Homework 3

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Q1

10 / 10

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MAT334 Problem Set 3 — Due Sunday November 6, 11pm
1006940802

1.

Let $f : D \to \mathbb{C}$, where D is an open domain with $0 \in D$. If f is holomorphic on D, then there exists an open disc $B_r(0) \subseteq D$ where the restriction of f to the disc $B_r(0)$ is also holomorphic. This implies that there exists a power series expansion of f centered at 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

If we differentiate f, then the k-th derivative of f at 0 is determined by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$

and hence $f^{(k)}(0) = k!a_k$, since all other terms vanish upon evaluation. Allow me to strive for a contradiction. If f is holomorphic and $f^{(k)}(0) = (k!)^2$, then $a_k = k!$ which follows from the derivation above. However, if $a_k = k!$, then the power series no longer converges at 0, and it's radius of convergence is simply the singleton $\{0\}$. Applying the ratio test,

Correct 10

$$\begin{aligned}\frac{1}{R} &= \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \\ &= \lim_{k\to\infty} \frac{(k+1)k!}{k!} \\ &= \infty,\end{aligned}$$

hence $R = 0$. This implies that f cannot be holomorphic since the power series expansion no longer converges on an open disc, alas we have obtained a contradiction. Therefore there is no holomorphic function f such that $f^{(k)}(0) = (k!)^2$.

1

Q2

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2.

Consider the function $F : (0, \infty) \setminus \{1\} \to \mathbb{C}$ defined by the integral $F(r) = \int_{C_r} \frac{z + e^z}{z^2(z-i)}$. To evaluate this integral, I will proceed by first invoking partial fraction decomposition to separate the denominator terms and then I will invoke Cauchy's integral formula on each term seperately. Firstly, we wish to determine coefficients A, B and C such that

$$\frac{1}{z^2(z-i)} = \frac{A}{z^2} + \frac{B}{z-i}.$$

Multiplying through,

$$\begin{aligned} 1 &= A(z-i) + Bz^2 \\ &= Bz^2 + Az - Ai \end{aligned}$$

which implies that $A = i$, and thus $B = -\frac{i}{z}$. Then $\frac{1}{z^2(z-i)} = \frac{i}{z^2} - \frac{i}{z(z-i)}$. Applying partial fraction decomposition again to the second term,

$$\begin{aligned} \frac{1}{z(z-i)} = \frac{A'}{z} + \frac{B'}{z-i} &\implies A'z - A'i + B'z = 1 \\ &\implies A' = i,\ B' = -i. \end{aligned}$$

Therefore the integrand is expanded by partial fraction decomposition as

$$\frac{1}{z^2(z-i)} = \frac{i}{z^2} - i\left(\frac{i}{z} - \frac{i}{z-i}\right) = \frac{i}{z^2} + \frac{1}{z} - \frac{1}{z-i}.$$

Now, applying Cauchy's integral formula,

$$F(r) = i\int_{|z|=r} \frac{z+e^z}{z^2}\,dz + \int_{|z|=r} \frac{z+e^z}{z}\,dz - \int_{|z|=r} \frac{z+e^z}{z-i}\,dz.$$

Since $g(z) = z + e^z$ is a holomorphic function on $\mathbb{C}$, then we may apply Cauchy's (Generalized) Integral Formula to each of the three terms. We may observe that whenever $r < 1$, then third integrand vanishes since $i \notin B_1(0)$. Since 0 is an element in any open ball centered at 0, then the first two integrands are non-zero.

- We have that $i\int_{|z|=r} \frac{z+e^z}{z^2}\,dz = i\frac{2\pi i}{1!}\frac{d}{dz}[z+e^z]\big|_{z=0} = -2\pi(1+e^{(0)}) = -4\pi$, by Cauchy's Generalized integral formula.

- Secondly, $\displaystyle\int_{|z|=r} \frac{z+e^z}{z}\, dz = 2\pi i(0+e^0) = 2\pi i.$
- Lastly, $\displaystyle\int_{|z|=r} \frac{z+e^z}{z-i}\, dz = \begin{cases} 0 & \text{if } r<1 \\ 2\pi i(i+e^i) & \text{if } r>1 \end{cases}$ since Cauchy's integral formula applies only when $i \in \mathrm{In}(C_r)$, and is zero otherwise.

