PHY250 PS3 — 03/25/2022

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I have read and understand the homework set policy in this course. Collaborators: none (for all problems).

Q1.

Since we are not concerned with the y-component of the particle's motion, the position and velocity vectors of the particle at time $t = 0$ are given by

$$\vec{r}(t=0) = (x_0, 0, 0) \qquad \dot{\vec{r}}(t=0) = (0, 0, v_0).$$

With the magnetic field in space being $\vec{B} = (by, bx, 0)$, we can apply the Lorentz force law and find the acceleration of the particle $\ddot{\vec{r}}$. We can proceed by solving it's equations of motion the re-parametrizing the motion to solve for $x(z)$:

$$\begin{aligned} \vec{F} = m\ddot{\vec{r}} &= q(\vec{v} \times \vec{B}) = q(\dot{\vec{r}} \times \vec{B}) \\ &= \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v_0 \\ by & bx & 0 \end{pmatrix} \\ &= \begin{pmatrix} -bv_0 x \\ bv_0 y \\ 0 \end{pmatrix}. \end{aligned}$$

Then taking the restriction of the motion to the x-component, $F_x = m\ddot{x} = -qbv_0\, x$, which is the equation of motion for the x position as a function of time, where the dots indicate a time derivative. Solving by taking $\omega^2 = \dfrac{qbv_0}{m}$, the x-position as a function of time is

$$x(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t) = A_1 \cos\left(\sqrt{\frac{qbv_0}{m}}\, t\right) + A_2 \sin\left(\sqrt{\frac{qbv_0}{m}}\, t\right).$$

The initial condition to determine A_1 is given when $x(t=0) = x_0$, the x-component of $\vec{r}(t=0)$. We can determine A_2 by reparametrizing the equation in terms of z, then taking that $x'(0) = \alpha_0$, where x' indicates a z derivative. Since $\dot{z} \approx v_0$, we can solve this ODE to determine that $z \approx v_0\, t$, or $t \approx \dfrac{z}{v_0}$. After re-parametrizing, our equation of motion for $x(z)$ then becomes

$$x(z) = x_0 \cos\left(\sqrt{\frac{qb}{mv_0}}\, z\right) + A_2 \sin\left(\sqrt{\frac{qb}{mv_0}}\, z\right).$$

To determine A_2, we can differentiate and set $z = 0$ and solve for A_2 in terms of α_0:

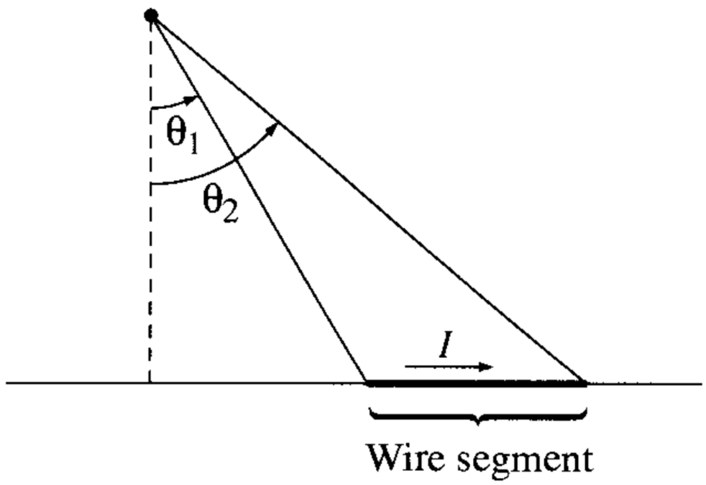
$$\begin{aligned} x'(z) &= -x_0 \sqrt{\frac{qb}{mv_0}} \sin\left(\sqrt{\frac{qb}{mv_0}}\, z\right) + A_2 \sqrt{\frac{qb}{mv_0}} \cos\left(\sqrt{\frac{qb}{mv_0}}\, z\right) \\ \implies x'(z=0) &= \alpha_0 = A_2 \sqrt{\frac{qb}{mv_0}} \end{aligned}$$

$$\implies A_2 = \alpha_0\sqrt{\frac{mv_0}{qb}}.$$

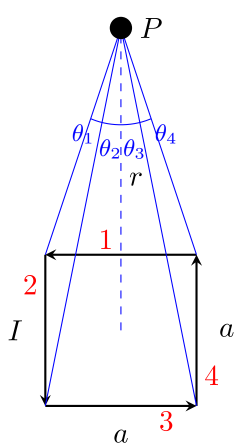
Therefore $\boxed{x(z) = x_0\cos\left(\sqrt{\frac{qb}{mv_0}}\,z\right) + \alpha_0\sqrt{\frac{mv_0}{qb}}\sin\left(\sqrt{\frac{qb}{mv_0}}\,z\right)}$ is the motion of the particle along x in terms of z.

