# PHY489 Problem Set 5

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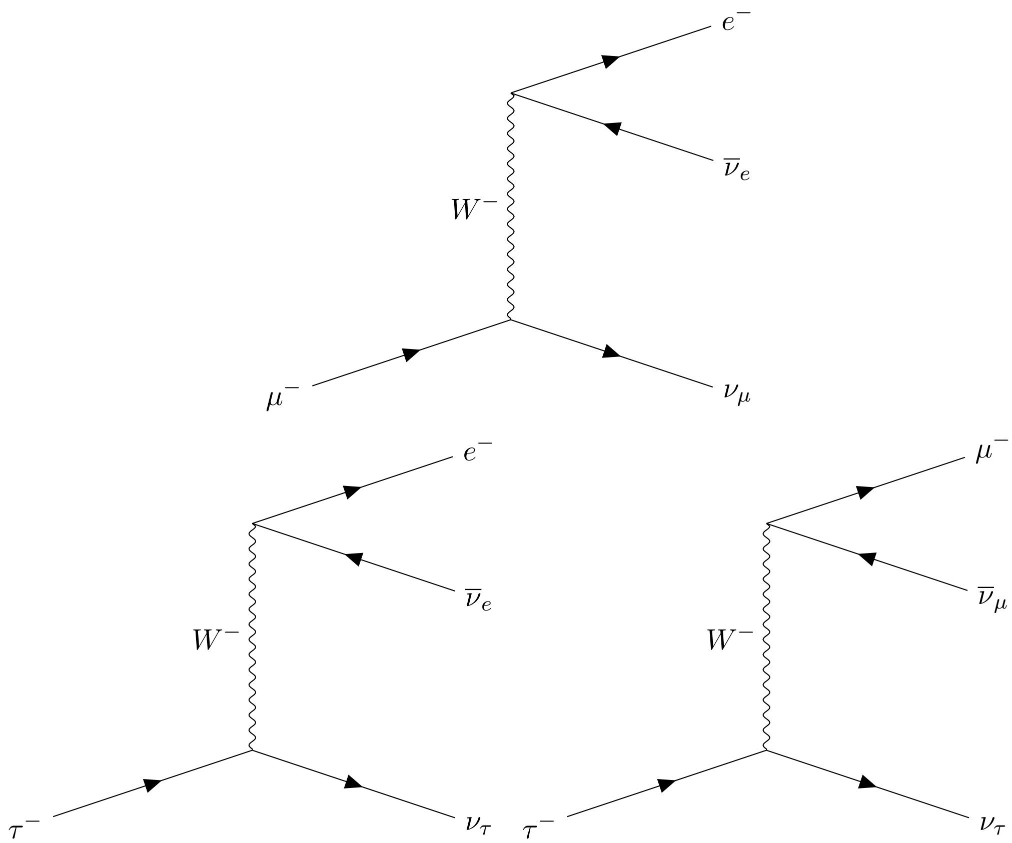
**Problem 1**

In lecture, we discussed the decay rate of the muon, given by

$$\Gamma(\mu^- \to \nu_\mu e^- \nu_e) = \left(\frac{m_\mu g_w}{M_W}\right)^4 \frac{m_\mu c^2}{12\hbar(8\pi)^3} \tag{1.1}$$

in the limit that $m_e \ll m_\mu$.

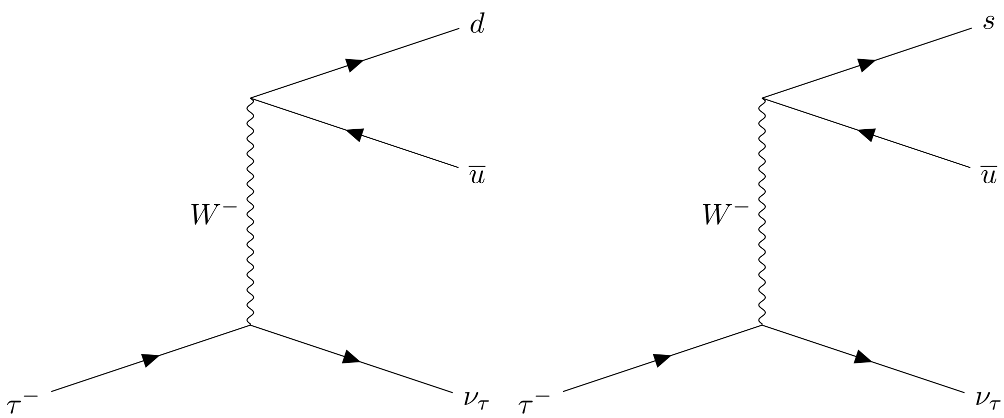
(**a**) We now consider the leptonic decay rate of the tau lepton, that is, the combine rate of the processes $\tau^- \to \nu_\tau \mu^- \nu_\mu$ and $\tau^- \to \nu_\tau e^- \nu_e$. We similarly approximate and neglect the masses of the electron and muon compared to that of the tau. First observe that all three processes (including the muon decay) have identical feynman diagrams:




which implies that each amplitude is also identical upon evaluation and spin-averaging. Since the amplitudes of contributing diagrams superpose, we have that $\langle|\mathcal{A}_{fi}|^2\rangle \to \langle|2\mathcal{A}_{fi}|^2\rangle = 4\langle|\mathcal{A}_{fi}|^2\rangle$, where we also write the mass of the tau instead of the muon. Since the constant $4$ is taken out of the integrand upon decay width calculation, then the rate of the combined processes is simply

$$\Gamma(\tau^- \to \nu_\tau \mu^- \nu_\mu \,+\, \tau^- \to \nu_\tau e^- \nu_e) = \left(\frac{m_\tau g_w}{M_W}\right)^4 \frac{m_\tau c^2}{3\hbar(8\pi)^3}. \tag{1.2}$$

(**b**) We now wish to extend this idea into the regime of the tau's hadronic decay modes, $\tau^- \to \nu_\tau d\overline{u}$ and $\tau^- \to \nu_\tau s\overline{u}$. Again, the relative masses of the quarks compared to that of the tau is negligible, and therefore we can approximate them as zero. The corresponding diagrams are




This is very similar to the leptonic processes, however, we now have flavour changing quark mechanisms occuring at each vertex, which just contribute to each vertex by a factor

$$-i\frac{g_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)V_{ij} \tag{1.3}$$

where $V_ij$ is now the CKM matrix elements corresponding to the change in mass bases of the quarks. The contribution from the first diagram is simply $V_{ud}$ at the end of the $W$ propagator (this is just a constant anyway and can be commuted through to the front) and $V_{us}$ for the second diagram. Therefore, upon summing the amplitude contributions and averaging over spins, we have

$$\langle|\mathcal{A}_1+\mathcal{A}_2|^2\rangle = \langle|V_{ud}\mathcal{A}_{fi}+V_{us}\mathcal{A}_{fi}|^2\rangle = (V_{ud}+V_{us})^2\,\langle|\mathcal{A}_{fi}|^2\rangle \tag{1.4}$$

which only works because $\mathcal{A}_{fi}$ can be factored out and the $V_{ij}$ elements are constants. The superposition of both contributing amplitudes for the hadronic decay modes therefore give

$$\Gamma(\tau^-\to\nu_\tau d\overline{u}\ +\ \tau^-\to\nu_\tau s\overline{u}) = \left(\frac{m_\tau g_w}{M_W}\right)^4\frac{m_\tau c^2}{12\hbar(8\pi)^3}(V_{ud}+V_{us})^2. \tag{1.5}$$

