mt1

My score

86% (43/50)

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mt1-9e618

#75 2 of 10

1. Determine whether the following limits exist. If they do, find their value. Justify your answers.

(a) (5 points)

$$\lim_{z\to 0} \frac{|z|^3}{z^2}$$

Consider an expression of z to be as $z = re^{i\theta}$. Then, as $z \to 0$, $r \to 0$ since $e^{i\theta}$ has no orientation, (r is outward)

Then $\lim_{z\to 0} \frac{|z|^3}{z^2} = \lim_{r\to 0} \frac{|re^{i\theta}|^3}{r^2 e^{i\cdot 2\theta}}$

$= \lim_{r\to 0} \frac{r^3 \cdot 1^3}{r^2 e^{2i\theta}}$

$= \lim_{r\to 0} \frac{r}{e^{i2\theta}}$

$\boxed{= 0}$

Q1a 5

Correct. 5

(b) (5 points)

$$\lim_{z\to i} \frac{z - i}{\bar{z}^2 + 1}$$

Now consider a similar approach.

As $z \to i$, $re^{i\theta} = z$ only [illegible] $r \to 1$. Then

This is not true.

$\lim_{r\to 1} \frac{re^{i\pi/2} - i}{(re^{-i\pi/2})^2 + 1} = \lim_{r\to 1} \frac{r\cdot i - i}{r^2 e^{-i\pi} + 1} = \lim_{r\to 1} \frac{i(r-1)}{-r^2+1}$.

These steps follow by the fact that $e^{-i\pi} = -1$ and $e^{i\pi/2} = i$.

Then $i \lim_{r\to 1} -\left(\frac{r-1}{r^2-1}\right) = -i \lim_{r\to 1} \left(\frac{\cancel{r-1}}{(r+1)\cancel{(r-1)}}\right) = \boxed{-\frac{i}{2}}$.

The limit does not exist.

Q1b 0

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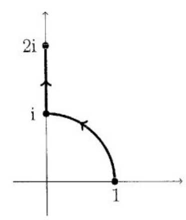
mt1-9e618

#75 3 of 10

Q2 10

2. (10 points) Let γ be the piecewise curve that consists of the union of the following two components: first, a quarter circle of radius 1 centered at the origin from 1 to i, and second, the straight line segment from i to 2i. Find the value of

$$\int_\gamma \mathrm{Im}(z)\mathrm{d}z$$



We must begin by parametrizing each line segment.

The first, given by $\gamma_1(t) = e^{it}$ for $0 \le t \le \frac{\pi}{2}$.

The second, $\gamma_2(t) = [i](1-t) + [2i]t$ for $0 \le t \le 1$.

By linearity of the integral across piecewise curves,

$$\int_\gamma \mathrm{Im}(z)\,dz = \int_{\gamma_1} \mathrm{Im}(z)\,dz + \int_{\gamma_2} \mathrm{Im}(z)\,dz.$$

The function we are integrating is defined by $f: \mathbb{C} \to \mathbb{R}$, $f(z) = \mathrm{Im}(z)$.

The derivatives: $\gamma_1'(t) = ie^{it}$, $\gamma_2'(t) = \frac{d}{dt}\left(i(1-t) + 2it\right) = -i + 2i = i$.

Then:

$$\int_0^{\pi/2} \mathrm{Im}(e^{it})\, ie^{it}\,dt + \int_0^1 \mathrm{Im}(i(1-t) + 2it)\cdot i\,dt$$

$$= \int_0^{\pi/2} \sin(t)\cdot i\cdot e^{it}\,dt + \int_0^1 ((1-t) + 2t)\,i\,dt$$

$$= \int_0^{\pi/2} \frac{e^{it} - e^{-it}}{2}\, e^{it}\,dt + i\left[t - \frac{1}{2}t^2 + t^2\right]_0^1$$

$$= \int_0^{\pi/2} \frac{1}{2}\cdot\left(e^{2it} - e^0\right)dt + i\left[1 - \frac{1}{2} + 1 - 0\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} e^{2it} - 1\,dt + \frac{3i}{2}$$

$$= \frac{1}{2}\left[\frac{1}{2i}e^{2it} - t\right]_0^{\pi/2} + \frac{3}{2}i$$

$$= \frac{1}{2}\left[\frac{1}{2}\cdot(-1) - \frac{\pi}{2} - \frac{1}{2} + 0\right] + \frac{3}{2}i = \frac{1}{4}$$

Parametrizing the quarter circle arc. 2

the line segment. 1

integral for the quarter circle arc using the parametrization. 2

Therefore $\boxed{\int_\gamma \ln(z)\,dz = 2i - \frac{\pi}{4}}$

using the parametrization. 2

ment) 1

Q3 6

3. (6 points) Find the radius of convergence of the following power series,

$$\sum_{n=0}^{\infty} \frac{(n+\mathrm{i})^5}{(1+\mathrm{i})^n} z^{3n}$$

Justify your answer.

