MAT224 Linear Algebra II
Term Test 1

Instructions:

Please read the Term Test 1 Information document for details on submission policies, permitted resources, how to ask a question, test announcements, and more. You were expected to read them in detail in advance of the test.

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2. **Submit your solutions using only this template PDF**. Your submission should be a single PDF with your full written solutions for each question. If your solution is not written using this template PDF (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided.
3. **Show your work and justify your steps** on every question unless otherwise indicated. Put your final answer in the box provided, if necessary.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for the test.

Academic Integrity Statement:

Full Name: Jace Alloway

Student number: 1006940802

I confirm that:

- I have not communicated with any person about the test other than a MAT224 teaching team member.
- I have not used any resources other than those that are listed as permitted in the Term Test 1 Information document at any point during the test.
- I have not participated in or enabled any MAT224 group chat during the test.
- I have not viewed the answers, solutions, term work, or notes of anyone.
- I have read and followed all of the rules described in the Term Test 1 Information document.
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Signature: Jace Alloway

1. Decide if the following statement is True or False:

Let $A_1, A_2, \ldots, A_k \in M_{n\times n}(\mathbb{R})$ be such that $M_{n\times n}(\mathbb{R}) = \text{span}\{A_1, A_2, \ldots, A_k\}$. Then $M_{n\times n}(\mathbb{R}) = \text{span}\{A_1^T, A_2^T, \ldots, A_k^T\}$.

Note; A^T denotes the transpose of A.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

This statement is true. I want to prove that for any matrices $A_1, A_2, \ldots, A_k \in M_{n\times n}(\mathbb{R})$ such that $M_{n\times n}(\mathbb{R}) = \text{span}\{A_1, A_2, \ldots A_k\}$, then $M_{n\times n}(\mathbb{R}) = \text{span}\{A_1^T, A_2^T, \ldots A_k^T\}$.

Proof. Suppose that $M_{n\times n}(\mathbb{R}) = \text{span}\{A_1, A_2, \ldots A_k\}$ for any matrices $A_1, A_2, \ldots, A_k \in M_{n\times n}(\mathbb{R})$. By the definition of span, we are able to write $\text{span}\{A_1, A_2, \ldots A_k\}$ as a linear combination of matrices $A_1, A_2, \ldots, A_k \in M_{n\times n}(\mathbb{R})$.

Let $c_1, c_2, c_3, \ldots c_k \in \mathbb{R}$ by any scalars such that $\text{span}\{A_1, A_2, \ldots A_k\} = c_1A_1 + c_2A_2 + \cdots + c_kA_k$.
Now consider the span of transposed matrices of A. Let $d_1, d_2, \ldots d_k \in \mathbb{R}$ such that
$\text{span}\{A_1^T, A_2^T, \ldots A_k^T\} = d_1A_1^T + d_2A_2^T + \cdots + d_kA_k^T$.

By the properties of transposed matrices, the linear combination $d_1A_1^T + d_2A_2^T + \cdots + d_kA_k^T = (d_1A_1 + d_2A_2 + \cdots + d_kA_k)^T$. Notice that the span of matrices $\text{span}\{A_1, A_2, \ldots A_k\}$ is contained in the span of transposed matrices, $\text{span}\{A_1^T, A_2^T, \ldots A_k^T\}$.
Furthermore, by the properties of transposed matrices, $c_1A_1 + c_2A_2 + \cdots + c_kA_k = ((c_1A_1 + c_2A_2 + \cdots + c_kA_k)^T)^T$, or equivalently, $c_1A_1 + c_2A_2 + \cdots + c_kA_k = (c_1A_1^T + c_2A_2^T + \cdots + c_kA_k^T)^T$. The span of transposed matrices, $\text{span}\{A_1^T, A_2^T, \ldots A_k^T\}$, is contained in the span of non-transposed matrices, $\text{span}\{A_1, A_2, \ldots A_k\}$.

It then follows that because $\text{span}\{A_1, A_2, \ldots A_k\} \subseteq \text{span}\{A_1^T, A_2^T, \ldots A_k^T\}$ and
$\text{span}\{A_1^T, A_2^T, \ldots A_k^T\} \subseteq \text{span}\{A_1, A_2, \ldots A_k\}$, and so we have that
$\text{span}\{A_1, A_2, \ldots A_k\} = \text{span}\{A_1^T, A_2^T, \ldots A_k^T\} = M_{n\times n}(\mathbb{R})$, which is what I needed to prove.

□

2. Decide if the following statement is True or False:

Let V be a vector space, and let $T : V \to \mathbb{R}$ be a linear transformation. Suppose that $\mathbf{x}, \mathbf{y}$ are linearly independent vectors in V, and that $T\mathbf{x} = 2$ and $T\mathbf{y} = \frac{4}{3}$. Then there exists a non-zero vector $\mathbf{z} \in V$ such that $T\mathbf{z} = 0$.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

This statement is true. I want to show that there exists a non-zero vector $\mathbf{z} \in V$ such that $T\mathbf{z} = 0$.