It is also important to discuss the lifetime of the tau, now taking into account both leptonic and hadronic decay modes. Since $\tau_\tau = \dfrac{1}{\Gamma_{\text{TOT}}}$, we sum the contributions from (1.2) and (1.5):

$$\Gamma(\tau^-\to\nu_\tau\mu^-\nu_\mu\ +\ \tau^-\to\nu_\tau e^-\nu_e)+\Gamma(\tau^-\to\nu_\tau d\overline{u}\ +\ \tau^-\to\nu_\tau s\overline{u}) = \left(\frac{m_\tau g_w}{M_W}\right)^4\frac{m_\tau c^2}{12\hbar(8\pi)^3}[4+(V_{ud}+V_{us})^2] \tag{1.6}$$

which implies that $\tau_{\text{TOT}} = 3.47\times10^{-13}$ s with $m_\tau = 1776.99$ MeV$/c^2$, $M_w = 80420$ MeV$/c^2$, and $g_w = 0.6295$. Using the same constants, we can determine the individual branching ratios for the muon, electron, and hadronic decay modes. Both the electron and muon branching ratios should be similar, determined by

$$\frac{\Gamma_{e/\mu}}{\Gamma_{\text{TOT}}} = \frac{1}{4+(V_{ud}+V_{us})^2} = 0.184\ [PDG: 0.178(e), 0.174(\mu)]. \tag{1.7}$$

Similarly, for the hadronic modes,

$$\frac{\Gamma_\pi}{\Gamma_{\text{TOT}}} = \frac{V_{ud}^2}{4 + (V_{ud} + V_{us})^2} = 0.174 \; [PDG : 0.108] \tag{1.8}$$

$$\frac{\Gamma_K}{\Gamma_{\text{TOT}}} = \frac{V_{us}^2}{4 + (V_{ud} + V_{us})^2} = 0.009 \; [PDG : 0.007] \tag{1.9}$$

It can be seen that the decay probabilities are similar to that of the PDG values and are good estimates, however not exact. This is because there are many more decay modes available to the $\tau^-$ than just the ones listed above, as well as leptonic modes involving the secondary emission of a photon. We furthermore assume the masses the distinguishable particles to be zero except from that of the tau, which will also contribute to the magnitude of the decay width.

### Problem 2

In this problem, we aim to determine the charged weak couplings for up and down type quarks. Given the coupling to the $Z^0$ boson and the electromagnetic/neutral currents

$$-ig_z(j^3_\mu - \sin^2\theta_w j^{\text{em}}_\mu)Z^\mu \to -\frac{ig_z}{2}\overline{\psi}\gamma_\mu(c^v - c^a\gamma^5)\psi Z^\mu \tag{2.1}$$

$$j^3_\mu = \frac{1}{2}\overline{u}_L\gamma_\mu u_L - \frac{1}{2}\overline{d}_L\gamma_\mu d_L \tag{2.2}$$

$$j^{\text{em}}_\mu = Q(\overline{u}_R\gamma_\mu u_R + \overline{u}_L\gamma_\mu u_L) \tag{2.3}$$

we may utilize the chiral spinor projections and the charges ofthe quarks $Q$ to determine the vector and pseudovector couplings $c^a$, $c^v$. Beginning with up-type quarks ($j^3_\mu$ doesn't include the second term) of charge $\frac{2}{3}$, we have

$$\begin{aligned}
-ig_z(j^3_\mu - \sin^2\theta_w j^{\text{em}}_\mu)Z^\mu &= -ig_z\left[\frac{1}{2}\overline{u}_L\gamma_\mu u_L - \left(\frac{2}{3}\overline{u}_R\gamma_\mu u_R + \frac{2}{3}\overline{u}_L\gamma_\mu u_L\right)\sin^2\theta_w\right]Z^\mu && (2.4)\\
&= -ig_z\left[\frac{1}{4}\overline{u}\gamma_\mu(1-\gamma^5)u - \left(\frac{1}{3}\overline{u}\gamma_\mu(1+\gamma^5)u + \frac{1}{3}\overline{u}\gamma_\mu(1-\gamma^5)u\right)\sin^2\theta_w\right]Z^\mu && (2.5)\\
&= -ig_z\left[\frac{1}{4}\overline{u}\gamma_\mu u - \frac{1}{4}\overline{u}\gamma_\mu\gamma^5 u - \frac{2}{3}\overline{u}\gamma_\mu u\sin^2\theta_w\right]Z^\mu && (2.6)\\
&= -\frac{ig_z}{2}\left[\overline{u}\gamma_\mu\left(\frac{1}{2} - \frac{4}{3}\sin^2\theta_w - \frac{1}{2}\gamma^5\right)u\right]Z^\mu && (2.7)
\end{aligned}$$

which immediately corresponds to vector and pseudovector couplings for the up-type quarks $c^v = \frac{1}{2} - \frac{4}{3}\sin^2\theta_w$, $c^a = \frac{1}{2}$ which correctly matches table 9.1 (Griffiths). We may perform the exact same calculation for down-type quarks, this time taking the second term in (2.2) and with $Q = -\frac{1}{3}$:

$$\begin{aligned}
-ig_z(j^3_\mu - \sin^2\theta_w j^{\text{em}}_\mu)Z^\mu &= -ig_z\left[-\frac{1}{2}\overline{d}_L\gamma_\mu d_L + \left(\frac{1}{3}\overline{d}_R\gamma_\mu d_R + \frac{1}{3}\overline{d}_L\gamma_\mu d_L\right)\sin^2\theta_w\right]Z^\mu && (2.7)\\
&= -ig_z\left[-\frac{1}{4}\overline{u}\gamma_\mu(1-\gamma^5)u + \left(\frac{1}{6}\overline{u}\gamma_\mu(1+\gamma^5)u + \frac{1}{6}\overline{u}\gamma_\mu(1-\gamma^5)u\right)\sin^2\theta_w\right]Z^\mu && (2.8)\\
&= -ig_z\left[-\frac{1}{4}\overline{u}\gamma_\mu u + \frac{1}{4}\overline{u}\gamma_\mu\gamma^5 u + \frac{1}{3}\overline{u}\gamma_\mu u\sin^2\theta_w\right]Z^\mu && (2.9)\\
&= -\frac{ig_z}{2}\left[\overline{u}\gamma_\mu\left(-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w + \frac{1}{2}\gamma^5\right)d\right]Z^\mu && (2.10)
\end{aligned}$$

thus corresponding to $c^v = -\frac{1}{2} + \frac{2}{3}\sin^2\theta_w$, $c^a = -\frac{1}{2}$, which are also consistent with Griffiths.

