PHY250 PS2 — 02/28/2022

Jace Alloway — 1006940802

I have read and understand the homework set policy in this course. Collaborators: none (for all problems).

Q1.1

Let $P = (x, y)$ be a point in the plane. The vector difference from the $+d$ charge is then $\mathbf{r}_1 = (x - d)\hat{\mathbf{x}} + y\hat{\mathbf{y}}$, and the difference from the $-d$ charge is $\mathbf{r}_2 = (x + d)\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. The electric field is then

$$\begin{aligned}
\mathbf{E}(x, y) &= \mathbf{E}_1(x, y) + \mathbf{E}_2(x, y) \\
&= \frac{q}{4\pi\varepsilon_0}\left[\frac{(x-d)\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{((x-d)^2 + y^2)^{3/2}} + \frac{(x+d)\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{((x+d)^2 + y^2)^{3/2}}\right] \\
&= \frac{q}{4\pi\varepsilon_0}\left[\frac{x-d}{((x-d)^2 + y^2)^{3/2}} + \frac{x+d}{((x+d)^2 + y^2)^{3/2}}\right]\hat{\mathbf{x}} + \frac{qy}{4\pi\varepsilon_0}\left[\frac{1}{((x-d)^2 + y^2)^{3/2}} + \frac{1}{((x+d)^2 + y^2)^{3/2}}\right]\hat{\mathbf{y}}.
\end{aligned}$$

Since we are only considering the $\hat{\mathbf{x}}$ component of the field,

$$E_x(x, y) = \frac{q}{4\pi\varepsilon_0}\left[\frac{x-d}{((x-d)^2 + y^2)^{3/2}} + \frac{x+d}{((x+d)^2 + y^2)^{3/2}}\right].$$

Since the particle is located on the x axis, then $y = 0$:

$$E_x(x, 0) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(x-d)^2} + \frac{1}{(x+d)^2}\right].$$

Taking $d \gg |x|$, $|x|/d \to 0$ and so

$$\begin{aligned}
E_x(x, 0) &\approx \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{d^2(x/d - 1)^2} + \frac{1}{d^2(x/d + 1)^2}\right] \\
&= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{d^2} + \frac{1}{d^2}\right] \\
&= \frac{q}{2\pi\varepsilon_0 d^2}.
\end{aligned}$$

Q1.2

Now, if we only consider the $\hat{\mathbf{y}}$ direction,

$$E_y(x, y) = \frac{qy}{4\pi\varepsilon_0}\left[\frac{1}{((x-d)^2 + y^2)^{3/2}} + \frac{1}{((x+d)^2 + y^2)^{3/2}}\right].$$

Again, since the particle is only located on the y axis, we can take $x \to 0$:

$$\begin{aligned}
E_y(0, y) &= \frac{qy}{4\pi\varepsilon_0}\left[\frac{1}{(d^2 + y^2)^{3/2}} + \frac{1}{(d^2 + y^2)^{3/2}}\right] \\
&= \frac{qy}{2\pi\varepsilon_0}\frac{1}{(d^2 + y^2)^{3/2}}.
\end{aligned}$$

Now, taking $d \gg |y|$, $|y|/d \to 0$ and thus

$$\begin{aligned} E_y(0,y) &= \frac{q}{2\pi\varepsilon_0}\frac{y}{d^3(1+y^2/d^2)^{3/2}} \\ &\approx \frac{q}{2\pi\varepsilon_0}\frac{y}{d^3}. \end{aligned}$$

Q2.1

(i) The enclosed charge inside the sphere is 0 (there is empty space). Therefore $E(r) = 0$ for $r < R_a$.

