MAT224 Linear Algebra II
Final Assessment

Instructions:

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3. **Show your work and justify your steps** on every question unless otherwise indicated. Put your final answer in the box provided, if necessary.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for the test.

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Full Name: Jace Alloway

Student number: 1006940802

I confirm that:

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- I have not used any resources other than those that are listed as permitted in the Final Assessment Information document at any point during the test.
- I have not participated in or enabled any MAT224 group chat during the test.
- I have not viewed the answers, solutions, term work, or notes of anyone.
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Signature: Jace Alloway

1. Decide if the following statement is True or False:

Let $\alpha = \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis for a vector space V over $\mathbb{C}$, and let $T \in \mathfrak{L}(V)$ be defined by

$$[T]_\alpha^\alpha = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let W be the set of vectors $\mathbf{x} \in V$ that satisfy $(T - 3I)^2\mathbf{x} = \mathbf{0}$ but $(T - 3I)\mathbf{x} \neq \mathbf{0}$, together with the zero vector. Then W is a subspace of V.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

I want to prove that W (given above) is a subspace of V.

Proof.

We have that $([T]_\alpha^\alpha - 3I) = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

It then follows that $([T]_\alpha^\alpha - 3I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

We need a vector $\mathbf{x} = (x_1, x_2, x_3)$ to satisfy $(T - 3I)^2\mathbf{x} = \mathbf{0}$ and $(T - 3I)\mathbf{x} \neq \mathbf{0}$. We have

$$([T]_\alpha^\alpha - 3I)^2\mathbf{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_3 = 0, \text{ by matrix multiplication.}$$

We also have

$$([T]_\alpha^\alpha - 3I)\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_2 \neq 0.$$

Therefore we can define W as $W = \{(x_1, x_2, 0) \in V \mid x_2 \neq 0\}$.
Now, it suffices to show that W is a subspace of V. Let $\mathbf{x}, \mathbf{y} \in W$ and let $a, b \in \mathbb{C}$. We have

$$\begin{aligned} (a\mathbf{x} + b\mathbf{y}) &= (ax_1 + by_1, ax_2 + by_2, 0) \\ &= (ax_1, ax_2, 0) + (by_1, by_2, 0) && \text{by addition of complex variables} \\ &= a(x_1, x_2, 0) + b(y_1, y_2, 0) && \text{by axiom } \mathbf{vii}\text{, distributivity} \\ &= a\mathbf{x} + b\mathbf{y} \end{aligned}$$

Thus W is closed under scalar multiplication and vector addition as defined in $\mathbb{C}$.
Since W is non-empty, $\mathbf{0} \in W$ (as given, "together with the zero vector") and W is closed under the operations of addition and multiplication defined in $\mathbb{C}$, then W is a subspace of V.

■

2. Decide if the following statement is True or False:

Let V be a vector space over $\mathbb{C}$, and let $T \in \mathfrak{L}(V)$ have characteristic polynomial $(\lambda+1)(\lambda+2)^3$. If T is not diagonalizable, then there exists a vector $\mathbf{x} \in V$ such that $(T+2I)^3\mathbf{x} = \mathbf{0}$ but $(T+2I)^2\mathbf{x} \neq \mathbf{0}$.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

○ True
● False

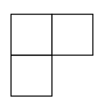
This is false, and I will provide a counter-example.

Consider the matrix $A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. The characteristic polynomial of A is given by

$$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 1 & 1 & 0 \\ 0 & -2-\lambda & 1 & 0 \\ 0 & 0 & -2-\lambda & 0 \\ 0 & 0 & 0 & -2-\lambda \end{bmatrix} = \lambda^4 + 7\lambda^3 + 18\lambda + 20\lambda + 8 = (\lambda+1)(\lambda+2)^3.$$

We have that $(A+2I)^2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and we also have that $(A-2I)^3 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^3 =$
$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ as well, so they are the same matrix.

