PHY356 Problem Set 5 — Due December 6th, 23:00

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6. Consider a system of angular momentum $l = 1$. A basis of its state space is formed by the three eigenvectors of L_z: $|+1\rangle$, $|0\rangle$, $|-1\rangle$, whose eigenvalues are, respectively, $+\hbar$, 0, and $-\hbar$, and which satisfy:

$$L_{\pm}|m\rangle = \hbar\sqrt{2}\,|m \pm 1\rangle$$
$$L_{+}|1\rangle = L_{-}|-1\rangle = 0$$

This system, which possesses an electric quadrupole moment, is placed in an electric field gradient, so that its Hamiltonian can be written:

$$H = \frac{\omega_0}{\hbar}(L_u^2 - L_v^2)$$

where L_u and L_v are the components of $\mathbf{L}$ along the two directions Ou and Ov of the xOz plane that form angles of 45° with Ox and Oz; ω_0 is a real constant.

a. Write the matrix representing H in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis. What are the stationary states of the system, and what are their energies? (These states are to be written $|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$, in order of decreasing energies.)

b. At time $t = 0$, the system is in the state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|+1\rangle - |-1\rangle]$$

What is the state vector $|\psi(t)\rangle$ at time t? At t, L_z is measured; what are the probabilities of the various possible results?

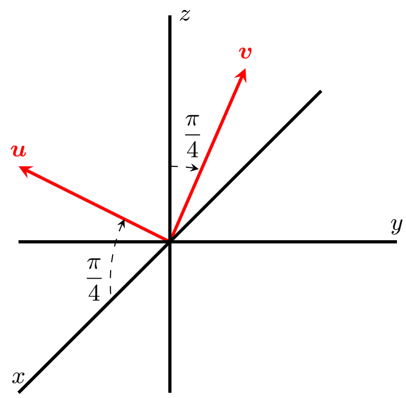
c. Calculate the mean values $\langle L_x\rangle(t)$, $\langle L_y\rangle(t)$ and $\langle L_z\rangle(t)$ at t. What is the motion performed by the vector $\langle\mathbf{L}\rangle$?

d. At t, a measurement of L_z^2 is performed.

(*i*) Do times exist when only one result is possible?

(*ii*) Assume that this measurement has yielded the result $\hbar^2$. What is the state of the system immediately after the measurement? Indicate, without calculation, its subsequent evolution.

(a) To begin, I will provide a diagram for the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Each of these vectors are rotated by $\frac{\pi}{4}$ in the xz-plane:



Projecting the components of L_u and L_v onto the cartesian coordinate plane allows us to writen them as

$$L_z = \cos\frac{\pi}{4}L_u + \sin\frac{\pi}{4}L_v = \frac{\sqrt{2}}{2}L_u + \frac{\sqrt{2}}{2}L_v$$
$$L_x = \cos\frac{\pi}{4}L_u + \sin\frac{\pi}{4}L_v = \frac{\sqrt{2}}{2}L_u - \frac{\sqrt{2}}{2}L_v.$$

Like every other pair of operators represented this way, one may quickly follow through the calculation to show that

$$L_u = \frac{\sqrt{2}}{2}\left(L_z + L_x\right)$$
$$L_v = \frac{\sqrt{2}}{2}\left(L_z - L_x\right).$$

However, this is not sufficient enough to determine the matrix representation of the Hamiltonian; we must calculate the squares of each respective operator. Note that we cannot expand L_u^2 or L_v^2 in operator form because L_z and L_x do not commute with each other - hence there is no correct way to multiply L_xL_z. Instead, let us first consider the matrix representations of such operators in the $\{|-1\rangle, |0\rangle, |+1\rangle\}$ basis:

$$L_x \equiv \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_z \equiv \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This implies that L_u and L_v can be written in matrix form as

$$L_u = \frac{\sqrt{2}}{2}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\sqrt{2}\hbar}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$= \frac{\hbar}{2}\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix},$$
$$L_v = -\frac{\sqrt{2}}{2}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\sqrt{2}\hbar}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix}.$$

