MAT237 Multivariable Calculus with Proofs
Problem Set 8
Due Friday April 1, 2022 by 13:00 ET

Instructions

This problem set is on Module H: Calculus with curves (H7 to H8) and Module I: Calculus with surfaces (I1 to I4). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 8 sessions are held on Tuesday March 29, 2022 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

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I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway

2) ________

Created and revised by C. Jonker, T. Janisse, and A. Zaman. Last updated March 31, 2022 at 17:35:08.

Problems

1. Let $F(x,y) = \left(\sin(y^2) + 2x, e^{3x} + 3xy\right)$ be a vector field in $\mathbb{R}^2$. Let C be the curve in $\mathbb{R}^2$ described in polar form by $r = \cos(2\theta)$ for $0 \le \theta \le \pi/4$. A particle moves along the curve C from $(1,0)$ to $(0,0)$.

   You will try two different ways to compute $\displaystyle\int_C F \cdot n\, ds$, the normal flow of F across C.

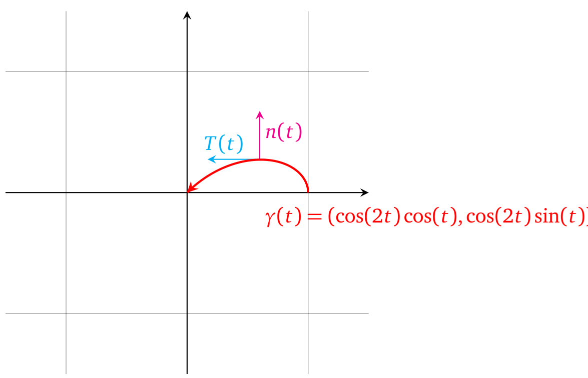
   (1a) Express the normal flow as a single variable integral using a parametrization of C. Do not compute it.

I will begin by establishing a parametrization of the curve C. The polar coordinate transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(r,\theta) = (r\cos\theta, r\sin\theta)$ suggests that the curve can be parametrized by $\gamma : \left[0, \frac{\pi}{4}\right] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos(2t)\cos(t), \cos(2t)\sin(t))$. This can be obtained by just substituting $r = \cos(2t)$ for $0 \le t \le \pi/4$. Note that when $t = 0$, $\gamma(0) = (0,1)$ and when $t = \pi/4$, $\gamma(\pi/4) = (0,0)$, which implies that $\gamma(t)$ is positively oriented.

I will proceed by finding the direction of the normal vector orthogonal to the tangent vector. I must insist that it is not necessary to calculate the norm of the normal vector, since it divides out of the integrand later in the process. The tangent vector direction is given by $\gamma'(t)$:

$$\gamma'(t) = (-2\sin(2t)\cos(t) - \cos(2t)\sin(t),\ -2\sin(2t)\sin(t) + \cos(2t)\cos(t)).$$

By the right-hand rule, we wish for the normal vector to be oriented away from the curve, since γ is a positively oriented parametrization.



The linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (y, -x)$ is the map which takes vectors and rotates them $\pi/2$ clockwise. Applying this transformation to $\gamma'(t)$ gives the direction of the normal vector to the curve:

$$N(t) = T(\gamma'(t)) = (-2\sin(2t)\sin(t) + \cos(2t)\cos(t),\ +2\sin(2t)\cos(t) + \cos(2t)\sin(t)).$$

Since the norm is just the change in direction of the tangent vector, the magnitudes stay the same: $||N(t)|| = ||\gamma'(t)||$. The normal flow of F across C can then be expressed by

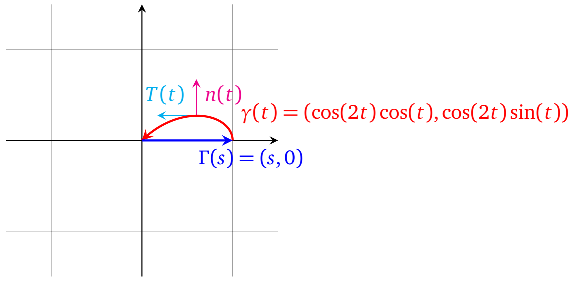
$$\begin{aligned}\int_C (F \cdot n)\, ds &= \int_0^{2\pi} \frac{1}{||\gamma'(t)||}(F \circ \gamma(t)) \cdot N(t) ||\gamma'(t)|| dt \\ &= \int_0^{2\pi} (F \circ \gamma(t)) \cdot N(t) dt.\end{aligned}$$