**Problem 3**

Consider a massive vector field whose plane wave propagates with momentum $\mathbf{p} = p\hat{z}$. According to the principle of least action, the massive vector field must satisfy the Proca equation:

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{3.1}$$

where $m$ parametrizes the mass of the field and $F^{\mu\nu}$ is the field tensor. As with the Dirac equation, we search for free plane wave solutions of the form $A^\mu = Ae^{-ip\cdot x/\hbar}\epsilon^{\mu(s)}$ with a polarization vector with 3-spin components $(s)$ $\epsilon^{\mu(s)}$ specifying direction, and $A$ is a normalization constant. Since $F^{\mu\nu}$ is constructed out of the derivatives of the fields $A^\mu$ (see definition) and is antisymmetric, then any contraction of two indices yields $\partial_\mu\partial_\nu F^{\mu\nu} = 0$ or that

$$\partial_\mu A^\mu = 0, \tag{3.2}$$

which is the choice of Gauge condition on the field. The expansion of the field implies that $p_\mu \epsilon^\mu = 0$ for any polarization vector, where the spatial components of polarization must obey normalization and mutual orthogonality

$$\epsilon_i^{(s)}\epsilon_i^{(r)*} = -\delta^{rs} \tag{3.3}$$

(the minus sign arises from the choice of metric signature, which here is $(+ - - -)$) where $i$ is a spatial index (no contraction). We wish to construct the completeness relation for these polarization vectors, which is different from the photon case because we must now consider a massive field. As specified, we choose the momentum four-vector of the field to lie along the z-axis direction:

$$p^\mu = (E/c,\ 0,\ 0, p). \tag{3.3}$$

It is easy to see that the first two vectors can be constant and lie anywhere in the $xy$ plane, as their $z$-component yields spatial orthogonality, and their temporal component of zero yields timelike orthogonality:

$$\epsilon\mu(1) = (0,\ 1,\ 0,\ 0) \tag{3.4}$$

$$\epsilon^{\mu(2)} = (0,\ 0,\ 1,\ 0). \tag{3.5}$$

The challenge is now constructing a third polarization vector orthogonal to the other two polarizations (3.4, 3.5) and $p^\mu$. Since the only nonzero components of $p^\mu$ are $p^0$ and $p^3$, then we can guess the same structure of the third polarization vector $\epsilon_\mu^{(3)}$ and force it to obey normalization and orthogonality:

$$\epsilon\mu(3) = (a,\ 0,\ 0,\ b) \tag{3.6}$$

from which $p^\mu \epsilon_\mu^{(3)} = 0$ gives

$$p^\mu \epsilon_\mu^{(3)} = \frac{aE}{c} - bp \tag{3.7}$$

$$\implies b = \frac{aE}{pc} \tag{3.8}$$

hence $\epsilon^{\mu(3)} = (a,\ 0,\ 0, aE/pc)$. We can then plug this into our normalization expression to obtain a condition on $a$:

$$\epsilon^{\mu(3)}\epsilon_\mu^{(3)} = a^2 - \frac{a^2E^2}{p^2c^2} \tag{3.9}$$

$$
\begin{aligned}
1 &= a^2\left(\frac{m^2c^4+p^2c^2}{p^2c^2}-1\right) && (3.10)\\
&= a^2\frac{m^2c^2}{p^2} && (3.11)\\
\implies a &= \frac{p}{mc}.
\end{aligned}
$$

Therefore $b = \dfrac{E}{mc^2}$ as well, so our third polarization vector is

$$
\epsilon^{\mu(3)} = \left(\frac{p}{mc},\, 0,\, 0,\, \frac{E}{mc^2}\right). \tag{3.12}
$$

We now proceed by constructing the completeness relation, which, simiarly to spinor completeness, is assumed to be of the form

$$
\sum_{s=1,2,3}\epsilon_\mu^{(s)}\epsilon_\nu^{(s)} = \epsilon_\mu^{(1)}\epsilon_\nu^{(1)} + \epsilon_\mu^{(2)}\epsilon_\nu^{(2)} + \epsilon_\mu^{(3)}\epsilon_\nu^{(3)} \tag{3.13}
$$

where the spatial indices are now lowered and each polarization is summed over. It is convenient to evaluate each term individually, as each vector was constructed independently. First observe that $\epsilon_\mu^{(1)}\epsilon_\nu^{(1)}$ is only non-zero if $\mu=\nu=1$ and yields a factor of $(-1)^2$. Similarly with $\epsilon_\mu^{(2)}\epsilon_\nu^{(2)}$, is only non-zero if $\mu=\nu=2$ (and gives 1 as well).
The third term is a bit more complicated, as we also must consider cross-terms. Thus

$$
\begin{aligned}
\epsilon_0^{(3)}\epsilon_0^{(3)} &= \frac{p^2}{m^2c^2} && (3.14)\\
&= \frac{p^2c^2+m^2c^4}{m^2c^4}-1 && (3.15)\\
&= -1+\frac{E^2}{m^2c^4} && (3.16)\\
\epsilon_3^{(3)}\epsilon_3^{(3)} &= \frac{E^2}{m^2c^4} && (3.17)\\
&= 1-\frac{p^2}{m^2c^2} && (3.18)\\
\epsilon_3^{(3)}\epsilon_0^{(3)} = \epsilon_0^{(3)}\epsilon_3^{(3)} &= -\frac{pE}{m^2c^2} && (3.19)\\
&= -\frac{1}{m^2c^2}\frac{pE}{c}. && (3.20)
\end{aligned}
$$

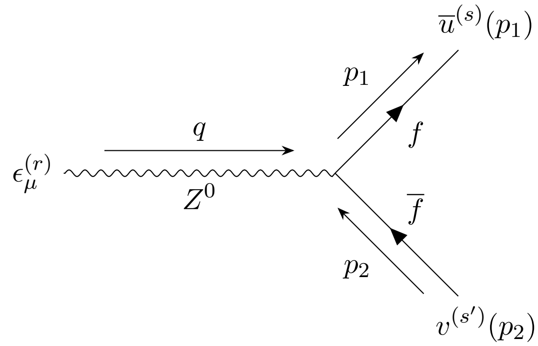
It is now easy to observe that when four-indices are the same, we obtain factors of $1$ and $-1$ corresponding to $-g_{\mu\nu}$. We also always subtract the term $\dfrac{p_\mu p_\nu}{m^2c^2}$, which can be seen from (3.14, 3.18, 3.20). Since the choice of axis of $\mathbf{p}$ loses no generality, this relation can generalized as

$$
\sum_{s=1,2,3}\epsilon_\mu^{(s)}\epsilon_\nu^{(s)} = -g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2c^2} \tag{3.21}
$$

which is the completeness relation for the massive vector field polarization, following the Lorentz gauge.

**Problem 4**

(**a**) We desire to calculate the decay rate for the $Z^0$ boson into each possible fermion, antifermion pair. We therefore aim to evaluate the Feynman amplitude for the decay diagram




where $f$ is either a lepton, lepton neutrino, or quark (excluding the top, antitop quark). The incoming boson has a polarization vector $\epsilon_\mu^{(r)}$, which will be contracted with the vertex factor $\frac{-ig_z}{2}\gamma^\mu(c_f^v - c_f^a\gamma^5)$, where $c_f^v$, $c_f^a$ are vector and pseudovector coupling constants determined by the electroweak interaction for each fermion with the electromagnetic current $j_\mu^{\text{em}}$ and the z-component of the weak curren $j_\mu^3$. Luckily, these were calculated in problem (2) for quarks and are listed in Table 9.1 in Griffiths:

| particle | $c_f^v$ | $c_f^a$ |
|---|---|---|
| $\nu_\ell$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\ell$ | $-\frac{1}{2} + 2\sin^2\theta_w$ | $-\frac{1}{2}$ |
| $u, c, t$ | $\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$ | $\frac{1}{2}$ |
| $d, s, b$ | $-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w$ | $-\frac{1}{2}$ |

