MAT237 Multivariable Calculus with Proofs

Problem Set 2

Due Friday October 15, 2021 by 13:00 ET

Instructions

This problem set is based on Module B: Topology (B3 to B8). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 2 sessions are held on Tuesday October 12, 2021 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope.** Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: **Jace Alloway** ________________

Student number: **1006940802** ________________

Full Name: ________________

Student number: ________________

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway ________________

2) ________________

Created and revised by C. Jonker, T. Janisse, and A. Zaman. Last updated October 15, 2021 at 00:26:50.

Problems

1. For each set S, determine whether it is open, closed, bounded, or path-connected. No justification is necessary. Fill in all boxes that apply. If none apply, leave them all blank. (filled ■ unfilled □)

   (1a) $S = \mathbb{R}^2 \setminus \operatorname{span}\{(1,1)\}$

   ■ **open** □ closed □ bounded □ path-connected

   (1b) $S = \mathbb{R}^3 \setminus \operatorname{span}\{(1,1,1)\}$

   ■ **open** □ closed □ bounded ■ **path-connected**

   (1c) $S = \{(x, \sin(1/x)) : x \in (0,1]\}$

   □ open □ closed ■ **bounded** ■ **path-connected**

   (1d) $S = \{(\frac{1}{m}, \frac{1}{n}) : m, n \in \mathbb{N}^+\} \cup \{(0,0)\}$

   □ open □ closed ■ **bounded** □ path-connected

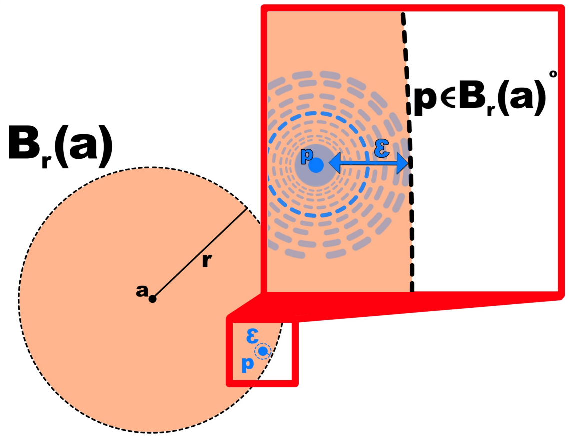
   (1e) $S = \{(wxyz, x^3 - yz, we^{xy}, \sin(wz)) : w^2 + x^2 \leq 2, 0 \leq y, z \leq 1\}$

   □ open ■ **closed** ■ **bounded** ■ **path-connected**

   (1f) $S = \mathbb{R} \times \mathbb{Q} \times \mathbb{Z} \times \mathbb{N}$

   □ open □ closed □ bounded □ path-connected

2. Fix $a \in \mathbb{R}^n$ and $r > 0$. Sketch a "picture proof" that the open ball $B_r(a)$ is open. Label your diagram with quantities that would be used in a direct proof by definition.



Notes:

- The picture proof is saying that every point $p \in B_r(a)$ is an interior point, that is, $\exists \varepsilon > 0$ such that $B_\varepsilon(p) \subseteq B_r(a)$.
- The point p is infinitesimal as you 'zoom in' on it, no matter how close you get to the boundary of $B_r(a)$, p will still be contained in the ball.

3. There are many strategies to verifying a set $S \subseteq \mathbb{R}^n$ is closed. For example, "show S contains its limits points" is one strategy. List as many other different strategies as you can. A diverse list is better than a long list.

My list:

- Show S contains all its limit points.
- (If S is bounded) show S is compact, by Bolzano-Weierstrass this proves that S is closed.
- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, and let $S = f^{-1}(V)$. You can show S is closed if every $V \subseteq \mathbb{R}^m$ is closed (Theorem 2.7.25c).
- Show that the complement of S, S^c, is open.
- Break down S into a finite number of closed subsets and show that each subset is closed. Then, a finite union of closed sets is closed.
- Prove that $S = \overline{S}$.
- Show that $\partial S \subseteq S$.

4. Limits can measure the rate at which functions tend to zero (or infinity) and polynomials are your favourite functions of all time. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and let $A \in \mathbb{N}^+$. You will compare the rate at which the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and the norm $||x||^A$ tend to zero as $x = (x_1, \ldots, x_n) \to 0$.