Therefore, summing each of the results of each of the three terms above, we obtain that $F(r) = -4\pi + 2\pi i = 2\pi(i-2)$ for $r<1$, and $F(r) = -4\pi + 2\pi i - 2\pi + 2\pi i e^i = 2\pi(i-3+ie^i)$ for $r>1$. Alas

$$F(r) = \begin{cases} 2\pi(i-2) & \text{if } r<1 \\ 2\pi(i-3+ie^i) & \end{cases}$$

Arithmetic error. -1

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Q3

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3.

Proof.

- Assume that $f : \mathbb{C} \to \{z \in \mathbb{C} : \mathbb{R}\mathrm{e}(z) > 0\}$ is an entire function. Consider the function composition $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = e^{-f(z)}$. Since $f(z)$ is entire and e^{-z} is also an entire function, then $g(z)$ is an entire function.

correct **10**

- Allow me to write f as a decomposition of real and complex components $f = u + iv$, and thus $g(z) = e^{-u-iv}$. If range$(f) \subset \{z \in \mathbb{C} : \mathbb{R}\text{e}(z) > 0\}$, then it must be that $u > 0$ since u represents the real component of f.
- Now, $|g(z)| = |e^{-u-iv}| = |e^{-u}| < e^0 = 1$, since $u > 0$, hence $g(z)$ is bounded above by a constant $M = 1 > 0$. By Liouville's theorem, $g(z)$ must be constant since g is entire and bounded.
- If $g(z)$ is constant, then it must be that $g'(z) = 0$: $\dfrac{d}{dz}[e^{-f(z)}] = -e^{-f(z)} f'(z) = 0$. However, since $e^{-f(z)}$ can never be zero, then it must be that $f'(z) = 0$.
- Thus $f'(z) = 0$, and therefore f must be constant as well, which is what I wanted to prove.

$\blacksquare$

3

Q4

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4.

For this question, consider the function f given by $f(z) = \dfrac{z^2 - z}{2z^2 - z - 1} + z^2 \sin\left(\dfrac{1}{z}\right)$.

(a) To determine the singularities of f and their types, I may begin by factoring the polynomial in the denominator of the first term, then expanding each term as a power series centered around each respective singularity. First note that $2z^2 - z - 1 = (z-1)(2z+1)$. It is now easier to observe that the singularities of f are $z = 1$, $z = -\dfrac{1}{2}$, and $z = 0$ due to the argument of the sin term. Then, by factoring,

$$f(z) = \frac{z^2 - z}{(z-1)(2z+1)} + z^2 \sin\left(\frac{1}{z}\right).$$

Theorem: For a Laurant series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$:

(i) If $a_n = 0$ for all $n < 0$, then the point z_0 constitutes a removable singularity.

(ii) If there exists an m such that $a_n = 0$ for all $n < -m$, then z_0 constitutes a pole of order m.

(iii) If there are infinitely many $a_n \neq 0$ for $n < 0$, then the point z_0 corresponds to an essential singularity.

**This theorem was proved in class.* To determine the types of these three singularites, I may proceed by analyzing the Laurant series of the expansion around each of these points, then apply the theorem to each z_0. For $f(z)$, we may examine each term separately then determine each of the singularity types individually.

$z_0 = 1:$ Allow me to write $z^2 - z$ in terms of a power series centered at $z_0 = 1$:

$$z^2 - z = A + B(z-1) + C(z-1)^2 = (A - B + C) + (B - 2C)z + Cz^2$$
$$\implies C = 1, \implies B = 1, \implies A = 0.$$

Therefore

$$\frac{z^2 - z}{(z-1)(2z+1)} = \frac{z-1}{(z-1)(2z+1)} + \frac{(z-1)^2}{(z-1)(2z+1)} = \frac{1}{2z+1} + \frac{z-1}{2z+1}.$$

As a series, we just have $\frac{1}{2z+1}\sum_{n=0}^{1}(z-1)^n$. Now, since the coefficients of the Laurant Series centered around $z_0 = 1$ are only a_0 and a_1, then $\boldsymbol{z_0 = 1}$ **is a removable singularity.**

$z_0 = -\frac{1}{2}:$ As in the previous case, allow me to begin by expanding $z^2 - z$ in terms of a power series centered at $z_0 = -\frac{1}{2}$:

$$z^2 - z = A + B(z + 1/2) + C(z + 1/2)^2 = (A + B/2 + C/4) + (B + C)z + Cz^2$$
$$\implies C = 1, \implies B = -2, \implies A = 3/4.$$

