MAT237 Multivariable Calculus with Proofs
Problem Set 7
Due Friday March 18, 2022 by 13:00 ET

Instructions

This problem set is based on Module H: Calculus with curves (H1 to H6). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 7 sessions are held on Tuesday March 15, 2022 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope.** Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

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I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway

2) ____

Created and revised by C. Jonker, T. Janisse, and A. Zaman. Last updated March 17, 2022 at 18:53:56.

Problems

1. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a parametrization of a curve $C \subseteq \mathbb{R}^n$, so γ is simple and regular. Define $L > 0$ to be the length of the curve C. Define $\psi : [a,b] \to [0,L]$ to be the arc length parameter of γ, so

$$\psi(t) = \int_a^t ||\gamma'(u)||du, \qquad a \leq t \leq b.$$

You may assume without proof that $||\gamma'(u)||$ is integrable on $[a,b]$. (Revised 2022-03-12)

(1a) Show that ψ is continuous on $[a,b]$ and C^1 with $\psi' > 0$ on (a,b). Conclude that ψ is bijective. *Hint:* This is mostly some single variable calculus.

Theorem A. If f is integrable on $[a,b]$, then $F : [a,b] \to \mathbb{R}$ given by $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$.

Fundamental Theorem of Calculus I. Let f be integrable on $[a,b]$. Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt.$$

Fix $c \in (a,b)$. If f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

Assume $||\gamma'(u)||$ is integrable on $[a,b]$.

Continuity

By **Theorem A**, since $||\gamma'(u)||$ is integrable on $[a,b]$, then the function $F : [a,b] \to \mathbb{R}$ defined by $F(t) = \int_a^t ||\gamma'(u)||\,du$ is continuous on $[a,b]$. Since the integrands of F and ψ are equivalent, $F = \psi$ and hence ψ is continuous on $[a,b]$

C^1

From continuity, ψ is continuous at every point $c \in (a,b) \subseteq [a,b]$. By the **Fundamental Theorem of Calculus I**, ψ is differentiable at c and $\psi'(c) = ||\gamma'(c)||$. Since γ is simple and regular and hence continuous by **Defintion 8.1.7**, then $||\gamma'(c)||$ is continuous. Therefore $\psi'(c)$ is continuous for every $c \in (a,b)$ and hence ψ is C^1 on (a,b).

$\psi' > 0$

From the C^1 property, we have that $\psi'(t) = ||\gamma'(t)||$ for every $t \in (a,b)$. It suffices to show that $||\gamma'(t)|| > 0$ on (a,b). Since γ is simple and regular, by **Definition 8.1.7**, γ is continuous on $[a,b]$ and $\gamma' \neq 0$. Since the norm of any vector is strictly positive, and $\gamma'(t) \neq 0$ for all t, then it must be that $||\gamma'(t)|| > 0$ for all $t \in [a,b]$. Therefore $||\gamma'(t)|| > 0$, so $\psi' > 0$.

Bijectivity

Note that $\psi' > 0$, so ψ is a monotonic, non-decreasing function. Every input $t \in [a,b]$ is mapped to a unique value in $[0,L]$ because ψ is non-decreasing, and hence ψ is injective. Since ψ is continuous on the interval $[a,b]$, then by the intermediate value theorem, for any $x \in [0,L]$, there exists a $y \in [a,b]$ such that $\psi(y) = x$, and thus ψ is surjective. With this concatenation, ψ is bijective.

(1b) Show that the inverse map ψ^{-1} is continuous on $[0, L]$ and C^1 with $(\psi^{-1})' > 0$ on $(0, L)$.
Hint: Start by showing the C^1 property. Then deal with the endpoints. You will need (1a).

$\underline{C^1}$

Consider the subsets $(a, b) \subseteq [a, b]$ and $(0, L) \subseteq [0, L]$. From **(1a)**, the restriction of ψ to (a, b), $\psi_{(a,b)} : (a, b \to (0, L))$ is C^1, bijective, and the derivative $\psi'(t) = \|\gamma'(t)\| > 0$ for all $t \in (a, b)$. By **Lemma 7.8.8**, ψ is a diffeomorphism. By **Definition 4.1.4**, the inverse ψ^{-1} is C^1 and hence continuous on $(0, L)$.