Therefore, we can just calculate the general amplitude for the process and then sum over every contributing coupling constant given above. From hereon, for no loss of generality, I will refer to $c_f^v \equiv v$ and $c_f^a \equiv a$. Working backwards along the fermion line, we have

$$i\mathcal{A}_{fi} = \overline{u}^{(s)}(p_1)\frac{-ig_z}{2}\gamma^\mu(v - a\gamma^5)v^{(s')}(p_2)\epsilon_\mu^{(r)} \tag{4.1}$$

where we contract the boson polarization vector with the vertex factor gamma matrix. At first order, this is therefore the only contrtibuting factor to the amplitude. Squaring, we find

$$|\mathcal{A}_{fi}|^2 = \frac{g_z^2}{4}[\overline{u}^{(s)}(p_1)\gamma^\mu(v - a\gamma^5)v^{(s')}(p_2)\epsilon_\mu^{(r)}] \times [\epsilon_\nu^{(r)*}\overline{v}^{(s')}(p_2)(v + a\gamma^5)\gamma^\nu u^{(s)}(p_1)] \tag{4.2}$$

which follows from the fact that $(v + a\gamma^5)^\dagger = v + a\gamma^5$ ($(\gamma^5)^\dagger = \gamma^5$ and the $v, a$ are real), and the fact that $\beta \equiv \gamma^0$ anticommutes with $\gamma^5$. We can invoke Casimir's trick and take the trace to sum over all initial and final spins (of both the polarization vector and the spinors) following the completeness relations

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} - mc \tag{4.3}$$

$$\sum_{r=1,2,3} \epsilon_\mu^{(r)} \epsilon_\nu^{(r)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2 c^2} \tag{4.4}$$

since the trace is invariant under cyclic permutations of the factors in the argument. We then have from (4.1)

$$\frac{1}{2} \sum_{s,s'=1,2} \sum_{r=1,2,3} |\mathcal{A}_{fi}|^2 = \frac{g_z^2}{8} \sum_{s,s'=1,2} \text{tr}(u^{(s)}(p_1)\overline{u}^{(s)}(p_1)\gamma^\mu(v - a\gamma^5)v^{(s')}(p_2)\overline{v}^{(s')}(p_2)(v + a\gamma^5)\gamma^\nu)$$
$$\times \sum_{r=1,2,3} \epsilon_\mu^{(r)} \epsilon_\nu^{(r)} \tag{4.5}$$
$$= \frac{g_z^2}{8} \text{tr}((\not{p}_1 - mc)\gamma^\mu(v - a\gamma^5)(\not{p}_2 + mc)(v + a\gamma^5)\gamma^\nu) \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2 c^2}\right] \tag{4.6}$$

from which we now must evaluate the trace of the spinors. Note that the polarization vectors were factored out of the spinor trace because the trace of spinors is just a number and therefore commutes with the contraction between the polarization vectors, which is also just a number. For brevity, let me denote the trace as $\mathcal{M}^{\mu\nu}$:

$$\mathcal{M}^{\mu\nu} = \text{tr}((\not{p}_1 - mc)(v + a\gamma^5)\gamma^\mu(\not{p}_2 + mc)(v + a\gamma^5)\gamma^\nu) \tag{4.7}$$
$$= \text{tr}([vp_{1\lambda}\gamma^\lambda + ap_{1\lambda}\gamma^\lambda\gamma^5 - mcv - mca\gamma^5]\gamma^\mu[vp_{2\sigma}\gamma^\sigma + ap_{2\sigma}\gamma^\sigma\gamma^5 + mcv + mca\gamma^5]\gamma^\nu) \tag{4.8}$$
$$= \text{tr}([vp_{1\lambda}\gamma^\lambda\gamma^\mu + ap_{1\lambda}\gamma^\lambda\gamma^5\gamma^\mu - mcv\gamma^\mu - mca\gamma^5\gamma^\mu]$$
$$\times [vp_{2\sigma}\gamma^\sigma\gamma^\nu + ap_{2\sigma}\gamma^\sigma\gamma^5\gamma^\nu + mcv\gamma^\nu + mca\gamma^5\gamma^\nu]) \tag{4.9}$$
$$= \text{tr}(v^2 p_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu + vap_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^5\gamma^\nu + (0+0)$$
$$+ vap_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^5\gamma^\mu\gamma^\sigma\gamma^\nu + vap_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^5\gamma^\mu\gamma^\sigma\gamma^5\gamma^\nu + (0+0)$$
$$+ (0+0) - m^2c^2v^2\gamma^\mu\gamma^\nu - m^2c^2av\gamma^\mu\gamma^5\gamma^\nu$$
$$+ (0+0) - m^2c^2av\gamma^5\gamma^\mu\gamma^\nu - m^2c^2a^2\gamma^5\gamma^\mu\gamma^5\gamma^\nu) \tag{4.10}$$
$$= \text{tr}(v^2 p_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu - vap_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^5 - vap_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^5$$
$$+ vap_{1\lambda}p_{2\sigma}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu - m^2c^2v^2\gamma^\mu\gamma^\nu + m^2c^2a^2\gamma^\mu\gamma^\nu) \tag{4.11}$$

where I have used the fact that the trace of an odd number of gamma matrices is zero in (4.10) and that $\text{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$ in (4.11). Since the trace is a linear operator on matrices, we can invoke the identities

$$\text{tr}(A\gamma^\mu\gamma^\nu) = 4Ag^{\mu\nu} \tag{4.12}$$
$$\text{tr}(A\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda) = 4A(g^{\mu\nu}g^{\sigma\lambda} - g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\sigma}) \tag{4.13}$$
$$\text{tr}(A\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^5) = 4Ai\epsilon^{\mu\nu\sigma\lambda} \tag{4.14}$$

to further simplify (4.11) into

$$\mathcal{M}^{\mu\nu} = 4v^2 p_{1\lambda}p_{2\sigma}(g^{\lambda\mu}g^{\sigma\nu} - g^{\lambda\sigma}g^{\mu\nu} + g^{\lambda\nu}g^{\mu\sigma}) - 8ivap_{1\lambda}p_{2\sigma}\epsilon^{\lambda\mu\sigma\nu}$$
$$+ 4vap_{1\lambda}p_{2\sigma}(g^{\lambda\mu}g^{\sigma\nu} - g^{\lambda\sigma}g^{\mu\nu} + g^{\lambda\nu}g^{\mu\sigma}) + 4m^2c^2(a^2 - v^2)g^{\mu\nu} \tag{4.15}$$
$$= 4v^2(p_1^\mu p_2^\nu - p_1 \cdot p_2 g^{\mu\nu} + p_1^\nu p_2^\mu) + 8ivap_{1\lambda}p_{2\sigma}\epsilon^{\lambda\sigma\mu\nu}$$
$$+ 4va(p_1^\mu p_2^\nu - p_1 \cdot p_2 g^{\mu\nu} + p_1^\nu p_2^\mu) + 4m^2c^2(a^2 - v^2)g^{\mu\nu} \tag{4.16}$$

