PHY454 Problem Set 5

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Problem 1

(a) We strive to prove the expression

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) \tag{1.1}$$

using index notation. First, consider the index expansion of each of the three terms:

$$\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u} &= (u_i\partial_j\,\mathbf{e_i}\cdot\mathbf{e_j})u_n\,\mathbf{e_n} \\
&= (u_i\partial_j\delta_{ij})u_n\,\mathbf{e_n} \\
&= u_i[\partial_i u_n]\,\mathbf{e_n} && (1.2)\\
(\nabla\times\mathbf{u})\times\mathbf{u} &= (\epsilon_{ijk}\partial_j u_k\,\mathbf{e_i})\times\mathbf{u} \\
&= (\epsilon_{ijk}[\partial_j u_k])\,\epsilon_{nim}u_m\,\mathbf{e_n} \\
&= \epsilon_{jki}\epsilon_{mni}\,[\partial_j u_k]u_m\,\mathbf{e_n} \\
&= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})[\partial_j u_k]u_m\,\mathbf{e_n} \\
&= u_m[\partial_m u_n]\,\mathbf{e_n} - u_m[\partial_n u_m]\,\mathbf{e_n} && (1.3)\\
\frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) &= \frac{1}{2}\nabla(u_i u_j\,\mathbf{e_i}\cdot\mathbf{e_j}) \\
&= \frac{1}{2}[\partial_n u_i u_j]\,\delta_{ij}\mathbf{e_n} \\
&= \frac{1}{2}\left([\partial_n u_i]u_j + [\partial_n u_j]u_i\right)\,\delta_{ij}\mathbf{e_n} \\
&= \frac{1}{2}\left(2[\partial_n u_i]u_i\right)\,\mathbf{e_n} \\
&= u_i[\partial_n u_i]\,\mathbf{e_n}. && (1.4)
\end{aligned}$$

Substituting these equations back into (1.1) and exchanging indices so that all unit vectors are identical, we obtain that

$$u_i[\partial_i u_n]\,\mathbf{e_n} = u_m[\partial_m u_n]\,\mathbf{e_n} - u_m[\partial_n u_m]\,\mathbf{e_n} + u_i[\partial_n u_i]\,\mathbf{e_n}. \tag{1.5}$$

By the summation linearity, we may exchange the m index into the i index, hence yielding that

$$u_i[\partial_i u_n]\,\mathbf{e_n} = u_i[\partial_i u_n]\,\mathbf{e_n} - u_i[\partial_n u_i]\,\mathbf{e_n} + u_i[\partial_n u_i]\,\mathbf{e_n}. \tag{1.6}$$

It is easy to see, upon rearranging, both side equate to zero:

$$\begin{aligned}
u_i[\partial_i u_n]\,\mathbf{e_n} - u_i[\partial_i u_n]\,\mathbf{e_n} &= u_i[\partial_n u_i]\,\mathbf{e_n} - u_i[\partial_n u_i]\,\mathbf{e_n} \\
\implies 0 &= 0,
\end{aligned}$$

and therefore equation (1.1) holds true, which is what I wanted to prove.

(**b**) Consider now the Navier-Stokes for a fluid with constant density and viscosity in a gravitational field:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{g} + \nu \nabla^2 \mathbf{u}. \tag{1.7}$$

We desire to take the curl of this equation to determine the vorticity equation. First, allow me to prove some intermediate identities which I will use in the derivation. First, the triple curl product:

$$\begin{aligned}
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \nabla \times (\epsilon_{ijk} a_j b_k \, \mathbf{e_i}) \\
&= \epsilon_{nmi}[\partial_m \epsilon_{ijk} a_j b_k] \, \mathbf{e_n} \\
&= \epsilon_n mi \epsilon_j ki [\partial_m a_j b_k] \, \mathbf{e_n} \\
&= [\delta_{nj}\delta_{mk} - \delta_{nk}\delta_{jm}] \partial_m a_j b_k \, \mathbf{e_n} \\
&= [\partial_k a_n b_k] \, \mathbf{e_n} - [\partial_j a_j b_n] \, \mathbf{e_n} \\
&= ([\partial_k a_n] b_k + [\partial_k b_k] a_n - [\partial_j a_j] b_n - [\partial_j a_n] b_j) \; \mathbf{e_n} \\
&= (\mathbf{b} \cdot \nabla)\mathbf{a} + (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \nabla)\mathbf{b}.
\end{aligned} \tag{1.8}$$