(4a) Prove that if $\alpha_1 + \cdots + \alpha_n \le A$ then $\lim_{x \to 0} \dfrac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{||x||^A}$ does not exist.

I want to prove that if $\alpha_1 + \cdots + \alpha_n \le A$, then the limit $\lim_{x \to 0} \dfrac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{||x||^A}$ does not exist.

Proof.

I will show that this limit does not exist by using the sequential definition

- Define the sequences $t(k) = \dfrac{1}{k}$ and $s(k) = \dfrac{2}{k}$. For both cases, $\lim_{k\to\infty} t(k) = \lim_{k\to\infty} s(k) = 0$.
- Define $x = (x_1, x_2, \ldots, x_n) = (t(k), t(k), \ldots, t(k))$. This is the sequence approaching zero on the line $(1, 1, \ldots, 1)$. Our limit then becomes

$$\lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^{\alpha_1}\left(\frac{1}{k}\right)^{\alpha_2}\cdots\left(\frac{1}{k}\right)^{\alpha_n}}{\left(\sqrt{\left(\frac{1}{k}\right)^2+\cdots+\left(\frac{1}{k}\right)^2}\right)^A} = \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^{\alpha_1+\cdots+\alpha_n}}{\left(\sqrt{n}\frac{1}{k}\right)^A} = \frac{1}{(\sqrt{n})^A} \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^{\alpha_1+\cdots+\alpha_n}}{\left(\frac{1}{k}\right)^A}.$$

- In the case where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = A$, then the sequences cancel and the limit computes to $\dfrac{1}{(\sqrt{n})^A}$. When $\alpha_1 + \alpha_2 + \cdots + \alpha_n < A$, the limit becomes

$$\frac{1}{(\sqrt{n})^A} \lim_{k\to\infty} \frac{1}{\left(\frac{1}{k}\right)^{A-\alpha_1-\alpha_1-\cdots-\alpha_n}},$$

  which does not exist because the denominator tends to zero.
- Now define $y = (y_1, y_2, \ldots, y_n) = (s(k), t(k)\, t(k), \ldots, t(k))$. This is the sequence approaching zero on the line $(2, 1, \ldots, 1)$, which is different from the first case. The limit becomes

$$\lim_{k\to\infty} \frac{\left(\frac{2}{k}\right)^{\alpha_1}\left(\frac{1}{k}\right)^{\alpha_2}\cdots\left(\frac{1}{k}\right)^{\alpha_n}}{\left(\sqrt{\left(\frac{2}{k}\right)^2+\cdots+\left(\frac{1}{k}\right)^2}\right)^A} = 2^{\alpha_1} \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^{\alpha_1+\cdots+\alpha_n}}{\left(\sqrt{n+4}\left(\frac{1}{k}\right)\right)^A} = \frac{2^{\alpha_1}}{(\sqrt{n+4})^A} \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^{\alpha_1+\cdots+\alpha_n}}{\left(\frac{1}{k}\right)^A}.$$

- In the case where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = A$, then the sequences cancel and the limit computes to $\dfrac{2^{\alpha_1}}{(\sqrt{n+4})^A}$. When $\alpha_1 + \alpha_2 + \cdots + \alpha_n < A$, the limit becomes

$$\frac{2^{\alpha_1}}{(\sqrt{n+4})^A} \lim_{k\to\infty} \frac{1}{\left(\frac{1}{k}\right)^{A-\alpha_1-\alpha_1-\cdots-\alpha_n}},$$

  which does not exist because the denominator tends to zero as $k \to \infty$.
- The limits using the sequential definition approach the same point on two different lines:$(1, 1, \ldots, 1)$ and $(2, 1, \ldots, 1)$. In either case, the limit either does not exist or both sequences, which both converge to zero as $k \to \infty$, compute values $\dfrac{1}{(\sqrt{n})^A}$ or $\dfrac{2^{\alpha_1}}{(\sqrt{n+4})^A}$, and thus the limit cannot exist.

■

(4b) Use the formal ε-δ definition of the limit to prove that if $\alpha_1 + \cdots + \alpha_n > A$, then $\lim_{x\to 0} \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}{||x||^A} = 0$.
That is, the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ tends to zero faster than $||x||^A$ when its degree $\alpha_1 + \cdots + \alpha_n$ exceeds A.