We may now take out the squares of such opeartors via matrix multiplication:

$$L_u^2 = \frac{\hbar^2}{4} \begin{pmatrix} 3 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 3 \end{pmatrix}$$

$$L_v^2 = \frac{\hbar^2}{4} \begin{pmatrix} 3 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 3 \end{pmatrix}.$$

Therefore, by taking the difference $L_u^2 - L_v^2$, the Hamiltonian can be written in the matrix representation in the L_z basis as

$$H = \frac{\omega_0}{\hbar} \cdot \frac{\hbar^2}{4} \begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 0 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\omega_0\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\omega_0\hbar A.$$

One may quickly diagonalize A and determine the respective eigenvalues and eigenkets of the system. Just considering the matrix, we have that $\mathrm{char}(A) = \det(A - \lambda I) = -\lambda^3 + 2\lambda = \lambda(2 - \lambda^2)$. The eigenfunctions are then given by the kernel of $A - \lambda I$ for each corresponding $\lambda = 0, \pm\sqrt{2}$:

$$\ker \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \ker \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

which has the corresponding eigenket $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. For $\lambda = \sqrt{2}$,

$$\ker \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} = \ker \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & \frac{\sqrt{2}}{2} & 1 \end{pmatrix},$$

which has corresponding eigenket $\begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}$. The eigenket for $\lambda = -\sqrt{2}$ is equivalently found to be $\begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}$. Therefore in the L_z basis, we have the stationary states of the Hamiltonian to be

$$\begin{aligned} \lambda &= 0, & |\varphi_0\rangle &= |1\rangle + |{-1}\rangle \\ \lambda &= \pm\sqrt{2}, & |\varphi_{2,3}\rangle &= |1\rangle \pm \sqrt{2}\,|0\rangle - |{-1}\rangle \end{aligned}$$

hence the energies are $E_0 = 0$ and $E_{2,3} = \pm\omega_0\hbar$.

(b) Now consider a general state $|\psi\rangle$, which can be expanded in terms of eigenfunctions of the Hamiltonian:

$$\begin{aligned}|\psi(t=0)\rangle &= c_1(|1\rangle + |-1\rangle) + c_2\left(|1\rangle + \sqrt{2}\,|0\rangle - |-1\rangle\right) + c_3\left(|1\rangle - \sqrt{2}\,|0\rangle - |-1\rangle\right)\\ &= (c_1+c_2+c_3)\,|1\rangle + \sqrt{2}(c_2-c_3)\,|0\rangle + (c_1-c_2-c_3)\,|-1\rangle\,.\end{aligned}$$

If the initial state corresponds to $|\psi(0)\rangle = \dfrac{1}{\sqrt{2}}\left(|1\rangle - |-1\rangle\right)$, we acquire the system of equations

$$\frac{1}{\sqrt{2}} = c_1 + c_2 + c_3 \tag{1}$$

$$0 = c_2 - c_3 \tag{2}$$

$$-\frac{1}{\sqrt{2}} = c_1 - c_2 - c_3. \tag{3}$$

This system is easy to solve. (2) yields $c_2 = c_3$ and (3) gives $c_1 = \dfrac{1}{\sqrt{2}} + 2c_2 = \dfrac{1}{\sqrt{2}} + 2c_3$. (1) then tells us that the magnitudes of the coefficients, which are real, are $c_1 = 0$ and $c_2 = c_3 = \dfrac{\sqrt{2}}{4}$. The Schrödinger equation then governs the time evolution of the system, which is then given by

$$\begin{aligned}|\psi(t)\rangle &= (0)\,e^{-i(0)t/\hbar}[|1\rangle + |-1\rangle]\\ &\qquad + \left(\frac{\sqrt{2}}{4}\right)e^{-i\omega_0 t}[|1\rangle + \sqrt{2}\,|0\rangle - |-1\rangle]\\ &\qquad\qquad + \left(\frac{\sqrt{2}}{4}\right)e^{i\omega_0 t}[|1\rangle - \sqrt{2}\,|0\rangle - |-1\rangle]\\ &= \frac{\sqrt{2}}{4}e^{-i\omega_0 t}[|1\rangle + \sqrt{2}\,|0\rangle - |-1\rangle] + \frac{\sqrt{2}}{4}e^{i\omega_0 t}[|1\rangle - \sqrt{2}\,|0\rangle - |-1\rangle]\end{aligned}$$