Endpoints

Since ψ is monotonic and continuous on a closed interval $[a, b]$ from **(1a)**, then both ψ and ψ^{-1} are bounded. For the endpoints, we wish to show that $\lim\limits_{t\to 0^+} \psi^{-1}(t) = \psi^{-1}(0) = a$ and $\lim\limits_{t\to L^-} \psi^{-1}(t) = \psi^{-1}(L) = b$. Now since ψ^{-1} is bounded above and below, the set $\{\psi^{-1}(t) : 0 < t < L\}$ has an infimum and supremum. The infimum is given by $\inf\limits_{0<t<L} \psi^{-1}(t) = \psi^{-1}(0)$ and the supremum $\sup\limits_{0<t<L} \psi^{-1}(t) = \psi^{-1}(L)$. It suffices to show that

$$\lim_{t\to 0^+} \psi^{-1}(t) = \inf_{0<t<L} \psi^{-1}(t) \quad \text{and} \quad \lim_{t\to L^-} \psi^{-1}(t) = \sup_{0<t<L} \psi^{-1}(t).$$

Allow us to strive for a contradiction. Assume that $\lim\limits_{t\to 0^+} \psi^{-1}(t) \neq \inf\limits_{0<t<L} \psi^{-1}(t)$. Since for all $t \in (0, L)$, $\psi^{-1}(t) \geq \inf\limits_{0<t<L} \psi^{-1}(t)$, then it must be that $\lim\limits_{t\to 0^+} \psi^{-1}(t) > \inf\limits_{0<t<L} \psi^{-1}(t)$. However, since ψ^{-1} would be non-increasing as $t \to 0^+$ over $[0, L]$, it is impossible for $\lim\limits_{t\to 0^+} \psi^{-1}(t) > \inf\limits_{0<t<L} \psi^{-1}(t)$. Using a similar argument on the supremum, if ψ^{-1} is non-decreasing as $t \to L^-$ over $[0, L]$, then it is impossible for $\lim\limits_{t\to L^-} < \sup\limits_{0<t<L} \psi^{-1}(t)$. We have reached a contradiction. Therefore it must be that

$$\lim_{t\to 0^+} \psi^{-1}(t) = \inf_{0<t<L} \psi^{-1}(t) = \psi^{-1}(0) \quad \text{and} \quad \lim_{t\to L^-} \psi^{-1}(t) = \sup_{0<t<L} \psi^{-1}(t) = \psi^{-1}(L).$$

Now $\psi(a) = 0 \implies \psi^{-1}(\psi(a)) = a = \psi^{-1}(0)$, and likewise $\psi(b) = L \implies \psi^{-1}(\psi(b)) = L = \psi^{-1}(b)$. Therefore ψ^{-1} is continuous at the endpoints with

$$\lim_{t\to 0^+} \psi^{-1}(t) = a \quad \text{and} \quad \lim_{t\to L^-} \psi^{-1}(t) = b.$$

Therefore ψ is continuous on $[0, L]$.

$\underline{(\psi^{-1})' > 0}$

Since $\psi'(t) = \|\gamma'(t)\| > 0$ for all $t \in (a, b)$, then the derivative of the inverse of ψ is given by the inverse function theorem, that is $(\psi^{-1})'(t) = \dfrac{1}{\|\gamma'(t)\|}$ for all $t \in (0, L)$. Since $\|\gamma'(t)\| > 0$ from **(1a)**, then $(\psi^{-1})' > 0$ on $(0, L)$.

(1c) Define $g = \gamma \circ \psi^{-1} : [0, L] \to \mathbb{R}^n$. Prove that g is an arc length parametrization of C.
Hint: You will need (1c) and assumptions on γ.