I want to prove that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } \forall x \in \mathbb{R}^n,\ 0 < \|x\| < \delta \implies \left\| \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}{||x||^A} \right\| < \varepsilon$$

where $\alpha_1 + \cdots + \alpha_n > A$ and $n \geq 2$.

Proof.

- Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^{1/(\alpha_1+\alpha_2+\cdots+\alpha_n-A)}$.
- Let $x \in \mathbb{R}^n$. Assume $0 < \|x\| < \delta$. This implies that for all $i = 1, 2, \ldots, n$,

$$|x_i| \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} < \delta.$$

  Then $|x_i| \leq \|x\| < \delta \implies |x_i|^{\alpha_i} \leq \|x\|^{\alpha_i} < \delta^{\alpha_i}$.
- This implies

$$\prod_{i=1}^{n} |x_i|^{\alpha_i} \leq \prod_{i=1}^{n} \|x\|^{\alpha_i} < \prod_{i=1}^{n} \delta^{\alpha_i},$$

  so then $|x_1|^{\alpha_1}|x_2|^{\alpha_2}\cdots \leq \|x\|^{\alpha_1}\|x\|^{\alpha_2}\cdots < \delta^{\alpha_1}\delta^{\alpha_2}\cdots = \delta^{\alpha_1+\alpha_2+\cdots+\alpha_n}$.
- Similarly, we have that $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} < \delta$, so then $\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right)^A < \delta^A$.
- By the properties of absolute values, for all $k \in \mathbb{R}$ then $k \leq |k|$. This implies

$$\left\| \frac{x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}}{\|x\|^A} \right\| \leq \left\| \frac{|x_1|^{\alpha_1}|x_2|^{\alpha_2}\ldots |x_n|^{\alpha_n}}{\|x\|^A} \right\|.$$

- With our bounds, our right hand side of the definition becomes

$$\begin{aligned}
\left\| \frac{x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}}{\|x\|^A} \right\| &\leq \left\| \frac{|x_1|^{\alpha_1}|x_2|^{\alpha_2}\ldots |x_n|^{\alpha_n}}{\|x\|^A} \right\| \\
&\leq \left\| \frac{\|x\|^{\alpha_1}\|x\|^{\alpha_2}\ldots \|x\|^{\alpha_n}}{\|x\|^A} \right\| \\
&= \left\| \|x\|^{\alpha_1+\alpha_2+\cdots+\alpha_n-A} \right\| \\
&< \delta^{\alpha_1+\alpha_2+\cdots+\alpha_n-A} \\
&< \left(\varepsilon^{1/(\alpha_1+\alpha_2+\cdots+\alpha_n-A)}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_n-A} \\
&< \varepsilon,
\end{aligned}$$

  as desired.

∎

5. Let f be an $\mathbb{R}^m$-valued function defined on the set $A \subseteq \mathbb{R}^n$. Define the function $F : A \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$F(x, y) = y - f(x).$$

Assume f is continuous on A. You will prove F is continuous on $A \times \mathbb{R}^m$.

(5a) Consider the flawed proof.

> 1. *The function $g(y) = y$ is continuous on $\mathbb{R}^m$.*
> 2. *The function $f(x)$ is continuous on A.*
> 3. *Since the difference of continuous functions is continuous, it follows that $g(y) - f(x)$ is continuous.*
> 4. *Therefore, $F(x, y) = y - f(x)$ is continuous on $A \times \mathbb{R}^m$.*

There is a fatal flaw in this proof. Identify the line with the flaw and briefly explain what is wrong.

The flaw in the proof is located in Line 3. Here, the proof is dealing with a difference of two continuous functions.
Lemma 2.7.16 states that if we let $A \subseteq \mathbb{R}^n$, $a \in A$, and f and g be $\mathbb{R}^m$-valued functions defined on A, then if f and g are continuous at a, then $f + g$ is continuous at a, where $\lambda \in \mathbb{R}$ is -1 in this case. This theorem does not hold for this proof, since $g(y) = y$ and $f(x)$ are defined on two completely different domains: $A \subseteq \mathbb{R}^n$ and $\mathbb{R}^m$.

(5b) Prove F is continuous by expressing F as a composition, product, and/or sum of continuous functions.

Proof.

