PHY454 Problem Set 2

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Problem 1

Consider the stress tensor T whose components are defined as

$$T = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}.$$

Now, if we have a surface plane whose orientation is given by $x+2y+2z=6$, then the unnormalized normal vector to this plane is $(1,2,2)$. Thus, any plane perpendicular to this one is of the form $x+2y+2z=d$ for any $d \in \mathbb{R}$. Now, on a surface at the point where T is defined, the force per unit area (or stress vector) on this part of the surface is given by $\mathbf{f} = \mathbf{n} \cdot T$ where $\mathbf{n}$ is the unit normal. In this case, $\mathbf{n} = \frac{1}{3}(1,2,2)$. Then,

$$\begin{aligned}
\mathbf{f} &= n_j T_{ij}\, \mathbf{x}_i \\
&= \mathbf{x}_1(n_1T_{11} + n_2T_{21} + n_3T_{31}) + \mathbf{x}_2(n_1T_{12} + n_2T_{22} + n_3T_{32}) + \mathbf{x}_3(n_1T_{13} + n_2T_{23} + n_3T_{33}) \\
&= \mathbf{x}_1(2\cdot 1 + (-1)\cdot 2 + 3\cdot 2) + \mathbf{x}_2((-1)\cdot 1 + 4\cdot 2 + 0\cdot 2) + \mathbf{x}_3(3\cdot 1 + 0\cdot 2 + (-1)\cdot 2) \\
&= 6\mathbf{x}_1 + 5\mathbf{x}_2 + \mathbf{x}_3.
\end{aligned}$$

This is the stress vector at this point, and therefore to determine the magnitude of the normal stress on the surface, we must project the stress vector onto the normal vector:

$$\begin{aligned}
|\mathbf{f}_n| &= f_i n_i \\
&= \frac{1}{3}(6\cdot 1 + 5\cdot 2 + 1\cdot 2) \\
&= \frac{1}{3}18 \\
&= 6\,\text{MPa}.
\end{aligned}$$

Problem 2

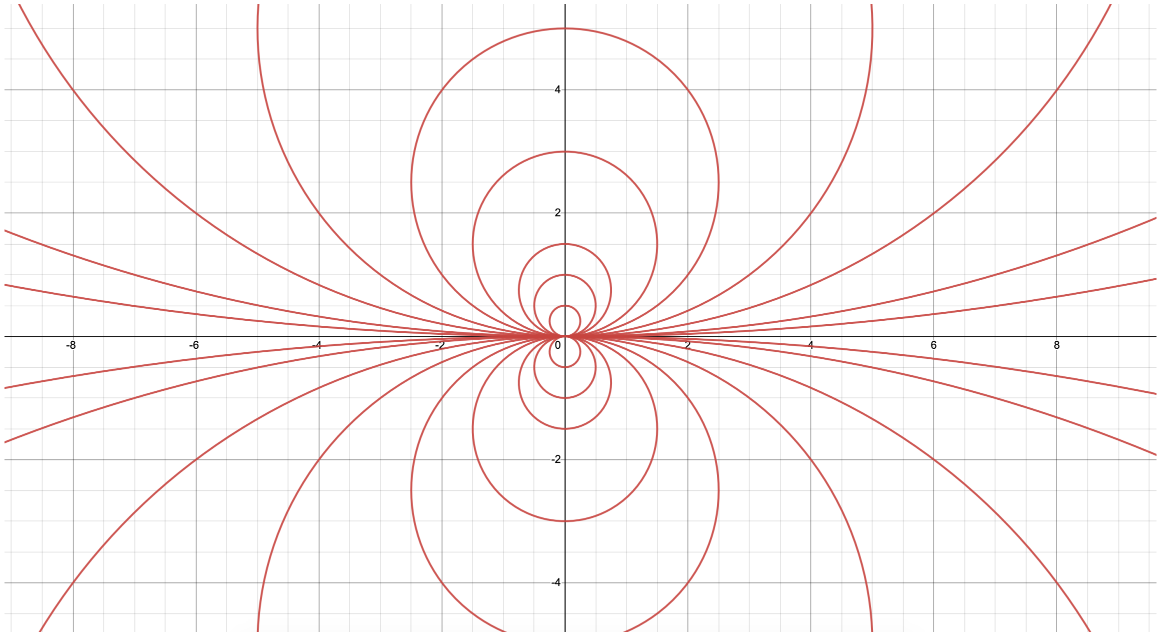
(a) To determine the simplest possible expression for the φ-component of the flow $\mathbf{u} = (u_r, u_\varphi)$, given $u_r = \dfrac{\Lambda}{r^2}\cos\varphi$, we must consider the fact that the two-dimensional flow is incompressible. Since $\mathbf{u}$ is incompressible, we must have that $\nabla \cdot \mathbf{u} = 0$. In polar coordinates, this yields that

