MAT244 Midterm 1

Main sitting, October 20, 18:00-20:00

Please print as legibly as possible:

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Instructions:

- This test is closed book. No calculators, phones or notes are permitted.
- You have 105 minutes to complete the test.
- Do not write on the top section of the pages. This area needs to be clear for the scanning and matching to be done correctly.
- Only the front of each page will be scanned and uploaded to Gradescope for grading. THE BACK OF EVERY PAGE IS FOR ROUGH WORK ONLY AND WILL NOT BE GRADED
- **ANY WORK WRITTEN ON THE BACK OF ANY PAGE WILL NOT BE GRADED OR CONSIDERED IN ANYWAY**
- This test consists of 14 pages including the cover page. The last two pages are extra space. If you want any work on these extra pages to be considered in grading you must indicate so on the page of the relevant question that work from that question is in the extra space.
- If you require extra space beyond that included please contact an invigilator. If you include extra pages, please set your exam aside at the end and do not include it with the main pile(s).
- Unless noted otherwise justify all solutions.
- The test is out of 90 points

1. Consider the ODE,

$$y'(t) + \frac{2t}{t^2+1}y(t) = 16(t^2+1)^2 t$$

(a) (8 points) Find the general solution via the method of integrating factors or otherwise

We have that $p(x) = \frac{2t}{t^2+1}$. Then $I(t) = \exp\left(\int \frac{2t}{t^2+1}\,dt\right)$
$= \exp(\log|t^2+1|)$.

Our integrating factor is then $I(t) = t^2+1$.
Multiplying through,

$(t^2+1)y'(t) + \cancel{(t^2+1)}\frac{2t}{\cancel{t^2+1}}y(t) = 16(t^2+1)^3 t$

$\Rightarrow \frac{d}{dt}[(t^2+1)y(t)] = 16(t^2+1)^3 t$ by the product rule.

Integrating both sides,

$\int \frac{d}{dt}\cdot[(t^2+1)y(t)] = 16\int (t^2+1)^3 t\,dt$ Let $u = t^2+1$ $du = 2t\,dt \Rightarrow t\,dt = \frac{du}{2}$
$= 8\int u^3\,du = \cancel{8}^{2}\frac{1}{\cancel{4}}u^4 + C$ for $C \in \mathbb{R}$

So $\boxed{y(t) = 2(t^2+1)^3 + \frac{C}{t^2+1}}$

(b) (5 points) Find the solution satisfying $y(0) = 1$

If $y(0) = 1$, then $1 = 2(1)^3 + \frac{C}{(0)+1}$

which implies that $1-2 = C$, so $C = -1$.

The solution to the IVP is then

$\boxed{y(t) = 2(t^2+1)^3 - \frac{1}{t^2+1}}$

2. Consider the ODE,
$$y''(t) + 4y'(t) + 8y(t) = 0$$

(a) (5 points) Find the characteristic equation (or characteristic polynomial) and its roots

Let $y = Ae^{rt}$. Then $y'' = r^2 e^{rt}$, $y' = re^{rt}$. We then have

$r^2 e^{rt} + 4re^{rt} + 8e^{rt} = 0$, This implies our characteristic eq$^{\underline{n}}$

is $r^2 + 4r + 8 = 0$, Factoring, we have

$r^2 + 4r + 8 \Rightarrow r = \frac{-4 \pm \sqrt{16 - 4 \cdot 8}}{2}$, or, for $r \in \mathbb{C}$,

$\boxed{r^2 + 4r + 8, \quad r = -2 \pm 2i}$

(b) (3 points) Find the general real solution of the equation

Our roots are $r = -2 \pm 2i$. Thus our general solution, complex, is

$y(t) = e^{-2t}\left[Ae^{2it} + Be^{-2it}\right]$. By De Moivres Theorem; we have

