PHY454 Problem Set 4

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Problem 1

(a) For the first problem, we may consider the Navier-Stokes equation given in cylindrical coordinates. This was chosen due to the symmetry of the problem. For steady, viscous, incompressible flows along the z-axis, the equation reduces to the simple form

$$0 = -\frac{\partial P}{\partial z} + \mu \nabla^2 u_z(r) \tag{1.1}$$

which follows since $\frac{\partial u_z}{\partial t} = 0$ and $\mathbf{u} \cdot \nabla \mathbf{u} = u_r \frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z \frac{\partial u_z}{\partial z} = 0$. Equation (1.1) is a partial differential equation which may be solved with the two boundary conditions $u(R_1) = u(R_2) = 0$ (the z has been omitted for brevity) by the non-slip condition at the boundaries. In cylindrical coordinates, the Laplacian transforms to

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{1.2}$$

since we only wish to consider the r-component of the flow magnitude. We obtain

$$\mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) u(r) = \frac{\partial P}{\partial z}, \tag{1.3}$$

which is an ordinary differential equation. We first solve the homogeneous equation prompted by (1.3), which is $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) u(r) = 0$, which can be taken out by letting $p(r) = \frac{\partial u}{\partial r}$ and solving for $p(r)$. $u(r)$ is then obtain by integration of p. We have

$$\begin{aligned} \frac{d^2}{dr^2} u &= -\frac{1}{r}\frac{d}{dr} u \\ \implies \frac{dp}{dr} &= -\frac{p}{r} \\ \implies \int \frac{dp}{p} &= -\int \frac{dr}{r} \\ \implies p(r) &= \frac{1}{r} + C_1. \end{aligned} \tag{1.4}$$

Therefore $u(r) = C_1 \log(r) + C_2$, which is thus the solution to the homogeneous equation. To solve the inhomogeneous equation, we can taken out a series solution of the form $u(r) = \sum_n a_n r^n$. Substituting the series into (1.3) produces the relation

$$\mu \sum_n a_n n(n-1) r^{n-2} + \mu \sum_n a_n n r^{n-2} = \frac{\partial P}{\partial z} \tag{1.5}$$

which can only hold if the r^{n-2} terms vanish, establishing that $n = 2$ is the only term in the sum. Concatenating the sum terms for $n = 2$ yield the relation for a_2:

$$\mu a_2[(2)(2-1)+2] = 4\mu a_2 = \frac{\partial P}{\partial z} \implies a_2 = \frac{1}{4\mu}\frac{\partial P}{\partial z}. \tag{1.6}$$

We thus have obtained the solution to the inhomogeneous equation. Superposing the homogeneous and inhomogeneous equations allows us to completely solve the dynamics of the system according to the given non-slip conditions, so we arrive at the expression

$$u(r) = a\log(R) + b + \frac{1}{4\mu}\frac{\partial P}{\partial z}r^2. \tag{1.7}$$

$u(R_1) = 0$ and $u(R_2) = 0$ imply that

$$0 = a\log(R_1) + b + a_2R_1^2 \tag{1.8.1}$$
$$0 = a\log(R_2) + b + a_2R_2^2, \tag{1.8.2}$$

which is a system of two equations with two unknowns. By (1.8.1), we find that $b = -a_2R_1^2 - a\log(R_1)$, which into (1.8.2) yields that

$$0 = a\log(R_2) - a_2R_1^2 - a\log(R_1) + a_2R_2^2 = a\log(R_2/R_1) + a_2(R_2^2 - R_1^2). \tag{1.9}$$

