JPE395 Problem Set 4

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Problem 1

We begin by considering a two-layered planet, of core radius r_c and outer radius R. By definition, the gravitational acceleration for a planet in terms of its density is given by

$$g(R) = \frac{4\pi G}{R^2} \int_0^R dr' \, \rho(r') r'^{\,2} \tag{1.1}$$

where G is the gravitational constant and R is the radius of the planet. This integral expression may be broken up into multiple components for planets with multiple layers. For our planet, for a density given by

$$\rho(r) = \begin{cases} \rho_c & 0 \leq r \leq r_c \\ \rho_m = \dfrac{R}{r}\rho_0 & r_c \leq r \leq R \end{cases} . \tag{1.2}$$

Upon substitution of equation (1.2) into (1.1), we may evaluate the integral and determine that the gravitational acceleration is

$$\begin{aligned} g(R) &= \frac{4\pi G}{R^2} \left[\int_0^{r_c} dr' \, \rho_c r'^{\,2} + R\rho_0 \int_{r_c}^R dr' \, r' \right] \\ &= \frac{4\pi G}{R^2} \left[\frac{1}{3}\rho_c r_c^3 + \frac{1}{2} R\rho_0 \left(R^2 - r_c^2 \right) \right] \\ &= \frac{2\pi G}{3R^2} \left[2r_c^3 \rho_c + 3R\rho_0 \left(R^2 - r_c^2 \right) \right] . \end{aligned} \tag{1.3}$$

For a given set of variables, such as the surface value of the gravitational acceleration $g(R)$, the core and planet radius, and the internal mantle density ρ_0, one may rearrange Equation (1.3) to solve for the core density ρ_c:

$$\rho_c = \frac{3}{2r_c^3} \left\{ \frac{g(R)R^2}{2\pi G} - R\rho_0 (R^3 - r_c^3) \right\} . \tag{1.4}$$

For $g(R) = 9.81\,\mathrm{ms}^{-2}$, $\rho_0 = 2000\,\mathrm{kgm}^{-3}$, $R = 6\,371\,000\mathrm{m}$ and $r_c = 3\,471\,000\mathrm{m}$, we find that the density of the core is $\rho_c = 21\,013\,\mathrm{kg\,m}^{-3}$. This was computed using Python.

Problem 2

(**a**) For this problem, we desire to determine the moment of inertia for a two-layered planet. By definition, the moment of inertia is defined as $I = \int_V dm\, r_\perp^2$, where $r_\perp$ is the radial distance to the mass element dm and the rotation axis. By symmetry, the moment of inertia is equivalent for any axis passing through the center of the sphere, and therefore $dm = dr\, d\varphi\, d\theta\, r^2 \sin\theta$ and $r_\perp = r\sin\theta$. For an inner density ρ_c for $0 \le r \le r_c$ and ρ_m for $r_c \le r \le R$, the moment of inertia integral becomes seperable by linearity:

$$
\begin{aligned}
I &= \int_V dm\, r^2 \sin^2\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} dr\, d\theta\, d\varphi\, r^4 \sin^3\theta \rho \\
&= 2\pi \int_0^{\pi} d\theta\, \sin^3\theta \left[\int_0^{r_c} dr\, r^4 \rho_c + \int_{r_c}^{R} dr\, r^4 \rho_m\right] \\
&= 2\pi \cdot \frac{4}{3} \cdot \frac{1}{5}\left[\rho_c r_c^5 + \rho_m (R^5 - r_c^5)\right] \\
&= \frac{8\pi}{15}\left[(\rho_c - \rho_m) r_c^5 + \rho_m R^5\right]. \qquad (2.1)
\end{aligned}
$$

By a similar calculation, the mass of the planet can be determined to calculate the ratio $\frac{I}{MR^2}$:

$$
\begin{aligned}
M &= \int_V dm \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} dr\, d\theta\, d\varphi\, r^2 \sin\theta \rho \\
&= 2\pi \int_0^{\pi} d\theta\, \sin\theta \left[\int_0^{r_c} dr\, r^2 \rho_c + \int_{r_c}^{R} dr\, r^2 \rho_m\right] \\
&= 2\pi \cdot 2 \cdot \frac{1}{3}\left[\rho_c r_c^3 + \rho_m (R^3 - r_c^3)\right] \\
&= \frac{4\pi}{3}\left[(\rho_c - \rho_m) r_c^3 + \rho_m R^3\right]. \qquad (2.2)
\end{aligned}
$$

