Homework 4

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87.7% (57/65)

Q1

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MAT334 Problem Set 4 — Due November 28, 23:00

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1.

I wish to evaluate the integral $\int_0^{2\pi} \log|1+ae^{i\theta}|\, d\theta$, and will proceed by invoking the residue theorem. To begin, I will first determine the power series expansion of $\log(1+x)$, for $|x|<1$. Then I will establish a change of variables to the integrand and evaluate accordingly.

First note that $\int_0^x \frac{dt}{1+t} = \log|1+t|$. For $|t|<1$, this is just the integral of a geometric series:

$$\int_0^x \frac{dt}{1+t} = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^k \, dt = \int_0^x \sum_{k=1}^{\infty} (-1)^{k-1} t^{k-1} \, dt = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} = \log|1+x|.$$

Once again note that this only holds for $|x|<1$. Here, $x \to ae^{i\theta}$, and since $|a|<1$, then $|ae^{i\theta}|<1$. We can therefore apply the power series expansion to our integrand.

Next I will introduce a change of variables $z = e^{i\theta}$, so $d\theta = \frac{dz}{zi}$. Then, when integrating from bound $\theta = 0$ to $\theta = 2\pi$, we are just integrating all values of z around the unit circle in the complex plane. Our integral then is

$$\frac{1}{i}\int_{|z|=1} \frac{\log|1+az|}{z}\, dz.$$

However, due to the singularity at $z=0$, the residue theorem yields that

$$\frac{1}{i}\int_{|z|=1} \frac{\log|1+az|}{z}\, dz = \frac{2\pi i}{i} \mathrm{Res}\left(\frac{\log|1+az|}{z}, 0\right).$$

Since we have already obtained the power series expansion for $\log|1+x|$, then for $x \to az$ has magnitude less than one. Therefore

$$\frac{\log|1+az|}{z} = \sum_{k=1}^{\infty} (-1)^{k-1} a^k \frac{z^{k-1}}{k} = \sum_{k=0}^{\infty} \frac{(-1^k) a^{k+1}}{k+1} z^k$$

The residue of this function is determined by the coefficient b_{-1}, which does not exist in this power series expansion (it is zero). Therefore $\mathrm{Res}\left(\frac{\log|1+az|}{z}, 0\right) = 0$, and hence

$$\int_0^2 \pi \log|1+ae^{i\theta}|\, d\theta = \frac{1}{i}\int_{|z|=1} \frac{\log|1+az|}{z}\, dz = 2\pi \mathrm{Res}\left(\frac{\log|1+az|}{z}, 0\right) = 0,$$

which is the value of the integral.

Q2a

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did not make the connection that the integrand here is only the real part of $f(z) = \log(1 - az)$ along the unit circle. You have to point this out and say that because the integral of f around the unit circle vanishes and our original integral is the real part of that integral we know that the original integral must also be zero. It is important to make that the distinction that you are applying Cauchy's integral theorem to f specifically and not to $\log|1 - ae^{i\theta}|$ because the latter is not an analytic function, which is a requirement for Cauchy's integral theorem (remember, an analytic function that is purely real or purely imaginary must be constant by the Cauchy-Riemann equations). The same problem arises if you solved this with the Residue Theorem. You can get around this if you use the Mean Value Theorem since it was stated that it also holds for the real parts of analytic functions but you would have to explicitly say that you are doing this & that you understand the precise result you are using. Always remember to actually verify that the hypotheses of the theorems you use are satisfied **-5**

2.

For the following problems, allow me to first introduce some conventions. Define the function $h_\varepsilon(z) = f(z) + \varepsilon g(z)$. Since f and g are both analytic on the open ball $B_2(0)$, then f and g are both an analytic on the closed disc of radius one $\overline{B_1(0)}$ and so is $h_\varepsilon(z)$. By the extreme value theorem, f and g must both attain maximum and minimum values on $\overline{B_1(0)}$, however by the maximum modulus principle, the extrema of f and g lie on the boundary of the ball $\partial B_1(0)$. Allow me to denote the maximum value of g on this boundary by M, so $|g(z)| \leq M$ $\forall z \in \partial B_1(0)$. Likewise, the minimum value of f by m so $m \leq |f(z)|$, $\forall z \in \partial B_1(0)$.