Under contraction with the polarization completeness,

$$\langle|\mathcal{A}_{fi}|^2\rangle = \frac{g_z^2}{8}\mathcal{M}^{\mu\nu}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2 c^2}\right] \tag{4.17}$$

we obtain a contribution from the metric tensor and one from the momentum terms:

$$\begin{aligned}\mathcal{M}^{\mu\nu}g_{\mu\nu} &= 4v^2(p_1^\mu p_2^\nu g_{\mu\nu} - p_1\cdot p_2 g^{\mu\nu}g_{\mu\nu} + p_1^\nu p_2^\mu g_{\mu\nu}) + 8ivap_{1\lambda}p_{2\sigma}\epsilon^{\lambda\sigma\mu\nu}g_{\mu\nu} \\ &\quad + 4va(p_1^\mu p_2^\nu g_{\mu\nu} - p_1\cdot p_2 g^{\mu\nu}g_{\mu\nu} + p_1^\nu p_2^\mu g_{\mu\nu}) + 4m^2c^2(a^2-v^2)g^{\mu\nu}g_{\mu\nu} \qquad (4.18)\\ &= 4v^2(p_1\cdot p_2 - 4p_1\cdot p_2 + p_1\cdot p_2) + 8ivap_{1\lambda}p_{2\lambda}g_{\mu\nu}\epsilon^{\lambda\sigma\mu\nu} \\ &\quad + 4va(p_1\cdot p_2 - 4p_1\cdot p_2 + p_1\cdot p_2) + 16m^2c^2(a^2-v^2) \qquad (4.19)\\ &= -8v(v+a)p_1\cdot p_2 + 16m^2c^2(a^2-v^2) \qquad (4.20)\end{aligned}$$

where the complex term was evaluated to be zero, because $g_{\mu\nu}$ is symmetric and the Levi-Civita four-tensor is anti-symmetric. Similarly, for the momentum contraction term,

$$\begin{aligned}\mathcal{M}^{\mu\nu}q_\mu q_\nu &= 4v^2(p_1^\mu p_2^\nu q_\mu q_\nu - p_1\cdot p_2 g^{\mu\nu}q_\mu q_\nu + p_1^\nu p_2^\mu q_\mu q_\nu) + 8ivap_{1\lambda}p_{2\sigma}q_\mu q_\nu\epsilon^{\lambda\sigma\mu\nu} \\ &\quad + 4va(p_1^\mu p_2^\nu q_\mu q_\nu - p_1\cdot p_2 g^{\mu\nu}q_\mu q_\nu + p_1^\nu p_2^\mu q_\mu q_\nu) + 4m^2c^2(a^2-v^2)g^{\mu\nu}q_\mu q_\nu \qquad (4.21)\\ &= 4v(v+a)(2(p_1\cdot q)(p_2\cdot q) - (p_1\cdot p_2)q^2) + 8ivap_{1\lambda}p_{2\sigma}q_\mu q_\nu\epsilon^{\lambda\sigma\mu\nu} \\ &\quad + 4m^2c^2(a^2-v^2)q^2 \qquad (4.22)\end{aligned}$$

where the complex term may or may not be zero, so we'll observe what the decay rate evaluation yields. We have, from the Golden rule,

$$\Gamma = \frac{g_z^2}{64\pi^2\hbar M_Z}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2\left(-\mathcal{M}^{\mu\nu}g_{\mu\nu} + \frac{\mathcal{M}^{\mu\nu}q_\mu q_\nu}{M_z^2c^2}\right)\frac{\delta^{(4)}(q-p_1-p_2)}{4E_1E_2} \tag{4.23}$$

where the evaluation of the incident momentum is in the *rest frame* of the particle, therefore giving $q_\mu = (E_Z/c, 0, 0, 0)$ as the incident four-momentum. The only nonzero terms are $q_0$ (so the complex term in (4.22) also vanishes), and the delta function in (4.23) simplifies to $\delta^{(3)}(p_1+p_2)\delta(E_z - E_1 - E_2)$. We then have that

$$\begin{aligned}\Gamma &= \frac{g_z^2}{64\pi^2\hbar M_Z}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2\Bigg(8v(v+a)p_1\cdot p_2 - 16m_f^2c^2(a^2-v^2) \\ &\quad + \frac{1}{M_Z^2c^2}[4v(v+a)(2E_1E_2M_Z^2 - M_Z^2c^2p_1\cdot p_2) + 4m_f^2M_Z^2c^4(a^2-v^2)]\Bigg)\frac{\delta^{(4)}(p_1+p_2)}{4E_1E_2} \qquad (4.24)\\ &= \frac{g_z^2}{64\pi^2\hbar M_z}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2\left(4v(v+a)p_1\cdot p_2 + 20m_f^2c^2(a^2-v^2) + 8v(v+a)\frac{E_1E_2}{c^2}\right) \\ &\quad \times\frac{\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2)\delta(M_Zc^2 - E_1 - E_2)}{4E_1E_2} \qquad (4.25)\\ &= \frac{g_z^2}{64\pi^2\hbar M_z}\int d^3\mathbf{p}_1\left(4v(v+a)(E_1^2/c^2 + \mathbf{p}_1\cdot\mathbf{p}_1) + 20m_f^2c^2(a^2-v^2) + 8v(v+a)\frac{E_1^2}{c^2}\right)\frac{\delta(M_Zc^2-2E_1)}{4E_1^2} \\ &\qquad (4.26)\\ &= \frac{g_z^2}{64\pi^2\hbar M_z}\int d^3\mathbf{p}_1\left(20m_f^2c^2(a^2-v^2) + 16v(v+a)|\mathbf{p}_1|^2\right)\frac{\delta(M_Zc^2-2E_1)}{4E_1^2} \qquad (4.27)\end{aligned}$$

where the angular components of the integral (4.26) can be taken out (the integrand only depends on the momentum magnitude). Furthermore, since we may assume that fermion masses can be neglected ($m_f^2 \approx 0$) when compared to that of the $M_Z$, (4.26) may be drastically simplified to a form where the $dp_1$ integral can be taken out (for brevity I will no longer write bolded vectors):

$$\Gamma = \frac{g_z^2}{64\pi\hbar M_z} \int p_1^2 dp_1 \frac{4v(v+a)p_1^2}{p_1^2} \delta(M_Z c^2 - 2p_1 c^2) \tag{4.28}$$

$$= \frac{4g_z^2 v(v+a)}{64\pi\hbar M_z} \cdot \frac{M_Z^2}{4} \tag{4.29}$$

$$= \frac{g_z^2 v(v+a) M_Z}{64\pi\hbar} \tag{4.30}$$

(the integral is then constant). For any fermion then, the decay rate is as follows:

$$\Gamma_{\nu_\ell} = \frac{g_z^2 M_Z}{128\pi\hbar} \qquad \text{[Neutrinos]} \tag{4.31}$$

$$\Gamma_\ell = \left(32\sin^4\theta_w - 3\sin^2\theta_w + \frac{1}{2}\right)\frac{g_z^2 M_Z}{64\pi\hbar} \qquad \text{[Leptons]} \tag{4.32}$$

$$\Gamma_{u/c} = \left(\frac{16}{9}\sin^4\theta_w - 2\sin^2\theta_w + \frac{1}{2}\right)\frac{g_z^2 M_Z}{64\pi\hbar} \qquad \text{[Up, Charm Quarks]} \tag{4.33}$$

$$\Gamma_{d/s/b} = \left(\frac{4}{9}\sin^4\theta_w - \sin^2\theta_w + \frac{1}{2}\right)\frac{g_z^2 M_Z}{64\pi\hbar} \qquad \text{[Down, Strange, Bottom Quarks]} \tag{4.34}$$

where the decay rates are identical regardless of quark color or lepton flavour, assuming the color of the quarks are distinguishable in producing one of the color doublets (so I have set $S = 1$, but in the case that quark colors are indistinguishable, we must include an additional factor of $\frac{1}{3!}$). I have explicitly chosen to leave the weak mixing angle in variable form, for generalization, although we may take the value $\sin^2\theta_w \approx 0.2314$ (Griffiths).