After substituting, the normal flow of F across C is then

$$\begin{aligned}\int_C (F \cdot n)\, ds = \int_0^{2\pi} &\left(\sin(\cos^2(2t)\sin^2(t)) + 2(\cos(2t)\cos(t)),\ e^{3\cos(2t)\cos(t)} + 3(\cos(2t)\cos(t))(\cos(2t)\sin(t))\right) \\ &\cdot (-2\sin(2t)\sin(t) + \cos(2t)\cos(t),\ +2\sin(2t)\cos(t) + \cos(2t)\sin(t))\, dt.\end{aligned}$$

(1b) Use Green's theorem to compute the normal flow. You may use WolframAlpha to calculate single variable integrals; indicate when you have done so. *Hint:* Close the loop.

I will begin by closing the loop with the positively oriented curve C' from $(0,0)$ to $(1,0)$, defined by the parametrization $\Gamma : [0,1] \to \mathbb{R}^2$ given by $\Gamma(s) = (s, 0)$:



The region bounded by the curves is $R = \{(r,\theta) \in \mathbb{R}^2 : 0 \le r \le \cos(2\theta),\, 0 \le \theta \le \pi/4\}$. R is certainly a regular region with each parametrization being positively oriented. Furthermore, $F(x,y)$ is a C^1 vector field in $\mathbb{R}^2$, and so I can proceed by applying Green's theorem:

$$\oint_{\partial R} (F \cdot n)\, ds = \iint_R \operatorname{div}(F)\, dA.$$

The divergence of F is simply $\operatorname{div}(F) = \partial_x(\sin(y^2) + 2x) + \partial_y(e^{3x} + 3xy) = 2 + 3x$. Integrating over the region R is polar coordinates, we have that

$$\begin{aligned}
\oint_{\partial R} (F \cdot n)\, ds &= \int_0^{\pi/4} \int_0^{\cos(2\theta)} (2 + 3r\cos\theta)\, r\, dr\, d\theta \\
&= \int_0^{\pi/4} (\cos^2(2\theta) + \cos^3(2\theta)\cos\theta)\, d\theta \\
&= \frac{\pi}{8} + \frac{8\sqrt{2}}{35} \qquad \text{by WolframAlpha.}
\end{aligned}$$

Now, this is the total flux through C and C', so to find the flux through C, we must subtract the flux through C'. With the unit normal pointing outwards from R, $n_{C'}(s) = (0,-1)$. Then the flux integral across the bottom line is

$$\begin{aligned}
\int_0^1 F(\Gamma(s)) \cdot n_{C'}(s)\, ds &= \int_0^1 \left(\sin(0^2) + 2(s),\, e^{3s} + 3(s)(0)\right) \cdot (0,-1)\, ds \\
&= -\int_0^1 e^{3s}\, ds \\
&= -\frac{1}{3}(e^3 - 1).
\end{aligned}$$

Therefore by the linearity of integrals over piecewise curves, the total flux across C_1 is

$$\begin{aligned}
\int_C (F \cdot n)\, ds &= \oint_{\partial R} (F \cdot n)\, ds - \int_{C'} (F \cdot n)\, ds \\
&= \frac{\pi}{8} + \frac{8\sqrt{2}}{35} - \left(-\frac{1}{3}(e^3 - 1)\right) \\
&= \frac{\pi}{8} + \frac{8\sqrt{2}}{35} + \frac{1}{3}(e^3 - 1).
\end{aligned}$$

2. Multivariable calculus has shown how you can do calculus with all of your linear algebra. Now, near the end of your journey, it is time to do linear algebra with all of your calculus (in two dimensions).