Second, the curl of a Laplacian:

$$\begin{aligned}
\nabla \times (\nabla^2 \mathbf{a}) &= \nabla \times (\partial_i^2 a_n \, \mathbf{e_n}) \\
&= \epsilon_{kmn} \partial_m (\partial_i^2 a_n) \, \mathbf{e_k} \\
&= \partial_i^2 (\epsilon_{kmn} \partial_m a_n \, \mathbf{e_k}) \\
&= \nabla^2 (\nabla \times \mathbf{a}).
\end{aligned} \tag{1.9}$$

Now, taking the curl of equation (1.7),

$$\nabla \times \frac{\partial \mathbf{u}}{\partial t} + \nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) = -\frac{1}{\rho} \nabla \times (\nabla P) + \nabla \times \mathbf{g} + \nu \nabla \times (\nabla^2 \mathbf{u}). \tag{1.10}$$

Note first that the curl of any gradient is zero, which therefore makes $\nabla \times (\nabla P) = 0$ and $\nabla \times \mathbf{g} = 0$, since gravity may be expressed in terms of a gravitational potential $\mathbf{g} = -\nabla \Phi$. In the first term, we may note that the spatial derivatives are independent of any temporal derivatives, and thus $\nabla \times \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \mathbf{u})$. From part (a) of this problem, we may re-write $(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u})$, which makes equation (1.10) of the form

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) + \nabla \times \left[(\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) \right] = \nu \nabla \times (\nabla^2 \mathbf{u}). \tag{1.11}$$

By a similar argument, the curl of the second term in the square braces vanishes because it is expressed as a gradient. Therefore

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) + \nabla \times [(\nabla \times \mathbf{u}) \times \mathbf{u}] = \nu \nabla \times (\nabla^2 \mathbf{u}). \tag{1.12}$$

Then, noting that $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, and by invoking relations (1.8) and (1.9), we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + (\nabla \cdot \mathbf{u})\boldsymbol{\omega} - (\nabla \cdot \boldsymbol{\omega})\mathbf{u} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \boldsymbol{\omega}. \tag{1.13}$$

It follows that $\nabla \cdot \mathbf{u} = 0$ by mass conservation in the Navier-Stokes equations, and that $\nabla \cdot \boldsymbol{\omega} = 0$ because the divergence of any curl vanishes. This vastly reduces the vorticity equation into the form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \boldsymbol{\omega}. \tag{1.14}$$

Rearranging the equation then yields the desired result,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{1.15}$$

Problem 2

In this problem, we consider a two-dimensional flow defined by $\mathbf{u}(x, y) = (u, v)(x, y)$. Under these circumstances, in three-dimensional space, there is no z-component to the flow. This should imply firstoff that the vorticity vector is only defined in the $\hat{\mathbf{z}}$ direction:

$$\begin{aligned}
\boldsymbol{\omega} &= \nabla \times \mathbf{u} = \epsilon_{ijk}\partial_j u_k \, \mathbf{e_i} \\
&= \epsilon_{1jk}[\partial_j u_k]\mathbf{e_1} + \epsilon_{2nm}[\partial_n u_m]\mathbf{e_2} + \epsilon_{3il}[\partial_i u_l]\mathbf{e_3} \\
&= [\epsilon_{123}\partial_2 u_3 + \epsilon_{132}\partial_3 u_2]\mathbf{e_1} + [\epsilon_{213}\partial_1 u_3 + \epsilon_{231}\partial_3 u_1]\mathbf{e_2} + [\epsilon_{312}\partial_1 u_2 + \epsilon_{321}\partial_2 u_1]\mathbf{e_3} \\
&= -\partial_3 u_2 \mathbf{e_1} - \partial_3 u_1 \mathbf{e_2} + [\partial_1 u_2 - \partial_2 u_1]\mathbf{e_3}.
\end{aligned} \tag{2.1}$$

