MAT237 Multivariable Calculus with Proofs
Problem Set 5
Due Friday January 28, 2021 by 13:00 ET

Instructions

This problem set is based on Module F: Integrals (F1 to F6). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 5 sessions are held on Tuesday January 25, 2021 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope.** Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: **Jace Alloway** ______

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Student number: ______

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway ______

2) ______

Created and revised by C. Jonker, T. Janisse, and A. Zaman. Last updated January 27, 2022 at 21:29:14.

Problems

1. Fix $a_1, b_1, a_2, c_2, b_2 \in \mathbb{R}$ such that $a_1 < b_1$ and $a_2 < c_2 < b_2$. Define the rectangles

$$R = [a_1, b_1] \times [a_2, b_2], \qquad R' = [a_1, b_1] \times [a_2, c_2], \qquad R'' = [a_1, b_1] \times [c_2, b_2]$$

so $R = R' \cup R''$ and the interiors of R' and R'' are disjoint.

(1a) Disprove that if P' is a partition of R' and P'' is partition of R'', then $P = P' \cup P''$ is a partition of R. (Revised 2022-01-14)

I will disprove this with a counterexample:

- Fix $a_1 = 0, b_1 = 6, a_2 = 0, b_2 = 6$, and $c_2 = 3$ and define the rectangle R as $R = [0,6] \times [0,6]$.
- Now define two subrectangles of R as $R' = [0,6] \times [0,3]$ and $R'' = [0,6] \times [3,6]$. Then $R'^o \cap R''^o = \varnothing$.
- Construct partitions P' of R' and P'' of R'' as

$$\begin{aligned} P' &= \{[0,3] \times [0,3], [3,6] \times [0,3]\} \\ P'' &= \{[0,6] \times [3,6]\}. \end{aligned}$$

- Now let $P = P' \cup P'' = \{[0,3] \times [0,3], [3,6] \times [0,3], [0,6] \times [3,6]\}$. This Lemma claims that P is also a partition of R.
- However, P is not a partition of R. The partition along the $[0,6]$ interval on the x-axis is $\{0,3,6\}$ and $\{0,6\}$ for different y values. The partition along $x = 3$ is only defined for $0 \leq y \leq 3$, so P is not a partition of R, which disproves this Lemma.

(1b) Prove that if P is a partition of R, then there exists a partition P' of R' and a partition P'' of R'' such that $P' \cup P''$ is a partition and is a refinement of P. (Revised 2022-01-14)

Proof. Let $P = \{R_{ij}\}_{i,j}$ be a partition of R where $1 \leq i \leq n$ and $1 \leq j \leq m$. Then there exists partitions $\{x_0, x_1, \dots, x_n\}$ of $[a_1, b_1]$ and $\{y_0, y_1, \dots, y_m\}$ of $[a_2, b_2]$ such that each $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Fix $N_1, N_2, N_3 \in \mathbb{N}$. Construct regular partitions of the intervals $[a_1, b_1]$, $[a_2, c_2]$ and $[c_2, b_2]$, such that $\{x'_0, x'_1, \dots, x'_{N_1}\}$ equally subdivides $[a_1, b_1]$ into N_1 subintervals each with equivalent length $\dfrac{b_1 - a_1}{N_1}$. Repeat this process likewise with $\{y'_0, y'_1, \dots, y'_{N_2}\}$ for $[a_2, c_2]$ and $\{y''_0, y''_1, \dots, y''_{N_3}\}$ for $[c_2, b_2]$.

We proceed by constructing the partitions of the intervals $[a_1, b_1], [a_2, c_2]$ and $[c_2, b_2]$ respectively as

$$\begin{aligned}
\{X_1, X_2, \dots, X_z\} &= \{x'_0, \dots, x'_{N_1}\} \cup \{x_0, \dots, x_n\} \\
\{Y'_1, Y'_2, \dots, Y'_{z'}\} &= \{y'_0, \dots, y'_{N_2}\} \cup ([a_2, c_2] \cap \{y_0, \dots, y_m\}) \\
\{Y''_1, Y''_2, \dots, Y''_{z''}\} &= \{y''_0, \dots, y''_{N_3}\} \cup ([c_2, b_2] \cap \{y_0, \dots, y_m\}).
\end{aligned}$$

Choose partition $P' = \{R'_{qr}\}_{q,r}$ as defined by $R'_{qr} = [X_{q-1}, X_q] \times [Y'_{r-1}, Y'_r]$ for $1 \leq q \leq z$ and $1 \leq r \leq z'$. Similarly, choose partition $P'' = \{R''_{qs}\}_{q,s}$ as defined by $R''_{qs} = [X_{q-1}, X_q] \times [Y''_{s-1}, Y''_s]$ for $1 \leq q \leq z$ and $1 \leq s \leq z''$.

