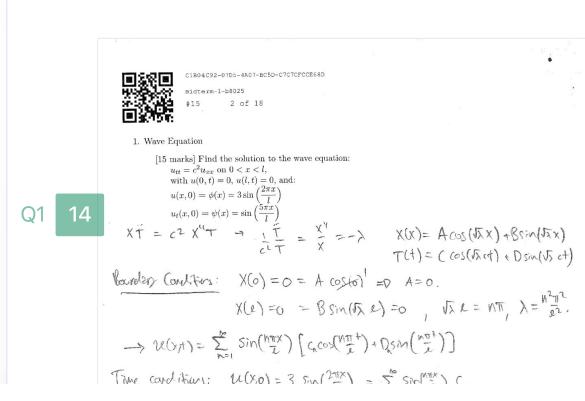
Crowdmark 2025-04-19, 5:19 PM

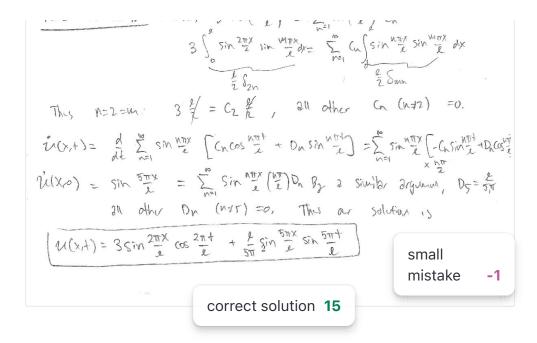
## Midterm 1

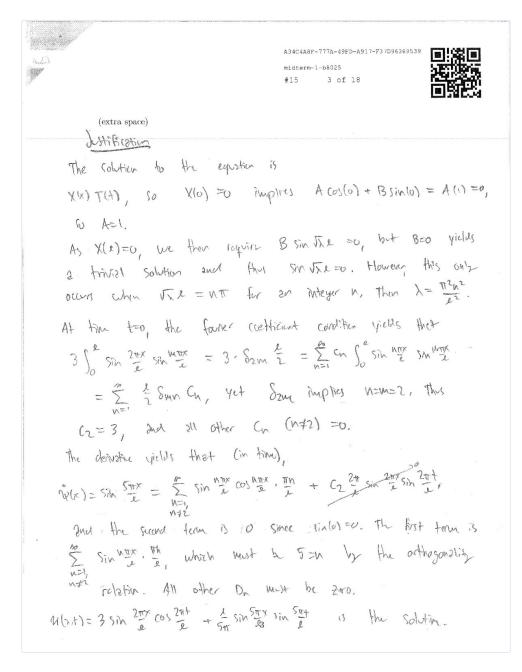


Class scores distribution Show

My score **86%** (86/100)









7C9BE9FA-BE5D-4F2B-96CC-7889F96576B7

midterm-1-b8025

#15 4 of 1

Q2 30

2. Separation of Variables

[30 marks] The **Laplace Equation** is given by  $u_{xx} + u_{yy} = 0$ , for u(x, y). Solve the laplace equation on the infinite rectangle:

$$S=\{0\leq x\leq l, 0\leq y<\infty\}$$

with the boundary conditions:

$$u(0, y) = 0 \quad u(l, y) = 0$$
$$\lim_{y \to \infty} u(x, y) = 0$$
$$u(x, 0) = \phi(x)$$

(Intuitively, we can view the variable y as the variable t.)

1. Using separation of variables u(x,y) = X(x)Y(y), define the separation constant with

$$\frac{X''}{X} = -\lambda$$

- 2. Recall for f(x), the solutions to the ODE f''(x) = kf(x) is given by  $f(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$
- 3. Apply the boundary conditions u(0,y)=0 and u(l,y)=0, to find  $\lambda$ .
- 4. Apply the boundary condition  $\lim_{y\to\infty} u(x,y) = 0$ .
- 5. Apply the boundary condition  $u(x,0) = \phi(x)$ , and write the solution u(x,y).

