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Empirical re-conceptualization: From empirical generalizations to insight and understanding

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ABSTRACT

Identifying patterns is an important part of mathematical reasoning, but many students struggle to justify their pattern-based generalizations. Some researchers argue for a de-emphasis on patterning activities, but empirical investigation has also been shown to support discovery and insight into problem structures. We introduce the phenomenon of empirical re-conceptualization, which is the development of an empirical generalization that is subsequently re-interpreted from a structural perspective. We define and elaborate empirical re-conceptualization by drawing on data from secondary and undergraduate students. We also identify four major instructional supports we found for empirical re-conceptualization: (1) experiencing need for verification, (2) fostering contextual interpretation, (3) fostering reflection and justification, and (4) fostering pattern exploration, as well as three processes facilitating the transition from empirical to deductive reasoning: (a) verification, (b) justification, and (c) creation / interpretation.

1. Introduction and motivation

Recognizing and developing patterns is a critical aspect of mathematical reasoning.

Curricular materials emphasize activities that foster students' abilities to develop empirical (typically pattern-based) generalizations (e.g., Lappan et al., 2014), and many students are adept at recognizing and formalizing patterns (Pytlak, 2015). However, students can also struggle to understand, explain, and justify the empirically-based findings they develop (e.g., Čadež & Kolar, 2015). One source of students' difficulties may rest with the empirical nature of their generalizations, by which we mean generalizations that are derived from observed patterns or regularities in data. Students can become overly reliant on examples and infer that a universal statement is true based on a few confirming cases (Knuth et al., 2009). Further, students tend not to analyze those examples strategically in order to develop insight into a justification for their generalization (Cooper et al., 2011; Knuth et al., 2012). Accordingly, researchers have advocated for instructional approaches that help students learn the limitations of empirical arguments (e.g., Mhlolo, 2016; Stylianides & Stylianides, 2009). These approaches have shown some success in helping students see the limitations of examples, but students nevertheless continue to struggle to produce deductive justifications (Bieda et al., 2013; Stylianides et al., 2017). Further, these approaches frame empirical reasoning strategies as stumbling blocks to overcome, rather than important and useful tools in the development of both generalizations and their justifications.

In contrast, we propose that empirical reasoning can play an essential role not only in the development, exploration, and

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understanding of generalizations, but also in subsequent attempts to develop justifications of those generalizations. Such reasoning, in turn, can help students transition from empirical to deductive argumentation (e.g., Tall et al., 2011). One way to do this may be through seeing an example as a generic example. Mason and Pimm (1984) introduced the notion of a generic example as a specific example that is presented in such a way as to bring out its intended role as the carrier of the general. Such an example can make explicit the reasons for the truth of an assertion by means of operations on an object that is a "characteristic representative of the class" (Balacheff, 1988, p. 219), and can therefore support the development of what Mason and Pimm (1984) called a generic proof. The characterization of a generic example as one that students can use to stand in for all objects in its class is reflected in other descriptions in the literature (e.g., Harel & Tall, 1991; Leron & Zaslavsky, 2013; Rowland, 1998, 2002), and researchers have offered guiding principles for developing such examples (Rowland, 2002), for judging an argument's use of a generic example (Yopp & Ely, 2016), and for using generic examples as vehicles to formal proof (Leron & Zaslavsky, 2013).

The above studies offer a promising avenue for helping students leverage their empirical reasoning, but more work is needed to better understand the nature of the transition from relying on empirical patterns to the development of examples that students use as generic examples, and more generally a deductive argument. In particular, there remains a need to further conceptualize this transition, and to identify the processes involved in such a transition. We introduce one such avenue for transitioning from empirical reasoning to deductive argumentation, which we call *empirical re-conceptualization*. Empirical re-conceptualization is the process of identifying a pattern or regularity, forming an associated generalization, and then re-interpreting those findings from a structural perspective. Through the act of empirical re-conceptualization, students can build their empirical generalizations into mathematically meaningful insights and arguments. The aim of the study reported here is to characterize the processes involved in leveraging empirical reasoning into more advanced forms of reasoning, and to identify the instructional supports that can foster empirical re-conceptualization.

2. Background: the challenge of transitioning from empirical to deductive reasoning

In the following sections we describe the benefits students can experience from

generalizing empirical patterns, as well as the documented challenges that exist when trying to move from empirical patterning to deductive reasoning and justification. Together, these works position empirically-based generalizing as having promise for offering a bridge to deductive reasoning, but there remains a need to better understand the manner in which this transition should occur. We then return to the idea of a generic example as one potential bridge between empirical and deductive reasoning, describing two key constructs related to students' use of the generic example.

2.1. The benefits and challenges of empirical patterning

In examining students' pattern generalization, researchers have found that students use multiple strategies (e.g., Barbosa et al., 2007; Becker & Rivera, 2006; Jurdak & El Mouhayar, 2014; Zazkis & Liljedhal, 2002). These works offer evidence that students can flexibly shift between different representations and can develop more sophisticated pattern strategies over time (e.g., Amit & Neria, 2008; Jurdak & El Mouhayar, 2014). There are a number of potential benefits from attending to empirical patterns. The act of developing empirically-based generalizations could potentially foster an understanding of a problem's structure, which could consequently support proof development (de Villiers, 2010; Tall, 2008). Indeed, there is some evidence that students can and do engage in a dynamic interplay between empirical patterning and justifying (e.g., Guven et al., 2010; Küchemann, 2010; Martinez & Pedemonte, 2014; Schoenfeld, 1986), just as research mathematicians do (de Villiers, 2010).

These works offer promise for positioning empirical patterning as a bridge to conceptual understanding and deduction, which reflects an expectation that students' reasoning will "proceed from inductive toward deductive and greater generality" (Simon & Blume, 1996, p. 9). Indeed, a number of mathematical reasoning hierarchies (e.g., Balacheff, 1987; Bell, 1976; Waring, 2000) and school mathematics curricula (e.g., Lappan et al., 2014) reflect this expected progression. However, as discussed above, students who generalize patterns can struggle to justify them (Čadež and Kolar, 2015; Hargreaves et al., 1998; Stylianides & Stylianides, 2009; Zazkis & Liljedahl, 2002), which has contributed to difficulties in multiple domains (e.g., Ellis & Grinstead, 2008; Lockwood & Reed, 2016; Vlahović-Štetić et al., 2010).

Given the documented difficulties students experience with transitioning from empirical to deductive reasoning, an overly strong emphasis on empirical patterns alone is unlikely to foster justification (Lin et al., 2004), and in fact can promote the learning of routine procedures and generalizations without understanding (Küchemann, 2010; MacGregor & Stacey, 1993). Furthermore, these difficulties are compounded by the fact that students typically receive little, if any, explicit instruction on how to strategically analyze examples in developing, understanding, exploring, and proving generalizations (Cooper et al., 2011). Thus, although a focus on empirical generalization can offer promise as a bridge to insight and deduction, this promise is yet to be broadly realized in instruction. In particular, what remains unclear is the manner in which students are supposed to actually make the transition from empirical to deductive arguments. As we described in Section 1, one potential avenue for using empirical patterning to develop deductive justification could be through the use of a generic example. In the section below, we introduce two constructs germane to generic examples, namely *generic abstraction* and *empirical versus structural generalizations*.

2.2. Generic abstraction and empirical versus structural generalizations

Harel and Tall (1991) introduced the notion of generic abstraction, in which a student comes to understand one or more examples as typical of a wider range of examples embodying an abstract concept. Such an abstraction entails a generalization because it embeds the example in a broader class embodied by the abstraction. Rowland (1998) linked this type of generic abstraction to two forms of generalization, drawing on Bills & Rowland's (1999) distinction between empirical and structural generalizations. Empirical generalizations, which are derived only from the form of results and observed relationships, can possess predictive potential but lack explanatory power. In contrast, structural generalizations, which are based on underlying meanings or structures, can provide explanatory insight. A generic example, then, that can successfully speak to a general process, has the quality of a structural generalization. Others have also described a similar difference in generalization, distinguishing it as empirical versus theoretical (Carraher et al., 2008; Doerfler, 1991). In each case, generalizations that are theoretical (or structural) are viewed as more mathematically powerful than their empirical counterparts.

As an example, Bills and Rowland (1999) discussed a pattern problem with a rectangle of matchsticks (Fig. 1), in which one had to determine "How many matches would be needed to make a rectangle with R rows and C columns?" (p. 106).

