MAT237 Multivariable Calculus with Proofs
Term Test 2

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- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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Question:	1	2	3	4	5	6	7	8	Total
Points:	4	5	3	7	4	6	6	5	40
Score:									

1. (4 points) The parts of this question are unrelated. No justification is necessary for any part.
Fill in EXACTLY ONE circle. (unfilled ○ filled ●)

(1a) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 map. Fix $a \in \mathbb{R}^n$. Which statement is FALSE?

- ○ There exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim_{h\to 0} \dfrac{F(a+h)-F(a)-L(h)}{\|h\|} = 0$.
- ○ The limit $\lim_{h\to 0} \dfrac{F(a+h)-F(a)}{\|h\|}$ exists.
- ○ The Jacobian of F at a exists and is an $m \times n$ matrix.
- ○ For $v \in \mathbb{R}^n$, $D_vF(a) = dF_a(v)$.
- ● **None of the above statements are false.**

(1b) Let $A \subseteq \mathbb{R}^n$ be a set and let $f : A \to \mathbb{R}$ be differentiable on the interior of A. Fix $a \in A$.
Which statement is TRUE?

- ○ If $\nabla f(a) = 0$, then f has a local extremum at a.
- ○ If $\nabla f(a) = 0$ and a is an interior point of A, then f has a local extremum at a.
- ● **If f has a local extremum at a, then $\nabla f(a) = 0$.**
- ○ If f has a global extremum at a, then $\nabla f(a) = 0$.
- ○ None of the above statements are true.

(1c) Let $A \subseteq \mathbb{R}^n$ be a set and let $f : A \to \mathbb{R}$ be a C^∞ real-valued function. Fix $a \in A$.
Which statement is EQUIVALENT to "f has a local maximum at a"?

- ○ $\forall x \in A,\ f(x) \le f(a)$.
- ○ $\exists \varepsilon > 0$ s.t. $\forall x \in \mathbb{R}^n,\ \|x-a\| \le \varepsilon \implies f(x) \le f(a)$.
- ○ $\exists \varepsilon > 0$ s.t. $\forall x \in A,\ \|x-a\| \le \varepsilon \implies f(x) < f(a)$.
- ○ $\nabla f(a) = 0$.
- ● **$\nabla f(a) = 0$ and $Hf(a)$ has only negative eigenvalues.**
- ○ None of the above statements are equivalent.

(1d) The SOCK-O company sells two styles of socks: Artemis Anklets and Luna Low-Cuts. They are trying to maximize their profit. They have determined that if they spend A hundreds of thousands of dollars on Artemis Anklets, spend L hundreds of thousands of dollars on Luna Low-Cuts, and sell all of the socks they produce, then they will achieve a profit of P hundreds of thousands of dollars. They know the following facts about P:

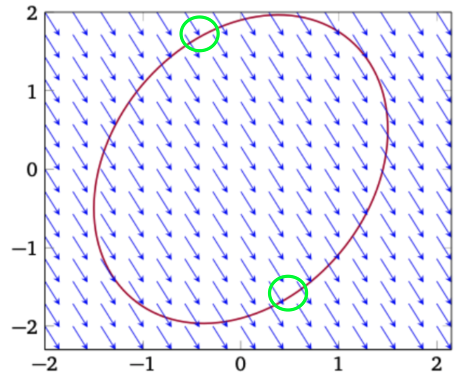
- P is differentiable on $\{(A, L) \in \mathbb{R}^2 : A > 0, L > 0\}$ and continuous on $\{(A, L) \in \mathbb{R}^2 : A \ge 0, L \ge 0\}$.
- The only critical points of P are $(A, L) = (40, 15)$ and $(A, L) = (12, 98)$.
- If $A + L \ge 100$, then $P \le 0$. Moreover, $P(40, 15) > 0$.

What is the *strongest* possible conclusion that SOCK-O can make about the maximum profit?