The cycle tableau for $\lambda = -2$ in the jth column is given by $\dim(\ker(A-\lambda I)^j) - \dim(\ker(A-\lambda I)^{j-1})$, for which we have $\operatorname{rank}(A+2I) = 2 \implies \dim(\ker(A+2I)) = 2$ and $\operatorname{rank}(A+2I)^2 = \operatorname{rank}(A+2I)^3 = 1 \implies \dim(\ker(A+2I)^2) = \dim(\ker(A+2I)^3) = 3$ by the Rank-Nullity Theorem, and so our cycle tableaux for the Jordan Block J_{-2} would look like



which yields the 3×3 Jordan Block matrix $J_{-2} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

It then follows that J_{-1} is a 1×1 Jordan block, and we have that the Jordan form for A is $J_A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$,

which is not a diagonal form, and thus A is not diagonalizable.

Notice that for all $n \geq 2$, we have that $(A+2I)^n = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which implies that there is no such vector $\mathbf{x} \in V$ where $(A+2I)^2\mathbf{x} \neq \mathbf{0}$ and $(A+2I)^3\mathbf{x} = \mathbf{0}$ simply because $(A+2I)^2 = (A+2I)^3$, which is a counter example to this statement.

3. Decide if the following statement is True or False:

Let V be a vector space over a field F, and let $S, T \in \mathfrak{L}(V)$ be such that $ST = TS$. For any $\lambda \in F$, $\ker(S - \lambda I)^2$ is T-invariant.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

This statement is true. I want to prove that if $ST = TS$, then for any $\lambda \in F$, $\ker(S - \lambda I)^2$ is T-invariant.

Proof.
Let $\mathbf{x} \in \ker(S - \lambda I)^2$ for any $\lambda \in F$. Then $(S - \lambda I)^2\mathbf{x} = \mathbf{0}$.
Since $ST = TS$, we have that $S = T^{-1}ST$. This implies that

$$(S - \lambda I)^2\mathbf{x} = (T^{-1}ST - \lambda I)^2\mathbf{x} = (T^{-1}ST - \lambda T^{-1}IT)\mathbf{x} = (T^{-1}(S - \lambda I)T)^2\mathbf{x} = \mathbf{0}.$$

Notice that $(T^{-1}(S - \lambda I)T)^2 = T^{-1}(S - \lambda I)TT^{-1}(S - \lambda I)T = T^{-1}(S - \lambda I)^2T$, and thus we have

$$(T^{-1}(S - \lambda I)T)^2\mathbf{x} = T^{-1}(S - \lambda I)^2T\mathbf{x} = \mathbf{0}.$$

By left multiplying by T, we have

$$TT^{-1}(S - \lambda I)^2T\mathbf{x} = T\mathbf{0} \implies (S - \lambda I)^2T\mathbf{x} = \mathbf{0},$$

and thus $T\mathbf{x} \in \ker(S - \lambda I)^2$.
Therefore $\ker(S - \lambda I)^2$ is invariant under T, which is what I wanted to prove.

■

4. Decide if the following statement is True or False:

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{F}_2)$ is similar to one of the matrices in $M_{2\times 2}(\mathbb{F}_2)$ listed below:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

○ True
● False

No, the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is not similar to any of the following 6 matrices.

Proof.
A matrix is similar to another matrix when there exists an invertible matrix $P \in M_{2\times 2}(\mathbb{F}_2)$ such that $A = P^{-1}BP$.

The only invertible matrices in $M_{2\times 2}(\mathbb{F}_2)$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

with respective inverses

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is easy to see that the three matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and their inverses in form $P^{-1}BP$ all compute the same matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

which is not one of the listed 6 matrices.
For the remaining 3 matrices, we have

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

which are all unlisted matrices above.
Therefore I have shown a counterexample, and thus the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is not similar to any of the 6 matrices listed above over the field $\mathbb{F}_2$.

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5. Consider the following statement, which is true:

"Let V be a vector space over a field F, and let $T \in \mathfrak{L}(V)$. Suppose that for some $\mathbf{x} \in V$, we have $T^3\mathbf{x} = \mathbf{0}$, and $V = C(\mathbf{x})$, where $C(\mathbf{x})$ is the cyclic subspace generated by $\mathbf{x}$. Then 1 is not an eigenvalue of T."

Read the following "proof" of the statement, which is incorrect.