(ii) By Gauss's Law, $\oiint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{encl}}{\varepsilon_0}$. The enclosed charge is given by

$$
\begin{aligned}
Q_{encl} &= \iiint_{R_a \leq r \leq R_b} \rho \, d\tau \\
&= \int_0^{\pi} \int_0^{2\pi} \int_{R_a}^{r} kr^{n\prime} r^{2\prime} \sin\theta dr' \, d\phi \, d\theta \\
&= k \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \int_{R_a}^{r} r^{n+2\prime} \, dr' \\
&= \frac{4\pi k}{n+3} [r^{n+3} - R_a^{n+3}].
\end{aligned}
$$

The area of the sphere of radius $r \in [R_a, R_b]$ is simply $A(r) = 4\pi r^2$, and thus

$$
\begin{aligned}
4\pi r^2 E &= \frac{4\pi k}{\varepsilon_0(n+3)} [r^{n+3} - R_a^{n+3}] \\
\implies E\left(R_a \leq r \leq R_b\right) &= \frac{k}{\varepsilon_0(n+3)} \frac{r^{n+3} - R_a^{n+3}}{r^2}.
\end{aligned}
$$

(iii) From (ii), enclosed charge is given by

$$
Q_{encl} = \frac{4\pi k}{n+3} [R_b^{n+3} - R_a^{n+3}].
$$

This is simply because $\rho(r > R_b) = 0$. By Gauss's Law,

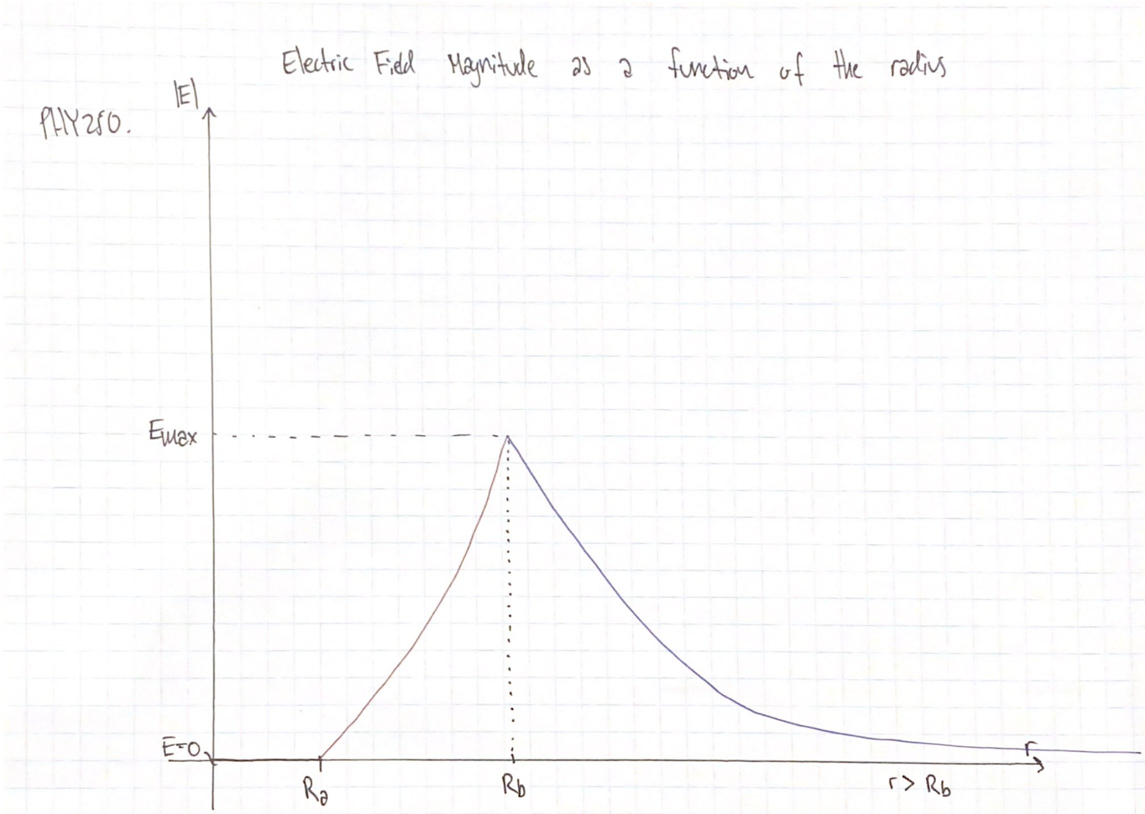
$$
\begin{aligned}
4\pi r^2 E &= \frac{4\pi k}{\varepsilon_0(n+3)} [R_b^{n+3} - R_a^{n+3}] \\
\implies E\left(r > R_b\right) &= \frac{k}{\varepsilon_0(n+3)} \frac{R_b^{n+3} - R_a^{n+3}}{r^2}.
\end{aligned}
$$