**(b)** To calculate the total decay width $\Gamma_{\text{TOT}}$, we must now account for every possible combination of fermions which could be produced from the decay. That is, for each of the 2+3 quark flavours we are examining, we must consider the fact that there are 3 quark flavours of each which could arise in the decay (assuming they're initially indistinguishable, but summing over all possible color combinations makes the decay width symmetric under the exchange of two or more quark colors). Since $\Gamma_{\text{TOT}} = \sum_i \Gamma_i$, then

$$\Gamma_{\text{TOT}} = (3)\Gamma_{\nu_\ell} + 3\Gamma_\ell + (2)(3)\Gamma_{u/c} + (3)(3)\Gamma_{d/s/b} \tag{4.34}$$

$$= \left(\frac{3}{2} + 3\left[4\sin^4\theta_w - 3\sin^2\theta_w + \frac{1}{2}\right] + 6\left[\frac{16}{9}\sin^4\theta_w - 2\sin^2\theta_w + \frac{1}{2}\right] + 9\left[\frac{4}{9}\sin^4\theta_w - \sin^2\theta_w + \frac{1}{2}\right]\right)\frac{g_z^2 M_Z}{64\pi\hbar} \tag{4.35}$$

$$= \left(\frac{21}{2} - 30\sin^2\theta_w + \frac{80}{3}\sin^4\theta_w\right)\frac{g_z^2 M_Z}{64\pi\hbar} \tag{4.36}$$

When taking values $\sin^2\theta_w = 0.2314$, $\hbar = 6.582 \times 10^{-19}$ MeV, $M_Z = 91.2 \times 10^3$ MeV$/c^2$, we find that the total decay width is

$$\Gamma_{\text{TOT}} = 3.43597 \times 10^{24}\ \text{s}^{-1} \tag{4.37}$$

which implies that

$$\tau_{Z^0} \approx 2.91 \times 10^{-25}\,\mathrm{s} \tag{4.38}$$

which falls within the expected value of $2.64 \times 10^{-25}$ s.

(**c**) The individual branching ratios may also be determined via $\dfrac{\Gamma_i}{\Gamma_{\mathrm{TOT}}}$. Since the fraction terms $\dfrac{g_z^2 M_Z}{64\pi\hbar}$ cancel out, only the $\sin^2\theta_w$ terms contribute:

$$\frac{\Gamma_{\nu_\ell}}{\Gamma_{\mathrm{TOT}}} = \frac{\frac{1}{2}}{\left(\frac{21}{2} - 30\sin^2\theta_w + \frac{80}{3}\sin^4\theta_w\right)} \tag{4.39}$$

$$\frac{\Gamma_\ell}{\Gamma_{\mathrm{TOT}}} = \frac{\left(32\sin^4\theta_w - 3\sin^2\theta_w + \frac{1}{2}\right)}{\left(\frac{21}{2} - 30\sin^2\theta_w + \frac{80}{3}\sin^4\theta_w\right)} \tag{4.40}$$

$$\frac{\Gamma_{u/c}}{\Gamma_{\mathrm{TOT}}} = \frac{\left(\frac{16}{9}\sin^4\theta_w - 2\sin^2\theta_w + \frac{1}{2}\right)}{\left(\frac{21}{2} - 30\sin^2\theta_w + \frac{80}{3}\sin^4\theta_w\right)} \tag{4.41}$$

$$\frac{\Gamma_{d/s/b}}{\Gamma_{\mathrm{TOT}}} = \frac{\left(\frac{4}{9}\sin^4\theta_w - \sin^2\theta_w + \frac{1}{2}\right)}{\left(\frac{21}{2} - 30\sin^2\theta_w + \frac{80}{3}\sin^4\theta_w\right)} \tag{4.42}$$

which yields

$$\frac{\Gamma_{\nu_\ell}}{\Gamma_{\mathrm{TOT}}} = 0.100 \tag{4.43}$$

$$\frac{\Gamma_\ell}{\Gamma_{\mathrm{TOT}}} = 0.305 \tag{4.44}$$

$$\frac{\Gamma_{u/c}}{\Gamma_{\mathrm{TOT}}} = 0.027 \tag{4.45}$$

$$\frac{\Gamma_{d/s/b}}{\Gamma_{\mathrm{TOT}}} = 0.058 \tag{4.46}$$

which once again corresponds to each individual lepton/quark flavour and color. To evaluate the expressions, we just took $\sin^2\theta_w = 0.2314$.

(**d**) If there were to exist a fourth generation of quarks (assuming their $a$, $v$ couplings are identical as before, and that their masses are light enough to be approximated as zero), the total decay rate would then add the extra widths corresponding to each quark (still neglecting top quark mass):

$$\Gamma_{\mathrm{TOT}} = (3)\Gamma_{\nu_\ell} + 3\Gamma_\ell + (3)(3)\Gamma_{u/c/} + (4)(3)\Gamma_{d/s/b/.} \tag{4.47}$$

$$= \left(\frac{3}{2} + 3\left[4\sin^4\theta_w - 3\sin^2\theta_w + \frac{1}{2}\right] + 9\left[\frac{16}{9}\sin^4\theta_w - 2\sin^2\theta_w + \frac{1}{2}\right] + 12\left[\frac{4}{9}\sin^4\theta_w - \sin^2\theta_w + \frac{1}{2}\right]\right)\frac{g_z^2 M_Z}{64\pi\hbar} \tag{4.48}$$

$$= \left(\frac{27}{2} - 39\sin^2\theta_w + \frac{100}{3}\sin^4\theta_w\right)\frac{g_z^2 M_Z}{64\pi\hbar} \tag{4.49}$$

which implies that the total rate increases, or that the lifetime of the $Z^0$ would decrease (to about $1.99 \times 10^{-25}s$ when evaluated). Due to the specific measurement of the true $Z^0$ lifetime, we know that a fourth generation of quarks cannot exist, because our lifetime measurement for the $Z^0$ is greater than that of the lifetime if four quarks were to exist. Furthermore, each branching ratio decreases (having to account now for 12 more particles, 4 quarks/antiquarks each with 3 colors):

$$\frac{\Gamma_{\nu_\ell}}{\Gamma_{\text{TOT}}} = 0.069 \tag{4.50}$$

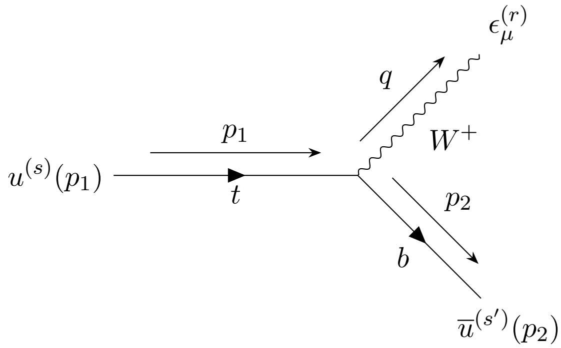
$$\frac{\Gamma_{\ell}}{\Gamma_{\text{TOT}}} = 0.209 \tag{4.51}$$

$$\frac{\Gamma_{u/c/}}{\Gamma_{\text{TOT}}} = 0.018 \tag{4.52}$$

$$\frac{\Gamma_{d/s/b/}}{\Gamma_{\text{TOT}}} = 0.040 \tag{4.53}$$

**Problem 5**

Assume that the top quark $t$ always decays into a $W$ boson and a bottom quark:




where we assume that the mass of the bottom quark is negligible compared to that of the top and $W^+$ boson. The arrangement of the amplitude is similar to that in problem 4, however the vertex factor must also contribute a flavour-changing CKM matrix element $V_{tb}$. Working backwards along the fermion line, we have that

$$i\mathcal{A}_{fi} = \overline{u}^{(s')}(p_2) \cdot \frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)V_{tb} \cdot u^{(s)}(p_1) \cdot \epsilon_\mu^{(r)*} \tag{5.1}$$

where the outgoing $W^+$ polarization vector is contracted with the gamma matrix at the vertex. We now work through the process of squaring the amplitude and averaging over the initial and final spins:

$$|\mathcal{A}_{fi}|^2 = \frac{g_w^2 V_{tb}^2}{8}[\overline{u}^{(s')}(p_2)\gamma^\mu(1-\gamma^5)u^{(s)}(p_1)][\overline{u}^{(s')}(p_2)\gamma^\nu(1-\gamma^5)u^{(s)}(p_1)]^\dagger \epsilon_\mu^{(r)*}\epsilon_\nu^{(r)} \tag{5.2}$$

$$= \frac{g_w^2 V_{tb}^2}{8}[\overline{u}^{(s')}(p_2)\gamma^\mu(1-\gamma^5)u^{(s)}(p_1)][\overline{u}^{(s)}(p_1)(1+\gamma^5)\gamma^\nu u^{(s')}(p_2)]\epsilon_\mu^{(r)*}\epsilon_\nu^{(r)} \tag{5.3}$$

$$\implies \frac{1}{2}\sum_{s,s'=1,2}\sum_{r=1,2,3}|\mathcal{A}_{fi}|^2 = \frac{g_w^2 V_{tb}^2}{16}\sum_{s,s'=1,2}\text{tr}(u^{(s')}(p_2)\overline{u}^{(s')}(p_2)\gamma^\mu(1-\gamma^5)u^{(s)}(p_1)\overline{u}^{(s)}(p_1)(1+\gamma^5)\gamma^\nu)$$

$$\times \sum_{r=1}^{3}\epsilon_\mu^{(r)*}\epsilon_\nu^{(r)} \tag{5.4}$$

$$\langle|\mathcal{A}_{fi}|^2\rangle = \frac{g_w^2 V_{tb}^2}{16}\text{tr}\left([\not{p}_2 - m_t c]\gamma^\mu(1-\gamma^5)[\not{p}_1 - m_t c](1+\gamma^5)\gamma^\nu\right)\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2 c^2}\right] \tag{5.5}$$

$$\langle|\mathcal{A}_{fi}|^2\rangle = \frac{g_w^2 V_{tb}^2}{16}\mathcal{M}^{\mu\nu}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2 c^2}\right] \tag{5.6}$$

where (5.5) follows from the completeness relations for spinors and polarization vectors (problem 3), and I have defined

$$\mathcal{M}^{\mu\nu} = \text{tr}\left([\not{p}_2 - m_b t]\gamma^\mu(1-\gamma^5)[\not{p}_1 - m_t c](1+\gamma^5)\gamma^\nu\right) \tag{5.7}$$

so the trace can be evaluated independently to keep things neat. We have that

$$\mathcal{M}^{\mu\nu} = \text{tr}\left(p_{2\lambda}\gamma^\lambda\gamma^\mu(1-\gamma^5)(p_{1\sigma}\gamma^\sigma - m_t c)(1+\gamma^5)\gamma^\nu\right) \tag{5.8}$$

$$\begin{aligned} &= \text{tr}\Big(p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu - p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu - p_{2\lambda}m_t c\gamma^\lambda\gamma^\mu\gamma^\nu + p_{2\lambda}m_t c\gamma^\lambda\gamma^\mu\gamma^5\gamma^\nu \\ &\quad + p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^5\gamma^\nu - p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^5\gamma^\nu - p_{2\lambda}m_t c\gamma^\lambda\gamma^\mu\gamma^5\gamma^\nu + p_{2\lambda}m_t c\gamma^\lambda\gamma^\mu\gamma^5\gamma^5\gamma^\nu \\ &\quad - p_{1\sigma}m_b c\gamma^\mu\gamma^\sigma\gamma^\nu + p_{1\sigma}m_b c\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu + m_b m_t c^2\gamma^\mu\gamma^\nu - m_b m_t c^2\gamma^\mu\gamma^5\gamma^\nu \\ &\quad - p_{1\sigma}m_b c\gamma^\mu\gamma^\sigma\gamma^5\gamma^\nu + p_{1\sigma}m_b c\gamma^\mu\gamma^5\gamma^\sigma\gamma^5\gamma^\nu + m_b m_t c^2\gamma^\mu\gamma^5\gamma^\nu - m_b m_t c^2\gamma^\mu\gamma^5\gamma^5\gamma^\nu\Big) \end{aligned} \tag{5.9}$$

$$= \text{tr}\left(-p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu + p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^5 + p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^5 - p_{1\sigma}p_{2\lambda}\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu\right) \tag{5.10}$$

where we have used anticommutation to rearrange the gamma matrices, as well as the fact that the trace of an odd number of gamma matrices is 0 in (5.10). Notice that the top and bottom quark masses both cancel and/or vanish due to odd numbered-products of gamma matrices. Since the trace is a linear operator on the matrices, we can add the like terms and use equations (4.13), (4.14) to find

$$\mathcal{M}^{\mu\nu} = -8p_{1\sigma}p_{2\lambda}(g^{\lambda\sigma}g^{\mu\nu} - g^{\lambda\mu}g^{\sigma\nu} + g^{\lambda\nu}g^{\sigma\mu}) + 8ip_{1\sigma}p_{2\lambda}\epsilon^{\lambda\sigma\mu\nu} \tag{5.11}$$

$$= -8(p_1\cdot p_2 g^{\mu\nu} - p_1^\nu p_2^\mu + p_1^\mu p_2^\nu) + 8ip_{1\sigma}p_{2\lambda}\epsilon^{\lambda\sigma\mu\nu}. \tag{5.12}$$

This may now be evaluated under the contraction of the $W$ boson completeness relation,

$$\mathcal{M}^{\mu\nu}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2c^2}\right] = \left[-8(p_1\cdot p_2 g^{\mu\nu} - p_1^\nu p_2^\mu + p_1^\mu p_2^\nu) + 8ip_{1\sigma}p_{2\lambda}\epsilon^{\lambda\sigma\mu\nu}\right]\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2c^2}\right] \tag{5.13}$$

$$\begin{aligned} &= 8(4p_1\cdot p_2 + p_1\cdot p_2 - p_1\cdot p_2) - 8ip_{1\sigma}p_{2\lambda}g_{\mu\nu}\epsilon^{\lambda\sigma\mu\nu} \\ &\quad - \frac{8}{M_W^2c^2}((p_1\cdot p_2)q^2 - (p_2\cdot q)(p_1\cdot q) + (p_1\cdot q)(p_2\cdot q)) \\ &\quad + \frac{8i}{M_W^2c^2}p_{1\sigma}p_{2\lambda}q_\mu q_\nu\epsilon^{\lambda\sigma\mu\nu} \end{aligned} \tag{5.14}$$

