MAT224 Linear Algebra II
Assignment 3

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Signatures: 1) Jace Alloway ____

2) ____

1. Consider the basis $\alpha = 1+x, \frac{1}{2}x, -\frac{3}{2}x^2+\frac{1}{2}$ for $P_2(\mathbb{R})$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the derivative transformation. That is $T(p(x)) = \frac{d}{dx}(p(x))$ for every $p(x) \in P_2(\mathbb{R})$.

You may use the that α is a basis for $P_2(\mathbb{R})$, and that T is a linear transformation without proof.

1(a) Determine $[x^2]_\alpha$.

Solution: We need a linear combination of vectors in the basis α to produce x^2 in terms of α.
Firstly, by computing $-\dfrac{2}{3}\left(-\dfrac{3}{2}x^2+\dfrac{1}{2}\right)$, we have

$$-\frac{2}{3}\left(-\frac{3}{2}x^2+\frac{1}{2}\right) = x^2 - \frac{1}{3}.$$

Next, we need to reduce $x^2 - \frac{1}{3}$ to x^2, so we need a linear combination of polynomials to produce $-\frac{1}{3}$. We have

$$\frac{1}{3}(1+x) = \frac{1}{3} + \frac{1}{3}x.$$

If we add another term in the basis $-\frac{2}{3}\left(\frac{1}{2}x\right)$, we have our linear combination of polynomials:

$$\frac{1}{3}(1+x) - \frac{2}{3}\left(\frac{1}{2}x\right) - \frac{2}{3}\left(-\frac{3}{2}x^2+\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{3}x - \frac{1}{3}x + x^2 - \frac{1}{3} = x^2.$$

Therefore $[x^2]_\alpha = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ where $\alpha = \{1+x, \frac{1}{2}x, -\frac{3}{2}x^2+\frac{1}{2}\}$, as needed.

1(b) Determine $[T]_\alpha^\alpha$.

Solution: To determine $[T]_\alpha^\alpha$, we need to determine what $T(1+x), T(\frac{1}{2}x)$, and what $T(-\frac{3}{2}x^2+\frac{1}{2})$ are. Computing each derivative, we see that

$$\begin{aligned} T(1+x) &= \frac{d}{dx}(1+x) = 1, \\ T\left(\frac{1}{2}x\right) &= \frac{d}{dx}\left(\frac{1}{2}x\right) = \frac{1}{2}, \text{ and} \\ T\left(-\frac{3}{2}x^2+\frac{1}{2}\right) &= \frac{d}{dx}\left(-\frac{3}{2}x^2+\frac{1}{2}\right) = -3x. \end{aligned}$$

In terms of α,

$$1 = (1+x) - 2\left(\frac{1}{2}x\right), \quad \text{so} \quad [1]_\alpha = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

$$\frac{1}{2} = \left(\frac{1}{2}(1+x) - \frac{1}{2}x\right), \quad \text{so} \quad \left[\frac{1}{2}\right]_\alpha = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} \text{ and}$$

$$-3x = -6\left(\frac{1}{2}x\right), \quad \text{so} \quad [-3x]_\alpha = \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix}.$$

Therefore $[T]_\alpha^\alpha = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -2 & -1 & -6 \\ 0 & 0 & 0 \end{bmatrix}$.

This matrix follows from the fact that computing the derivative of each polynomial in the basis α represents the transformation of the basis. Every polynomial $p(x) \in P_2(\mathbb{R})$ can be created from a linear combination of vectors in α, which implies that $T(1+x), T\left(\dfrac{1}{2}x\right)$, and $T\left(-\frac{3}{2}x^2+\frac{1}{2}\right)$ establish the derivatives of the basis vectors. Finding these transformations in terms of α yields $[T]_\alpha^\alpha$.

1(c) Use your answers from 1(a) and 1(b) to calculate $\frac{d}{dx}(x^2)$.