The authors noted that one could hold the number of rows R at 2 and increase the number of columns to determine the number of matchsticks for a given column, producing a pattern yielding 5 more matchsticks for each new column, which can be described as $M = 5C + 2$. Then, one could increase the number of rows R to 3 to yield $M = 7C + 3$. Similarly, for $R = 4$ and $R = 5$, one can find the rules $M = 9C + 4$, and $M = 11C + 5$, respectively. Thus, a pattern emerges that suggests the general rule to be $M = (2R + 1)C + R$. This generalization is *empirical*, as it is the consequence of attending to the pattern of outcomes of counting the number of matchsticks for given row and column values. As Bills and Rowland described, however, one can also consider the number of vertical and horizontal matches in relation to a generic rectangle, specifically, recognizing that "there are $C + 1$ columns of vertical matches, each containing R matches; there are $R + 1$ rows of horizontal matches, each containing C matches" (p. 107), therefore, altogether there will be $(C + 1)R + (R + 1)C$ matches. In contrast to the empirical generalization, this is a *structural* generalization, because it relies on considering the number of matchsticks in relation to the number of rows and columns. Bills and Rowland (1999) noted that the type of pattern-based reasoning that produced the initial generalization $M = (2R + 1)C + R$ can be a useful way to produce conjectures, but ultimately students must shift to other sources of conviction.

The above example highlights a meaningful distinction between empirical and structural generalization, but it does not necessarily account for the processes by which one progresses from empirical to structural generalization. It is in exploring how students can transition from empirical to structural generalizations (which ultimately may lead to deductive reasoning) that we position the construct of empirical re-conceptualization. In the Conceptual Framework section below, we define generalization and then outline the perspectives that guided our development of empirical re-conceptualization.

3. Conceptual framework

Generalization is commonly characterized as the establishment of a claim that a property holds for a set of mathematical objects or conditions that is larger than the set of individually verified cases (Carraher et al., 2008; Dreyfus, 1991; Ellis, 2007; Kaput, 1999). For instance, Radford (1996) argued that generalization involves identifying a commonality based on particulars, and then extending it to all terms, and Harel and Tall (1991) defined generalization as the process of applying a given argument to a broader context. As a complement to these cognitively-based definitions, others have positioned generalization as a social phenomenon (e.g., Jurow, 2004), arguing that generalization is an activity that can occur collectively, distributed across multiple agents (Ellis, 2011; Reid, 2002; Tuomi-Gröhn & Engeström, 2003). For the purposes of this paper, we define generalizing as an activity in which learners in specific sociocultural and instructional contexts either (a) identify commonality across cases, (b) extend their reasoning beyond the range in which it originated, or (c) derive broader results from particular cases.

3.1. Result pattern generalization (RPG) and process pattern generalization (PPG)

Harel (2001) described RPG as a way of thinking about the process of generalization in which one attends solely to regularities in the result. Generalizing through RPG led to the rule $M = (2R + 1)C + R$ from the above matchstick problem (Bills & Rowland, 1999), and it is often observed in students' activity when they then struggle to shift from recursive to explicit relationships or justify their patterns (e.g., Čadež & Kolar, 2015, Moss et al., 2006). In contrast, PPG entails attending to regularity in the process. To extend the matchstick example, one might consider a generic starting row made up of R vertical matches, and then reason that when adding a column, one must add another R vertical matches plus $R + 1$ horizontal matches, i.e., $2R + 1$ total matches. Thus, for an additional C columns, the number of matchsticks will be $M = R + (2R + 1)C$. This distinction between RPG and PPG helped us think about qualitatively different kinds of generalizations, and helped to frame our development of empirical re-conceptualization.

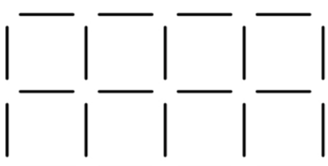

Fig. 1. Rectangle matchstick problem (Bills & Rowland, 1999, p. 106).

3.2. Figurative and operative activity

The second construct we draw on is a distinction between figurative and operative activity (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985). Figurative activity involves attending to similarity in perceptual or sensorimotor characteristics. As such, it is not a form of thought that we should necessarily interpret as deficient, as it can support the development of generalizations. In the matchstick rectangle problem, figurative activity supported the development of the rule $M = (2R + 1)C + R$. In contrast, operative activity entails attending to similarity in structure or function through the coordination and transformation of mental operations. Considering the number of vertical and horizontal matches in relation to a generic rectangle entailed operative activity, as it required mentally coordinating rows, columns, and matchsticks in relation to one another for any given size rectangle. A shift from RPG to PPG often co-occurs with a shift from figurative to operative activity, and we consider operative activity to be a hallmark of the tendency to reason structurally.

3.3. Symbolic interactionism and instructional supports

In order to attend to the features of instruction and interaction that can foster empirical re-conceptualization, we consider learning situations through the lens of multiple processes of interactions, in which the students and the teacher-researcher co-contribute to the development of meaning through their conversation, shared problem-solving activity, negotiated meaning of tasks, and engagement with tools and artifacts. The symbolic interactionist perspective (Blumer, 1969) privileges both individual students' reasoning and the social processes supporting that reasoning (Voigt, 1995). As such, it allows us to attend to how teachers, students, tasks, and tools can mutually interact to support shared ways of reasoning. It is with this framing that we sought out *instructional supports* for empirical re-conceptualization, examining how the teacher-researcher, students, task structures, and tools worked together. We use the term instructional supports rather than teacher moves as a reflection of our view of the learning environment as a system, made of up mutually interacting agents. We then consider how that system supports students' abilities to re-conceptualize their generalizations from a structural perspective. This perspective is a useful way to make sense of student learning within teaching-experiment settings, which can be more complex and entail more forms of interaction than individual clinical interviews.

4. The process of empirical re-conceptualization

We now define empirical re-conceptualization as the process of re-interpreting an empirical generalization from a structural perspective. By structural perspective, we follow Bills and Rowland's (1999) characterization of "looking at the underlying meanings, structures, or procedures" (p. 106). More specifically, we consider this to entail either (a) a shift from RPG to PPG, (b) a shift from figurative to operative mental activity, and/or (c) a shift from considering specific numbers or patterns to considering a case as a generic example. Re-interpreting a generalization from a structural perspective thus entails a shift that involves recognizing, exploring, and reasoning with general structures rather than with particular cases alone. The matchstick rectangle example described in Bills and Rowland (1999) could therefore be a case of empirical re-conceptualization. If a student developed the rule $M = (2R + 1)C + R$ from exploring and generalizing number patterns for specific row values, it would be an empirical generalization. In then considering a generic rectangle of matchsticks with R rows and C columns, one would be re-interpreting the rule in light of the relationship between rows and columns. This would entail both a shift from figurative to operative mental activity and a shift to the use of a generic example.

In this paper, our aim is to characterize and elaborate what occurs when students engage in empirical re-conceptualization. We also address the following research questions:

1 What instructional supports exist to foster empirical re-conceptualization?
2 How does empirical re-conceptualization facilitate students' transition from empirical reasoning to deductive reasoning and justification? What characterizes this shift?

5. Methods

This study was part of a larger three-year project investigating the processes of students' generalizing activity. For that project, we first implemented a series of individual, semi-structured interviews with 29 middle-school students, 24 high-school students, and 40 undergraduate students in order to explore their generalizations in the topics of algebra, advanced algebra, trigonometry, and combinatorics. We then conducted a series of 13 small and medium-scale videoed teaching experiments (Steffe & Thompson, 2000) with a total of 18 middle-school students, 5 high-school students, and 29 undergraduate students (see Ellis, Lockwood, Tillema, & Moore, 2021) for more details about the broader project). The data for the study reported in this paper come from the teaching-experiment phase. For the purposes of illustrating the phenomenon of empirical re-conceptualization, we highlight two teaching experiments, one with secondary students and one with undergraduate students.

