MAT244 PS2

Q1/

$y' = t+y \qquad y(0) = 1. \qquad \phi_0(t) = 1.$

We have $\phi_{i+1}(t) = \int_0^t f(s, \phi_i(s))\, ds$

Find $\phi_3(t)$.

Here, $f(t,y) = t+y$. Then $\boxed{\phi_0(t) = 1}$

$$\phi_1(t) = \int_0^t f(s, 1)\, ds$$
$$= \int_0^t (s+1)\, ds$$
$$= \left[\tfrac{1}{2}s^2 + s\right]\Big|_0^t \qquad \boxed{\phi_1(t) = \tfrac{1}{2}t^2 + t}$$

$$\phi_2(t) = \int_0^t f(s, \tfrac{1}{2}s^2 + s)\, ds$$
$$= \int_0^t (s + \tfrac{1}{2}s^2 + s)\, ds$$
$$= \int_0^t (2s + \tfrac{1}{2}s^2)\, ds$$
$$= \left[s^2 + \tfrac{1}{6}s^3\right]\Big|_0^t \qquad \boxed{\phi_2(t) = \tfrac{1}{6}t^3 + t^2}$$

$$\phi_3(t) = \int_0^t f(s, \tfrac{1}{6}s^3 + s^2)\, ds$$
$$= \int_0^t (s + \tfrac{1}{6}s^3 + s^2)\, ds$$
$$= \left[\tfrac{1}{2}s^2 + \tfrac{1}{24}s^4 + \tfrac{1}{3}s^3\right]\Big|_0^t \qquad \boxed{\phi_3(t) = \tfrac{1}{24}t^4 + \tfrac{1}{3}t^3 + \tfrac{1}{2}t^2}$$

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Q2/ $y'' + y' + y = A\sin t$

Find every $A \in \mathbb{R}$ such that $\phi(t) = \phi(t+\pi)$ could be a solution of the ODE.

If $\phi(t) = \phi(t+\pi)$, then $\frac{d}{dt}[\phi(t)] = \frac{d}{dt}[\phi(t+\pi)]$.

The equation then becomes

$$\phi''(t+\pi) + \phi'(t+\pi) + \phi(t+\pi) = \phi''(t) + \phi'(t) + \phi(t)$$

Which implies that

$$A\sin(t+\pi) = A\sin(t).$$

$\sin(t+\pi) = -\sin(t)$ by trig identities, and

Thus

$$A(-\sin t) = A\sin t$$

$$-A\cancel{\sin t} = A\cancel{\sin t}$$

$$\Rightarrow \quad 0 = 2A$$

$$\Rightarrow \boxed{A=0.}$$

Therefore $A=0$ is the only real number for which there exists a solution $\phi(t)$ such that

$$\phi(t) = \phi(t+\pi)$$

defined for all t.

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Q3/ a) $\boxed{\text{True.}}$ If we fix $a, b, c \in \mathbb{R}$, then there must be a unique general solution of the form

$$y = c_1 y_1 + c_2 y_2$$

in order to satisfy the initial conditions as well as the equation $ay'' + by' + cy = 0$.

The initial values are what make the solution unique.

b) $\boxed{\text{False.}}$ Let $a = 1$, $b = -1$, and $c = -2$.

Then $y'' - y' - 2y = 0$, guess $y = e^{rt}$.

$\implies r^2 - r - 2 = 0$

which factors into $(r-2)(r+1) = 0$, so our general solution becomes

$$y = c_1 e^{2t} + c_2 e^{-t}, \quad \text{for some } c_1, c_2 \in \mathbb{R}$$

which is <u>not constant</u> or unique (because no initial conditions were given. General solutions are not unique).

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Q4) $(2x^2+1)y'' - 4xy' + 4y = 0.$

a) $y(x) = x$, then $y'(x) = 1$ and $y''(x) = 0.$

Hence

$$(2x^2+1)(0) - 4x(1) + 4(x) = 0$$

$$\Longrightarrow \quad -4x + 4x = 0$$

$$0 = 0$$

Thus $y(x) = x$ is a solution.

b) Let $y(x) = v(x)\,x$.