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \frac{\partial u_i}{\partial x_i} = 0 \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \varphi} u_\varphi \\
\implies \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Lambda}{r}\cos\varphi\right) &= -\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} \\
\implies -\frac{\Lambda}{r^3}\cos\varphi &= -\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} \\
\implies \frac{\partial u_\varphi}{\partial \varphi} &= \frac{\Lambda}{r^2}\cos\varphi \\
\implies u_\varphi &= \frac{\Lambda}{r^2}\sin\varphi.
\end{aligned}$$

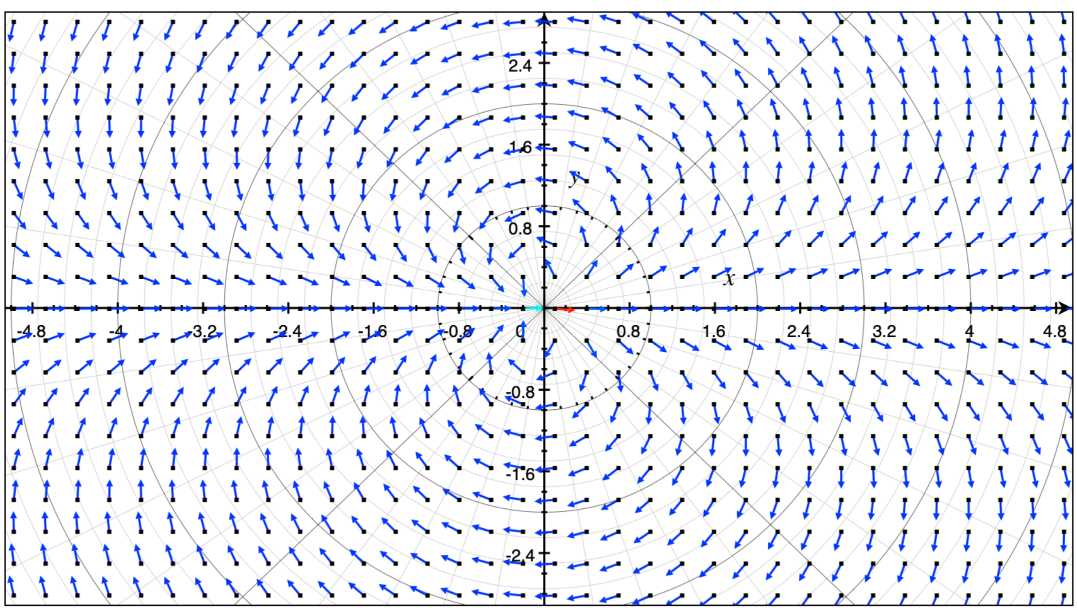
(b) In determining the form of the streamfunction, recall that $u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \varphi}$ and $u_\varphi = -\dfrac{\partial \psi}{\partial r}$ in polar coordinates. We may substitute our expressions from part (a) and integrate with respect to each variable to determine a common expression for ψ:

$$\begin{aligned}
u_r &= \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \\
&= \frac{\Lambda}{r^2}\cos\varphi \\
\implies \frac{\partial \psi}{\partial \varphi} &= \frac{\Lambda}{r}\cos\varphi \\
\implies \psi(r, \varphi) &= \frac{\Lambda}{r}\sin\varphi, \\
u_\varphi &= -\frac{\partial \psi}{\partial r} \\
&= \frac{\Lambda}{r^2}\sin\varphi \\
\implies \psi(r, \varphi) &= \frac{\Lambda}{r}\sin\varphi,
\end{aligned}$$

and thus the simplest expression for the streamfunction is $\psi(r, \varphi) = \dfrac{\Lambda}{r}\sin\varphi$. The streamlines of this flow are given by lines of constant ψ, and may be plotted in polar coordinates by rearranging $r(\varphi) = \dfrac{\Lambda}{C}\sin\varphi$ for constants Λ and C:



The direction of this flow is circular (counterclockwise for $x > 0$ and clockwise for $x < 0$... it was difficult to plot direction arrows) and therefore can be determined by examining the vector field generated by $\mathbf{u} = (u_r, u_\varphi)$. When $\varphi = 0$, the components of $\mathbf{u}$ are $(\Lambda/r^2, 0)$ which is always positive since $\Lambda > 0$. This points to the right. As φ increases to $\pi/4$, $\pi/2, \ldots$, the components of $\mathbf{u}$ are $(0, \Lambda/r^2)$ and $(-\Lambda/r^2, 0)$, respectively. Plotting this vector field, it is easier to visualize:



Problem 3

(a) For this part of the problem, we prove some lemmas regarding coordinate axis rotations for tensors:

(1) For an arbitrary nonzero rotation of the axes, we are given that $\mathbf{x}' = C^T \cdot \mathbf{x}$ and $\mathbf{x} = C \cdot \mathbf{x}'$. Thus, to transform a vector into the rotated frame and then transform it backto the unrotated initial frame, we must have that $\mathbf{x} = C \cdot \mathbf{x}' = C \cdot C^T \cdot \mathbf{x}$ which therein implies that $C \cdot C^T = I$. Similarly, $\mathbf{x}' = C^T \cdot \mathbf{x} = C^T \cdot C \cdot \mathbf{x}'$, which is only true if $C^T \cdot C = I$.

(2) Consider the scaled united tensor aI for a scalar a. Under a rotation of the axes, $(aI)' = C^T \cdot (aI) \cdot C = a(C^T \cdot I \cdot C) = a(C^T \cdot C) = aI$ from part (a), and therefore the scaled unitary tensor is invariant under any arbitrary axis rotation.

(3) Note that for any vector $\mathbf{u}$ with magnitude $|u|$, $|u|$ is a scalar and therefore the tensor $|u|I$ remains invariant under axis rotation, as proved in part (b). Therefore $|u| = |u'|$, for the rotated vector $\mathbf{u}'$ in the rotated frame.

(b) Now consider an arbitrary rotation by the angle θ around the $\mathbf{e}_3$ axis.

(1) This is equivalent to a rotation in the xy-plane, and thus we may determine the two-dimensional rotation tensor representation by examining the transformation of basis vectors. We require:

$$\begin{pmatrix}1\\0\end{pmatrix} M = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \qquad \text{and}$$
$$\begin{pmatrix}0\\1\end{pmatrix} M = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}.$$

This implies that M is the matrix $M = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ in the xy-plane. Therefore since the rotation is independent of $\mathbf{e}_3$, we have that $[C] = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$. Now, to determine the tensor matrix representation $[C]$ which transforms B into the diagonal form B', we must determine the eigenvectors of the matrix tensor representation $B = \begin{pmatrix}0 & B & 0\\ B & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$.

It is easy to determine the characteristic equation $\lambda(\lambda^2 - B^2) = 0$, and thus the eigenvalues are $0, \pm B$. By subtracting eigenvalues and determining the kernel, the corresponding eigenvectors can be determined (I really don't want to typeset this, after all this is first year linear algebra):

$$\begin{aligned}\lambda &= 0, & \mathbf{v}_0 &= (0,0,1)\\ \lambda &= B, & \mathbf{v}_B &= (1,1,0)\\ \lambda &= -B, & \mathbf{v}_{-B} &= (-1,1,0).\end{aligned}$$

All of these eigenvectors are orthogonal, we form the matrix $\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ in order of $\lambda = B, -B, 0$. Now, consider again the matrix tensor representation $[C]$ with rotation angle $\theta =$

$-\pi/4$:

$$[C(\pi/4)] = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

This matrix has almost an identical form to that of the diagonalization matrix above, and the diagonalization process can determine the diagonal components B'_{ii} of the transformed tensor.

