PHY454 Problem Set 3

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Problem 1

The stress in a medium is given by

$$\mathbf{T} = \alpha \begin{pmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T_{33} \end{pmatrix} \tag{1.1}$$

subject to the body force $\boldsymbol{F}_b = -g\mathbf{e}_3$, where α is a constant. To determine an expression for T_{33} required for equilibrium, we must consider the equation of motion of the body, given by Newton's law:

$$\rho \frac{D\mathbf{u}}{Dt} = 0 = \rho\mathbf{g} + \nabla \cdot \mathbf{T}, \tag{1.2}$$

since the medium is assumed to be at rest. The constant ρ may be absorbed into the constant α, yielding

$$\nabla \cdot \mathbf{T} = g\mathbf{e}_3. \tag{1.3}$$

The divergence of the stress tensor is given by $\dfrac{\partial T_{ij}}{\partial x_i}$, in which we only consider the $j = 3$ component, since $\mathbf{e}_3$ is the only non-zero body force component specified. Thus we have

$$\begin{aligned} \frac{\partial T_{i3}}{\partial x_i} &= \alpha \left[\frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \right] \mathbf{e_3} \\ g &= 0 - \alpha \frac{\partial x_2}{\partial x_2} + \alpha \frac{\partial T_{33}}{\partial x_3} \\ \implies g &= -\alpha + \alpha \frac{\partial T_{33}}{\partial x_3} \end{aligned}$$

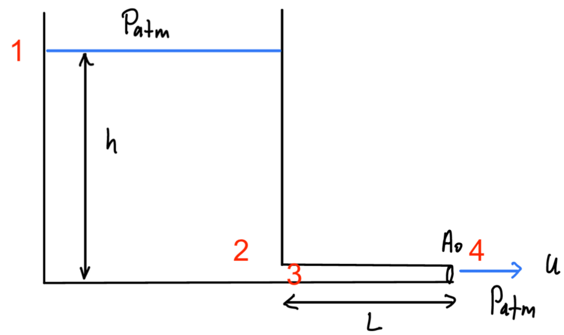
which may be determine by integration with respect to x_3:

$$T_{33} = \frac{(\alpha + g)}{\alpha} x_3 + f(x_1, x_2), \tag{1.4}$$

where T_{33} is unique up to an arbitrary function of the positons x_1 and x_2.

Problem 2

For this problem, we can consider the irrotational, unsteady flow of a fluid out of a tank, as shown below:



For brevity, consider the equation of motion for the fluid motion u_s along the streamline from point (1) to (2) to (3) then to (4), as given by the Navier-Stoke's equation:

$$\rho \frac{Du_s}{Dt} = -\nabla P + \rho g + \mu \nabla^2 u_s. \tag{2.1}$$

Equation (2.1) reduces by noting that $\nabla^2 u_s = 0$, the flow is uniform in the tank and in the tube. Furthermore, by the definition of the particle derivative, thus we obtain

$$\rho \frac{\partial u_s}{\partial t} + \rho u_s \frac{\partial u_s}{\partial s} = -\frac{\partial P}{\partial s} + \rho g, \tag{2.2}$$

where $'s'$ indicates the streamline direction. Since integration yields the form of Bernoulli's equation, we can integrate the momentum equation along points (1) to (4), which will yield the speed of the fluid along the streamline which is exiting the tube:

$$\begin{aligned} \int_{(1)}^{(4)} \rho \frac{\partial u_s}{\partial t}\, ds + \int_{(1)}^{(4)} \rho u_s \frac{\partial u_s}{\partial s}\, ds &= -\int_{(1)}^{(4)} \frac{\partial P}{\partial s}\, ds + \int_{(1)}^{(4)} \rho g \, ds \\ \implies \int_{(1)}^{(4)} \rho \frac{\partial u_s}{\partial t}\, ds + \frac{1}{2}\rho u_s^2 \bigg|_{(1)}^{(4)} &= -P((4)) + P((1)) + \rho g h. \end{aligned} \tag{2.3}$$