Since $\mathbf{u} = \mathbf{u}(x, y)$, any derivative in the z-direction of $\mathbf{u}$ vanishes. Equation (2.1) follows from the previous line because $u_3 = 0$. Therefore $\boldsymbol{\omega}$ is only defined in the z-direction, and

$$\omega_z = \partial_x u_y - \partial_y u_x. \tag{2.2}$$

If we consider any stretching of the defined vortex tube, then,

$$\begin{aligned}
\boldsymbol{\omega} \cdot \nabla \mathbf{u} &= w_z \partial_z \mathbf{u} \\
&= 0
\end{aligned} \tag{2.3}$$

since any derivative $\partial_z \mathbf{u} = 0$. This follows from $\boldsymbol{\omega}$ being only in the z-direction. If we implement this finding into the vorticity equation from question 1, we find that

$$\begin{aligned}
\frac{\partial \omega_z}{\partial t} + (\mathbf{u} \cdot \nabla)\omega_z &= (\boldsymbol{\omega} \cdot \nabla)u_z + \nu \nabla^2 \omega_z \\
&= \nu \nabla^2 \omega_z.
\end{aligned} \tag{2.4}$$

Now, consider writing the two-dimensional flow in terms of the streamfunction ψ, from which the flow components are given by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{2.5}$$

Under these conditions, the vorticity equation (2.4) simplifies further:

$$\begin{aligned}
\nu \nabla^2 \omega_z &= \frac{\partial \omega_z}{\partial t} + (u\partial_x + v\partial_y)\omega_z \\
&= \frac{\partial \omega_z}{\partial t} + (\partial_x \psi \partial_x - \partial_y \psi \partial_y)\omega_z \\
&= \frac{\partial \omega_z}{\partial t} + \partial_x \psi \partial_x \omega_z - \partial_y \psi \partial_y \omega_z \\
&= \frac{\partial \omega_z}{\partial t} + \det \begin{pmatrix} \partial_x \omega_z & \partial_y \omega_z \\ \partial_x \psi & \partial_y \psi \end{pmatrix},
\end{aligned}$$

or similarly,

$$\frac{\partial \omega_z}{\partial t} + J(\omega_z, \psi) = \nu \nabla^2 \omega_z \tag{2.6}$$

which is what I was trying to show. Furthermore, by equation (2.2), the expression for ω_z becomes the Poisson equation in terms of the streamfunction:

$$\omega_z = \partial_x(-\partial_x \psi) - \partial_y(\partial_x \psi)$$

$$
\begin{aligned}
&= -\partial_x^2\psi - \partial_y^2\psi \\
&= -\nabla^2\psi \\
\implies \nabla^2\psi &= -\omega_z
\end{aligned} \tag{2.7}
$$

as desired.

Problem 3

(**a**) Consider the flow given by the components in cylindrical coordinates $(u_r, u_\varphi, u_z) = \left(-\frac{1}{2}\alpha r, 0, \alpha z\right)$. First, note that the flow is incompressible by taking the divergence in cylindrical coordinates:

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left[-\frac{1}{2}\alpha r^2\right] + \frac{\partial}{\partial z}[\alpha z] \\
&= -\frac{1}{r}\alpha r + \alpha \\
&= 0.
\end{aligned} \tag{3.1}$$