"Proof": Suppose $T\mathbf{v} = \mathbf{v}$ for some $\mathbf{v} \in V$. We will show $\mathbf{v} = \mathbf{0}$ and therefore 1 is not an eigenvalue of T.

We are given $V = C(\mathbf{x}) = \text{span}\{\mathbf{x}, T\mathbf{x}, T^2\mathbf{x}, \dots\}$ and $T^3\mathbf{x} = \mathbf{0}$. Since $T^3\mathbf{x} = \mathbf{0}$, then $T^k\mathbf{x} = \mathbf{0}$ for all $k \geq 3$, and $V = \text{span}\{\mathbf{x}, T\mathbf{x}, T^2\mathbf{x}\}$.

Therefore, there exist scalars $c_0, c_1, c_2 \in F$ such that

$$\mathbf{v} = c_0\mathbf{x} + c_1T\mathbf{x} + c_2T^2\mathbf{x} \tag{1}$$

Applying T to both sides of this expression, together with our assumption that $T\mathbf{v} = \mathbf{v}$, gives

$$\mathbf{v} = c_0T\mathbf{x} + c_1T^2\mathbf{x} \tag{2}$$

Equating the expressions (1) and (2) for $\mathbf{v}$, and rearranging gives

$$c_0\mathbf{x} + (c_1 - c_0)T\mathbf{x} + (c_2 - c_1)T^2\mathbf{x} = \mathbf{0}$$

which implies $c_0 = 0, c_1 = c_0 = 0$, and $c_2 = c_1 = 0$. Therefore, $\mathbf{v} = c_0\mathbf{x} + c_1T\mathbf{x} + c_2T^2\mathbf{x} = \mathbf{0}$ which is what we wanted to show.

What is wrong with this "proof"? Clearly identify any incorrect assumptions or conclusions in the "proof", and briefly explain why they are incorrect. Your answer should be no longer than one paragraph. [4 marks]

One error in the proof is the assumption that proving that if $\mathbf{v} = \mathbf{0}$, then the statement is proved and fully justified (it's not). It is still possible for $\mathbf{v} = \mathbf{0}$ and 1 to be an eigenvalue of T, since we would have $T\mathbf{0} = 1\mathbf{0} = \mathbf{0}$. This is the property of the zero vector in a linear transformation.
In the proof, once the author assumed $T\mathbf{v} = \mathbf{v}$, they had that $T\mathbf{v} - \mathbf{v} = \mathbf{0}$ which actually implies that $(T - I)\mathbf{v} = \mathbf{0}$, which just shows that $\mathbf{v} \in \ker(T - I)$. By definition, $\mathbf{v}$ cannot be an eigenvector in general because $\mathbf{v} = \mathbf{0}$, which still does not justify why 1 could not be an eigenvalue of T because $\mathbf{0}$ is contained in every subspace (in this case, $\ker(T-I)$).
The other error in the proof is what the author was initially trying to show. When proving 'for some' $\mathbf{v} \in V$ and 'there exists scalars $c_0, c_1, c_2 \in F$', they are just showing that the single vector $\mathbf{v} = \mathbf{0}$ satisfies $T\mathbf{v} = \mathbf{v}$ (which obviously it does), once again not justifying why 1 could not be an eigenvalue for T for the generalization of all vectors $\mathbf{v} \in V$.

6. Let V be a finite dimensional vector space over a field F, and let $R, T \in \mathfrak{L}(V)$. Suppose that $S \in \mathfrak{L}(V)$ is invertible, and that $T = SRS^{-1}$. Let $K_\lambda(T)$ denote the λ-generalized eigenspace of T; and $K_\lambda(R)$ the λ-generalized eigenspace of R.

6(a) Prove that $\mathbf{x} \in K_\lambda(T)$ if and only if $S^{-1}\mathbf{x} \in K_\lambda(R)$. [4 marks]

I want to prove that for $R, T \in \mathfrak{L}(V)$, if $T = SRS^{-1}$ for an invertible S, then $\mathbf{x} \in K_\lambda(T) \iff S^{-1}\mathbf{x} \in K_\lambda(R)$.

