PHY454 Problem Set 1

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Jace Alloway - 1006940802

Problem 1

In the first problem, we determine the pressure in the core of a planet by applying the expression for hydrostatic balance, given by $\dfrac{\partial P}{\partial z} = -\rho g$. In the case of a planet with radius R and uniform density ρ, with variable gravity as a function of depth (or radius), our expression becomes $\dfrac{\partial P}{\partial r} = -\rho g(r)$.

One may recall that the expression for gravitational strength in the presence of a uniform mass distribution in a planet, as a function of the radius, is $g(r) = \dfrac{Gm(r)}{r^2}$. For a uniform density ρ, the mass distribution is directly proportional to r^3, due to the volume of the enclosed mass inside a sphere: $m(r) = \dfrac{4\pi}{3}\rho r^3$. Therefore

$$\frac{\partial P}{\partial r} = -\frac{4\pi}{3} G\rho^2 r,$$

which is the equation we wish to solve. We can do this by integrating both sides from the center of the planet $(r = 0)$ to its radius $(r = R)$:

$$\begin{aligned}
\int_0^R \frac{\partial P}{\partial r}\, dr &= -\frac{4\pi}{3} G\rho^2 \int_0^R r\, dr \\
\implies P(R) - P(0) &= -\frac{2\pi}{3} G\rho^2 r^2 \Big|_0^R \\
\implies P(0) &= \frac{2\pi}{3} G\rho^2 R^2 - P(R),
\end{aligned}$$

which is thus the expression for the pressure at the center of the planet.

In the case that we assume $P(R) = 0$ (no atmosphere), then we just obtain that $P(0) = \dfrac{2\pi}{3} G\rho^2 R^2$. For earth, however, the pressure at sea level exerted by the atmosphere is $1013.25\,\text{mb}$, which is equivalent to $P(R_{\text{earth}}) = 101325\,\text{Nm}^{-2}$ by a simple unit conversion. Furthermore, assuming earth's density is uniformly $\rho = 5500\,\text{kgm}^{-3}$ with $R = 6378100\,\text{m}^2$, we obtain the pressure inside the core of earth:

$$\begin{aligned}
P_{\text{center}} &= \frac{2\pi}{3}(6.67 \times 10^{-11}\,\text{kg}^{-1}\,\text{m}^3\text{s}^{-2})(5500\,\text{kgm}^{-3})^2(6378100\,\text{m})^2 \\
&= 1.719065699 \times 10^{11}\text{kg}^{-1}\text{m}^3\text{s}^{-2}\text{kg}^2\text{m}^{-6}\text{m}^2 \\
&\approx 1.719 \times 10^{11}\,\text{kg}\,\text{m}^{-1}\text{s}^{-2} \\
&= 1.719 \times 10^{11}\,\text{N}\,\text{m}^{-2},
\end{aligned}$$

as desired.

Problem 2

(a) In this problem, we evaluate calculus identities using index notation. For the sake of clarity, instead of using $\mathbf{x} = (x_1, x_2, x_3)$ as the vector, I will use $\mathbf{v} = (v_1, v_2, v_3)$ to avoid confusion with spatial coordinates (x_1, x_2, x_3). I am doing this to generalize cases that vector components may depend on spatial coordinates. The Einstein summation convention is assumed.

(i)

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x_i} v_j \, \mathbf{e_i} \cdot \mathbf{e_j} \\
&= \frac{\partial v_j}{\partial x_i} \delta_{ij} \\
&= \frac{\partial v_i}{\partial x_i} \\
&= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}
\end{aligned}$$

(ii)

$$\begin{aligned}
\nabla \times \mathbf{v} &= \epsilon_{ijk} \frac{\partial}{\partial x_i} v_j \, \mathbf{e_k} \\
&= \epsilon_{ijk} \frac{\partial v_j}{\partial x_i} \, \mathbf{e_k} \\
&= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \mathbf{e_1} \\
&\quad + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \mathbf{e_2} \\
&\quad\quad + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e_3}
\end{aligned}$$

