PHY354 PS3 — 03/01/2022

1006940802

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1. (This is a problem from a previous year's midterm.) A particle is constrained to move on a bowl-shaped surface shown in the figure. The height z of the surface of the bowl satisfies $z = a\rho^3$, where ρ is the distance from the point on the bowl's surface to the z axis (i.e. cylindrical polar coordinates). The particle is subject to a constant gravitational force $-mg\hat{z}$.

   (a) Find the Lagrangian of the particle.

Since $F = -\nabla V$, then $V = mgz$. The kinetic energy of the particle is simply $T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right)$. Therefore the Lagrangian of the particle is given by

$$\boxed{L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - mgz.}$$

(b) What are the integrals of motion? Express them in terms of the appropriate generalized coordinates. Is the system integrable?

There are only two integrals of motion. The Lagrangian is not explicitly dependent on t, and φ. Thus, the Hamiltonian H and the angular momentum M_z is conserved. They are

$$\boxed{\begin{aligned} &H = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) + mgz \\ &M = m\rho^2\dot{\varphi}. \end{aligned}}$$

There are 3 degrees of freedom: ρ, φ, and z. There are only two integrals of motion, which are H and M. Since $3 > 2$, the $\boxed{\text{system is not integrable.}}$

(c) Reduce the problem to an effective one-dimensional problem and find the differential equation satisfied by $\rho(t)$. Write down an implicit solution of the equations of motion which would in principle allow you to determine the form of the trajectory (i.e. reduce the problem to quadratures).

With $\dot{\varphi} = \dfrac{M}{m\rho^2}$ and $z = a\rho^3$, then $\dot{z} = 3a\rho^2\dot{\rho}$ and

$$\boxed{L = \frac{1}{2}m\dot{\rho}^2 + \frac{M^2}{2m\rho^2} + \frac{9}{2}ma^2\rho^4\dot{\rho}^2 - mga\rho^3,}$$

which reduces the problem down to one dimension. The Hamiltonian becomes

$$E = \frac{1}{2}m\dot{\rho}^2 + \frac{M^2}{2m\rho^2} + \frac{9}{2}ma^2\rho^4\dot{\rho}^2 + mga\rho^3$$

$$= \dot{\rho}^2 \frac{m}{2} \left[1 + 9a^2\rho^4\right] + U_{eff}(\rho),$$

with $U_{eff}(\rho) = \dfrac{M^2}{2m\rho^2} + mga\rho^3$. This gives the implicit differential equation that ρ must satisfy, $\boxed{E = \frac{1}{2}m\dot{\rho}^2[1 + 9a^2\rho^4] + U_{eff}(\rho).}$ Reducing the problem to quadrature,

$$\boxed{\begin{aligned} \frac{d\rho}{dt} &= \left[\frac{2}{m}\frac{[E - U_{eff}(\rho)]}{[1 + 9a^2\rho^4]}\right]^{1/2} \\ \implies T &= \int \sqrt{\frac{2m[1 + 9a^2\rho^4]}{[E - U_{eff}(\rho)]}}\, d\rho, \end{aligned}}$$

which is the time for one period of an orbit. Taking $\dfrac{d\rho}{dt} = \dfrac{d\rho}{d\varphi}\dfrac{d\varphi}{dt} = \dfrac{d\rho}{d\varphi}\dfrac{M}{m\rho^2}$, then

$$\boxed{\begin{aligned} \frac{d\rho}{d\varphi}\frac{M}{m\rho^2} &= \left[\frac{2}{m}\frac{[E - U_{eff}(\rho)]}{[1 + 9a^2\rho^4]}\right]^{1/2} \\ \implies \varphi &= \int \frac{M}{\rho^2}\sqrt{\frac{[1 + 9a^2\rho^4]}{2m[E - U_{eff}(\rho)]}}\, d\rho, \end{aligned}}$$

which gives the implicit solution of the trajectory of the particle (to the equations of motion).

(d) What is the condition on the conserved quantities for the particle to undergo circular motion at constant $\rho = \rho_0$?

