Midterm 2

Class scores distribution Show

My score

100% (100/100)

Q1 25

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$\frac{\partial y}{\partial x} = e^x$ $y(x) = e^x + c$

$c = e^x - y$

1. First Order Linear Equation

[25 marks] Consider $u = u(x, y)$.
$e^{-x}u_x + u_y = 0$

1. Using a suitable change of variables, find the general solution to the PDE.
2. Sketch some characteristic lines on the xy-plane.
3. Let the initial condition be given by $u(x, 0) = \phi(x)$.
   Find the particular solution $u(x, y)$, based on the initial condition.
   In which region of the xy-plane is the particular solution $u(x, y)$ uniquely determined by the initial condition?

1 Let $t = e^x + y$ Then $u_x = u_t(e^x) + u_s(e^x)$

$s = e^x - y$. $u_y = u_t(1) + u_s(-1)$

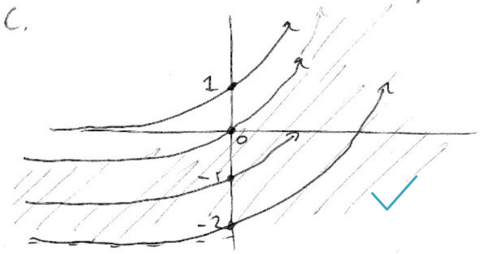
Then, in the equation, one finds

$e^{-x}u_x + u_y = e^{-x}e^x(u_t + u_s) + u_t - u_s = 2u_t = 0.$

Hence $u_t = 0 \Rightarrow u(s,t) = f(s)$ since $f(s)$ is independent of t. In terms of x, y,

$$\boxed{u(x,y) = f(e^x - y)}$$ Gen. Sol^n.

③ Characteristic lines are determined by $y(x) = e^x + C$ for a constant C.



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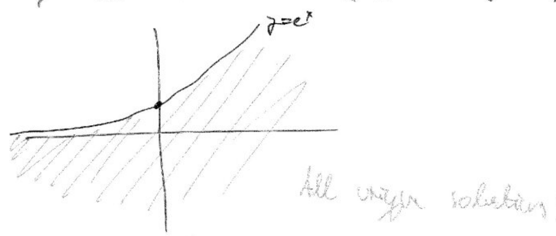
The characteristic lines are only valid for $c<0$, since $e^x + c$ has no solutions for $c \geq 0$. That is, the light blue region shaded.

③ If the initial condition is $u(x,0) = \varphi(x)$, then $u(x,0) = f(e^x) = \varphi(x)$, so the function which yields the inverse is $\log(x)$, so $\log(e^x) = \log(\varphi(x))$ hence $\varphi \circ \log(x) = f(x)$ yields the initial condition $u(x,0) = \varphi(x)$: hence the unique solution is

$$\boxed{u(x,y) = \varphi \circ \log\left(e^x - y\right)}.$$

As before, these solutions are unique for any constant $c<0$, where $y = e^x + c$ is defined: $\varphi \circ \log(c)$ only defined for $c<0$. That is, the set $S = \{(x,y) \in \mathbb{R}^2 : y < e^x\}$.



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correct 25

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Q2 10

2. Duhamel's Principle

[10 marks] View $y = t$, consider $u = u(x,t)$, the inhomogeneous equation from the previous question.
$e^{-x}u_x + u_t = f(x,t)$
with initial conditions $u(x,0) = \phi(x)$.

Find the **Solution Operator** S^t for this first order PDE.
Find the solution to the inhomogeneous equation using it.

As from the previous question, the general unique solution is given by $u(x,t) = \varphi \circ \log(e^x - t)$, which is the solution of the homogeneous equation given above.
The solution operator S^t is then defined to be $\log(e^x - t)$, and S^t acting on the initial condition is the composition
$S^t \cdot \varphi(x) = \varphi(\log(e^x - t))$. Thus the solution for the inhomogeneous equation is

S^t correct 5

$$u(x,t) = S^t \cdot \varphi(x) + \int_0^t S^{t-s} \cdot f(x,s)\, ds$$
$$= \varphi(\log(e^x - t)) + \int_0^t f(\log(e^x - t + s, s))\, ds$$

For a function f.