$$= p_1\cdot p_2\left(32 - \frac{8q^2}{M_W^2c^2}\right) + \frac{8i}{M_W^2c^2}p_{1\sigma}p_{2\lambda}q_\mu q_\nu\epsilon^{\lambda\sigma\mu\nu} \tag{5.15}$$

$$= 24p_1\cdot p_2 + \frac{8i}{M_W^2c^2}p_{1\sigma}p_{2\lambda}q_\mu q_\nu\epsilon^{\lambda\sigma\mu\nu} \tag{5.16}$$

where the last few lines followed from the fact that the metric is symmetric while the Levi-Civita four-tensor is anti-symmetric (the product of a symmetric and antisymmetric tensor gives zero) and that $q^2 = M_W^2c^2$. Eventually, we should expect the complex term to resolve into a real number upon integration, since the decay rate must be real, so I will label it $8E(p_1, p_2, q)$ for now. The total squared amplitude is then

$$\langle|\mathcal{A}_{fi}|^2\rangle = \frac{g_w^2V_{tb}^2}{2}\left[3p_1\cdot p_2 + E(p_1,p_2,q)\right]. \tag{5.17}$$

We now proceed by evaluating the decay rate:

$$\Gamma = \frac{g_w^2V_{tb}^2}{16\pi^2\hbar m_t}\int d^3\mathbf{p}_2 d^3\mathbf{q}[3p_1\cdot p_2 + E(p_1,p_2,q)]\frac{\delta^{(4)}(p_1 - q - p_2)}{4E_2E_W} \tag{5.18}$$

where we choose the momentum of the top quark to be zero so that we examine the decay in the frame of the top quark. Seperating the energy and momentum delta functions,

$$\Gamma = \frac{g_w^2 V_{tb}^2}{64\pi^2 \hbar m_t} \int d^3\mathbf{p}_2 d^3\mathbf{q}[3p_1 \cdot p_2 + E(p_1, p_2, q)] \frac{\delta^{(3)}(\mathbf{q}+\mathbf{p}_2)\delta(m_t c^2 - E_W - E_2)}{E_2 E_W} \tag{5.19}$$

$$= \frac{g_w^2 V_{tb}^2}{64\pi^2 \hbar m_t} \int d^3\mathbf{p}_2 d^3\mathbf{q}[3m_t E_2 + E(p_1, p_2, q)] \frac{\delta^{(3)}(\mathbf{q}+\mathbf{p}_2)\delta(m_t c^2 - E_W - E_2)}{E_2 E_W} \tag{5.20}$$

$$= \frac{g_w^2 V_{tb}^2}{64\pi^2 \hbar m_t} \int d^3\mathbf{q}[-3m_t|\mathbf{q}|c + E(p_1, \mathbf{q}, \mathbf{q})] \frac{\delta(m_t c^2 - E_W + |\mathbf{q}|c)}{-E_W|\mathbf{q}|c}. \tag{5.21}$$

where I have invoked the approximation $m_b \approx 0$ in (5.21) and taken out the $\mathbf{p}_2$ integral. Before taking out the last integral, however, we must note how $E(p_1, q, q)$ reduces. Notice that, upon integration of $p_2$ in (5.20) and choosing $\mathbf{p}_1 = \mathbf{0}$ due to the tau rest frame (only nonzero components are spatial in $p_2, q$),

$$\begin{aligned} E(p_1, \mathbf{q}, \mathbf{q}) &= \frac{i}{M_W^2 c^2} p_{10} q_\lambda q_\mu q_\nu \epsilon^{0\lambda\mu\nu} \\ &= 0 \end{aligned} \tag{5.22}$$

since the contraction of permutations cancels the terms to zero:

$$E(p_1, \mathbf{q}, \mathbf{q}) = \frac{ip_{10}}{M_W^2 c^2} q_1 q_2 q_3 \left[\epsilon^{0123} + \epsilon^{0132} + \epsilon^{0312} + \epsilon^{0321} + \epsilon^{0231} + \epsilon^{0213}\right] \tag{5.23}$$

$$= \frac{ip_{10}}{M_W^2 c^2} q_1 q_2 q_3 \left[+1 - 1 + 1 - 1 + 1 - 1\right] \tag{5.24}$$

$$= \frac{ip_{10}}{M_W^2 c^2} q_1 q_2 q_3 [0] \tag{5.25}$$

Therefore

$$\Gamma = \frac{g_w^2 V_{tb}^2}{64\pi^2 \hbar m_t} \int d^3\mathbf{q} 3m_t|\mathbf{q}|c \frac{\delta(m_t c^2 - E_W + |\mathbf{q}|c)}{E_W|\mathbf{q}|c} \tag{5.26}$$

$$= \frac{3g_w^2 V_{tb}^2}{16\pi\hbar} \int q^2 dq\, \frac{\delta(m_t c^2 - E_W + qc)}{E_W} \tag{5.27}$$

(I won't bold spatial vectors from now on, since we are only radially integrating from here). We now must integrate the last delta function in $q$. Note that the $q$ integral sends $m_t c^2 - E_W + qc \to 0$, so we can solve for $q$ this way:

$$m_t c^2 - E_W + qc = 0 \tag{5.28}$$

$$\implies E_W^2 = M_W^2 c^4 + q^2 c^2 = (m_t c^2 + qc)^2 = m_t^2 c^4 + q^2 c^2 + 2m_t q c^3 \tag{5.29}$$

$$\implies (M_W^2 - m_t^2)c^4 = 2m_t c^3 q \tag{5.30}$$

$$\implies q = \frac{(M_W^2 - M_t^2)c}{2m_t} \tag{5.31}$$

which then implies that

$$E_w = \sqrt{M_W^2 c^4 + \frac{c^2}{4m_t^2}(M_W^2 - M_t^2)^2 c^4} \tag{5.32}$$

$$= \sqrt{\frac{(4M_W^2 m_t^2 + M_W^4 + m_t^4 - 2M_w^2 m_t^2)c^4}{4m_t^2}} \tag{5.33}$$

$$= \sqrt{\frac{(M_W^2 + m_t^2)^2 c^4}{4m_t^2}} \tag{5.34}$$

$$= \frac{(M_W^2 + m_t^2)c^2}{2m_t} \tag{5.35}$$

and therefore (5.27) integrates to

$$\Gamma = \frac{3g_w^2 V_{tb}^2}{16\pi\hbar} \cdot \frac{(M_W^2 - m_t^2)^2 c^2}{4m_t^2} \cdot \frac{2m_t}{(M_W^2 + m_t^2)c^2} \tag{5.36}$$

$$= \frac{3g_w^2 V_{tb}^2}{32\pi\hbar m_t} \frac{(M_W^2 - m_t^2)^2}{(M_W^2 + m_t^2)} \tag{5.37}$$

which is the decay width of the top quark. The lifetime is then $\frac{1}{\Gamma}$:

$$\tau_t = \frac{32\pi\hbar m_t}{3g_w^2 V_{tb}} \frac{(M_W^2 + m_t^2)}{(M_W^2 - m_t^2)^2} \tag{5.38}$$

which, if taking $M_W = 80420$ MeV/$c^2$, $m_t = 174000$ MeV/$c^2$ and $V_{tb} = 1.014$, we have that $\tau_t \approx 6.19 \times 10^{-25}$ s, which is close enough to the expected lifetime of $\sim 5 \times 10^{-25}$ s considering the approximations which we made.