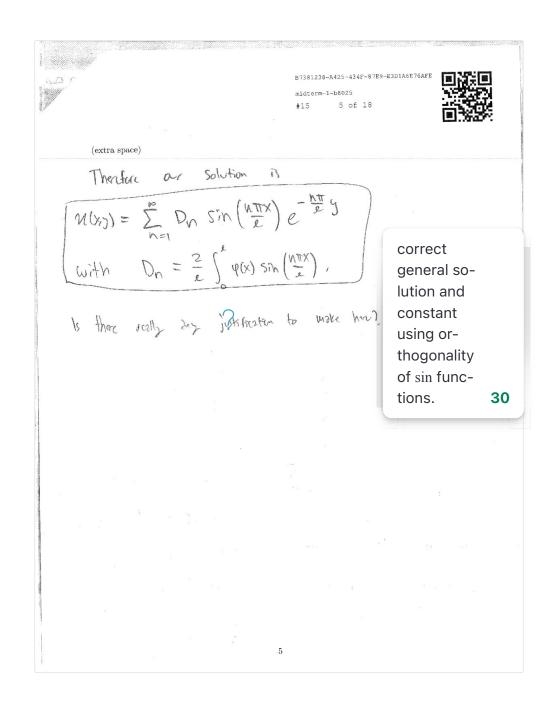
1) 
$$X''J + g''X = 0 \rightarrow \frac{X''}{X} = \frac{g''}{J} = -\lambda$$

2) 
$$X'' = -\lambda \times = 0$$
  $X(x) = A sin(\sqrt{x}x) + B cos(\sqrt{x}x)$  by solving the ODE.  
 $Y'' = \lambda y = 0$   $Y(y) = C e^{ixy} + D e^{-ixy}$ 

3) 
$$\chi(0) = BG = 0$$
 =  $0$   $B = 0$ .  $\chi(2) = 0 = ASIN(J\lambda 2)$   $\lambda = \frac{R^2 4^2}{R^2}$ 

t) We have that 
$$u(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{n} \left( \frac{1}{2} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{n} \right)$$
.

5) At y=0, 
$$N(x_3) = \varphi(x)$$
,  $2(x_1,0) = \varphi(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{x} D_n$ , belowing  $D_n$ :
$$\sum_{n=1}^{\infty} D_n \int_0^x \sin \frac{n\pi x}{x} dx = \int_0^x \varphi(x) \sin \frac{n\pi x}{x} dx = D_n \cdot \frac{\pi}{2} S_{mn}$$
. By Fourier orthogonality.









3. Schrödinger Equation

[20 marks] Consider a quantum particle in a box of length  $\frac{l}{2}$ , obeying the Schrödinger Equation:

 $i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}$   $0 < x < \frac{l}{2}$ 

Initially, its probability density is **uniformly distributed** in the box (of size  $\frac{l}{2}$ ). At time t = 0, the box suddenly doubled in size to length l.

Thus the domain has increased to 0 < x < l, while the initial condition is:

$$\Psi(x,0) = \phi(x) = \begin{cases} N & 0 < x < \frac{l}{2} \\ 0 & \frac{l}{2} < x < l \end{cases}$$

- 1. Find the normalization constant N. 2. Find the solution  $\Psi(x,t)=\sum_{n=0}^{\infty}A_n\sin\frac{n\pi x}{l}e^{\frac{-iE_nt}{\hbar}}$
- 3. Using Parseval's Equality, evaluate  $\sum_{n=1}^{\infty} \frac{1}{n^2} (1 \cos \frac{n\pi}{2})^2$

Explain how this series is different to the standard p-series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