- ● **They maximize profit by spending \$4.0 million on anklets and \$1.5 million on low-cuts.**
- ○ They maximize profit by spending \$1.2 million on anklets and \$9.8 million on low-cuts.
- ○ They maximize profit by spending either \$4.0 million on anklets and \$1.5 million on low-cuts, or \$1.2 million on anklets and \$9.8 million on low-cuts.
- ○ They can maximize profit, but there is not enough information to decide how.
- ○ They cannot maximize profit.
- ○ Nothing can be concluded since a maximum profit may or may not exist.

2. (5 points) The parts of this question are unrelated. No justification is necessary for any part.

(2a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be C^1 real-valued functions. The graph below shows the gradient vector field $\nabla f(x,y)$ and the constraint curve $g(x,y) = 237$.



Label approximately where are the *possible* local extrema of f on the curve.

(2b) You are optimizing a function $f : \mathbb{R}^3 \to \mathbb{R}$ on the closed ball $B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 81\}$. You have done many calculations and determined all of the following.

- f is C^1 on $\mathbb{R}^3$.
- $\nabla f(x,y,z) = (0,0,0)$ if and only if $(x,y,z) = (2,3,5)$, and $(6,6,6)$.
- $f(2,3,5) = 2022$, and $f(6,6,6) = -\pi$.
- The only solutions to the system

$$\nabla f(x,y,z) = (2\lambda x, 2\lambda y, 2\lambda z) \qquad x^2 + y^2 + z^2 = 81$$

  are $(x,y,z) = (9,0,0)$ with $\lambda = -2$ and $(0,9,0)$ with $\lambda = 3$.
- $f(9,0,0) = 137$ and $f(0,9,0) = 224$

What is the maximum of f on B? **Fill in EXACTLY ONE circle.** (unfilled ○ filled ●)

● **2022** ○ 224 ○ 137 ○ $-\pi$ ○ Not enough information to decide

What is the minimum of f on B? **Fill in EXACTLY ONE circle.** (unfilled ○ filled ●)

○ 2022 ○ 224 ● **137** ○ $-\pi$ ○ Not enough information to decide

(2c) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 map. Fix $a, b, c \in \mathbb{R}^3$. You have computed that

$$DF(a) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad DF(b) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad DF(c) = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 3 & 7 \\ 3 & 3 & 7 \end{bmatrix}.$$

At which points is F a local diffeomorphism?
Fill in ALL boxes that apply. If none apply, leave it blank. (unfilled □ filled ■)

■ a □ b □ c

Which statement is TRUE?
Fill in EXACTLY ONE circle. (unfilled ○ filled ●)

○ F is a diffeomorphism.
● **F is not a diffeomorphism.**
○ There is not enough information to decide whether F is a diffeomorphism.

3. (3 points) The parts of this question are unrelated. No justification is necessary.

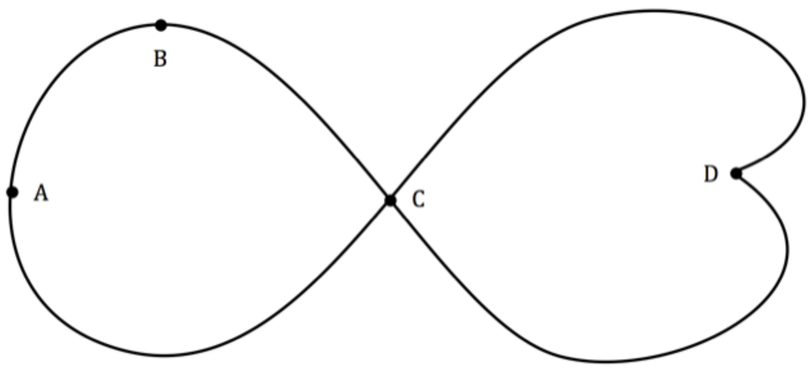
Fill in ALL boxes that apply. If none apply, leave it blank. (unfilled ☐ filled ■)

(3a) Let $G : \mathbb{R}^4 \to \mathbb{R}^2$ be a C^1 function. Fix $p \in \mathbb{R}^4$. Assume $dG_p(x) = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix} x$.

Based on this information, which of these four statements must be TRUE?