We find that $a = \dfrac{a_2(R_2^2 - R_1^2)}{\log(R_2/R_1)}$, and therefore $b = -a_2R_1^2 - \dfrac{a_2(R_2^2 - R_1^2)}{\log(R_2/R_1)}\log(R_1)$. Therefore the solution to the Navier-Stokes equation for this system is

$$\begin{aligned}
u(r) &= \frac{a_2(R_2^2 - R_1^2)}{\log(R_2/R_1)}\log(r) - a_2R_1^2 - \frac{a_2(R_2^2 - R_1^2)}{\log(R_2/R_1)}\log(R_1) + a_2r^2 \\
&= a_2\left[r^2 - R_1^2 + \frac{(R_2^2 - R_1^2)}{\log(R_2/R_1)}(\log(r) - \log(R_1))\right] \\
&= a_2\left[r^2 - R_1^2 + \frac{(R_2^2 - R_1^2)}{\log(R_2/R_1)}\log(r/R_1)\right] \\
&= \frac{1}{4\mu}\frac{\partial P}{\partial z}\left[r^2 - R_1^2 + \frac{(R_2^2 - R_1^2)}{\log(R_2/R_1)}\log(r/R_1)\right] \qquad (1.10)
\end{aligned}$$

which is the solution as desired.

(b) The shear stress in the z-direction of the system is given by the stress tensor in cylindrical coordinates. Logically, the shear stress in the system corresponds to the S_{rz} component of the tensor, since we are examining the shear stress radially in the z direction. There should be no stress in the φ direction due to the symmetry of the problem. The cylindrical shear stress component is given by $S_{rz} = \mu\left(\dfrac{\partial u_z}{\partial r} + \dfrac{\partial u_r}{\partial z}\right)$, from which the latter term vanishes since $u_r = 0$ everywhere.

Since the maximum velocity of the fluid in the tube is given when $\left.\dfrac{\partial u}{\partial r}\right|_{r=r_0} = 0$, then it is easy to see that the shear stress at the maximum velocity positon $r = r_0$ is in fact zero. This value of r_0 can be found by differentiation of equation (1.10):

$$\frac{du}{dr} = \frac{1}{4\mu}\frac{\partial P}{\partial z}\left[2r + \frac{1}{r}\cdot\frac{R_2^2 - R_1^2}{\log(R_2/R_1)}\right] = 0$$

$$
\begin{aligned}
0 &= 2r_0 + \frac{1}{r_0}\frac{R_2^2 - R_1^2}{\log(R_2/R_1)} \\
\implies r_0 &= \sqrt{\frac{R_2^2 - R_1^2}{2\log(R_2/R_1)}} \qquad (1.11)
\end{aligned}
$$

where the '$\pm$' has been omitted due to radial symmetry. This is therefore the location of the maximum velocity of the fluid, and hence the location when the shear stress is a minimum. Note that, upon examination, r_0 does not occur at the center of the annulus. If it did, then $r_0 = \dfrac{R_2 - R_1}{2} + R_1$.

Problem 2

(**a**) For this problem, we consider the components of a velocity field in terms of an angular momentum vector $\mathbf{\Omega}$. Invoking the Levi-Civita tensor to calculate cross products, the expression of fluid flow can be re-written as

$$u_i = \epsilon_{ijk}\Omega_j x_k. \tag{2.1}$$

The derivative of such a quantity is also easy to determine, since the components of $\mathbf{\Omega}$ are constant, and $\dfrac{\partial x_i}{\partial x_k} = \delta_{ik}$. We have that

$$\frac{\partial u_i}{\partial x_n} = \epsilon_{ijk}\Omega_j \frac{\partial x_k}{\partial x_n} = \epsilon_{ijn}\Omega_j = -\epsilon_{inj}\Omega_j. \tag{2.2}$$

An immediate result from this differentiation is that $\dfrac{\partial u_m}{\partial x_m} = 0$, since a repeated index appears in the Levi-Civita permutation. This implies that the fluid is divergenceless $\nabla \cdot \mathbf{u} = 0$. The stress tensor is given as

$$T_{ij} = -p\delta_{ij} + \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \mu_v \delta_{ij}\frac{\partial u_k}{\partial x_k}, \tag{2.3}$$

which reduces to

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.4}$$

when the divergence properties of $\mathbf{u}$ are invoked. Furthermore, the shear stress terms can also be determined by substituting in (2.2):

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \mu\left(-\epsilon_{ijk}\Omega_k + (-\epsilon_{jik}\Omega_k)\right) \\ &= -p\delta_{ij} + \mu\left(\epsilon_{jik}\Omega_k - \epsilon_{jik}\Omega_k\right) \\ &= -p\delta_{ij}. \end{aligned} \tag{2.5}$$

Therefore T_{ij} is just a unit pressure tensor of normal stresses to a fluid particle. $T_{11} = T_{22} = T_{33} = -p$, and $T_{ij} = 0$ for $i \neq j$.