Upon substitution into the expression $\frac{I}{MR^2}$,

$$
\begin{aligned}
\frac{I}{MR^2} &= \frac{8\pi/15}{4\pi/3}\frac{(\rho_c - \rho_m) r_c^5 + \rho_m R^5}{[(\rho_c - \rho_m) r_c^3 + \rho_m R^3]R^2} \\
&= \frac{2}{5}\frac{(\rho_c - \rho_m)(r_c^5/(\rho_m R^5)) + 1}{(\rho_c - \rho_m)(r_c^3/(\rho_m R^3) + 1)} \\
&= \frac{2}{5}\frac{(\rho_c/\rho_m - 1)\,(r_c/R)^5 + 1}{(\rho_c/\rho_m - 1)\,(r_c/R)^3 + 1}. \qquad (2.3)
\end{aligned}
$$

Note that, upon setting $\rho_m = \rho_c$, the expression for the moment of inertia of a sphere reduces to $\frac{I}{MR^2} = \frac{2}{5}$, which is the expected coefficient for a sphere. For a multi-layered planet, the dimensionless moment of inertia expression given in Equation (2.3) is as desired.

(**b**) The following data in this problem was taken from https://solarsystem.nasa.gov/moons/earths-moon/in-depth/ for the moon and Mercury. For the remainder of this problem, let me simplify Equation (2.3) (and the equations which follow) by introducing the substitutions

$$a = \frac{5}{2}\frac{I}{MR^2}, \qquad b = \frac{\rho_c}{\rho_m} - 1, \qquad x = \frac{r_c}{R}. \tag{2.4}$$

Then, Equation (2.3) becomes

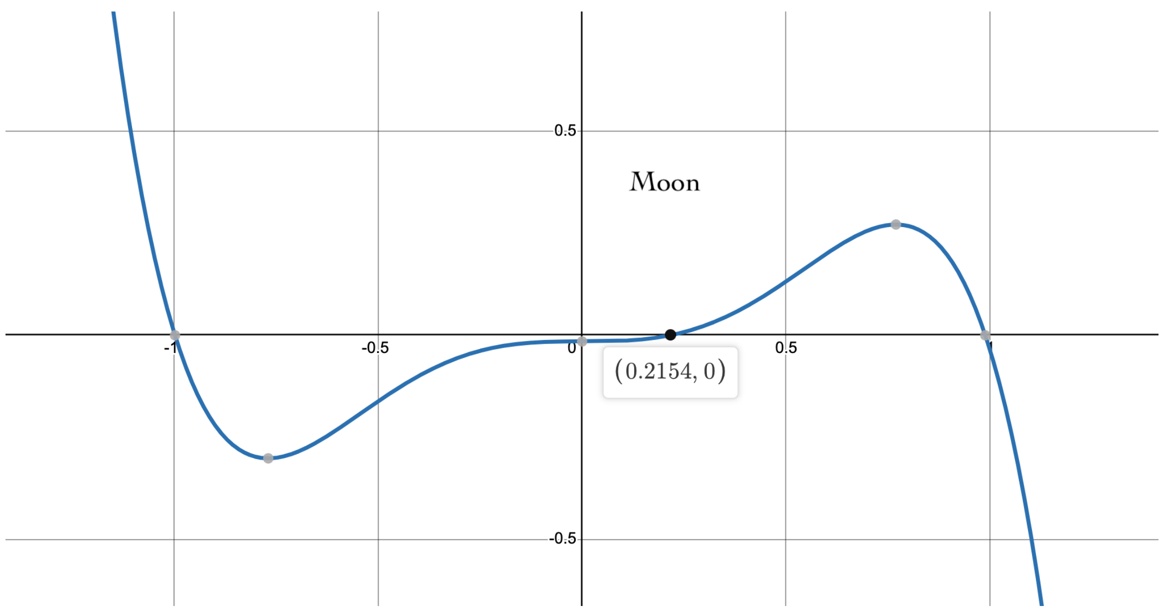
$$\begin{aligned} a &= \frac{bx^5 + 1}{bx^3 + 1} \\ \implies 0 &= -bx^5 + abx^3 + a - 1. \end{aligned} \tag{2.5}$$