(iii)

$$\begin{aligned}
\nabla \mathbf{v} &= \nabla \otimes \mathbf{v} \\
&= \frac{\partial}{\partial x_i} v_j \, \mathbf{e_i} \otimes \mathbf{e_j} \\
&= \frac{\partial v_j}{\partial x_i} \, \mathbf{e_i}\mathbf{e_j} \\
&= \begin{pmatrix} \partial_{x_1} v_1 & \partial_{x_1} v_2 & \partial_{x_1} v_3 \\ \partial_{x_2} v_1 & \partial_{x_2} v_2 & \partial_{x_2} v_3 \\ \partial_{x_3} v_1 & \partial_{x_3} v_2 & \partial_{x_3} v_3 \end{pmatrix}
\end{aligned}$$

(iv)

$$\begin{aligned}
\nabla(\mathbf{v} \cdot \mathbf{v}) &= \nabla(v_i v_j \, \mathbf{e_i} \cdot \mathbf{e_j}) \\
&= \frac{\partial}{\partial x_k} v_i v_j \delta_{ij} \, \mathbf{e_k} \\
&= \left(\frac{\partial v_i}{\partial x_k} v_i + \frac{\partial v_i}{\partial x_k} v_i \right) \mathbf{e_k}
\end{aligned}$$

$$= 2(\nabla \mathbf{v}) \cdot \mathbf{v}$$

(v)

$$\begin{aligned}
\nabla^2 \mathbf{v} &= \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \, \mathbf{e_i} \cdot \mathbf{e_j} \right) \mathbf{v} \\
&= \frac{\partial^2}{\partial x_i \partial x_j} \delta_{ij} v_k \, \mathbf{e_k} \\
&= \frac{\partial^2 v_k}{\partial x_i^2} \, \mathbf{e_k}
\end{aligned}$$

where it is assumed to sum over i as well, since $\partial x_i^2 = \partial x_i \partial x_i$ is repeated over i. This expression is equivalent to

$$\nabla^2 \mathbf{v} = (\nabla^2 v_1,\, \nabla^2 v_2,\, \nabla^2 v_3).$$

Furthermore,

$$\begin{aligned}
\nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}) &= \nabla \left(\frac{\partial v_i}{\partial x_i} \right) - \nabla \times \left(\epsilon_{ijk} \frac{\partial v_j}{\partial x_i} \, \mathbf{e_k} \right) \\
&= \frac{\partial^2 v_i}{\partial x_j \partial x_i} \, \mathbf{e_j} - \left[\epsilon_{\ell mn} \frac{\partial}{\partial x_\ell} \left(\epsilon_{ijk} \frac{\partial v_j}{\partial x_i} \, \mathbf{e_k} \cdot \mathbf{e_m} \right) \mathbf{e_n} \right] \\
&= \frac{\partial^2 v_i}{\partial x_j \partial x_i} \, \mathbf{e_j} - \left[\epsilon_{n\ell m} \epsilon_{mij} \frac{\partial^2 v_j}{\partial x_\ell \partial x_i} \, \mathbf{e_n} \right] \\
&= \frac{\partial^2 v_i}{\partial x_j \partial x_i} \, \mathbf{e_j} - \left[(\delta_{ni} \delta_{\ell j} - \delta_{nj} \delta_{\ell i}) \frac{\partial^2 v_j}{\partial x_\ell \partial x_i} \, \mathbf{e_n} \right] \\
&= \frac{\partial^2 v_i}{\partial x_j \partial x_i} \, \mathbf{e_j} - \frac{\partial^2 v_j}{\partial x_j \partial x_i} \, \mathbf{e_i} + \frac{\partial^2 v_j}{\partial^2 x_i} \, \mathbf{e_j} \\
&= \frac{\partial^2 v_k}{\partial x_i^2} \, \mathbf{e_k}
\end{aligned}$$

by index changes. Therefore

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}).$$

(b) Now consider the square magnitude of the position vector $r^2 = x_j^2$, where $r = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

(i) First note that $\dfrac{\partial r^2}{\partial x_i} = 2r \dfrac{\partial r}{\partial x_i}$ and $\dfrac{\partial r^2}{\partial x_i} = \dfrac{\partial x_j^2}{\partial x_i} = 2x_j \dfrac{\partial x_j}{\partial x_i}$ by the product rule. Therefore $r \dfrac{\partial r}{\partial x_i} = x_j \dfrac{\partial x_j}{\partial x_i}$. Multiplying through by the derivative $\dfrac{\partial x_i}{\partial x_j}$ then yields

$$\begin{aligned}
r \frac{\partial r}{\partial x_i} \frac{\partial x_i}{\partial x_j} &= x_j \\
\implies \frac{\partial r}{\partial x_j} &= \frac{x_j}{r},
\end{aligned}$$

which is what I wanted to show.