4

Expanding as a power series in terms of $(z - 1/2)$, then

$$\frac{z^2 - z}{(z-1)(2z+1)} = \frac{3}{8(z-1)(z+1/2)} - \frac{(z+1/2)}{(z-1)(z+1/2)} + \frac{(z+1/2)^2}{2(z-1)(z+1/2)}$$
$$= \frac{3}{8(z-1)(z+1/2)} - \frac{1}{(z-1)} + \frac{(z+1/2)}{2(z-1)}.$$

It is not difficult to see that there are infinitely many $a_n = 0$ for $n < -1$, since the lowest index is portrayed by the $\frac{}{z+1/2}$ (the $n=-1$) term. Thus $\boldsymbol{z_0 = -\frac{}{2}}$ **is a pole of order** $\boldsymbol{m=1}$.

$z_0 = 0:$ For the $z = 0$ singularity, consider the second term $z^2 \sin\left(\frac{1}{z}\right)$. To determine the type of singularity, I will determine a Laurant Series expansion of this term. The power series expansion of $\sin\left(\frac{1}{z}\right)$ is given by $\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\frac{1}{z^{2n+1}}$. This implies, by multiplying through z^2,

$$\begin{aligned} z^2 \sin\left(\frac{1}{z}\right) &= \sum_{n=0}^{\infty}(-1)^n \frac{z^2}{(2n+1)!}\frac{1}{z^{2n+1}} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\frac{1}{z^{2n-1}} \\ &= \sum_{n=-\infty}^{0}(-1)^n \frac{1}{(2|n|+1)!} z^{2n+1}, \end{aligned}$$

where the last step is just determined by changing the index of the sum into a Laurant sum from $-\infty$ to 0. In this representation, it is obvious that there exist infinitely many coefficients $a_n \neq 0$ with $n < 0$, which then implies that $\boldsymbol{z_0 = 0}$ **is an essential singularity**.

(b) I now wish to determine $\int_{|z|=2} f(z)\,dz$, which can be determined by an appropriate application of the residue theorem. In this problem I will be utilizing the Laurant Series expansions of f which I determined in the previous part of this question. First off, by the linearity of the integral,

$$\int_{|z|=2} f(z)\,dz = \int_{|z|=2} \frac{z^2 - z}{(z-1)(2z+1)}\,dz + \int_{|z|=2} z^2 \sin\left(\frac{1}{z}\right)\,dz.$$

If I let $g(z) = \frac{z^2 - z}{(z-1)(2z+1)}$ and $h(z) = z^2 \sin\left(\frac{1}{z}\right)$, then since each singularity $0, -1/2, 1$ is located inside $\mathrm{In}(B_2(0))$, the residue theorem yields that

$$\int_{|z|=2} f(z)\,dz = 2\pi i\,[\mathrm{Res}(g,1) + \mathrm{Res}(g,-1/2)] + 2\pi i\,\mathrm{Res}(h,0).$$

5

This follows from the fact that both g and h are analytic on their domains. Now, since the residue at a point z_0 is just the value of the a_{-1} coefficient of the Laurant series of each function evaluated at the z_0 singularity, then

$$\begin{aligned} \mathrm{Res}(g,1) &= 0 \qquad \text{(the } a_{-1} \text{ term is zero)} \\ \mathrm{Res}(g,-1/2) &= \frac{3}{8(-1/2-1)} = -\frac{3\cdot 2}{8\cdot 3} = -\frac{1}{4} \end{aligned}$$

$$\mathrm{Res}(h,0) = -\frac{1}{3!} = -\frac{1}{6},$$

which again are just simply the coefficients of the $n = -1$ term in each Laurant series expansion. Therefore

$$\begin{aligned}\int_{|z|=2} f(z)\,dz &= 2\pi i\left(0 - \frac{1}{4}\right) + 2\pi i\left(-\frac{1}{6}\right)\\ &= -\pi i\left(\frac{1}{2} + \frac{1}{3}\right)\\ &= -\frac{5}{6}\pi i,\end{aligned}$$

which is the value of the integral I wished to determine.

Q5

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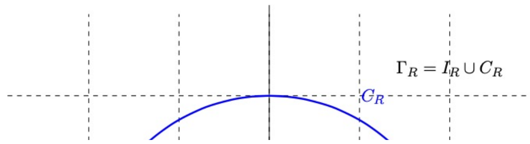
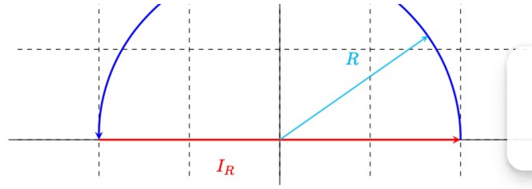
5.