Furthermore, it may be more convenient to write $\psi(t)$ by grouping like-states together, such as

$$\begin{aligned}|\psi(t)\rangle &= \frac{\sqrt{2}}{4}\left(e^{-i\omega_0 t} + e^{i\omega_0 t}\right)|1\rangle + \frac{1}{2}\left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right)|0\rangle - \frac{\sqrt{2}}{4}\left(e^{-i\omega_0 t} + e^{i\omega_0 t}\right)|-1\rangle\\ &= \frac{\sqrt{2}}{2}\cos(\omega_0 t)\,|1\rangle + i\sin(\omega_0 t)\,|0\rangle - \frac{\sqrt{2}}{2}\cos(\omega_0 t)\,|-1\rangle\\ &= \frac{\sqrt{2}}{2}\cos(\omega_0 t)\,[|1\rangle - |-1\rangle] + i\sin(\omega_0 t)\,|0\rangle\,.\end{aligned}$$

Suppose we measure L_z at time t, which acts on states as $L_z\,|\pm 1\rangle = \hbar\,|\pm 1\rangle$ and $L_z\,|0\rangle = 0$. We have that we measure $|0\rangle$ with eigenvalue 0 with a probability $\sin^2(\omega_0 t)$, and each state $|1\rangle$ and $|-1\rangle$ with eigenvalues $\pm\hbar$, respectively, with equal probability $\dfrac{1}{2}\cos^2(\omega_0 t)$. From this, we easily verify the normalization condition since

$$\frac{1}{2}\cos^2(\omega_0 t) + \frac{1}{2}\cos^2(\omega_0 t) + \sin^2(\omega_0 t) = 1$$

for all time.

(c) One may also determine the time-dependence of the expectation values $\langle L_x \rangle (t), \langle L_y \rangle (t), \langle L_z \rangle (t)$. It may be useful to first expand the L_x and L_y operators in terms of the raising and lowering operators L_+ and L_-. Simply, they are given as

$$L_\pm = L_x \pm i L_y \implies L_x = \frac{1}{2}(L_+ + L_-), \quad L_y = \frac{1}{2i}(L_+ - L_-).$$

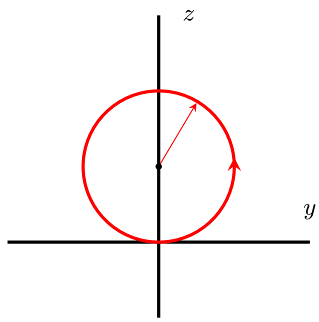
The action of these operators is given as above. We have that

$$\begin{aligned}
\langle \psi(t) | L_x | \psi(t) \rangle &= \frac{1}{2} \langle \psi(t) | (L_+ + L_-) \left[\frac{\sqrt{2}}{2} \cos(\omega_0 t)(|1\rangle - |-1\rangle) + i \sin(\omega_0 t) |0\rangle \right] \\
&= \frac{1}{2} \langle \psi(t) | \left(-\hbar \cos(\omega_0 t) |0\rangle + \sqrt{2}\hbar i \sin(\omega_0 t) |1\rangle + \hbar \cos(\omega_0 t) |0\rangle + \sqrt{2}\hbar i \sin(\omega_0 t) |-1\rangle \right) \\
&= \frac{1}{2} \left[\frac{\sqrt{2}}{2} \cos(\omega_0 t)(\langle 1| - \langle -1|) - i \sin(\omega_0 t) \langle 0| \right] \left[\sqrt{2}\hbar i \sin(\omega_0 t)(|1\rangle + |-1\rangle) \right] \\
&= \frac{1}{2} \left[\hbar i \cos(\omega_0 t) \sin(\omega_0 t) - \hbar i \cos(\omega_0 t) \sin(\omega_0 t) \right] \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle \psi(t) | L_y | \psi(t) \rangle &= \frac{1}{2i} \langle \psi(t) | (L_+ - L_-) \left[\frac{\sqrt{2}}{2} \cos(\omega_0 t)(|1\rangle - |-1\rangle) + i \sin(\omega_0 t) |0\rangle \right] \\
&= \frac{1}{2i} \langle \psi(t) | \left(-\hbar \cos(\omega_0 t) |0\rangle + \sqrt{2}\hbar i \sin(\omega_0 t) |1\rangle - \hbar \cos(\omega_0 t) |0\rangle - \sqrt{2}\hbar i \sin(\omega_0 t) |-1\rangle \right) \\
&= -\frac{1}{2i} \left[\frac{\sqrt{2}}{2} \cos(\omega_0 t)(\langle 1| - \langle -1|) - i \sin(\omega_0 t) \langle 0| \right] \left[2\hbar \cos(\omega_0 t) |0\rangle \right] \\
&= \hbar \cos(\omega_0 t) \sin(\omega_0 t) \\
&= \frac{\hbar}{2} \sin(2\omega_0 t),
\end{aligned}$$

