PHY256 PS3

Q1) $\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$, $R = 1.09737 \times 10^7 \frac{1}{m}$

$E = h\nu = \frac{hc}{\lambda}$, $h_{ev} = 4.1357 \times 10^{-15}$ eV·s

a) We have that $E = hcR\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$, where $n_f = 1$ and $n_i = 3$.

Then $E = (4.1357 \times 10^{-15} \text{ eV·s})(3 \times 10^8 \frac{m}{s})(1.09737 \times 10^7 \frac{1}{m})\left(1 - \frac{1}{9}\right)$

$\boxed{E = 12.10238 \text{ eV}}$

b) To find E in Joules, we can convert $\left[\frac{1.6 \times 10^{-19} J}{1 eV}\right]$, so

$E = 12.10238 \text{ eV} \times \left[\frac{1.6 \times 10^{-19} J}{1 eV}\right]$

$\boxed{E = 1.93638 \times 10^{-18} \text{ J}}$

c) $1.9\cdots \times 10^{-18}$ J is not enough Energy to apply the relativistic K.E. formula, thus we have

$E_K = \frac{1}{2} m_e v^2$, where $m_e = 9.11 \times 10^{-31}$ kg.

Then $v = \sqrt{\frac{2E_K}{m_e}}$,

$v = \sqrt{\frac{2(1.93638 \times 10^{-18} \text{ J})}{9.11 \times 10^{-31} \text{ kg}}}$

$\boxed{v = 2.06182 \times 10^6 \text{ m/s}}$

To find the deBroglie wavelength, we have that

$$p = m_e v = \frac{h}{\lambda}, \quad \text{thus} \quad \lambda_{dB} = \frac{h}{m_e v} = \frac{h}{m_e \sqrt{\frac{2E_K}{m_e}}}$$

so $$\lambda_{dB} = \frac{h_J}{\sqrt{2 E_K m_e}}$$

$$= \frac{6.6 \times 10^{-34} \text{ J·s}}{\sqrt{2(1.93638106 \times 10^{-18} \text{ J})(9.11 \times 10^{-31} \text{ kg})}}$$

$$\boxed{\lambda_{dB} = 3.5138 \times 10^{-10} \text{ m}}$$

d) Similarly, if we approximate the average mass of a human by 62 kg, we have

$$v = \sqrt{\frac{2(1.93638106 \times 10^{-18} \text{ J})}{62}}$$

$$\boxed{v = 2.499 \times 10^{-10} \text{ m/s}}$$

$$\lambda_{dB} = \frac{6.6 \times 10^{-34} \text{ J·s}}{\sqrt{2(1.93638106 \times 10^{-18} \text{ J})(62)}}$$

$$\boxed{\lambda_{dB} = 4.2593 \times 10^{-26} \text{ m}}$$

e) $E = h\nu$, so $\nu = \frac{c}{\lambda} = \frac{E}{h}$, thus

$$\nu = \frac{1.93638106 \times 10^{-18} \text{ J}}{6.6 \times 10^{-34} \text{ J·s}}$$

$$\boxed{\nu = 2.9339 \times 10^{15} \text{ Hz}}$$

and

$$E = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E}, \quad \text{so} \quad \lambda = \frac{(6.6 \times 10^{-34} \text{ J·s})(3 \times 10^{8} \text{ m/s})}{1.93638106 \times 10^{-18} \text{ J}}$$

$$\boxed{\lambda = 102 \times 10^{-9} \text{ m}}$$ (or 102 nm)

(which is ultraviolet radiation.)

PHY256 PS3

Q2) a) $[A, BC] = ABC - BCA$

$= ABC - BAC + BAC - BCA$

$= (AB - BA)C + B(AC - CA)$

$\boxed{= [A, B]C + B[A, C]}$

Analogue: $[AB, C] = [A, C]B + A[B, C]$

b) Know $[\hat{P}, \hat{X}] = -i\hbar \mathbb{I}$

Then $[\hat{P}^2, \hat{X}^2] = [\hat{P}^2, \hat{X}\hat{X}]$

$= [\hat{P}^2, \hat{X}]\hat{X} + \hat{X}[\hat{P}^2, \hat{X}]$

$= (\hat{P}[\hat{P}, \hat{X}] + [\hat{P}, \hat{X}]\hat{P})\hat{X} + \hat{X}(\hat{P}[\hat{P}, \hat{X}] + [\hat{P}, \hat{X}]\hat{P})$

$= -2i\hbar \hat{P}\hat{X} - 2i\hbar \hat{X}\hat{P}$

Since $\hat{P}\hat{X} - \hat{X}\hat{P} = -i\hbar\mathbb{I}$, $\hat{X}\hat{P} = i\hbar\mathbb{I} + \hat{P}\hat{X}$,

which implies

$= -2i\hbar \hat{P}\hat{X} - 2i\hbar(i\hbar\mathbb{I} + \hat{P}\hat{X})$

$= -2i\hbar \hat{P}\hat{X} + 2\hbar^2\mathbb{I} - 2i\hbar\hat{P}\hat{X}$

$\boxed{= 2\hbar^2\mathbb{I} - 4i\hbar\hat{P}\hat{X}}$

PHY256 PS3

Q3) We have $B^2|\Psi\rangle = \lambda|\Psi\rangle$.

