MAT224 Linear Algebra II
Assignment 4

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: Jace Alloway

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Full Name:

Student number:

I confirm that:

- I have read and followed the policies described in the document Assignment Policies & FAQ.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1)

2)

1. Let

$$W = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = 0 \right\}$$

1(a) Show that W is a subspace of $\mathbb{R}^4$.

I want to show that W is a subspace of $\mathbb{R}^4$.

I will begin by computing the determinant of the 4×4 matrix:

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = x_1 \cdot \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - x_2 \cdot \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + x_3 \cdot \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - x_4 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We then have

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = x_1 \cdot \left[\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right] - x_2 \cdot \left[\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right]$$

$$+\mathrm{x}_3 \cdot \left[\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] - x_4 \cdot \left[\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right] + \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \text{- x}_1 - x_2 + 2x_3 - x_4 = 0.$$

We can re-write W as $W = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | - x_1 - x_2 + 2x_3 - x_4 = 0 \right\}$.

I will now show that W is a subspace of $\mathbb{R}^4$. There is a condition on the vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, which is $-x_1 - x_2 + 2x_3 - x_4$ for each $x_1, x_2, x_3, x_4 \in \mathbb{R}$. This implies that $W \subset \mathbb{R}^4$. If all $x_1 = x_2 = x_3 = x_4 = 0$, then $\mathbf{0} \in W$, which implies that W is non-empty.
Now, I need to show that $\forall a, b \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in W$, we have that $a\mathbf{x} + b\mathbf{y} \in W$. That is, that W is closed under the scalar multiplication and vector addition axioms from $\mathbb{R}^4$. We have

$$\begin{aligned} -(ax_1 + by_1) - (ax_2 + by_2) + 2(ax_3 + by_3) - (x_4 + y_4) &= -ax_1 - by_1 - ax_2 - by_2 + 2ax_3 + 2by_3 - x_4 - y_4 \\ &= -ax_1 - ax_2 + 2ax_3 - x_4 - by_1 - by_2 + 2by_3 - y_4 \\ &= a(-x_1 - x_2 + 2x_3 - x_4) + b(-y_1 - y_2 + 2y_3 - y_4) \\ &= a(0) + b(0) = 0 + 0 = 0. \end{aligned}$$

Therefore since the same vector addition and scalar multiplication axioms from $\mathbb{R}^4$ hold in W, and since $a\mathbf{x} + b\mathbf{y} \in W$, then W is a vector subspace of $\mathbb{R}^4$, which is what I needed to show.

1. Let

$$W = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & -1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 3 & 7 & 0 & 1 \end{bmatrix} = 0 \right\}$$

1(b) Determine $\dim W$.

To determine $\dim W$, I need to compute the determinant of the matrix $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & -1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 3 & 7 & 0 & 1 \end{bmatrix}$. We have

$$\begin{aligned}
\det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & -1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 3 & 7 & 0 & 1 \end{bmatrix} &= x_1 \cdot \det \begin{bmatrix} -1 & 2 & 3 \\ 3 & 1 & 2 \\ 7 & 0 & 1 \end{bmatrix} - x_2 \cdot \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} + x_3 \cdot \det \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 7 & 1 \end{bmatrix} - x_4 \cdot \det \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 3 & 7 & 0 \end{bmatrix}. \\
&= x_1 \cdot [(-1)(1)(1) + (2)(2)(7) + (3)(3)(0) - (7)(1)(3) - (0)(2)(-1) - (1)(3)(2)] \\
&\quad + x_2 \cdot [(1)(1)(1) + (2)(2)(3) + (3)(2)(0) - (3)(1)(3) - (0)(2)(1) - (1)(2)(2)] \\
&\quad - x_3 \cdot [(1)(3)(1) + (-1)(2)(3) + (3)(2)(7) - (3)(3)(3) - (7)(2)(1) - (1)(2)(-1)] \\
&\quad + x_4 \cdot [(1)(3)(0) + (-1)(1)(3) + (2)(2)(7) - (3)(3)(2) - (7)(1)(1) - (0)(2)(-1)] \\
&= x_1[0] + x_2[0] + x_3[0] + x_4[0] = 0.
\end{aligned}$$

From here, it appears we cannot conclude anything about the properties of the subspace W because each of the cofactors of x_1, x_2, x_3, x_4 are 0:

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & -1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 3 & 7 & 0 & 1 \end{bmatrix} = x_1[0] + x_2[0] + x_3[0] + x_4[0] = 0.$$

A basis for W could be the standard basis of $\mathbb{R}^4$, that is, the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Therefore, $\dim W = 4$.

2. Let V be a vector space, and let $T \in \mathfrak{L}(V)$.

2(a) Prove that if T is injective, then 0 is not an eigenvalue of T.

I want to prove that for an injective linear map $T \in \mathfrak{L}(V)$, 0 is not an eigenvalue of T.