(**b**) For an angular momentum vector with components $\Omega_1 = \Omega_2 = 0$, $\Omega_3 \neq 0$, the fluid motion (according to equation (2.1)) becomes simply

$$(u_1, u_2) = (-\Omega_3 x_2, \Omega_3 x_1). \tag{2.6}$$

We may now consider the Cauchy momentum equation. In the absence of body forces (such as gravity) acting on the fluid, the equation reduces to

$$\rho\frac{D\mathbf{u}}{Dt} = \nabla \cdot \mathbf{T}. \tag{2.7}$$

Since the fluid has no time dependence, one may expand the particle derivative term on the left hand side of (2.6) to obtain

$$\rho\left(u_1\frac{\partial u_i}{\partial x_1} + u_2\frac{\partial u_i}{\partial x_2} + u_3\frac{\partial u_i}{\partial x_3}\right) = \frac{\partial T_{ij}}{\partial x_i}, \tag{2.8}$$

where the equation has been generalized to the i-th component of fluid flow. Since the stress tensor T_{ij} is strictly pressure-dependent, and the pressure is in terms of radius $p = p(r) = p(x_1^2 + x_2^2)$ (with

$p(0) = p_0$), the application of the chain rule to the right hand side expands the equation in terms of p:

$$\rho\left(u_1\frac{\partial u_i}{\partial x_1} + u_2\frac{\partial u_i}{\partial x_2} + u_3\frac{\partial u_i}{\partial x_3}\right) = \frac{\partial T_{ij}}{\partial p}\frac{\partial p}{\partial r}\frac{\partial r}{\partial x_i}. \tag{2.9}$$

It is easy to observe that $\dfrac{\partial T_{ij}}{\partial p} = -\delta_{ij}$ by equation (2.5). Since $u_3 = 0$, we may now explicity determine an expression for $p(r)$. For $i = 1$, the derivative term $\dfrac{\partial u_1}{\partial x_1} = 0$ since u_1 is x_1-independent. We have:

$$\begin{aligned}
\rho\Omega_3 x_1(-\Omega_3) &= -\delta_{1j}\frac{\partial p}{\partial r}\cdot\frac{\partial}{\partial x_1}(x_1^2 + x_2^2)\\
\implies -\rho x_1\Omega_3^2 &= -(2x_1\delta_{11} + 2x_2\delta_{12})\frac{\partial p}{\partial r}\\
\implies \rho x_1\Omega_3^2 &= 2x_1\frac{\partial p}{\partial r}\\
\implies \frac{\partial p}{\partial r} &= \frac{1}{2}\rho\Omega_3^2.
\end{aligned} \tag{2.10}$$

Integration of (2.10) yields that $p(r) = \dfrac{1}{2}\rho\Omega_3^2 r + C_1$, by which $p(0) = C_1 = p_0$. One may quickly find that the equivalent solution can be determined for $i = 2$ as well:

$$\begin{aligned}
\rho\Omega_3 x_2(-\Omega_3) &= -\delta_{2j}\frac{\partial p}{\partial r}\cdot\frac{\partial}{\partial x_2}(x_1^2 + x_2^2)\\
\implies -\rho x_2\Omega_3^2 &= -(2x_1\delta_{21} + 2x_2\delta_{22})\frac{\partial p}{\partial r}\\
\implies \rho x_2\Omega_3^2 &= 2x_2\frac{\partial p}{\partial r}\\
\implies \frac{\partial p}{\partial r} &= \frac{1}{2}\rho\Omega_3^2.
\end{aligned} \tag{2.11}$$

The pressure in terms of the radial distance away from the origin is therefore $p(r) = \dfrac{1}{2}\rho\Omega_3^2 r + p_0$, which is linear. The physical interpretation of this solution, according to the Cauchy momentum equation, is that the pressure in a rotating fluid increases, which is as expected for rotating fluids. If a fluid was rotating a constant angular velocity $\Omega_3\hat{\boldsymbol{z}}$ such that the radial component of the flow is constant for each radius $r = R$, then one would expect that there must exist a force or pressure keeping the fluid accelerating to the center of the rotation axis. As the radius increases, the pressure must increase because the fluid is moving faster. A classic example of such a phenomenon is Newton's rotating bucket problem.

