MAT224 Linear Algebra II
Assignment 1

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: Jace Nicklin Alloway

Student number: 1006940802

Full Name: __________

Student number: __________

I confirm that:

- I have read and followed the policies described in the document Assignment Policies & FAQ.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway

2) __________

In this assignment we will work with two new definitions. The first is:

Definition: A vector system is a V together with two operations called *vector addition* and *scalar multiplication* such that the following nine *axioms* hold:

(i) For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x}+\mathbf{y} \in V$

(ii) For all vectors $\mathbf{x} \in V$, and scalars $c \in \mathbb{R}$, $c\mathbf{x} \in V$

(iii) For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$

(iv) There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{0}+\mathbf{x} = \mathbf{x}+\mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$

(v) For each vector $\mathbf{x} \in V$, there exists a vector $-\mathbf{x} \in \mathbf{V}$ with the property that $-\mathbf{x}+\mathbf{x} = \mathbf{x}+(-\mathbf{x}) = \mathbf{0}$

(vi) For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$

(vii) For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{x}+\mathbf{y}) = c\mathbf{x}+c\mathbf{y}$

(viii) For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c+d)\mathbf{x} = c\mathbf{x}+d\mathbf{x}$

(ix) For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$

1. Prove that commutativity holds in a vector system. That is, prove for all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$.

I want to show that for all vectors $\vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} = \vec{y}+\vec{x}$.

proof:

- Consider $2\cdot(\vec{x}+\vec{y}) = 2\cdot(\vec{x}+\vec{y})$. (given)
- On one hand, we have:

$$\begin{aligned} 2\cdot(\vec{x}+\vec{y}) &= (1+1)\cdot(\vec{x}+\vec{y}) && \text{(definition of } 2=1+1\text{)} \\ &= 1\cdot(\vec{x}+\vec{y}) + 1\cdot(\vec{x}+\vec{y}) && \text{(by axiom 8)} \\ &= 1\cdot\vec{x}+1\cdot\vec{y}+1\cdot\vec{x}+1\cdot\vec{y} && \text{(by axiom 7)} \\ &= \vec{x}+\vec{y}+\vec{x}+\vec{y} && \text{(by axiom 9)} \end{aligned}$$

- On the other hand, we have that

$$\begin{aligned} 2\cdot(\vec{x}+\vec{y}) &= 2\cdot\vec{x} + 2\cdot\vec{y} && \text{(by axiom 7)} \\ &= (1+1)\cdot\vec{x} + (1+1)\cdot\vec{y} && \text{(definition of } 2=1+1\text{)} \\ &= 1\cdot\vec{x}+1\cdot\vec{x}+1\cdot\vec{y}+1\cdot\vec{y} && \text{(by axiom 8)} \\ &= \vec{x}+\vec{x}+\vec{y}+\vec{y} && \text{(by axiom 9)} \end{aligned}$$

- Thus, we have that $\vec{x}+\vec{y}+\vec{x}+\vec{y} = \vec{x}+\vec{x}+\vec{y}+\vec{y}$
- By right-addition, I will add $(-\vec{y})$ to both sides:

$$\begin{aligned} (\vec{x}+\vec{y}+\vec{x}+\vec{y})+(-\vec{y}) &= (\vec{x}+\vec{x}+\vec{y}+\vec{y})+(-\vec{y}) \\ (\vec{x}+\vec{y}+\vec{x})+(\vec{y}+(-\vec{y})) &= \vec{x}+\vec{x}+\vec{y}+(\vec{y}+(-\vec{y})) && \text{(by axiom 3)} \\ (\vec{x}+\vec{y}+\vec{x})+\vec{0} &= (\vec{x}+\vec{x}+\vec{y})+\vec{0} && \text{(by axiom 5)} \\ \vec{x}+\vec{y}+\vec{x} &= \vec{x}+\vec{x}+\vec{y} && \text{(by axiom 4)} \end{aligned}$$

- By left-addition, I will add $(-\vec{x})$ to both sides:

$$\begin{aligned} (-\vec{x})+(\vec{x}+\vec{y}+\vec{x}) &= (-\vec{x})+(\vec{x}+\vec{x}+\vec{y}) \\ ((-\vec{x})+\vec{x})+(\vec{y}+\vec{x}) &= ((-\vec{x})+\vec{x})+(\vec{x}+\vec{y}) && \text{(by axiom 3)} \\ \vec{0}+(\vec{y}+\vec{x}) &= \vec{0}+(\vec{x}+\vec{y}) && \text{(by axiom 5)} \\ \vec{y}+\vec{x} &= \vec{x}+\vec{y} && \text{(by axiom 4)} \end{aligned}$$

- Therefore $\vec{x}+\vec{y} = \vec{y}+\vec{x}$, which is what I needed to show.