5.1. Secondary teaching experiment

Barney, a 7th-grade student, and Homer, a 9th-grade student, participated in a paired teaching experiment. (The students chose their own pseudonyms.) At the time of the teaching experiment, Barney had just completed a pre-algebra course, which was part of the advanced mathematics track at his school, and Homer had completed Algebra I. The two students had a strong familiarity with one another and communicated well together, which was evident in their discussions throughout the teaching experiment. The teaching experiment occurred across five sessions, with each session averaging about 75 min (and ranging in time from 60−90 min). The first author was the teacher-researcher. In between each session, the first author met with the second author to review the data, debrief the session, and revise or develop new tasks for the next session. Consequently, the tasks were created and revised on a session-by-session basis in response to the teacher-researcher's hypothesized models about the students' mathematics.

We developed tasks to first foster an understanding of linear growth as a constant rate of change, and quadratic growth as a constantly-changing rate of change. Following the design principles addressed in covariational reasoning frameworks (Carlson et al., 2002; Thompson & Carlson, 2017), we created a task sequence exploring rates of change within the contexts of speed, area, and volume. The tasks were organized to create opportunities for students to develop a foundation of thinking about functional growth from a covariation perspective that the students could then leverage when exploring cubic and higher-order polynomial functions (Ellis et al., 2020). We then transitioned to volume tasks to explore the students' generalizations about volumes of growing cubes and rectangular prisms. For instance, one such task was the growing cube task, in which the students examined how the volume of an n by n by n cube would grow if each side grew by 1 cm. The students manipulated a dynamic geometry software image of a smoothly growing cube, and also worked with physical cubes to build up models to think about the constituent parts of the added volume (Fig. 2). The emphasis on volume then provided a conceptual metaphor (Lakoff & Núñez, 2000) for exploring growing hypervolumes of hypercubes and other objects in four dimensions and beyond. This metaphor gave meaning to the students' representations when they transitioned to reasoning combinatorially about n-dimensional figures and binomial coefficients. Table 1 provides the mathematical topics and contexts addressed each day during the teaching experiment. With respect to these tasks, Homer had encountered quadratic growth in his mathematics classes, but Barney had not. Neither student had encountered higher-order polynomials, and Barney had not yet learned algebraic manipulations such as $(n+1)^2$ or $(n+1)^3$. Homer was familiar with these manipulations, but did not use them during the teaching experiment.

5.2. Undergraduate teaching experiment

For the undergraduate teaching experiment, we recruited four students who were novice counters, without formal instruction counting problems. This criterion was important because we wanted to observe reasoning about combinatorial ideas, rather than the recall of formulas. We assigned the pseudonyms Carson, Aaron, Anne-Marie, and Josh to the four participants. The teaching experiment consisted of nine 90-minute videotaped sessions, totaling 13.5 h, over a 15-day span. The second author was the teacher-researcher. Other members of the research team filmed the sessions, and the team members met in between each session to review the data and to revise or develop new tasks.

The teaching experiment included a number of combinatorial activities aimed at facilitating students' generalizing, including counting tasks and the development of combinatorial proofs of binomial identities. For the purposes of this paper, we focus on a set of tasks that we call the Passwords Activity. The overall aim of the activity was to guide students through the development and articulation of a general statement of the binomial theorem $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$, in our case where n is a nonnegative integer. In order to do so, we developed a sequence of tasks in which students explore what would happen as x, k, and n vary (in terms of the number of letters, the number of numbers, and the total length of the passwords). To get there, we began with passwords with only one type of character (letters), asking students to count the number of passwords of length 3, 4, 5, and eventually length n, that consisted of the characters A or B, where repetition was allowed. We prompted students to create tables to organize passwords with a certain number of As, reasoning about the total number of passwords of a given length. In subsequent tasks, the students counted passwords of various lengths from the characters 1, A, or B, and then eventually counted various length passwords that consisted of x numbers and y letters, ultimately generalizing a statement of the binomial theorem.

The purpose of asking the students to complete tables was to draw their attention to certain mathematical relationships, such as the fact that the sum of the rows in the table is equal to the total number of passwords (which relate to the two sides of the expression of the binomial theorem). In this paper, we focus on student work related to the 5-character AB table, in this case, $\sum_{i=0}^{5}\binom{5}{i} = 2^5$ (Table 2). Here the number of 5-character AB passwords with exactly three As can be found by selecting three of the five distinct positions in which to place As (and the rest of the positions will be Bs).[1] Thus, for example, the 10 in the fourth row of Table 2 can be thought of as $\binom{5}{3}$, or $\frac{5!}{3!2!}$.

5.3. Analysis

Our aim in analysis was to address our research questions by first identifying and characterizing students' processes of empirical re-conceptualization, and then by identifying any instructional supports that fostered empirical re-conceptualization and examining the

[1] Generally, the number of n-character AB passwords with exactly k As (the right column of the table) can be found through listing or by using the formula for sections or combinations $\binom{n}{k}$.

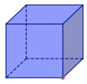

Fig. 2. Screen capture of a dynamically growing cube and physical model of a growing cube.

Table 1
Overview of the secondary teaching experiment unit.

Day	Mathematical Topics	Contexts
1	Covariation, linear growth, average rate of change	Growing rectangles
2	Piecewise linear growth, quadratic growth	Growing stair steps, triangles
3	Cubic growth	Growing cubes, rectangular prisms
4	Higher-order polynomials, combinations, Pascal's triangle	Growing hypercubes, *n*-dimensional figures
5	Binomial coefficients	Combinations and listing

Table 2
The 5-character AB table.

Number of As	Number of Passwords
0	1
1	5
2	10
3	10
4	5
5	1

processes involved in transitioning from empirical to deductive reasoning. In order to accomplish this, we first transcribed all teaching sessions. Using the constant-comparative method (Strauss & Corbin, 1990), we then analyzed both data sets in order to identify the participants' generalizations; we did this in order to find those generalizations that were empirical (Bills & Rowland, 1999). We drew on Ellis et al.'s (2017; Ellis, Lockwood, Tillema, & Moore, 2021) RFE Framework, which identifies types and forms of generalizations. Once we had identified all of the empirical generalizations, we then shifted to open coding to trace the development of each empirical generalization. In particular, we asked the following: (a) Did the generalization change, and if so, how? Specifically, did the students re-interpret their empirical generalization from a structural perspective? (b) Did the generalization lead to a justification? (c) Did the student(s) experience a subsequent mathematical insight? Open coding yielded types of insight such as the re-interpretation of a previously-developed formula, generalization, or statement, the development of a new pattern, or the establishment of a new piece of knowledge or form of understanding. This enabled us to characterize what occurred as students made sense of their empirical generalizations from a structural perspective. By answering question a), we gained insight into the actual shifts that occurred in empirical re-conceptualization and what might be occurring as students engage in this process of empirical re-conceptualization (research question #1). Then, answering questions b) and c) provided insight into the processes by which empirical re-conceptualization facilitated the transition from empirical to deductive argumentation (research question #3).

Once we identified each instance of empirical re-conceptualization, we began a new round of open coding with the secondary data to identify what instructional supports foster empirical re-conceptualization (research question #2). Beginning with each case of empirical re-conceptualization, we examined the teacher moves, student interactions and use of artifacts, and task features that preceded the instance and supported the students' structural re-conceptualization of the empirical generalization. This led to an initial set of four types of instructional supports, which we present in Section 6.1. We then coded the undergraduate teaching experiment data with our initial categories, refined and adjusted the categories in light of the undergraduate data, and then re-coded both data sets with our final coding scheme.

6. Results

Recall that we defined empirical re-conceptualization as a shift, that entails beginning with an empirical generalization, and then shifting (a) from RPG to PPG, (b) from figurative to operative mental activity, and/or (c) from considering specific numbers or patterns to considering a case as a generic example. Our two research questions concerned (1) what instructional supports foster empirical re-conceptualization, and (2) how empirical re-conceptualization facilitates the transition from empirical to deductive reasoning. In the next section, we first provide an overview of the four instructional supports we found for empirical re-conceptualization. In Section 6.2, we then introduce the three processes that students engaged in as they leveraged empirical re-conceptualization to transition to deductive reasoning and justification. Then, in Sections 6.3 and 6.4, we present two extended data vignettes to further elaborate both the instructional supports and the processes. For each vignette we (a) characterize the phenomenon of empirical re-conceptualization in terms of its shifts, (b) analyze the relevant instructional supports, and (c) analyze the relevant mathematical processes in transitioning to deductive reasoning. By providing vignettes across two different age ranges and involving multiple content areas, we highlight the fact that empirical re-conceptualization is a phenomenon that may occur in a variety of mathematical settings.