   Let $U \subseteq \mathbb{R}^2$ be an open set. Let $C^\infty(U)$ be the set of real-valued functions $f : U \to \mathbb{R}$ with infinitely many partial derivatives; that is, $\partial^\alpha f$ exists and is continuous on U for all multi-indices $\alpha \in \mathbb{N}^2$. The space of C^∞ **scalar functions** $V = C^\infty(U)$ and space of C^∞ **vector fields** $V^2 = V \times V$ can each be thought of as a space of vectors. For example, the zero function belongs to V and acts like the zero vector. Moreover, any linear combination in V also belongs to V. Similar statements hold true for V^2.

   (2a) You can view the differential operators 'grad' and 'curl' as linear transformations on these spaces.

   - Gradient is a linear map of C^∞ scalar functions to C^∞ vector fields. That is, $\text{grad} : V \to V^2$ is a linear map. Hence, if $f \in V$ then $\text{grad}(f) \in V^2$.
   - Two-dimensional curl is a linear map of C^∞ vector fields to C^∞ scalar functions. That is, $\text{curl} : V^2 \to V$ is a linear map. Hence, if $F \in V^2$ then $\text{curl}(F) \in V$.

   Prove that curl is a linear map from V^2 to V. You may assume that the partial derivative operators $\partial_1 : V \to V$ and $\partial_2 : V \to V$ are linear maps.

Proof. It suffices to show that, for $F, G \in V^2$ and $\lambda \in \mathbb{R}$, that $\text{curl}(F + \lambda G) = \text{curl}(F) + \lambda\,\text{curl}(G)$, which shows that curl is linear. Now $\text{curl} : V^2 \to V$ is given by $\text{curl}(J) = \partial_1 J_2 - \partial_2 J_1$ for $J \in V^2$. Since $F, G \in V^2$, then each component of F and G are real valued: $F_1, F_2, G_1, G_2 \in V$. Assuming the linearity of the partial derivative operators, we have that

$$\begin{aligned}
\text{curl}(F + \lambda G) &= \partial_1(F_2 + \lambda G_2) - \partial_2(F_1 + \lambda G_1) && \text{by operator definition} \\
&= \partial_1 F_2 + \lambda \partial_1 G_2 - \partial_2 F_1 - \lambda \partial_2 G_1 && \text{by linearity of partial derivative operators} \\
&= (\partial_1 F_2 - \partial_2 F_1) + \lambda(\partial_1 G_2 - \partial_2 G_1) && \text{by rearranging, factoring} \\
&= \text{curl}(F) + \lambda\,\text{curl}(G) && \text{by operator definition}
\end{aligned}$$

which is what I wanted to show. Note that the operator takes in functions $F, G \in V^2$ and maps them to the codomain V, which is preserved under the partial derivative operators which also map to V. Therefore
$\text{curl} : V^2 \to V$ is linear.

□

   (2b) The image of the gradient is contained in the kernel of curl. That is, $\text{img(grad)} \subseteq \text{ker(curl)}$. There are two ways to prove this fact: by direct boring calculation or by the "one true proof".

   Prove that $\text{img(grad)} \subseteq \text{ker(curl)}$ by a direct boring calculation with partial derivatives.

Proof. I will begin by defining the sets

$$\text{img(grad)} = \{\nabla f = (\partial_1 f, \partial_2 f) : f \in V\} \qquad \text{ker(curl)} = \{F \in V^2 : \partial_1 F_2 - \partial_2 F_1 = 0.\}$$

The kernel of the curl is the set of all vector fields which are taken to zero when the operator is applied. Since the functions in the vector spaces V and V^2 are C^∞ (infinitely differentiable), then Clairaut's theorem applies. We have that

$$\text{curl}(\text{grad}(f)) = \text{curl}(\partial_1 f, \partial_2 f) = \partial_1(\partial_2 f) - \partial_2(\partial_1 f) = 0.$$

Therefore the curl of any gradient vector field is zero, so $\text{img(grad)} \subseteq \text{ker(curl)}$.

□

(2c) The "one true proof" of img(grad) $\subseteq$ ker(curl) relies upon the fundamental theorem of line integrals and Green's theorem. Here is such an attempt to prove this containment.