Duhamel's principle and substitute S^t 5

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3. Second Order Linear Equation

[25 marks] Find the general solution $u = u(x,t)$ to the PDE:
$u_{xx} + 2u_{xt} - 8u_{tt} = 0$

by "factoring" the second order differential operator into 2 first order differential operators, to attain 2 sets of characteristic lines, which will correspond to the suitable change of variables.

Q3 25

We begin by factoring the differential operator $\left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial t} - 8\frac{\partial^2}{\partial t^2}\right)u = 0$ by $(x^2 + 2x - 8) = (x+4)(x-2)$. Then the PDE becomes

$$\left(\frac{\partial}{\partial x} + 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)u = 0.$$

The characteristic lines are $\frac{\partial t}{\partial x} = 4$, $\frac{\partial t}{\partial x} = -2$ hence $t - 4x = c_1$, $t + 2x = c_2$.

Let $p = t - 4x$, $q = t + 2x$. Then,

$$u_x = u_p(-4) + u_q(2), \quad u_t = u_p(1) + u_q(1).$$

In the first equation,

$$-4u_p + 2u_q + 4u_p + 4u_q = 6u_q = 0,$$

and the second

$$-4u_p + 2u_q - 2u_p - 2u_q = -6u_p = 0.$$

That is, $u = f(p)$ and $u = g(q)$, thus the general solution is

$$\boxed{u(x,t) = f(t-4x) + g(t+2x)}$$

correct 25

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4. Higher Order Chain Rule

[20 marks] Let $u = u(x, y), x = r\cos\theta, y = r\sin\theta$. Find $\frac{\partial^2 u}{\partial r^2}$.

Q4 20

First derivative

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= u_x(\cos\theta) + u_y(\sin\theta)$$

Second derivative

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(u_x\cos\theta + u_y\sin\theta\right)$$

$$= \frac{\partial u_x}{\partial r}\cos\theta + \frac{\partial u_y}{\partial r}\sin\theta$$

$$= \left(\frac{\partial u_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial u_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial r}\right)\sin\theta$$

$$= \left[u_{xx}(\cos\theta) + u_{xy}(\sin\theta)\right]\cos\theta + \left(u_{xy}(\cos\theta) + u_{yy}(\sin\theta)\right)\sin\theta$$

$$\boxed{\frac{\partial u^2}{\partial r^2} = u_{xx}\cos^2\theta + u_{yy}\sin^2\theta + 2u_{xy}\sin\theta\cos\theta}$$

correct **20**

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Q5 20

5. Duhamel's Principle

[20 marks] Consider the **inhomogeneous** heat equation with constant drift speed coefficient V, on the whole line $-\infty < x < \infty$,

$$u_t = cu_{xx} + Vu_x + f(x,t)$$

where $c > 0$ is a constant, with **initial condition**:

$$u(x,0) = \phi(x)$$

Define the **Solution Operator** S^t,

$$\left(\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2} - V\frac{\partial}{\partial x}\right)\left(S^t \cdot (u_0(x))\right) = 0$$

For $u(x,t) = S^t \cdot (u_0(x))$, satisfies the initial condition $u(x,0) = u_0(x)$.

Find the solution to the **inhomogeneous** heat equation.

Using the **Solution Operator**, the linear differential operator above, multivariable chainrule, fundamental theorem of calculus, verify that the solution for the inhomogeneous wave equation is correct.

Note: **You are NOT asked to solve S^t.**

The general solution is $u(x,t) = S^t \cdot \phi(x) + \int_0^t S^{t-s} \cdot f(x,s)\, ds$

solution to inhomogeneous equation is given by Duhamel's principle 5

First, note that by definition $S^t \cdot \phi(x)$ ... homogeneous equation. Thus, we wish to examine the second term $I = \int_0^t S^{t-s} \cdot f(x,s)\, ds$. The time derivative operator $\frac{\partial}{\partial t}$ suggest the change of

the term $S^t \phi(x)$ is the solution of the homogeneous equation 2.5

variables $v = t$ (in upper bound of integral) and $w = t$ (in integrand).