For the particle to undergo a circular trajectory, we must require that $U'_{eff}(\rho_0) = 0$.

$$U'_{eff}(\rho) = -\frac{M}{m\rho^3} + 3mga\rho^2 = 0,$$

which implies that $\boxed{M = 3m^2ga\rho_0^5}$. This is the condition which M must satisfy for the particle to undergo circular motion at r_0. From this, we also can derive that $\dot{\varphi}$ must satisfy

$$\dot{\varphi} = 3mga\rho_0^3 = 3mgz(\rho_0).$$

2. * A particle of mass m moves in a "singular" central potential,

$$V(r) = -\frac{k}{r^n}$$

with $n > 2$ and $k > 0$. Reduce the equations of motion to an equivalent one-dimensional problem and discuss the qualitative nature of the orbit for different values of the energy. For the orbits where this is relevant, find expressions for the amount of time for the particle to spiral to the origin as well as the number of revolutions during this time, and show that both are finite.

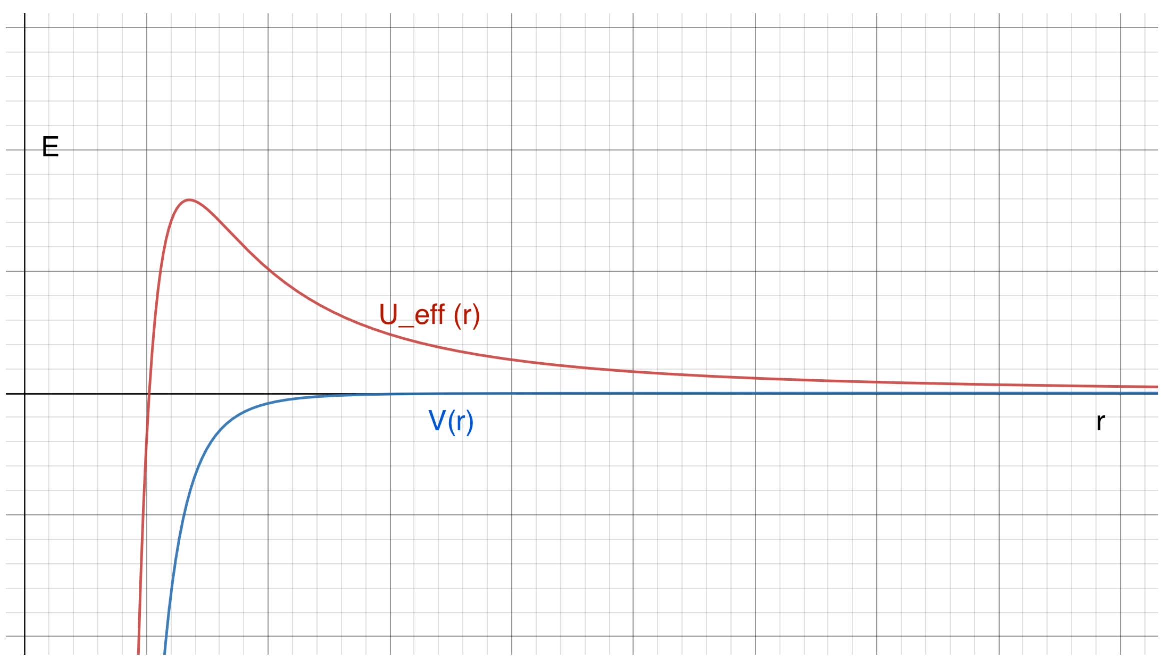
The Lagrangian of the system is given by

$$L = \frac{1}{2}mv^2 - V(r),$$

which is not explicitly time dependent, so the Hamiltonian is conserved. Furthermore, $H = E$ since T is quadratic in v. Since we are working with a central potential, the system is symmetric radially about its origin, so the angular momentum is conserved as well, with $M = M_z = mr^2\dot{\varphi}$.

$$\begin{aligned} H &= \frac{1}{2}mv^2 + V(r) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} - \frac{k}{r^n} \\ &= \frac{1}{2}m\dot{r}^2 + U_{eff}(r), \end{aligned}$$

where $U_{eff}(r) = \dfrac{M^2}{2mr^2} - \dfrac{k}{r^n}$. With this, the problem has been reduced to one dimension. Plotting the potential for arbitrary values of M, m, k, and n:



Reducing the problem to quadrature, we have

$$\begin{aligned}
E &= \frac{1}{2}m\dot{r}^2 + U_{eff}(r) \\
\implies \frac{dr}{dt} &= \sqrt{\frac{2}{m}[E - U_{eff}]} \\
\implies T &= \int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{m}[E - U_{eff}]}} \\
&= \int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{m}[E - M^2/(2mr^2) + k/r^n]}},
\end{aligned}$$

which gives the amount of time the particle will take to traverse two different radii r_1 from r_2. It now suffices to show the convergence of this integral for different cases. The integral varies for different values of n, k, M, and E, however with our assumptions that $n > 2$ and $k > 0$, the convergence of the integral depends on the value of E.
Let $U_{max} = U_{eff}(r_{max})$ be the maximum value of the effective potential on its domain. By differentiating the effective potential, it can be shown that $r_{max} = \left(\frac{knm}{M^2}\right)^{\frac{1}{n-2}}$

- For $E > U_{max}$, the orbiting particle is unbound. It will come in from $r = \infty$ then spiral into the origin. Then $T = \displaystyle\int_0^\infty \frac{dr}{\sqrt{\frac{2}{m}[E - M^2/(2mr^2) + k/r^n]}}$ diverges.
- For $E = U_{max}$, the particle will converge to a circular orbit of radius r_{max}. The integral diverges so $T = \infty$ because the integrand is undefined. Here, r_{max} is an unstable equilibrium - any 'nudge' will send the partcile to $r = \infty$ or into the origin.
- For $E < U_{max}$, the initial radius at which the particle is released will determine its behaviour. For $r_0 < r_{max}$, the particle will orbit and spiral into the origin with a finite T, since $r(t)$ would be monotonically decreasing function. For $r_0 > r_{max}$, the particle will diverge away from the origin and $T = \infty$.

Now for $E < U_{max}$ where the integral of T converges ($r < r_{max}$), there exists a maximum radius r_0 which the particle will obtain throughout its orbit. I will give an argument for the convergence of this integral. Since $E < U_{max}$ and $r < r_{max}$, the particle physically does not have enough energy to become 'stuck' in the infinite, unstable orbit at $r = r_{max}$. The only option would be for the particle to spiral inwards until $r = 0$, since the dominating term in the potential is $-\frac{k}{r^n}$, not the centripetal energy $\frac{M^2}{2mr^2}$. Intuitively, if the particle cannot become 'stuck', then T is finite. If we apply a convergence test, we can consider the function $g(r) = \frac{c}{\sqrt{r_0 - r}}$:

$$0 \leq \frac{1}{\sqrt{\frac{2}{m}[E - M^2/(2mr^2) + k/r^n]}} \leq \frac{1}{\sqrt{r_0 - r}}$$

for $0 \leq r \leq r_0 < r_{max}$. Now $\displaystyle\int_0^{r_0} \frac{dr}{\sqrt{r_0 - r}} = 2\sqrt{r_0}$, which is convergent. Therefore

$$T = \int_0^{r_0} \frac{dr}{\sqrt{\frac{2}{m}[E - U_{eff}(r)]}}$$

is convergent and hence finite. We proceed by analyzing the shape of the orbit. With $d\varphi = \dfrac{M}{mr^2}dt$, then with substituing dt,

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}[E - U_{eff}]} \implies \varphi(r) = \int \frac{M}{r^2} \frac{dr}{\sqrt{2m[E - U_{eff}]}},$$

which yields an expression for the path. It was derived during lecture that during the amount of time (T) which the particle takes to orbit from $r_0 \to r = 0$, the radius vector turns an angle $\Delta\varphi$, given by

$$\Delta\varphi = \int_0^{r_0} \frac{M}{r^2} \frac{1}{\sqrt{2m[E - U_{eff}(r)]}}\, dr.$$

Applying the same argument as the convergence of T, the integral for $\Delta\varphi$ is also finite. Intuitively, if the particle is spiraling into the origin with a finite time, it must require a finite distance to travel to converge. Since the potential in singular, once the particle enters the singularity (at $r = 0$), it does not come back out of the singularity. This implies that the change in the radial angle $\Delta\varphi$ is the total angle for which the particle sweeps out, and thus

$$N = \frac{\Delta\varphi}{2\pi}$$

gives the number of revolutions, which will most likely be a fraction. This too is finite (which is a direct result of $\Delta\varphi$ being finite). Therefore

$$\boxed{T = \int_0^{r_0} \frac{dr}{\sqrt{\frac{2}{m}[E - M^2/(2mr^2) + k/r^n]}} \text{ and } N = \frac{1}{2\pi} \int_0^{r_0} \frac{M}{r^2} \frac{1}{\sqrt{2m[E - U_{eff}(r)]}}\, dr}$$

3. * Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \ln\left(\frac{r^2}{r^2+b^2}\right)$$

where b is a constant with dimensions of length.