Q2.

By the Biot-Savart law, the magnetic field a perpendicular distance s away from a wire is given by $B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$, where θ_2 indicates the endpoint and θ_1 indicates the start point:



For the square of side length a centered at the origin and the point P a distance r away from the origin, we are able the break up the individual magnetic fields from each of the wire segments then superpose them to find the net field at P.



Notice that $\theta_4 = -\theta_1$ and $\theta_3 = -\theta_2$. Along wire 1, the distance to point P is $r - a/2$. For wires 2 and 4, the perpendicular distance is $a/2$ and for wire 3, it is $r + a/2$. The magnetic fields from each of the 4 wires are

$$B_1 = \frac{\mu_0 I}{4\pi}\frac{1}{r - a/2}[\sin\theta_1 - \sin\theta_4] \qquad B_2 = \frac{\mu_0 I}{4\pi}\frac{2}{a}[\sin\theta_2 - \sin\theta_1]$$

$$B_3 = \frac{\mu_0 I}{4\pi}\frac{1}{r + a/2}[\sin\theta_3 - \sin\theta_2] \qquad B_4 = \frac{\mu_0 I}{4\pi}\frac{2}{a}[\sin\theta_4 - \sin\theta_3].$$

To find the leading order magnetic field, we can apply a Taylor expansion to two of the angles, due to the symmetry of the system. Since $\sin\varphi = \dfrac{\text{opposite}}{\text{hypotenuse}}$, we have that $\sin\theta_1 = \dfrac{-a/2}{\sqrt{(a/2)^2+(r-a/2)^2}} = -\sin\theta_4$ and $\sin\theta_2 = \dfrac{-a/2}{\sqrt{(a/2)^2+(r+a/2)^2}} = -\sin\theta_3$.

Since $a/r \ll 1$, for $(r \pm a/2)^2 = r^2(1 \pm a/2r)^2$, the leading order approximating in the Taylor expansion is just r^2. Then

$$\sin\theta_1 \approx \sin\theta_2 \approx \frac{-a/2}{\sqrt{(a/2)^2+r^2}} \approx -\sin\theta_3 \approx -\sin\theta_4.$$

Making another approximation to the square root in the denominator,

$$\sqrt{(a^2/4+r^2)} = r\sqrt{a^2/4r^2+1} \approx r.$$

Therefore

$$\sin\theta_1 \approx \sin\theta_2 \approx -\frac{a}{2r} \approx -\sin\theta_3 \approx -\sin\theta_4$$

gives the leading or approximation of the magnetic field at P. Then the magnitude of the magnetic field along each wire is

$$B_1 \approx -\frac{\mu_0 I}{4\pi}\frac{2}{2r-a}\left[\frac{a}{r}\right] \qquad B_2 \approx \frac{\mu_0 I}{4\pi}\frac{2}{a}\left[-\frac{a}{2r}+\frac{a}{2r}\right] = 0$$

$$B_3 \approx \frac{\mu_0 I}{4\pi}\frac{2}{2r+a}\left[\frac{a}{r}\right] \qquad B_4 \approx \frac{\mu_0 I}{4\pi}\frac{2}{a}\left[\frac{a}{2r}-\frac{a}{2r}\right] = 0.$$

Superposing the four components of the magnetic field at P,

$$\begin{aligned}
B &= B_1 + 0 + B_3 + 0 \\
&= \frac{\mu_0 I}{4\pi}\frac{2a}{r}\left[-\frac{1}{2r-a}+\frac{1}{2r+a}\right] \\
&= \frac{\mu_0 I}{2\pi}\frac{a}{r}\left[\frac{-2a}{4r^2-a^2}\right] \\
&= -\frac{\mu_0 I}{\pi}\cdot\frac{a}{r}\cdot\frac{a}{4r^2-a^2}.
\end{aligned}$$

Again, we can apply a Taylor approximation to the denominator. Note that

$$(4r^2-a^2) = 4r^2(1-a^2/4r^2) \approx r^2,$$

and thus our magnetic field again approximates to $B = -\dfrac{\mu_0 I}{4\pi}\dfrac{a^2}{r^3}$.

Taking the magnitude of the field (B to be positive), then the leading order magnetic field at P is $\boxed{B = \dfrac{\mu_0 I}{4\pi}\dfrac{a^2}{r^3}}$ for $r \gg a$.