Since our power series contains z^{3n}, we must substitute $u = z^3$ and find the radius of convergence for the new series $\sum_{n=0}^{\infty} \frac{(n+i)^5}{(1+i)^n} u^n$.

Let $a_n = \frac{(n+i)^5}{(1+i)^n}$. I will apply the root test:

$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, we have that

$$\frac{1}{R} = \lim_{n\to\infty} \left[\left| \frac{(n+i)^5}{(1+i)^n} \right| \right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|n+i|^{5/n}}{|1+i|^{n/n}} = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+1}\right)^{5/n}}{\sqrt{2}}.$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} (n^2+1)^{\frac{5}{2n}}.$$

Now here, $\frac{5}{2n} \to 0$ as $n \to \infty$ faster than n^2+1 grows due to the exponential, thus $\lim_{n\to\infty} (n^2+1)^{5/2n} \to 1$. This is because

$\lim_{n\to\infty} (n^2+1)^{5/2n} = \lim_{n\to\infty} e^{5/2n \cdot \log(n^2+1)}$, and $\frac{5}{2n}$ decreases faster to 0 than $\log(u^2+1)$ grows.

Therefore $R = \sqrt{2}$ for the substitution $u = z^3$. To find the radius of convergence of the initial series, it is equivalent to the exponentiation ratio of $u = z^3$, which then is $\boxed{R = [2]^{1/6}}$.

This is because $R^3 = [2]^{3/6} = \sqrt{2}$, which is the radius of convergence for the second series with u.

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4. Let $S = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) \geq -1\}$ and $f : S \to \mathbb{C}$ be the function given by

$$f(z) = \exp\left((z+\mathrm{i})^4\right), \qquad = e^{(z+i)^4}$$

where $\exp(z) := \mathrm{e}^z$. The goal of this question is to find the range of f.

(a) (6 points) Show that for every $w \in \mathbb{C} \setminus \{0\}$ there exists $z \in S$ such that

$$(z+\mathrm{i})^4 = w.$$

Draw S:

Q4a 6

- Let $w \in \mathbb{C} \setminus \{0\}$ be arbitrary. Let us express $w = re^{i\theta}$, where $r > 0$. Certainly $w \in \mathbb{C} \setminus \{0\}$ since $r > 0$ and $\theta \in [0, 2\pi)$.
- Choose $z = r^{1/4} e^{i\theta/4} - i$, This can be expressed in polar form as
  $z = r^{1/4}[\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}] - i$.

  It suffices to show $z \in S$. Since $r^{1/4} > 0$, the proof to show that $z \in S$ only relies on showing how $0 \leq \theta \leq \frac{\pi}{2}$, or that θ remains in the top right quadrant where $\mathrm{Re}(z) > 0$, $\mathrm{Im}(z) \geq -1$, we have that

  $0 \leq \theta \leq 2\pi$ by the requirement of $w \in \mathbb{C} \setminus \{0\}$, so

  $0 \leq \theta \leq 2\pi \Rightarrow 0 \leq \frac{\theta}{4} \leq \frac{2\pi}{4} \Rightarrow 0 \leq \frac{\theta}{4} \leq \frac{\pi}{2}$, so

  $\frac{\theta}{4}$ remains in the first quadrant. Therefore $r^{1/4}e^{i\theta/4} \in S$, however since $\mathrm{Im}(z) \geq -1$, this implies $r^{1/4}e^{i\theta/4} - i \in S$ too! Thus $z \in S$.
- Then $(z+i)^4 = \left(r^{1/4}e^{i\theta/4} + i - i\right)^4 = \left(r^{1/4}e^{i\theta/4}\right)^4 = re^{i\theta} = w$,

<u>This is what I wanted to show.</u>

$\square$

Q4b 2

(b) (4 points) Using part (a), show that the range of $f(z)$ is $\mathbb{C} \setminus \{0\}$.