Proof. It suffices to prove that g is parametrized by arc length, that is that g satisfies **Definition 8.2.7**, or that $\|g'(t)\| = 1$ for all $0 \leq t \leq L$. We have that γ is differentiable since it is regular and simple, and ψ^{-1} is differentiable since it is a diffeomorphism, and hence C^1. This implies that g is differentiable. Recall from **(1a)** and **(1b)** that $\psi'(t) = \|\gamma'(t)\|$ and $(\psi^{-1})'(t) = \dfrac{1}{\|\gamma'(t)\|}$. Applying the chain rule on g,

$$\begin{aligned} Dg &= \gamma'(\psi^{-1}) \circ (\psi^{-1})' \\ &= \frac{\gamma'(\psi^{-1})}{\|\gamma'\|}. \end{aligned}$$

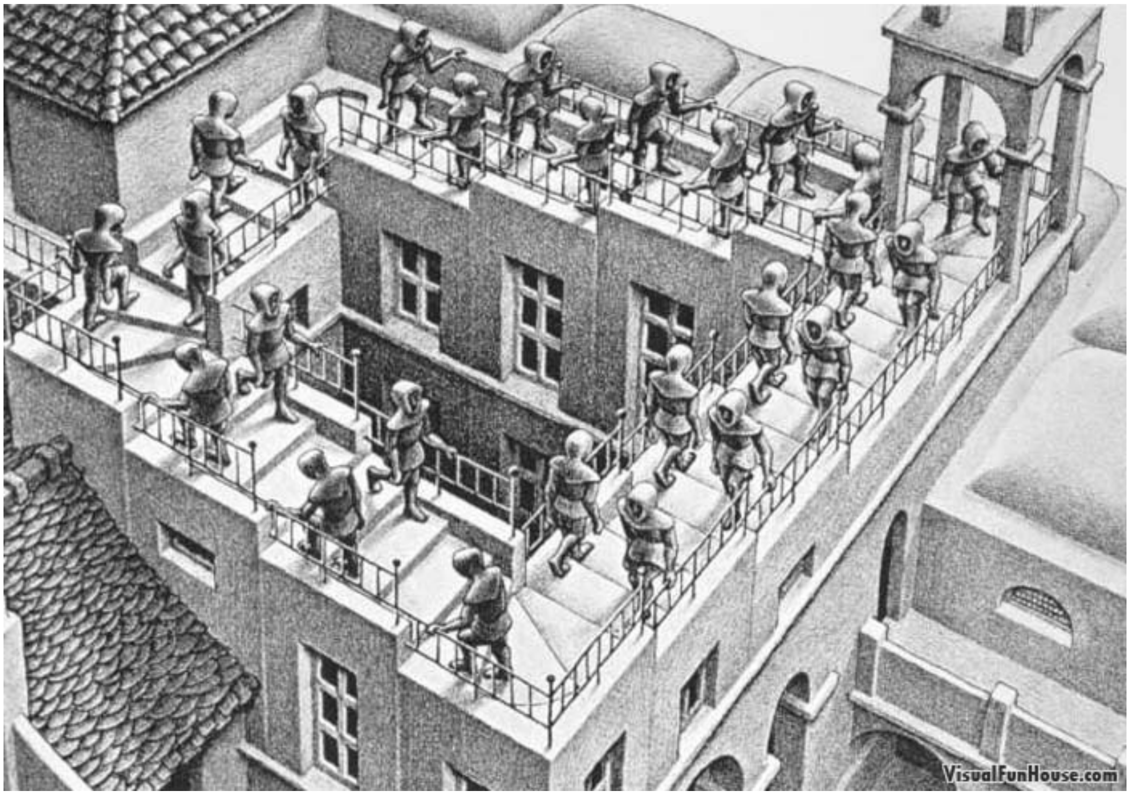
The derivative of g is a unit vector - the direction of upon the norm of the vector. Thus

$$\|Dg\| = \|\frac{\gamma'(\psi^{-1})}{\gamma'}\| = 1.$$

Therefore, from definition, g is parametrized by arc length over the curve C parametrized by γ. This completes the proof.

$\square$

2. Building upon work of Penrose in 1959, artist M.C. Escher in 1960 made the remarkable lithograph *Ascending and Descending* illustrating an infinite staircase.



A person could seemingly climb them forever and never get any higher. Using full sentences, concisely explain the visual contradiction in M.C. Escher's work using your newfound language of line integrals.