Problem 3

(**a**) In this problem we consider a cylindrical flow in the $\hat{\boldsymbol{\varphi}}$ direction, $\mathbf{u} = u(r)\hat{\boldsymbol{\varphi}}$. Since the fluid is rotation about the z-axis in the φ-direction, the only stress tensor component contributing to the fluid motion would be the $S_{r\varphi}$ component (the component of stress in the φ direction, radially outward). The expression for $S_{r\varphi}$ is given by the equation sheet:

$$\begin{aligned} S_{r\varphi} &= \mu\left(\frac{\partial u_\varphi}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r}\right) \\ &= \mu\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right). \end{aligned} \tag{3.1}$$

We wish to simplify the terms in bracket in equation (3.1). This can be done by noting that, by factoring out an r, establishes a relation which appears to have been taken out by the product rule (the φ index has been omitted for brevity):

$$\begin{aligned} S_{r\varphi} &= \mu \cdot r\left(\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) \\ &= \mu r\left(\frac{1}{r}\frac{\partial u}{\partial r} + u\frac{\partial}{\partial r}\left[\frac{1}{r}\right]\right) \\ &= \mu r\frac{\partial}{\partial r}\left[\frac{u}{r}\right]. \end{aligned} \tag{3.2}$$

We may now consider the action of the shear stress in terms of force. Since force is equivalent to the product of pressure and area, we have that a differential force component is in fact

$$d\mathbf{f} = S_{r\varphi}\, da\, \hat{\boldsymbol{\varphi}}. \tag{3.3}$$

The radial area element da is simply $rd\varphi dz$, and $S_{r\varphi}$ is given by equation (3.2). We therefore obtain that the shear force acting on the fluid is only in the φ-direction,

$$d\mathbf{f} = \mu r^2\left[\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right] d\varphi dz\, \hat{\boldsymbol{\varphi}}, \tag{3.4}$$

which is what I wanted to show.

(**b**) Since the force due to the viscous stress has been determined, we may not consider the total torque exerted on the fluid by such a force. For a cylinder of length L, one may integrate (3.4) around one ring of the cylinder to obtain the total force on one ring:

$$\begin{aligned} \mathbf{f} &= \mu r^2\left[\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right]\int_0^L dz\int_0^{2\pi} d\varphi \\ &= 2\pi\mu L r^2\left[\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right]. \end{aligned} \tag{3.5}$$

Torque is defined as $\mathbf{M} = \mathbf{r} \times \mathbf{f}$. Since the radius vector $\mathbf{r}$ and the shear forcing of the fluid $\mathbf{f}$ are perpendicular to each other, the magnitude of the torque simply reduces to $M = rf$, which upon substitution of (3.5), becomes

$$M = 2\pi\mu L r^3\left[\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right] \tag{3.6}$$

which is what I wanted to show.

(**c**) Since the outer cylinder rotates at a constant angular frequency and the inner cylinder is 'fixed' in position (by the application of a torque in the opposite direction), it must be that the torque expression for M in equation (3.6) is indeed constant. The non-slip condition of the fluid motion along the inner and outer cylinders indicate the boundary conditions $u(R_1) = u(R_2) = 0$. We now strive to solve the differential equation given in (3.6). First consider the homogeneous solution to $\frac{d}{dr}\left[\frac{u}{r}\right] = 0$ (the φ index has been omitted for brevity).