The flow is therefore incompressible. If we were to consider the deformation of a particular fluid element in the flow, such as the stretching of the flow, we may consider the gradient of the fluid field $\nabla \mathbf{u}$. If we are to consider first the radial component of the flow,

$$\begin{aligned}
\nabla u_r &= \left[\frac{1}{r}\frac{\partial}{\partial r} r\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right] u_r \\
&= -\frac{1}{2}\alpha\left[-\frac{1}{2}\alpha\frac{\partial}{\partial r} r\,\hat{\mathbf{r}}\right] r \\
&= \frac{1}{4}\alpha^2\frac{\partial}{\partial r}[r^2]\,\hat{\mathbf{r}} \\
&= \frac{1}{2}\alpha^2 r\,\hat{\mathbf{r}},
\end{aligned} \tag{3.2}$$

and the z-component,

$$\begin{aligned}
\nabla u_z &= \left[\frac{1}{r}\frac{\partial}{\partial r} r\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right] u_z \\
&= \alpha\left[\frac{\partial}{\partial z}\right] z\,\hat{\mathbf{z}} \\
&= \alpha\,\hat{\mathbf{z}},
\end{aligned} \tag{3.3}$$

which implies that the fluid velocity increases quadratically in the radial direction and homogeneously in the z-direction. Note that because $u_\varphi = 0$, there is no rotation in the fluid flow, so $\boldsymbol{\omega} = 0$ everywhere.

(**b**) Now, consider the same flow but with non-zero circulation component $u_\varphi \to u_\varphi(r)$. Because u_φ is only dependent on the radial distance from the origin, the flow should remain incompressible following equation (3.1):

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left[-\frac{1}{2}\alpha r^2\right] + \frac{\partial}{\partial z}[\alpha z] \\
&= -\frac{1}{r}\alpha r + \alpha \\
&= 0.
\end{aligned} \tag{3.4}$$

It follows that the vorticity is now nonzero only if $u_\varphi(r) \neq 0$, and therefore in cylindrical coordinates we obtain

$$\begin{aligned}
\boldsymbol{\omega} &= \nabla \times \mathbf{u} \\
&= \left(\frac{1}{r}\frac{\partial u_z}{\partial \varphi} - \frac{\partial u_\varphi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{r}\left(\frac{\partial [r u_\varphi]}{\partial r} - \frac{\partial u_r}{\partial \varphi}\right)\hat{\boldsymbol{z}} \\
&= \frac{1}{r}\left(\frac{\partial}{\partial r}[r u_\varphi]\right)\hat{\mathbf{z}} \\
&= \left[\frac{u_\varphi}{r} + \frac{\partial u_\varphi}{\partial r}\right]\hat{\mathbf{z}},
\end{aligned} \tag{3.5}$$

which implies that the vorticity is strictly along the z-axis. For brevity, let $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ from this point onwards. If we are to consider the equations of motion which govern u_φ, we may simplify the Navier-Stokes equation for the $\hat{\boldsymbol{\varphi}}$ component of the flow (in cylindrical coordinates):

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t}\cdot\hat{\boldsymbol{\varphi}} + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\hat{\boldsymbol{\varphi}} &= \nu\nabla^2\mathbf{u}\cdot\hat{\boldsymbol{\varphi}} \\
u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_r u_\varphi}{r} &= \nu\left[\nabla^2 u_\varphi - \frac{u_\varphi}{r^2}\right] \\
&= \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\varphi}{\partial r}\right) - \frac{u_\varphi}{r^2}\right]
\end{aligned} \tag{3.6}$$

which follows from the Navier-Stokes equation and the differential vector operators in cylindrical coordaintes (namely, the advective derivative and vector Laplacian), noting that (i) $\dot{u}_\varphi = 0$ (u_φ is only position-independent and does not evolved explicity through time), (ii) there is no gravitational influence in the azimuthal direction, and (iii) there are no pressure gradients in the azimuthal direction because the magnitude of the flow is only dependent on the radial direction. By the aid of equation (3.5), (3.6) may be re-written in terms of ω as