> *1. Let* $F \in \operatorname{img}(\operatorname{grad})$ *and* $p \in U$*. Then* $\forall \varepsilon > 0, \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds = 0.$
>
> *2.* $\implies \forall \varepsilon > 0, \iint\limits_{B_\varepsilon(p)} (\operatorname{curl} F)\, dA = 0 \implies \lim\limits_{\varepsilon \to 0^+} \left[\frac{1}{\operatorname{vol}(\overline{B_\varepsilon(p)})} \iint\limits_{B_\varepsilon(p)} (\operatorname{curl} F)\, dA \right] = 0$
>
> *3.* $\implies (\operatorname{curl} F)(p) = 0 \implies F \in \ker(\operatorname{curl})$

There are no serious errors but it is terribly written. Rewrite this into a well-written justified proof.

1. Let $F \in \operatorname{img}(\operatorname{grad})$ and $p \in U$. Then F can be written as the gradient of a scalar function $f : U \to \mathbb{R}$, so $F = \nabla f$. By the fundamental theorem for line integrals **(Theorem 8.4.1)**, then the line integral of any closed curve is zero. Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the positively oriented parametrization of the boundary of the epsilon ball centered at p. Then

$$\forall \varepsilon > 0, \oint_{\partial B_\varepsilon(p)} (F \cdot T) ds = \int_0^{2\pi} \nabla f \cdot d\gamma = f(\gamma(2\pi)) - f(\gamma(0)) = 0.$$

2. By Green's Theorem **(Theorem 8.7.6)**, since $\overline{B_\varepsilon(p)}$ is a regular region, then

$$\forall \varepsilon > 0, \oint_{\partial B_\varepsilon(p)} (F \cdot T) ds = \iint\limits_{\overline{B_\varepsilon(p)}} \operatorname{curl}(F)\, dA = 0.$$

   This implies that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\operatorname{vol}((\overline{B_\varepsilon(p)}))} [0] = 0 = \lim_{\varepsilon \to 0^+} \left[\frac{1}{\operatorname{vol}(\overline{B_\varepsilon(p)})} \iint\limits_{\overline{B_\varepsilon(p)}} (\operatorname{curl} F)\, dA \right].$$

3. By the Integral mean value theorem **(Corollary 6.8.13)**, since $\operatorname{curl}(F)$ is real valued and is smooth on U, then $\operatorname{curl}(F)$ is continuous on an open set containing p, which implies

$$\operatorname{curl}(F)(p) = \lim_{\varepsilon \to 0^+} \frac{1}{\operatorname{vol}(\overline{B_\varepsilon(p)})} \iint\limits_{\overline{B_\varepsilon(p)}} (\operatorname{curl} F)\, dA = 0.$$

   Since the kernel of the curl is the set of all vector fields which are taken to zero when the operator is applied, and $\operatorname{curl}(F)(p) = 0$ is true for any $p \in U$, then $F \in \ker(\operatorname{curl})$. This completes the proof.

(2d) The images may or may not equal the kernels in (2b). It depends on the topology of $U \subseteq \mathbb{R}^2$.

- Give an example of a set $U = U_1$ where $\operatorname{img}(\operatorname{grad}) = \ker(\operatorname{curl})$.
- Give an example of a set $U = U_2$ where $\operatorname{img}(\operatorname{grad}) \neq \ker(\operatorname{curl})$.

Briefly justify each of your examples. *Hint:* See Problem Set 7.

Choose $U = U_1 = \mathbb{R}^2$. Here, smooth vector fields are then defined on U_1. From the previous parts, we know that the curl of any gradient vector field is zero. And thus the kernel of the curl is every vector field defined on $\mathbb{R}^2$, since this is where every gradient vector field is taken to zero. Therefore for U_1, $\operatorname{img}(\operatorname{grad}) = \ker(\operatorname{curl})$.