Q2.2

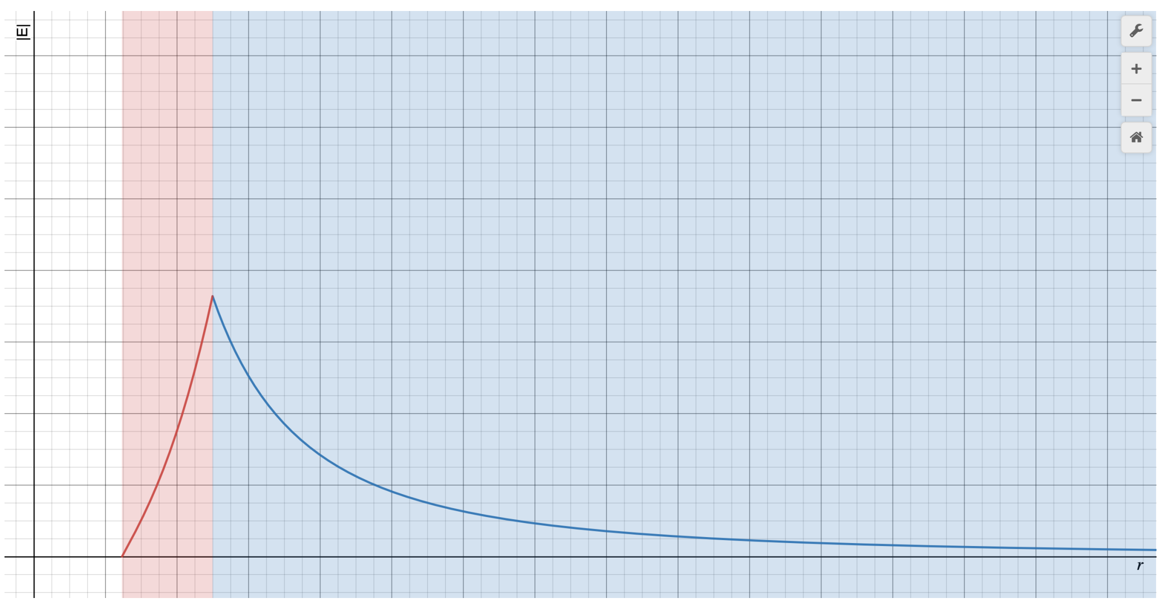
For $n = 2$ and arbitrary values of k, R_a, R_b :

$$
\begin{aligned}
E\left(R_a \leq r \leq R_b\right) &= \frac{k}{5\varepsilon_0} \frac{r^5 - R_a^5}{r^2} \qquad \text{(red)} \\
E\left(r > R_b\right) &= \frac{k}{5\varepsilon_0} \frac{R_b^5 - R_a^5}{r^2} \qquad \text{(blue)}
\end{aligned}
$$

Hand-drawn plot:



Desmos plot:



In the white region, the magnitude of the electric field is 0. This is because $r < R_a$, so there is no electric field inside of a closed region. By Gauss's law, $Q_{encl} = 0$, so $E = 0$.

In the red region, notice how the magnitude of the electric field increases as the radius increases. This is because the enclosed charge increases as the radius increases from $R_a \rightarrow R_b$.

In the blue region, the magnitude of the electric field decreases to 0 as the radius decreases. The enclosed charge remains constant (since there is no solid in the $r > R_b$ region), and so the electric field decreases proportional to $1/r^2$. Furthermore, notice how the boundary condition is satisfied when $r = R_b$. This point is the direct surface of the sphere and is the radius of maximum amplitude of the electric field.

Q3.