Since the pressures at points (1) and (4) are identically P_{atm}, then $P((4)) - P((1)) = 0$. The last term in (2.3) is obtain since $\rho g(z_4 - z_1) = \rho g h$. Secondly, note that $\frac{1}{2}\rho u_s^2 \Big|_{(1)}^{(4)} = \frac{1}{2}\rho u_4$, since it is assumed the velocity of the fluid at point (1) is zero ($\dot{h} = 0$).
The first term in (2.3) may be broken up into the individual path components by integral linearity. This term expands into

$$\begin{aligned} \int_{(1)}^{(4)} \rho \frac{\partial u_s}{\partial t}\, ds &= \int_{(1)}^{(2)} \rho \frac{\partial u_s}{\partial t}\, ds + \int_{(2)}^{(3)} \rho \frac{\partial u_s}{\partial t}\, ds + \int_{(3)}^{(4)} \rho \frac{\partial u_s}{\partial t}\, ds \\ &= 0 + \int_{(2)}^{(3)} \rho \frac{\partial u_s}{\partial t}\, ds + \rho L \frac{\partial u_s}{\partial t}. \end{aligned} \tag{2.4}$$

The first integral term along (1) to (2) vanishes becuase the velocity in the tank is assumed to be zero. The last integral along the outflow tube of length L is given because the flow is uniform in this region, thus $\dot{u}_s$ is position-indepdent. For the integral along (2) to (3), we must consider the area components of the flow. Into the tube, the tank area changes from A to A_0, with which $A_0 \ll A$. By the approximation $\frac{A_0}{A} \ll 1$, the total momentum along the streamline depicted by the second integral contributes very little to the momentum contribution from the fluid in the tube, and therefore we may assume $\int_{(2)}^{(3)} \rho \frac{\partial u_s}{\partial t}\, ds \simeq 0$.

Therefore, the momentum equation (2.3) reduces to the nonlinear time-dependent ordinary differential equation

$$L\frac{\partial u}{\partial t} + \frac{1}{2}u^2 = gh, \tag{2.5}$$

where the density ρ has been cancelled out on both sides of the expression. This ODE may be solved via the method of variable seperation, by which we obtain

$$\begin{aligned} \frac{gh - 0.5u^2}{L} &= \frac{du}{dt} \\ \implies \frac{L}{gh - u^2/2}\, du &= dt \end{aligned} \tag{2.6}$$

which can be integrated from the initial time 0, where $u(t = 0) = 0$ and the current velocity $u(t)$ at time t:

$$\begin{aligned} \int_0^t dy &= \int_0^u \frac{L}{gh - x^2/2}\, dx \\ \implies t &= 2L \int_0^u \frac{dx}{2gh - x^2} \\ q = \frac{ix}{2gh} \implies dq &= \frac{i}{2gh}\, dx \\ &= 2L \cdot \frac{2gh}{i} \int_0^{q(u)} \frac{dq}{2gh + 4g^2h^2q^2} \\ &= \frac{4Lgh}{i} \cdot \frac{1}{2gh} \int_0^{q(u)} \frac{dq}{1 + 2ghq^2} \\ &= \frac{4L}{i} \left[\arctan\left(\sqrt{2gh}q\right)\right] \Big|_0^{q(u)} \\ &= \frac{4L}{i} \arctan\left(\frac{iu}{\sqrt{2gh}}\right) \\ &= 2L\left[\log\left(1 + \frac{u}{\sqrt{2gh}}\right) - \log\left(1 - \frac{u}{\sqrt{2gh}}\right)\right], \end{aligned}$$

We may now solve for $u(t)$ by raising each side to the power of e:

$$\begin{aligned} e^{t/2L} &= \exp\left(\log\left(\frac{1 + u/\sqrt{2gh}}{1 - u/\sqrt{2gh}}\right)\right) \\ &= \frac{1 + u/\sqrt{2gh}}{1 - u/\sqrt{2gh}} \\ \implies \sqrt{2gh}u &= \frac{e^{t/2L} - 1}{e^{t/2L} + 1} = \tanh\left(\frac{t}{4L}\right) \end{aligned}$$

which therefore implies that $u(t) = \frac{1}{\sqrt{2gh}} \tanh\left(\frac{t}{4L}\right)$, which is the velocity of the fluid out of the tube.