Allow us to consider the path C in three dimensions which traverses the staircase once counterclockwise. This curve is positively oriented and is decreasing as you follow its path, since you are going down the stairs. Its start point and end point are the same. The negatively oriented curve, $-C$, which follows clockwise, also has the same endpoints as C but is increasing as you traverse it, because you are walking up the stairs. This is the first contradiction: how could one ascending path and descending path have the same endpoints? It cannot. Now, allow us to consider a vector field or 'wind' which points uniformly downwards along all parts of the curves. In this case, the field is conservative, because the uniform vector wind field could be written as a potential. For instance, the field $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (0, 0, -1)$. This field is equivalent to the gradient of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = -z$, hence F is conservative by **Definition 8.4.5**. The second contradiction is obtained through **Theorem 8.5.2**, which states that the line integrals of F over the curve C and $-C$, with the same endpoints, are equivalent. This is not the case, since the line integral of the 'wind' around the counterclockwise curve C is positive since the 'wind' is helping you (or pushing you to) walk down the stairs. The line integral of F around the clockwise curve is negative, since you are walking upwards, against the downward force of the wind. This is the second contradiction, which lies in the curves, because we have already showed that F is conservative.

3. Irrotational vector fields are gradient vector fields in some cases.

> **Theorem A.** *If $U \subseteq \mathbb{R}^2$ is an open convex set and F is a C^1 irrotational vector field on U, then F is a gradient vector field on U. That is, $F = \nabla f$ on U for some C^2 scalar function f on U.*

On the other hand, consider the vector field $F(x, y) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right)$.

(3a) By direct calculation, show that F is irrotational.

The domain of F is $V = \mathbb{R}^2 \setminus \{(0,0)\}$, since F is undefined when $x = y = 0$. V is an open set by **Lemma 2.4.13** because $V^C = \{(0,0)\}$ is closed (the singleton). By **Definition 8.4.10**, F is irrotational on V if $\partial_1 F_2 = \partial_2 F_1$. By the quotient rule, we have that

$$\begin{aligned} \partial_x F_2 &= \frac{1}{(x^2+y^2)^2}\left[\frac{\partial}{\partial x}[x](x^2+y^2) - \frac{\partial}{\partial x}[x^2+y^2](x)\right] \\ &= \frac{1}{(x^2+y^2)^2}\left[(x^2+y^2) - 2x^2\right] \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned} \qquad \begin{aligned} \partial_y F_1 &= \frac{1}{(x^2+y^2)^2}\left[\frac{\partial}{\partial y}[-y](x^2+y^2) - \frac{\partial}{\partial y}[x^2+y^2](-y)\right] \\ &= \frac{1}{(x^2+y^2)^2}\left[-(x^2+y^2) + 2y^2\right] \\ &= \frac{y^2-x^2}{(x^2+y^2)^2}. \end{aligned}$$

Therefore F is irrotational on V.

(3b) By direct calculation, show that $\int_C (F \cdot T)\, ds = 2\pi$ where C is the circle $(\cos t, \sin t)$ for $0 \le t \le 2\pi$.

Recall that for a simple regular parameterization $\gamma(t)$ with unit tangent vector $\dfrac{\gamma'(t)}{\|\gamma'(t)\|}$, we have that

$$\int_C (F \cdot T)\, ds = \int_C F \cdot \frac{\gamma'(t)}{\|\gamma(t)\|} \cdot \|\gamma'(t)\|\, dt = \int_C F \cdot d\gamma.$$

Define the parameterization of the unit circle $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin t)$. Note that γ is simple and regular, with $\gamma'(t) = (-\sin t, \cos t)$. With $0 \le t \le 2\pi$, the line integral becomes

$$\begin{aligned} \int_C F \cdot d\gamma &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} 1\, dt \\ &= 2\pi, \end{aligned}$$

which is what I wanted to show.

(3c) Explain why F is not a gradient vector field on its domain and why this does not contradict Theorem A.