(a) *proof.* I will begin by applying Rouché's theorem. Since f and h_ε are both analytic on an open set $B_2(0)$ containing $\partial B_1(0)$, then if $|f(z) - h_\varepsilon(z)| < |f(z)|$, then f and h_ε have an equal number of zeroes inside $B_1(0)$. Since f has a zero at $z = 0$ with multiplicity one, then if the inequality holds, then h_ε has a unique zero in $\{z : |z| < 1\}$.
We have that

$$
\begin{aligned}
|f(z) - h_\varepsilon(z)| &= |f(z) - f(z) - \varepsilon g(z)| \\
&= \varepsilon |g(z)| < |f(z)|.
\end{aligned}
$$

The above implies that this is true for $\varepsilon < \dfrac{|f(z)|}{|g(z)|}$, which is minimized by the minimum value of f and maximum value of g on $\partial B_1(0)$ (provided they need not be the same values of z, only their maximum and minimum values contribute). That is, $\varepsilon < \dfrac{m}{M} \equiv \varepsilon_0$. Therefore there exists an $\varepsilon_0 = \dfrac{m}{M}$ such that for every $0 \leq \varepsilon < \varepsilon_0$, $h_\varepsilon(z)$ has a unique zero z_ε inside the ball $\overline{B_1(0)}$, which is what I wanted to prove. qed.

Correct 10

Q2b

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(b) Here, I wish to prove that $\forall\alpha > 0$, $\exists\delta > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $|\varepsilon - \varepsilon'| < \delta \implies |z_\varepsilon - z_{\varepsilon'}| < \alpha$. That is, the function $\varepsilon \to z_\varepsilon$ is continuous. First, allow me to show some of my rough work.

Rough work. First consider Rouché's theorem applied to the functions $h_\varepsilon(z)$ and $h_{\varepsilon'}(z)$. From part (a), we know that $h_\varepsilon(z)$ has a unique zero inside the unit ball at the origin. For points inside $\partial B_\alpha(z_\varepsilon)$, if $|h_\varepsilon(z) - h_{\varepsilon'}(z)| < |h_\varepsilon(z)|$, then $h_{\varepsilon'}(z)$ has one unique zero inside $B_\alpha(z_\varepsilon)$. We can do this because we know any $h_\varepsilon(z)$ is analytic on this ball, because $B_\alpha(z_\varepsilon) \subset B_2(0)$ and $h_\varepsilon(z)$ is analytic on $B_2(0)$.
Now, this assumption implies that $|f(z)+\varepsilon g(z) - f(z) - \varepsilon' g(z)| < |f(z)+\varepsilon g(z)|$, hence $|g(z)||\varepsilon - \varepsilon'| < |f(z)+\varepsilon g(z)|$. Since $h_\varepsilon(z)$ is analytic on $\overline{B_\alpha(z_\varepsilon)}$, $h_\varepsilon(z)$ must attain a minimum on $\partial B_\alpha(z_\varepsilon)$ by the maximum modulus principle. Allow me to denote this minimum by m_ε. This minimum value depends on ε, and here $\varepsilon > 0$. If $\varepsilon = 0$, then $f(z)+\varepsilon g(z)$ will have no zero in $B_\alpha(z_\varepsilon)$, because the only zero of f is when $z = 0$ which may be inside or outside this ball; and Rouché's theorem will no longer hold. Thus $\varepsilon > 0$.
Since the maximum of $g(z)$ lies alone the boundary of the unit ball, every other $|g(z)| \leq M$. Therefore I choose $\delta = \dfrac{m_\varepsilon}{M}$, which then minimizes δ for the minimum and maximum values of $h_\varepsilon(z)$ and $g(z)$, respectively.
Lastly, I want to note that from part (a), a zero of $h_\varepsilon(z)$ can only exist for $\varepsilon < \varepsilon_0 = \dfrac{m}{M}$, as defined previously. For any $\varepsilon \geq \varepsilon_0$, a zero of $h_\varepsilon(z)$ will not exist, hence a solution of $h_{\varepsilon'}(z)$ will not exist either. We may now begin the proof.