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent vectors in V.
Take $\mathbf{z} = -\frac{3}{2}\mathbf{y} + \mathbf{x}$. Because $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, and because V is a vector space, then $\mathbf{z} \in V$ as well, as V satisfies the addition and scalar multiplication vector space axioms.
$\mathbf{x}$ and $\mathbf{y}$ are linearly independent vectors in V, so $\mathbf{z} = -\frac{3}{2}\mathbf{y} + \mathbf{x}$ can not be zero. If $\mathbf{x}$ or $\mathbf{y}$ was 0, then $\mathbf{x} \neq \mathbf{y}$ because $T\mathbf{x} = 2 \neq \frac{4}{3} = T\mathbf{y}$ and thus $\mathbf{z} = -\frac{3}{2}\mathbf{y} + \mathbf{x}$ would still not be zero.

Because T is linear, then

$$T\mathbf{z} = T\left(-\frac{3}{2}\mathbf{y} + \mathbf{x}\right) = -\frac{3}{2}T\mathbf{y} + T\mathbf{x} = -\frac{3}{2} \cdot \frac{4}{3} + 2 = (-2) + 2 = 0.$$

Therefore there exists a vector $\mathbf{z} \in V$ such that $T\mathbf{z} = 0$, which is what I needed to prove.

□

3. Decide if the following statement is True or False:

Let V and W be vector spaces, and let $S, T : V \to W$ be linear transformations. The subset $U = \{\mathbf{x} \in V \mid S\mathbf{x} = T\mathbf{x}\}$ is a subspace of V.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

● True
○ False

I want to prove that the subset $U = \{\mathbf{x} \in V \mid S\mathbf{x} = T\mathbf{x}\}$ is a subspace of V.

Proof. I will begin by showing that U is non-empty. Consider the zero vector in V, $\mathbf{0}_V$. Because T and S are linear mappings, then both T and S always map $\mathbf{0}$ in the domain to $\mathbf{0}$ in the codomain.
To show this, consider any vectors $\mathbf{a}, \mathbf{b} \in V$ and any scalars $c, d \in \mathbb{R}$. Suppose $c = d = 0$. This implies that $T(c\mathbf{a} + d\mathbf{b}) = T(0\mathbf{a} + 0\mathbf{b}) = T(\mathbf{0}_V) = \mathbf{0}_W$ and $S(c\mathbf{a} + d\mathbf{b}) = S(0\mathbf{a} + 0\mathbf{b}) = S(\mathbf{0}_V) = \mathbf{0}_W$, so $T\mathbf{0}_V = S\mathbf{0}_V$.
Therefore U is non-empty and contains $\mathbf{0}_V$.

I will now show that U is closed under scalar multiplication and vector addition.
Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in V such that $T\mathbf{v}_1 = S\mathbf{v}_1$ and $T\mathbf{v}_2 = S\mathbf{v}_2$, and let $\lambda \in \mathbb{R}$ be a scalar.
Since $\mathbf{v}_1, \mathbf{v}_2 \in V$ and V is a vector space, then $\lambda\mathbf{v}_1 + \mathbf{v}_2 \in V$ as well.
Because T and S are linear, we then have that

$$T(\lambda\mathbf{v}_1 + \mathbf{v}_2) = \lambda T\mathbf{v}_1 + T\mathbf{v}_2 = \lambda S\mathbf{v}_1 + S\mathbf{v}_2 = S(\lambda\mathbf{v}_1 + \mathbf{v}_2),$$

which implies that U is closed under scalar multiplication and vector addition from V, and therefore U is a subspace of V, which is what I needed to prove.

□

4. Decide if the following statement is True or False:

Let U, W, and X be subspaces of a finite dimensional vector space V such that $U \cap W = \{\mathbf{0}\}$ and $U \cap X = \{\mathbf{0}\}$. If $U + W = U + X$, then $W = X$.

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●) and justify your choice with a proof or counter-example. [4 marks]

○ True
● False

This is false. I will provide a counterexample. Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ where $\mathbf{v}_1 = (1,0), \mathbf{v}_2 = (0,1)$ and $\mathbf{v}_3 = (2,-3)$. Now, consider their subspaces of $\mathbb{R}^2$, $U = \text{span}\{\mathbf{v}_1\}, W = \text{span}\{\mathbf{v}_2\}, X = \text{span}\{\mathbf{v}_3\} \subseteq \mathbb{R}^2$.
We have that $U + W = \text{span}\{\mathbf{v}_1\} \cup \text{span}\{\mathbf{v}_2\} = \mathbb{R}^2$ and $U + X = \text{span}\{\mathbf{v}_1\} \cup \text{span}\{\mathbf{v}_3\} = \mathbb{R}^2$, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span $\mathbb{R}^2$ as well as $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly independent and span $\mathbb{R}^2$.
Furthermore, we have that $U \cap W$, or $\text{span}\{\mathbf{v}_1\} \cap \text{span}\{\mathbf{v}_2\} = \{\mathbf{0}\}$. Similarly, we also have that $U \cap X$, or $\text{span}\{\mathbf{v}_1\} \cap \text{span}\{\mathbf{v}_3\} = \{\mathbf{0}\}$.

