MAT224 Linear Algebra II
Assignment 2

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Academic Integrity Statement:

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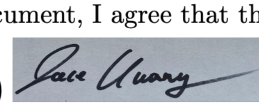
Signatures: 1) 

2) Mallory Bond

1. Let V be a vector space, and let $\mathbf{x} \in V$. Let $W = \{c\mathbf{x} \mid c \in \mathbb{R}\}$.

1(a)Show that W is a vector space.

We need to show that all vectors in W satisfy the 8 vector space axioms.

To show that W is a vector space, we will first show that W is a subspace of V which implies that W is closed and inherits the axioms of scalar multiplication and vector addition from V. From this we will have proven that axioms 1,2, 5-8 are satisfied. It then suffices to show that W also satisfies the additive identity axiom (axiom 3) and additive inverse axiom (axiom 4).

Let $c\mathbf{x} \in W$ where $\mathbf{x} \in V$. By definition of a vector space, $c\mathbf{x} \in V$ because V is closed under scalar multiplication. Thus, all elements of W are contained in V so $W \subseteq V$.
Suppose that $c = 0$. We then have that $c\mathbf{x} = 0\mathbf{x} = \mathbf{0} \in W$ and that W is non-empty, which implies W is a subspace of V because $\{\mathbf{0}\} \subset V$.
We now need to check if W is closed under the scalar multiplication and vector addition axioms of V. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vectors in W and let $\lambda, \mu \in \mathbb{R}$.

For any $\mathbf{w}_1, \mathbf{w}_2 \in W$, $\mathbf{w}_1 = a\mathbf{x}$ and $\mathbf{w}_2 = b\mathbf{x}$, we have that

$$\begin{aligned}
\lambda\mathbf{w}_1 + \mu\mathbf{w}_2 &= \lambda(a\mathbf{x}) + \mu(b\mathbf{x}) \\
&= \lambda(ax_1, ..., ax_n) + \mu(bx_1, ..., bx_n) \\
&= (\lambda ax_1, ..., \lambda ax_n) + (\mu bx_1, ..., \mu bx_n) \\
&= (\lambda ax_1 + \mu bx_1, ..., \lambda ax_n + \mu bx_n) \\
&= ((\lambda a + \mu b)x_1, ..., (\lambda a + \mu b)x_n) \\
&= (\lambda a + \mu b)(x_1, ..., x_n) \\
&= (\lambda a + \mu b)\mathbf{x}
\end{aligned}$$

$(\lambda a + \mu b)\mathbf{x} \in W$, so we know that $\lambda\mathbf{w}_1 + \mu\mathbf{w}_2 \in W$. Thus, W inherits vector space axioms from V. W is closed under the vector addition and scalar multiplication operations from V. Hence, vector space axioms 1,2,5-8 are satisfied.

Suppose $\mathbf{w}_1, \mathbf{w}_2 \in W$ where $\mathbf{w}_1 = c\mathbf{x}, \mathbf{w}_2 = d\mathbf{x}$ such that $d = 0$. In this case, $\mathbf{w}_2 = \mathbf{0}$ is an additive identity element in W because by Proposition 1.1.6(b), $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{w}_i \in W$. We then have that

$$\mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{x} + d\mathbf{x} = c\mathbf{x} + 0 \cdot \mathbf{x} = c\mathbf{x} + \mathbf{0} = c\mathbf{x} = \mathbf{w}_1,$$

thus axiom 3, the existence of an additive identity element in W, is satisfied.

Furthermore, by Definition 5.1.4 (iv), there exists an additive inverse $-c \in \mathbb{R}$ such that $c + (-c) = 0$. Let $\mathbf{w}_1, \mathbf{w}_2 \in W$ such that $\mathbf{w}_1 = c\mathbf{x}, \mathbf{w}_2 = d\mathbf{x}$, where $d = -c$. It then follows that

$$\mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{x} + d\mathbf{x} = c\mathbf{x} + (-c)\mathbf{x} = (c + (-c))\mathbf{x} = 0 \cdot \mathbf{x} = \mathbf{0} \text{ (by proposition 1.1.6(b))},$$

which implies that axiom 4, existence of an additive inverse, is satisfied.