The domain of F is $V = \mathbb{R}^2 \setminus \{(0,0)\}$, because the components of F are undefined at $(0,0)$. After some trial and error integrating and differentiating, it can be quickly shown that the potential function of F on V is

$$f(x,y) = -\arctan\left(\frac{x}{y}\right).$$

Now here lies a contraction: the domain of f is not V, but rather the set $W = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$. This is because whenever $y = 0$, the argument of arccos is undefined - you are dividing by zero. Since the domains $V \neq W$, f cannot be the potential function of F; F is not a gradient vector field on its domain due to the discontinuites of the scalar potential f when $y = 0$.

However, this does not contradict Theorem A. Note that V (the domain of F) is not a convex set, since V has a 'hole' at $(0,0)$. The line $\ell : \mathbb{R} \to \mathbb{R}^2$ defined by $\ell(t) = (t,0)$ lies outside of V when $t = 0$. The requirements of Theorem A are not satisfied, hence we are unable to conclude whether F is a gradient vector field on its domain.

(3d) Choose as large as possible of an open set $X \subseteq \mathbb{R}^2$ such that the restriction $F|_X$ is a gradient vector field. You do not need to verify that it is as large as possible, but you should exhibit its potential function.

Choose the set $X \subseteq \mathbb{R}^2$ defined by $X = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, or the upper half of the xy plane. Note that X is open and convex, and F is C^1 irrotational on X, as proven in **(3a)**. There are no discontinuities of F on X. By **Theorem A**, F is a gradient vector field on X. Define the function $f : X \to \mathbb{R}^2$ by $f(x,y) = -\arctan\left(\frac{x}{y}\right)$. On X, f has no discontinuities since $y > 0$ and arctan is continuous on X, so

$$\begin{aligned}
\nabla f &= \nabla\left(-\arctan\left(\frac{x}{y}\right)\right) = \left(-\frac{\partial}{\partial x}\arctan\left(\frac{x}{y}\right), -\frac{\partial}{\partial y}\arctan\left(\frac{x}{y}\right)\right) \\
&= \left(-\frac{1}{x^2/y^2+1}\cdot\frac{1}{y}, -\frac{1}{x^2/y^2+1}\left(-\frac{x}{y^2}\right)\right) \\
&= \left(-\frac{y^2}{x^2+y^2}\frac{1}{y}, \frac{y^2}{x^2+y^2}\frac{x}{y^2}\right) \\
&= \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \\
&= F|_X(x,y).
\end{aligned}$$

Therefore the restriction $F|_X$ is a gradient vector field on X, since the scalar potential f is defined on the largest open convex set, which is X.

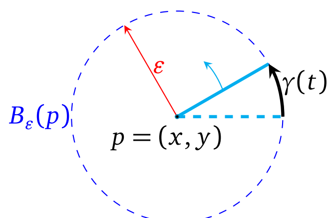
4. Let $F = (f, g)$ be a vector field in $\mathbb{R}^2$ with C^1 components f and g. Fix a point $p = (x, y) \in \mathbb{R}^2$. For $\varepsilon > 0$, let $B_\varepsilon(p) \subseteq \mathbb{R}^2$ be the disk of radius ε centred at p. Orient its boundary $\partial B_\varepsilon(p)$ counterclockwise. **Do not use Green's theorem for any part of this question.**

   (4a) For $\varepsilon > 0$, show that the circulation of F along $\partial B_\varepsilon(p)$ may be expressed as

$$\oint_{\partial B_\varepsilon(p)} (F \cdot T) ds = \int_0^{2\pi} -f(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \sin t + g(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \cos t dt.$$

Let $\varepsilon > 0$ be given. The boundary of the epsilon ball centered at p is continuous, so it can be parametrized. The center of the ball is located at the point $p = (x, y)$ with radius ε, so then define the function $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(t) = (x + \varepsilon \cos t, y + \varepsilon \sin t)$.

It follows to show that γ is a simple regular curve. The curve we are wanting to traverse with a parametrization is the positively oriented boundary $\partial B_\varepsilon(p)$. Note that it is true that $\partial B_\varepsilon(p) = \gamma([0, 2\pi])$ because the radius ε is centered at (x, y) while t 'traces' out the angle around the circular path, which is $\partial B_\varepsilon(p)$ (see the figure below). γ is continuous and C^1 because each of the components $\gamma_1(t) = x + \varepsilon \cos t$ and $\gamma_2(t) = y + \varepsilon \sin t$ are continuous and C^1. The derivative $\gamma'(t) = (-\varepsilon \sin t, \varepsilon \cos t)$ exists and is continuous, and for all $t \in (0, 2\pi)$, $\gamma'(t) \neq 0$ because neither $\sin t$ or $\cos t$ are simultaneously zero. Lastly, γ is injective (except with $\gamma(0) = \gamma(2\pi)$) because both of its components are injective, and therefore γ is a simple regular parametrization.



