QUIZ-12 SOLUTION

1) → Store all the first s element of the stream to S.

→ Suppose we have seen $n-1$ elements, now the nth element arrives $(n > s)$

- with probability $\frac{s}{n}$, keep the nth element else discard it.
- if we picked the nth element, then it replaces one of the s elements in the sample S, picked uniformly at random.

Assumption :- [0.5]

- We assume that ~~positions~~ after n elements, the sample contains each element seen so far with probability $\frac{s}{n}$

- **Our goal :-** We need to show that after seeing element $n+1$ the sample maintains the property i.e the sample contains each element seen so far with probability $\frac{s}{n+1}$

[0.5] **Base Case** - ~~when~~ when $n=s$ elements, the sample S has the desired property $\Rightarrow$ Probability $= \frac{s}{n+1}$

[0.5] **Inductive Hypothesis** - After n elements, the sample S contains each element seen so far with probability : $\frac{s}{n}$

[0.5] **Inductive step** -
Now element $n+1$ arrives, for elements already in S, probability that algorithm keeps it in s is :-

$$\left(1 - \frac{s}{n+1}\right) + \left(\frac{s}{n+1}\right) \cdot \left(\frac{s-1}{s}\right) = \frac{n}{n+1}$$

Example :- [1 POINT]

stream: $a, b, c, d, e, f, g \ldots$ and sample size 2 :-

At the very beginning, b is in the sample since $\frac{2}{2} = 1$

when time $n = 3$,

the probability of putting c into the sample is $\frac{2}{3}$,

so that the probability of keeping a is

$$P(\text{do not add } c) + P(b \text{ is removed} \mid \text{add } c) \times P(\text{add } c)$$

$$= \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{2}{3} = \frac{s}{n+1}$$

2) Suppose we have y darts and x target [1.5 POINTS].

— The prob. of a specific dart cannot hit the specific target is $\frac{x-1}{x}$

— y darts all fail to hit a specific target $\left(\frac{x-1}{x}\right)^y$

$$= \left(1 - \frac{1}{x}\right)^{x \frac{y}{x}} \simeq e^{-y/x}$$

Now,

— suppose we have n bits in array, m elements in the set S, k hash function

— we have $x = n$ targets and $y = k \times m$ darts.

— $\therefore$ the probability that a bit is still not hit by any dart is $e^{-\frac{km}{n}}$

— false positive rate — $\left(1 - e^{-\frac{km}{n}}\right)^k$

To get optimal k, we take the derivate of $f(k) = \left(1 - e^{-\frac{km}{n}}\right)^k$

we can calculate optimal value of k when $f(k)$ reaches the minimal value

$$k = \frac{n}{m} \ln 2 \quad \longrightarrow \quad [1.5 \text{ POINTS}]$$

3) — The probability (p) that some elements have at least r tailing 0 is $(1 - e^{-\frac{m}{2^r}})$

— The probability (p') that none of m distinct elements has tail length at least r is $(1 - 2^{-r})^m \simeq e^{-m2^{-r}}$

Therefore, we can observe that

[1.5] ⓐ if $2^r \gg m$, $p = \frac{m}{2^r} \to 0$; $p' = 1$

In this case, the hashed result is never likely with R trailing 0s. So the R cannot be too large.

[1.5] ⓑ if $2^r \ll m$, $p = 1 - e^{-m/2^r} \to 1$; $p' = 0$.

In this case, every hashed result is likely with R trailing 0s. So the R cannot too small.

In general, R should be neither too large or too small. 2^R ~~around~~ should be around m.

4) Because the kth moment is very powerful and informative tool when we wish to measure some features about a stream.

→ 0th moment : shows the number of distinct elements in the stream

→ 1th moment : tells the length of the stream

→ 2th moment : rep. the elements evenen of distribution the S.

[1 POINT]