frac{\lambda}{3}$$
$$\sqrt{1-\frac{4\lambda}{3}} \simeq 1 - \frac{1}{2}\frac{4\lambda}{3} = 1 - \frac{2\lambda}{3}.$$

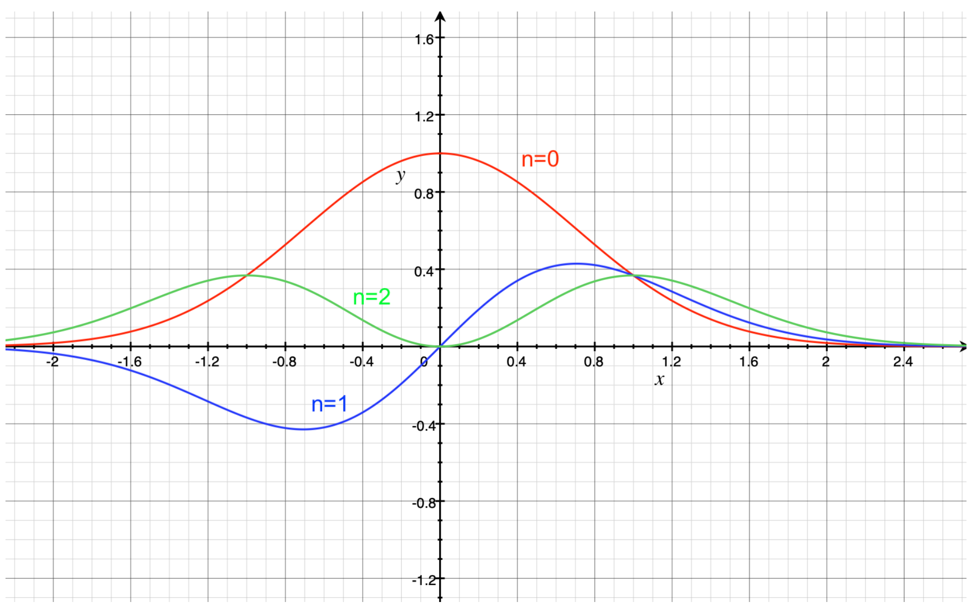
This implies that

$$E_{n_x,n_y,n_z} \simeq \hbar\omega\left(n_x + n_y + n_z + 1 + \frac{\lambda}{3} + \frac{1}{2} - \frac{\lambda}{3}\right)$$
$$= \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right),$$

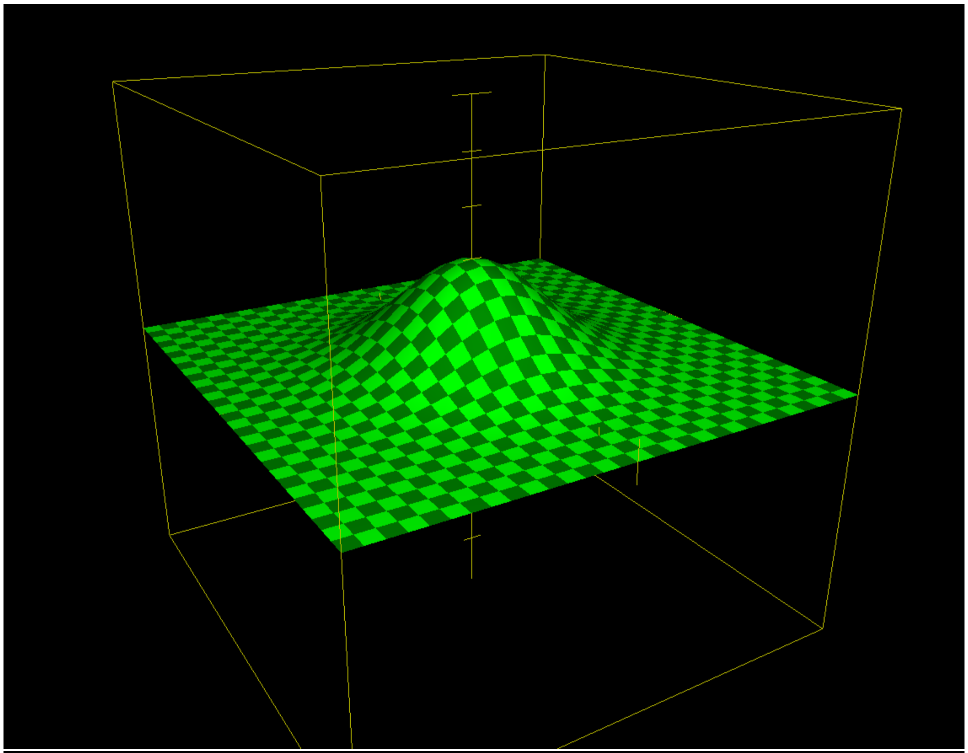
which is equivalently the energy of the isotropic three-dimensional harmonic oscillator! Therefore