(2) To determine the diagonal components of the transformed tensor B'_{ii}, we take the transpose of $[C(\pi/4)]$ and multiply the matrix through:

$$\begin{aligned} B' &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & B & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} B & B & 0 \\ B & -B & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2B & 0 & 0 \\ 0 & -2B & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

which implies that $B'_{11} = B$, $B'_{22} = -B$, and $B'_{33} = B'_{ij} = 0$, $i \neq j$. This could have been determine from the eigenvalues found in part (1) of this problem, however verification of this claim was deemed more sufficient.

Problem 4

For this problem, we consider the flow with particle trajectories given by the vector form $\mathbf{q} = (X_0 e^{\alpha(t-t_0)}, Y_0, Z_0)$. The flow has the density field $\rho(t) = \rho_0 e^{-\alpha(t-t_0)}$.

(a) To determine the Eulerian velocity, it suffices to determine the time derivative of the position vector previously specified, $\mathbf{u} = \dot{\mathbf{q}} = \frac{d}{dt}(X_0 e^{\alpha(t-t_0)}, Y_0, Z_0) = (X_0 \alpha e^{\alpha(t-t_0)}, 0, 0)$. This velocity may be written in component form as $\mathbf{u} = \alpha x(t)\, \mathbf{x}$, which is what I wanted to show.

(b) We now consider the total mass and time rate of change of mass (mass flux) throughout a horizontal cylinder aligned parallel to the x-axis with ends at $x = 0$, $x = 3$ of cross-sectional area A. The mass density field is given by $\rho = \rho_0 e^{-\alpha(t-t_0)} = \frac{m}{V}$, and for $V = 3A$, we have that the total mass inside the cylinder at a time t is $m(t) = 3A\rho_0 e^{-\alpha(t-t_0)}$.
Then, the rate of change of mass inside the cylinder is $\frac{dm}{dt} = -3A\alpha\rho_0 e^{-\alpha(t-t_0)} = -\alpha m(t)$, which is given by differentiation.

(c) For the cylinder as a control volume, we now consider the mass flux through the front ($x = 0$) side of the cylinder. This is given by the conservation relation

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S \rho \mathbf{u} \cdot \mathbf{n}\, dA,$$

where $\mathbf{n}$ is the unit normal pointing outward from the surface. The left hand side of this relation was determined in part (b), and the left hand side is given by integration over the two sides of the cylinder. First, note that the amount of mass entering the cylinder is equivalent to the amount of mass exiting the other end of the cylinder since $\rho(t)$ is spatially independent, and therefore the flux of mass through the front cylinder face is

$$\begin{aligned}\Phi &= A\rho(t)\frac{dx(t)}{dt} \\ &= A\rho_0 e^{-\alpha(t-t_0)} X_0 \alpha e^{\alpha(t-t_0)} \\ &= A\rho_0 X_0 \alpha\end{aligned}$$

which is constant, and only depending on A, X_0, ρ_0, and α, the decay constant.

(d) To show that a fluid element of mass M of a material volume (volume moving with the fluid) is time-independent at exactly t_0 when coincident with the control cyclinder indicated in the previous parts, we must take the material derivative of M:

$$\frac{DM}{dt} = \frac{\partial M}{\partial t} + \mathbf{u} \cdot \nabla M,$$

which at t_0 is $M = 3A\rho_0$, which is spatial and time independent. Then

$$\begin{aligned}\frac{DM}{dt} &= \frac{\partial M}{\partial t} + \mathbf{u} \cdot \nabla M, \\ &= \frac{d}{dt}(3A\rho_0) + \left.\frac{dx(t)}{dt}\right|_{t=t_0} \frac{d}{dx}(3A\rho_0) \\ &= 0 + \alpha X_0 \frac{d}{dx}(3A\rho_0)\end{aligned}$$

$$= 0,$$

and therefore the mass in a material volume element coincident with the control volume at $t = t_0$ does not change with time, as required.