Now choose $U = U_2 = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. This correlates to every smooth vector field defined on the first quadrant of the xy plane in $\mathbb{R}^2$. The kernel of the curl is every vector field defined on U_2, however it is not always the case that the image of the gradient is every gradient vector field defined on U_2. If we consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \log(x) + \log(y)$, we can easily see that f is defined on U_2. However, the gradient of f is $\nabla f = \left(\frac{1}{x}, \frac{1}{y}\right)$, which is actually defined on the set $\{(x, y) \in \mathbb{R}^2 : x \neq 0, y \neq 0\} \subseteq \mathbb{R}^2$. Here, $\operatorname{img}(\operatorname{grad}) \neq \ker \operatorname{curl}$.

(2e) All of your observations about grad and curl can be beautifully encapsulated in this elegant diagram.

$$V \xrightarrow{\text{grad}} V^2 \xrightarrow{\text{curl}} V$$

At first glance, this appears to just be a composition of maps but if you dig a bit deeper, you will notice it actually captures all of vector calculus in $\mathbb{R}^2$. Take an arbitrary element at the leftmost V in the diagram. Map it once to V^2 and then map it again to the next V. The element has moved two stages to the right. What happened to this element? How do the two core theorems of vector calculus in $\mathbb{R}^2$ relate to this phenomenon? Explain in two to three full sentences using the previous parts of this question.

Consider an element in V, ie a real valued function $f \in C^\infty(U)$. The gradient operator first takes f into the gradient of f, that is $f \xrightarrow{\text{grad}} \nabla f = (\partial_1 f, \partial_2 f)$. From here, the curl operator takes the gradient vector field and maps it to zero, which is a direct result of smooth functions and Clairaut's theorem: $\operatorname{curl}(\partial_1 f, \partial_2 f) = \partial_1(\partial_2 f) - \partial_2(\partial_1 f) = 0$.

The two core theorems used to describe this phenomenon are Green's theorem and the Fundamental Theorem of Line Integrals. Respectively, Green's theorem relates a surface integral over curl to a closed line integral in $\mathbb{R}^2$, while the fundamental theorem of line integrals tells us that for a gradient vector field over a closed curve, the line integral is zero.

3. Fix $0 < b < a$. Define the map $G : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$G(s, t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t).$$

Define the set $T = \operatorname{img} G \subseteq \mathbb{R}^3$ so T is a torus. See this Math3D demo for an illustration.

(3a) Show that T is a surface parametrized by G.

To show that T is a surface parametrized by G, it suffices to show that G is simple and regular.

- Note that $G(s, t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t)$. Since $\sin x$ and $\cos x$ are continuous functions, then G is continuous since each component is continuous. Furthermore,

$$\partial_s G(s, t) = (-(a + b\cos t)\sin s,\ (a + b\cos t)\cos s,\ 0) \quad \partial_t G(s, t) = (-b\sin t\cos s,\ -b\sin t\sin s,\ b\cos t)$$

  which are continuous derivatives. Therefore G is C^1. For linear dependence, it suffices to check one component. Note that $\partial_s G(s, t) \cdot e_3 = 0 \neq \partial_t G(s, t) \cdot e_3 = b\cos t$, only if $t = 0$, which is a set of zero Jordan measure. With this, we also have that $\{\partial_s G(s, t),\ \partial_t G(s, t)\}$ is linearly independent only if $(s, t) \neq (0, 0)$, which again defines the set of zero Jordan measure $\{(0, 0)\}$. By **Definition 9.1.6**, G is regular.
- Let $(s_1, t_1), (s_2, t_2) \in (0, 2\pi)^2$. Assume that $G(s_1, t_1) = G(s_2, t_2)$. That is,

$$\begin{pmatrix} (a + b\cos t_1)\cos s_1 \\ (a + b\cos t_1)\sin s_1 \\ b\sin t_1 \end{pmatrix} = \begin{pmatrix} (a + b\cos t_2)\cos s_2 \\ (a + b\cos t_2)\sin s_2 \\ b\sin t_2 \end{pmatrix}$$

  By direct fallout of the e_3 component, we have that $\sin t_1 = \sin t_2$, which implies that $t_1 = t_2$ since sin is injective. Likewise, with the e_1 and e_2 componenets, we have that $\cos s_1 = \cos s_2$ and $\sin s_1 = \sin s_2$. Since sin and cos are again both injective functions, then $s_1 = s_2$. Therefore $(s_1, t_1) = (s_2, t_2)$, and by **Definition 9.1.11**, G is injective on $(0, 2\pi)^2$. The boundary gives non-injectivity ($G(0, 0) = G(2\pi, 2\pi)$), however still satisfied the definition of G being simple.
- By **Definition 9.1.1**, since $T = \operatorname{img}(G)$ and G is both regular and simple, then T is a surface parametrized by G.