Evaluate $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+1}\,dx$. For this problem, I will define a simple closed curve Γ_R consisting of an interval line lying on the real-axis I_R, and a semicircle counterclockwise, positively oriented curve C_R:





Correct contour. **1**

Now, instead of our initial integral, consider the integral defined by $\int_{\Gamma_R} g(z)\,dz$ in the limit where $R \to \infty$, where $g(z)$ is the function defined by $g(z) = \frac{e^{iz}}{z^4+1}$. Then, it is easy to see that

Correct integrand. **2**

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+1}\,dx = \mathbb{R}\mathrm{e}\left\{\lim_{R\to\infty} \int_{I_R} \frac{e^{iz}}{z^4+1}\,dz\right\},$$

where the integral is taken over I_R and not all of Γ_R. I will return to this later, but for now I will evaluate $\int_{\Gamma_R} g(z)\,dz$.

Allow me to begin by factoring the denominator. By the fundamental theorem of algebra, $z^4 = -1$ has four solutions. These are not hard to determine and I will not take up more space finding a simple solution. They are $z = \pm\sqrt{i}$ and $z = \pm i\sqrt{i}$. By factoring, $g(z)$ then becomes

$$\frac{e^{iz}}{z^4+1} = \frac{e^{iz}}{(z-\sqrt{i})(z+\sqrt{i})(z-i\sqrt{i})(z+i\sqrt{i})}.$$

$g(z)$ has now four easily identifyable singularities, all of which are poles of order 1. It is imporant to notice that in the limit as $R \to \infty$, the only singularities located inside $\mathrm{In}(\Gamma_R)$ are the ones whose imaginary components is positive: $z = \sqrt{i}$ and $z = i\sqrt{i}$. This is determined in the following way:

Correct singularities. **1**

$$\sqrt{i} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$i\sqrt{i} = i\left(e^{i\pi/2}\right)^{1/2} = ie^{i\pi/4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

and is easy to show that $-\sqrt{i}, -i\sqrt{i}$ both have negative imaginary parts (just flip the sign). Therefore, by the residue theorem, since $g(z)$ is entire on $\mathbb{C} \setminus \{\sqrt{i}, -\sqrt{i}, i\sqrt{i}, -i\sqrt{i}\}$, we have that

$$\int_{\Gamma_R} g(z)\,dz = 2\pi i\left[\mathrm{Res}(g,\sqrt{i}) + \mathrm{Res}(g, i\sqrt{i})\right].$$

Since each singularity is a pole of order 1, its respective residue is determined by $\lim_{z\to z_0} \frac{d^0}{dz^0}[(z-z_0)g(z)]$:

Correct appli- cation of the

residue theorem. 1

$$
\begin{aligned}
\operatorname{Res}(g,\sqrt{i}) &= \lim_{z\to\sqrt{i}} \frac{(z-\sqrt{i})e^{iz}}{(z-\sqrt{i})(z+\sqrt{i})(z-i\sqrt{i})(z+i\sqrt{i})} \\
&= \lim_{z\to\sqrt{i}} \frac{e^{iz}}{(z+\sqrt{i})(z-i\sqrt{i})(z+i\sqrt{i})} \\
&= \frac{e^{i\sqrt{i}}}{(2\sqrt{i})(\sqrt{i}(1-i))(\sqrt{i}(1+i))} \\
&= \frac{e^{i\sqrt{i}}}{4i\sqrt{i}}
\end{aligned}
$$

Mostly correct calculations for the residues. 2

$$
\begin{aligned}
\operatorname{Res}(g,i\sqrt{i}) &= \lim_{z\to i\sqrt{i}} \frac{(z-i\sqrt{i})e^{iz}}{(z-\sqrt{i})(z+\sqrt{i})(z-i\sqrt{i})(z+i\sqrt{i})} \\
&= \lim_{z\to i\sqrt{i}} \frac{e^{iz}}{(z-\sqrt{i})(z+\sqrt{i})(z+i\sqrt{i})} \\
&= \frac{e^{-\sqrt{i}}}{(\sqrt{i}(i-1))(\sqrt{i}(i+1))(2i\sqrt{i})} \\
&= \frac{e^{-\sqrt{i}}}{4\sqrt{i}}.
\end{aligned}
$$