a) If $|\Psi\rangle$ is an eigenstate of B^2 with definite eigenvalue λ, then $\Delta B^2 = 0$.

b) $|\Psi\rangle = \sum_n c_n |b_n\rangle$

Then
$$B^2|\Psi\rangle = B[B|\Psi\rangle]$$
$$= B\left[B\sum_n c_n|b_n\rangle\right]$$
$$= B\sum_n c_n b_n |b_n\rangle$$

(since $|b_n\rangle$ is an eigenstate of B with eigenvalue b_n)

Thus $\boxed{B^2|\Psi\rangle = \sum_n c_n b_n^2 |b_n\rangle}$

c) The eigenvalue equation is then $B^2|\Psi\rangle = \lambda \cdot \sum_n c_n|b_n\rangle$.

$$\Rightarrow \langle b_j|B^2|\Psi\rangle = \langle b_j|\sum_n c_n b_n^2 |b_n\rangle$$
$$= \sum_n c_n b_n^2 \langle b_j|b_n\rangle$$
$$\lambda\langle b_j|\Psi\rangle = \sum_n c_n b_n^2 \delta_{jn}$$
$$\lambda\sum_n c_n \delta_{jn} = \sum_n c_n b_n^2 \delta_{jn}$$
$$\lambda c_j = c_j b_j^2 \implies \boxed{\lambda = b_j^2}, \quad \boxed{c_j \neq 0.}$$

If you measured B, the eigenvalue you would get is a specific eigenvalue of B corresponding to an eigenvector $|b_i\rangle$ in the B basis, but it would be $\pm$

ie. $B|b_i\rangle = b_i|b_i\rangle$, then $B|\Psi\rangle = \pm b_i|\Psi\rangle$ for instance.

PHY256 PS3

Q4) a) We know $[\sigma_x, \sigma_y] = 2i\sigma_z$, $[\sigma_y, \sigma_z] = 2i\sigma_x$, $[\sigma_x, \sigma_z] = 2i\sigma_y$

Evaluate $[\sigma_x, \sigma_x^2] = [\sigma_x, \sigma_x]\sigma_x + \sigma_x[\sigma_x, \sigma_x]$

$$= (\sigma_x^2 - \sigma_x^2)\sigma_x + \sigma_x(\sigma_x^2 - \sigma_x^2)$$

$$\boxed{= 0}$$

$$[\sigma_x, \sigma_y^2] = [\sigma_x, \sigma_y]\sigma_y + \sigma_y[\sigma_x, \sigma_y]$$

$$= 2i\sigma_z\sigma_y + 2i\sigma_y\sigma_z$$

$$= 2i\sigma_z\sigma_y - 2i\sigma_z\sigma_y$$

$$\boxed{= 0}$$

$$[\sigma_x, \sigma_z^2] = [\sigma_x, \sigma_z]\sigma_z + \sigma_z[\sigma_x, \sigma_z]$$

$$= 2i\sigma_y\sigma_z + 2i\sigma_z\sigma_y$$

$$= 2i\sigma_y\sigma_z - 2i\sigma_y\sigma_z$$

$$\boxed{= 0}$$

Simpler argument: all pauli matrices have the property that $\sigma_i^2 = \mathbb{I}$ for $i = x, y, z$. Then, easily by inspection, $[\sigma_x, \sigma_x^2] = [\sigma_x, \sigma_y^2] = [\sigma_x, \sigma_z^2] = [\sigma_x, \mathbb{I}] = 0$

b) $S^2 = S_x^2 + S_y^2 + S_z^2 = \left(\frac{\hbar}{2}\right)^2\mathbb{I} + \left(\frac{\hbar}{2}\right)^2\mathbb{I} + \left(\frac{\hbar}{2}\right)^2\mathbb{I} = \frac{3\hbar^2}{4}\mathbb{I}$.

Then $[S_x, S^2] = \frac{3\hbar^3}{8}\sigma_x\mathbb{I} - \frac{3\hbar^3}{8}\mathbb{I}\sigma_x \boxed{= 0}$,

(c) and thus yes, S_x and S^2 are compatible, since $[S_x, S^2] = 0$.

d) $[S_x, S_y] = \frac{\hbar^2}{4}\sigma_x\sigma_y - \frac{\hbar^2}{4}\sigma_y\sigma_x$

$$= \frac{\hbar^2}{4}[\sigma_x, \sigma_y]$$

$$= \frac{\hbar^2}{2} i\,\sigma_z \quad \neq 0,$$

and thus no, S_x and S_y are not compatible.

e) As before, $S^2 = \frac{3\hbar^2}{4}\mathbb{I}$, which implies that

the eigenvalues of S^2 for a spin-$\frac{1}{2}$ particle is only

$\boxed{\lambda = \frac{3\hbar^2}{4}}$.