Solution: We know that

$$[x^2]_\alpha = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

and that

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -2 & -1 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Computing the matrix multiplication $[T]_\alpha^\alpha [x^2]_\alpha$ will give $T(x^2) = \dfrac{d}{dx}\left(x^2\right)$. We have

$$[T]_\alpha^\alpha [x^2]_\alpha = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -2 & -1 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \left(-\frac{2}{3}\right)\begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} + \left(-\frac{2}{3}\right)\begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

We have that $[T]_\alpha^\alpha [x^2]_\alpha = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = 4\left(\dfrac{1}{2}x\right) = 2x$, and so $\dfrac{d}{dx}\left(x^2\right) = 2x$, as needed.

2. Let V and W be vector spaces, and let $T \in \mathfrak{L}(V, W)$. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v_n}$ be a basis for V.

2(a) Prove that if T is an isomorphism, then $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n$ is a basis for W.

I want to prove that if a linear map $T \in \mathfrak{L}(V, W)$ is an isomorphism, and if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ is a basis for the domain V, then $\{T\mathbf{v}_1, T\mathbf{v}_2, , T\mathbf{v}_n\}$ is a basis for the codomain W.

Proof. Assume that $T \in \mathfrak{L}(V, W)$ is an isomorphism. We then have that $T : V \to W$ is invertible, that is, both injective and surjective. By [**Proposition 2.6.7**], T being an isomorphism implies that $\dim V = \dim W$.
We then have that

$$\textbf{[Injective]:}\ \forall \mathbf{x}_1, \mathbf{x}_2 \in V,\ \ \mathbf{x}_1 \neq \mathbf{x}_2 \implies T(\mathbf{x}_1) \neq T(\mathbf{x}_2),$$

$$\textbf{[Surjective]:}\ \forall \mathbf{w} \in W,\ \ \exists \mathbf{x} \in V \text{ such that } T(\mathbf{x}) = \mathbf{w}.$$

Since $T : V \to W$ is injective, then for any vector $\mathbf{x} \in V$ that is unique, $T(\mathbf{x})$ is also unique. This means that there exists only one vector $T(\mathbf{x}) \in W$ for every $\mathbf{x} \in V$, where $\mathbf{x}$ is expressed as a unique linear combination of vectors in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$.
Since $T : V \to W$ is surjective, then for every $\mathbf{w} \in W$, there is at least one vector $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{w}$. This means that $\operatorname{Im}(T) = W$, that is, every vector $\mathbf{w} \in W$ is being mapped to by the set of vectors $T\mathbf{x}$, where $\mathbf{x} \in V$ is expressed as a unique linear combination of vectors in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$.

It then follows that because T is injective, we can conclude that since the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ of V is unique, then every vector in the set $\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\}$ is also unique. Furthermore, since T is an isomorphism, by **Proposition 2.6.7**, the requirement for $\dim V = \dim W = n$ is satisfied.
Because T is surjective, then we can conclude that the image of T generates W, and thus any vector $T\mathbf{x} \in W$ can be expressed as a unique linear combination of vectors in the basis $\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\}$ of W, that is, $\operatorname{span}\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\} = W$.

Because $\operatorname{span}\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\} = W$ and the set $\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\}$ is unique, then $\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\}$ is a basis for W, which is what I needed to prove.

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2. Let V and W be vector spaces, and let $T \in \mathfrak{L}(V, W)$. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v_n}$ be a basis for V.

2(b) Prove that if $T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n$ is a basis for W, then T is an isomorphism.

To prove that the transformation $T \in \mathfrak{L}(V, W)$ is an isomorphism, it suffices to show that T is invertible. That is, both injective and surjective.

I want to prove that for any transformation $T \in \mathfrak{L}(V, W)$ if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for the domain W and if $\{T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n\}$ is a basis for the codomain W, then T is an isomorphism.

Proof. Assume that the set $\{T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n\}$ is a basis for W. This implies that every element in the set $\{T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n\}$ is unique.