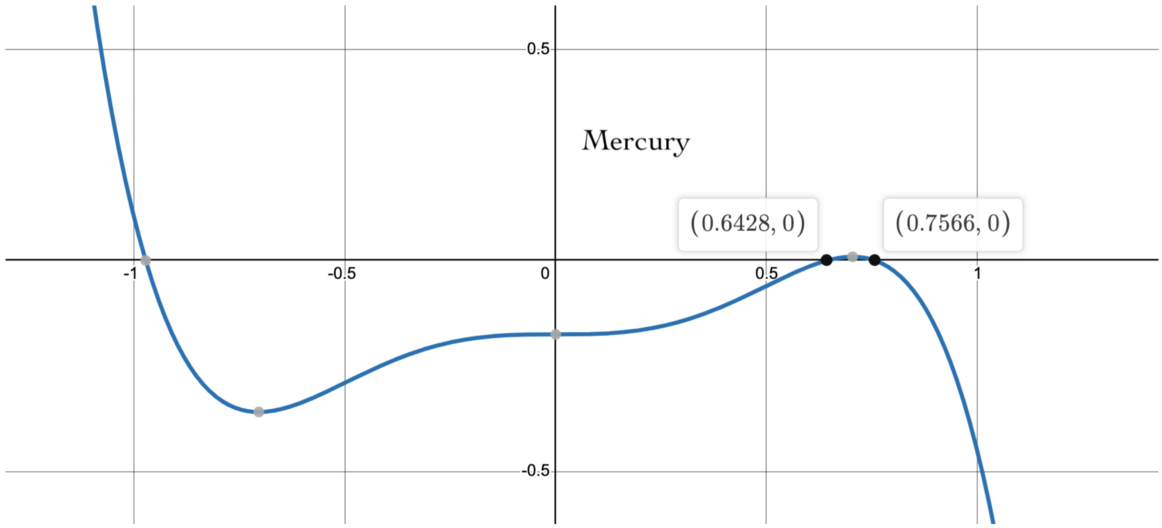
As taken from the NASA website, the values of core radius and planet/moon radius for Mercury and the mooon are

$$\text{Mercury: } \begin{cases} R = 2\,439\,700\,\text{m} \\ r_c = 2\,047\,000\,\text{m} \end{cases} \implies r_c/R \approx 0.8390 \tag{2.6}$$

$$\text{Moon: } \begin{cases} R = 1\,740\,000\,\text{m} \\ r_c = 330\,000 - 489\,000\,\text{m} \end{cases} \implies r_c/R \approx 0.1897 - 0.2810 \tag{2.7}$$

Note that NASA provided a range of values for the core radius. Furthermore, assuming the density distribution inside both masses are equivalent, with $\rho_c = 7\,800$ kg m^{-3} and $\rho_m = 3\,000$ kg m^{-3}, we may determine and compare the computed values with expected values. Since this is a quintic polynomial, it may not be entirely factored analytically, so I used Desmos to determine the solutions of x to compare with the values given by (2.6), (2.7):





Observe that the value of r_c/R for the moon is within the expected range as defined in Equation (2.7), so this agrees with the expected value. However, upon examination of Mercury's computed value for r_c/R, it appears as though it is much less than the expected value outlined in Equation (2.6). Since the values of r_c/R for Mercury were taken from a credible source, it is expected that the value of 0.33 for I/MR^2 to be slightly lower than actual. By 'playing around' with the I/MR^2 value (in Desmos), it was determined that the closer-to-expected value for Mercury to be 0.35, which implies a greater density distribution of the core.