(a) For what values of the relative angular momentum ℓ does a circular orbit exist? Find the radius r_0 of the circular orbit.

We begin by finding the Lagrangian of the two particle system. As derived in lecture, $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$. With the reduced mass $\mu = \dfrac{m_1 m_2}{m_1+m_2}$, then

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|) \quad \longrightarrow \quad L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U_0\log\left(\frac{r^2}{r^2+b^2}\right).$$

The Lagrangian is not explicity dependent on t and φ, and so H and M are consvered quantities. Furthermore, $H = E$ since the kinetic energy is quadratic in $\dot{\mathbf{r}}$. We have

$$\begin{aligned}
L &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U_0\log\left(\frac{r^2}{r^2+b^2}\right) \\
\implies E &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + U_0\log\left(\frac{r^2}{r^2+b^2}\right) \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2 + U_0\log\left(\frac{r^2}{r^2+b^2}\right) \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{M^2}{2\mu r^2} + U_0\log\left(\frac{r^2}{r^2+b^2}\right) \\
&= \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r),
\end{aligned}$$

with $\dot{\varphi} = \dfrac{M}{\mu r^2}$ and $U_{eff}(r) = \dfrac{M^2}{2\mu r^2} + U_0\log\left(\dfrac{r^2}{r^2+b^2}\right)$. Now, any elliptical orbit is given when the particle traverses paths between two different radii r_1 and r_2. These are the turning points. For a perfect circle, the particle 'sits' in a well of the effective potential, given when $\dfrac{dU_{eff}}{dr} = 0$:

$$\begin{aligned}
\frac{dU_{eff}}{dr} &= -\frac{M^2}{\mu r^3} + 2U_0\left(\frac{1}{r} - \frac{r}{r^2+b^2}\right) = 0 \\
\implies \frac{M^2}{\mu r^3} &= \frac{2U_0}{r}\left(\frac{b^2}{r^2+b^2}\right) \\
\implies M^2r^2 + M^2b^2 &= 2\mu U_0 r^2 \\
\implies r &= \frac{Mb}{\sqrt{2\mu b^2 U_0 - M^2}} \equiv r_0.
\end{aligned}$$

We wish to analyze the values of M for which this r_0 is defined. For a defined value of r_0 ($M \neq 0$), we require the value under the square root of the denominator to be non-zero and non-negative:

$$2\mu b^2 U_0 - M^2 > 0 \implies \boxed{0 < |M| < b\sqrt{2\mu U_0}}$$

This yields all the values of M for which a circular orbit exists, with radius $\boxed{r_0 = \dfrac{Mb}{\sqrt{2\mu b^2 U_0 - M^2}}}$.

(b) Suppose the orbit is nearly circular, with $r(t) = r_0 + \eta(t)$, where $|\eta(t)| \ll r_0$. To find the shape of the orbit, we need to find $\eta(\phi)$, not $\eta(t)$. Show that $\eta(\phi)$ obeys the equation

$$\frac{d^2\eta(\phi)}{d\phi^2} = -\beta^2\eta(\phi) + \dots$$

(where the dots denote terms suppressed by powers of η/r_0), and hence $\eta(\phi) = A\cos(\beta\phi + \delta)$, where A and δ are constants. Find β in terms of ℓ and ℓ_c, where ℓ_c is the maximum value of ℓ for which a critical orbit exists (from the previous part). Sketch the shape of the orbit.
(NOTE: since $|\eta| \ll r_0$, you are going to need to use Taylor expansions wherever possible.)