(3b) Find the surface area of T.

Most of this solution is just algebraic manipulations. The surface area of the surface T is defined as $A(T) = \iint_{[0,2\pi]^2} \|\partial_s G \times \partial_t G\| \, dA$. The partial derivatives $\partial_s G$ and $\partial_t G$ are given from the previous part. They are

$$\partial_s G(s,t) = (-(a + b\cos t)\sin s,\ (a + b\cos t)\cos s,\ 0) \qquad \text{and} \qquad \partial_t G(s,t) = (-b\sin t\cos s,\ -b\sin t\sin s,\ b\cos t).$$

Then, taking the cross product, we have that

$$\begin{aligned}
\partial_s G \times \partial_t G(s,t) &= \det\begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_s G_1 & \partial_s G_2 & 0 \\ \partial_t G_1 & \partial_t G_2 & \partial_t G_3 \end{pmatrix} = e_1(\partial_2 G_2 \partial_t G_3) - e_2(\partial_2 G_1 \partial_t G_3) + e_3(\partial_s G_1 \partial_t G_2 - \partial_t G_1 \partial_s G_2) \\
&= (a + b\cos t)b\left(e_1[\cos s\cos t] - e_2[\sin s\cos t] + e_3\left[\sin^2 s\sin t + \cos^2 s\sin t\right]\right).
\end{aligned}$$

Then, taking the norm,

$$\begin{aligned}
\|\partial_s G \times \partial_t G(s,t)\| &= (a + b\cos t)b\left[\cos^2 s\cos^2 t + \sin^2 s\cos^2 t + \sin^2 t\sin^4 s + \sin^2 t\cos^4 s + 2\sin^2 t\sin^2 s\cos^2 s\right]^{1/2} \\
&= (a + b\cos t)b\left[\cos^2 t(\cos^2 s + \sin^2 s) + \sin^2 t(\sin^4 s + \cos^4 s + 2\sin^2 s\cos^2 s)\right] \\
&= (a + b\cos t)b\left[\cos^2 t + \sin^2 t\left(\sin^2 s + \cos^2 s\right)^2\right] \\
&= (a + b\cos t)b,
\end{aligned}$$

which is obtained by applying the pythagorean identity $\sin^2 x + \cos^2 x = 1$ in lines 3 and 4. The expression for surface area then becomes

$$\begin{aligned}
A(T) &= \int_0^{2\pi}\int_0^{2\pi} (a + b\cos t)b\, ds dt \\
&= 2\pi b\int_0^{2\pi} a + b\cos t\, dt \\
&= 2\pi b \cdot (2\pi a + 0) \\
&= 4\pi^2 ab.
\end{aligned}$$

Therefore the surface area of the torus is $SA = 4\pi^2 ab$ where $0 < b < a$ are the inner and outer radii of the torus, respectively.

4. Let $G : U \to \mathbb{R}^3$ and $H : V \to \mathbb{R}^3$ be parametrizations of a surface $S \subseteq \mathbb{R}^3$. Assume G and H are C^1 and $\{\partial_1 G, \partial_2 G\}$ are linearly independent on the interior of their domains. Assume there exists a diffeomorphism $\psi : A \to B$ such that $U \subseteq A$ and $V \subseteq B$ and $G = H \circ \varphi$, where $\varphi = \psi|_U : U \to V$. If $\det D\psi > 0$, then show

$$\forall u \in U^{\circ}, v \in V^{\circ}, G(u) = H(v) \implies \frac{\partial_1 G \times \partial_2 G(u)}{||\partial_1 G \times \partial_2 G(u)||} = \frac{\partial_1 H \times \partial_2 H(v)}{||\partial_1 H \times \partial_2 H(v)||}.$$