Therefore, we have concluded that this statement is false, as $\text{span}\{\mathbf{v}_1\} \cap \text{span}\{\mathbf{v}_2\} = \{\mathbf{0}\}$, $\text{span}\{\mathbf{v}_1\} \cap \text{span}\{\mathbf{v}_3\} = \{\mathbf{0}\}$ and $\text{span}\{\mathbf{v}_1\} \cup \text{span}\{\mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\} \cup \text{span}\{\mathbf{v}_3\} = \mathbb{R}^2$, and $\text{span}\{\mathbf{v}_2\} \neq \text{span}\{\mathbf{v}_3\}$.
More generally, $U \cap W = \{\mathbf{0}\}$, $U \cap X = \{\mathbf{0}\}$, $U + W = U + X$ and $W \neq X$, as needed.

5. Let V be a vector space, and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in V$.

5(a) Prove that if either $\mathbf{x}_1 = \mathbf{0}$ or $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent. [4 marks]

I want to prove that for a list of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in V$, where V is a vector space, that if either $\mathbf{x}_1 = \mathbf{0}$ or if $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$, for some $j = 1, 2, \ldots, k-1$, then the list $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is linearly dependent.

Proof. Let $a_1, a_2, \ldots a_k \in \mathbb{R}$ be scalars such that $\mathbf{0} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ is a linear combination of the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$.
By [Definition 1.4.2], we define a list of linearly dependent vectors to be a linear combination of vectors such that there exists at least one scalar a_i, for $1 \leq i \leq k$, such that $a_i \neq 0$.

Suppose that $\mathbf{x}_1 = \mathbf{0}$. If $\mathbf{x}_1 = \mathbf{0}$, then suppose that $a_1 \neq 0$ is the only non-zero scalar in the linear combination. We then have that

$$\mathbf{0} = a_1\mathbf{0} + 0\mathbf{x}_2 + 0\mathbf{x}_3 + \cdots + 0\mathbf{x}_k = a_1\mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}(a_1 + (k-1)).$$

Because there is one scalar $a_1 \neq 0$ and because $\mathbf{0}$ can be created from a linear combination of the other vectors in the list, then if $\mathbf{x} = \mathbf{0}$, the list $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is linearly dependent. Furthermore, by [Example 1.4.3, (a)], any vector $\mathbf{x} = \mathbf{0}$ is linearly dependent in any vector space or any linear combination.

Now suppose that $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$. By the definition of span, $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$ can be expressed as a linear combination of vectors in the span, so $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_j\mathbf{x}_j$ for any scalars $b_1, b_2, \ldots, b_j \in \mathbb{R}$.
Because $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$, then $\mathbf{x}_{j+1}$ is an element in the linear combination $b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_j\mathbf{x}_j$, thus $\mathbf{x}_{j+1}$ is already contained in and can be created from the linear combination of vectors in the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$.
By [Definition 1.4.2], this implies that there exists at least one scalar in the linear combination
$b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_j\mathbf{x}_j + b_{j+1}\mathbf{x}_{j+1}$ that is not zero, and so the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \mathbf{x}_{j+1}\}$ is linearly dependent.
However, $j = 1, 2, \ldots, k-1$, and so the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \mathbf{x}_{j+1}\}$ is equivalent to $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k\}$, and therefore if $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$, then the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent, which is what I needed to prove.

$\square$

5. Let V be a vector space, and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in V$.

5(b) Prove that if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent, then either $\mathbf{x}_1 = \mathbf{0}$ or $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$. [4 marks]

I want to prove that if the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent, then either $\mathbf{x}_1 = \mathbf{0}$ or $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$.

Proof. Suppose that the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent. By [Definition 1.4.2], there exists at least one scalar $a_i \in \mathbb{R}$ in the linear combination $\mathbf{0} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ such that $a_i \neq 0$. This implies that at least one vector in the list can be created from a linear combination of the other vectors in that list.

