MAT244 PS3

Q1) From (i) - The period is $T = \frac{2\pi}{\omega} = 2$, thus

$$\omega = \pi.$$

Solve the Eqn: $\ddot{y} + b\dot{y} + Ky = 0.$

$$\Rightarrow r^2 + br + K = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4K}}{2}$$

Underdamping $\Rightarrow 4K > b^2$, thus $r = -\frac{b}{2} \pm i \frac{\sqrt{4K - b^2}}{2}$

This implies the general solution is of the form

$$y = e^{-\frac{b}{2}t}\left[c_1 \cos \frac{\sqrt{4K-b^2}}{2} t + c_2 \sin \frac{\sqrt{4K-b^2}}{2} t\right]$$

The solution is periodic, so $y(t+2) = y(t)$ with period $T=2$.

- Find b:

$$\frac{e^{-\frac{b}{2}t - b}\left[\cancel{c_1 \cos \cdots + c_2 \sin \cdots}\right]}{\cancel{e^{-\frac{b}{2}t}}\left[\cancel{c_1 \cos \cdots + c_2 \sin \cdots}\right]} = \frac{1}{2}$$

$$\Rightarrow e^{-b} = \frac{1}{2} \Rightarrow \boxed{b = -\log\left(\tfrac{1}{2}\right)}$$

- Find K: If $\omega = \frac{\sqrt{4K - b^2}}{2} = \pi$, then $4K - b^2 = 4\pi^2$

Which implies $K = \pi^2 + \frac{b^2}{4} = \pi^2 + \frac{\log\left(\frac{1}{2}\right)^2}{4}$

So $\boxed{K = \pi^2 + \frac{1}{4}\left[\log\left(\tfrac{1}{2}\right)\right]^2}$

- Numerically,

$$\boxed{\begin{array}{l} b \approx 0.6931\ldots \\ K \approx 9.9897\ldots \end{array}}$$

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Q2) a) $\ddot{y} + b\dot{y} + ky = -G$

→ We can find the final resting height for any initial condition if we find the particular solution y_p to this equation.

If $y_p = \text{Constant}$, then $\ddot{y}_p = \dot{y}_p = 0$. Hence $y_p = -\frac{G}{k}$.

Thus for any initial condition,

$$\boxed{\lim_{t\to\infty} y(t) = -\frac{G}{k},}$$

which is the mass's final resting height.

b) From (a), we know $y_p = -\frac{G}{k}$.

Solve $\ddot{y} + b\dot{y} + ky = 0$ (y_h)

$\Longrightarrow$ $r^2 + br + k = 0$ $\Longrightarrow$ $r = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4k}}{2}$ $(b=0)$

$r = \pm i\sqrt{k}$. Then $y_1 = e^{i\sqrt{k}t}$, $y_2 = e^{-i\sqrt{k}t}$.

Our general solution is then

$$y = c_1 \cos\sqrt{k}t + c_2 \sin\sqrt{k}t - \frac{G}{k}.$$

For $\lim_{t\to\infty} y$ to exist, $c_1 = c_2 = 0$ because the individual limits $\lim_{x\to\infty} \cos x$, $\lim_{x\to\infty} \sin x$ DNE.

$$\boxed{\text{Thus } c_1 = c_2 = 0 \text{ and } \lim_{t\to\infty} y(t) = -\frac{G}{k}.}$$

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3) a) If E conserved, E constant. Then $\dot{E}=0$.

$$\dot{E} = \dot{E}_k + \dot{E}_p = \dot{y}(t)\cdot\ddot{y}(t) + V'(y(t))\cdot\dot{y}(t).$$

By the equation of motion, $V'(y(t)) = -\ddot{y}(t)$,

which implies

$$\dot{E} = \ddot{y}(t)\dot{y}(t) + (-\ddot{y}(t))\dot{y}(t) = 0$$

$\boxed{\text{So } \dot{E}=0,}$ thus E is conserved.

b) If E is decreasing with respect to time, then $\dot{E}<0$.

(by Derivative).

From (a), we know $\dot{E} = \ddot{y}(t)\dot{y}(t) + V'(y(t))\dot{y}(t)$.

If $V'(y) = -\ddot{y}(t) - b\dot{y}(t)$, then

$$\dot{E} = \ddot{y}(t)\dot{y}(t) - (\ddot{y}(t) + b\dot{y}(t))\dot{y}(t)$$
$$= \ddot{y}(t)\dot{y}(t) - \ddot{y}(t)\dot{y}(t) - b[\dot{y}(t)]^2$$
$$= -b[\dot{y}(t)]^2.$$

Since $b>0$, then $\boxed{\dot{E} = -b[\dot{y}(t)]^2 < 0,}$ so

<u>E is therefore a decreasing function with respect to time.</u>

c) If E constant, $\dot{E}=0$. This implies

$$\dot{E}=0=-b\,[\dot{y}(t)]^2 .$$

The only way for $\dot{E}=0$ is for $b=0$ or $\dot{y}(t)=0$. However $b>0$, so it must be that $\dot{y}(t)=0$.

This directly implies that $y(t)=$ constant, so

$\boxed{y(t)=y_0 .}$

Furthermore if $y(t)=y_0$, then $\dot{y}(t)=0$ and $\ddot{y}(t)=0$.