Q3.1

For a single ring of radius R, the magnetic field a distance z along its axis of symmetry is given by $B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z'^2)^{3/2}}$. Now if we have two rings, located distances $-d/2$ and $+d/2$ away from $z = 0$, the individual magnetic fields are given when $z' = z - \frac{d}{2}$ and when $z' = z + \frac{d}{2}$:

$$B_1(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + (z - d/2)^2)^{3/2}} \qquad B_2(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + (z + d/2)^2)^{3/2}}.$$

Superposing these fields give the expression for the total magnetic field contribution from the two rings for any z:

$$\boxed{B(z) = \frac{\mu_0 I R^2}{2} \left[\frac{1}{(R^2 + (z - d/2)^2)^{3/2}} + \frac{1}{(R^2 + (z + d/2)^2)^{3/2}} \right].}$$

Q3.2

To determine whether $B(z)$ is an odd or even function, we can examine the symmetries of the expression. Notice the squared term in the denominator: $(z \pm d/2)^2$. For any value of z, this term will always be positive. It follows that $B(z) = B(-z)$, which implies that $\boxed{B(z) \text{ is an even function of } z.}$

In terms of the Taylor expansion, this means that only even powers contribute around $z = 0$. That is, $B(z) = \sum_{k=0}^{\infty} B^{(2k)}(0) \frac{z^{2k}}{(2k)!}$.

The property of an even function f is that for all x, $f(x) = f(-x)$. Assuming this f is differentiable,

$$\begin{aligned} f'(x) &= f'(x) \cdot (-1) = -f'(x) \\ \implies 0 &= f'(x) + f'(-x). \end{aligned}$$

$x = 0$, we have that $2f'(0) = 0$, and hence $f'(0) = 0$ for any even function f. This idea can be applied to $B(z)$, and therefore $\boxed{\frac{\partial B}{\partial z}(0) = 0.}$

Q3.3

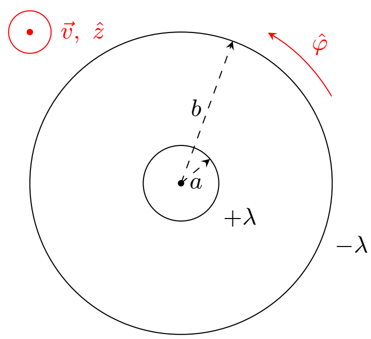
When $d = R$, $B(z)$ varies very little in a neighbourhood around $z = 0$. This must imply that the curvature of B as a function of z is very small around $z = 0$. If we examine the first two terms in the Taylor expansion of $B(z)$ around $z = 0$ as found in **(Q3.2)**, we have that

$$B(z) \approx B(0) + B''(0) \frac{z^2}{2} + H.O.T.$$

for $z \ll 1$. Around $z = 0$, the best approximation for $B(z)$ is the leading term: $B(0)$, because $B(z)$ varies very little around $z = 0$. This implies that $B''(0)\frac{z^2}{2}$ is very small, hence having a very little contribution. Since $z \neq 0$, this term can essentially be neglected in the approximation by allowing $B''(0) = 0$. This is why $B(z)$ varies so little when $d = R$ - <u>because the second derivative of $B(z)$ at $z = 0$ is zero.</u> Physically, highly uniform magnetic fields along the axis of symmetry is produced, which is what allows experimentalists to move objects in and out of that portion of space.

Q4.1

As stated in the question, I am assuming the shells are infinitely thin and hollow with radii a and b.



By the right hand rule, $\vec{B}$ must be in the $\hat{\varphi}$ direction, pointing counterclockwise if $\vec{v}$ is out of the page. I will invoke Ampere's law: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$. Recall that the enclosed current is given by the charge density in product with the moving speed of said charge: $\vec{I} = \lambda \vec{v}$.

$s < a$: Since the inner cylinder is hollow, there is no interior moving charge. This implies that the enclosed current in the $s < a$ region is 0, that is $I_{s<a} = 0$. By Ampere's law, the magnetic field inside the inner cylinder is zero. Thus $\boxed{\vec{B}_{s<a}(s) = 0.}$