Suppose that the scalar $a_1 \neq 0$ and that the other scalars $a_2, a_3, \ldots, a_k = 0$. We then have the linear combination

$$\mathbf{0} = a_1\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k = a_1\mathbf{x}_1 + \mathbf{0} + \cdots + \mathbf{0} = a_1\mathbf{x}_1.$$

However, if $a_1\mathbf{x}_1 = \mathbf{0}$ and because $a_1 \neq 0$, then it must follow that $\mathbf{x} = \mathbf{0}$.
Therefore if the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent then $\mathbf{x}_1 = \mathbf{0}$. Furthermore, by [Example 1.4.3 (a)], any vector $\mathbf{x} = \mathbf{0}$ is linearly dependent in any linear combination.

Now I want to prove that if the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent, then $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$.

Consider the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}\} \subseteq \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ which is equivalent to $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\} \subseteq \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ for $j = 1, 2, \ldots, k-1$. Because $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent, then we can conclude that the list $\{\mathbf{x}_{j+1}\} \cup \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$ is linearly independent, where $\mathbf{x}_{j+1} = \mathbf{x}_k$. We cannot conclude anything about the linear dependence of the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$.
Since $\{\mathbf{x}_{j+1}\} \cup \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$ is linearly independent, we know that $\mathbf{x}_{j+1}$ is contained in the list $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$, which directly implies that $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_j\}$.

Therefore if the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent, then either $\mathbf{x}_1 = \mathbf{0}$ or $\mathbf{x}_{j+1} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$ for some $j = 1, 2, \ldots, k-1$, which is what I needed to prove.

$\square$

6. Let U and W be 3-dimensional subspaces of a 4-dimensional vector space V, and suppose that $U \neq W$.

6(a) Prove that $U + W \neq U$, and $U + W \neq W$. [4 marks]

I want to show that for any 3-dimensional vector subspace U and W of V where $U \neq W$, $U + W \neq U$ and $U + W \neq W$.

Proof. Because U and W are both of dimension 3, then each subspace has at least 3 linearly independent vectors in its basis.
Let $U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for U and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for W.
Suppose that $U \neq W$. We then have that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.
By [Proposition 1.3.8], the sum of subspaces is the union of their spans, so we then have that

$$U + W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \cup \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}.$$

We then have that $W + U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \neq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = U$ and that
$U + W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \neq \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = W$,
primarily because $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$; their bases both contain different elements.
Therefore I have proven that $U + W \neq U$ and that $U + W \neq W$, as needed.

$\square$

6. Let U and W be 3-dimensional subspaces of a 4-dimensional vector space V, and suppose that $U \neq W$.

6(b) Prove that $U + W = V$. [4 marks]

I want to prove that for the 3-dimensional subspaces U and W of a 4-dimensional vector space V, $V = U + W$.

Proof. Because U and W are both of dimension 3, then each subspace has at least 3 linearly independent vectors in its basis.
Let $U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for U and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for W.
Suppose that $U \neq W$. We then have that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.
By [Proposition 1.3.8], the sum of subspaces is the union of their spans, so we then have that

$$U + W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \cup \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}.$$

By [Theorem 1.6.10], we have that $\dim(U + W)$ can be at most 4 because no linearly independent set spanning V can have more than 4 elements, and thus the list $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent and has at most 4 linearly independent elements.
Suppose that $\mathbf{u}_3 = c\mathbf{w}_1$ and $\mathbf{w}_3 = d\mathbf{u}_1$ are linearly dependent vectors in the list for some $c, d \in \mathbb{R}$. We then have that the list $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2\}$ is a linearly independent set for the sum of U and W, and thus $U + W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2\}$.
We then have that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2\} \subseteq V$, and $\dim(\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2\}) = 4$, so the list $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2\}$ is a basis for V and that $U + W = V$, which is what I needed to prove.

$\square$

6. Let U and W be 3-dimensional subspaces of a 4-dimensional vector space V, and suppose that $U \neq W$.

6(c) Is V the direct sum of U and W? [4 marks]

No, V is not a direct sum of U and W. This directly follows from the dimension theorem,

$$\dim(U) + \dim(W) - \dim(U \cap W) = \dim(V).$$

By contradiction. Suppose V is a direct sum of U and W. Then $\dim(U \cap W) = \dim(\{\mathbf{0}\}) = 0$, because if V is a direct sum of U and W, then $U \cap W = \{\mathbf{0}\}$. We then have that

$$\dim(U) + \dim(W) - \dim(U \cap W) = \dim(U) + \dim(W) - 0 = \dim(V),$$

however $\dim(V) = 4$ and $\dim(U) = \dim(W) = 3$, so we then have that

$$\dim(U) + \dim(W) = 3 + 3 = 6 \neq 4 = \dim(V).$$

We have reached a contradiction. This implies that $\dim(U \cap W) \neq 0$ because $\dim(U) + \dim(W) \neq \dim(V)$, so because $\dim(U \cap W) \neq 0$, V cannot be a direct sum of U and W, which is what I needed to show.