Since electric fields superpose, we can consider the electric fields from the bottom face disk and the outside cylinder separately. Let the centre of the bottom face of the cylinder be the origin. The electric field from the disk, in the $\hat{\mathbf{z}}$ direction, is

$$E_z = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right].$$

Taking $z \to 0$, then $E_{disk} = \dfrac{\sigma}{2\varepsilon_0}$, which is the electric field magnitude on the surface of the centre of the disk. Intuitively, the electric field produced by the infinite cylinder should be strictly in the $\hat{\mathbf{z}}$ direction, since the radial components cancel by symmetry. The electric field of a ring in the $\hat{\mathbf{z}}$ direction is given by

$$E_z = \frac{\sigma}{2\varepsilon_0}\frac{a\,z}{(a^2+z^2)^{3/2}}.$$

Integrating z from $0 \to \infty$ will give the electric field contribution from the cylinder in the z direction:

$$\begin{aligned}
E_{cylinder} &= \frac{\sigma}{2\varepsilon_0}\int_0^\infty \frac{a\,z}{(a^2+z^2)^{3/2}}\,dz \\
&= \frac{\sigma\,a}{2\varepsilon_0}\left[-\frac{1}{\sqrt{a^2+z^2}}\right]_0^\infty \\
&= -\frac{\sigma\,a}{2\varepsilon_0}\left[0 - \frac{1}{a}\right] \\
&= \frac{\sigma}{2\varepsilon_0}.
\end{aligned}$$

By symmetry, the electric field at this selected origin is strictly in the $\hat{\mathbf{z}}$ direction, and so

$$E = \frac{\sigma}{\varepsilon_0}.$$

Q4.1

I will begin by finding the electric field of the sphere. From **Q2.1**, the electric field outside of the sphere is given by

$$\mathbf{E}\left(r > R_b\right) = \frac{k}{\varepsilon_0(n+3)}\frac{R_b^{n+3} - R_a^{n+3}}{r^2}\hat{\mathbf{r}}.$$

since $\rho = kr^n$. Taking $R_a = 0$ (no inner 'shell'), then

$$\mathbf{E}\left(r > R\right) = \frac{k}{\varepsilon_0(n+3)}\frac{R^{n+3}}{r^2}\hat{\mathbf{r}},$$

which is the electric field of a sphere. The potential outside the sphere is given by

$$\begin{aligned} V(R) - V(\infty) &= -\int_{\infty}^{R} \mathbf{E}\cdot\hat{\mathbf{r}}\,dr' \\ &= -\frac{k\,R^{n+3}}{\varepsilon_0(n+3)}\int_{\infty}^{R}\frac{dr'}{r'^2} \\ &= -\frac{k\,R^{n+3}}{\varepsilon_0(n+3)}\left[-\frac{1}{r'}\right]_{\infty}^{R} \\ V(R) &= \frac{k}{\varepsilon_0(n+3)}R^{n+2}, \end{aligned}$$

which is taken from the fact that $V(r=\infty) = 0$. By equation (2.43) in Griffiths,

$$\begin{aligned} W &= \frac{1}{2}\iiint \rho V\,d\tau \\ &= \frac{1}{2}\iiint kr^n\frac{k}{\varepsilon_0(n+3)}R^{n+2}\,d\tau. \end{aligned}$$

Integrating in spherical coordinates, $d\tau = r^2\sin\theta\,drd\phi d\theta$:

$$\begin{aligned} W &= \frac{1}{2}\iiint kr^n\frac{k}{\varepsilon_0(n+3)}R^{n+2}\,d\tau \\ &= \frac{k^2R^{n+2}}{2\varepsilon_0(n+3)}\int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^n\,r^2\sin\phi drd\phi d\theta \\ &= \frac{k^2R^{n+2}}{2\varepsilon_0(n+3)}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}\,d\phi\int_0^{R} r^{n+2}\,dr \\ &= \frac{k^2R^{n+2}}{2\varepsilon_0(n+3)}(1)(2\pi)\frac{R^{n+3}}{n+3} \\ &= \frac{\pi k^2}{\varepsilon_0(n+3)^2}R^{2n+5}. \end{aligned}$$

Q4.2

Using equation (2.45),

$$
\begin{aligned}
W &= \frac{\varepsilon_0}{2} \iiint E^2 \, d\tau \\
&= \frac{\varepsilon_0}{2} \int_0^{\pi} \int_0^{2\pi} \int_R^{\infty} \frac{k^2}{\varepsilon_0^2 (n+3)^2} \frac{(R^{n+3})^2}{r^4} \, r^2 \sin\theta \, dr d\phi d\theta \\
&= \frac{k^2 (R^{n+3})^2}{2\varepsilon_0 (n+3)^2} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \, d\phi \int_R^{\infty} \frac{dr}{r^2} \\
&= \frac{k^2 \, R^{2n+6}}{2\varepsilon_0 (n+3)^2} (1)(2\pi) \left(\frac{1}{R} \right) \\
&= \frac{\pi k^2}{\varepsilon_0 (n+3)^2} R^{2n+5}.
\end{aligned}
$$