Now, the infinitessmial distance ds is equivalent to $ds = \|\gamma'(t)\| dt$, and the unit tangent vector of γ exists and is equal to $T = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$. Since F is a vector field in $\mathbb{R}^2$ with C^1 components, F is integrable over $\mathbb{R}^2$. Following from **Definition 8.2.12**, the circulation of F along $\partial B_\varepsilon(p)$ can be expressed as

$$\oint_{\partial B_\varepsilon(p)} (F \cdot T) ds = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) dt.$$

Algebra follows:

$$\begin{aligned}
\int_{\partial B_\varepsilon(p)} (F \cdot T) ds &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) dt \\
&= \int_0^{2\pi} (f(\gamma(t)), g(\gamma(t))) \cdot \gamma'(t) dt \\
&= \int_0^{2\pi} -f(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \sin t + g(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \cos t \, dt,
\end{aligned}$$

which is what I wanted to show.

(4b) Since f is C^1 on U, differentiability implies that there exists $\delta_f > 0$ and $E_f : B_{\delta_f}(0,0) \to \mathbb{R}$ such that

$$\forall(\Delta x, \Delta y) \in B_{\delta_f}(0,0), \quad f(x+\Delta x, y+\Delta y) = f(x,y) + \partial_1 f(x,y)\Delta x + \partial_2 f(x,y)\Delta y + E_f(\Delta x, \Delta y),$$

where $\lim_{(a,b)\to(0,0)} \frac{|E_f(a,b)|}{\|(a,b)\|} = 0$. The analogous statement holds for g with $\delta_g > 0$ and $E_g : B_{\delta_g}(0,0) \to \mathbb{R}$.

Prove that for $0 < \varepsilon < \frac{\min\{\delta_f, \delta_g\}}{2}$,

$$\frac{1}{\text{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds = (\text{curl}\, F)(p) + \frac{1}{\pi\varepsilon} \int_0^{2\pi} -E_f(\varepsilon\cos t, \varepsilon \sin t) \cdot \sin t + E_g(\varepsilon \cos t, \varepsilon \sin t) \cdot \cos t\, dt.$$

Proof. Assume that $0 < \varepsilon < \dfrac{\min\{\delta_f, \delta_g\}}{2}$. From **(4a)**, the circulation of F along $\partial B_\varepsilon(p)$ can be expressed as

$$\int_{\partial B_\varepsilon(p)} (F \cdot T)\, ds = \int_0^{2\pi} -f(x+\varepsilon\cos t, y + \varepsilon \sin t) \cdot \varepsilon \sin t + g(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \cos t\, dt.$$

Since $0 < \varepsilon < \dfrac{\min\{\delta_f, \delta_g\}}{2}$, allow us to consider a small deviation from the point p: $(\Delta x, \Delta y) = (\varepsilon \cos\alpha, \varepsilon \sin \alpha) \in B_\varepsilon(0,0) \subseteq B_{\min\{\delta_f,\delta_g\}/2}(0,0) \subseteq B_{\delta_f}(0,0)$ for any $\alpha \in [0, 2\pi]$. With this assumption, the C^1 function f can be expressed as

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \partial_1 f(x,y)\Delta x + \partial_2 f(x,y)\Delta y + E_f(\Delta x, \Delta y).$$

The same argument applies to the function g. Then,

$$\frac{1}{\text{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds = \frac{1}{\text{area}(B_\varepsilon(p))} \int_0^{2\pi} -\varepsilon \sin t(f(x,y) + \partial_1 f(x,y)\varepsilon\cos t + \partial_2 f(x,y)\varepsilon \sin t + E_f(\varepsilon\cos t, \varepsilon \sin t))$$
$$+ \varepsilon \cos t(g(x,y) + \partial_1 g(x,y)\varepsilon \cos t + \partial_2 g(x,y)\varepsilon \sin t + E_g(\varepsilon\cos t, \varepsilon\sin t))\, dt.$$