Problem 5

Let u_0 be the velocity of the fluid exiting the bowl, and let u_1 be the velocity of the fluid at the top height z. To begin solving this problem, we can first consider the flux relationship between u_0 and u_1, which is given by $2\pi r_0 u_0 = 2\pi u r$. In the assumption that $\frac{r_0}{r} \ll 1$, then $u_1 \approx 0$ when compared to u_0.

In the Bernoulli relationship, then, we have that

$$
\begin{aligned}
\frac{1}{2}\rho u_0^2 + p_0 + p_{\text{air}} + \rho g z_0 &= \frac{1}{2}\rho \overbrace{\cancel{u_1^2}}^{\approx 0} + p_{\text{air}} + \rho g z \\
\implies \frac{1}{2}\rho u_0^2 &= \rho g z \\
\implies u_0 &= \sqrt{2gz},
\end{aligned}
$$

where z indicates the height of the fluid in the bowl. We must now consider the amount of volume which is inside the bowl for a certain height z, and this is simply done by integrating the volume of the bowl. In this problem I am assuming the bowl is three dimensional, and thus

$$
\begin{aligned}
V(z) &= \int_0^{2\pi} d\varphi \int_0^z \int_0^{\sqrt{R^2-z^2}} r\, dr dz \\
&= 2\pi \int_0^z \frac{1}{2}(R^2 - z^2)\, dz \\
&= \pi\left(zR^2 - \frac{1}{3}z^3\right).
\end{aligned}
$$

The flux relationship for this fluid is given by

$$
\frac{d}{dt}\int_V dV = -\int_S \mathbf{u}\cdot\mathbf{n}\, dA,
$$

which is re-written as $\frac{dV}{dt} = -uA_0$ for the hole at the base of the bowl of radius A_0. Since the height z is a function of t, then we can apply chain rule to $\frac{dV}{dt}$ to determine how V changes with respect to $\frac{dz}{dt}$, the speed of the fluid at the top of the bowl. By chain rule,

$$
\begin{aligned}
\frac{dV}{dt} &= \pi\left(\frac{dz}{dt}R^2 - z^2\frac{dz}{dt}\right) \\
&= \pi\left(R^2 - z^2\right)\frac{dz}{dt}.
\end{aligned}
$$

Equating this relationship with the above relation to $u_0 A_0$, an implicit expression for $z(t)$ is determined:

$$
-A_0\sqrt{2gz} = \pi(R^2 - z^2)\frac{dz}{dt} \implies \frac{dz}{dt} = \frac{A_0\sqrt{2g}}{\pi}\frac{\sqrt{z}}{z^2 - R^2},
$$

and this ODE may be solved via integration. Here, we can evaluate z between z_1 and z_2, and t between $t_0 = 0$ and t, hence determining the amount of time it would take for the fluid to drain from z_2 to z_1:

$$
dt = \frac{\pi}{A_0\sqrt{2g}}\frac{z^2 - R^2}{\sqrt{z}}\, dz
$$

$$
\begin{aligned}
\int_0^t dt' &= \frac{\pi}{A_0\sqrt{2g}} \int_{z_2}^{z_1} \frac{z^2 - R^2}{\sqrt{z}}\, dz \\
t &= \frac{\pi}{A_0\sqrt{2g}} \left[\frac{2}{5} z^{5/2} - 2R^2\sqrt{z} \right]_{z_2}^{z_1} \\
&= \frac{\pi}{A_0\sqrt{2g}} \left[\frac{2}{5} z_1^{5/2} - 2R^2\sqrt{z_1} - \frac{2}{5} z_2^{5/2} + 2R^2\sqrt{z_2} \right] \\
&= \frac{\pi}{A_0\sqrt{2g}} \left[\frac{2}{5} (z_1^{5/2} - z_2^{5/2}) + 2R^2(\sqrt{z_2} - \sqrt{z_1}) \right],
\end{aligned}
$$

which is the amount of time that the bowl will take to drain between heights z_1 and z_2. Since $R \geq z_2 > z_1 \geq 0$, it is easy to verify that this is always a positive quantity, since

$$
10R^2(\sqrt{z_2} - \sqrt{z_1}) \geq (z_2^{5/2} - z_1^{5/2}),
$$

as it should be.