6.1. Overview of instructional supports

The four major categories of instructional supports supporting empirical re-conceptualization that emerged from our analysis of the data sets are (1) *experiencing need for verification*, (2) *fostering contextual interpretation*, (3) *fostering reflection and justification*, and (4) *fostering pattern exploration* (Table 3). The first category addresses a need to verify, which occurred in two forms: A need to verify a hypothesis about the context, or a need to verify the developed generalization or statement. Our participants experienced these needs arising from their own mathematical activity, but it is possible that these needs could also be deliberately fostered through task design, instructional moves, or classroom norms.

The remaining three categories and their subcategories do not specify an actor; consistent with the symbolic interactionist perspective, we found that the teacher-researcher, the tasks, the other students, and the students' interaction with the tasks or the task-related tools mutually interacted to foster the processes. Contextual interpretation was fostered either by encouraging students to consider the context, or by sharing a contextual interpretation. Fostering reflection and justification occurred either through pushing for an explanation of a student's thinking, or through specifically asking for a justification or a proof. Lastly, pattern exploration was fostered either through directing students' attention to patterns, or through task organization, talk, or instructional moves that encouraged the development or extension of a pattern.

6.2. Overview of processes in the transition to deductive reasoning

When engaging in empirical re-conceptualization, our participants often leveraged that activity to develop deductive arguments and justifications. Our second research question addresses this transition, namely, how did empirical re-conceptualization facilitate students' transition from empirical to deductive reasoning? In addressing this question, we identified three major types of processes facilitated by empirical re-conceptualization that supported this transition: (a) verification, (b) justification, and (c) creation / interpretation (Table 4).

Verification addresses the role that empirical re-conceptualization plays in serving as a source of verification for the correctness of a pattern, generalization, formula, or idea. In contrast to verification, *justification* goes a step farther: By engaging in the process of empirical re-conceptualization, students can develop an argument to justify the generalization or statement they developed. For more advanced students, this argument can take the role of a formal proof. C*reation / interpretation* occurred in four distinct ways. Firstly, students can re-interpret a developed generalization or statement from a contextual perspective, enabling them to understand a new aspect of their developed generalization. The second form of creation / interpretation is the construction of a new representation, which may then support a new form of insight into the extant generalization. The third form is the creation or extension of a new pattern or generalization, and the last form is the establishment of a new piece of knowledge or understanding. Through re-conceptualizing, students may develop new connections, understanding, relations, or develop new knowledge about the mathematical ideas that led to the initial generalization; these can then foster a transition to deductive reasoning.

Across both data sets, we identified a total of 64 instances of empirical re-conceptualization, 39 in the undergraduate data set and 25 in the secondary data set. Of those instances, 50 (78 %) had at least one of the above associated processes. Twelve instances had two processes, and one had three. Fourteen percent were verification, 42 % were justification, and 44 % were creation / interpretation; for this latter category, 54 % of those cases were re-interpretation, 14 % were representation, 18 % were pattern or generalization, and 14 % were knowledge or understanding. The percentage of instances of these processes across the data sets were similar, with one notable exception: We found a greater percentage of processes for the empirical re-conceptualization in the secondary data set compared to the undergraduate data set (22 out of 25 secondary instances had at least one process, compared to 28 out of 39 undergraduate instances).

6.3. Secondary vignette: growing volume

In the following vignette the secondary students, Homer and Barney, engaged in empirical re-conceptualization when generalizing about growing cubes and then generalizing about growing n-dimensional figures. The students' processes of empirical re-conceptualization occurred in relation to multiple, sometimes interacting, instructional supports, and they engaged in a number of different processes as they leveraged their empirical re-conceptualization and transitioned toward deductive reasoning. In presenting this vignette, we highlight two major episodes - first generalizing about growing cubes, and then generalizing about growing n-

Table 3
The four categories of instructional supports for empirical re-conceptualization.

Experiencing Need for Verification	Context Verification: Students may use the generalization or statement to verify a hypothesis about the context. Generalization Verification: Students may use the context to verify the generalization or statement.
Fostering Contextual Interpretation	Encouraging: Encouraging students to engage in a contextual interpretation of a generalization or statement. Sharing: Sharing a contextual interpretation of a generalization or statement.
Fostering Reflection and Justification	Prompting Explanation: Asking for an explanation of one's thinking, actions, or mathematical activity. Prompting Justification: Asking for a justification or proof of one's generalization or statement.
Fostering Pattern Exploration	Encouraging: Encouraging students to generate or extend a pattern. Directing: Directing students' attention to a pattern or patterns.

Table 4
The three major categories of processes facilitated by empirical re-conceptualization.

Verification (14 %)	Students are able to verify the correctness of a pattern, generalization, formula, or idea.
Justification (42 %)	Students develop an argument justifying the generalization or statement they developed.
Creation / Interpretation (44 %)	Re-interpretation: Students re-interpret a developed generalization or statement from a contextual perspective.
	Representation: Students create a new representation.
	Pattern or Generalization: Students create or extend a new pattern or generalization.
	Knowledge or Understanding: Students establish a new piece of knowledge or understanding.

dimensional figures. In each case we first offer the data excerpt, and then we discuss (a) the shifts involved in the students' empirical re-conceptualizations, (b) the instructional supports that fostered those empirical re-conceptualizations, and then (c) the processes by which they leveraged those empirical re-conceptualizations to transition beyond empirical reasoning.

6.3.1. Generalizing about growing cubes

When considering a cube that grew 1 cm in height, width, and length, the students initially worked with blocks. As an example, Fig. 3 depicts a yellow cube. The teacher-researcher asked Homer and Barney to build a new cube that would be 1 cm larger in height, width, and length. The students added the three n^2 pieces (depicted as the blue, pink, and green squares in Fig. 3), then the three n pieces (as depicted by the three orange rectangles), and finally one 1 piece (as depicted by the small purple cube). Originally, Homer and Barney thought about these pieces in terms of the specific dimensions of the original cube. The teacher-researcher pushed them to instead express the additional volume algebraically, if "it was an $n \times n \times n$ [cube]". Barney immediately responded, "n squared times three. Plus $3n$, no, that's the wrong way of writing it. Three n squared plus $3n$ plus one equals volume gained", writing the expression "$3n^2 + 3n + 1$". Barney's expression was a generalization of the students' process of decomposing the added volume into the constituent parts. Both Homer and Barney could identify each of the parts of the expression $3n^2 + 3n + 1$ in terms of the additional pieces on the new cube.

The students then began to investigate the added volume of a cube that would grow x cm in each direction. In attempting to extend to this new case, they generalized from their prior result of $3n^2 + 3n + 1$. Homer wrote "$(3x)n^2 + (3x)n + x^2$", replacing the three in the first two terms of his original expression with a $3x$, and replacing the 1 in the last term, which he had conceived as 1^2, with an x^2. Note that this is an empirical generalization: Homer and Barney derived it from the form of the prior generalization, $3n^2 + 3n + 1$, rather than from the structure of that generalization as representing, for instance, three n^2 pieces, each of which has the dimensions $n \times 1 \times 1$ because the prior cube grew by 1 cm. As such, the students' mental activity was figurative, as they were attempting to create a new expression that was perceptually similar to the prior expression. Further, replacing the constants with xs was based on the result of their prior activity, rather than attending to the process by which they grew the cube's volume. Unsure about the correctness of their generalization, Barney said, "Let me model on the cube." Working with the cube pieces depicted in Fig. 3, Barney decided that the first term, $3xn^2$, was correct because it represented three additional rectangular prisms, each with a volume of xn^2. Both students then realized errors in the next two terms. Barney explained that the second term should actually be $3x^2n$, "because you're adding three of x by x by n." As Barney continued to build the additional volume on the cube, both students also realized that the final term should be x^3, rather than x^2, because the final purple cube should have the dimensions x by x by x.