- Define the functions $g(x, y) : A \times \mathbb{R}^m \to \mathbb{R}^m$ as $g(x, y) = y$ and $h : A \times \mathbb{R}^m \to \mathbb{R}^m$ as $h(x, y) = f(x)$.
- The function $g(x, y) = y$ is a projection map from $A \times \mathbb{R}^m \to \mathbb{R}^m$ and can be described as a matrix $G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and therefore it is a linear map. This can be applied to $h(x, y)$ as well, and therefore by **Lemma 2.4.17**, g and h are continuous.
- By **Lemma 2.7.16**, since h and g are $\mathbb{R}^m$ valued functions defined on the same domain $A \times \mathbb{R}^m$ and are continuous on such domain, then

$$F\left(f(x, y), g(x, y)\right) = g(x, y) - f(x, y) = y - f(x)$$

is continuous on $A \times \mathbb{R}^m$, as desired.

■

6. Let $a, b \in \mathbb{R}$ with $a < b$. Let f and g be real-valued functions. Assume f and g are continuous on $[a, b]$ and satisfy $f \leq g$ on $[a, b]$. Define the set

$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\},$$

which often arises through integration. You will show S is compact.

Here is an attempted proof to show that S is closed using sequences.

> 1. *Let $(s, t) \in \mathbb{R}^2$ be a limit point of S.*
> 2. *There exists a sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ in $S \setminus \{(s, t)\}$ converging to (s, t).*
> 3. *Since $(x_k, y_k) \in S$, it follows that for $k \in \mathbb{N}^+$,*
>
> $$a \leq x_k \leq b \quad \textit{and} \quad f(x_k) \leq y_k \leq g(x_k).$$
>
> 4. *Taking $k \to \infty$ in the first inequality, it follows that $a \leq s \leq b$.*
> 5. *Taking $k \to \infty$ in the second inequality, it follows that $f(s) \leq t \leq g(s)$.*
> 6. *Therefore, $(s, t) \in S$ by definition, so S is closed.*

The proof is missing details. You will study this proof in (6a) and (6b).

(6a) Identify which lines, if any, use that f and g are continuous on $[a, b]$. No justification required. Fill in all boxes that apply. If none apply, leave them blank.

☐ Line 1 ☐ Line 2 ■ **Line 3** ☐ Line 4 ■ **Line 5** ☐ Line 6

(6b) Both lines 4 and 5 rely on a limit law for sequences in $\mathbb{R}$. Precisely state that limit law as a lemma. No proof is required.

Lines 4 and 5 rely on the definition:
Let $p \in \mathbb{R}$ be a point. Let $\{x_n\}_n$ be a sequence in $\mathbb{R}$. x_n is said to converge to p if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, if $k \geq M$, then we must have $|x_k - p| < \varepsilon$. In this case we write "$\lim_{n\to\infty} x_n = p$".

(6c) Writing alternate proofs of the same fact can be insightful. Use Question 5 and properties of continuous functions to show that S is closed. Do not use sequences.

- Define the functions $F : [a,b]\times\mathbb{R} \to \mathbb{R}$ by $F(x,y) = y - f(x)$ and $H : [a,b]\times\mathbb{R} \to \mathbb{R}$ by $H(x,y) = y - g(x)$. By question 5, F and H are continuous (in this case, the set $A = [a,b] \subseteq \mathbb{R}$).
- Let $U = [0,\infty) \subseteq \mathbb{R}$ and $V = (-\infty, 0] \subseteq \mathbb{R}$. Then $\mathbb{R} = U \cup V$, which is closed.
- Then

$$\begin{aligned} F^{-1}(U) &= \left\{(x,y) \in \mathbb{R}^2 : F(x,y) \in U\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : y - f(x) \in [0,\infty)\right\} \\ &\implies y - f(x) \geq 0 \implies y \geq f(x) \end{aligned}$$

  and

$$\begin{aligned} H^{-1}(V) &= \left\{(x,y) \in \mathbb{R}^2 : H(x,y) \in V\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : y - g(x) \in (-\infty,0]\right\} \\ &\implies y - g(x) \leq 0 \implies y \leq g(x). \end{aligned}$$

- Then

$$\begin{aligned} F^{-1}(U) \cup H^{-1}(V) &= \left\{(x,y) \in \mathbb{R}^2 : F(x,y) \in U \wedge H(x,y) \in V\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : x \in [a,b] \wedge f(x) \leq y \leq g(x)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\, f(x) \leq y \leq g(x)\right\} \\ &= S. \end{aligned}$$

- S has been expressed as a preimage of the closed set $\mathbb{R} = (-\infty,0] \cup [0,\infty)$ by functions F and H, and thus since F and H are continuous, then by **Lemma. 2.7.25**, S is closed.