Let $\{Y_1, Y_2, \dots, Y_v\} = \{Y'_1, \dots, Y'_{z'}\} \cup \{Y''_1, \dots, Y''_{z''}\}$. Then $P' \cup P'' = \{R_{qt}\}_{q,t}$ is also a partition of R constructed by $R_{qt} = [X_{q-1}, X_q] \times [Y_{t-1}, Y_t]$ for $1 \leq q \leq z$ and $1 \leq t \leq v$. Furthermore, $P' \cup P''$ is a refinement of P since $\{x_0, x_1, \dots, x_n\} \subseteq \{X_1, X_2, \dots, X_z\}$ and $\{y_0, y_1, \dots, y_m\} \subseteq \{Y_1, Y_2, \dots, Y_v\}$.

□

(1c) Let $f : R \to \mathbb{R}$ be bounded. Use (1b) to show that if P is a partition of R, then there exists a partition P' of R' and a partition P'' of R'' such that

$$L_P(f) \le L_{P'}(f) + L_{P''}(f), \qquad U_P(f) \ge U_{P'}(f) + U_{P''}(f).$$

Proof. Let $P = \{R_{ij}\}_{ij}$ be a partition of R. By **(1b)**, there exists partitions P' and P'' such that $P' \cup P''$ is a refinement of P. Let $P' = \{R'_{kl}\}_{k,l}$ and $P'' = \{R''_{mn}\}_{m,n}$. Since $P' \cup P''$ is a refinement of P, then for each k, l and m, n, there exists unique indices i', j' and i'', j'' such that $R'_{kl} \subseteq R_{i'j'}$ and $R''_{mn} \subseteq R_{i''j''}$.

Fix subrectangles $R_{i'j'} \subseteq R'$ and $R_{i''j''} \subseteq R''$ of P for some i', j' and i'', j''. As before there exists a finite set of indices $I'_{i',j'} = \{k', l'\}_{i',j'}$ and $I''_{i'',j''} = \{m'', n''\}_{i'',j''}$ such that for every $k', l' \in I'_{i',j'}$ and $m'', n'' \in I''_{i'',j''}$, $R'_{k'l'} \subseteq R_{i'j'}$ and $R''_{m''n''} \subseteq R_{i''j''}$. Then

$$\operatorname{area}(R_{i'j'}) = \sum_{k',l' \in I'_{i',j'}} \operatorname{area}(R'_{k'l'}) \qquad \text{and} \qquad \operatorname{area}(R_{i''j''}) = \sum_{m'',n'' \in I''_{i'',j''}} \operatorname{area}(R''_{m''n''}).$$

Like the proof of **Lemma 6.2.7**, it suffices to prove the lower sum inequality. The upper sum in equality quickly follows, since each $' \le'$ is replaced with a $' \ge'$ and m with M.

If each $R'_{k'l'} \subseteq R_{i'j'}$, then $m_{i'j'} = \inf_{x \in R_{i'j'}} f(x) \le \inf_{x \in R_{k'l'}} f(x) = m_{k'l'}$. Using a similar argument for each $R''_{m''n''} \subseteq R_{i''j''}$, we have that

$$m_{i'j'} \operatorname{area}(R_{i'j'}) = \sum_{k',l' \in I'_{i',j'}} m_{i'j'} \operatorname{area}(R'_{k'l'}) \le \sum_{k',l' \in I'_{i',j'}} m_{k'l'} \operatorname{area}(R'_{k'l'}) \text{and}$$

$$m_{i''j''} \operatorname{area}(R_{i''j''}) = \sum_{m'',n'' \in I''_{i'',j''}} m_{i''j''} \operatorname{area}(R''_{m''n''}) \le \sum_{m'',n'' \in I''_{i'',j''}} m_{m''n''} \operatorname{area}(R''_{m''n''}).$$

Notice that

$$L_P(f) = \sum_{i,j} m_{ij} \operatorname{area}(R_{ij}) = \sum_{i',j'} m_{i'j'} \operatorname{area}(R_{i'j'}) + \sum_{i'',j''} m_{i''j''} \operatorname{area}(R_{i''j''}).$$