$$\begin{aligned}
\langle \psi(t) | L_z | \psi(t) \rangle &= \langle \psi(t) | \left[\frac{\sqrt{2}}{2} \hbar \cos(\omega_0 t)(|1\rangle - |-1\rangle) + 0 |0\rangle \right] \\
&= \left[\frac{\sqrt{2}}{2} \cos(\omega_0 t)(\langle 1| - \langle -1|) - i \sin(\omega_0 t) \langle 0| \right] \left[\frac{\sqrt{2}}{2} \hbar \cos(\omega_0 t)(|1\rangle - |-1\rangle) + 0 |0\rangle \right] \\
&= \frac{1}{2} \hbar \cos^2(\omega_0 t) + \frac{1}{2} \hbar \cos^2(\omega_0 t) \\
&= \hbar \cos^2(\omega_0 t).
\end{aligned}$$

To show the motion of $\langle \boldsymbol{L} \rangle (t)$, we can consider the parametric motion in the $yz-$plane, since $\langle L_x \rangle (t) = 0$. Note that $\langle L_z \rangle (t) \geq 0$, while $\langle L_y \rangle (t)$ traces a sinusoid. The corresponding parametrization is a circle of radius $\frac{\hbar}{2}$ centered at $\left(x = 0, y = 0, z = \frac{\hbar}{2} \right)$. Geometrically,



where the radius is $\hbar/2$.

(d) First note that since $[L_z^2, L_z] = 0$, then L_z^2 and L_z can be simultaneously diagonalized and hence share a common set of eigenfunctions, which are previously given as $|1\rangle, |0\rangle, |-1\rangle$ in the L_z basis. Their corresponding eigenvalues are $\hbar^2, 0$, and $\hbar^2$ ($\hbar^2$ is two-fold degenerate), which can easily be found by squaring the matrix representaion of L_z in the L_z basis.
Observe that one result is possible to measure whenever $\cos(\omega_0 t) = 0$, since $\cos(\omega_0 t)$ is the time-dependent coefficient for two eigenstates $|1\rangle$ and $|-1\rangle$. Whenever $\cos(\omega_0 t) = 0$, $\sin(\omega_0 t) \neq 0$, and this occurs whenever $\omega_0 t = \pi \left(k + \frac{1}{2}\right)$ for an integer k. Here, the only result possible is measuing 0, associated with the eigenstate $|0\rangle$. Thus, one result is possible to be measured at every $t = \frac{\pi}{\omega_0}\left(k + \frac{1}{2}\right)$ for $k \in \mathbb{Z}$. Furthermore, since $|1\rangle$ and $|-1\rangle$ have the same eigenvalue of L_z^2 which is $\hbar^2$, another time at where only one value can be measured is whenever $\sin(\omega_0 t) = 0$, which is given for all $t = \frac{m\pi}{\omega_0}$ for an $m \in \mathbb{Z}$. Upon measurement of $|\psi(t)\rangle$ at this time, we obtain

$$L_z^2 |\psi(t = m\pi/\omega_0)\rangle = \frac{\sqrt{2}}{2}\cos(m\pi) L_z^2(|1\rangle - |-1\rangle) = \pm\frac{\sqrt{2}}{2}\hbar^2(|1\rangle - |-1\rangle),$$