(6d) Prove that S is compact.

To prove S is compact, it suffices to prove that S is bounded. Then by Question 6c and Bolzano-Weierstrass, S must be compact.

Proof.

- Define the points $x_0 = \dfrac{a+b}{2}$ and $y_0 = \dfrac{f(\frac{a+b}{2}) + g(\frac{a+b}{2})}{2}$, so (x_0, y_0) is the 'midpoint' of S. To show S is bounded, we must show that there exists an open ball of radius $r > 0$ such that $S \subseteq B_r((x_0, y_0))$.
- The boundary of S is

  $$\partial S = \left\{(x,y) \in \mathbb{R}^2 : a < x < b,\ y = f(x) \vee y = g(x)\right\} \cup \left\{(x,y) \in \mathbb{R}^2 : x = a \vee x = b,\ f(x) \leq y \leq g(x)\right\}.$$

- Let $(p_1, p_2) \in \partial S$ and let $(z_1, z_2) \in \partial S$ be the point that satisfies $\max \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|$.
- Choose $r = \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| + 1$. I now need to show that for any point $(x,y) \in S$, then $(x,y) \in B_r((x_0, y_0))$.
- Fix a point $(x,y) \in S$. Then

  $$\begin{aligned} \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| &\leq \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| \\ &< \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| + 1 \\ &< r, \end{aligned}$$

  which directly implies that $(x,y) \in B_r((x_0, y_0))$.
- Therefore S is bounded. From Question 6c, S is also closed, and therefore by **Bolzano-Weierstrass**, S is compact.

■

7. Let $A \subseteq \mathbb{R}^n$ be a compact set. Let a be a limit point of A. Let f be a real-valued function that is continuous on the set $A \setminus \{a\}$. Prove that if $\lim_{x \to a} f(x) = -\infty$, then f attains a maximum on $A \setminus \{a\}$.
   Hint: Use the extreme value theorem for compact sets.

To prove that f attains a maximum on the set $A \setminus \{a\}$, it suffices to prove ???

Proof.

- Assume that $\lim_{x \to a} f(x) = -\infty$:

$$\forall M < 0, \exists \delta > 0 \text{ s.t } \forall x \in A \setminus \{a\}, 0 < |x - a| < \delta \implies f(x) < M.$$

- Choose $M < 0$ as $M = f(p)$ for a certain value of $p \in A \setminus \{a\}$. By definition, there will exist a $\delta > 0$ such that the above definition is satisfied.
- Restrict the domain of f to $A \setminus B_\delta(a)$, where $B_\delta(a) \subseteq A$ is the open ball of radius δ centred at a.
- The set $B_\delta(a)$ is open and thus by **Lemma 2.4.13**, $\mathbb{R}^n \setminus B_\delta(a)$ is closed. Notice that $A \setminus B_\delta(a) = A \cap (\mathbb{R}^n \setminus B_\delta(a))$, so by **Lemma 2.4.17 (d)**, $A \setminus B_\delta(a)$ is closed.
  If A is compact then A is bounded. Since $B_\delta(a) \subseteq A$, then $B_\delta(a)$ is bounded by the same ball as A.
  By **Bolzano-Weierstrass**, $A \setminus B_\delta(a)$ is compact.
- By the **Extreme Value Theorem**, since the restricted domain $A \setminus B_\delta(a)$ is compact and f is continuous on $A \setminus B_\delta(a) \subseteq A \setminus \{a\}$, then f attains a maximum and minimum at points in $A \setminus B_\delta(a)$.
- Let $f(x_0)$ denote the maximum of f on $A \setminus B_\delta(a)$, so $\forall x \in A \setminus \{a\},\ f(x_0) \geq f(x)$.
- Now let m denote the maximum of f on $A \setminus \{a\}$. If $f(p) \neq f(x_0)$, then let $m = \max\{f(p), f(x_0)\}$. If $f(p) = f(x_0)$, let $m = f(x_0)$.
- Therefore f attains a maximum on $A \setminus \{a\}$, as needed.

■