$y(t) = e^{-2t}\left[A\cos(2t) + Ai\sin(2t) + B\cos(2t) - Bi\sin(2t)\right]$

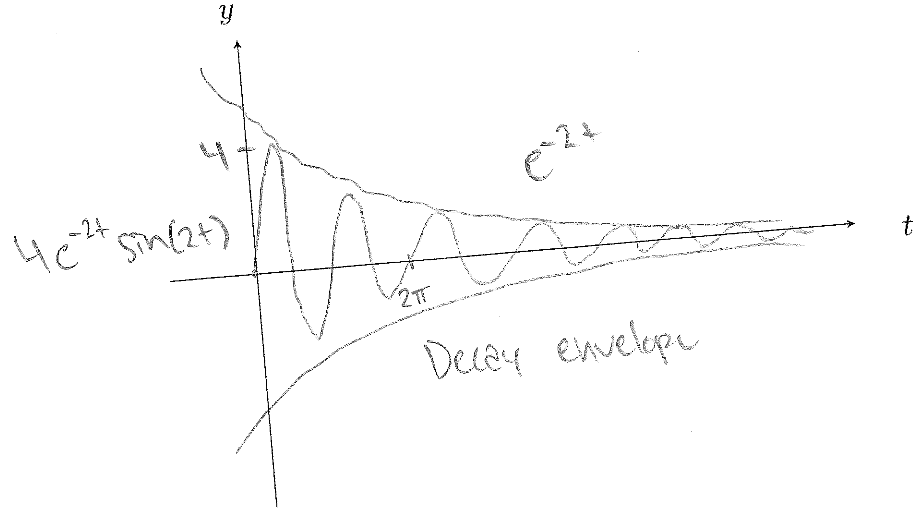
$= e^{-2t}\left[(A+B)\cos(2t) + i(A-B)\sin(2t)\right]$

for a constant A and B,

Thus the general real solution,

$\boxed{y(t) = e^{-2t}(A+B)\cos(2t)}$

(c) (3 points) Roughly sketch the solution that satisfies $y(0) = 0$ and $y'(0) = 2$ on the provided axes



$y = e^{-2t}[(A+B)\cos(2t) + i(A-B)\sin(2t)]$

Then $y' = -2e^{-2t}[(A+B)\cos(2t) + i(A-B)\sin(2t)]$
$+ e^{-2t}[-(A+B)\sin(2t) + i(A-B)\cos(2t)]$

$0 = \cancel{e^0}[(A+B)\cancel{\cos(0)}] \Rightarrow A = -B.$

$2 = -2[(A+B)\cancel{\cos(0)} + i(A-B)\cancel{\sin(0)}]$
$+ \cancel{e^{-2t\cdot 0}}[-(A+B)(0) + i(A-B)\cancel{\cos(0)}]$

$\cancel{2} = -2 + i(A-B)$
$\cancel{2} = -\cancel{2} + i(-2B) \Rightarrow -\frac{2}{i} = B,$ so $B = 2i$. Then $A = -2i$.

Can't draw imaginary numbers. :)
$y = e^{-2t}[+i(-2i-2i)\sin(2t)] = e^{-2t}[4\sin(2t)]$

3. Consider the ODE,

$$y''(t) - 6y'(t) + 9y(t) = 0$$

(a) (5 points) Find the characteristic equation (or characteristic polynomial) and its roots

Let $y = e^{rt}$. Then $r^2 - 6r + 9 = 0$. This factors into $(r-3)^2$, so we have a repeated root of $r=3$ with multiplicity two. Thus

$$\boxed{r^2 - 6r + 9 = 0 \Rightarrow r = 3 \text{ is a repeated root}}$$

(b) (3 points) Find the general real solution of the equation

By reduction of order, one solution is given by $y_1 = Ae^{3t}$. Another solution is then $y_2 = Bte^{3t}$.

Thus our general solution is then

$$\boxed{y(t) = Ae^{3t} + Bte^{3t}.}$$

4. Consider the ODE,

$$y''(t) + 2y'(t) - 3y(t) = -10\sin(t)$$

(a) (10 points) Find the general solution

Let $G(t) = A\sin t + B\cos t$.
$\Rightarrow G'(t) = A\cos t - B\sin t$. plugging into our ODE, we have
$\Rightarrow G''(t) = -A\sin t - B\cos t$.

$-A\sin t - B\cos t + 2A\cos t - 2B\sin t - 3A\sin t - 3B\cos t = -10\sin t$.

Thus cos: $-B + 2A - 3B = -4B + 2A = 0$
sin: $-A - 2B - 3A = -4A - 2B = -10$.