which has eigenvalue $\pm\hbar^2$, depending on the value of m at the time. To put everything together, the times at which one value can be measured are

$$\begin{cases} t = \frac{\pi}{\omega_0}\left(k + \frac{1}{2}\right), & k \in \mathbb{Z}, & \text{measurement of } 0 \\ \\ t = \frac{m\pi}{\omega_0}, & m \in \mathbb{Z},\ m \text{ even}, & \text{measurement of } +\hbar^2 \\ \\ t = \frac{m\pi}{\omega_0}, & m \in \mathbb{Z},\ m \text{ odd}, & \text{measurement of } -\hbar^2 \end{cases}$$

Once a measurement of $\hbar^2$ has been made, the state must collapse to the result, as if the ket had always been in such a state at the time origin of the measurement. This is the projection postulate. Since the result $+\hbar^2$ has been measured, this must have been made at a time $t = \frac{2m\pi}{\omega_0}$ for any positive integer m (it was m, but I substituted $m \to 2m$ since m must be even at this time for a $+\hbar^2$, as indicated above). The state has now collapsed to

$$|\psi'(t = 0)\rangle = \frac{\sqrt{2}}{2}(|1\rangle - |-1\rangle),$$

which is already normalized. The state will now evolve at its new energy E' as determined by the time-evolution of the Schrödinger equation. These new energies are just the corresponding energies of the respective kets $|1\rangle$ and $|-1\rangle$ in the L_z basis, which are $E_{\pm 1} = \pm\hbar$. We have that

$$|\psi'(t)\rangle = \frac{\sqrt{2}}{2}\left(|1\rangle\, e^{-it} - |-1\rangle\, e^{it}\right).$$

Once again notice that a measurement of L_z^2 at $t = 0$ still yields the result $\hbar^2$, as required.

1. Particle in a cylindrically symmetric potential

Let ρ, φ, z be the cylindrical coordinates of a spinless particle ($x = \rho\cos\varphi$, $y = \rho\sin\varphi$; $\rho \geq 0$, $0 \leq \varphi < 2\pi$). Assume that the potential energy of this particle depends only on ρ, and not on φ and z. Recall that:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

a. Write, in cylindrical coordinates, the differential operator associated with the Hamiltonian. Show that H commutes with L_z and P_z. Show that this allows writing the wave functions associated with the stationary states of the particle as:

$$\varphi_{n,m,k}(\rho,\varphi,z) = f_{n,m}(\rho)\, \mathrm{e}^{im\varphi}\, \mathrm{e}^{ikz}$$

where the values that can be taken on by the indices m and k are to be specified.

b. Write, in cylindrical coordinates, the eigenvalue equation of the Hamiltonian H of the particle. Derive from it the differential equation that $f_{n,m}(\rho)$ obeys.

c. Let Σ_y be the operator whose action, in the $\{|\mathbf{r}\rangle\}$ representation, is to change y to $-y$ (reflection with respect to the xOz plane). Does Σ_y commute with H? Show that Σ_y anticommutes with L_z, and show that, as a result, $\Sigma_y|\varphi_{n,m,k}\rangle$ is an eigenvector of L_z. What is the corresponding eigenvalue? What can be concluded concerning the degeneracy of the energy levels of the particle? Could this result be predicted directly from the differential equation established in (b)?

(a) We begin by writing the Hamiltonian differential operator in cylindrical coordinates. In cartesian coordinates, it is given by

$$H = \frac{1}{2M}\left(P_x^2 + P_y^2 + P_z^2\right) + V(X,Y,Z).$$

Here I have used M as the mass variable instead of m to avoid any confusion later in the problem. Due to the cylindrical symmetry of the potential, P_z remains the momentum operator in the $z-$direction, meanwhile P_x and P_y change into the form as described in the problem. We have that

$$\begin{aligned}
H &= \frac{1}{2M}\left(P_x^2 + P_y^2\right) + \frac{1}{2M}P_z^2 + V(X,Y,Z) \\
&= \frac{1}{2M}\left((-i\hbar)^2\frac{\partial^2}{\partial x^2} + (-i\hbar)^2\frac{\partial^2}{\partial y^2}\right) + \frac{(-i\hbar)^2}{2M}\frac{\partial^2}{\partial z^2} + V(x,y,z) \\
&= -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} + V(x,y,z) \\
&= -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) + V(\rho),
\end{aligned}$$

where the last step follows because the potential $V(\rho,\varphi,z) = V(\rho)$ so the potential is independent of φ and z.