(this is because I can choose a V such that all of the enclosed charge is accounted for, so I integrate from $R \to \infty$). Therefore

$$W = \frac{\pi k^2}{\varepsilon_0 (n+3)^2} R^{2n+5}.$$

Q5.

For the $x = a$ line, $\mathbf{r} = s\cos\phi\hat{\mathbf{x}} + s\sin\phi\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and $\mathbf{r}_1' = a\hat{\mathbf{x}} + \ell\hat{\mathbf{z}}$. Thus $\boldsymbol{\mathscr{r}}_1 = (s\cos\phi - a)\hat{\mathbf{x}} + s\sin\phi\hat{\mathbf{y}} + (z-\ell)\hat{\mathbf{z}}$. Since the charge density λ is uniform,

$$V_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{\lambda}{|\boldsymbol{\mathscr{r}}|}\,d\ell.$$

With $p_1^2 = (s\cos\phi - a)^2 + (s\sin\phi)^2$, then

$$\begin{aligned}
V_1(\mathbf{r}) &= \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{1}{\sqrt{p_1^2 + (z-\ell)^2}\,d\ell} \\
&= \frac{\lambda}{4\pi\varepsilon_0}\int_{z+L}^{z-L} \frac{d\ell}{\sqrt{p_1^2 + u_1^2}} \qquad \text{with } u = z - \ell, \\
&= \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{z - L + \sqrt{p_1^2 + (z-L)^2}}{z + L + \sqrt{p_1^2 + (z+L)^2}}\right).
\end{aligned}$$

For the $x = -a$ line, the charge density is also $-\lambda$. Taking $p_2^2 = (s\cos\phi + a)^2 + (s\sin\phi)^2$ and thus the potentials superpose to

$$\begin{aligned}
V_{tot}(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{z - L + \sqrt{p_1^2 + (z-L)^2}}{z + L + \sqrt{p_1^2 + (z+L)^2}}\right) - \log\left(\frac{z - L + \sqrt{p_2^2 + (z-L)^2}}{z + L + \sqrt{p_2^2 + (z+L)^2}}\right)\right] \\
&= V(s, \phi, z).
\end{aligned}$$

For $L \gg a$, $L \gg s$, and $L \gg z$, we can take the limit as $z \to 0$.

$$V_{tot}(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{-L + \sqrt{p_1^2 + L^2}}{L + \sqrt{p_1^2 + L^2}}\right) - \log\left(\frac{-L + \sqrt{p_2^2 + L^2}}{L + \sqrt{p_2^2 + L^2}}\right)\right].$$

By the binomial approximation, $\sqrt{1+x} \approx 1 + \dfrac{1}{2}x$. Then

$$\begin{aligned}
V_{tot}(\mathbf{r}) &\approx \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{-1 + p_1^2/2L^2 + 1}{1 + p_1^2/2L^2 + 1}\right) - \log\left(\frac{-1 + p_2^2/2L^2 + 1}{1 + p_2^2/2L^2 + 1}\right)\right] \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{p_1^2}{2L^2}\right) - \log\left(2 + \frac{p_1^2}{2L^2}\right) - \log\left(\frac{p_2^2}{2L^2}\right) + \log\left(2 + \frac{p_2^2}{2L^2}\right)\right] \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{p_1^2}{4L^2}\right) - \log\left(1 + \frac{p_1^2}{4L^2}\right) - \log\left(\frac{p_2^2}{4L^2}\right) + \log\left(1 + \frac{p_2^2}{4L^2}\right)\right]
\end{aligned}$$