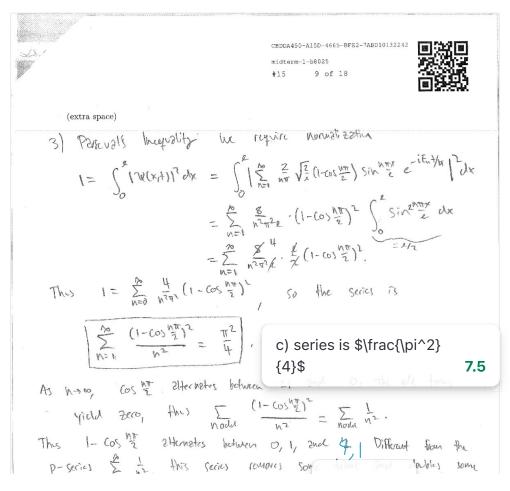
i.e. How is the extra term  $(1-\cos\frac{n\pi}{2})^2$  affecting the series?

Sinc series definition.

This is just:  $An = \frac{2}{L} \int_{0}^{L} N \sin \frac{\pi n}{L} dx$  (q(x) had defined for  $\frac{1}{L}(x, LR)$   $= \frac{2}{a} N \int_{0}^{R} \sin \frac{n\pi x}{L} = -\frac{2}{L} N_{HI}^{L} \cos(\frac{n\pi}{L}x) \Big|_{0}^{HZ} = -\frac{2}{n\pi} N_{I}^{L} (\cos(\frac{n\pi}{L}) - 1)$ 

This  $\sqrt{2(x+)} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sqrt{\frac{2}{x}} \left(1 - \cos\left(\frac{n\pi}{2}\right)\right) \sin\left(\frac{n\pi x}{x}\right) e^{-i\frac{x}{2}n\pi}$ 

b) correct value of \$A\_n\$ and solution 7.5



```
\int_{N=1}^{\infty} \frac{1}{n^2} (1-\cos(\frac{n\pi}{2}))^2 = \frac{1}{12} + \frac{2}{2^2} + \frac
```



D9A291CC-935P-4FC7-9BEE-36BZA3ZDB1D9
midterm-1-b8025
#15 11 of 18

(extra space)

10

11



E3B7945E-4A22-4D59-879E-179894EDCDAC

12 of 18

4. Fourier Series

[20 marks] Let f(x)=x with  $x\in(0,l)$ . Consider its Fourier Sine Series F(x), and its Fourier Cosine Series G(x), on  $x\in(0,l)$ ,

$$x=F(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\sin\left(\frac{n\pi x}{l}\right) \qquad x=G(x)=\frac{l}{2}+\sum_{\substack{n\text{ odd}}}^{\infty}\frac{-4l}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

For each of the above 2 series, explain whether we are able to take derivatives on both sides of the equation to attain the Fourier Series of f'(x).

If so, take derivatives on both sides, and find f'(x) with its Fourier Series on the appropriate domain. If not, explain why not.

For the F. sine series F(x), we way not take a derivative.

The series does not come a) arguing that derivative of series diverges

For the Coine series, we work obtain need to look at extensions of f(x) need to look at extensions of f(x) need to look at extensions of f(x) onto f(x) and think about if the a) arguing that derivative of series series becomes a diverges

For the Coine series, we have the converge of the converge of

& -40 MARCOMETER:

$$G'(x) = \sum_{n \text{ odd}} \frac{1}{n^2 \pi^2} \cdot (-\sin(\frac{n\pi}{x})) \cdot z = \sum_{n \text{ odd}} \frac{1}{n \pi} \cdot (-2).$$
We may check this: for  $f(x) = 1$  on  $(0, e)$ : its  $F$ . Since series is
$$A_n = \frac{2}{e} \int_0^e (1) \sin(\frac{n\pi}{x}) dx = \frac{2}{e} \cdot \frac{e}{n\pi} \cdot [-\cos(\frac{n\pi}{x})]_0^e = \frac{2}{n\pi} \cdot (-\cos(n\pi) + 1)$$
We for odd  $m$ , we have that  $\cos(m\pi) = -1$ , thu  $A_n = \frac{4}{n\pi}$ .
This is as obtained above therefore
$$1 = G'(x) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(\frac{n\pi}{x}) = \frac{1}{12} \text{ is the derivative of } G(x) \text{ and it}$$