Solving, $A=2$ and $B=1$. Then $y_p(t) = 2\sin t + \cos t$.
Now let $y = Ae^{rt}$. Then $r^2 + 2r - 3 = 0$. This factors into $(r-1)(r+3)$

Our general solution is then $X = X_h + X_p$ so

$\boxed{y(t) = Ae^{t} + Be^{-3t} + 2\sin t + \cos t.}$

(b) (4 points) Find the solution with $y(0) = 1$ and $y'(0) = 6$.

$0 = A + B + 1$ (y) $\Rightarrow$ $A + B = -1$
$6 = A - 3B + 2$ (y') $\quad A - 3B = 4$.

$A = -1 - B$, so $-1 - B - 3B = 4$, or $-4B = 5$, so $B = -\frac{5}{4}$

Then $A = -1 + \frac{5}{4} = \frac{-4+5}{4} = \frac{1}{4}$, So $A = \frac{1}{4}$.

The solution to the IVP is then

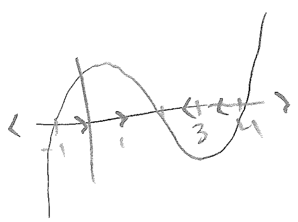
$\boxed{y(t) = \frac{1}{4}e^{t} - \frac{5}{4}e^{-3t} + 2\sin t + \cos t}$

5. Consider the autonomous ODE,

$$y' = (y+1)(y-2)(y-4)$$

(a) (5 points) Find and classify all equilibrium points as stable or unstable

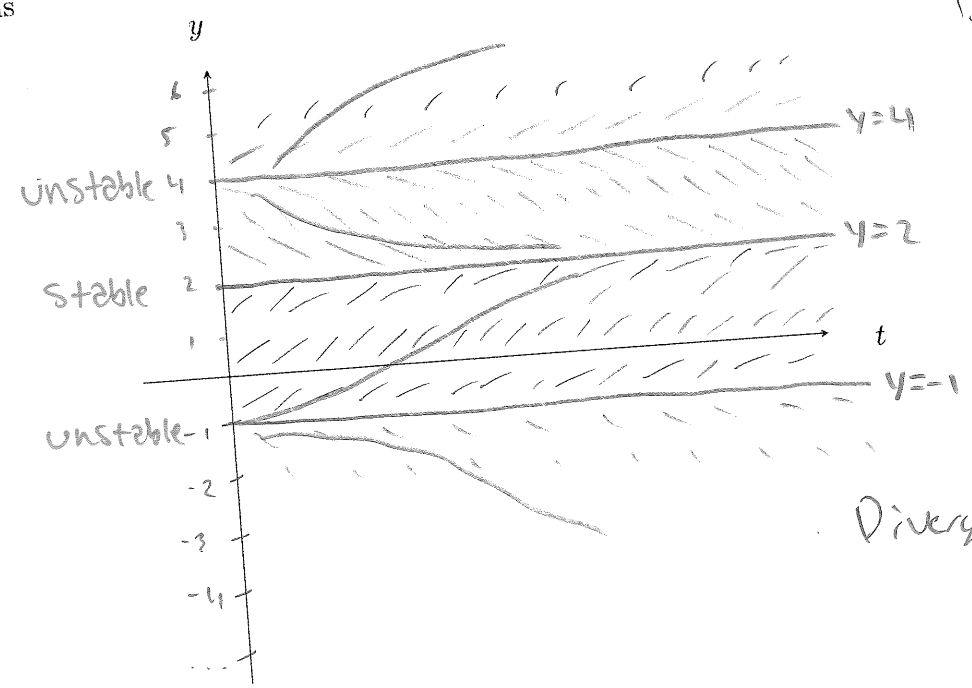
Our equilibrium points are when $y = -1, 2, 4$.
This is a positive cubic function, we can graph it:



When $y=0$, $y' = 1 \cdot (-2) \cdot (-4) > 0$, so the solution would want to tend to $y=2$.
$y=1$, $y' = 2 \cdot (-1) \cdot (-3) > 0$. Also tend to $y=2$.
$y=-2$, $y' = (-1) \cdot (-4) \cdot (-6) < 0$, tend away from $y=-$
$y=3$, $y' = (4)(1)(-1) < 0$, tend toward $y=2$.
$y=5$, $y' = 6 \cdot 4 \cdot 1 > 0$, tend away from $y=4$.

Thus $y=-1$ unstable, $y=2$ stable, $y=4$ unstable.