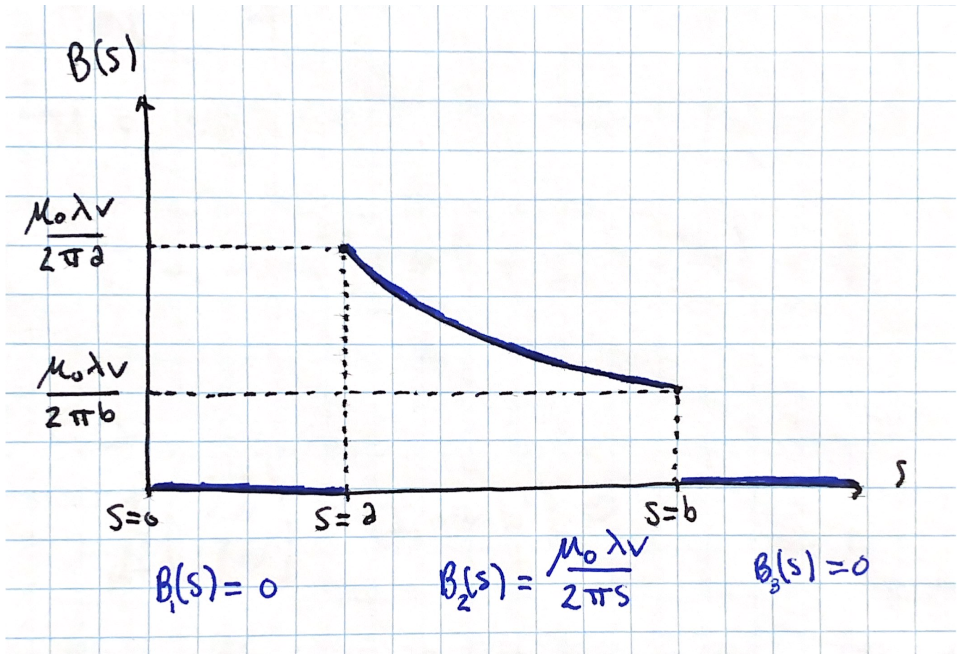
$a < s < b$: As in the $s < a$ case, the current contribution from the outer cylindrical shell is zero. Therefore the only enclosed current is given by the inner cylidrical shell, $I_{inner} = I_{encl} = \lambda v$. Drawing an Amperian loop,

$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} = B \int_0^{2\pi} s d\varphi = 2\pi B s \\ = \mu_0 I_{encl} = \mu_0 \lambda v. \end{aligned}$$

And thus the magnetic field in between the two cylinders is $\boxed{\vec{B}(s) = \frac{\mu_0 \lambda v}{2\pi s}\hat{\varphi}.}$

$s > b$: Following a similar process as the previous two cases, I will proceed by finding the enclosed current. From the inner shell, its contribution is $I_{inner} = +\lambda v$. The contribution from the outer shell is $I_{outer} = -\lambda v$, since the currents move in the same direction but with opposite charge signs. For $s > b$, the enclosed charge is $I_{encl} = I_{inner} + I_{outer} = v(+\lambda - \lambda) = 0$. Just as in the first case, since $I_{encl} = 0$, then the magnetic field outside the cylinder is zero as well: $\boxed{\vec{B}_{s>b}(s) = 0.}$

The plot of the magnitude of $B(s)$ is discontinuous at the boundaries: $s = a$ and $s = b$.



Q4.2

The magnetic energy stored in the cable for a section of length L is given by the integral across all space: $W = \dfrac{1}{2\mu_0}\iiint B^2\, d\tau$. From question **(Q4.1)**, we have that the magnetic field for $s < a$ and $s > b$ is zero. Meanwhile, the magnetic field in between the cylindrical shells is given by $B_{a<s<b}(s) = \dfrac{\mu_0 \lambda v}{2\pi s}$. I will proceed by integrating in cylidrical coordinates:

$$
\begin{aligned}
W &= \frac{1}{2\mu_0}\int_0^L \int_0^{2\pi} \int_0^{\infty} B^2 s\, ds d\varphi dz \\
&= \frac{1}{2\mu_0}\int_0^L \int_0^{2\pi} \left[\int_0^a B^2_{s<a} s\, ds + \int_a^b B^2_{a<s<b} s\, ds + \int_b^{\infty} B^2_{s>b} s\, ds\right] d\varphi dz \\
&= \frac{1}{2\mu_0}\int_0^L \int_0^{2\pi} \left[\int_0^a (0) s\, ds + \int_a^b B^2_{a<s<b} s\, ds + \int_b^{\infty} (0) s\, ds\right] d\varphi dz \\
&= \frac{1}{2\mu_0}\int_0^L dz \int_0^{2\pi} d\varphi \int_a^b \frac{\lambda^2 v^2 \mu_0^2}{4\pi^2 s^2} s\, ds \\
&= \frac{1}{2\mu_0} \cdot \frac{\lambda^2 v^2 \mu_0^2}{4\pi^2} \cdot L \cdot 2\pi \cdot \int_a^b \frac{ds}{s} \\
&= \frac{\lambda^2 v^2 \mu_0 L}{4\pi} \log\left(\frac{b}{a}\right).
\end{aligned}
$$

Since $b > a$, the natural logarithm $\log\left(\dfrac{b}{a}\right) > 0$, and thus the magnetic energy stored in the cable for a section of length L is $\boxed{W = \dfrac{\lambda^2 v^2 \mu_0 L}{4\pi} \log\left(\dfrac{b}{a}\right).}$

Q5.