Show that T is injective:

$$\forall \mathbf{x}_1, \mathbf{x}_2 \in V, \quad \mathbf{x}_1 \neq \mathbf{x}_2 \implies T\mathbf{x}_1 \neq T\mathbf{x}_2.$$

Since every element $\mathbf{x} \in V$ can be expressed as a unique linear combination of vectors in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then the hypothesis of the definition is satisfied for two different vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ where $\mathbf{x}_1 \neq \mathbf{x}_2$. Secondly, every element $\mathbf{w} \in W$, where $\mathbf{w} = T\mathbf{x}$ for each $\mathbf{x} \in V$, can be expressed as a unique linear combination of vectors in the basis $\{T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n\}$, which makes $T\mathbf{x} \in W$ unique for any $\mathbf{x} \in V$ since T is linear. Therefore for all vectors $\mathbf{x}_1, \mathbf{x}_2 \in V, \mathbf{x}_1 \neq \mathbf{x}_2 \implies T\mathbf{x}_1 \neq T\mathbf{x}_2$, and therefore $T : V \to W$ is injective.

Show that T is surjective:

$$\forall \mathbf{w} \in W, \quad \exists \mathbf{x} \in V \text{ such that } T(\mathbf{x}) = \mathbf{w}.$$

Since every element $\mathbf{w} \in W$ can be expressed as a linear combination of vectors in the set $\{T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n\}$ because $\{T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n\}$ is a basis for W, then every element $\mathbf{w} \in W$ is uniquely determined by some vector $T\mathbf{x} \in W$, where $\mathbf{x}$ is expressed as a unique linear combination of vectors in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Therefore $T : V \to W$ is surjective.

Lastly, since $\dim V = \dim W = n$, and because $T \in \mathfrak{L}(V, W)$ is both injective and surjective, then by [**Proposition 2.6.7**], $T : V \to W$ is an isomorphism, which is what I needed to prove.

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3. Let V and W be finite dimensional vector spaces, and let $T \in \mathfrak{L}(V, W)$. Prove that, for any choice of bases α for V and β for W,

$$\dim \operatorname{im} T = \operatorname{rank} [T]_\alpha^\beta$$

Note: It is also true that $\dim \ker T = \dim \operatorname{null} [T]_\alpha^\beta$ but you do not need to prove this here. We include it for completeness sake.

I want to prove that for a transformation $T \in \mathfrak{L}(V, W)$ and any bases α and β of V and W, respectively, that $\dim \operatorname{im} T = \operatorname{rank}[T]_\alpha^\beta$.

Proof.
Suppose $\dim V = n$ and $\dim W = m$. Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be bases of the vector spaces V and W, respectively. I will note that it is possible for $n = m$.
We define a *pivot column* of a matrix as a column in a matrix with a 1 in the i^{th} row (that is, the i^{th} entry of the column), for $1 \leq i \leq m$, and having zeros in every other entry. A column containing a pivot is defined as a linearly independent column of the matrix, which can be found by computing $[T]_\alpha^\beta[\mathbf{x}]_\alpha = \mathbf{0}$ for any $\mathbf{x} \in V$. If any other column can be created from a linear combination of the pivot columns, then the columns of the matrix are said to be linearly dependent.
Furthermore, we define the *rank* of a matrix by dimension of the column space of $[T]_\alpha^\beta$. That is, the number of linearly independent pivot columns in the matrix $[T]_\alpha^\beta$.

Now, let $\mathbf{t}_j$ be the j^{th} column of the matrix $[T]_\alpha^\beta$ for $1 \leq j \leq n$. From this, it follows that if $\mathbf{t}_j$ is a pivot column, then the $\mathbf{t}_j$ represents an element in the basis α and hence is a coordinate vector of the image of T, that is, $T(\mathbf{v}_j)$. We then have $\mathbf{t}_j = [T(\mathbf{v}_j)]_\beta$ because $\mathbf{t}_j$ is a pivot column.
Any pivot column in the matrix $[T]_\alpha^\beta$ is unique because it is linearly independent, and thus together they form a basis for the image of T.
Lastly, if the rank of the matrix $[T]_\alpha^\beta$ is k, then there are k linearly independent pivot columns in the matrix $[T]_\alpha^\beta$, which is the dimension of the column space of $[T]_\alpha^\beta$. Therefore there are k elements in the basis of the image of T, which is what I needed to prove. This proof follows from the ideas presented in **[(2.3.14) Procedure 1]** in the textbook.