b) derivative of \$G(x)\$ 4





33C352E8-A6F3-4B15-A0C2-D920A85B64B4
midterm-1-b8025
#15 15 of 18



## 5. Fourier Series

[15 marks] Consider real valued functions defined on [-l,l]. Define the **inner product** (or "dot product") between 2 functions f,g to be

$$\langle f,g \rangle = \int_{-l}^{l} f(x)g(x) \, dx$$
 (when this integral exists)

This in turn, gives,  $\|f\|,$  the  ${\bf norm}$  (or length) of a function f, as

$$\|f\|^2 = \langle f,f \rangle = \int_{-l}^l f(x)^2 \, dx \qquad \text{ (when this integral exists)}$$

This in turn, gives d(f,g), the "distance" between 2 functions f,g, as

$$d(f,g)^2 = \|f-g\|^2 = \int_{-l}^l \left(f(x) - g(x)\right)^2 dx \qquad \text{ (when this integral exists)}$$

This space of functions is typically called  $L^2$ , with the above distance (metric). (Each function f, is a "point" in this space of functions  $L^2$ .)

Take a function f in this space. We would like to investigate how the Fourier Series approximate the function f, in this distance (metric).

Consider a Fourier Series g(x) written in general form with coefficients  $a_k$ :

$$g(x) = \sum_{k=1}^{\infty} a_k X_k(x)$$

where  $X_k(x)$  are orthogonal functions:  $\langle X_k, X_m \rangle = 0$  if  $k \neq m$ .

1. Consider the first n terms of the series:

$$g_n(x) = a_1 X_1(x) + a_2 X_2(x) + \dots + a_n X_n(x) = \sum_{k=1}^n a_k X_k(x)$$

Consider the (squared) distance between f and  $g_n$ :

$$D_n = d(f, g_n)^2 = ||f - g_n||^2 = \langle f - g_n, f - g_n \rangle = \langle f - \sum_{k=1}^n a_k X_k, f - \sum_{m=1}^n a_m X_m \rangle$$

Expand and simplify this expression using the orthogonality relation.

Expand and simplify this expression using the energy  $a_k = \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle}$ . Choose the coefficients  $a_k = \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle}$ . Using the fact that  $0 \le D_n$ , show Bessel's Inequality:

$$\sum_{k=1}^n |a_k|^2 \int_{-l}^l |X_k(x)|^2 \leq \int_{-l}^l |f(x)|^2$$

15



3. Recall the definition of a series, given  $\{s_k\}$ :

$$\sum_{k=1}^{\infty} s_k = \lim_{n \to \infty} \sum_{k=1}^{n} s_k$$

We say the Fourier Series g(x) converges to f(x) in the  $L^2$  distance when:

$$\lim_{n\to\infty} D_n = 0$$

Show that if this is the case, then we attain Parseval's Equality:

$$\sum_{k=1}^{\infty} |a_k|^2 \int_{-l}^{l} |X_k(x)|^2 = \int_{-l}^{l} |f(x)|^2$$

 $D_{p} = \langle f - \sum_{k=1}^{N} 3k x_{k}(x) \rangle, f - \sum_{k=1}^{N} 3m x_{k}(x) \rangle = \langle f, f \rangle - \sum_{k=1}^{N} 3k \langle f, x_{k}(x) \rangle$   $= \langle f - \sum_{k=1}^{N} 3k x_{k}(x) \rangle$ 

The last step follows by 2 change in index since they 22

the same series, and the best form follows since 
$$\sum_{k=1}^{p}\sum_{m=1}^{p}a_ka_m\sum_{k=1}^{p}Nx_mN^2$$
, the dele function removes an early

16

Crowdmark

2025-04-19, 5:19 PM