Observe that the angular momentum operator $L_z = -i\hbar\dfrac{\partial}{\partial\varphi}$ actually appears in the Hamiltonian, as well as $P_z = -i\hbar\dfrac{\partial}{\partial z}$. Allow me to briefly define the differential operator $Q = -\hbar^2\left(\dfrac{\partial^2}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\right)$, that I may quickly compute the commutation relations. The Hamiltonian becomes

$$H = \frac{1}{2M}\left(Q + \frac{L_z^2}{\rho^2} + P_z^2\right) + V.$$

First note that $[L_z^2, L_z] = [P_z^2, P_z] = 0$ from direct observation. Furthermore, the commutator of L_z and P_z is

$$\begin{aligned}
[L_z, P_z] &= [XP_y - YP_x, P_z] = [XP_y, P_z] - [YP_x, P_z] \\
&= X[P_y, P_z] + [X, P_z]P_y - Y[P_x, P_z] - [Y, P_z]P_x \\
&= 0,
\end{aligned}$$

if we recall the commutation identities $[X_i, X_j] = [P_i, P_j] = 0$ and $[X_i, P_j] = i\hbar\delta_{ij}$. Since the operators $P_\rho = -i\hbar\dfrac{\partial}{\partial\rho}$ and P_z act on orthognal axes in the position space, we have that $[P_\rho, P_z] = 0$, and thus $[Q, P_z] = 0$. To show $[P_\rho, L_z] = 0$, allow me consider the action of the commutator on a state in such position space:

$$\begin{aligned}
\langle \boldsymbol{r}|[P_\rho, P_z]|\psi\rangle &= (P_\rho P_z - P_z P_\rho)\,\psi(\boldsymbol{r}) \\
&= \left[-\hbar^2\frac{\partial}{\partial\rho}\frac{\partial}{\partial\varphi} + \hbar^2\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\rho}\right]\psi(\boldsymbol{r}) \\
&= -\hbar^2\left[\frac{\partial}{\partial\rho}\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\rho}\frac{\partial}{\partial\varphi}\right]\psi(\boldsymbol{r}) \\
&= 0.
\end{aligned}$$

Since $[P_\rho, L_z] = 0$, then $[Q, L_z] = 0$ as well since Q is comprised of P_ρ operators. Therefore

$$\begin{aligned}
[H, P_z] &= \frac{1}{2M}\left[Q + \frac{L_z^2}{\rho^2} + P_z^2 + 2MV(\rho), P_z\right] \\
&= \frac{1}{2M}[Q, P_z] + \frac{1}{2M}\left[\frac{L_z^2}{\rho^2}, P_z\right] + \frac{1}{2M}\left[P_z^2, P_z\right] + [V(\rho), P_z] \\
&= (0) + (0) + (0) + (V(\rho)P_z - P_z V(\rho)) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
[H, L_z] &= \frac{1}{2M}\left[Q + \frac{L_z^2}{\rho^2} + P_z^2 + 2MV(\rho), L_z\right] \\
&= \frac{1}{2M}[Q, L_z] + \frac{1}{2M}\left[\frac{L_z^2}{\rho^2}, L_z\right] + \frac{1}{2M}\left[P_z^2, L_z\right] + [V(\rho), L_z] \\
&= (0) + (0) + (0) + (V(\rho)L_z - L_z V(\rho)) \\
&= 0.
\end{aligned}$$