(ii) Now, using the expression from part (i) and from part (a-v), we obtain that

$$\begin{aligned}
\nabla^2 r &= \frac{\partial^2}{\partial x_i^2}\, r \\
&= \frac{\partial}{\partial x_i}\left(\frac{\partial r}{\partial x_i}\right) \\
&= \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) \\
&= \frac{1}{r}\frac{\partial x_i}{\partial x_i} + x_i \frac{\partial}{\partial x_i}\left[\frac{1}{r}\right] \\
&= \frac{1}{r} - \frac{x_i}{r^2}\frac{\partial r}{\partial x_i} \\
&= \frac{1}{r} - \frac{x_i}{r^2}\frac{x_i}{r} \\
&= \frac{1}{r} - \frac{x_i^2}{r^3} \\
&= \frac{1}{r} - \frac{r^2}{r^3} \\
&= 0.
\end{aligned}$$

Problem 3

(a) Consider a symmetric tensor $\mathbf{B}$ and a vector $\mathbf{a}$. Suppose that $B_{ij} = B_{ji}$ (that is, $\mathbf{B}$ is symmetric). Then, the dot product between $\mathbf{a}$ and $\mathbf{B}$ is

$$\begin{aligned}\mathbf{a} \cdot \mathbf{B} &= a_i B_{ij} \\ &= a_i B_{ji} \\ &= B_{ji} a_i \\ &= \mathbf{B} \cdot \mathbf{a},\end{aligned}$$

which is what I wanted to show. In the case where $\mathbf{B}$ is not symmetric ($B_{ij} \neq B_{ji}$), then $a_i B_{ij} \neq a_i B_{ji}$, which proves that this relation holds only if $\mathbf{B}$ is symmetric.

(b) Now suppose that $\mathbf{B}$ is antisymmetric (that is, $B_{ij} = -B_{ji}$). Then, for two dot products with $\mathbf{a}$,

$$\begin{aligned}\mathbf{a} \cdot \mathbf{B} \cdot \mathbf{a} &= a_i B_{ij} a_j \\ &= a_j B_{ji} a_i \\ &= -a_i B_{ij} a_j,\end{aligned}$$

which must be zero, since for any number $y = -y$, then $y = 0$. This primarily holds because the diagonals of an antisymmetric tensor are zero. Therefore $\mathbf{a} \cdot \mathbf{B} \cdot \mathbf{a} = 0$, as desired.

Problem 4

For these problems, I will just show my steps for completion. Many proofs of the following identities do not call for much justification.

(a)

$$\begin{aligned}
\nabla \times (\nabla f) &= \nabla \times \left(\frac{\partial f}{\partial x_i} \, \mathbf{e_i} \right) \\
&= \epsilon_{\ell mn} \frac{\partial}{\partial x_\ell} \left(\frac{\partial f}{\partial x_i} \, \mathbf{e_i} \cdot \mathbf{e_m} \right) \mathbf{e_n} \\
&= \epsilon_{\ell mn} \frac{\partial^2 f}{\partial x_\ell \partial x_m} \, \mathbf{e_n} \\
&= \epsilon_{\ell mn} \frac{\partial^2 f}{\partial x_m \partial x_\ell} \, \mathbf{e_n} \\
&= 0
\end{aligned}$$

By Clairaut's theorem, assuming f is a C^2 function. The order of derivation indices may be exchanged which, when summed over, superpose to zero.