It is clear that any function $u(r) = ar$ satisfies this expression, and we may now consider the inhomogeneous differential equation. Consider a series solution $u(r) = \sum_n a_n r^n$, which yields the relationship

$$\begin{aligned} M &= 2\pi\mu L r^3 \frac{d}{dr}\sum_n a_n r^{n-1} \\ &= 2\pi\mu L \sum_n a_n (n-1) r^{n+1} \end{aligned}$$

which can only be satisfied for $n = -1$:

$$M = -4\pi\mu L a_{-1} \implies a_{-1} = -\frac{M}{4\pi\mu L}. \tag{3.7}$$

The contenation of the series solution for the inhomogeneous equation to the homogeneous equation solution yields the general solution to the equation,

$$u(r) = ar - \frac{M}{4\pi\mu L}\frac{1}{r}. \tag{3.8}$$

Invoking boundary conditions,

$$\begin{aligned} u(R_1) = 0 &= aR_1 - \frac{M}{4\pi\mu L R_1} \\ \implies a &= \frac{M}{4\pi\mu L R_1^2} \\ u(R_2) = R_2\Omega &= aR_2 - \frac{M}{4\pi\mu L R_2} \\ u(R_2) = R_2\Omega &= \frac{MR_2}{4\pi\mu L R_1^2} - \frac{M}{4\pi\mu L R_2} \\ &= \frac{M}{4\pi\mu L}\left(\frac{R_2}{R_1^2} - \frac{1}{R_2}\right) \\ \implies M &= 4\pi\mu L\Omega\frac{R_1^2 R_2^2}{R_2^2 - R_1^2} && (3.9) \\ \implies a &= \frac{4\pi\mu L\Omega}{4\pi\mu L R_1^2}\frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \\ &= \frac{\Omega R_2^2}{R_2^2 - R_1^2}. && (3.10) \\ \implies a_{-1} &= -\Omega\frac{R_1^2 R_2^2}{R_2^2 - R_1^2} && (3.11) \end{aligned}$$

Therefore the particular solution to the differential equation is

$$u_\varphi(r) = \Omega \frac{R_2^2}{R_2^2 - R_1^2}\, r - \Omega \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \cdot \frac{1}{r}$$

which simplifies to

$$u_\varphi(r) = \frac{\Omega R_2^2}{R_2^2 - R_1^2} \left(r - \frac{R_1^2}{r} \right), \tag{3.12}$$

which matches all boundary conditions, as desired. This is therefore the fluid velocity inside the cylinders, only dependent on Ω, R_1, R_2 and r.

Problem 4

(**a**) Consider a flow determined by the streamfunction $\psi = \alpha xy$ for $\alpha \in \mathbb{R}^{>0}$. The streamfunction characterizes the flow, which may be found by taking the derivates $u_x = \dfrac{\partial \psi}{\partial y}$, $u_y = -\dfrac{\partial \psi}{\partial x}$. This implies that $\mathbf{u} = (\alpha x, -\alpha y)$.

To determine the strain rate tensor components, first note that $S_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$. The components may then be calculated explicitly:

$$\begin{aligned}
S_1 1 &= \frac{\partial u_x}{\partial x} \\
&= \alpha \\
S_{12} = S_{21} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
&= 0 \\
S_{22} &= \frac{\partial u_y}{\partial y} \\
&= -\alpha.
\end{aligned}$$

The tensor may be expressed in terms of a matrix: $\mathbf{S} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$. Following a similar calculation, the antisymmetric rotation tensor may also be determined for the flow, $R_{ij} = \dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}$:

$$\begin{aligned}
R_1 1 &= 0 \\
R_{12} = -R_{21} &= \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \\
&= 0 \\
R_{22} &= 0
\end{aligned}$$

which is equivalently the zero matrix. Therefore $\mathbf{R} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so there is no rotation in the flow, and only elongation and compression of a fluid element along its principle axes.

(**b**) The pressure fields may now be determined by solving the Navier-Stokes equation in cartesian coordinates. In the absence of gravity, the individual components of the equation break up into

$$\rho\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}\right) u_x = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_x \tag{4.1}$$

$$\rho\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}\right) u_y = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_y. \tag{4.2}$$

Substituting the expressions for u_x and u_y found in the previous part, equations (4.1) and (4.2) reduce to

$$\begin{aligned}
\rho \alpha x \frac{\partial}{\partial x}(\alpha x) &= -\nabla P + \mu \frac{\partial^2}{\partial x^2}(\alpha x) \\
\implies \rho \alpha^2 x &= -\frac{\partial P}{\partial x}
\end{aligned} \tag{4.3}$$