Lastly, to justify this proof, we can examine the rank-nullity theorem for linear transformations. The theorem states that

$$\dim V = \operatorname{rank}[T]_\alpha^\beta + \operatorname{nullity}[T]_\alpha^\beta = \dim \operatorname{im} T + \dim \ker T,$$

since we have assumed that $\operatorname{nullity}[T]_\alpha^\beta = \dim \ker T$ without proof. Because $\dim V$ does not change for a fixed vector space V, it them follows that $\dim \operatorname{im} T = \operatorname{rank}[T]_\alpha^\beta$.

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4. Let $A, B \in M_{n\times n}(\mathbb{R})$ be similar. Use the result from question 3 to prove that

$$\operatorname{rank} A = \operatorname{rank} B$$

Note: It is also true that $\dim \operatorname{null} A = \dim \operatorname{null} B$ but you do not need to prove this here. We include it for completeness sake.

I want to prove that for similar matrices $A, B \in M_{n\times n}(\mathbb{R})$, that rank $A =$ rank B.

Proof.
We call two matrices similar if there exists an invertible matrix $P \in M_{n\times n}(\mathbb{R})$ such that $A = P^{-1}BP$.
To begin, by multiplying the matrix P on the left hand side of both side of the equation, we have

$$A = P^{-1}BP \implies PA = PP^{-1}BP = IBP \implies PA = BP,$$

since any matrix multiplied by the identity yields itself and since any matrix multiplied by its inverse yields the identity.

To prove that $\operatorname{rank} A = \operatorname{rank} B$, it suffices to show that since P is an invertible matrix, then $\operatorname{rank} PA = \operatorname{rank} A$ and $\operatorname{rank} AB = \operatorname{rank} B$.

1. Proof that $\operatorname{rank} PA = \operatorname{rank} A$.

   Let $\mathbf{a} \in \operatorname{null} A$. We then have that $A\mathbf{a} = \mathbf{0}$. By the multiplicative associativity property of matrices, it then follows that, $PA\mathbf{a} = P(A\mathbf{a}) = P\mathbf{0} = \mathbf{0}$, which implies that $\mathbf{a} \in \operatorname{null}(PA)$. Therefore $\operatorname{null}(A) \subseteq \operatorname{null}(AB)$ since $\operatorname{null}(PA) = P^{-1} \operatorname{null} A$ by [**Proposition 6.7, 6.8**] in Quiri Li's linear transformation slides.

   Now suppose $\mathbf{a} \in \operatorname{null}(PA)$, and so $PA\mathbf{a} = \mathbf{0}$. This implies that

   $$PA\mathbf{a} = \mathbf{0} \implies P(A\mathbf{a}) = \mathbf{0} \implies A\mathbf{a} = P^{-1}\mathbf{0} = \mathbf{0}.$$

   By the same proposition, $\operatorname{null}(PA) \subseteq \operatorname{null} A$, and therefore $\operatorname{null}(A) = \operatorname{null}(PA) \implies \dim \operatorname{null}(PA) = \dim \operatorname{null} A$.

   By the rank-nullity theorem,

   $$\operatorname{rank}(PA) + \dim \operatorname{null}(PA) = \operatorname{rank} A + \dim \operatorname{null} A \implies \operatorname{rank}(AB) = \operatorname{rank}(A).$$

2. By the same argument, we have that $\operatorname{rank}(BP) = \operatorname{rank} B$ because $\operatorname{null} B \subseteq \operatorname{null}(BP)$ and $\operatorname{null}(BP) \subseteq \operatorname{null} B$.

Therefore since $\operatorname{rank}(PA) = \operatorname{rank} A$ and $\operatorname{rank}(BP) = \operatorname{rank} B$, and since $A = P^{-1}BP \implies PA = BP$, then

$$\operatorname{rank}(PA) = \operatorname{rank} A = \operatorname{rank} B = \operatorname{rank}(BP),$$

which is what I needed to prove.

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