In other words, the unit normal is invariant under reparametrization. (Revised 2022-03-26)

I will begin by applying the chain rule to $G = H \circ \varphi$, and then apply the cross product. Assume that $G = G(s,t)$, so $H = H(\varphi(s,t))$. Since G, H and φ are C^1, then by the chain rule we have that

$$\frac{\partial G}{\partial s} = \frac{\partial H}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial s} + \frac{\partial H}{\partial \varphi_2}\frac{\partial \varphi_2}{\partial s} \qquad \text{and} \qquad \frac{\partial G}{\partial t} = \frac{\partial H}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial t} + \frac{\partial H}{\partial \varphi_2}\frac{\partial \varphi_2}{\partial t}.$$

Since $\{\partial_s G, \partial_t G\}$ are linearly independent on the interior of their domains, then evaluated at (s,t) their cross product is non-zero and is given by

$$\begin{aligned}\frac{\partial G}{\partial s} \times \frac{\partial G}{\partial t}(s,t) &= \left(\frac{\partial H}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial s} + \frac{\partial H}{\partial \varphi_2}\frac{\partial \varphi_2}{\partial s}\right) \times \left(\frac{\partial H}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial t} + \frac{\partial H}{\partial \varphi_2}\frac{\partial \varphi_2}{\partial t}\right)(\varphi(s,t)) \\ &= \left[\frac{\partial \varphi_1}{\partial s}\frac{\partial \varphi_1}{\partial t}\left(\frac{\partial H}{\partial \varphi_1} \times \frac{\partial H}{\partial \varphi_1}\right) + \frac{\partial \varphi_1}{\partial s}\frac{\partial \varphi_2}{\partial t}\left(\frac{\partial H}{\partial \varphi_1} \times \frac{\partial H}{\partial \varphi_2}\right) + \frac{\partial \varphi_2}{\partial s}\frac{\partial \varphi_1}{\partial t}\left(\frac{\partial H}{\partial \varphi_2} \times \frac{\partial H}{\partial \varphi_1}\right) + \frac{\partial \varphi_2}{\partial s}\frac{\partial \varphi_2}{\partial t}\left(\frac{\partial H}{\partial \varphi_2} \times \frac{\partial H}{\partial \varphi_2}\right)\right](\varphi(s,t)).\end{aligned}$$

Since for any vectors a and b, $a \times a = 0$ and $a \times b = -b \times a$, then this reduces to

$$\begin{aligned}\frac{\partial G}{\partial s} \times \frac{\partial G}{\partial t}(s,t) &= \left[\frac{\partial \varphi_1}{\partial s}\frac{\partial \varphi_2}{\partial t}\left(\frac{\partial H}{\partial \varphi_1} \times \frac{\partial H}{\partial \varphi_2}\right) + \frac{\partial \varphi_2}{\partial s}\frac{\partial \varphi_1}{\partial t}\left(\frac{\partial H}{\partial \varphi_2} \times \frac{\partial H}{\partial \varphi_1}\right)\right](\varphi(s,t)) \\ &= \frac{\partial H}{\partial \varphi_1} \times \frac{\partial H}{\partial \varphi_2}(\varphi(s,t)\left[\frac{\partial \varphi_1}{\partial s}\frac{\partial \varphi_2}{\partial t} - \frac{\partial \varphi_2}{\partial s}\frac{\partial \varphi_1}{\partial t}\right].\end{aligned}$$

Now, since ψ is bijective with $\det D\psi > 0$, then $\varphi = \psi|_U : U \to V$ is bijective with $\det D\varphi > 0$. This implies that $G(u) = G(s,t) = H(\varphi(s,t)) = H(v)$, so we can perform a brief change of notation to find that

$$\partial_1 G \times \partial_2 G(u) = \partial_1 H \times \partial_2 H\,[\partial_1\varphi_1\partial_2\varphi_2 - \partial_1\varphi_2\partial_2\varphi_1](v) = \partial_1 H \times \partial_2 H(v)\det(D\varphi).$$