In another instance of empirical re-conceptualization, when considering the corrected expression $3xn^2 + 3x^2n + x^3$, Barney said, "It's interesting how there's one x here (in the first term), two x's here (in the second term), three x's here (in the third term)." The degree to which Barney saw this statement as a generalization is unclear, but the statement was empirically-based, as Barney was attending to the form of the expression. He then re-conceptualized this statement by justifying it geometrically:

> Barney: But I think that's because, here (pointing to an imaginary x by n by n added volume), you're only doing x once. But once you do these (points to the imaginary x by x by n piece), it's this amount x (gestures along the width) and this amount x (gestures along the height). And then once you do all of that, it's this amount x (gestures to the remaining cube of added volume), you already have this x, this x (gestures along the three dimensions, height, width, and length).
> Homer: Oh, it's because in the first one it's n squared, and then one n –
> Barney: – And then no ns.
> Homer: So it's two ns, one n, and then no ns. And then the next one it would…so the first one is two ns and one x, the second one is one n and two xs, and the last one is no ns and three xs.

6.3.1.1. The shift of empirical re-conceptualization. In the first instance of empirical re-conceptualization, by deciding to model on the cube, Barney engaged in a shift from figurative to operative activity. Specifically, he took the expression they had produced, and re-interpreted it geometrically in terms of volumes with specific dimensions. In the process, Barney shifted to operative activity by

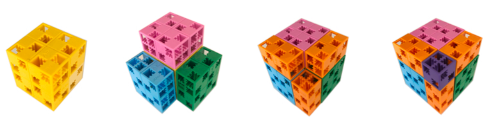

Fig. 3. Constituent parts of a cube that has grown by 1 cm in height, length, and width.

coordinating his mental actions of constructing component volumes and translating those quantities to algebraic representations. Barney's second appeal to the constituent parts of the cube was another empirical re-conceptualization. He again shifted from figurative to operative activity, coordinating and transforming mental operations on the cube to justify the pattern he saw. Although Barney was working with a specific cube, he was able to use it as a generic example with dimensions n and x.

6.3.1.2. Instructional supports. We found two supports that fostered Barney's first instance of empirical re-conceptualization. The first was the task itself, which *fostered pattern exploration* by encouraging the students to extend the action of growing a cube by 1 cm in every direction to growing by x cm in every direction. This led to the empirical generalization, which then caused Barney to experience a *need for verification*, a second support. He wanted to verify the accuracy of their produced generalization, and the physical cubes provided the tools Barney needed to re-conceptualize an algebraic expression geometrically.

The *need to verify* was again a support for Barney's second instance of empirical re-conceptualization, which then caused Homer to notice another pattern. Gesturing along the dimensions of the cube, Homer re-conceptualized Barney's pattern about *n*s into the context not only of volume, but also of dimensions. Engaging in operative activity, Homer could use the specific cube to reason in terms of a general $n \times n \times n$ structure, but more importantly, he began to think about any general cube that could be considered in terms of its three dimensions. Barney's activity supported Homer's empirical re-conceptualization by *fostering a contextual interpretation*, in this case, an interpretation of dimensions and not just volume.

6.3.1.3. Processes for transition to deductive reasoning. The students' first empirical re-conceptualization supported several processes in the transition to deductive reasoning. First, it enabled a *re-interpretation* of the meaning of the coefficients of n as representing volumes. It also served as a *source of verification*; Barney and Homer realized that their first generalization was incorrect, and they were able to correct it. Finally, it led to a new *creation / interpretation*, specifically an *identification of a new pattern* when Barney realized that the new expression had one, two, and three *x*s in each of its three terms, respectively.

In the second instance of empirical re-conceptualization, Homer experienced a new creation / interpretation, specifically a *new understanding* that each added piece of volume could be thought of in terms of the dimensions height, width, and length. Evidence that the students did shift to thinking about dimensions is present in their engagement with the subsequent task, in which they explored the volume of a new cube and specifically referred to the cube's height, width, and length in determining its growth in volume. Thus, Homer's re-conceptualization productively influenced the manner in which the students engaged with further tasks.

6.3.2. Generalizing about growing n-dimensional figures

The next day, the teacher-researcher introduced the students to the idea of a hypercube, and asked them to determine the amount of added hypervolume when adding 1 cm to every dimension of an $n \times n \times n \times n$ hypercube. Reasoning from their prior activity, Homer and Barney knew that their answer should have the structure of "$a_1n^3 + a_2n^2 + a_3n + 1$", but they had to determine the coefficients. They also realized that when adding 1 cm to the $n \times n \times n$ slice of a hypercube, they would have four outcomes: a) $1 \times n \times n \times n$; b) $n \times 1 \times n \times n$; c) $n \times n \times 1 \times n$; and d) $n \times n \times n \times 1$. Using this reasoning, both students could easily justify why the n term and the n^3 term should both have a coefficient of four. Building on the prior day's expression, $3n^2 + 3n + 1$, the students generalized the new expression to be "$4n^3 + 4n^2 + 4n + 1^4$". This expression is incorrect – the coefficient of the n^2 term should be six – but it is an unsurprising generalization. It was the result of the students generalizing both from their operative activity of determining the coefficient of n^3 to be four, as well as from a figurative extension of the numeric structure of the three-dimensional case to the four-dimensional case (reasoning that each coefficient would be four, because each coefficient in the prior case was three). This aspect of their generalizing was empirical, as it was an attempt to extend the form of the prior expression.

The teacher-researcher then asked the students to list out all of the combinatorial options for the n^2 coefficient. Both students correctly listed the six options for arranging two *n*s and two 1s into four slots, and the teacher-researcher pushed them to explain why the number of combinations must be six. Homer explained that there must be three possible combinations with an n in the first slot, another two combinations with n in the second slot, and a final remaining combination with an n in the third slot. Barney followed up on this idea, adjusting it to remark, "So basically, if you have n in the first spot, there's only three places the n can go. So, it's n and then there's three that start with n and then there's three that start with one."

Once the students correctly determined expressions for the 2nd, 3rd, and 4th dimensions, the teacher-research wrote those expressions on a piece of paper (Fig. 4), which prompted Homer to recognize that each coefficient could be determined by adding the sum of the coefficients of the consecutive terms in the prior line. Referring to the coefficients, he explained, "Two plus one is three, and two plus one is three, three plus three is six, three plus one is four, one plus three is four [writes the red numbers in Fig. 4]." Barney responded, "Wow. It's just that one triangle, Pascal's triangle, right?"

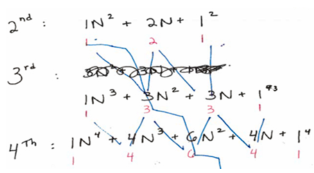

Fig. 4. Expressions for added volume in the 2nd, 3rd, and 4th dimensions.

This is again an empirical generalization, based on a recognition of the form of the pattern of the coefficients. Pascal's triangle then became a mechanism for determining the additional volume of a 5th-dimensional solid. The students used the pattern to write "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$", another empirical generalization. However, both students wanted to verify their expression, and they decided to first check the $10n^3$ term by listing out the arrangements of three *n*s and two 1s. They successfully listed the 10 cases, and as the students reflected on their activity, Homer explained that he did not want to do the tedious listing that would be required to double check every coefficient. Barney then realized that since that they had verified the $10n^3$ case, they did not need to check the $10n^2$ case: "We can basically just take this and switch all the *n*s to 1s and 1s to *n*s." Explaining further, Barney said, "It will be the same combinations here, just substituting I for *n* and *n* for I" (he inadvertently started calling the 1s "I"s).

6.3.2.1. The shifts of empirical re-conceptualization. The students experienced two major shifts in the above episode. First, when they initially determined six options for arranging two *n*s and two 1 s into four slots, they did so empirically through listing. However, they then both shifted to begin to think structurally about six general slots, and then imagined placing the available *n*s in an organized manner. By shifting from listing to mentally organizing the outcomes according to *n*s and slots, both students shifted from RPG to PPG, as well as from figurative to operative activity. This was their first instance of empirical re-conceptualization.

In the second instance, when the students later realized that they did not need to check the $10n^2$ case because they had already verified the $10n^3$ case, they again shifted from the empirical activity of listing combinations and counting the outcomes to imagining a structural relationship between the number of *n*s and the number of 1s in the list of outcomes - shifting from figurative to operative activity. Barney was able to reflect on his coordination of operations in listing the 10 outcomes and realize that there was nothing special about the characters *n* and 1, and they could simply be reversed in the case of determining the combinations of two *n*s and three 1s. Thus, Barney was able to see the list of outcomes for the $10n^3$ case as a generic example.