Then, summing over i', j' and i'', j'' respectively,

$$\sum_{i',j'} m_{i'j'} \operatorname{area}(R_{i'j'}) \le \sum_{i',j'} \left(\sum_{k',l' \in I'_{i',j'}} m_{k'l'} \operatorname{area}(R'_{k'l'}) \right) = \sum_{k,l} m_{kl} \operatorname{area}(R'_{kl}) = L_{P'}(f), \qquad \text{and}$$

$$\sum_{i'',j''} m_{i''j''} \operatorname{area}(R_{i''j''}) \le \sum_{i'',j''} \left(\sum_{m'',n'' \in I''_{i'',j''}} m_{m''n''} \operatorname{area}(R'_{m''n''}) \right) = \sum_{m,n} m_{mn} \operatorname{area}(R'_{mn}) = L_{P''}(f).$$

Reintroducing the inequalities for the upper sums, therefore

$$L_P(f) = \sum_{i',j'} m_{i'j'} \operatorname{area}(R_{i'j'}) + \sum_{i'',j''} m_{i''j''} \operatorname{area}(R_{i''j''}) \le \sum_{k,l} m_{kl} \operatorname{area}(R'_{kl}) + \sum_{m,n} m_{mn} \operatorname{area}(R'_{mn}) = L_{P'}(f) + L_{P''}(f) \qquad \text{and}$$

$$U_P(f) = \sum_{i',j'} M_{i'j'} \operatorname{area}(R_{i'j'}) + \sum_{i'',j''} M_{i''j''} \operatorname{area}(R_{i''j''}) \ge \sum_{k,l} M_{kl} \operatorname{area}(R'_{kl}) + \sum_{m,n} M_{mn} \operatorname{area}(R''_{mn}) = U_{P'}(f) + U_{P''}(f),$$

as required.

$\square$

(1d) Let $f : R \to \mathbb{R}$ be bounded. Use (1c) to prove that if f is integrable on R, then f is integrable on R' and integrable on R''. Moreover,

$$\int_R f\,dV = \int_{R'} f\,dV + \int_{R''} f\,dV.$$

Proof. Assume f is integrable on R. By the ε-characterization for integrability **[Lemma 6.3.10]**, there exists a partition P of R such that $\forall \varepsilon > 0$, $U_P(f) - L_P(f) < \varepsilon$. Fix $\varepsilon > 0$. By **(1a)** and **(1b)**, there exists partitions P' and P'' of R' and R'', respectively, such that $P' \cup P''$ is a refinement of P and

$$L_P(f) \leq L_{P'}(f) + L_{P''}(f), \quad U_P(f) \geq U_{P'}(f) + U_{P''}(f).$$

It then follows that $-L_P(f) \geq -[L_{P'}(f) + L_{P''}(f)]$, so

$$U_{P'}(f) + U_{P''}(f) - [L_{P'}(f) + L_{P''}(f)] \leq U_P(f) - L_P(f) < \varepsilon.$$

This implies that

$$U_{P'}(f) - L_{P'}(f) < \frac{\varepsilon}{2} \quad \text{and} \quad U_{P''}(f) - L_{P''}(f) < \frac{\varepsilon}{2},$$

so again by **Lemma 6.3.10**, since there exists partitions P' of R' and P'' of R'' and $\varepsilon > 0$ was arbitrary, then f is integrable on R' and R''. Since $R = R' \cup R''$ and $(R')^o \cap (R'')^o = \varnothing$, then it follows from the two lines above that since f is integrable on all of R, R', and R'', then

$$\int_R f\ dV = \int_{R'} f\ dV + \int_{R''} f\ dV.$$

$\square$

2. Let $A \subseteq \mathbb{R}^n$ be open and convex. Let $f : A \to \mathbb{R}$ be differentiable. Assume there exists $M > 0$ such that $||\nabla f(x)|| \leq M$ for all $x \in A$. Prove that f is uniformly continuous on A. *Hint:* Use the mean value theorem.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{M}$. Fix $x, y \in A$. Since A is convex, then for any two points $x, y \in A$, there exists a line segment L from x to y which is contained in A. Furthermore, since A is open and f is differentiable, then by the Mean Value Theorem **(Theorem 5.1.1)**, there exists a $c \in L$ such that $f(x)-f(y) = \nabla f(c)\cdot(x-y)$. Since for any $x \in A$, $\|\nabla f(x)\| \leq M$, then $\|\nabla f(c)\| \leq M$. Now assume $\|x-y\| < \delta = \dfrac{\varepsilon}{M}$. Then $\|x-y\|M < \varepsilon$. Therefore

$$\|f(x)-f(y)\| = \|x-y\|\|\nabla f(c)\| \leq \|x-y\|M < \varepsilon,$$

and therefore f is uniformly continuous on A.