Proof.
$\rightarrow$: Assume $\mathbf{x} \in K_\lambda(T)$. Then $(T - \lambda I)^k \mathbf{x} = \mathbf{0}$ for some integer $k \geq 0$.
Since $T = SRS^{-1}$, we have that

$$(T - \lambda I)^k \mathbf{x} = (SRS^{-1} - \lambda SIS^{-1})^k \mathbf{x} = (S(R - \lambda I)S^{-1})^k \mathbf{x} = S(R - \lambda I)^k S^{-1}\mathbf{x} = \mathbf{0}.$$

This follows from the fact that the matrix S cancels from $S^{-1}S = I$, which is

$$(S(R - \lambda I)S^{-1})^k = S(R - \lambda I)S^{-1}S(R - \lambda I)S^{-1}S \cdots = S(R - \lambda I)^k S^{-1}.$$

By left-multiplying by S^{-1}, we have

$$S^{-1}S(R - \lambda I)^k S^{-1}\mathbf{x} = I(R - \lambda I)^k S^{-1}\mathbf{x} = S^{-1}\mathbf{0} \implies (R - \lambda I)^k S^{-1}\mathbf{x} = \mathbf{0},$$

and therefore so $S^{-1}\mathbf{x} \in K_\lambda(R)$.

$\leftarrow$: Now assume that $S^{-1}\mathbf{x} \in K_\lambda(R)$. Then $(R - \lambda I)^k S^{-1}\mathbf{x} = \mathbf{0}$ for some integer $k \geq 0$.
As before, we have that $SRS^{-1} = T$. By left-multiplying by S, we have

$$S(R - \lambda I)^k S^{-1}\mathbf{x} = S\mathbf{0} = \mathbf{0}.$$

Because $(S(R - \lambda I)S^{-1})^k = S(R - \lambda I)^k S^{-1}$ as before, we have

$$S(R - \lambda I)^k S^{-1}\mathbf{x} = (S(R - \lambda I)S^{-1})^k \mathbf{x} = (SRS^{-1} - \lambda SIS^{-1})^k \mathbf{x} = (T - \lambda I)^k \mathbf{x} = \mathbf{0},$$

and so $\mathbf{x} \in K_\lambda(T)$.

Therefore $\mathbf{x} \in K_\lambda(T) \iff S^{-1}\mathbf{x} \in K_\lambda(R)$, which is what I needed to prove.

$\blacksquare$

6. Let V be a finite dimensional vector space over a field F, and let $R, T \in \mathfrak{L}(V)$. Suppose that $S \in \mathfrak{L}(V)$ is invertible, and that $T = SRS^{-1}$. Let $K_\lambda(T)$ denote the λ-generalized eigenspace of T; and $K_\lambda(R)$ the λ-generalized eigenspace of R.

6(b) Prove that $\dim K_\lambda(T) = \dim K_\lambda(R)$. [4 marks]

I want to prove that for $R, T \in \mathfrak{L}(V)$, if $T = SRS^{-1}$ for an invertible S, then $\dim K_\lambda(T) = \dim K_\lambda(R)$.

Proof.
Suppose that $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a basis for the generalized eigenspace $K_\lambda(T)$.
Then α consists of linearly independent vectors $\mathbf{x}_i$, for $i = 1, \ldots, k$.

From part **6(a)**, we know that each vector $\mathbf{x} \in K_\lambda(T)$ is also $S^{-1}\mathbf{x} \in K_\lambda(R)$. Since S is invertible, then S is an isomorphism, so it preserves linear independence (bases!).
Therefore the list $\alpha' = \{S^{-1}\mathbf{x}_1, S^{-1}\mathbf{x}_2, \ldots, S^{-1}\mathbf{x}_k\}$ is also linearly independent, and thus it is a basis for $K_\lambda(R)$ for $\mathbf{x}_i \in \alpha$.

The bases α and α' both have k elements, and thus $\dim K_\lambda(T) = \dim K_\lambda(R)$, which is what I wanted to prove.

$\blacksquare$

7. Let V be a finite dimensional vector space over $\mathbb{C}$, and let $T \in \mathfrak{L}(V)$. Prove that if $\dim \operatorname{im} T = 1$, then either T is diagonalizable or T is nilpotent. [4 marks]

I want to prove that if dim im $T = 1$, then T is either diagonalizable or nilpotent.