- ☐ G locally defines (x_1, x_2) as a C^1 function of (x_3, x_4) near p.
- ■ **G locally defines (x_1, x_3) as a C^1 function of (x_2, x_4) near p.**
- ■ **G locally defines (x_2, x_3) as a C^1 function of (x_1, x_4) near p.**
- ☐ G locally defines (x_3, x_4) as a C^1 function of (x_1, x_2) near p.

(3b) Let $g : \mathbb{R}^2 \to \mathbb{R}$. The curve below is defined by the equation $g(x, y) = 0$.



Four points $A, B, C, D \in \mathbb{R}^2$ are labelled on the curve.
At which of these points can x be described locally as a C^1 function of y?

■ A ☐ B ☐ C ☐ D

(3c) Eseosa wishes to prove that curve $x^3 + y^3 = 1$ is a regular curve. She gives the following argument:

1. *Let $S = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 = 1\}$ and let $(x, y) \in S$.*
2. *Let $f : \mathbb{R} \to \mathbb{R}$ be the C^1 function given by $f(x) = (1 - x^3)^{\frac{1}{3}}$.*
3. *Notice that $(x, y) \in S$ if and only if $f(x) = y$.*
4. *Therefore, S is the graph of a C^1 function and hence is regular.*

Select all valid critiques of this argument. If none apply, do not select any.

- ☐ Line 1 is flawed since the set S is empty.
- ■ **Line 2 is flawed since f is not C^1.**
- ☐ Line 3 is flawed since there exists at least one point $(x, y) \in S$ which does not satisfy $f(x) = y$.
- ☐ Line 4 is flawed since a regular curve does not need to be a graph.

4. (7 points) Find the global extrema of the function $f(x,y) = x^2 + 3xy + y^2 - 6x + 6y$ on the right halfdisk $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 16, x \ge 0\}$. As always, remember to justify your argument.

- We begin by checking whether global extrema of f exist on A.
  Notice that $(0,0) \in A$, so A is nonempty. Since A is bounded and closed, A is compact by Bolzano-Weierstrass.
  Since $f : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial in 2 variables, f is continuous by Lemma 2.7.24.
  Thus, by the Extreme Value Theorem (Theorem 2.9.7), f attains points of extremum on A.
- Next we wish to check the interior of A for points of extrema when $\nabla f(x,y) = 0$. On A^o,
  $$\nabla f(x,y) = (2x + 3y - 6, 2y + 3x + 6).$$
  Then
  $$\begin{pmatrix} 2x+3y-6 \\ 2y+3x+6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \end{pmatrix},$$
  which can be calculated simply by row reducing.
  However, $(x,y) = (-6,6) \notin A$, since $(-6)^2 + (6)^2 = 72 > 16$.
- It now remains to check the boundary of A, which can be paratremized.
  Define two functions $g : [0,\pi] \to \mathbb{R}^2$ and $h : [-4,4] \to \mathbb{R}^2$ by
  $$g(t) = (4\sin t, -4\cos t), \qquad h(s) = (0,s),$$
  respectively. Since f is differentiable and g is differentiable, it follows that $f \circ g$ is differentiable, and likewise with $f \circ h$.