Therefore H, L_z and P_z all commute with each other and hence share a common set of eigenstates. Since the common eigenstates for P_z are $\varphi_k(z) = e^{ikz}$ with eigenvalue k, and L_z are $\varphi_m(\varphi) = e^{im\varphi}$ with eigenvalue m, then the tensor product between such seperable state spaces yield the common eigenstate $\varphi_{m,k}(\varphi, z) = e^{im\varphi}e^{ikz}$. Since the Hamiltonian is then radially dependent due to the potential, we say a function $f_{n,m}(\rho)$ is the common eigenfunction which satisfies the radial component of the Hamiltonian eigenvalue equation, which will be derived in part (b). Therefore the eigenstates of the Hamiltonian can be written as

$$\varphi_{n,m,k}(\rho,\varphi,z) = f_{n,m}(\rho)e^{im\varphi}e^{ikz}.$$

(b) The Hamiltonian eigenvalue equation is given by $H\varphi = E\varphi$, hence in position space can be written in cylindrical form as

$$\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial z^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)+V(\rho)\right] f_{n,m}(\rho)e^{im\varphi}e^{ikz} = E_{n,m,k}\, f_{n,m}(\rho)e^{im\varphi}e^{ikz}.$$

One may follow through the derivation to determine the ordinary differential equation in ρ which $f_{n,m}(\rho)$ must satisfy. Let us focus on the left hand side of the relation:

$$\begin{aligned}
E_{n,m,k}\varphi_{n,m,k} &= e^{im\varphi}e^{ikz}\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\right] f_{n,m}(\rho) + \frac{\hbar^2k^2}{2M}e^{im\varphi}e^{ikz}f_{n,m}(\rho)\\
&\qquad + \frac{\hbar^2m^2}{2M}e^{im\varphi}e^{ikz}\frac{1}{\rho^2}f_{n,m}(\rho) + e^{im\varphi}e^{ikz}V(\rho)f_{n,m}(\rho)\\
&= e^{im\varphi}e^{ikz}\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\right] f_{n,m}(\rho) + e^{im\varphi}e^{ikz}f_{n,m}(\rho)\frac{\hbar^2k^2}{2M}\\
&\qquad + e^{im\varphi}e^{ikz}\left[\frac{\hbar^2m^2}{2M}\frac{1}{\rho^2}+V(\rho)\right] f_{n,m}(\rho)\\
&= e^{im\varphi}e^{ikz}\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+V(\rho)\right] f_{n,m}(\rho) + e^{im\varphi}e^{ikz}f_{n,m}(\rho)\frac{\hbar^2k^2}{2M}.
\end{aligned}$$

One may clearly see the differential equation $f_{n,m}(\rho)$ must satisfy:

$$E_{n,m}f_{n,m}(\rho) = \left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+V(\rho)\right] f_{n,m}(\rho).$$

Furthermore, it is easy to see the dependence of f on the indices n and m. The n index comes from the relation of partial differentials with respect to ρ which $f_{n,m}(\rho)$ must satisfy, while the m index comes from the angular momentum.

(c) Let $S_y \equiv \Sigma_y$. First note the action of S_y on the momentum operators. Since S_y deptices a reflection in the $xz-$plane, only the $y-$dependent components are affected. That is,

$$S_yP_x = P_x, \qquad S_yP_y = -P_y \qquad S_yP_z = P_z.$$

However, when each of these operators are squared to determine their magnitude, it is easy to geometrically see that

$$S_yP_x^2 = P_x^2, \qquad S_yP_y^2 = P_y^2 \qquad S_yP_z^2 = P_z^2,$$

hence the S_y operator acts as an identity to momentum magnitudes. Consider the commutators $[S_y, P_x]$ and $[S_y, P_z]$. Since these momentum operators are orthogonal to the components of S_y, they commute with each other. This implies that $[S_y, P_x^2] = [S_y, P_z^2] = 0$. To consider the commutator $[S_y, P_y^2]$, first note the action of S_y on P_y^2 is an identity, since

$$S_yP_y^2 = (-1)^2P_y^2 = P_y^2.$$

This implies that $[S_y, P_y^2] = 0$ as well. For this problem, considering the Hamiltonian in cartesian coordinates will allow me to easily find the commutator $[S_y, H]$, yet I must begin by showing that $[S_y, V(X,Y,Z)] = 0$. Consider the action of such a commutator on a state in the cartesian position representation:

$$\begin{aligned}\langle \boldsymbol{r}|[S_y, V(X,Y,Z)]|\psi\rangle &= (S_y V(x,y,z) - V(x,y,z)S_y)\,\psi(x,y,z) \\ &= S_y\,(V(x,y,z)\psi(x,y,z)) - V(x,y,z)\psi(x,-y,z) \\ &= V(x,-y,z)\psi(x,-y,z) - V(x,y,z)\psi(x,-y,z)\end{aligned}$$

however since V is a cylindrically symmetric potential, we have that any relfection in the x or y axes is equivalent to if there were none. Hence $V(x,y,z) = V(x,-y,z)$ and $[S_y, V(X,Y,Z)] = 0$. For the Hamiltonian itself,

$$\begin{aligned}[S_y, H] &= \frac{1}{2M}[S_y, P_x^2 + P_y^2 + P_z^2] + [S_y, V(X,Y,Z)] \\ &= 0.\end{aligned}$$

Now consider the angular momentum in the z-direction, given by $L_z = XP_y - YP_x$. Consider the anticommutator of L_z and S_y acting on an arbitrary state ψ in the cartesian position representation:

$$\begin{aligned}\langle \boldsymbol{x}|\{S_y, L_z\}|\psi\rangle &= (S_y L_z + L_z S_y)\,\psi(x,y,z) \\ &= [S_y\,(XP_y - YP_x) + (XP_y - YP_x)S_Y]\,\psi(x,y,z) \\ &= (-XP_y + YP_x)\,\psi(x,-y,z) + (XP_y - YP_x)\psi(x,-y,z) \\ &= -L_z\psi(x,-y,z) + L_z\psi(x,-y,z) \\ &= 0.\end{aligned}$$

Now Consider the representation of $|\varphi_{n,m,k}\rangle$ in the cylindrical coordinate system, then allow me to change coordinate systems into the cartesian basis:

$$f_{n,m}(\rho)e^{im\varphi}e^{ikz} \longrightarrow f_{n,m}(\sqrt{x^2+y^2})e^{im\arctan(y/x)}e^{ikz}.$$

It is clear to see that $S_y f_{n,m}(\sqrt{x^2+y^2}) = f_{n,m}(\sqrt{x^2+y^2})$, hence $f_{n,m}(\rho)$ is unchanged by symmetry. Further, notice that $S_y \arctan(y/x) = -\arctan(y/x)$, so $S_y e^{im\varphi} = e^{-im\varphi}$. This implies that

$$S_y\, f_{n,m}(\rho)e^{im\varphi}e^{ikz} = f_{n,m}(\rho)e^{-im\varphi}e^{ikz}.$$

If we act on this state with the differential operator L_z in the cylindrical coordinate basis, notice that

$$\begin{aligned}L_z(S_y\, f_{n,m}(\rho)e^{im\varphi}e^{ikz}) &= -i\hbar\frac{\partial}{\partial\varphi}\, f_{n,m}(\rho)e^{-im\varphi}e^{ikz} \\ &= -\hbar m\, f_{n,m}(\rho)e^{-im\varphi}e^{ikz},\end{aligned}$$

which shows that $S_y\,|\varphi_{n,m,k}\rangle$ is also an eigenvector of L_z with eigenvalue $-im$. Acting within the Hamiltonian, due to the L_z^2 operator acting on $|\varphi_{n,m,k}\rangle$, one can see that the degeneracy of these eigenvalues are affected. The differential equation which $f_{n,m}(\rho)$ must satisfy,

$$E_{n,m}f_{n,m}(\rho) = \left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\right) + V(\rho)\right] f_{n,m}(\rho),$$

holds both for $|\varphi_{n,m,k}\rangle$ and $S_y\,|\varphi_{n,m,k}\rangle$. In such a case, there are two values of m (which are $\pm m$) which satisfy this energy eigenvalues differential equation for $f_{n,m}(\rho)$. This is because $(\pm im)^2 = -m^2$, so each energy level is at least two-fold degenerate. From an angular momentum perspective, this is reasonable because the values of m range from $-l$ to l, so the $-m^2$ term has already accounted for two-fold degeneracy of $E_{n,m}$. This result could have been predicted from the differential equation which $f_{n,m}(\rho)$ must satisfy.