Because W is closed under the scalar multiplication and vector addition operations from V, axioms 1,2, and 5-8 are satisfied.
Furthermore, because W satisfies vector space axioms 3 and 4, W is a vector space, which is what we needed to show.

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1(b) Determine *all* subspaces of W.

Final Answer: $\{\mathbf{0}\}$ and W are the only two subspaces of W.

By definition, any vector space is a subspace of itself. So, W is a subspace of W.
We also know that $\{\mathbf{0}\}$ is a subspace of W.

Proof.
Since W is a vector space, we know that $\mathbf{0} \in W$ so $\{\mathbf{0}\}$ is a subset of W.
We know that for any $\mathbf{x}, \mathbf{y} \in \{\mathbf{0}\}$, $x = y = 0$. Therefore, for all $\mathbf{x}, \mathbf{y} \in \{\mathbf{0}\}$ and for all $c \in \mathbb{R}$,

$$\begin{aligned} c\mathbf{x} + \mathbf{y} &= c\mathbf{0} + \mathbf{0} \text{ (by assumption)} \\ &= \mathbf{0} + \mathbf{0} \text{ (by proposition 1.1.6(b): } 0\mathbf{x} = \mathbf{0}) \\ &= \mathbf{0} \text{ (by axiom 3)} \end{aligned}$$

Since $c\mathbf{x} + \mathbf{y} \in \{\mathbf{0}\}$, by Theorem 1.2.8, $\{\mathbf{0}\}$ is a subspace of W.

□

There are only two subspaces of W, $\{\mathbf{0}\}$ and W itself. We will prove this by contradiction.

Proof.
Suppose that there exists a subspace S of W but $S \neq \{\mathbf{0}\}$ and $S \neq W$. This means that there exists $c\mathbf{x} \in S$ where $c \neq 0$ and there exists $\alpha\mathbf{x} \in W$ such that $\alpha\mathbf{x} \notin S$.
Let $c\mathbf{x}, d\mathbf{x} \in S$. Let $\alpha = c + d$.

$$c\mathbf{x} + d\mathbf{x} = (c + d)\mathbf{x} \text{ (by axiom 6)}$$

$(c + d)\mathbf{x} \notin S$ so S is not closed under vector addition. Therefore, S is not a subspace.

□

2. Let $U = \{a + bx + cx^2 + dx^3 \in P_3(\mathbb{R}) \mid a - b + d = 0\}$, and let $W = \left\{a + bx + cx^2 + dx^3 \in P_3(\mathbb{R}) \mid a + 2c - d = 0\right\}$.

2(a) Find a finite subset S of $P_3(\mathbb{R})$ such that $\operatorname{span} S = U + W$.

$S = \{1, x, x^2, x^3\}$

We want to show that $P_3(\mathbb{R}) = U + W$. Then, any set S that spans $P_3(\mathbb{R})$ will also span $U + W$.

Proof.
Relabel $W = \left\{e + fx + gx^2 + hx^3 \in P_3(\mathbb{R}) \mid e + 2g - h = 0\right\}$.
Let $p(x) \in U + W$. Then,

$$p(x) = (a + bx + cx^2 + dx^3) + (e + fx + gx^2 + hx^3)$$
$$p(x) = (a + e) + (b + f)x + (c + g)x^2 + (d + h)x^3$$

Thus, $p(x) \in P_3(\mathbb{R})$.
Let $q(x) \in P_3(\mathbb{R})$. Then,

$$q(x) = \alpha_0(1) + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

This can be rewritten as

$$q(x) = (a + e) + (b + f)x + (c + g)x^2 + (d + h)x^3$$
$$q(x) = (a + bx + cx^2 + dx^3) + (e + fx + gx^2 + hx^3)$$

Thus, $q(x) \in U + W$.
We have proven that $P_3(\mathbb{R}) = U + W$ which means that any finite subset S that spans $P_3(\mathbb{R})$ will span $U + W$.
We know that $S = \{1, x, x^2, x^3\}$ spans $P_3(\mathbb{R})$ so $\operatorname{span} S = U + W$.

$\square$

2(b) Is $U + W$ a direct sum?

Final Answer: No, $U + W$ is not a direct sum.

For $U + W$ to be a direct sum, $U \cap W = \{\mathbf{0}\}$ would need to be true.