proof. From part (a), a zero of $h_\varepsilon(z)$ exists for $0 \leq \varepsilon < \varepsilon_0$, and denote this zero by z_ε. Fix $\alpha > 0$ and take $\delta = \dfrac{m_\varepsilon}{M}$, where m_ε denotes the minimum value of $h_\varepsilon(z)$ on $\overline{B_\alpha(z_\varepsilon)}$, as previously shown and exists by the extreme value theorem and only for values of $\varepsilon > 0$. This value of z must lie in $\partial B_\alpha(z_\varepsilon)$ by the maximum modulus principle. Fix $0 < \varepsilon < \varepsilon_0 = \dfrac{m}{M}$, which must be less than ε_0 for a solution z_ε to exist by part (a). Therefore for points $z \in \partial B_\alpha(z_\varepsilon)$, we have that $|\varepsilon - \varepsilon'| < \dfrac{m_\varepsilon}{M}$. This implies that

$$\begin{aligned}
&|g(z)||\varepsilon - \varepsilon'| \leq M|\varepsilon - \varepsilon'| < m_\varepsilon \leq |f(z)+\varepsilon g(z)| \\
\implies &|\varepsilon g(z) - \varepsilon' g(z) + f(z) - f(z)| < |f(z)+\varepsilon g(z)| \\
\implies &|h_\varepsilon(z) - h_{\varepsilon'}(z)| < |h_\varepsilon(z)|,
\end{aligned}$$

which therein implies that $h_\varepsilon(z)$ and $h_{\varepsilon'}(z)$ have the same number of zeros inside $B_\alpha(z_\varepsilon)$ by Rouché's theorem. This is equivalent to $|z_\varepsilon - z_{\varepsilon'}| < \alpha$, which is what I wanted to prove.
Since this holds for any $0 < \varepsilon < \varepsilon_0$, the function $\varepsilon \to z_\varepsilon$ is continuous, which is what I wanted to prove. qed.

In either case (a) and (b), there exists an $\varepsilon_0 = \dfrac{m}{M}$ such that for all $0 < \varepsilon < \varepsilon_0$, the claim holds true. Which is what I wanted to show.

Correct. 10

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Q3a

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3.

(a) By the maximum modulus principle, since f is nonconstant on the open set $\mathbb{D}$ and f is analytic, then $|f|$ can have no local maximum on $\mathbb{D}$. However, the local extreme value theorem implies that the maximum values of f must occur along the boundar $\partial\mathbb{D}$. Since the boundary of $\mathbb{D}$ is just the unit circle with $|z| = 1$, then the maximum value of $f(z)$ is one, which occurs for points $|z| = 1$. Therefore for every other points $z \in \mathbb{D}$, we obtain that $|f(z)| \leq 1 = \max_{|z|=1} f(z)$, which is what I wanted to show.

Good! **5**

Q3b

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(b) Assume for a contradiction that f does not vanish for anywhere inside $\mathbb{D}$. Consider the analytic, nonconstant function defined by function $\frac{1}{f(z)}$. Since f does not vanish anywhere in $\mathbb{D}$, $\frac{1}{f(z)}$ is well defined in $\mathbb{D}$. By the maxium modulus principle, $\frac{1}{|f(z)|}$ does not have a maximum on $\mathbb{D}$. However, from part (a), since $|f(z)| \leq 1$, then $1 \leq \frac{1}{|f(z)|}$, which here shows that the maximum of f does not lie along any boundary of $\mathbb{D}$, and since f does not vanish, the maximum must lie on $\partial\mathbb{D}$. We have obtained a contradiction. Since this violates the maximum modulus principle, f must vanish somewhere in $\mathbb{D}$. That is, $\exists z_0 \in \mathbb{D}$ such that $f(z_0) = 0$, which is what I wanted to show.

Good! **5**

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Q3c

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(c) I will begin by defining the function h given by $h(z) = f(z) - w$, where w is given as in the problem. Since f is an analytic function on an open set containing the closure of $\mathbb{D}$, so is $h(z)$. By Rouché's theorem, if $|f(z) - h(z)| < |f(z)| = |w|$, then $f(z)$ and $h(z)$ have an equal number of roots in $\mathbb{D}$. We have that

$$|f(z) - h(z)| = |f(z)||1 - h(z)/f(z)| \implies \left|1 - 1 + \frac{w}{f(z)}\right| < 1,$$

This condition should be checked on boundary

so [illegible] < 1. This implies that $|w| < |f(z)| \le 1$ [illegible] every $z \in \mathbb{D}$, and therefore the image of $f(z)$ is $\mathbb{D}$ since $|w| < 1$ as given. This implies that $f(z)$ and $h(z)$ have an equal number of zeroes inside $\mathbb{D}$, and therefore there exists a z_1 such that $h(z_1) = 0 \implies f(z_1) = w$, just has $f(z_0) = 0$. This is what I wanted to show.

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Q4

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4.