Since the Lagrangian does not explicitly depend on time, the Hamiltonian is a conserved quantity. From **(3a)**,

$$H = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + U_0 \log\left(\frac{r^2}{r^2+b^2}\right).$$

Or, writing H in terms of the effective potential $U_{eff}(r) = \dfrac{M^2}{2\mu r^2} + U_0 \log\left(\dfrac{r^2}{r^2+b^2}\right)$,

$$H = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r).$$

With $r \to r_0 + \eta$ for $\eta \ll r_0$, we can apply a Taylor expansion to $U_{eff}(r_0 + \eta)$. Note that since r_0 is constant, the $\dot{r}_0 = 0$, and thus $\dot{r} = \dot{\eta}$

$$H \approx \frac{1}{2}\mu\dot{\eta}^2 + U_{eff}(r_0) + \eta U'_{eff}(r_0) + \frac{1}{2}\eta^2 U''_{eff}(r_0) + \dots$$

Since r_0 is a critical orbit, we have that $U'_{eff}(r_0) = 0$. By the chain rule, since H is conserved, $\dfrac{dH}{dt} = \dfrac{dH}{d\varphi}\dfrac{d\varphi}{dt} = 0$, which implies that $\dfrac{dH}{d\varphi} = 0$. This also implies that $\dfrac{d\eta}{dt} = \dfrac{d\eta}{d\varphi}\dot{\varphi}$. The Hamiltonian becomes

$$\begin{aligned}
H &= \frac{1}{2}\mu\dot{\eta}^2 + U_{eff}(r_0) + \frac{1}{2}\eta^2 U''_{eff}(r_0) + \dots \\
&= \frac{1}{2}\mu\left(\dot{\varphi}\frac{d\eta}{d\varphi}\right)^2 + U_{eff}(r_0) + \frac{1}{2}\eta^2 U''_{eff}(r_0) + \dots \\
&= \frac{1}{2}\mu\frac{\ell^2}{\mu^2[r_0+\eta]^4}\left(\frac{d\eta}{d\varphi}\right)^2 + U_{eff}(r_0) + \frac{1}{2}\eta^2 U''_{eff}(r_0) + \dots \\
&\approx \frac{1}{2}\frac{\ell^2}{\mu}\left[\frac{1}{r_0^4} - \frac{4\eta}{r_0^5} + \frac{10\eta^2}{r_0^6} - \dots\right]\left(\frac{d\eta}{d\varphi}\right)^2 + U_{eff}(r_0) + \frac{1}{2}\eta^2 U''_{eff}(r_0) + \dots \\
\implies \frac{dH}{d\varphi} &= \frac{1}{2}\frac{\ell^2}{\mu}\left[-\frac{4}{r_0^5} + \frac{20\eta}{r_0^6} - \dots\right]\left(\frac{d\eta}{d\varphi}\right)^3 + \frac{\ell^2}{\mu}\left[\frac{1}{r_0^4} - \frac{4\eta}{r_0^5} + \frac{10\eta^2}{r_0^6} - \dots\right]\frac{d\eta}{d\varphi}\frac{d^2\eta}{d\varphi^2} + \eta\frac{d\eta}{d\varphi}U''_{eff}(r_0) + \dots \\
\implies 0 &= \frac{\ell^2}{\mu r_0^4}\frac{d^2\eta}{d\varphi^2} + \eta U''_{eff}(r_0) + \dots
\end{aligned}$$

This directly implies that $\boxed{\dfrac{d^2\eta}{d\varphi^2} = -\beta^2\eta}$, with $\boxed{\beta = \sqrt{\dfrac{U''_{eff}(r_0)\mu r_0^4}{\ell^2}}.}$ The second derivative of the effective potential simplifies to

$$U''_{eff}(r_0) = \frac{3\ell^2 b^4 + (6\ell^2 b^2 - 2U_0\mu b^4)r_0^2 + (3\ell^2 - 6U_0 b^2 \mu)r_0^4}{\mu r_0^4 (r_0^2 + b^2)^2}.$$

By Wolfram Alpha, this simplifies β^2 to give

$$\beta^2 = -\frac{\ell^2}{U_0 \mu b^2} + 2.$$

For a critical orbit to exist, $\frac{\ell_c^2}{2} < U_0 \mu b^2$ from **3a**, which implies that

$$\beta^2 = 2\left(1 - \frac{\ell^2}{\ell_c^2}\right).$$

Below I have included a sample sketch, as well as 3 sample plots for different values of β.