Therefore $\int_{\Gamma_R} g(z)\,dz = 2\pi i\left[\frac{e^{i\sqrt{i}}}{4i\sqrt{i}} + \frac{e^{-\sqrt{i}}}{4\sqrt{i}}\right]$ which, I should note, holds true for any $R > \frac{1}{\sqrt{2}}$ (this doesn't necessarily matter, since we are taking $R \to \infty$ in the end anyway). Simplifying,

$$
\begin{aligned}
2\pi i\left[\frac{e^{i\sqrt{i}}}{4i\sqrt{i}} + \frac{e^{-\sqrt{i}}}{4\sqrt{i}}\right] &= \frac{\pi e^{i\sqrt{i}}}{2\sqrt{i}} + \frac{\pi i e^{-\sqrt{i}}}{2\sqrt{i}} \\
&= -i\sqrt{i}\frac{\pi}{2}\left[e^{i\sqrt{i}} + ie^{-\sqrt{i}}\right] \\
&= \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\frac{\pi}{2}\left[e^{-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} + e^{-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}+\frac{i\pi}{2}}\right] \\
&= \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\frac{\pi}{2}e^{-\frac{1}{\sqrt{2}}}\left[e^{\frac{i}{\sqrt{2}}} + e^{i\left(\frac{\pi}{2}-\frac{1}{\sqrt{2}}\right)}\right].
\end{aligned}
$$

Now, consider the integral over the curve C_R: $\int_{C_R} g(z)\,dz$. By ML-Estimation, it must be that

$$
\begin{aligned}
\left|\int_{C_R} g(z)\,dz\right| &\le \max_{z\in C_R}\{|g(z)|\}\cdot \text{Length}(C_R) \\
&= \pi R \max_{z\in C_R}\left\{\left|\frac{e^{iz}}{z^4+1}\right|\right\} \\
&\le \pi R \max_{z\in C_R}\left\{\left|\frac{1}{z^4+1}\right|\right\} && \text{(since } |e^{iz}| = e^{-y} \le 1,\ y = \mathbb{Im}(z) \ge 0 \text{ on } C_R) \\
&\le \pi R \max_{z\in C_R}\left\{\frac{1}{|z|^4-1}\right\} && \text{(by triangle inequality)}
\end{aligned}
$$

$$= \frac{\pi R}{R^4 - 1}. \qquad \text{(since } |z| = R\text{)}$$

However, as $R \to \infty$, the term $\dfrac{\pi R}{R^4-1} \to 0$, and therefore the integral around C_R is zero as R becomes infinitely large:

Correct bounds for the arc. 3

$$\lim_{R\to\infty} \int_{C_R} g(z)\,dz = 0.$$

This now has huge implications on our initial integral! Note that, since Γ_R is a piecewise curve, then by linearity

$$\int_{\Gamma_R} g(z)\,dz = \int_{I_R} g(z)\,dz + \int_{C_R} g(z)\,dz$$

hence $\lim\limits_{R\to\infty} \int_{\Gamma_R} g(z)\,dz = \lim\limits_{R\to\infty} \int_{I_R} g(z)\,dz + 0$. However, the integral $\lim\limits_{R\to\infty} \int_{\Gamma_R} g(z)\,dz$ has already been determined via the Residue theorem. Alas, by our initial claim,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+1}\,dx &= \mathbb{R}\mathrm{e}\left\{\lim_{R\to\infty} \int_{I_R} \frac{e^{iz}}{z^4+1}\,dz\right\} = \mathbb{R}\mathrm{e}\left\{\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{z^4+1}\,dz\right\} \\
&= \mathbb{R}\mathrm{e}\left\{\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}}\left[e^{\frac{i}{\sqrt{2}}} + e^{i\left(\frac{\pi}{2} - \frac{1}{\sqrt{2}}\right)}\right]\right\} \\
&= \mathbb{R}\mathrm{e}\left\{\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}}\left[\cos\left(\frac{1}{\sqrt{2}}\right) + i\sin\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right) + i\cos\left(\frac{1}{\sqrt{2}}\right)\right]\right\} \\
&= \frac{\pi}{2\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}\left[2\cos\left(\frac{1}{\sqrt{2}}\right) + 2\sin\left(\frac{1}{\sqrt{2}}\right)\right].
\end{aligned}$$

Therefore the value of the integral is

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+1}\,dx = \frac{\pi e^{-\frac{1}{\sqrt{2}}}}{\sqrt{2}}\left[\cos\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)\right],$$

which is what I wanted to determine.