For a current density $\vec{J}$, the direction in which the volume charge density ρ is moving is the same. That is, $\vec{J} = \rho\vec{v}$ by definition. Assuming $\vec{J}$ uniform, by the right hand rule, $\vec{B}$ is purely in the $\hat{\varphi}$ direction: $\vec{B} = B\hat{\varphi}$. There exists a vector potential $\vec{A}$ by Helmholtz decomposition, since $\vec{\nabla} \cdot \vec{B} = 0$ always. Therefore we can write $\vec{B}$ as $\vec{B} = \vec{\nabla} \times \vec{A}$.

It follows that if $\vec{B}$ is purely in the $\hat{\varphi}$ direction, then $\vec{\nabla} \times \vec{A}$ must also be in the $\hat{\varphi}$ direction. Now, by definition of $\vec{A}$, we have that

$$\vec{A} = \frac{\mu_0}{2\pi} \int \frac{\vec{J}}{|\mathscr{r}|}\, d\tau',$$

which implies that $\vec{A}$ must be in the same direction (or at least on the same axis) as $\vec{J}$. Since $\vec{J}$ is along the $\hat{z}$ direction, we can express the volume current as $\vec{J} = J\hat{z}$. Therefore $\vec{A} = A\hat{z}$, so the z-component of $\vec{A}$ is the only contributing quantity. Computing the curl of $\vec{A}$ in cylindrical coorinates yields

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \left[\frac{1}{s}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\varphi} + \frac{1}{s}\left[\frac{\partial (sA_\varphi)}{\partial s} - \frac{\partial A_s}{\partial \varphi}\right]\hat{z} \\
&= -\frac{\partial A_z}{\partial s}\hat{\varphi}
\end{aligned}$$

since the other components are zero and the only contributing component of $\vec{B} = \vec{\nabla} \times \vec{A}$ is the $\hat{\varphi}$ component.

Now that we have obtained a relation between $\vec{B}$ and $\vec{A}$, I will proceed by finding $\vec{B}$ using Ampere's law.

$r \leq a$: The enclosed volume current is $\vec{J}$ at a radius $r \leq a$ is given by $I_{encl} = J\pi r^2$. Since $\vec{B}$ is only in the $\hat{\varphi}$ direction, the line integral across the amperian loop of $\vec{B}$ is just $B \cdot 2\pi r$.

$$\begin{aligned}
&\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi r = \mu_0 I_{encl} = \mu_0 J\pi r^2 \\
&\implies \vec{B}_{in} = \frac{\mu_0 J}{2} r\hat{\varphi}.
\end{aligned}$$

$r > a$: For $r > a$, the enclosed current is given at a radius $r > a$, which is just $I_{encl} = J\pi a^2$ since there is no current flowing outside of $r = a$. Analogous to the previous case, we have that

$$\begin{aligned}
&\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi r = \mu_0 I_{encl} = \mu_0 J\pi a^2 \\
&\implies \vec{B}_{out} = \frac{\mu_0 J a^2}{2}\frac{1}{r}\hat{\varphi}.
\end{aligned}$$

With the relation $B = -\dfrac{\partial A_z}{\partial s}$, we can solve each differential equation for each case by integrating:

$r \leq a$:

$$A_{z,\,inner} = -\int B_{in}\, dr = -\int \frac{\mu_0 J}{2} r\, dr = -\frac{\mu_0 J}{4} r^2 + C_1$$

$r > a$:

$$A_{z,\,outer} = -\int B_{out}\,dr = -\int \frac{\mu_0 J a^2}{2}\frac{1}{r}\,dr = -\frac{\mu_0 J a^2}{2}\log(r) + C_2.$$

Since the constants of integration C_1 and C_2 are arbitrary (they vanish when taking any derivative), I will set them to be zero. As before, $\vec{A}$ must point in the $\hat{z}$ direction. Therefore the vector potentials for the cylinder are

$$\boxed{\vec{A}_{r\leq a}(r) = -\frac{\mu_0 J}{4} r^2 \hat{z} \qquad \text{and} \qquad \vec{A}_{r>a}(r) = -\frac{\mu_0 J a^2}{2}\log(r)\hat{z}.}$$