6.3.2.2. Instructional supports. Three instructional supports fostered the instances of empirical re-conceptualization. The first was the teacher-researcher's request to list the options for n^2 after they had produced the expression "$4n^3 + 4n^2 + 4n + 1^4$". In doing so, she *encouraged a contextual interpretation*, but this time, the context was combinatorial rather than the context of volume. Secondly, the teacher-researcher *prompted a justification* by asking them to explain why the number of combinations must be six. Additionally, when Barney and Homer imagined a structural relationship between the number of *n*s and the number of 1s in the list of outcomes for $10n^2$, their ability to shift to operative activity was fostered by their *need for verification*. Homer, in particular, wanted to verify that all of the coefficients for the expression "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$" were correct, but he did not want to have to list for each term. Thus, the combination of the students' need to verify with the desire to avoid tedious listing encouraged them to consider instead the relationship between the number of *n*s and the number of 1s in the list of outcomes.

6.3.2.3. Processes for transition to deductive reasoning. Barney and Homer's first instance of empirical re-conceptualization engendered a *justification*: They could both justify why the coefficient of the term $6n^2$ had to be six, because there were three possible combinations with an *n* in the first slot, another two combinations with *n* in the second slot, and a final remaining combination with an *n* in the third slot. They also produced a justification as part of their second empirical re-conceptualization, explaining why the coefficient for the $10n^3$ case had to be the same as for the $10n^2$ case. This activity also helped Barney develop a new *creation / interpretation* by establishing a *new understanding* of symmetry, which then caused Homer to extend that finding to new cases: "Oh, and you know what? You can do the same for these (pointing to the $5n^4$ and the $5n^1$ terms)…you can just replace these 1s for *n*s." Thus, Homer took up Barney's practice of attending to symmetry, and was able to form a *new generalization*.

6.4. Undergraduate vignette: counting 5-character AB passwords

6.4.1. Making sense of symmetry combinatorially

In the following episode, we begin with the undergraduate students' work when they created tables for 5-character AB passwords and highlight the students' discussion of the symmetric nature of the second column of the 5-character AB table in Table 2. The symmetry is important for two reasons. Firstly, it underscores a number of important relationships that can be understood and justified combinatorially, one of which is that the number of As and Bs in the passwords are related, and a 5-character password with three As must have two Bs (and vice versa). It also highlights the combinatorial identity $\binom{n}{k} = \binom{n}{n-k}$.

The students had previously established that there were 2^5 total 5-character AB passwords by drawing on the multiplication principle and arguing that there were two choices, A or B, for each of the five positions. They then began to create tables for 5-character AB passwords, which yielded the column of 1, 5, 10, 10, 5, 1 (Fig. 5). As seen in the excerpt below, both Aaron and Carson indicated that they had used numerical patterns to fill out the middle entries of the table. Namely, they knew the table entries were 1, 5, _, _, 5,

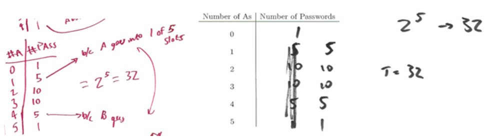

Fig. 5. Aaron's (left) and Carson's (right) 5-character tables.

1, they knew the total had to be 32, and they hypothesized that the two middle entries would be the same number, 10, because tables from prior tasks had followed that pattern. Thus, as Aaron explained, “I just knew that it had to add up to 32, so, you know, 20 added up to what was already there.” Carson agreed with Aaron:

> Carson: I did the exact same thing. I started with the symmetry again where you know you have one option for each of the monogamous sets, and then five options for sets where one of the things is different than all of the others, and then two empty slots. And because it’s symmetric, then we know that those two slots need to be the same number, and knew that the total had to go up to 32 just based off of, you know, if you [have] five slots and two options for each slot, you’re going to get 2^5.
> TR2: Okay.
> Carson: So, there was a remainder of 20, so half of 20 is 10, so 10 for each of those slots.

Aaron and Carson’s initial determination that the middle values of the table would be 10 was the result of an empirical generalization. The students had observed a pattern of symmetry across prior tables, but their observation was figurative and an RPG, based on the outcome of each table, and they did not articulate why the tables had to be symmetric. The teacher-researcher the asked the students, “Can you just talk about how you remember becoming aware of the symmetry and then why it might make sense?” In response, two students, Anne-Marie and Carson, introduced the idea of considering what was happening in the list of passwords that would be generated. In particular, they talked about noticing a regularity in their process of writing outcomes - they began by writing outcomes with fewer As, and “moving the A” through the Bs. Then, at a certain point in the listing process, they switched to “moving the B instead of the A”; which helped them view the symmetry in a different light:

> Anne-Marie: But, I just see it as, like, it’s where the As and Bs flip. So, like, when I was writing out the list, I was moving the A through the matrix of Bs. And, like, so that’s when…the symmetry…I think it was kind of when you were moving the B instead of the A.
> TR2: Okay, nice.
> Carson: Yeah, so just, it’s…asking to move one A around in a group of Bs is the same thing asking to move one B around a group of As. It’s going to return the same number of results.
> TR2: Okay.
> Carson: So, it’s like that flip she was talking about, you know, where you were going down this list of how many As you have. Eventually it’d be easier to ask how many Bs you have, because it’ll be the same list, just backwards.

6.4.1.1. The shifts of empirical re-conceptualization. The students’ discussion of passwords and moving As and Bs to generate certain arrangements suggests that Anne-Marie and Carson re-conceptualized the prior empirical generalization, articulating a combinatorial interpretation of the symmetric pattern. The students initially generalized that the table entries would be symmetric by observing that regularity in the outcome of prior tables, a figuratively-based RPG. Further, Anne-Marie indicated that she had not previously developed a combinatorial justification for the symmetry in terms of passwords, just as Carson’s initial generalization had also been based on a numerical pattern. Carson then shifted from conceiving of the symmetry as a regularity based on numerical results, to now conceiving of it as being rooted in symmetric processes (moving an A through Bs or moving a B through As). The students justified this symmetric pattern in terms of a regularity in process, a shift to PPG. Further, Carson built on Anne-Marie’s idea by explaining that moving one A around versus moving one B around returns the same number of results, indicating a shift to operative activity. Carson could see the two processes as identical: the specific number of As and Bs did not matter, and instead he could imagine a generic process.

6.4.1.2. Instructional supports. The teacher-researcher explicitly asked the students to consider why the symmetry they had observed might make sense, thereby *fostering reflection and justification (prompting explanation)*. This episode also highlights how interaction can influence empirical re-conceptualization. Carson’s shift to operative thinking built upon Anne-Marie’s initial re-interpretation, showing that the students’ discussions supported empirical re-conceptualization. Anne-Marie *fostered a contextual interpretation* (sharing), as she shared the particular contextual interpretation of a generalization, connecting the symmetry to the combinatorial context of listing passwords.

6.4.1.3. Processes for transition to deductive reasoning. In this case, the students’ process of empirical re-conceptualization facilitated *justification*. In particular, the students justified the symmetry in a manner rooted in the context of passwords, which in turn supported a transition from empirical reasoning to deeper conceptual understanding.

6.4.2. Connecting to the interpretation of counting arrangements with restricted repetition

Still wanting to reinforce a combinatorial interpretation of the symmetry in the 5-character AB table, the teacher-researcher asked the rest of the students, “Someone else just tell me a little bit more about why that’s the case. If you have, you know, two As and three Bs, three Bs two As – that’s the same thing. Two As and three Bs, three As and two Bs, like, yeah, why is it the same number of things?” In response, Carson introduced another way to view the problem, suggesting that one can think of counting AB passwords as counting arrangements with repetition. Earlier in the teaching experiment the students had solved “arrangement with restricted repetition” problems, which involve arranging some number of characters, some of which are identical (or repeated) – one such problem is, “How many arrangements are there of the word BANANA?” However, although they had solved these problems previously, the students had not connected the activity of counting AB passwords to those prior problems. Instead, they thought about counting AB passwords in terms of selecting places to place As (so, as a selection or combination problem). These are both valid approaches, and they involve the same formulas, but they result in different interpretations of the formulas. Thus, the approach Carson describes below is novel, distinct from prior work on the Passwords task up to this point. To make his point, Carson started with the third row and re-interpreted the original empirical solution $\binom{5}{3}$ in terms of arrangements with repetition, specifically arranging three identical As and two identical Bs:

> Carson: Another way you could look at this [the fourth row in the 5-character AB table] is if you have the A, A, A, B, B, how many different ways can you arrange the letters in that word? So that's going to 5!, which is the total number of letters over the repeat letters, so 3!, 2! (writes "AAABB" and "$\frac{5!}{3!2!}$"). Right?
> TR2: Mm hmm.
> Carson: And that's the same equation there. That will hold true whether the repeat letters are three As and two Bs, or three Bs and two A, or three Cs and two Ds. Right?
> TR2: Okay.
> Carson: And I mean, that's a mathematical relationship we've talked about before, because you're just pulling out the redundant, um, redundant arrangements given repeated letters.
> TR2: Okay, good.
> Carson: So, I mean, yeah, it doesn't really matter what the letters are as long as they're as those ratios.