To begin this problem, allow me to first prove the generalized argument principle. Consider a function $h(z)$. Let n_j be the multiplicity of a zero z_j of h for $j = 1, \ldots, N$. That is, there is a function $g_j(z)$ which is analytic near z_j with $g(z_j) \neq 0$. h can be expressed as

$$h(z) = (z - z_j)^{n_j} g_j(z).$$

This implies that $\frac{h'(z)}{h(z)} = \frac{n_j}{z - z_j} + \frac{g_j'(z)}{g_j(z)}$. For the generalization, we have that $\frac{h'(z)}{h(z)} f(z) = \frac{n_j f(z)}{z - z_j} + \frac{g'(z)}{g(z)} f(z)$. The residue of $\frac{h'(z)}{h(z)} f(z)$ at z_j is just $n_j f(z_j)$. Similarly if $h(z)$ has a pole at w_l of order m_l, for $l = 1, \ldots, M$, then there is an analytic function $G_l(z)$ near w_l with $G_l(w_l) \neq 0$. Therefore, applying a similar argument,

$$h(z) = (z - w_l)^{-m_l} G_l(z) \implies \frac{h'(z)}{h(z)} = \frac{-m_l}{z - w_l} + \frac{G_l'(z)}{G_l(z)}.$$

The equivalent generalization yields that the residue of $\frac{h'(z)}{h(z)} f(z)$ due to the number of poles of h is just $-m_l f(w_l)$. Therefore, like the non-generalized argument principle, we have that

$$\frac{1}{2\pi i} \int_\gamma \frac{h'(z)}{h(z)} f(z)\, dz = \sum_{j=1}^{N} n_j f(z_j) - \sum_{l=1}^{M} m_l f(w_l),$$

which only holds for zeros and poles inside the piecewise simple smooth closed curve γ. Now back to the problem.

Since $f(z)$ is analytic on an open set containing $\overline{B}_1(0)$, then $f(z) - 1$ is also analytic and therefore contains no poles inside $\overline{B_1(0)}$. The curve we wish to integrate over is the boundary $|z| = 1$. I will note that since f is analytic on a domain which contains the closed unit ball, then the zeros of the function h given by $h(z) = f(z) - 1$ may or may not lie inside the unit disc. Allow me to define the subset $\{Z_1, \ldots, Z_l\} \subseteq \{z_1, z_2, \ldots, z_k\}$ which denotes the zeroes lying inside $B_1(0)$, with respective multiplicities $M_1, \ldots, M_l$. We are able to count such zeroes because they are countable finite and distinguished by the topology. Since $h(z) = f(z) - 1$ has l roots lying inside $B_1(0)$ with none on the boundary, and $h'(z) = f'(z)$, then by the generalized argument principle proved above,

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^2 f'(z)}{f(z) - 1}\, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^2 h'(z)}{h(z)}\, dz = \sum_{n=1}^{l} M_n Z_n^2,$$

which is what I wanted to determine. Again note that the Z_n's only correspond to zeroes inside the unit ball, since no zeroes lie on the boundary, which may be different from the total number of zeroes $f(z) - 1$ has on it's whole domain.

Correct. **10**

Q5a

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5.

Similarly to question 2, allow me to define a convention which I will use for the remainder of this problem. Define the fucntion h_a by $h_a(z) = \dfrac{z-a}{1-\bar{a}z}$. Since $f : \mathbb{D} \to \mathbb{D}$ and $z_0 \in D$, then every value of z we will be working with will have magnitude $|z| < 1$. This implies that the linear fractional transformation h_a sends the unit disc to the unit disc (see example 1 of section 3.3). Note that $h_a^{-1}(\omega)$ exists (I'm using ω here to avoid confusion with w as a range variable).

(a) Consider the composition of functions $g : \mathbb{D} \to \mathbb{D}$ given by $g(z) = h_w \circ f \circ h_{z_0}^{-1}(z)$. Since the functions $h_a(z)$, $h_a^{-1}(z)$ and f are both analytic on $\mathbb{D}$, then so is g. More explicitly, one may write

$$\begin{aligned} g(z) = h_w(f(h_{z_0}^{-1}(z))) &= h_w\left(f\left(\frac{-z-z_0}{-1-\bar{z}_0 z}\right)\right) \\ &= \frac{f\left(\frac{-z-z_0}{-1-\bar{z}_0 z}\right) - w}{1 - \bar{w} f\left(\frac{-z-z_0}{-1-\bar{z}_0 z}\right)}. \end{aligned}$$