Problem 3

In this problem, we define the crossflow over a semicircular structure in radial components as

$$u_r(r,\theta) = U_0\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{3.1}$$

$$u_\theta(r,\theta) = -U_0\left(1 + \frac{R^2}{r^2}\right)\sin\theta, \tag{3.2}$$

where R is the radius of the semicircle. The goal of this problem is to achive an appropriate location for a window at θ_i such that the force on the structure is minimized. Inuitively, since every pressure component acts on the structure in the $\hat{\boldsymbol{r}}$ direction, we expect the force acting on the structure to be purely in the $\hat{\boldsymbol{y}}$ direction, since the pressure components should cancel along the $\hat{\boldsymbol{x}}$ direction.

Along the surface of the structure, the flow magnitude is given by

$$[u(r=R,\theta)]^2 = 4U_0^2\sin^2\theta. \tag{3.3}$$

To determine the force components on the structure, we must consider the pressures along the streamlines on the surface of the structure whose reference points are in the $r\to\infty$ limit:

$$\frac{p_e}{\rho} + \frac{1}{2}[u(r=0,\theta)]^2 + gh = \frac{1}{2}U_0^2 + \frac{P_0}{\rho} \tag{3.4}$$

$$\frac{p_i}{\rho} + \frac{1}{2}[u(r=0,\theta=\theta_i)]^2 + gh = \frac{1}{2}U_0^2 + \frac{P_0}{\rho}. \tag{3.5}$$

The force on the structure is therefore the pressure difference between the inside cavity and the exterior of the structure, integrated over the entirety of the area of the structure:

$$\mathbf{F} = \int_0^L\int_0^\pi \hat{\boldsymbol{r}}[p_e - p_i]\,Rd\theta dz, \tag{3.6}$$

which acts radially due to the radial action of the pressure on the structure. From equation (3.3) and, using the fact that $\hat{\boldsymbol{r}} = \cos\theta\hat{\boldsymbol{x}} + \sin\theta\hat{\boldsymbol{y}}$, (3.5) becomes

$$\mathbf{F} = RL\hat{\boldsymbol{y}}\int_0^\pi [p_e - p_i]\sin\theta\,d\theta,$$

which follows since any integral from 0 to π of $\cos\theta$ vanishes by odd-function properties. Implementing equations (3.4), (3.5), we obtain

$$\mathbf{F} = RL\hat{y}\int_0^\pi [2\rho U_0^2(\sin^2\theta_i - \sin^2\theta) + \rho g R(\sin\theta_i - \sin\theta)]\sin\theta\,d\theta,$$

where equation (3.3) has also been used. Therefore, integrating, and using the integration facts

$$\int_0^\pi \sin\theta\,d\theta = 2, \qquad \int_0^\pi \sin^2\theta\,d\theta = \frac{\pi}{2}, \qquad \int_0^\pi \sin^3\theta\,d\theta = \frac{4}{3},$$

we have

$$\mathbf{F} = RL\hat{\boldsymbol{y}}\left[2\rho U_0^2\sin^2\theta_i\cdot 2 - 2\rho U_0^2\cdot\frac{4}{3} + \rho g R\sin\theta_i\cdot 2 - \rho g R\cdot\frac{\pi}{2}\right]. \tag{3.7}$$

This is the force on the structure depending on θ_i, and it may be positive $\hat{\mathbf{y}}$ (creating lift) or negative $\hat{\mathbf{y}}$ (pushing down), and hence is minimized when $|\mathbf{F}| = 0$. By doing this, a second-order quadratic is obtained in $\sin\theta_i$:

$$4U_0^2 \sin^2\theta_i + 2gR\sin\theta_i - \frac{8}{3}U_0^2 - \frac{\pi}{2}gR = 0. \tag{3.8}$$