Carson then made another observation, stating, "And that's the same equation there. That will hold true whether the repeat letters are three As and two Bs, or three Bs and two As, or three Cs and two Ds. Right?" He quickly realized, "I mean, yeah, it doesn't really matter what the letters are, as long as they're those ratios." Carson realized an important phenomenon: While they were counting AB passwords with a certain number of As, they were actually doing something that could be considered much more broadly as involving any two kinds of characters.

6.4.2.1. The shifts of empirical re-conceptualization. Carson created a new representation for this context, $\frac{5!}{3!2!}$, that represented his thinking about the AB-passwords as involving a new type of counting process - specifically arranging all letters and dividing by duplicates. We interpret this representation to be a consequence of Carson shifting from viewing the table entry as choosing places for the As to go (a selection problem), to viewing it as arranging the five letters A, A, A, B, B. Thus, Carson shifted from an initial focus on regularity based on results (when he justified the symmetry based on the fact that the numerical pattern in the tables was symmetric) to regularity based on a process - here an entirely new process that they had not yet discussed. This also suggests a shift from figurative to operative thought. Carson went from focusing on the symmetry in the tables as being based on an observable figural pattern to justifying that regularity based on structure and sophisticated combinatorial operations.

6.4.2.2. Instructional supports. The teacher-researcher employed the instructional supports of *fostering contextual interpretation (encouraging)* and *fostering reflection and justification (prompting justification)*. Specifically, she attempted to reinforce, in the particular context of AB passwords, why this particular relationship made sense. It was contextual because it was specific to a combinatorial situation, where the teacher-researcher drew particular attention to the combinatorial reasons why the relationship might be interpreted and make sense. This instance of empirical re-conceptualization was thus supported by the teacher-researcher's repeated prompts to explain, justify, and contextualize the symmetric pattern in the table.

6.4.2.3. Processes for transition to deductive reasoning. Through empirical re-conceptualization, Carson developed a new *creation / interpretation*, creating a new *representation*, $\frac{5!}{3!2!}$. He also *reinterpreted* his generalization from a structural perspective, recognizing the more general structure of repeated characters, even explicitly noting that it did not matter what the letters were as long as some were repeated. Carson understood that it was not the As and Bs themselves that were important, but rather that they were dealing with two types of objects. In this manner, empirical re-conceptualization contributed to a more coherent picture of all of the students' combinatorial work to that point. Indeed, Carson was able to reinforce a connection with prior work the students had done, and thus this exchange constitutes an important *creation / interpretation,* that of *establishing a new understanding.*

7. Discussion and conclusion

A growing body of research has moved beyond discouraging empirical reasoning to recognizing its value, suggesting that students can learn to develop structural understanding from patterns (e.g., Küchemann, 2010), and can leverage pattern-based thinking to support mathematical deduction and proof (e.g., Tall, 2008). What remains to be elaborated, however, is how students actually make that transition from empirical reasoning to structural and deductive reasoning. This paper has introduced empirical re-conceptualization as one promising mechanism for this transition.

7.1. Summary of findings

We have characterized the construct of empirical re-conceptualization by elaborating the specific shifts that mark a progression from empirical generalization to structural generalization. Empirical re-conceptualization occurs when students shift from RPG to

PPG, when they shift from figurative to operative mental activity, and/or when they shift from considering specific numbers or patterns to engaging in operations on a generic mental object. Through empirical re-conceptualization, we found that students can leverage their empirical investigations and generalizations into more powerful forms of reasoning, supporting the processes of verification, justification, and insight generation. Similar to the work exploring students' use of generic examples (Harel & Tall, 1991; Leron & Zaslavsky, 2013; Rowland, 2002), we found that with empirical re-conceptualization, students are able to think with particular patterns or cases, but in a structural manner that affords explanatory power (Bills & Rowland, 1999).

Other researchers have similarly distinguished empirical from structural (or theoretical) generalizations (Bills & Rowland, 1999; Carraher et al., 2008; Doerfler, 1991). What our study offers, however, is not only evidence that these distinct forms of generalizing exist, but an elaboration of how students can shift from one form to the other. In identifying the types of instructional supports that can foster this shift, our findings lend credence to the assertion that students, like mathematicians, can engage in a dynamic interplay between empirical patterning and deductive argumentation (de Villiers, 2010; Schoenfeld, 1986), leveraging their empirical investigations into more sophisticated knowledge structures (Tall et al., 2011). By taking the symbolic interactionist perspective, we could see how the established norms, students' participation, teacher moves, and task structures mutually interacted to foster empirical re-conceptualization. Our findings suggest that one element alone may not be sufficient to appropriately foster empirical re-conceptualization, but rather, teachers should enact carefully-designed tasks within a broader context of establishing and encouraging a norm of explaining and justifying one's reasoning.

We found that empirical re-conceptualization served as a source of verification, as seen when Barney and Homer used combinatorial reasoning to verify the accuracy of the expression for added "volume" of a 5th-dimensional cube. It also served as a source of justification, for instance when Anne-Marie and Carson justified why the entries for exactly two As and the entries for three As must both be 10. Finally, we found that empirical re-conceptualization enabled mathematical understanding. Both groups of students came to understand number patterns from a contextual perspective, such as when Homer made geometric sense of the constant second differences in the table of triangle areas. The students also identified new patterns and generalizations and created new representations, such as when Carson introduced and leveraged the representation $\frac{5!}{3!2!}$ as a link between AB-passwords and the process of arranging all letters and dividing by duplicates. Overall, our findings help to frame empirical re-conceptualization as one potential avenue by which students can transition from empirical to deductive reasoning and argumentation.

7.2. The role of context and domain in empirical re-conceptualization

Empirical re-conceptualization itself is a process that is more likely to occur in domains that afford pattern investigation. Although one can explore empirical patterns across many domains, those that lend themselves naturally to patterning activities, or to collecting multiple examples to explore a conjecture or a phenomenon, will offer more opportunities for re-conceptualizing those empirical findings. Furthermore, contexts that move beyond visualization, such as N-dimensional hypercubes, may play a special role in encouraging generalizations that go beyond empirical validation. Nevertheless, as research on mathematicians' problem-solving processes has revealed (e.g., de Villiers, 2010; Lockwood et al., 2016; Lynch & Lockwood, 2019), leveraging empirical patterns to support proof development can occur even with problems that are not explicitly about patterning. Similarly, the instructional supports we identified could occur in a variety of content domains. One common element across both teaching experiments was an emphasis on situating students' exploration within meaningful contexts, whether those were geometric or combinatorial. This set an important foundation for fostering empirical re-conceptualization, because it provided students with the means to make structural sense of empirical findings.

In both the secondary and undergraduate teaching experiments, multiple elements of the instructional setting, including the tasks, teacher moves, student talk and activity, and the processes of justification, verification, and creation/interpretation, interacted to co-construct opportunities for empirical re-conceptualization. When reasoning about the growing cubes, the secondary students engaged with tasks that fostered pattern exploration, encouraging them to make extensions to more general cases. Without the established norm that it was desirable to justify empirical patterns contextually, however, the cubes may have been of little importance, and the tasks encouraging pattern development may not have fostered empirical re-conceptualization. We also saw instances of both the teacher-researcher and the students fostering a contextual interpretation, either by sharing an interpretation, such as when Barney interpreted coefficients in terms of volume, or by encouraging an interpretation, such as when the teacher-researcher asked the students to check a coefficient by thinking about it as a number of combinations. The teacher-researcher and the students also both fostered reflection and justification, either by asking for explanations, or by prompting justifications, as when the teacher-researcher asked the students to explain why the number of combinations must be six. We saw that these supports interacted with and reinforced the students' processes of justifying, verifying, re-interpreting, and developing new patterns, generalizations, and knowledge or understanding.