Proof.
Let $p(x) \in U \cap W$. Then $p(x) \in U$ and $p(x) \in W$. So,

$$p(x) = a + bx + cx^2 + dx^3$$
$$p(x) = e + fx + gx^2 + hx^3$$

For any polynomial in U, $a - b + d = 0$ which means that $a = b - d$. For any polynomial in W, $e + 2g - h = 0$ which means that $e = -2g + h$. This means that

$$p(x) = b - d + bx + cx^2 + dx^3$$
$$p(x) = -2g + h + fx + gx^2 + hx^3$$

$$b - d + bx + cx^2 + dx^3 = p(x) = -2g + h + fx + gx^2 + hx^3$$

$$b - d = -2g + h$$
$$b = f$$
$$c = g$$
$$d = h$$

From this we can gather that

$$b - d = -2g + h$$
$$b - d = -2c + d$$
$$b = -2c + 2d$$

Let $c = d = 1$. Then $b = -2(1) + 2(1) = 0$. From our definition of U we find that $a = 0 - 1 = -1$ and from our definition of W we find that $a = -2(1) + 1 = -1$.

$$p(x) = -1 + x^2 + x^3$$

We have found a polynomial $p(x) \in U \cap W$ that is not $\mathbf{0}$.

$\square$

3(a) Prove that if the list $\mathbf{x}_1, \mathbf{x}_2$ is a basis for a vector space V, then the list $a\mathbf{x}_1, b\mathbf{x}_1 + c\mathbf{x}_2$ is also a basis for V for any $a \neq 0$ and $c \neq 0$. Why does the conclusion fail if $a = 0$ or $c = 0$?

We want to show that if $\{\mathbf{x_1}, \mathbf{x_2}\}$ is a basis for V, then the list $\{a\mathbf{x_1}, b\mathbf{x_1} + c\mathbf{x_2}\}$ is also a basis for V.

Proof. If $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for V, then we have that $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\} = V$. Because the list $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for V, then we have that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent:

$$0 = g_1\mathbf{x}_1 + g_2\mathbf{x}_2, \text{ where } g_1 = g_2 = 0.$$

For any scalars $a, b, c \in \mathbb{R}$, where $a \neq 0$ and $c \neq 0$, we need to show that the list $\{a\mathbf{x_1}, b\mathbf{x_1}+c\mathbf{x_2}\}$ is also linearly independent, which would imply that the list $\{a\mathbf{x_1}, b\mathbf{x_1} + c\mathbf{x_2}\}$ is also a basis for V.
We have that

$$0 = h_1(a\mathbf{x}_1) + h_2(b\mathbf{x}_1 + c\mathbf{x}_2)$$
$$0 = h_1(a\mathbf{x}_1) + h_2(b\mathbf{x}_1) + h_2(c\mathbf{x}_2)$$

By the multiplicative associativity axiom (axiom 7) and the distributivity over scalar addition axiom (axiom 6), we have that

$$0 = (h_1a)\mathbf{x}_1 + (h_2b)\mathbf{x}_1 + (h_2c)\mathbf{x}_2$$
$$0 = (h_1a + h_2b)\mathbf{x}_1 + (h_2c)\mathbf{x}_2.$$

Because the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent, then it follows that $0 = (h_1a+h_2b)\mathbf{x}_1+(h_2c)\mathbf{x}_2$ is also linearly independent, which implies that

$$h_1a + h_2b = 0 \text{ and}$$
$$h_2c = 0.$$

Since $c \neq 0$, it follows that h_2 must be 0:

$$h_1a + 0b = h_1a + 0 = h_1a = 0$$
$$h_2c = 0c = 0$$

Since $a \neq 0$, it then follows that h_1 must be 0 as well.

Thus, we have shown that $0 = h_1(a\mathbf{x}_1)+h_2(b\mathbf{x}_1+c\mathbf{x}_2)$ only has a trivial solution which implies that the list $\{a\mathbf{x_1}, b\mathbf{x_1} + c\mathbf{x_2}\}$ is linearly independent and also a basis for V, which is what we needed to prove.