$$\begin{aligned}
u_r\left[\omega - \frac{u_\varphi}{r}\right] + \frac{u_r u_\varphi}{r} &= \frac{\nu}{r}\frac{\partial}{\partial r}\left[\omega r - u_\varphi\right] - \nu\frac{u_\varphi}{r^2} \\
\implies u_r\omega - \frac{u_r u_\varphi}{r} + \frac{u_r u_\varphi}{r} &= \frac{\nu}{r}\left[\omega + r\frac{\partial \omega}{\partial r} - \frac{\partial u_\varphi}{\partial r}\right] - \nu\frac{u_\varphi}{r^2} \\
\implies u_r\omega &= \frac{\nu}{r}\left[\omega + r\frac{\partial \omega}{\partial r} - \omega + \frac{u_\varphi}{r}\right] - \nu\frac{u_\varphi}{r^2} \\
&= \nu\frac{\partial \omega}{\partial r} + \nu\frac{u_\varphi}{r^2} - \nu\frac{u_\varphi}{r^2} \\
\implies -\frac{1}{2}\alpha r\omega &= \nu\frac{\partial \omega}{\partial r},
\end{aligned} \tag{3.7}$$

which will yield an equation of motion for $\omega(r)$ onced solved. This will be the next step.

(c) We now consider equation (3.7) and follow through to solve the differential equation. Simply, by variable separation, one obtains

$$-\frac{\alpha}{2\nu}r\,dr = \frac{d\omega}{\omega} \tag{3.8}$$

which, by integration, yields that

$$\log|\omega| = -\frac{\alpha}{4\nu}r^2 + C \implies \omega(r) = Ce^{-\alpha r^2/(4\nu)}. \tag{3.9}$$

Furthermore, we may obtain an expression for $u_\varphi(r)$ by the relation given in Equation (3.5), which can be written in terms of $\omega(r)$:

$$\begin{aligned}
\omega(r) &= \frac{1}{r}\frac{\partial}{\partial r}[ru_\varphi] \\
\implies ru_\varphi &= \int dr\, r\omega(r) \\
&= C\int dr\, re^{-\alpha r^2/(4\nu)} \\
&= -\frac{2C\nu}{\alpha}e^{-\alpha r^2/(4\nu)} + D \\
\implies u_\varphi(r) &= -\frac{2\nu C}{\alpha r}e^{-\alpha r^2/(4\nu)} + \frac{D}{r}. \qquad (3.10)
\end{aligned}$$

Note that, upon integrating, we obtain a second integration constant D. Upon the substitution of u_φ into (3.5), observe that the second term D/r vanishes and is therefore just the solution to the homogenous equation given in (3.5). Therefore, for simplicity, I will let $D = 0$. Therefore our expressionn for u_φ becomes

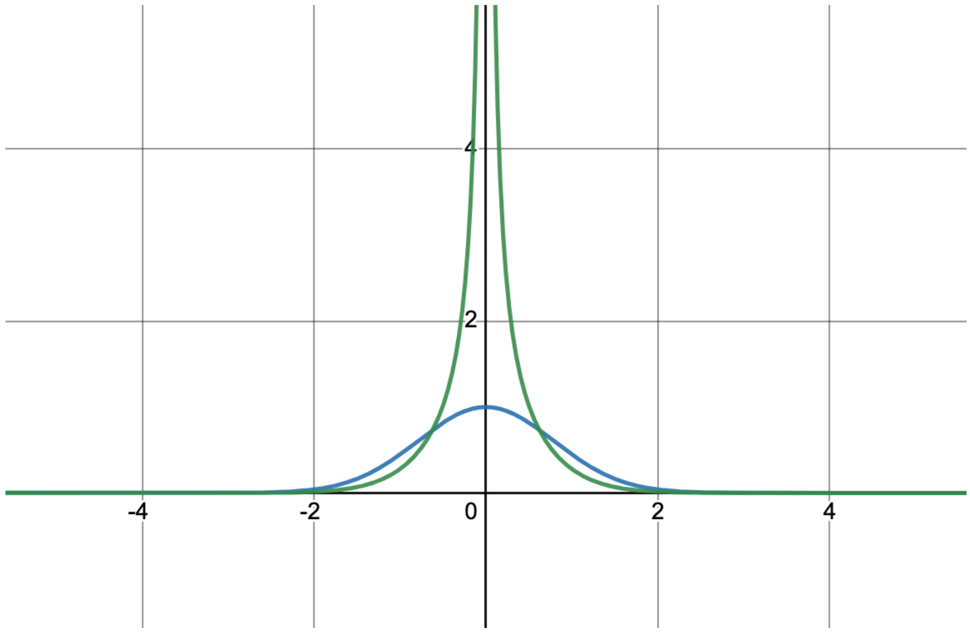
$$u_\varphi(r) = -\frac{2\nu C}{\alpha r}e^{-\alpha r^2/(4\nu)} = -\frac{2\nu}{\alpha r}\omega(r). \qquad (3.11)$$