PHY256 PS3

Q5) a) Let $|\Psi\rangle = \frac{1}{\sqrt{2}}|S1\rangle + \frac{1}{\sqrt{2}}|S2\rangle$, where $|S1\rangle := |Slit\ 1\rangle$ and $|S2\rangle := |Slit\ 2\rangle$.

Let $|\Psi'\rangle = \frac{1}{\sqrt{2}}|S1\rangle|A\rangle + \frac{1}{\sqrt{2}}|S2\rangle|B\rangle$ be the spy's failed measurement.

- We want to find $\langle\Psi'|x\rangle\langle x|\Pi_{spy}|\Psi'\rangle$, or $\langle M_{part}\, \Pi_{spy}\rangle$,

where $|x\rangle\langle x| = (|S1\rangle + |S2\rangle e^{i\phi})(\langle S1| + \langle S2|e^{-i\phi})$

$$= |S1\rangle\langle S1| + |S1\rangle\langle S2|e^{-i\phi} + |S2\rangle\langle S1|e^{i\phi} + |S2\rangle\langle S2|$$

and $\Pi_{spy} = |S1\rangle|A\rangle\langle S1|\langle A| + |S2\rangle|B\rangle\langle S2|\langle B|$.

- We then have

$$\left(\frac{1}{\sqrt{2}}\langle S1|\langle A| + \frac{1}{\sqrt{2}}\langle S2|\langle B|\right)\Big[\left(|S1\rangle\langle S1| + |S1\rangle\langle S2|e^{-i\phi} + |S2\rangle\langle S1|e^{i\phi} + |S2\rangle\langle S2|\right)$$
$$\left(|S1\rangle|A\rangle\langle S1|\langle A| + |S2\rangle|B\rangle\langle S2|\langle B|\right)\Big]\left(\frac{1}{\sqrt{2}}|S1\rangle|A\rangle + \frac{1}{\sqrt{2}}|S2\rangle|B\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2}}\langle S1|\langle A| + \frac{1}{\sqrt{2}}\langle S2|\langle B|\right)\left(|S1\rangle\langle S1| + |S1\rangle\langle S2|e^{-i\phi} + |S2\rangle\langle S1|e^{i\phi} + |S2\rangle\langle S2|\right)$$
$$\left(\frac{1}{\sqrt{2}}|S1\rangle|A\rangle + \frac{1}{\sqrt{2}}|S2\rangle|B\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2}}\langle S1|\langle A| + \frac{1}{\sqrt{2}}\langle S2|\langle B|\right)\left[\frac{1}{\sqrt{2}}|S1\rangle|A\rangle + \frac{1}{\sqrt{2}}|S2\rangle|A\rangle e^{i\phi} + \frac{1}{\sqrt{2}}|S1\rangle|B\rangle e^{-i\phi} + \frac{1}{\sqrt{2}}|S2\rangle|B\rangle\right]$$

$$= \frac{1}{2} + \frac{1}{2}\langle A|B\rangle e^{-i\phi} + \frac{1}{2}\langle B|A\rangle e^{i\phi} + \frac{1}{2}$$

- The max of $\langle M_{part}\, \Pi_{spy}\rangle$ is given when $\phi = 0$ $(e^{\pm i\phi} = 1)$ and min when $\phi = \pi$ $(e^{\pm i\pi} = -1)$.

- The visibility is then $\frac{max - min}{max + min} = \frac{1 + \frac{1}{2}\langle A|B\rangle + \frac{1}{2}\langle B|A\rangle - (1 - \frac{1}{2}\langle A|B\rangle - \frac{1}{2}\langle B|A\rangle)}{2 + \frac{1}{2}\langle A|B\rangle + \frac{1}{2}\langle B|A\rangle - \frac{1}{2}\langle A|B\rangle - \frac{1}{2}\langle B|A\rangle}$

$$= \frac{\langle A|B\rangle + \langle B|A\rangle}{2}.$$

- $|A\rangle$ and $|B\rangle$ may not be orthogonal, and we may assume $\langle A|B\rangle = \langle B|A\rangle$, and thus the visibility function reduces to

$$V = \frac{\langle A|B\rangle + \langle B|A\rangle}{2}$$

$$\Rightarrow \boxed{V = \langle A|B\rangle}$$

b) If the spy was successful in his measurement, $|A\rangle$ and $|B\rangle$ would have definite values because the measured state would collapse to either $|\text{Slit } 1\rangle$ or $|\text{Slit } 2\rangle$, whether or not the spy observed the particle to go through either slit (if its in $|S1\rangle$, the we know its not in $|S2\rangle$ and if the particle is not in $|S2\rangle$, we know with 100% certainty that it is in $|S1\rangle$).

Then this would imply that there is a definite value of $\langle A|B\rangle$, implying that (from part (a)) we would no longer see a visibility distribution on the screen but rather just a 'dot' where the particle landed on the screen.