  Now
  $$\begin{aligned} f(g(t)) &= 16\sin^2(t) - 48\cos t \sin t + 16\cos^2 t - 24\sin t - 24\cos t \\ &= -48\cos t \sin t - 24\sin t - 24\cos t + 16 \\ f(h(s)) &= s^2 + 6s \\ &= s(s+6), \end{aligned}$$
  and so $\frac{d}{dt} f(g(t)) = 48(\sin^2 t - \cos^2 t) - 24(\cos t \sin t)$ and $\frac{d}{ds} f(h(s)) = 2s + 6$. We find (by Wolfram Alpha) that $f'(g(t)) = 0 \iff t = \frac{\pi}{4}, 2.7176$ and $f'(h(s)) = 0 \iff s = -3$.
  These values correspond to the points $(2\sqrt{2}, -2\sqrt{2}), (1.6456, 3.6458)$, and $(0,-3)$ respectively.
- Then by plugging and chugging,
  $$\begin{aligned} f(2\sqrt{2}, -2\sqrt{2}) &= -8 - 24\sqrt{2} \\ f(1.6456, 3.6458) &\approx 253.94 \\ f(0,-3) &= 27. \end{aligned}$$
- Overall, the minimum and maximum values of f occur on the circular portion of the boundary of A at the points $f(2\sqrt{2}, -2\sqrt{2}) = -8 - 24\sqrt{2}$ and $f(1.6456, 3.6458) \approx 253.94$, respectively. This is because there are no points of f on A^o where $\nabla f(x,y) = 0$, and the critical values of f on the line segment $[-4,4]$ are not as great or as less as the ones found on the circular portion of the boundary of A.
- Therefore the maximum value of f on A is $f(1.6456, 3.6458) \approx 253.94$ and the minimum value of f on A is $f(2\sqrt{2}, -2\sqrt{2}) = -8 - 24\sqrt{2}$.

5. (4 points) You are classifying the local extrema of a C^3 real-valued function f on $\mathbb{R}^3$. You find it has exactly three critical points $a, b, c \in \mathbb{R}^3$. You compute the Hessian matrices

$$Hf(a) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -\pi & 0 \\ 0 & 0 & 7 \end{bmatrix}, \qquad Hf(b) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

You compute the second Taylor polynomial of f at c to be

$$P(x, y, z) = 237 - 2x^2 - z^2.$$

For each part below, no justification necessary. **Fill in EXACTLY ONE circle.** (unfilled ○ filled ●)

(5a) Classify the critical point a.

○ local max
○ local min
● **saddle point**
○ not enough information to decide
○ none of these

(5b) Classify the critical point b.

○ local max
● **local min**
○ saddle point
○ not enough information to decide
○ none of these

(5c) Classify the critical point c.

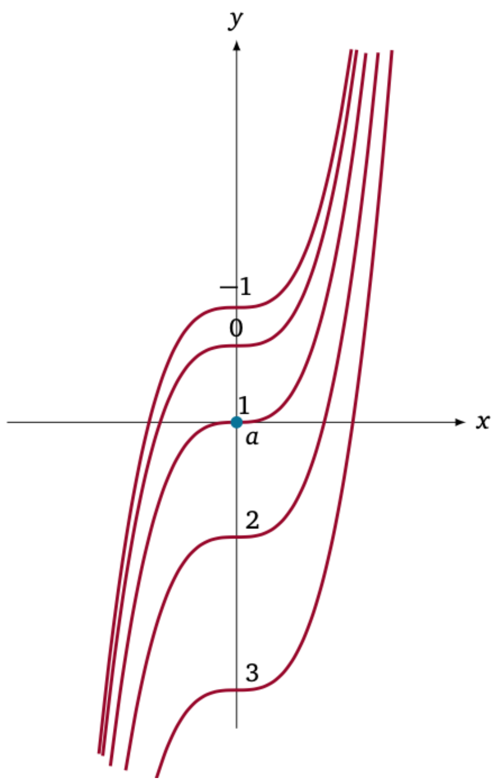
○ local max
○ local min
○ saddle point
● **not enough information to decide**
○ none of these

(5d) You verify that $f(a) = 2022$ and $f(b) = 244$. At which point does f achieve a global minimum?

○ a
○ b
● c
○ not enough information to decide

6. (6 points) Below is a contour plot of the C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ and the origin $a = (0,0) \in \mathbb{R}^2$.