Respectively, we may further analyze the behaviour of $\omega(r), u_\varphi(r)$ be expressing these equations in terms of the outward radial velocity $u_r(r) = -\frac{1}{2}\alpha r$:

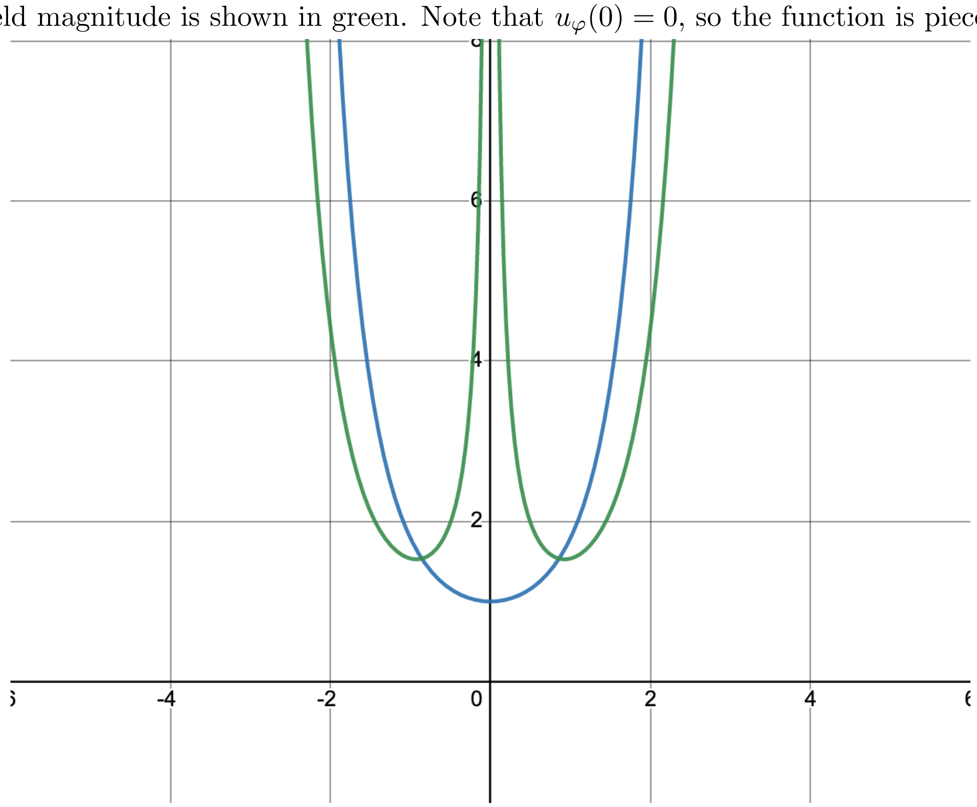
$$\omega(r) = C\exp\left(\frac{u_r}{2\nu}r\right) \qquad (3.12)$$

$$u_\varphi(r) = \frac{\nu}{u_r}C\exp\left(\frac{u_r}{2\nu}r\right). \qquad (3.13)$$

First note the independence of these expressions of the φ- and z-directions. This flow is axisymmetric, so there are azimuthal pressure or density components. Instead, due to the rotation of the flow, the vorticity is only dependent on the radial distance by the construction of the flow. Since the vorticity contains only this r-dependence, we therefore expect and observe that the $\omega(r)$ and $u_\varphi(r)$ components are z-independent.