Now $\{\partial_1 G, \partial_2 G\}$ is linearly independent, so $\partial_1 G \times \partial_2 G \neq 0$. Since $\det D\varphi > 0$, then it must also be that $\partial_1 H \times \partial_2 H \neq 0$. This implies that we can normalize each vector, so $\dfrac{\partial_1 G \times \partial_2 G(u)}{||\partial_1 G \times \partial_2 G(u)||} = \dfrac{\det(D\varphi)\partial_1 H \times \partial_2 H(v)}{|\det(D\varphi)|||\partial_1 H \times \partial_2 H(v)||}$. It follows that since $\dfrac{x}{|x|} = 1$ for $x > 0$ and $\dfrac{x}{|x|} = -1$ for $x < 0$, then since $\det D\varphi > 0$,

$$\frac{\partial_1 G \times \partial_2 G(u)}{||\partial_1 G \times \partial_2 G(u)||} = (1)\frac{\partial_1 H \times \partial_2 H(v)}{||\partial_1 H \times \partial_2 H(v)||},$$

which is what I wanted to show. Therefore the unit normal is invariant under reparametrization.

5. An electric field generated by a wire along the z-axis is given by $F(x,y,z) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\right)$. Fix $H, R > 0$. Compute the flux of F across the cylinder

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = R^2, 0 \leq z \leq H\}$$

oriented with unit normal pointing radially away from the z-axis. (Revised 2022-03-25)

To begin, we can define a parametrization of the cylinder. Using spherical coordinates to integrate will simplify the calculations, so that is what I will do (lol just kidding, I will use cylindrical coordinates). The radius is fixed, while the height of the cylinder varies between 0 and H. Define the parametrization $☺ : [0, 2\pi] \times [0,1] \to \mathbb{R}^3$ by $☺(s,t) = (R\cos s, R\sin s, Ht)$. Now, $\operatorname{img}(☺) = S$ and $☺$ is C^1, simple, and injective (hence regular), and therefore $☺$ is a two-variable parametrization of S.

To find the flux of F through the cylinder, it follows to compute the unit normal vector for $☺$, which we want to be pointing outwards from the cylinder. The unit normal is given by the cross product of the partial derivatives or $☺$, which are

$$\partial_s ☺(s,t) = (-R\sin s, R\cos s, 0) \qquad \text{and} \qquad \partial_t ☺(s,t) = (0,0,H).$$

$$\partial_s ☺ \times \partial_t ☺(s,t) = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ -R\sin s & R\cos & 0 \\ 0 & 0 & H \end{pmatrix} = e_1(RH\cos s) + e_2(RH\sin s).$$

Note that when $s = 0$, the direction of the normal vector is $(RH, 0, 0)$, which points outwards along the x-axis, as desired. The norm $\|\partial_s ☺ \times \partial_t ☺(s,t)\| = \sqrt{R^2H^2\cos^2 s + R^2H^2\sin^2 s} = RH$, and thus the unit normal is given by $☹ : [0,2\pi] \times [0,1] \to \mathbb{R}^3$ defined as $☹(s,t) = (RH\cos s, RH\sin s, 0)$. The flux of F across S is then

$$\begin{aligned}
\iint_{[0,2\pi]\times[0,1]} (F \circ ☺) \cdot ☹ \, dA &= \int_0^1 \int_0^{2\pi} \left(\frac{R\cos s}{R^2\cos^2 s + R^2\sin^2 s}, \frac{R\sin s}{R^2\cos^2 s + R^2\sin^2 s}, 0\right) \cdot (RH\cos s,\, RH\sin s, 0)\, ds\, dt \\
&= \frac{R^2H}{R^2} \int_0^1 \int_0^{2\pi} (\cos s,\, \sin s,\, 0) \cdot (\cos s,\, \sin s,\, 0)\, ds\, dt \\
&= H \int_0^1 \int_0^{2\pi} \cos^2 s + \sin^2 s \, ds\, dt \\
&= H \int_0^1 \int_0^{2\pi} 1 \, ds\, dt \\
&= H(1)(2\pi).
\end{aligned}$$

Therefore the total electric flux of F across the cylinder is simply $2\pi H$.