(b)

$$\begin{aligned}
\nabla \times (f\mathbf{v}) &= \epsilon_{ijk} \frac{\partial}{\partial x_i} f v_j \, \mathbf{e_k} \\
&= e_{ijk} \left(\frac{\partial f}{\partial x_i} v_j + \frac{\partial v_j}{\partial x_i} f \right) \mathbf{e_k} \\
&= e_{ijk} \frac{\partial f}{\partial x_i} v_j \, \mathbf{e_k} + \epsilon_{ijk} \frac{\partial v_j}{\partial x_i} f \, \mathbf{e_k} \\
&= e_{ijk} (\nabla f)_i v_j \, \mathbf{e_k} + f \left(\epsilon_{ijk} \frac{\partial v_j}{\partial x_i} \, \mathbf{e_k} \right) \\
&= -e_{jik} (\nabla f)_i v_j \, \mathbf{e_k} + f (\nabla \times \mathbf{v}) \\
&= f (\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla f),
\end{aligned}$$

as desired.

(c)

$$\begin{aligned}
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla \times (\epsilon_{ijk} u_i v_j \, \mathbf{e_k}) \\
&= \epsilon_{\ell mn} \frac{\partial}{\partial x_\ell} \, \mathbf{e_n} \, \epsilon_{ijm} u_i v_j \\
&= \epsilon_{n\ell m} \epsilon_{mij} \frac{\partial u_i v_j}{\partial x_\ell} \, \mathbf{e_n} \\
&= (\delta_{ni}\delta_{\ell j} - \delta_{nj}\delta_{\ell i}) \left(\frac{\partial u_i}{\partial x_\ell} v_j + \frac{\partial v_j}{\partial x_\ell} u_i \right) \mathbf{e_n} \\
&= \left[\delta_{ni}\delta_{\ell j} \frac{\partial u_i}{\partial x_\ell} v_j + \delta_{ni}\delta_{\ell j} \frac{\partial v_j}{\partial x_\ell} u_i - \delta_{nj}\delta_{\ell i} \frac{\partial u_i}{\partial x_\ell} v_j - \delta_{nj}\delta_{\ell i} \frac{\partial v_j}{\partial x_\ell} u_i \right] \mathbf{e_n} \\
&= \frac{\partial u_i}{\partial x_j} v_j \, \mathbf{e_i} + \frac{\partial v_j}{\partial x_j} u_i \, \mathbf{e_i} - \frac{\partial u_i}{\partial x_i} v_j \, \mathbf{e_j} - \frac{\partial v_j}{\partial x_i} u_i \, \mathbf{e_j}
\end{aligned}$$

$$
\begin{aligned}
&= \left(v_j \frac{\partial}{\partial x_j} \right) u_i \, \mathbf{e_i} + \left(\frac{\partial v_j}{\partial x_j} \right) u_i \, \mathbf{e_i} - \left(\frac{\partial u_i}{\partial x_i} \right) v_j \, \mathbf{e_j} - \left(u_i \frac{\partial}{\partial x_i} \right) v_j \, \mathbf{e_i} \\
&= (\mathbf{v} \cdot \nabla) \, \mathbf{u} + (\nabla \cdot \mathbf{v}) \, \mathbf{u} - (\nabla \cdot \mathbf{u}) \, \mathbf{v} - (\mathbf{u} \cdot \nabla) \, \mathbf{v},
\end{aligned}
$$

which is what I wanted to show.

Problem 5

In this last problem, we consider the two-dimensional flow governed by the equations given by $(u, v) = (U_0, at)$ for $U_0, a > 0$.

(a) To determine the streamlines, we must first determine the streamfunction ψ for this flow. The streamfunction $\psi(x, y)$ should recover the equations of the flow, provided

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Solving for ψ from these relations yields streamlines, given for values of constant ψ. We have that

$$U_0 = \frac{\partial \psi}{\partial y} \implies \psi(x, y) = U_0 y + f(x), \qquad at = -\frac{\partial \psi}{\partial x} \implies \psi(x, y) = -atx + g(y).$$

Quick comparison of the above to relations provides that $\psi(x, y) = -xat + Uy$. The streamlines are then explicitly given by $C = -xat + Uy$, hence $y(x) = \dfrac{at}{U_0}x + C$ are the streamlines for this flow.

One may determine visual parametric trajectories for the streamlines of this flow, given by $(x(t), y(t)) = \left(t, \dfrac{at^2}{U_0} + C\right)$ in two dimensions.