Then $y'(x) = v'(x)\,x + v(x)$

and $y''(x) = v''(x)\,x + v'(x) + v'(x)$

$= v''(x)\,x + 2v'(x).$

→ Substitute into ODE:

$$(2x^2+1)(v''(x)x + 2v'(x)) - 4x(v'(x)x + v(x)) + 4v(x)x = 0$$

$$2x^3v''(x) + v''(x)x + \cancel{4x^2v'(x)} + 2v'(x) - \cancel{4x^2v'(x)} - \underbrace{\cancel{4xv(x) + 4xv(x)}}_{0} = 0$$

$$(2x^3 + x)v''(x) + 2v'(x) = 0.$$

Let $p(x) = v'(x)$.

Then $$p'(x) + \frac{2}{2x^3+x}p(x) = 0.$$

$$\Rightarrow \quad \frac{p'(x)}{p(x)} = -\frac{2}{2x^3+x}$$

Then $$\int \frac{p'(x)}{p(x)}\,dx = \log|p(x)| = -2\int \frac{dx}{2x^3+x}$$

$$= -2 \int \frac{dx}{x^3\left(2+\frac{1}{x^2}\right)}$$

Let $u = 2 + \frac{1}{x^2}$

$du = -\frac{2}{x^3}\,dx$

$$= \int \frac{du}{u}$$

$$= \log\left|2 + \frac{1}{x^2}\right| + C_1$$

This implies that $p(x) = e^{C_1}\left(2 + \frac{1}{x^2}\right) = v'(x)$

Then $v(x) = \int e^{C_1}\left(2+\frac{1}{x^2}\right)dx$

$$= 2e^{C_1}x - \frac{e^{C_1}}{x} + C_2$$

Our general solution then becomes

$$y(x) = v(x)\,y_1(x) = v(x)\,x$$

$$= 2e^{C_1}x^2 - e^{C_1} + C_2 x$$

$$y(x) = e^{C_1}(2x^2-1) + C_2 x$$

where $C_1, C_2 \in \mathbb{R}$ are constants.

Another solution to this ODE is

$$\boxed{y_2(x) = e^{C_1}(2x^2-1)}$$

c) Show that $W[y_1, y_2] \neq 0$.

We have $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$

$$= C_2 x\left(4e^{C_1}x\right) - (C_2)\left(e^{C_1}(2x^2-1)\right)$$

$$= 4C_2e^{C_1}x^2 - 2C_2e^{C_1}x^2 - C_2e^{C_1}$$

$$= 2C_2e^{C_1}x^2 - C_2e^{C_1}$$

$$\boxed{= C_2e^{C_1}(2x^2-1) \neq 0}$$

Therefore y_1, y_2 are a set of fundamental solutions to the ODE.

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Q5/ $t^2y'' + ty' - y = 0$, assume $t>0$.

a) $t = t(x) = e^x$.

Then $y(t) \to y(e^x)$.

By chain rule, $\boxed{\frac{d}{dx}[y(e^x)] = y'(e^x)\cdot e^x}$.

Again, by product (& chain) rule,

$$\frac{d^2}{dx^2}[y(e^x)] = \frac{d}{dx}[y'(e^x)\cdot e^x]$$

$$= y''(e^x)\cdot e^x \cdot e^x + y'(e^x)\cdot e^x$$

$$\boxed{\frac{d^2}{dx^2}[y(e^x)] = y''(e^x)\cdot e^{2x} + y'(e^x)e^x}$$

b) When substiting $t = t(x) = e^x$, our ODE becomes

$$\underbrace{(e^x)^2 y''(e^x) + (e^x)y'(e^x)} - y(e^x) = 0$$

$$= y''(e^x)\cdot e^{2x} + y'(e^x)e^x$$

$$= \frac{d^2}{dx^2}[y(e^x)]$$

Thus $\boxed{\frac{d^2y}{dx^2} - y = 0.}$

c) Solving the IVP:

$y'' - y = 0$ Guess $y = e^{rt}$

$\Rightarrow r^2\cancel{e^x} - \cancel{e^x} = 0$

$\Rightarrow (r^2-1) = (r-1)(r+1) = 0$

So our general solution becomes $\boxed{y = c_1 e^x + c_2 e^{-x}}$

for some constants $c_1, c_2 \in \mathbb{R}$.

d) Since our general solution is $y = c_1 e^x + c_2 e^{-x}$, this is equivalent to $y = c_1 e^x + \frac{c_2}{e^x}$.

However since $e^x = t(x) = t$, then the general solution to our initial ODE is

$$\boxed{y = c_1 t + \frac{c_2}{t}}$$

for some constants c_1, c_2.

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Q6/ $(t^2-2t)y'' + 4(t-1)y' + 2y = e^{2t}$

a) Let $F(t) = (t^2-2t)y' + (2t-2)y$.

Then $\frac{dF}{dt} = y''(t^2-2t) + y'(2t-2) + y'(2t-2) + y(2)$

$= y''(t^2-2t) + 4y'(t-1) + 2y$

$\boxed{= e^{2t}}$

Since $\frac{dF}{dt}$ is our initial ODE.

b) $\frac{dF}{dt} = \frac{1}{2}e^{2t}$ from part (a) Then

$\int \frac{dF}{dt}\, dt = \int e^{2t}\, dt$

$F(t) = \frac{1}{2}e^{2t} + C$ for some constant C

which implies that

$\boxed{(t^2-2t)y' + (2t-2)y = \frac{1}{2}e^{2t} + C}$

as desired.

c) Solve the equation from part (b).

Let $M(t,y) = (2t-2)y$

and $N(t,y) = (t^2-2t)$

- This equation is exact since

$\underline{\partial_y M(t,y) = 2t-2 = \partial_t N(t,y)}$.

Clearly, by the product rule,

$$(t^2-2t)y' + (2t-2)y = \frac{d}{dt}[(t^2-2t)y] = \frac{e^{2t}}{2} + c.$$

This implies

$$(t^2-2t)y = \int\left(\frac{e^{2t}}{2} + c\right)dt$$

$$= \frac{e^{2t}}{4} + c_1 t + c_2 \quad \text{for some } c_1, c_2 \in \mathbb{R}.$$

Our general solution is then given implicitly by

$$\boxed{(t^2-2t)y = \frac{1}{4}e^{2t} + c_1 t + c_2}$$

d) Find the unique solution if $y(0)=1$ and $y'(0)=1$?

$$\cancel{(0^2-2(0))(1)}^{0} = \frac{1}{4}e^{2(0)} + c_1(0) + c_2$$

$$0 = \frac{1}{4} + c_2$$

Then $c_2 = -\frac{1}{4}$.

Similarly, when we differentiate, (implicitly),

$$\frac{d}{dt}[(t^2-2t)y] = \frac{d}{dt}\left[\frac{1}{4}e^{2t} + c_1 t - \frac{1}{4}\right]$$

$$(2t-2)y + (t^2-2t)y' = \frac{1}{2}e^{2t} + c_1$$

$$(2(0)-2)(1) + ((0)^2-2(0))(1) = \frac{1}{2}e^{2(0)} + c_1$$

$$-2 = \frac{1}{2} + c_1 \quad \text{So then } c_1 = -4.$$

Our solution to the IVP is then implicitly given by

$$\boxed{(t^2-2t)y = \frac{1}{4}e^{2t} - 4t - \frac{1}{4}}$$

Note: I am giving this solution implicitly because the explicit solution $y = \dfrac{\frac{1}{4}e^{2t} - 4t - \frac{1}{4}}{(t^2 - 2t)}$

would not be defined for all t, more formally, $y(0)$ and $y(2)$ are both undefined solutions.

Hence I am giving the solution implicitly.