Furthermore, since $\log(1+x) \approx x$,

$$\begin{aligned}
&\approx \frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{p_1^2}{4L^2} + \log\left(\frac{p_1^2}{4L^2}\right) - \log\left(\frac{p_2^2}{4L^2}\right) + \frac{p_2^2}{4L^2}\right] \\
&= \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{p_1}{p_2}\right) \\
&= \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{s_+}{s_-}\right)
\end{aligned}$$

And thus $V(s,\phi,z) = \dfrac{\lambda}{2\pi\varepsilon_0}\log\left(\dfrac{\sqrt{s^2+a^2-2as\cos\phi}}{\sqrt{s^2+a^2+2as\cos\phi}}\right) = \dfrac{\lambda}{2\pi\varepsilon_0}\log\left(\dfrac{s_+}{s_-}\right)$ when $L \gg s$, $L \gg z$, which is equivalent to the electric potential produced by two infinite charged lines a distance a apart. When $a \to 0$ or $L \gg a$, $V(s,\phi,z) \to 0$ because the charged lines essentially overlap, and intuitively this makes sense that the electric fields and potentials cancel $(+\lambda + (-\lambda) = 0)$.

Q6.

Let Q be the charge of the inner conductor. Then the outer shell will have a charge $-Q$ since it is a capacitor. The electric field at the surface of the inner conductor (radius $r = b$), is simply

$$E(b) = \frac{Q}{4\pi\varepsilon_0 b^2} \equiv E_0.$$

This implies that $Q = 4\pi\varepsilon_0 E_0 b^2$. To find the capacitance, we must calculate the potential difference between the two shells:

$$\begin{aligned} V_{ba} &= -\int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}\,dr \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) \\ &= \frac{4\pi\varepsilon_0 E_0 b^2}{4\pi\varepsilon_0}\left(\frac{a-b}{ab}\right) \\ &= E_0 b^2\left(\frac{a-b}{ab}\right). \end{aligned}$$

Then the capacitance is

$$\begin{aligned} C &\equiv \frac{Q}{V} \\ &= \frac{4\pi\varepsilon_0 E_0 b^2}{E_0 b^2\left(\frac{a-b}{ab}\right)} \\ &= 4\pi\varepsilon_0 \frac{ab}{a-b}. \end{aligned}$$

The total work (or energy) stored in the capacitor is given by equation (2.55) in Griffiths:

$$\begin{aligned} W &= \frac{1}{2}\frac{Q^2}{C} \\ &= \frac{1}{2}\frac{16\pi^2\varepsilon_0^2 E_0^2 b^4}{4\pi\varepsilon_0\frac{ab}{a-b}} \\ &= \frac{2\pi\varepsilon_0 E_0 b^3(a-b)}{a}. \end{aligned}$$

Letting $b = ka$ for some constant k to be determined,

$$\begin{aligned} W &= 2\pi\varepsilon_0 E_0 \frac{k^3a^3(a-ka)}{a} \\ &= 2\pi\varepsilon_0 E_0 k^3 a^3(1-k). \end{aligned}$$

We wish to maximize energy with respect to k:

$$\frac{dW}{dk} = 2\pi\varepsilon_0 a^3 \frac{d}{dk}[k^3(1-k)]$$

$$
\begin{aligned}
&= 2\pi\varepsilon_0 a^3 \left(3k^2(1-k) - k^3\right) \\
&= 2\pi\varepsilon_0 a^3 k^2 (3(1-k) - k) \\
\implies 0 &= 2\pi\varepsilon_0 a^3 k^2 (3(1-k) - k) \\
\implies 3 &= 4k \\
\implies k &= \frac{3}{4}.
\end{aligned}
$$

To find the energy stored in the capacitor with $b = \dfrac{3}{4}a$,

$$
\begin{aligned}
W &= 2\pi\varepsilon_0 E_0 \left(\frac{3}{4}\right)^3 a^3 \left(1 - \frac{3}{4}\right) \\
&= 2 \cdot \frac{27}{64} \cdot \frac{1}{4}\, \pi\varepsilon_0 E_0 a^3 \\
&= \frac{27}{128} \pi\varepsilon_0 E_0 a^3.
\end{aligned}
$$

Therefore $b = \dfrac{3}{4}a$ and the maximum amount of energy stored in the capacitor is $E = \dfrac{27}{128}\pi\varepsilon_0\, E_0\, a^3$.