Problem 3

(**a**) Consider the gravitational potential as derived in lecture,

$$V_g(r,\theta) = -\frac{GM_E}{r^2}\left[1 - J_2\left(\frac{R_e}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right)\right]. \tag{3.1}$$

First, note that the second term in (3.1) described the potential deformation of the oblate spheriod defined by the second-order Legendre polynomial, $P_2(\theta)$. Because of this deformation and the symmetry of $P_2(\theta)$, there must exist at least two points θ_0 where the gravitational potential conforms to that of a spherical potential, ie. the values of θ when $P_2(\theta) = 0$ so that $V_g(r,\theta) = -\frac{GM_E}{r^2}$. By quick rearrangement, we find that

$$\cos^2\theta = \frac{1}{3} \implies \theta = \arccos\left(\pm\frac{1}{\sqrt{3}}\right). \tag{3.2}$$

Distinctly, these θ values (in radians) are 0.955 and 2.186. In degrees, $\theta = 54.7°, 125.3°$. Note that, while converted back to latitudal coordinates, these angles convert to $\theta = \pm 35.3°$, which is symmetric about the equatorial line, as expected.

(**b**) To determine the equatorial bulge, we may utilize the assumption that the spheroid deformation is proportional to $P_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1)$. In this assumption, we may defined a deformation function $d(\theta) = \frac{A}{2}(3\cos^2\theta - 1) + B$, where A, B are constants to be determined by intial boundary conditions on the function. Note that, at $\theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$, $d = 0$, which implies that $B = 0$. When $\theta = 0$ (the northern/southern poles), the deformation is given to be $d_p \approx -14$ km (this is inward, towards the sphere center). Then, since $\cos(0) = 1$, we have that $d(0) = d_p = \frac{A}{2}(3-1) = A$, and therefore

$$d(\theta) = \frac{d_p}{2}(3\cos^2\theta - 1). \tag{3.3}$$

Evaluating $\theta = \frac{\pi}{2}$ (at the equatorial line), we may determine the bulging distance of the planet. First, note that $\cos^2(\pi/2) = 0$, and therefore $d(\pi/2) = -\frac{d_p}{2}$, making the bulging distance at the equator approximately 7 km.

Problem 4

(**a**) This question was solved primarily using Python. First, to determine the free-air anomaly and Bouguer anomaly of all five data points, the following relations were used:

$$g_F = g_{\text{obs}} - g_0\left(1 - \frac{2h}{r_0}\right) \tag{4.1}$$

$$g_B = g_{\text{obs}} - g_0\left(1 - \frac{2h}{r_0}\right) - \delta g_B \tag{4.2}$$

where r_0 is the radius of the reference ellipsoid, given by

$$r_0 = R_e\left[1 + \frac{2f - f^2}{(1-f)^2}\cos^2\theta\right]^{-1/2}. \tag{4.3}$$

In Equation (4.3), f is the flattening parameter, given by $f = \frac{1}{298.247}$. The values of h in Equations (4.1), (4.2) are the ice thicknesses, measured from a height from the reference spheroid (Equation (4.3)). Lastly, δg_B is the Bouguer anomaly, given by

$$\delta g_B = 2\pi G \int_0^{h'} \rho \, dy. \tag{4.4}$$

It is important to note that, h' is Equation (4.4) is different from the reference ellipsoid height h in Equations (4.1)/(4.2), and instead is the thickness of the 'hill' / mass distribution. h was taken to be the measured reference height, $h = 1\,650$ m, θ was taken to be the colatitude value of the site (in radians) $\theta = 90^\circ + 83^\circ \approx 3.019$ rad, and g_0 was determined by θ in the relation