□

If $a = 0$ or $c = 0$, then the list $\{a\mathbf{x_1}, b\mathbf{x_1} + c\mathbf{x_2}\}$ is not a basis for V.
By Definition 1.6.12, the number of elements in a basis for a vector space V must be the same as the dimension of V.
Suppose $a = 0$. We then have the list $\{0\mathbf{x}_1, b\mathbf{x}_1 + c\mathbf{x}_2\}$. $\{0\mathbf{x}_1, b\mathbf{x}_1 + c\mathbf{x}_2\}$ is not a basis for V because $\dim(\operatorname{span}\{0\mathbf{x}_1, b\mathbf{x}_1 + c\mathbf{x}_2\}) = 1$, since $\operatorname{span}\{0\mathbf{x}_1\} = \operatorname{span}\{\mathbf{0}\} = \{\mathbf{0}\}$ and $\dim\{\mathbf{0}\} = 0$.
Furthermore, $\dim V = 2 \neq 1 = \dim(\operatorname{span}\{0\mathbf{x}_1, b\mathbf{x}_1 + c\mathbf{x}_2\})$, which implies that, $\operatorname{span}\{\mathbf{0}, b\mathbf{x}_1 + c\mathbf{x}_2\} = \operatorname{span}\{b\mathbf{x}_1 + c\mathbf{x}_2\}$.
Now suppose that $c = 0$. We then have that the list $\{a\mathbf{x}_1, b\mathbf{x}_1 + 0\mathbf{x}_2\} = \{a\mathbf{x}_1, b\mathbf{x}_1 + \mathbf{0}\}$, which is also not a basis for V because this list is linearly dependent and only contains one element, $\{\mathbf{x}_1\}$. Using the same argument that $\operatorname{span}\{0\mathbf{x}_2\} = \operatorname{span}\{\mathbf{0}\} = \{\mathbf{0}\}$ and $\dim\{\mathbf{0}\} = 0$, we have that the $\operatorname{span}\{a\mathbf{x}_1, b\mathbf{x}_1 + \mathbf{0}\} = \operatorname{span}\{a\mathbf{x}_1, b\mathbf{x}_1\} = \operatorname{span}\{\mathbf{x}_1\}$, as $a\mathbf{x}_1$ and $b\mathbf{x}_1$ which, by the definition of span being all linear combinations of the elements in a set, can be written as linear combinations of $\mathbf{x}_1$. Therefore $\dim V = 2 \neq 1 = \dim(\operatorname{span}\{\mathbf{x}_1\})$.
Therefore if $a = 0$ or $c = 0$, the list $\{a\mathbf{x_1}, b\mathbf{x_1} + c\mathbf{x_2}\}$ does not generate V and is not a basis for V.

3(b) Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a basis for a vector space V. By generalizing the methodology in 3(a), explain how you could produce an infinite number of different bases for V.

Because the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a basis for V, then the set of all linear combinations of vectors in this list is trivial because every element in this list is linearly independent. By following the methodology in 3(a), sets $\mathcal{S}$ and $\mathcal{S}'$ can be created using elements in the basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of V, which implies that as long as a basis has dimension n, meaning that the dimension is the same as $\dim V$, it will be a basis for V regardless if previous elements are repeated in the construction of new elements in the basis. To provide an example, consider the basis $\mathcal{S}$, where each new term k contains vectors from the $k^{th} - 1$ term, where $2 \leq k \leq n$:

$$\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_1 + \mathbf{x}_2, \ldots, \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n\}.$$

Furthermore, we can conclude that any basis for V can be also constructed using any real scalars as long as the basis has at most dimension n, meaning that there are n linearly independent vectors in the basis and the dimension of the basis is equal to $\dim V$. We are able to create an infinite number of bases because there are an infinite number of scalars used to create bases consisting of linear combinations.

We are able to produce an infinite number of bases for V by constructing a list of linearly independent vectors using scalars. Consider the basis $\mathcal{S}'$ of n vectors for V, where for any scalar $a_{ij} \in \mathbb{R}$, where $a_{ij} \neq 0$ and $1 \leq i \leq n$, $1 \leq j \leq n$, we are able construct another new basis for V:

$$\mathcal{S}' = \{a_{11}\mathbf{x}_1, a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2, \ldots, a_{n1}\mathbf{x}_1 + a_{n2}\mathbf{x}_2 + \cdots + a_{nn}\mathbf{x}_n\}.$$

To add, notice that this basis fails if there exists one $a_{ij} = 0$, as the dimension of the span of the basis will not be the same as the dimension of the vector space the basis is generating.