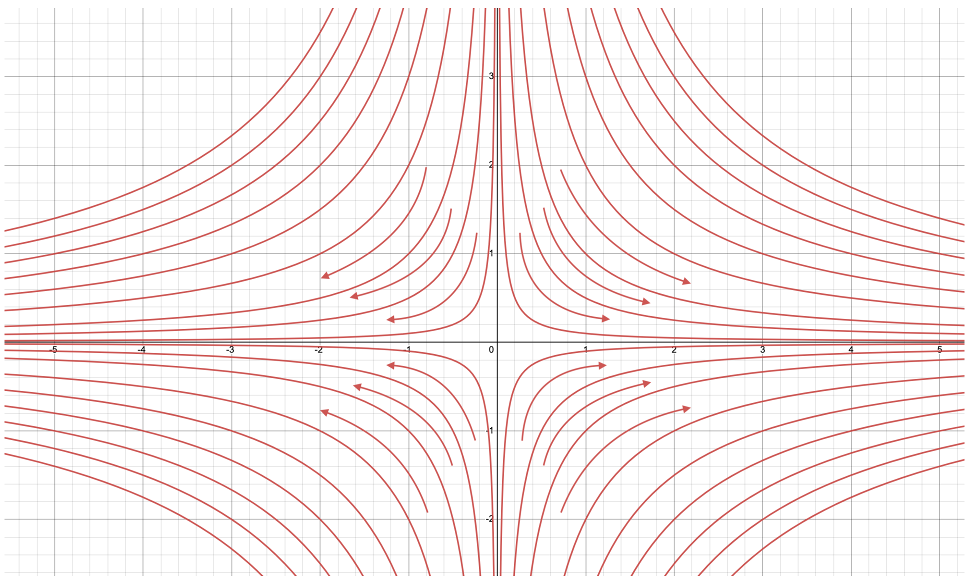
$$-\rho\alpha y \frac{\partial}{\partial x}(-\alpha y) = -\nabla P + \mu \frac{\partial^2}{\partial y^2}(-\alpha y)$$
$$\implies \rho\alpha^2 y = -\frac{\partial P}{\partial y}. \tag{4.4}$$

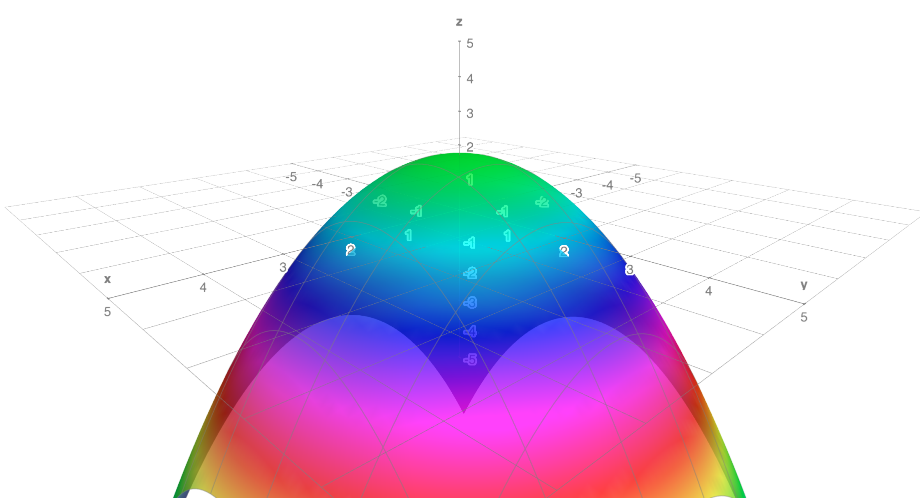
Integrating equations (4.3) and (4.4) yields an expression for the spatial pressure function,

$$P(x,y) = -\frac{1}{2}\rho\alpha^2(x^2+y^2) + P_0, \tag{4.5}$$

where P_0 is the reference pressure at the origin of the system. The pressure expression in (4.5) is equivalently the scalar pressure field in the flow, as desired.

(c) For $\alpha = 1$, the corresponding streamlines and pressure field may be plotted. The streamlines were plotted using Desmos, for the lines $y = \dfrac{C}{x}$ for $C \in \mathbb{R}$. The pressure field was plotted using Math3D, with the assumption that $\rho = 1$ and $P_0 = 1.7$.