In the equation of motion,

$$\ddot{y}(t)=0=-V'(y(t))-b\,\dot{y}(t)=-V'(y_0)-b(0)$$

$$=-V'(y_0),$$

$\boxed{\text{So } V'(y_0)=0 \text{ and } y_0 \text{ is a critical point of } V.}$

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Q4) If $y(t) = u(t)\,y_1(t)$ solves

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t),$$

then

$$y'(t) = u'(t)\,y_1(t) + u(t)\,y_1'(t) \quad \text{and}$$

$$y''(t) = u''(t)\,y_1(t) + 2u'(t)\,y_1'(t) + u(t)\,y_1''(t),$$

where $y_1(t)$ solves the homogeneous equation.

We have:

$$u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t) + p(t)\,u'(t)y_1(t)$$
$$+ p(t)\,u(t)\,y_1'(t) + q(t)u(t)y_1(t) = g(t).$$

This is equal to

$$u''(t)y_1(t) + u'(t)\left[2y_1'(t) + p(t)\,y_1(t)\right] +$$
$$u(t)\underbrace{\cancel{\left[y_1''(t) + p(t)\,y_1'(t) + q(t)\,y_1(t)\right]}}_{0 \text{ Since } y_1(t) \text{ solves homogeneous ODE.}} = g(t)$$

Thus $u(t)$ must satisfy

$$\boxed{u''(t)\,y_1(t) + u'(t)\left[2y_1'(t) + p(t)\,y_1(t)\right] = g(t),}$$

as desired.

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Q5) a) $y_1 = \frac{1}{t}$, then $y_1''(t) + \frac{7}{t} y_1'(t) + \frac{5}{t^2} y_1(t) = 0.$

Then $y_1'(t) = -\frac{1}{t^2}$ and $y_1''(t) = \frac{2}{t^3}$.

$$\Rightarrow \frac{2}{t^3} + \frac{-7}{t^3} + \frac{5}{t^3} = \frac{2+5-7}{t^3} = \frac{0}{t^3} \boxed{=0},$$

thus y_1 solves the homogeneous ODE.

b) $y_1(t) = \frac{1}{t}$, $g(t) = \frac{1}{t}$, $p(t) = \frac{7}{t}$.

Solve for $u(t)$:

$$u''(t)\, y_1(t) + u'(t)\left[2y_1'(t) + p(t)\, y_1(t)\right] = g(t).$$

Let $u'(t) = s$. Then

$$\frac{s'}{t} + s\left[\frac{-2}{t^2} + \frac{7}{t^2}\right] = \frac{s'}{t} + s\left(\frac{5}{t^2}\right) = \frac{1}{t}.$$

$$\Rightarrow s' + 5s\frac{1}{t} = 1.$$

$$I(t) = \exp\left(\int \frac{5}{t}\, dt\right) = \exp(5\log|t|)$$
$$= t^5.$$

$$\Rightarrow t^5 s' + 5t^4 s = t^5$$

$$= \frac{d}{dt}\left[t^5 s\right] = t^5 \Rightarrow t^5 s = \frac{1}{6}t^6 + C_1$$

Thus $s = \frac{t}{6} + \frac{C_1}{t^5}$. Since $s = u'(t)$ then

$$u(t) = \int \frac{t}{6} + \frac{C_1}{t^5}\, dt = \frac{t^2}{12} - \frac{C_1}{4t^4} + C_2,$$

so $\boxed{u(t) = \frac{t^2}{12} + C_2 - \frac{C_1}{4t^4}}$.

c) Since $y(t) = u(t)\, y_1(t)$,

$$y(t) = \left(\frac{t^2}{12} + c_2 - \frac{c_1}{4t^4}\right) \cdot \frac{1}{t}$$

$$= \frac{t}{12} + \frac{c_2}{t} - \frac{c_1}{4t^5}.$$

Define new constants α, β to replace $c_2, -\frac{c_1}{4}$ respectively. Then

$$\boxed{y(t) = \frac{\alpha}{t} + \frac{\beta}{t^5} + \frac{t}{12}}$$

d) This implies:

$$y_1 = \frac{1}{t} \qquad y_p = \frac{t}{12}.$$

$$y_2 = \frac{1}{t^5}$$

To check y_2 solves the homogeneous ODE,

$$y_2' = -\frac{5}{t^6}, \quad y_2'' = \frac{30}{t^7}.$$

Then

$$\frac{30}{t^7} - \frac{(7)(5)}{t^7} + \frac{5}{t^7} = \frac{30 - 35 + 5}{t^7} = \frac{0}{t^7} = 0$$

Thus $y_2(t) = \frac{1}{t^5}$ also solves the homogeneous equation.

· Checking the Wronskian:

$$W[y_1, y_2] = \det \begin{pmatrix} \frac{1}{t} & \frac{1}{t^5} \\ -\frac{1}{t^2} & -\frac{5}{t^6} \end{pmatrix}$$

$$= -\frac{5}{t^7} + \frac{1}{t^7} = \frac{4}{t^7} \boxed{\neq 0},$$

Hence $y_1(t) = \frac{1}{t}$ and $y_2(t) = \frac{1}{t^5}$

are a set of fundamental solutions to the ODE.

· Check the particular solution $y_p(t) = \frac{t}{12}$:

$y_p'(t) = \frac{1}{12}$, $y_p''(t) = 0$.

Then

$$0 + \frac{7}{12t} + \frac{5t}{12t^2} = \frac{7}{12t} + \frac{5}{12t} = \frac{12}{12t} \boxed{= \frac{1}{t}} \checkmark.$$

Therefore our general solution is

$$\boxed{y(t) = \frac{c_1}{t} + \frac{c_2}{t^5} + \frac{t}{12}}$$