By linearity, this integral can be broken into two parts: that of which contains the trigonometric components, and the error functions:

$$\begin{aligned}\frac{1}{\text{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds = \frac{1}{\text{area}(B_\varepsilon(p))} &\int_0^{2\pi} -\varepsilon \sin t(f(x,y) + \partial_1 f(x,y)\varepsilon \cos t + \partial_2 f(x,y)\varepsilon \sin t) \\ &+ \varepsilon \cos t(g(x,y) + \partial_1 g(x,y)\varepsilon\cos t + \partial_2 g(x,y)\varepsilon \sin t)\, dt \\ &+ \frac{1}{\text{area}(B_\varepsilon(p))} \int_0^{2\pi} -\varepsilon \sin t E_f(\varepsilon\cos t, \varepsilon \sin t) + \varepsilon\cos t E_g(\varepsilon \cos t, \varepsilon \sin t)\, dt.\end{aligned}$$

Since the line integral is evaluated only with respect to t, only the trigonometric integrals contribute. The integrals of $\sin x$ and $\cos x$ over their periods 2π evaluate to 0 by symmetry, while the integrals $\int_0^{2\pi} \cos^2 dx$ and $\int_0^{2\pi} \sin^2 dx$ evaluate to π. $\int_0^{2\pi} \sin x \cos x\, dx$ evaluates to 0 by the orthogonality of the trigonometric functions sin and cos. With the area of the epsilon ball being $\pi\varepsilon^2$, we have that

$$\begin{aligned}\frac{1}{\pi\varepsilon^2} \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds &= \frac{1}{\pi\varepsilon^2}\left[-\pi\varepsilon^2 \partial_2 f(x,y) + \pi\varepsilon^2 \partial_1 g(x,y)\right] + \frac{\varepsilon}{\pi\varepsilon^2}\int_0^{2\pi} -\sin t E_f(\varepsilon\cos t, \varepsilon \sin t) + \cos t E_g(\varepsilon \cos t, \varepsilon \sin t)\, dt \\ &= \partial_1 F_2(x,y) - \partial_2 F_1(x,y) + \frac{1}{\pi\varepsilon}\int_0^{2\pi} -\sin t E_f(\varepsilon \cos t, \varepsilon \sin t) + \cos t E_g(\varepsilon \cos t, \varepsilon\sin t)\, dt \\ &= \text{curl}(F)(p) + \frac{1}{\pi\varepsilon}\int_0^{2\pi} -\sin t E_f(\varepsilon\cos t, \varepsilon \sin t) + \cos t E_g(\varepsilon\cos t, \varepsilon \sin t)\, dt,\end{aligned}$$

which is what I wanted to prove. $\square$

(4c) Use the limit definition to prove that $\lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon}\int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\sin t\, dt = 0$ and conclude that

$$(\operatorname{curl} F)(p) = \lim_{\varepsilon\to 0^+} \frac{1}{\operatorname{area}(B_\varepsilon(p))}\oint_{\partial B_\varepsilon(p)} (F\cdot T)\, ds.$$

For sake of sanity, relabel $\varepsilon = \alpha$. I want to show that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall \alpha \in \mathbb{R}^+,\ 0 < \alpha < \delta \implies \left\| \frac{1}{\pi\alpha}\int_0^{2\pi} E_f(\alpha\cos t, \alpha\sin t)\sin t\, dt \right\| < \varepsilon.$$

Proof. By assumption, from **(4b)**, $\lim_{(a,b)\to(0,0)} \frac{\|E_f(a,b)\|}{\|(a,b)\|} = 0$. Taking $a = \alpha\cos t$ and $b = \alpha\sin t$ for any $t \in [0, 2\pi]$, as $(a,b) \to (0,0)$, $\alpha \to 0^+$. Formally,

$$\forall \varepsilon' > 0,\ \exists \delta' > 0 \text{ such that } \forall \alpha \in \mathbb{R}^+,\ 0 < \alpha < \delta' \implies \frac{1}{\alpha}\|E_f(\alpha\cos t, \alpha\sin t)\| < \varepsilon'. \tag{I}$$

Since this is assumed, for any value of $\varepsilon' > 0$ that I pick, there exists a $\delta' > 0$ such that **(I)** is satisfied.