$\square$

The second new definition we will work with is:

Definition: A *Peano system* is a set V together with two operations, *addition* and *scalar multiplication* such that the following axioms hold:

1. For all $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} \in V$
2. For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
4. For all $\mathbf{x} \in V$, and scalars $c \in \mathbb{R}$, $c\mathbf{x} \in V$
5. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$
6. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
7. For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
8. For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$
9. There exists a vector $\mathbf{0} \in V$ with the property that $0\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in V$.

This was actually the original set of axioms given by Giuseppe Peano for a vector space in 1888. Cool!

2(a) Which of the axioms in a *Peano system* are not present in the definition of a *vector system*? You may answer using the numbering of the axioms as given above, e.g., 1., 3., etc. No justification is required.

Answer: 2, 9

2(b) Which of the axioms in the definition of a *vector system* are not present in a *Peano system*? You may answer using the numbering of the axioms as given in the definition of a *vector system*, e.g., (i), (iii), etc. No justification is required.

Answer: 4, 5

3. Prove that a *Peano system* is a *vector system*.

I need to show that all vector system axioms hold under the Peano System axioms.

proof:

- I will begin by proving a lemma: for all vectors $\vec{x} \in V$, $(-1)\vec{x} = -\vec{x}$.

$-\vec{x} = (-1)\vec{x}$ (given)

$-\vec{x} + \vec{x} = (-1)\vec{x} + \vec{x}$ (add $\vec{x}$ to both sides)

$-1\vec{x} + 1\vec{x} = (-1)\vec{x} + 1\vec{x}$ (by axiom 8)

$((-1)+(1))\vec{x} = ((-1)+1)\vec{x}$ (by axiom 6)

$0\vec{x} = 0\vec{x}$ (by definition 5.1.4, axiom 4)

$\vec{0} = \vec{0}$ (by axiom 4)

- Because the left and right side are the same after adding $\vec{x}$ to both sides, we know that $-\vec{x}$ is unique and the same as $(-1)\vec{x}$, so thus $-\vec{x} = (-1)\vec{x}$, as needed. $\square$
- From the definitions of Peano System and vector system, Peano axioms 1, 2, 3, 6, 7, 8, and 9 are already axioms included in the Vector System definition, so I will have to show that Vector System axioms 4 and 5 can be derived from the Peano System axioms.

(iv) There exists a vector $\vec{0} \in V$ with the property that $\vec{0} + \vec{x} = \vec{x} + \vec{0} = \vec{x}$ for all vectors $\vec{x} \in V$.

$\vec{x} = \vec{x}$ (given)

$\vec{x} + \vec{0} = \vec{x} + \vec{0}$ (right-add $\vec{0}$ to both sides)

$= \vec{x} + 0\vec{x}$ (by axiom 9)

$= 1\vec{x} + 0\vec{x}$ (by axiom 8)

$= (1+0)\vec{x}$ (by axiom 6)

$= (1)\vec{x}$ (by definition 5.1.4, axiom 3)

$= 1\vec{x}$ (by axiom 4)

$= \vec{x}$ (by axiom 8)

Similarly,

$\vec{x} = \vec{x}$ (given)

$\vec{0} + \vec{x} = \vec{0} + \vec{x}$ (left-add $\vec{0}$ to both sides)

$= 0\vec{x} + \vec{x}$ (by axiom 9)

$= 0\vec{x} + 1\vec{x}$ (by axiom 8)

$= (0+1)\vec{x}$ (by axiom 6)

$= (1)\vec{x}$ (by definition 5.1.4, axiom 3)

$= 1\vec{x}$ (by axiom 4)

$= \vec{x}$ (by axiom 8)

- Therefore $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ holds under the Peano System axioms.