Proof. By contradiction. Assume that $T \in \mathfrak{L}(V)$ is injective. Then, by **Proposition 24, Quiri Li Linear Transformation, Slide 86**, we can conclude that $\ker(T) = \{\mathbf{0}\}$.
Now, suppose that 0 is an eigenvalue of T. By [**Proposition 4.1.5**], a non-zero vector $\mathbf{x}$ is an eigenvector of T with eigenvalue λ if and only if $\mathbf{x} \in \ker(T - \lambda I)$. If 0 is an eigenvalue of T, then we have that

$$\mathbf{x} \in \ker(T - \lambda I) \implies \mathbf{x} \in \ker(T - 0I) \implies \mathbf{x} \in \ker(T).$$

Since T is injective, then $\mathbf{x} \in \ker(T) = \{\mathbf{0}\}$, which implies that $\mathbf{x} = \mathbf{0}$. However, if $\mathbf{x}$ is an eigenvector of T, then $\mathbf{x} \neq \mathbf{0}$, which is contradictory.
Therefore 0 cannot be an eigenvalue of T, which is what I needed to prove.

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2(b) Prove that if 0 is not an eigenvalue of T then T is injective.

I want to prove that if 0 is not an eigenvalue of T, then T is injective.

Proof. To prove that T is injective, by **Proposition 24, Quiri Li Linear Transformation, Slide 86**, it suffices to show that $\ker(T) = \{\mathbf{0}\}$.
By contrapositive. Then, I need to show that if 0 is an eigenvalue of T, then T is not injective. That is, that $\ker(T) \neq \{\mathbf{0}\}$.
Assume that 0 is an eigenvalue of T. By [**Proposition 4.1.5**], every eigenvector $\mathbf{x} \neq \mathbf{0}$ with eigenvalue λ is in $\ker(T - \lambda I)$.
We then have

$$\mathbf{x} \in \ker(T - \lambda I) \implies \mathbf{x} \in \ker(T - 0I) \implies \mathbf{x} \in \ker(T).$$

Thus, since $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \in \ker(T)$, then $\ker(T) \neq \{\mathbf{0}\}$.
I have proved that the contrapositive is true, and thus if 0 is not an eigenvalue of T, then T is injective.

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3. Let V be a finite dimensional vector space, and let $S, T \in \mathfrak{L}(V)$.

3(a) Suppose that if $\lambda \neq 0$ is an eigenvalue of ST. Show that λ is also an eigenvalue of TS.

Let $\mathbf{x} \in V$ be an eigenvector of ST with non-zero eigenvalue $\lambda \neq 0$, so $S(T(\mathbf{x})) = \lambda\mathbf{x}$. By definition, if $\mathbf{x}$ is an eigenvector of ST, then $\mathbf{x} \neq \mathbf{0}$.
I want to show that if $\lambda \neq 0$ is an eigenvalue of ST, then it is also an eigenvalue of TS.

Suppose that $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of ST with non-zero eigenvalue λ. Let $\mathbf{v} = T(\mathbf{x})$ such that $\mathbf{v} \neq \mathbf{0}$. Then

$$ST(\mathbf{x}) = S\mathbf{v} = \lambda\mathbf{x}.$$

We then have

$$(TS)(T\mathbf{x}) = T(ST(\mathbf{x})) = T(\lambda\mathbf{x}) = \lambda T\mathbf{x}.$$

This implies that $T\mathbf{x} \neq \mathbf{0}$ is an eigenvector of TS with the same eigenvalue λ. Therefore ST and TS have the same eigenvalues (but they may not have the same eigenvectors), which is what I needed to show.

3. Let V be a finite dimensional vector space, and let $S, T \in \mathfrak{L}(V)$.

3(b) Suppose that 0 is an eigenvalue of ST. Show that 0 is also an eigenvalue of TS.

I want to show that if 0 is an eigenvalue of ST, then 0 is also an eigenvalue of TS.

If 0 is an eigenvalue of ST, then we know that ST is not an isomorphism, that is, that ST cannot be invertible. It then follows that since ST is not an isomorphism, then both T and S individually cannot be isomorphisms. Hence TS cannot be an isomorphism, and thus 0 is also an eigenvalue of ST.

For further justification, suppose that there exists an eigenvector $\mathbf{x} \in V$ with eigenvalue 0 when under the transformation ST: $\quad ST(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$. By definition, if $\mathbf{x} \in V$ is an eigenvector of ST, then $\mathbf{x} \neq \mathbf{0}$.
Let $\mathbf{v} = T\mathbf{x}$. Using the same argument used in question **3(a)**, then

$$(TS)(T\mathbf{x}) = T(ST(\mathbf{x})) = T(\lambda\mathbf{x}) = T(\mathbf{0}) = 0T\mathbf{x} = \mathbf{0}.$$

It is ok if $\mathbf{v} = T\mathbf{x} = \mathbf{0}$ because here, $\lambda = 0$. Therefore if 0 is an eigenvalue for ST, then it is also an eigenvalue of TS, which is what I needed to show.