Determine whether each quantity is positive, negative, or zero. Select the most plausible answer. No justification is necessary. **Fill in EXACTLY ONE circle.** (unfilled ○ filled ●)



(6a) $f(a)$ ● **positive** ○ negative ○ zero

(6b) $\dfrac{\partial f}{\partial y}(a)$ ○ positive ● **negative** ○ zero

(6c) $\dfrac{\partial f}{\partial x}(a)$ ○ positive ○ negative ● **zero**

(6d) $\dfrac{\partial^2 f}{\partial y^2}(a)$ ○ positive ● **negative** ○ zero

(6e) $\dfrac{\partial^2 f}{\partial y \partial x}(a)$ ○ positive ○ negative ● **zero**

(6f) $\dfrac{\partial^2 f}{\partial x \partial y}(a)$ ○ positive ○ negative ● **zero**

7. (6 points) **No justification is necessary for any part below.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^∞ and fix $a = (2,0) \in \mathbb{R}^2$. Suppose $f(a) = 5$, $\nabla f(a) = (-1,2)$, and $Hf(a) = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}$. Let P_N be the Nth Taylor polynomial of f at a.

(7a) Give a formula for P_1.

$P_1(x,y) = 5 + (2-x) + (2y) = 7 - x + 2y$

(7b) Evaluate $P_2(2,-0.2)$.

$P_2(2,-0.2) = 6.12$

(7c) Estimate $f(2,-0.2)$ using a quadratic approximation.

$f(2,-0.2) \approx 6.12$

(7d) If possible, evaluate $\partial^{(1,1)} P_3(2,0)$. Otherwise, write "N/A".

$\partial^{(1,1)} P_3(2,0) = 2$

(7e) Evaluate the limits. If it does not exist, write "DNE". If it is not possible, write "N/A".

$$\lim_{(x,y)\to(2,0)} \frac{f(x,y) - P_2(x,y)}{(x-2)^2 + y^2} = 0$$

$$\lim_{(x,y)\to(2,0)} \frac{f(x,y) - P_3(x,y)}{(x-2)^2 + y^2} = 0$$

8. (5 points) Let $f:\mathbb{R}^n\to\mathbb{R}$ be a C^∞ real-valued function. Assume for every integer $N\in\mathbb{N}^+$ that $\lim_{h\to 0}\frac{f(h)}{||h||^N}=0$. Prove that $\partial^\alpha f(0)=0$ for every multi-index $\alpha\in\mathbb{N}^n$. *Hint:* What is the Nth Taylor polynomial for every N?

Proof. Assume that for every integer $N\in\mathbb{N}^+$ that $\lim_{h\to 0}\frac{f(h)}{||h||^N}=0$. By Definition 5.4.9 and Taylor's Theorem (Theorem 5.4.11), the function $P_N(h)$ is the N-th order approximation of f at 0 if $\lim_{h\to 0}\frac{f(h)-P_N(h)}{\|x\|^N}=0$. We know the polynomial defined by

$$P_N(h)=\sum_{\alpha\in\mathbb{N}^n,\ |\alpha|\le N}\frac{\partial^\alpha f(0)}{\alpha!}h^\alpha$$

yields the N-th order approximation of f at 0.
By the linearity of limits, we have that

$$\lim_{h\to 0}\frac{f(h)-P_N(h)}{||h||^N}=\lim_{h\to 0}\frac{f(h)}{||h||^N}-\lim_{h\to 0}\frac{P_N(h)}{||h||^N}=0-\lim_{h\to 0}\frac{P_N(h)}{||h||^N}=0,$$

which implies that we must have that

$$\lim_{h\to 0}\frac{P_N(h)}{||h||^N}=0.$$

Now, for any multi-index $\alpha\in\mathbb{N}^n$, it must be that

$$\lim_{h\to 0}\frac{\sum_{\alpha\in\mathbb{N}^n,\ |\alpha|\le N}\frac{\partial^\alpha f(0)}{\alpha!}h^\alpha}{||h||^N}=0.$$

For the Taylor polynomial $P_N(h)$, the term with the highest degree will have degree N, while every other term will have degree $<N$. The limit with any term degree N may not vanish.
This implies that the only way for the limit to be zero is if the coefficient $\frac{\partial^\alpha f(0)}{\alpha!}=0$ for every α. By Lemma 5.4.8, the limit is zero for any multi-index with $|\alpha|>N$ on the term h.
Now, since $\alpha!\neq 0$ for any multi-index α, it must be that $\partial^\alpha f(0)=0$ for every multi-index $\alpha\in\mathbb{N}^n$, as required. ∎