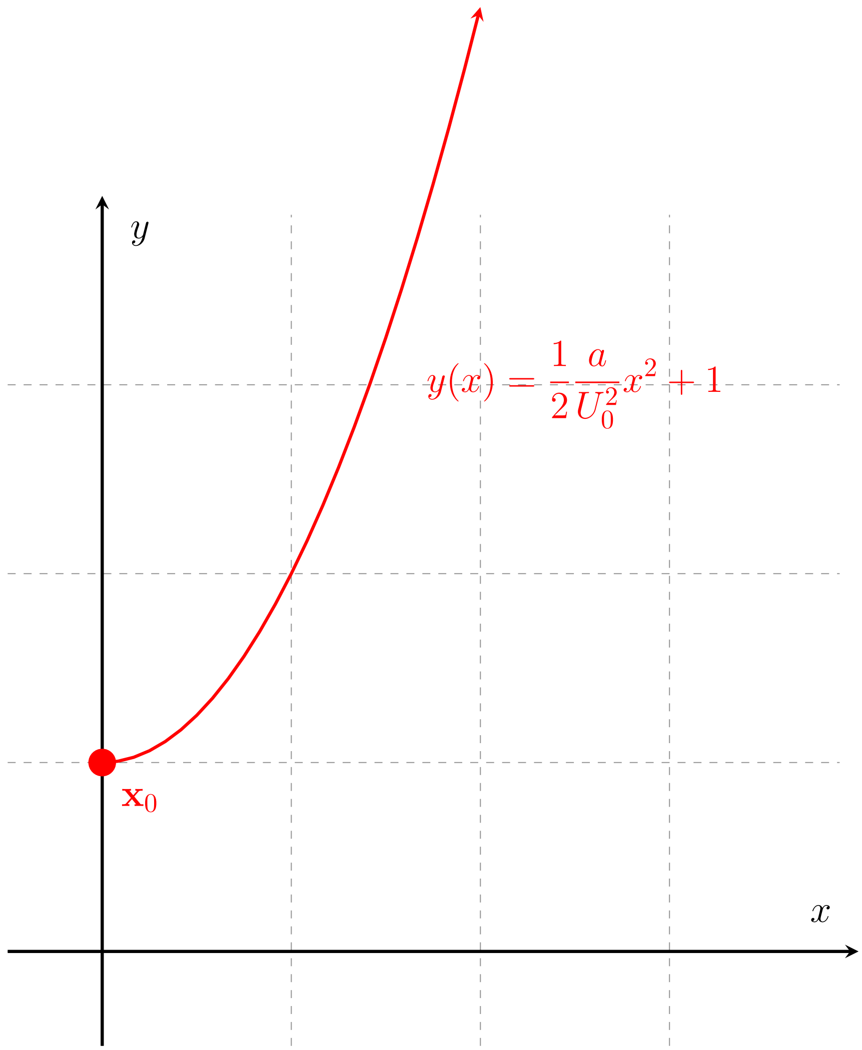
(b) Different from the streamline, the pathline is the trajectory of a fluid element from an initial point and time. To determine the pathline at $\mathbf{x}_0 = (0, 1)$ at time $t = 0$, we may integrate over the flow, thus 'following' the fluid element along its path. From the initial conditions, we have that

$$\begin{aligned} \int_{x_0}^{x} dx' &= \int_0^t U_0 \, dt' \\ \implies x - x_0 &= U_0 t \\ \implies x &= U_0 t \end{aligned}$$

for the x component, and

$$\begin{aligned} \int_{y_0}^{y} dy' &= \int_0^t at' \, dt' \\ \implies y - y_0 &= \frac{1}{2}at^2 \\ \implies y(t) &= \frac{1}{2}at^2 + 1 \end{aligned}$$

for the y component. With $t = \dfrac{x}{U_0}$, the explicit equation for the pathline becomes $y(x) = \dfrac{1}{2}\dfrac{a}{U_0^2}x^2 + 1$. This may be plotting and visualized in two-dimensional space of the flow:


y
$y(x) = \frac{1}{2}\frac{a}{U_0^2}x^2 + 1$
$\mathbf{x}_0$
x