By part (a), we have shown that $\forall w \in \mathbb{C}\setminus\{0\}$, $\exists z \in S$ where $(z+i)^4 = w$. This implies that for any $z \in S$, we can express $f(z)$ as

$$f(z) = \exp\left((z+i)^4\right) = \exp(w) = e^w.$$

The range of f is then determined by all output values of e^w. Utilizing the same representation of w as in (a), then

$$f(z) = \exp[re^{i\theta}].$$

Once again, $f(z)$ can be broken into real and complex components.

$$f(z) = e^{r[\cos\theta + i\sin\theta]} = \underbrace{e^{r\cos\theta}\left[\cos(r\sin\theta) + i\sin(r\sin\theta)\right]}_{\mathrm{Re}(f(z)),\ \mathrm{Im}(f(z))}.$$

Since $e^{r\cos\theta}$ is never zero and both $\cos(r\sin\theta) + i\sin(r\sin\theta)$ can never both be zero simultaneously, then <u>the range of $f(z)$ is justified to be $\mathbb{C}\setminus\{0\}$</u>,

It is true that the exponential

$$\exp: \mathbb{C} \to \mathbb{C} \setminus \{0\}$$

is surjective, but here we do *not* have 0 in our domain. Why is this still surjective?

5. Let $f(x+iy) = u(x,y) + iv(x,y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

(a) (2 points) State the Cauchy-Riemann equations

$$\boxed{\begin{array}{l} \frac{\partial}{\partial x} u - \frac{\partial}{\partial y} v = 0 \\ \frac{\partial}{\partial y} u + \frac{\partial}{\partial x} v = 0. \end{array}}$$

or

$u_x = v_y$

$u_y = -v_x$.

Q5a 2

(b) (5 points) Explicitly show that the Cauchy-Riemann equations hold for the function

$$f(z) = e^{z^2}$$

Now $f(z) = f(x+iy) = e^{(x+iy)^2} = e^{x^2-y^2+2ixy}$

We have that $f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$

Thus
$u = e^{x^2-y^2}\cos(2xy)$ and $v = e^{x^2-y^2}\sin(2xy)$.

Then:

$\partial_x u = e^{x^2-y^2}\cos(2xy)\cdot 2x - e^{x^2-y^2}\sin(2xy)\cdot 2y$

$\partial_y u = -e^{x^2-y^2}\cos(2xy)\cdot 2y - e^{x^2-y^2}\sin(2xy)\cdot 2x$

$= \cdot(-1)$ $=$

and

$\partial_x v = e^{x^2-y^2}\sin(2xy)\cdot 2x + e^{x^2-y^2}\cos(2xy)\cdot 2y$

$\partial_y v = -e^{x^2-y^2}\sin(2xy)\cdot 2y + e^{x^2-y^2}\cos(2xy)\cdot 2x$.

It is easy to see that $\underline{\partial_x v = -\partial_y u}$, and $\underline{\partial_y v = \partial_x u}$.

Therefore the CRE hold.

Q5b 5

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Q6 7

6. (7 points) Let $u(x, y) = x^2 - y^2 + 3y + 2$. Find v so that $f(x + iy) = u(x, y) + iv(x, y)$ is analytic on $\mathbb{C}$ and $f(0) = 2 + i$.

We require a harmonic conjugate to u. Since $\nabla^2 u = 2-2=0$, u is harmonic. We can determine the harmonic conjugate v by the CRE.

$u_x = 2x$, $u_y = -2y + 3$.
$= v_y$ $= -v_x$

Integrating, $v = 2xy + f_1(x)$, so $v_y = 2x$. And
$v = +2xy - 3x + f(y)$. Thus $f_1(x) = -3x$ and
the harmonic conjugate to u is

$$V(x,y) = 2xy - 3x + C$$ for any $C \in \mathbb{R}$, since $\frac{d}{dx} C = 0$.
The constant would vanish under any derivative.

Then $f(x+iy) = (x^2 - y^2 + 3y + 2) + i(2xy - 3x + C)$.
$f(0) = 2 + iC = 2 + i$, so $C = 1$.
Therefore $\boxed{V = 2xy - 3x + 1}$, and f is analytic on $\mathbb{C}$ by the CRE.

Correct 7

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#75 9 of 10

Extra work for Question ____
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#75 10 of 10

Extra work for Question ____

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