Here $I = \int_0^v S^{w-s} \cdot f(x,s)\, ds$ which implies that

$$\frac{\partial I}{\partial t} = \frac{\partial I}{\partial w}(1) + \frac{\partial I}{\partial v}(1) = \frac{\partial}{\partial w}\int_0^v S^{w-s} \cdot f(x,s)\, ds + \frac{\partial}{\partial v}\int_0^v S^{w-s} \cdot f(x,s)\, ds$$

(extra space)

We may swap integration/differentiation order in the first term, and invoke the FTC in the second:

$$\frac{\partial I}{\partial t} = \int_0^v \frac{\partial}{\partial w} S^{w-s} \cdot f(x,s)\,ds + S^{v-v} \cdot f(x,t)$$

$$= \int_0^t \frac{\partial}{\partial t} S^{t-s} \cdot f(x,s)\,ds + S^0 \cdot f(x,t).$$

Liebniz rule when differentiating the integral 5

FTC to get the inhomogeneous term $f(x,t)$ 2.5

The second term yields $f(x,t)$ ... output of the composition at ...

$$\frac{\partial I}{\partial t} = \int_0^t \frac{\partial}{\partial t} S^{t-s} \cdot f(x,s)\,ds + f(x,t).$$

Now, we find that

$$\frac{\partial}{\partial t} \int_0^t S^{t-s} \cdot f(x,s)\,ds = c\frac{\partial^2}{\partial x^2} \int_0^t S^{t-s} \cdot f(x,s)\,ds + V\frac{\partial}{\partial x} \int_0^t S^{t-s} \cdot f(x,s)\,ds$$

$$\Rightarrow \int_0^t \frac{\partial}{\partial t} S^{t-s} \cdot f(x,s)\,ds + f(x,t) = \int_0^t c\frac{\partial^2}{\partial x^2} S^{t-s} \cdot f(x,s)\,ds + \int_0^t v\frac{\partial}{\partial x} S^{t-s} \cdot f(x,s)\,ds$$

(we may swap $\frac{\partial}{\partial x}$ and $\int_0^t$ since x and t are independent of each other), thus

$$f(x,t) = -\int_0^t \frac{\partial}{\partial t} S^{t-s} \cdot f(x,s)\,ds + \int_0^t \left(c\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x}\right) S^{t-s} \cdot f(x,s)\,ds$$

$$= \int_0^t \left(c\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\left(S^{t-s} f(x,s)\right) ds$$

(extra space)

which is what I wanted to show, since

$$f(x,t) = \int_0^t \left(c\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) S^{t-s} \cdot f(x,s)\, ds$$ only when

$f(x,t)$ satisfies the inhomogeneous equation. Note that the solution

operator $\left(c\frac{\partial^2}{\partial x^2} \right.$ [illegible]

$S^{t-s} \cdot f(x,s)$ satis[illegible]

using definition of S^t to deal with differentiation inside the integral 5

$$f(x,t) = \int_0^t (0)\, ds = 0,$$ but because $\frac{\partial I}{\partial t} = \int_0^t \frac{\partial}{\partial t} S^{t-s} \cdot f(x,s)\, ds$

$+ f(x,t)$ is derivated $\left(\frac{\partial I}{\partial t}\right)$, then $\frac{\partial I}{\partial t} \sim f(x,t)$

which then must imply that $f(x,t)$ satisfies the PDE as well.

(it may be more noticeable to write

$$f(x,t) = \int_0^t \left(c\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)(S^{t-s} \cdot f(x,s))\, ds$$

$$\Rightarrow \int_0^t \left(\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2} - v\frac{\partial}{\partial x} \right)(S^{t-s} \cdot f(x,s))\, ds + f(x,t) = 0$$

Hence $f(x,t)$ also solves the PDE)

(extra space)

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