Letting $x = \sin\theta_i$, and applying the quadratic formula on (3.8), a non-complex expression is obtained for x:

$$\begin{aligned} x &= \frac{-2gR \pm \sqrt{4g^2R^2 - 4(4U_0^2)(-8/3U_0^2 - \pi/2gR)}}{8U_0^2} \\ &= \frac{-2gR \pm \sqrt{4g^2R^2 + 128/3U_0^4 + 8\pi U_0^2 gR}}{8U_0^2}. \end{aligned}$$

To obtain a value of θ_i between 0 and π, then $\sin\theta_i > 0$, so we must take $'+'$ in the argument of x. By flow symmetry, the position of the window yields the same value of $\mathbf{F}$ regardless of being placed on the left or the right, thus it is ok to only take $\arcsin(x)$ instead of $\pi - \arcsin(x)$. Therefore, the optimal position of the window to minimize the force on the structure is given by

$$\theta_i = \arcsin\left[\frac{\sqrt{4g^2R^2 + 128/3\,U_0^2 + 8\pi U_0^2 gR} - 2gR}{8U_0^2}\right].$$

Problem 4

For this problem, we can first consider Bernoulli's equation to determine the speed of the flow out of the rear tubes. Note that, while neglecting the effects of gravity, and noting that the pressure is equivalent at all entries and exits, we obtain:

$$\begin{aligned}
&\frac{1}{2}u^2 + \frac{P_0}{\rho} = \frac{1}{2}u_1^2 + \frac{P_0}{\rho} = \frac{1}{2}u_2^2 + \frac{P_0}{\rho} \\
&\implies u = u_1 = u_2,
\end{aligned} \tag{4.1}$$

which is further verified by the flux condition $Au = \frac{1}{2}Au_1 + \frac{1}{2}Au_2$. We may hence move forward with the principle of momentum conservation on the control volume,

$$\frac{d}{dt}\int_{V^*} \rho\mathbf{u}\,dV + \oint_{A^*} \rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}})\,dA = \int_{V^*} \mathbf{F}_b\,dV + \oint_{A^*} \mathbf{F}_s\,dA. \tag{4.2}$$

Our flow is time-independent and is therefore steady, and there are no body forces acting on the control volume, since gravity is ignored. Let $\mathbf{F} = \oint_{A^*} \mathbf{F}_s\,dA$ be the total force acting on the surface of the structure. We therefore have that, from equation (4.2),

$$\begin{aligned}
\mathbf{F} &= \oint_{A^*} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dA \\
&= \int_A \rho u\hat{\mathbf{x}}(u\hat{\mathbf{x}}\cdot(-\hat{\mathbf{x}}))\,dA + \int_{A_1} \rho u\hat{\mathbf{y}}(u\hat{\mathbf{y}}\cdot\hat{\mathbf{y}})\,dA + \int_{A_2} \rho(-u\hat{\mathbf{y}})(-u\hat{\mathbf{y}}\cdot(-\hat{\mathbf{y}}))\,dA \\
&= \int_A \rho u\hat{\mathbf{x}}(u\hat{\mathbf{x}}\cdot(-\hat{\mathbf{x}}))\,dA \\
&= -A\rho u^2\hat{\mathbf{x}}
\end{aligned}$$

since the $\hat{\mathbf{y}}$ momentum components oppose each other, cancelling each other out. Integrating the flux in yields $-A\rho u^2\hat{\mathbf{x}} = \mathbf{F}$, which corresponds to the force required to keep the cart at a steady speed.