Note that $g(0)$ implies that $f(h_{z_0}(0)) = f(z_0) = w$, and therefore $g(0) = 0$. Since g is analytic, $|g(z)| < 1$ and $g(0) = 0$, one may apply Schwartz's lemma. That is, for $z \in \mathbb{D}$ ($|z| < 1$),

$$|g(z)| = \left| \frac{f\left(\frac{-z-z_0}{-1-\bar{z}_0 z}\right) - w}{1 - \bar{w} f\left(\frac{-z-z_0}{-1-\bar{z}_0 z}\right)} \right| \leq |z|.$$

Furthermore this is equivalently $\left| \dfrac{f\left(h_{z_0}^{-1}(z)\right) - w}{1 - \bar{w} f\left(h_{z_0}^{-1}(z)\right)} \right| \leq |z|$. If we now substitute $z \to h_{z_0}(z')$, we obtain

$$\begin{aligned} \left| \frac{f\left(h_{z_0}^{-1}(h_{z_0}(z'))\right) - w}{1 - \bar{w} f\left(h_{z_0}^{-1}(h_{z_0}(z'))\right)} \right| &= \left| \frac{f(z') - w}{1 - \bar{w} f(z')} \right| \\ &\leq |h_{z_0}(z')| = \left| \frac{z' - z_0}{1 - \bar{z}_0 z'} \right|. \end{aligned}$$

which with the arbitrary z' yields the result

$$\left| \frac{f(z) - w}{1 - \bar{w} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|,$$

which is what I wanted to show.

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Q5b

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(b) To begin, I will first take the derivative of $h_a(z)$ and $h_a^{-1}(z)$:

$$\begin{aligned}
h_a'(z) &= \frac{d}{dz}\left[\frac{z-a}{1-\bar{a}z}\right] \\
&= \frac{(1)(1-\bar{a}z)-(-\bar{a})(z-a)}{(1-\bar{a}z)^2} \\
&= \frac{1-|a|^2}{(1-\bar{a}z)^2} \\
h_a^{-1\prime}(z) &= \frac{d}{dz}\left[\frac{-z-a}{-1-\bar{a}z}\right] \\
&= \frac{(-1)(-1-\bar{a}z)-(-\bar{a})(-z-a)}{(1+\bar{a}z)^2} \\
&= \frac{1+|a|^2}{(1+\bar{a}z)^2},
\end{aligned}$$

which follows from the quotient rule. Next again allow me to consider the composition of functions which I had defined in part (a), $g : \mathbb{D} \to \mathbb{D}$ given by $g(z) = h_w \circ f \circ h_{z_0}^{-1}(z)$. The product rule on g yields

$$g'(z) = h_w'(f(h_{z_0}^{-1}(z))) \cdot f'(h_{z_0}^{-1}(z)) \cdot h_{z_0}^{-1\prime}(z)$$

which implies that

$$\begin{aligned}
g'(0) &= h_w'(f(z_0)) \cdot f'(z_0) \cdot (1+|z_0|^2) \\
&= \frac{1-|w|^2}{(1-|w|^2)^2} \cdot f'(z_0) \cdot (1+|z_0|^2) \\
&= \frac{1}{1-|f(z_0)|^2} \cdot f'(z_0) \cdot (1+|z_0|^2).
\end{aligned}$$

Now, I wish to apply Schwartz's inequality on $g(0)$. Note that from part (a), since $|g(z)| \leq |z|$ on $\mathbb{D}$, this implies that $|g'(z)| \leq 1$ on $\mathbb{D}$ and therefore $|g'(0)| \leq 1$. From the equality obtained above, this implies that

This implication is not true. -2

$$\left|\frac{1}{1-|f(z_0)|^2} \cdot f'(z_0) \cdot (1+|z_0|^2)\right| \leq 1$$

and therefore

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{|1+|z|^2|},$$

since $z_0 \in \mathbb{D}$ was arbitrary. Now, the triangle inequality notes that $||a|-|b|| \leq |a+b| \leq |a|+|b|$, which means that

$$\frac{1}{1+|z|^2} \leq \frac{1}{|1-|z|^2|} = \frac{1}{1-|z|^2},$$

since $|z| < 1$ for $z \in \mathbb{D}$, $0 < \frac{1}{1-|z|^2}$. This implies that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2},$$

which is what I wanted to show.