[Figure 1] The radial inflow, $\alpha > 0$. The vorticity magnitude is shown in blue, while the azimuthal velocity field magnitude is shown in green. Note that $u_\varphi(0) = 0$, so the function is piecewise.



[Figure 2] The radial outflow, $\alpha < 0$. The vorticity magnitude is shown in blue, while the azimuthal velocity field magnitude is shown in green. Note that $u_\varphi(0) = 0$, so the function is piecewise.

The interpretation of these equations are that for $\alpha > 0$, there is radial inflow, and outflow for $\alpha < 0$. In terms of the vorticity magnitude, then, we expect to be small for $r \to \infty$ for inflow and large as $r \to \infty$ for outflow, because the vorticity is increasing as a function of r. This is reflected in the relations for $u_\varphi(r)$, since as the vorticity goes to 0, so does the azimuthal velocity magnitude because there is no rotation. If $\omega \to \infty$, so does the azimuthal velocity magnitude, because the fluid is rotating faster and faster for greater values of r. As expected, the origin $r = 0$ is an unstable equilibrium, so any perturbation causes fluid to quickly spiral outwards. This is why $u_\varphi(r) \to \infty$ as $r \to 0$. Note that, upon varying the viscosity ν, the width of these curves becomes greater (or less) because the fluid becomes more (or less) viscous. This implies that, for large ν, more torque is generated which shears more of the fluid layers in the r-direction. This corresponds to a widening of the curves shown in Figures (1) and (2). Equivalently, for small ν, the fluid is less viscous and less shearing torque is generated, making the curve widths in figures (1) and (2) smaller.

(d) Since the vorticity is a function of r only, we are able to compute the circulation of the flow around a closed curve containing the origin. Since this curve is closed, then, the start and end-points are the same, which implies that there shall be no overall change in the radial or vertical path direction. Due to this, our infinitessimal line integration vector is given by $d\boldsymbol{\ell} = rd\varphi\,\hat{\boldsymbol{\varphi}}$. The circulation, in terms of the flow at a fixed radius, is given by

$$\Gamma(r) = \oint \mathbf{u} \cdot d\boldsymbol{\ell}. \tag{3.14}$$

Upon substitution of $\mathbf{u}$, the dot product confirms that we only integrate around the azimuthal direction, fromm 0 to 2π. We therefore have, upon substitution of Equation (3.11), that the circulation evaluated at a fixed radius r is

$$\begin{aligned} \Gamma(r) &= \oint \mathbf{u} \cdot d\boldsymbol{\ell} \\ &= -\frac{2\nu}{\alpha r}\omega(r) \int_0^{2\pi} rd\varphi \\ &= -\frac{4\pi\nu}{\alpha}\omega(r). \end{aligned} \tag{3.15}$$

The total circulation is then given by integrating across the entire radial flow, $\Gamma = \int_0^\infty dr\,\Gamma(r)$. This is an improper integral, however due to the Gaussian-shape of $\omega(r)$, we are sure that the integral converges:

$$\begin{aligned} \Gamma &= -\frac{4\pi\nu C}{\alpha} \int_0^\infty dr\, e^{-\alpha r^2/(4\nu)} \\ &= -\frac{8\pi\nu C}{\alpha}\sqrt{\frac{\pi\nu}{\alpha}} \\ &= -8C\left(\frac{\pi\nu}{\alpha}\right)^{3/2} \end{aligned} \tag{3.16}$$

where the integral taken out is just the Gaussian integral $\int_0^\infty dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$. Note that, as the integration constant C increases, the total circulation does as well. As expected, the circulation is only dependent on the properties of the viscosity, and magnitude of the vorticity (C), and the constant α which characterizes the radial flow velocity.