Problem 5

This last problem consists of a laminar boundary layer exerting drag on a fluid parallel to a wall. To analyze the problem for determining the integral expression for the drag force, we proceed with a control volume analysis, invoking both mass and momenutm conservation:

$$\frac{d}{dt}\int_{V^*}\rho\mathbf{u}\,dV + \oint_{A^*}\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dA = \frac{d}{dt}\int_{V^*}\mathbf{F}_b\,dV + \oint_{A^*}\mathbf{F}_s\,dA, \tag{5.1}$$

$$\frac{d}{dt}\int_{V^*}\rho\,dV + \oint_{A^*}\rho(\mathbf{u}\cdot\mathbf{n})\,dA = 0. \tag{5.2}$$

Note that the flow over the layer is steady, resulting is no time-dependent momentum or mass change. Then, by (5.2), we obtain the flux condition that mass in = mass out. While neglecting the effects of gravity on the flow, there are no body forces acting on the fluid, hence $\int_{V^*}\mathbf{F}_b\,dV = 0$. Lastly, by integrating the surface forces, we obtain the drag on the fluid. Therefore (5.1) and (5.2) become

$$\oint_{A^*}\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dA = \mathbf{F}_d \tag{5.3}$$

$$\oint_{A^*}\rho(\mathbf{u}\cdot\mathbf{n})\,dA = 0. \tag{5.4}$$

Consider first the momentum flux into and out of the control volume: sides h and δ, respectively, of base L and width D, bounded above by the top streamline as shown in the figure. Note that, since streamlines cannot cross, there can be no momentum (or mass) flux in or out of the top boundary. Hence we are only concerned with the momentum flux through the left and right hand sides of the volume. (5.3) yields

$$\begin{aligned}\mathbf{F}_d &= \int_0^h\int_0^D \rho U_0\hat{\mathbf{x}}(U_0\hat{\mathbf{x}}\cdot(-\hat{\mathbf{x}}))\,dydz + \int_0^\delta\int_0^D \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dydz \\ &= -\rho U_0^2 hD\hat{\mathbf{x}} + \int_0^\delta\int_0^D \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dydz. \end{aligned}\tag{5.5}$$

Now, considering the mass flux,

$$\begin{aligned} 0 &= \int_0^h\int_0^D \rho(U_0\hat{\mathbf{x}}\cdot(-\hat{\mathbf{x}}))\,dydz + \int_0^\delta\int_0^D \rho(\mathbf{u}\cdot\mathbf{n})\,dydz \\ &= -\rho U_0 hD + \int_0^\delta\int_0^D \rho(\mathbf{u}\cdot\mathbf{n})\,dydz \\ \Longrightarrow\ \rho U_0 hD &= \int_0^\delta\int_0^D \rho(\mathbf{u}\cdot\mathbf{n})\,dydz. \end{aligned}\tag{5.6}$$

The net drag force may be determined by then combining equation (5.6) into the first term in equation (5.5):

$$\begin{aligned}\mathbf{F}_d &= -U_0\hat{\mathbf{x}}\int_0^\delta\int_0^D \rho(\mathbf{u}\cdot\mathbf{n})\,dydz + \int_0^\delta\int_0^D \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dydz \\ &= \int_0^\delta\int_0^D \left\{-\rho u_L U_0\hat{\mathbf{x}} + \rho u_L\mathbf{u}_L\right\}\,dydz \end{aligned}$$

$$
\begin{aligned}
\implies \mathbf{F}_d &= -\hat{\mathbf{x}}\rho \int_0^\delta \int_0^D u_L \left(u_L - U_0\right)\, dydz \\
&= -\hat{\mathbf{x}}\rho D \int_0^\delta u_L \,\left(u_L - U_0\right)\, dz
\end{aligned}
$$

where $u_L = u(x = L, y, z) = u_L(z)$ is the unknown velocity distribution (in terms of z only) in the $\hat{\mathbf{x}}$ direction at the right hand side of the boundary layer at $x = L$, and it is assumed that $\mathbf{F}_d$ acts in the negative $\hat{\mathbf{x}}$ direction (dragging the fluid) on the lower surface. Note that if $u_L = U_0$, no drag acts on the fluid.