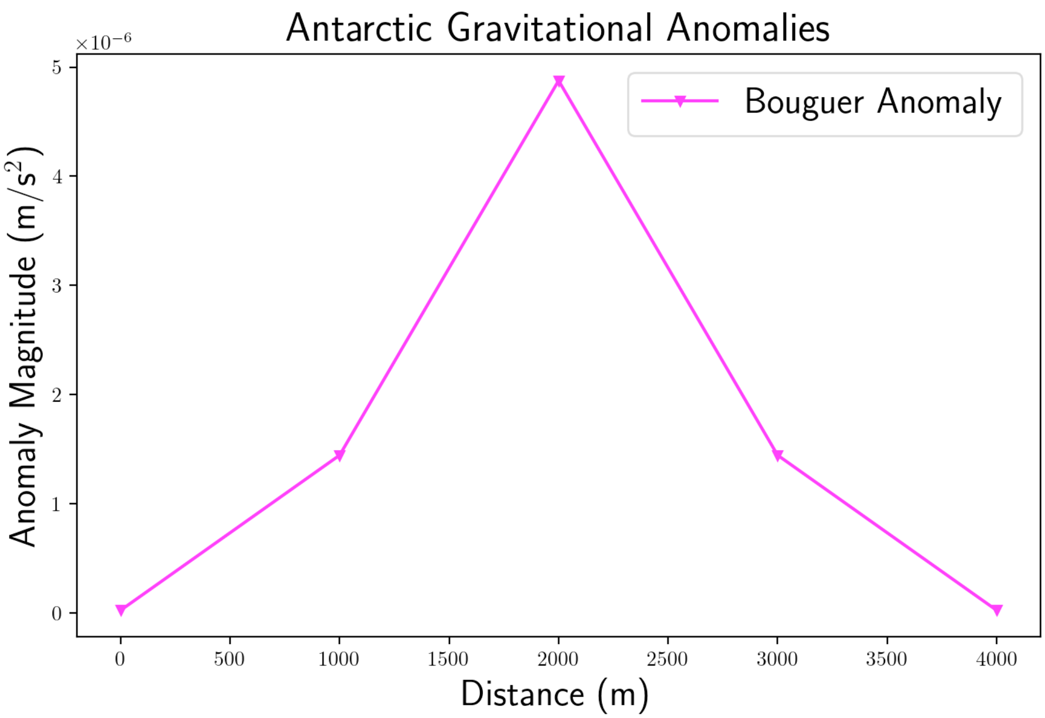
$$g_0 = 9.7803267715(1 + 0.0052790414\cos^2\theta). \tag{4.5}$$

For the Bouguer anomaly calculation, h' was taken to be h since this represented the 'hill height', ie. the thickness of the ice from the reference spheroid. The g_F and g_B values were calculated and arranged in a table:

Data	Position (km)	g_{obs} (m/s^2)	g_F (m/s^2)	g_B (m/s^2)
1	0	9.82677911	6.920×10^{-4}	2.593×10^{-8}
2	1	9.82678053	6.934×10^{-4}	1.446×10^{-6}
3	2	9.82678396	6.968×10^{-4}	4.876×10^{-6}
4	3	9.82678053	6.934×10^{-4}	1.450×10^{-6}
5	4	9.82677911	6.920×10^{-4}	2.593×10^{-8}

It is important to notice the symmetry in the datapoints around the measurement distances.

(**b**) The data processed from part (a) may be plotted for the Bouguer anomaly in terms of the position measurement. This will show a visual depiction of the anomaly detected below the ice as well as it's relative size. Once again, this was plotted using Python.



First note the positive correlation in the plot, ie. the gravitational anomaly is positive. This implies that the mass located underneath the ice is heavier than usual, which implies a denser material. Therefore there should be a mass excess beneath the ice.

(c) From part (b), we know that the Bouguer anomaly indicates a mass excess beneath the ice. This may only be due to iron underneath the ice, since the density of iron is approximately 1.3 times that of ice ($\rho_{\text{ice}} = 1000$ kg m^{-3}, $\rho_{\text{iron}} = 1300$ kg m^{-3}). Note that, upon noting a maximum drilling a depth of 2650 m, the minimum radius for economic viability of iron is about 500 m. This implies a minimum drilling depth of 1650 m, and a drilling depth of 2150 m to the center of the anomaly, and these are the plausible drilling depths for the iron located underneath the ice.
From part (b), one may determine the average falloff (slope) from the maximum and minimum data point to be $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4.876 \times 10^{-6} - 2.593 \times 10^{-8}}{2000 - 0} \approx 2 \times 10^{-9}$ s^{-2}. Using this information, and assuming the minimum size of $500m$, we find that the minimum gravitational Bouguer anomaly at the center (250m mark) to be $250 \cdot m = 6.06 \times 10^{-7}$ m s^{-2}. Compared with the maximum value determined in part (a), $g_{B,\text{max}} = 4.876 \times 10^{-6}$ m s^{-2}, it may be concluded that this source of iron is in fact economically viable in terms of profit.