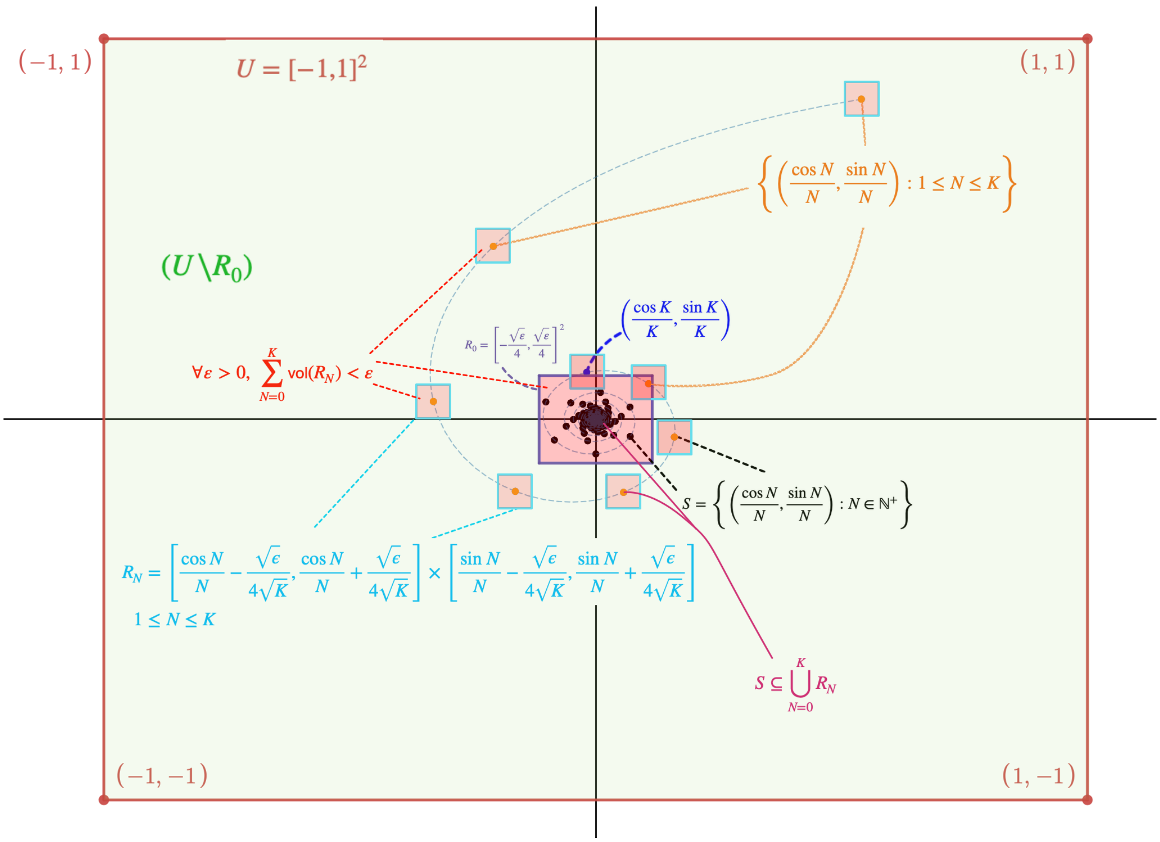
□

3. Here you will prove by definition that the set

$$S = \left\{ \left(\frac{\cos n}{n}, \frac{\sin n}{n} \right) : n \in \mathbb{N}^{+} \right\} \subseteq \mathbb{R}^2$$

has zero Jordan measure.

(3a) Provide a "picture proof" by drawing a two-dimensional sketch which illustrates the formal argument. Label your diagram with points in S, numbered rectangles covering S, and all relevant quantities that you will use in your proof. Add some expressions or brief phrases to explain your picture but do not write formal arguments or calculations. You want a reader (who is familiar with standard notation) to understand the key ideas of your formal proof using only your well-labelled illustration.