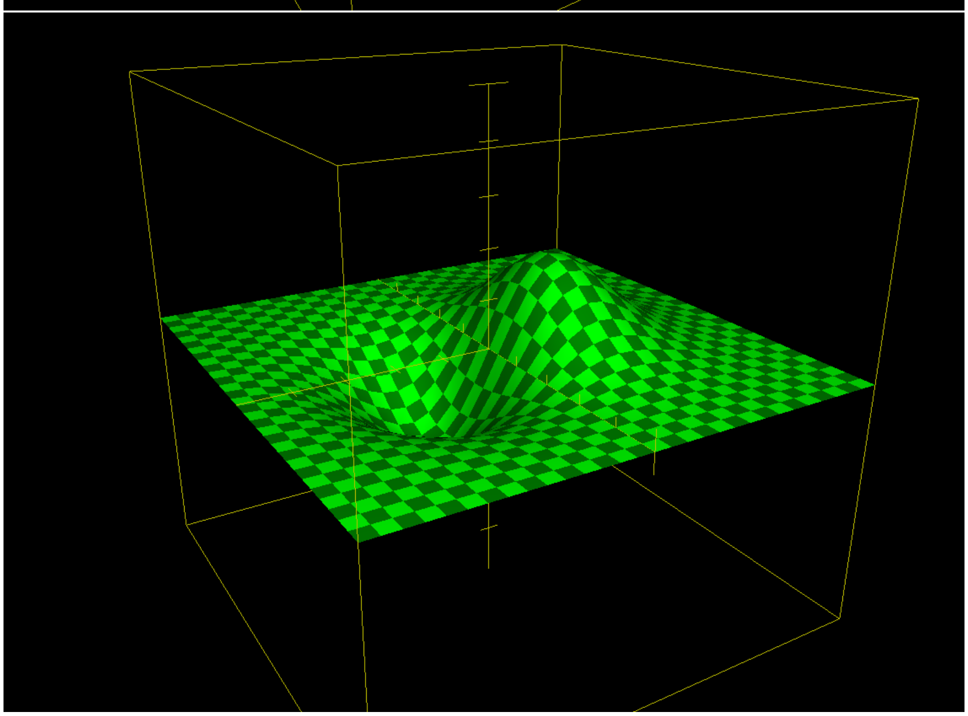
$$\psi_{n_x,n_y,n_z}(x,y,z,t) = \varphi_{n_x}(x)\varphi_{n_y}(y)\varphi_{n_z}(z)e^{-i\omega(n_x+n_y+n_z+3/2)t}.$$