One may wonder, why does the circulation and/or u_φ depend on the radial flow magnitude α? Note that, by the nature of the vorticity function $\omega(r)$, the vorticity magnitude is a maximum at the origin when $r = 0$. This implies that, if $\alpha > 0$ for instance, fluid is flowing towards the origin and therefore gaining vorticity. If $\alpha < 0$, we then expect $\omega(r)$ to decrease near the origin and increase as $r \to \infty$. For the circulation, then, α characterizes a circulation density (ie, more circulation in the center for $\alpha > 0$ and more circulation on the outer region $r > 0$ for $\alpha < 0$), which should therefore influence the total circulation.

(e) Let me now show the constancy of the vorticity field at a fixed point in the flow. From the vorticity equation (Equation 1.15), we may determine the local time derivative $\dfrac{\partial \omega}{\partial t}$ which is independent of position. Firstly, since $\boldsymbol{\omega} = \omega(r)\,\hat{\mathbf{z}}$, then the only component of $\mathbf{u}$ which we need to examine is the z-component. By evaluating each term in Equation (1.15), we find that

$$\begin{aligned}
(\boldsymbol{\omega}\cdot\nabla)u_z &= \omega\frac{\partial}{\partial z}u_z \\
&= \alpha\omega \qquad (3.17)\\
(\mathbf{u}\cdot\nabla)\omega &= u_r\frac{\partial}{\partial r}\omega \\
&= u_r\frac{\partial}{\partial r}\left[Ce^{-\alpha r^2/(4\nu)}\right] \\
&= -\frac{1}{2}\alpha r\left(-\frac{1}{2}\frac{\alpha}{\nu}r\omega\right) \\
&= \frac{1}{4}\frac{\alpha^2}{\nu}r^2\omega \qquad (3.18)\\
\nu\nabla^2\omega &= \frac{\nu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[Ce^{-\alpha r^2/(4\nu)}\right]\right] \\
&= \frac{\nu}{r}\frac{\partial}{\partial r}\left[-\frac{C\alpha r^2}{2\nu}e^{-\alpha r^2/(4\nu)}\right] \\
&= \frac{\nu}{r}\left[-\frac{\alpha r}{\nu}\omega + \frac{1}{4}\frac{\alpha^2}{\nu^2}r^3\omega\right] \\
&= -\alpha\omega + \frac{1}{4}\frac{\alpha^2}{\nu}r^2\omega \qquad (3.19)
\end{aligned}$$

so that the vorticity equation becomes

$$\frac{\partial \omega}{\partial t} + \frac{1}{4}\frac{\alpha^2}{\nu}r^2\omega = \alpha\omega - \alpha\omega + \frac{1}{4}\frac{\alpha^2}{\nu}r^2\omega \qquad (3.20)$$

which therefore implies that $\dfrac{\partial \omega}{\partial t} = 0$, and thus the vorticity at a fixed point in the flow remains constant. For a fluid element, however, the vorticity is not constant in time, since by the means of the vorticity equation and the particle derivative,

$$\begin{aligned}
\frac{D\omega}{Dt} &= (\boldsymbol{\omega}\cdot\nabla)u_z + \nu\nabla^2\omega \\
&= \alpha\omega - \alpha\omega + \frac{1}{4}\frac{\alpha^2}{\nu}r^2\omega \\
&= \frac{C}{4}\frac{\alpha^2}{\nu}r^2e^{-\alpha r^2/(4\nu)} \qquad (1.16)
\end{aligned}$$

and so the vorticity of a fluid element in the flow decreases with r for $\alpha > 0$ and increases with r for $\alpha < 0$. Intuitively, this makes sense, because if $\alpha > 0$ the vorticity is zero at the center and goes to zero at $r = \infty$, because fluid is flowing into the origin in the r-direction and out in the z-direction. For $\alpha < 0$, fluid flows into the origin from the z-axis and out of the origin in the radial direction. Since this is the case, we find that fluid vorticity should increase as $r \to \infty$.