Notice that S is a set with infinitely many elements. As $N \to \infty$, the elements in S converge to $(0,0)$. My proof consists of fixing a rectangle centered at $(0,0)$ of arbitrary width, $\frac{\sqrt{\varepsilon}}{2}$ for a fixed $\varepsilon > 0$. I will show that there are finitely many points located outside of this rectangle, since the elements in S converge to $(0,0)$. From here, I will create more rectangles of arbitrary width around each point outside of the rectangle centered at $(0,0)$. I will then prove that every element in S is covered by these rectangles, and the total volume of these rectangles will be less than ε. My picture proof is not necessarily to scale, however every element I utilize in my proof is there.

(3b) Prove by definition that S has zero Jordan measure.

Proof. Fix $\varepsilon > 0$. Define the rectangle centred at $(0,0)$ by $R_0 = \left[-\frac{\sqrt{\varepsilon}}{4}, \frac{\sqrt{\varepsilon}}{4}\right]^2$. Note that S is bounded by the rectangle $U = [-1,1]^2$ as the elements in S converge to $(0,0)$ as $N \to \infty$, and thus there will exist a finite number of points in $U \setminus R_0$.

Define the sequence $S_N : \mathbb{N}^+ \to \mathbb{R}^2$ by $(a(N), b(N)) = \left(\frac{\cos N}{N}, \frac{\sin N}{N}\right)$. Then

$$S = \left\{(a(N), b(N)) \in \mathbb{R}^2 : N \in \mathbb{N}^+\right\}.$$

Since $(a(N), b(N))$ converges to $(0,0)$ as $N \to \infty$, then

$$\exists K \in \mathbb{N}^+ \text{ such that } \forall N \in \mathbb{N}^+, N > K \implies \left\|\frac{\cos N}{N}\right\| < \frac{\sqrt{\varepsilon}}{4} \wedge \left\|\frac{\sin N}{N}\right\| < \frac{\sqrt{\varepsilon}}{4}.$$

Informally, $\exists K \in \mathbb{N}^+$ such that $\forall N \in \mathbb{N}^+, N > K \implies \left(\frac{\cos N}{N}, \frac{\sin N}{N}\right) \in R_0$. Take this value of K. Then

$$S \cap (U \setminus R_0) \subseteq \left\{\left(\frac{\cos N}{N}, \frac{\sin N}{N}\right) : 1 \leq N \leq K\right\}.$$

Now construct finitely many individual rectangles around each of these K points, namely

$$R_N = \left[\frac{\cos N}{N} - \frac{\sqrt{\varepsilon}}{4\sqrt{K}}, \frac{\cos N}{N} + \frac{\sqrt{\varepsilon}}{4\sqrt{K}}\right] \times \left[\frac{\sin N}{N} - \frac{\sqrt{\varepsilon}}{4\sqrt{K}}, \frac{\sin N}{N} + \frac{\sqrt{\varepsilon}}{4\sqrt{K}}\right] \qquad \text{for } N \in \{1, \ldots, K\}.$$

Notice that for each $N \in \{1, \ldots, K\}$, R_N contains the elements $\left(\frac{\cos N}{N}, \frac{\sin N}{N}\right)$ from S. Thus, each of these K subrectangles contain at least every point in $S \cap (U \setminus R_0)$. Therefore

$$S \subseteq \bigcup_{N=0}^{K} R_N.$$

Furthermore,

$$\begin{aligned}
\sum_{N=0}^{K} \operatorname{vol}(R_N) &= \operatorname{vol}\left(\left[-\frac{\sqrt{\varepsilon}}{4}, \frac{\sqrt{\varepsilon}}{4}\right]\right) + \sum_{N=1}^{K} \operatorname{vol}\left(\left[\frac{\cos N}{N} - \frac{\sqrt{\varepsilon}}{4\sqrt{K}}, \frac{\cos N}{N} + \frac{\sqrt{\varepsilon}}{4\sqrt{K}}\right] \times \left[\frac{\sin N}{N} - \frac{\sqrt{\varepsilon}}{4\sqrt{K}}, \frac{\sin N}{N} + \frac{\sqrt{\varepsilon}}{4\sqrt{K}}\right]\right) \\
&= 4\left(\frac{\sqrt{\varepsilon}}{4}\right)^2 + \sum_{N=1}^{K} 4\left(\frac{\sqrt{\varepsilon}}{4\sqrt{K}}\right)^2 \\
&= \frac{\varepsilon}{4} + \sum_{N=1}^{K} \frac{\varepsilon}{4K} = \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Therefore S is contained in finitely many compact, non-trivial rectangles whose total volume is $< \varepsilon$. Therefore S has zero Jordan measure by definition.