It is crucial to note the direction of the streamlines. For $\alpha = 1$, at $(x,y) = (1,1)$ for instance, the flow has direction $\mathbf{u}(1,1) = (1,-1)$, and therefore by symmetry, the flow moves outward from the y-axis and towards the x-axis.

(d) Now, consider the relationship between the streamlines and flow to the pressure field. We may directly employ Bernoulli's equation, which relate the flow magnitude to the pressure difference. Explicitly, $u_1^2 - u_2^2 = \dfrac{2}{\rho}(P_2 - P_1)$, so if $u_2 > u_1, P_2 > P_1$. This means that as the flow approaches the origin, the pressure due to the flow increases because the flow speed is slower. As the flow magnitude approaches $\pm\infty$, the pressure drops according to Bernoulli's equation. This is why the pressure field attains a maximum at the origin, since this is the location where the flow magnitude is the lowest.
The flow magnitude is given by $|\mathbf{u}(x,y)| = \alpha\sqrt{x^2+y^2}$, which can be seen to have direct inverse overlap with the pressure field. This is how the pressure and flow fields are related.

Problem 5

(**a**) For this problem, we will consider the dimensionless form of the Navier-Stokes equation:

$$\mathrm{St}\cdot\frac{\partial\mathbf{u}^*}{\partial t^*}+(\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^*=-\nabla^*P^*+\frac{1}{\mathrm{Fr}^2}\mathbf{g}^*+\frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^* \tag{5.1}$$

where

$$\mathrm{St}=\frac{L\Omega}{U}\propto\frac{\text{unsteady acceleration}}{\text{advective acceleration}}$$
$$\mathrm{Fr}=\frac{U}{\sqrt{gL}}\propto\left(\frac{\text{inertial force}}{\text{gravitational force}}\right)^{1/2}$$
$$\mathrm{Re}=\frac{\rho UL}{\mu}\propto\frac{\text{inertial force}}{\text{viscous force}}$$
$$\nabla^*=\nabla L,\quad x_i^*=x_i/L,\quad u_i^*=u_i/U,\quad t^*=\Omega t,\quad g_i^*=g_i/g,\quad P^*=(P-P_\infty)/(\rho U^2). \tag{5.2}$$

In class, we examined the dimensional form of (5.1) for when the Reynold's $\mathrm{Re}\to\infty$, which correlates to highly turbulent flow. In such a case, the viscous term has minimal contribution to the resulting forces, and therefore may be neglected. Now, consider the limit for when $\mathrm{Re}\to 0$. If we multiply (5.1) through by Re, we find that

$$\mathrm{St}\cdot\mathrm{Re}\cdot\frac{\partial\mathbf{u}^*}{\partial t^*}+\mathrm{Re}\,(\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^*=-\mathrm{Re}\,\nabla^*P^*+\frac{\mathrm{Re}}{\mathrm{Fr}^2}\,\mathbf{g}^*+\nabla^{*2}\mathbf{u}^*. \tag{5.3}$$

Note that, upon taking the viscous force to infinity, we have that the fluid becomes more and more incompressible. This implies that the advective acceleration force approaches zero, which means that the St·Re term does not vanish ($\mathrm{St}\to\infty$ and $\mathrm{Re}\to 0$), while the advective acceleration term $(\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^*\to 0$. The pressure gradient term vanishes by taking $\mathrm{Re}\to 0$, and thus so does any gravitational contribution. The dimensionless equation may then be appropriately approximated to

$$\mathrm{St}\cdot\frac{\partial\mathbf{u}^*}{\partial t^*}=\frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^*. \tag{5.4}$$

We may then re-dimensionalize the equation according to the relations given in (5.2):

$$\begin{aligned}\frac{L\Omega}{U}\cdot\frac{1}{U\Omega}\cdot\frac{\partial\mathbf{u}}{\partial t}&=\frac{\mu}{\rho UL}\frac{L^2}{U}\nabla^2\mathbf{u}\\ \implies\ \rho\frac{\partial\mathbf{u}}{\partial t}&=\mu\nabla^2\mathbf{u}.\end{aligned} \tag{5.5}$$

This is the dimensionalized form of the Navier Stokes equation for when viscous forces are most prominent, and this is as expected because the flow velocity evolves strictly according to the viscosity term $\mu\nabla^2\mathbf{u}$.