(V) For each $\vec{x} \in V$, there exists a vector $-\vec{x} \in V$ with the property that $-\vec{x} + \vec{x} = \vec{x} + (-\vec{x}) = \vec{0}$.

$\vec{0} = \vec{0}$ (given)

$= 0\vec{x}$ (by axiom 9)

$= (0)\vec{x}$ (by axiom 4)

$= (1+(-1))\vec{x}$ (by definition 5.1.4, axiom 4)

$= 1\vec{x} + (-1)\vec{x}$ (by axiom 6)

$= \vec{x} + (-1)\vec{x}$ (by axiom 8)

$= \vec{x} + (-\vec{x})$ (by lemma)

Similarly,

$\vec{0} = \vec{0}$ (given)

$= \vec{x}0$ (by axiom 9)

$= \vec{x}(0)$ (by axiom 4)

$= \vec{x}(1+(-1))$ (by definition 5.1.4, axiom 4)

$= \vec{x}((-1)+1)$ (by axiom 2)

$= (-1)\vec{x} + 1\vec{x}$ (by axiom 6)

$= (-1)\vec{x} + \vec{x}$ (by axiom 8)

$= (-x) + \vec{x}$ (by lemma)

- Therefore $\vec{x} + (-\vec{x}) = -\vec{x} + \vec{x} = \vec{0}$ holds under the Peano system axioms.

- Thus all vector system axioms hold under the Peano System axioms, which is what I needed to show.
- Therefore a Peano System is a vector system, as needed. $\square$

4. Explain why a *Peano system* is a vector space. Your should include a definition of a vector space, either from your lecture notes, or the textbook to support your answer.

- Recall that a vector space is a set V, whose elements are called vectors, along with two operations:

  - Vector addition, or vector sum, for which each pair of vectors $\vec{x}, \vec{y} \in V$ produce another vector in V, $\vec{x}+\vec{y}$ and;
  - Scalar Multiplication, or scaling, by a real number for which every vector $\vec{x} \in V$ and for every $c \in \mathbb{R}$ produce another another vector in V, denoted $c\vec{x}$;

satisfy the following 8 axioms:

1) [Associativity] For all vectors $\vec{x}, \vec{y}$, and $\vec{z} \in V$, $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
2) [Commutativity] For all vectors $\vec{x}$ and $\vec{y} \in V$, $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
3) [Additive Identity] There exists a vector $\vec{0} \in V$ such that for all vectors $\vec{x} \in V$,
$$\vec{x}+\vec{0} = \vec{x}$$
4) [Additive Inverse] For each vector $\vec{x} \in V$, there exists an additive inverse element, $-\vec{x}$, such that $\vec{x}+(-\vec{x}) = \vec{0}$
5) [Distributivity] For all vectors $\vec{x}$ and $\vec{y} \in V$ and for all scalars $c \in \mathbb{R}$,
$$c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$$
6) [Distributivity] For all vectors $\vec{x} \in V$ and for all scalers c and $d \in \mathbb{R}$,
$$(c+d)\vec{x} = c\vec{x}+d\vec{x}$$
7) [Multiplicative Associativity] For all vectors $\vec{x} \in V$ and for all scalars c and $d \in \mathbb{R}$,
$$(cd)\vec{x} = c(d\vec{x})$$
8) [Multiplicative Identity] For all vectors $\vec{x} \in V$, $1\vec{x} = \vec{x}$.

- Like a vector space, a Peano System is defined by two operations: vector addition and scalar multiplication (indicated by axioms 1 and 4) satisfy 9 axioms.
- A Peano system, defined by 9 axioms (already indicated on page 11 of this assignment), satisfies all 8 axioms which define a vector space.
  Peano axioms 2, 3, 5, 6, 7, and 8 are all included in the definition of vector space.
  Furthermore, vector space axioms 3 and 4 (indicated above) can be derived from the other Peano axioms, as proved in question 3 of this assignment.
- Thus the 9 Peano axioms satisfy all 8 vector space axioms as following the defined vector addition and scalar multiplication operations.
- Therefore, the Peano System is also a vector space.