□

4. Let $S, T \subseteq \mathbb{R}^n$ be sets.

(4a) Prove that if S and T are Jordan measurable, then $S \cup T$ and $S \cap T$ are Jordan measurable and

$$\operatorname{vol}(S \cup T) = \operatorname{vol}(S) + \operatorname{vol}(T) - \operatorname{vol}(S \cap T).$$

Proof. Assume S and T are Jordan measurable. It follows from the definition that both S and T are bounded and ∂S, ∂T have zero Jordan measure. Since the union and intersection of two sets is bounded, then $S \cup T$ and $S \cap T$ are both bounded. Furthermore, if $S \cap T = \varnothing$, it is bounded trivially.

By **Lemma 6.5.9**, $(\partial S \cup \partial T)$ has zero Jordan measure. Furthermore, from MAT237 PS1, $\partial(S \cup T) \subseteq (\partial S \cup \partial T)$ and $\partial(S \cap T) \subseteq (\partial S \cup \partial T)$. It follows again by **Lemma 6.5.9** that since $\partial(S \cup T)$ and $\partial(S \cup T)$ are subsets of two sets with zero Jordan measure, then both $\partial(S \cup T)$ and $\partial(S \cup T)$ have zero Jordan measure.

Therefore both $S \cup T$ and $S \cap T$ are bounded with $\partial(S \cup T)$ and $\partial(S \cap T)$ having zero Jordan measure, then $S \cup T$ and $S \cap T$ are Jordan measurable.

Let R be the rectangle which bounds all of $S, T, S \cup T$, and $S \cap T$. This rectangle exists because each of the sets $S, T, S \cup T$, and $S \cap T$ are bounded (this follows from letting each be bounded by a rectangle R_i, then taking the union $R = \bigcup_{i=1}^{4} R_i$).

By **Theorem 6.6.9**, since $S \cup T$ and $S \cap T$ are Jordan measurable sets bounded by the rectangle R then their indicator functions, $\chi_{S \cup T}$ and $\chi_{S \cap T}$ are integrable on R. This argument also applies to the indicator functions χ_S and χ_T of S and T, respectively, and so they too are integrable on R.

Notice that

$$\begin{aligned} \chi_{S \cap T} &= \min\{\chi_S, \chi_T\} = \chi_S \cdot \chi_T \qquad \text{and} \\ \chi_{S \cup T} &= \max\{\chi_S, \chi_T\} = \chi_S + \chi_T - \chi_S \cdot \chi_T \\ &\phantom{= \max\{\chi_S, \chi_T\}} = \chi_S + \chi_T - \chi_{S \cap T}. \end{aligned}$$

As before since $\chi_{S \cup T}$ and $\chi_{S \cap T}$ are integrable on R, respectively, then by the linearity of the integral (**Theorem 6.3.13**),

$$\int_R \chi_{S \cup T}\, dV = \int_R \chi_S\, dV + \int_R \chi_T\, dV - \int_R \chi_{S \cap T}\, dV.$$

By definition (**Definition 6.6.10**), then

$$\operatorname{vol}(S \cup T) = \operatorname{vol}(S) + \operatorname{vol}(T) - \operatorname{vol}(S \cap T).$$

□

(4b) Prove or disprove: if $S \cup T$ and $S \cap T$ are Jordan measurable, then S and T are Jordan measurable and

$$\mathrm{vol}(S \cup T) = \mathrm{vol}(S) + \mathrm{vol}(T) - \mathrm{vol}(S \cap T).$$

I will disprove this with a counterexample:

- Let $S = [0,1] \cap \mathbb{Q}$ and let $T = [0,1] \cap \mathbb{Q}^c$. Neither S or T are Jordan measurable because $\partial S = \partial T = [0,1]$, which does not have zero Jordan measure as $[0,1]$ contains an infinite number of elements.
- Note that $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$, which is the set of irrational numbers, disjoint from the set of rationals. Then

$$S \cup T = [0,1] \qquad \text{and} \qquad S \cap T = \varnothing.$$

- $S \cup T = [0,1]$ is Jordan measurable because it is bounded and it's boundary $\{0,1\}$ has zero Jordan measure, which is easy to verify. Similarly, the empty set $S \cap T = \varnothing$ is Jordan measurable because it is bounded (trivially) and has zero volume (trivially).
- Therefore it is false that if $S \cup T$ and $S \cap T$ are Jordan measurable, then S and T are Jordan measurable. Furthermore, by **Theorem 6.6.9**, $\mathrm{vol}(S)$ and $\mathrm{vol}(T)$ cannot be computed because S and T both are not Jordan measurable.