(b) (5 points) Sketch the behavior of solutions on the provided axes, labelling equilibrium solutions



Diverge as $t \to \infty$

Converge to $y=2$ as $t \to \infty$

Diverge as $t \to \infty$

(c) (2 points) Suppose $y(0) = 3$. Use your sketch to find $\lim_{t\to\infty} y(t)$.

Solution, as $t \to \infty$, would want to converge to $y=2$ as $t \to \infty$. Thus

$\boxed{\lim_{t\to\infty} y(t) = 2.}$

(d) (2 points) Suppose $y(0) = 0$. Use your sketch to find $\lim_{t\to\infty} y(t)$.

Solution, as $t \to \infty$, would want to diverge from $y=-1$ to $y=2$ as $t \to \infty$, thus

$\boxed{\lim_{t\to\infty} y(t) = 2.}$

6. (a) (4 points) Determine whether the following equation exact.

$$(\sin(y) + e^x - 1) + \frac{dy}{dx}(x\cos(y) - e^y) = 0$$

Let $M(x,y) = \sin y + e^x - 1$ and
$N(x,y) = x\cos y - e^y$.

If the equation is exact, then

$\partial_y M(x,y) = \partial_x N(x,y)$.

We have

$\partial_y M(x,y) = \cos y$ and
$\partial_x N(x,y) = \cos y$.

Therefore this equation is exact, since $\partial_y M = \partial_x N$.

(b) (8 points) The following equation is exact. Find the general solution in implicit form.

$$(y+1)+(x+1)\frac{dy}{dx}=0$$

Let $M(x,y) = y+1$ and $N(x,y) = x+1$.

We need a solution $\Psi(x,y)$ such that

$M(x,y) = \partial_x \Psi(x,y)$ and $N(x,y) = \partial_y \Psi(x,y)$.

Integrating with respect to x,

$\int \partial_x \Psi(x,y)\, dx = \int M(x,y)\, dx = \int (y+1)\, dx = xy + x + h(y)$. (+c

On the other hand,

$\int \partial_y \Psi(x,y)\, dy = \int N(x,y)\, dy = \int (x+1)\, dy = xy + y + h(x)$. (+c

By comparing the solutions, clearly $h(y) = y$ and $h(x) = x$,

thus the general solution, given implicitly is

$\boxed{\Psi(x,y) = xy + x + y + C}$ for a constant C.

7. (8 points) The following equation is separable. Find an implicit equation for the general solution.

$$y' = \frac{1}{e^t(y^4+1)}$$

$$y' = \frac{dy}{dt} = \frac{1}{e^t(y^4+1)}.$$

$$\Rightarrow (y^4+1)\,dy = \frac{dt}{e^t}.$$

$$\int (y^4+1)\,dy = \int e^{-t}\,dt$$

$$\frac{1}{5}y^5 + y = -e^{-t} + c \quad \text{for a constant } c \in \mathbb{R}.$$

Thus our implicit solution is

$$\boxed{y\left(\frac{1}{5}y^4+1\right) = e^{-t} + c} \quad \text{for a constant } c.$$

8. (10 points) Let $p(t)$ and $q(t)$ be continuous, real-valued functions with domain $\mathbb{R}$. Can $y = \sin(t^2)$ be a solution on an interval containing 0 of the equation $y'' + p(t)y' + q(t)y = 0$? Explain.

Hint: plug $y = \sin(t^2)$ into the LHS of the equation and evaluate at $t = 0$.

Taking the hint,

$y' = \cos(t^2) \cdot 2t$, so $y'' = -\sin(t^2) \cdot 2t \cdot 2t + \cos(t^2) \cdot 2$ by the product rule.

Then

$-4t^2 \sin(t^2) + 2\cos(t^2) + p(t)\, 2t \cos(t^2) + q(t) \sin(t^2) = 0.$

At $t=0$, this ODE satisfies $0=0$, thus yes, $y=\sin(t^2)$ can be a solution on an interval containing zero.

However, by the existence & uniqueness theorems for 2nd order ODE's $y=\sin(t^2)$ may only be a single valid solution to the ODE depending on $p(t)$ or $q(t)$; $y=\sin(t^2)$ can be a solution on an interval containing 0, but y may not be a valid solution to the rest of the interval depending on $p(t)$ or $q(t)$.

Extra work for Question ____
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