(**b**) Now, consider a viscous-dependent pressure non-dimensionalization. The mechanical pressure is given by ρU^2, however by re-evaluation of the dimensionless pressure in terms of a viscous stress $\mu U/L$, we obtain $P^*=\dfrac{L(P-P_\infty)}{\mu U}$.

We may now carry out a non-dimensionalization of the Navier-Stokes equation with the new dimensionless parameters:

$$\rho U\Omega\frac{\partial\mathbf{u}^*}{\partial t^*}+\rho U^2L^{-1}(\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^*=-\mu UL^{-2}\nabla^*P^*+\mu UL^{-2}\nabla^{*2}\mathbf{u}^*+\rho g\mathbf{g}^*. \tag{5.6}$$

Multiplying through by $\dfrac{L}{\rho U^2}$ returns familiar dimensionless variables:

$$
\begin{aligned}
\frac{L\Omega}{U}\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* &= -\frac{\mu}{\rho U L}({\nabla^*}^2\mathbf{u}^* - \nabla^* P^*) + \frac{Lg}{U^2}\mathbf{g} \\
\implies \mathsf{St}\cdot\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* &= -\frac{1}{\mathrm{Re}}({\nabla^*}^2\mathbf{u}^* - \nabla^* P^*) + \frac{1}{\mathrm{Fr}^2}\mathbf{g}^*. \qquad (5.7)
\end{aligned}
$$

Equation (5.7) is the dimensionless form of the Navier-Stokes equation, with the assumption that the mechanical pressure is viscosity-dependent. The primary result due to this assumption is the dependence of the dimensionless pressure gradient on Reynold's number.

(c) Applying the exact same arguments made in part (a), one may observe that the advective acceleration term vanishes along with the gravitational forcing term as $\mathrm{Re} \to 0$. As a result, the dimensionless equation reduces to a strict pressure and viscosity related equation of motion:

$$
\mathsf{St}\cdot\frac{\partial \mathbf{u}^*}{\partial t^*} = -\frac{1}{\mathrm{Re}}({\nabla^*}^2\mathbf{u}^* - \nabla^* P^*). \qquad (5.8)
$$

Undoing the dimensionless substitution reveals the equation of motion when the visosity term is dominant:

$$
\begin{aligned}
\frac{\rho U L}{\mu}\frac{\Omega L}{U}\frac{\partial \mathbf{u}}{\partial t} &= \frac{L^2}{U}\nabla^2\mathbf{u} - \frac{L^2}{\mu U}\nabla(P - P_\infty) \\
\implies \rho\frac{\partial \mathbf{u}}{\partial t} &= \mu\nabla^2\mathbf{u} - \nabla P. \qquad (5.9)
\end{aligned}
$$

The significant difference between equations (5.9) and (5.5) is that the pressure term was non-dimensionalized with a visous stress instead of a mechanical pressure. The interpretation of this is that pressure is purley kinematic when non-dimensionalized with a mechanical pressure and can exist everywhere in space. When flow pressures are viscous-generated, we therefore wish to nondimensionalize the pressure with a viscosity-dependent stress. This exists when the flow is unsteady, which is why the pressure term appears on the right hand side of (5.9) along with the viscous term. An instance of this appears in the Cauchy stress tensor, when comparing the diagonal stress terms due to dynamic viscosity μ_v:

$$
T_{ij} = \underbrace{-p\delta_{ij}}_{\text{normal pressures}} + \mu\underbrace{\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]}_{\text{viscous shear stresses and strains}} + \underbrace{\mu_v\delta_{ij}\frac{\partial u_k}{\partial x_k}.}_{\text{pressures due to dynamic viscosity}} \qquad (5.10)
$$

Therefore equation (5.9) physically makes sense when accounting for pressure gradients generated by viscous fluid motion. It is therefore assumed that the fluid is compressible, since $\dfrac{\partial u_k}{\partial x_k} \neq 0$ in the dynamic viscosity pressure term.