Fix $\varepsilon > 0$ and take $\varepsilon' = \frac{\varepsilon}{2}$. Then there exists a δ' such that **(I)** is satisfied. Take this value of δ: $\delta = \delta'$. Fix $\alpha \in \mathbb{R}^+$. From **(4b)**, we can assume that the function E_f is bounded and integrable on $[0, 2\pi]$. We have that

$$\left\| \frac{1}{\pi\alpha}\int_0^{2\pi} E_f(\alpha\cos t, \alpha\sin t)\sin t\, dt \right\| \le \frac{1}{\pi\alpha}\int_0^{2\pi} \|E_f(\alpha\cos t, \alpha\sin t)\sin t\|\, dt \tag{II}$$

by the triangle inequality **(Theorem 6.3.15)**. Since $0 \le |\sin t| < 1$ for $0 \le t \le 2\pi$, then $\|E_f(\alpha\cos t, \alpha\sin t)\sin t\| \le \|E_f(\alpha\cos t, \alpha\sin t)\|$. By monotonicity **(Theorem 6.3.14)**,

$$\frac{1}{\pi\alpha}\int_0^{2\pi} \|E_f(\alpha\cos t, \alpha\sin t)\sin t\|\, dt \le \frac{1}{\pi\alpha}\int_0^{2\pi} \|E_f(\alpha\cos t, \alpha\sin t)\|\, dt = \frac{1}{\pi}\int_0^{2\pi} \left\|\frac{E_f(\alpha\cos t, \alpha\sin t)}{\alpha}\right\| dt \tag{III}$$

also by linearity **(Theorem 6.3.13)**, since α is constant with respect to the integration variable t. Now, since $\left\|\frac{E_f(\alpha\cos t, \alpha\sin t)}{\alpha}\right\| < \varepsilon'$, then again by monotonicity,

$$\frac{1}{\pi}\int_0^{2\pi} \left\|\frac{E_f(\alpha\cos t, \alpha\sin t)}{\alpha}\right\| dt < \frac{1}{\pi}\int_0^{2\pi} \varepsilon'\, dt = 2\varepsilon'. \tag{IV}$$

Therefore for $0 < \alpha < \delta$, cumulating **(II)**, **(III)** and **(IV)**,

$$\left\| \frac{1}{\pi\alpha}\int_0^{2\pi} E_f(\alpha\cos t, \alpha\sin t)\sin t\, dt \right\| \le \frac{1}{\pi\alpha}\int_0^{2\pi} \|E_f(\alpha\cos t, \alpha\sin t)\sin t\|\, dt \le \frac{1}{\pi}\int_0^{2\pi} \left\|\frac{E_f(\alpha\cos t, \alpha\sin t)}{\alpha}\right\| dt < 2\varepsilon' = \varepsilon,$$

which is what I wanted to prove. □

The proof that $\lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon}\int_0^{2\pi} E_g(\varepsilon\cos t, \varepsilon\sin t)\cos t\, dt = 0$ is analogous. From **(4b)**,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon^2}\oint_{\partial B_\varepsilon(p)} (F\cdot T)\, ds &= \lim_{\varepsilon\to 0^+} \operatorname{curl}(F)(p) + \lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon}\int_0^{2\pi} -\sin t E_f(\varepsilon\cos t, \varepsilon\sin t) + \cos t E_g(\varepsilon\cos t, \varepsilon\sin t)\, dt, \\
&= \operatorname{curl}(F)(p) + 0.
\end{aligned}$$

Therefore $\operatorname{curl}(F)(p) = \lim_{\varepsilon\to 0^+} \frac{1}{\operatorname{area}(B_\varepsilon(p))}\oint_{\partial B_\varepsilon(p)} (F\cdot T)\, ds$, which completes the conclusion.