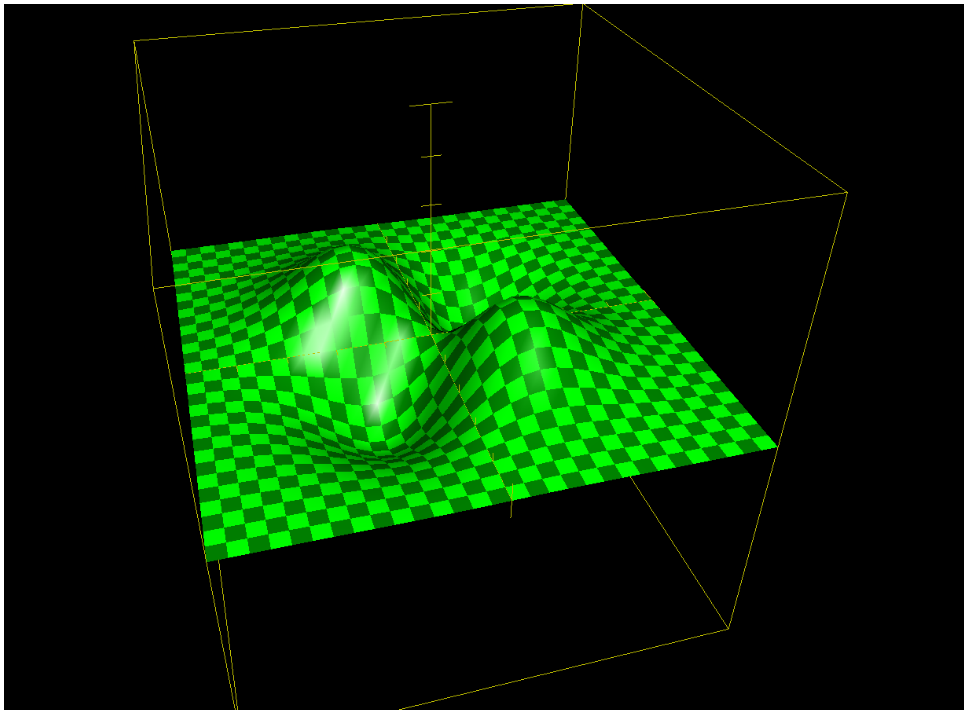
The degeneracies of the energies and the eigenvectors of the states remain the same as previously determined. Plotting $\varphi_{0_x}(x)$, $\varphi_{1_x}(x)$, and $\varphi_{2_x}(x)$ thus show the shape of the wavefunction as a cross section for one of the three dimensions. The stationary states evolve at constant rates, independent of λ and given by their energy level $E_{n_x} = \hbar\omega(n + 0.5)$. We have that



Furthermore, we can plot the ground state, first excited state, and second excited state in $x - y$ space (since we cannot visualize a four-dimensional plot):







where each of the plots are the respective eigenstates (I don't really know what else this question is asking for).

(b-iv) Now, for the special case when $\lambda = \frac{3}{4}$. Here, we may first note that the potential $V_z \to 0$ as $\lambda \to \frac{3}{4}$, and thus the state becomes a two dimensional anisotropic oscillator along x and y coordinate space, but a free particle along z coordinate space. The corresponding wave function is thus

$$\psi_{n_x,n_y}(x,y,z,t) = \varphi_{n_x}(x)\varphi_{n_y}(y)\chi(z)e^{-i\omega(n_x+n_y+\sqrt{3/2})t-i\omega' t},$$

where $\xi(z)e^{-i\omega' t}$ is the wavefunction of a wavepacket in the z-direction. The corresponding parities of the eigenstates are still equivalent (ie, odd for odd values of $n = n_x + n_y$ and even for even n), however the parity of the wavepacket is odd with respect to it's origin. Furthermore, the degree of degeneracy is now new as well, since n_z is no longer contributing to the energy value. Using the formula for stars and bars as I did before, with $k = 2$ instead of 3, the degree of degeneracy of the state is given by

$$\begin{aligned} D(n) = \frac{(2+n-1)!}{n!(2-1)!} &= \frac{(n+1)!}{n!} \\ &= n+1, \end{aligned}$$

hence for $n = 0$, the state is not degenerate, but for $n = 1$, $D(1) = 2$, $D(2) = 3$, and so on. The eigenstates are as determined before, however this time with $q = \sqrt{\frac{3}{2}}$ and $k = 0$ (except the z-direction inhabits a free particle equation, there are no eigenstates. Only eigenstates are in x and y). Lastly, for $\lambda < \frac{3}{4}$, the energy levels evolve according to the frequency

$$\Omega(t) = \omega\left(n_x + n_y + n_z + \sqrt{1+\frac{2\lambda}{3}} + \frac{1}{2}\sqrt{1-\frac{4\lambda}{3}}\right)t$$

which is in fact just proportional to the energy function which I had plotted earlier in this question. The integral values n_x, n_y, n_z just increase the value of the energy by integer values, however also increases the frequency at which the states evolve at. The eigenstates are still the same which I had determined in part (a) of this problem.