Proof.
Let $\dim V = n$, and assume that dim $T = 1$. I will break this proof into two cases: when T is diagonalizable, it cannot be nilpotent, and when T is nilpotent, it cannot be diagonalizable.

[T nilpotent $\implies$ T not diagonalizable]:
Assume the map $T \in \mathfrak{L}(V)$ is nilpotent. Then since $\dim V = n$, $T^n = 0$. If T were diagonalizable, we would have to have that there exists an invertible matrix P such that $T = P^{-1}DP$, which implies that $T^n = P^{-1}D^nP$.

However, since $T^n = 0$, then it must be that $P^{-1}D^nP = 0 \implies D^n = 0 \implies D = 0$. If $\mathbf{v}$ is a non-zero eigenvector for T, we also have that $T^n\mathbf{v} = \lambda^n\mathbf{v} = 0 \implies \lambda = 0$. Thus, the only eigenvalues of T are zero, since T is nilpotent.

This implies that the only diagonalizable nilpotent matrix is the zero matrix, however since $\dim \operatorname{im} T = 1$, then $\operatorname{rank} T = 1 \neq 0$, so T cannot be the zero matrix. We have reached a contradiction.

Therefore T cannot be diagonalizable if it is nilpotent.

[T diagonalizable $\implies$ T not nilpotent]:
Assume T is diagonalizable. Then there must be n distinct eigenvectors, one with eigenvalue $\lambda \neq 0$ because $\dim \operatorname{im} T = 1$.

By the Rank-Nullity Theorem, since $\dim \operatorname{im} T = 1$, and $\dim V = n$, then $\dim \ker T = n - 1$, which directly implies that there must be $n - 1$ distinct eigenvectors with eigenvalues $\lambda = 0$.

Since there exists at least one eigenvalue $\lambda \neq 0$, T cannot be nilpotent. This is because every eigenvalue would have to be zero if T is nilpotent, as previously shown.

Therefore T cannot be nilpotent if it is diagonalizable.

Therefore if $\dim \operatorname{im} T = 1$, then T is either diagonalizable or nilpotent, but not both, which is what I needed to prove.

■

8. Let V be an n-dimensional vector space over $\mathbb{C}$, and let $T \in \mathfrak{L}(V)$.

In your own words, give a sketch a proof that the following two statements are equivalent:

1. V can be decomposed into a direct sum of two, non-trivial, T-invariant subspaces U and W.
2. The Jordan form of T consists of more than one Jordan block.

[4 marks]

Explanation:
Suppose α is a basis for the subspace U, where $\dim U = a$, and β is a basis for the subspace W, where $\dim W = b$. Then $a + b = n$, where $n = \dim V$. Then, the basis for V is given by $\nu = \alpha \cup \beta$ because $V = U + W$ and $\alpha \cap \beta = \{\mathbf{0}\}$ since $U \cap W = \{\mathbf{0}\}$.
For the linear map $T : V \to V$, we then have that the restriction of T to U is $T|_U$, which restricts V to the basis α because U is invariant. I will denote the matrix $[T|_U] = A$. Similarly, we have $T|_W$ restricts V to the basis β, whose matrix I will denote $[T|_W] = B$.
For any $\mathbf{x} \in \alpha$, $T\mathbf{x} \in \alpha$ and for any $\mathbf{y} \in \beta$, $T\mathbf{y} \in \beta$. Thus, our matrix for T is composed of blocks corresponding to each basis for each subspace:

$$[T]_\nu^\nu = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where again A is the matrix for $T|_U$ and B the matrix for $T|_W$. Therefore $[T]_\nu^\nu = A \oplus B$.

This is equivalent to **(2)** because the matrix $[T]_\nu^\nu = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is comprised of two block matrices A and B. When in Jordan Form, both matrices A and B will have at least one Jordan Block each, hence *at least* two Jordan Blocks for $[T]_\nu^\nu = A \oplus B$ (which is more than one).

Therefore these statements are equivalent.