Similarly, in the undergraduate teaching experiment, the tasks themselves were explicitly designed to allow for students to identify and articulate generalizations, even empirical ones, through the creation of tables and the intentional build up to larger cases. But, both prompts by the teacher-researcher and interactions among the students encouraged movement toward more structural reasoning and practices such as verification and interpretation. In particular, the interaction between multiple students encouraged them to discuss their work together, and they could build upon each other's ideas and add to mathematical contributions. The teacher-researcher and the students regularly fostered reflection and justification by asking for explanations, or by prompting justifications, as when the teacher-researcher asked the students to explain the symmetry they observed in the tables. Instructional supports thus interacted with and reinforced the students' practices of justifying and re-interpreting, and developing knowledge or understanding. Across both data sets, it was by considering the teaching-experiment setting to be a system made up of mutually interacting agents that we could better characterize how empirical re-conceptualization can occur in instruction.

7.3. Instructional supports and fostering intellectual need

Some of the instructional supports are related to Harel's (2013) notion of intellectual need, in which one experiences perturbation through encountering a situation or problem unsolvable by their current knowledge. Harel identified five categories of intellectual need, which are the need for certainty, need for causality, need for computation, need for communication, and need for structure. Our instructional support *experiencing need for verification* is compatible with Harel's need for certainty, which is the desire to know whether a conjecture is true. Harel's need for causality can also occur within the instructional supports. This need addresses one's desire to determine and explain the cause of a phenomenon, which can drive one's need for verification or even the desire to share a contextual interpretation of a generalization, as contextual interpretations often offer a causal explanation. Finally, the need for structure, which Harel (2013) described as "the need to reorganize the knowledge one has learned into a logical structure" (p. 140) can undergird students' pattern exploration and justification, and indeed is broadly compatible with the very notion of empirical re-conceptualization. We found that task design that can facilitate intellectual need for students, particularly a need for verification, offers a useful route towards supporting empirical re-conceptualization.

It is worth noting, however, that students' need for verification may not always be an intellectual need; in our data, particularly early in the teaching experiments, it was a social need engendered by the teacher-researchers' instructional moves to encourage the students to verify and justify their results. As the students progressed in their respective teaching experiments, they then began to take ownership of the norms of verification and justification, even reproducing prompts and encouragements within their dialogue with one another. In this manner, the symbolic interactionist perspective helped clarify how the interacting agents of the teacher-researcher's instructional moves, the task design, and the students' engagement and uptake of the teaching-experiment norms worked together to mutually facilitate the need to verify. It is useful to create tasks that can necessitate knowledge creation for students, but it is also helpful to understand that these tasks always exist within a system of interaction. Our aim was to support the students' journey in becoming intellectually curious, to experience perturbation and intellectual need, and to engage in justification to understand why their new knowledge resolved problematic situations.

In conclusion, reasoning empirically was something all of our participants found natural and expressed a need to engage in. This mirrors the body of research showing that students are adept at identifying and developing patterns (e.g., Blanton & Kaput, 2002; Pytlak, 2015; Rivera & Becker, 2008). Our data suggest that when students are allowed to explore empirical findings within a context of tasks and instructional moves that explicitly encourage contextual interpretation and re-conceptualization, they are able to leverage their empirical generalizations to ultimately build more powerful mathematical insights. This paper offers one mechanism for characterizing how students can do this, and for articulating the supports that can foster this process. Empirical re-conceptualization is surely not the only mechanism for doing so, but it is an important mechanism that can be explicitly nurtured. Our findings show that it is possible to support students' meaning making and deductive argumentation without discouraging a reliance on empirical patterns, and without requiring students to prematurely shift to fully abstract representations. Instead, empirical re-conceptualization can be a way to foster students' emerging abilities to bridge from inductively-developed patterns and arguments to more deductive reasoning, justification, and proof.

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References

Amit, M., & Neria, D. (2008). Rising to the challenge": Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM Mathematics Education, 40*, 111–129.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pinun (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder & Stoughton.

Balacheff, N. (1987). Processes of proving and situations of validation. *Educational Studies in Mathematics, 18*(2), 147–176.

Barbosa, A., Palhares, P., & Vale, I. (2007). Patterns and generalization: The influence of visual strategies. *Proceedings of the fifth congress of the European society for research in mathematics education* (Vol. 6,, 844–851.

Becker, J. R., & Rivera, F. (2006). Sixth graders' figural and numerical strategies for generalizing patterns in algebra. November *Proceedings of The 28th Annual Meeting of The North American chapter of the international group for the psychology of mathematics education* (Vol. 2,, 95–101.

Bell, A. W. (1976). A study of pupils' proof-explanation in mathematical situations. *Educational Studies in Mathematics, 7*, 23–40.

Bieda, K. N., Ji, X., Drwencke, J., & Picard, A. (2013). Reasoning-and-proving opportunities in elementary mathematics textbooks. *International Journal of Education Research, 64*, 71–80.

Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. *Research in Mathematics Education, 1*(1), 103–116.

Blanton, M., & Kaput, J. (2002). Developing elementary teachers' algebra "eyes and ears": Understanding characteristics of professional development that promote generative and self-sustaining change in teacher practice. April *Paper presented at the Annual Meeting of the American Educational Research Association*.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method.* Englewood Cliffs, NJ: Prenticel Hall.

Čadež, T. H., & Kolar, V. M. (2015). Comparison of types of generalizations and problem-solving schemas used to solve a mathematical problem. *Educational Studies in Mathematics, 89*(2), 283–306.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*, 352–378.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education, 40*, 3–22.

Cooper, J., Walkington, C., Williams, C., Akinsiki, O., Kalish, C., Ellis, A. B., & Knuth, E. (2011). Adolescent reasoning in mathematics: Exploring middle school students' strategic approaches in empirical justifications. In L. Carlson, C. Hölscher, & T. Shipley (Eds.), *Proceedings of the 33rd annual conference of the cognitive science society* (pp. 2188–2193). Austin, TX: Cognitive Science Society.

de Villiers, M. (2010). Experimentation and proof in mathematics. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics* (pp. 205–221). Boston, MA: Springer.

Doerfler, W. (1991). Forms and means of generalization in mathematics. In A. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 63–85). Mahwah, NJ: Erlbaum.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordrecht, the Netherlands: Kluwer.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308–345.

Ellis, A. B. (2007). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences, 16*(2), 221–262.

Ellis, A. B., & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *Journal of Mathematical Behavior, 27*(4), 277–296.

Ellis, A. B., Ely, R., Tasova, H., & Singleton, B. (2020). Scaling continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics, 104*(1), 87–103.

Ellis, A. B., Tillema, E., Lockwood, E., & Moore, K. (2017). Generalization across domains: The relating-forming-extending framework. In E. Galindo, & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 677–684).

Ellis, A. B., Lockwood, E., Tillema, E., & Moore, K. (2021). Generalization across multiple mathematical domains: Relating, forming, and extending. *Cognition and Instruction*. https://doi.org/10.1080/07370008.2021.2000989

Guven, B., Cekmez, E., & Karatas, I. (2010). Using empirical evidence in the process of proving: The case of Dynamic Geometry. *Teaching Mathematics and Its Applications: International Journal of the IMA, 29*(4), 193–207.

Harel, G. (2013). Intellectual need. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 119–151). NY: Springer.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell, & R. Zaskis (Eds.), *Learning and teaching number theory* (pp. 185–212). Norwood, NJ: Ablex.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic. *For the Learning of Mathematics, 11*, 38–42.

Hargreaves, M., Shorrocks-Taylor, D., & Threlfall, J. (1998). Children's strategies with number patterns. *Educational Studies, 24*(3), 315–331.

Jurdak, M. E., & El Mouhayar, R. R. (2014). Trends in the development of student level of reasoning in pattern generalization tasks across grade level. *Educational Studies in Mathematics, 85*(1), 75–92.

Jurow, S. (2004). Generalizing in interaction: Middle school mathematics students making mathematical generalizations in a population-modeling project. *Mind, Culture, and Activity, 11*, 279–300.

Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah, NJ: Erlbaum.

Knuth, E. J., Choppin, J., & Bieda, K. (2009). Middle school students' production of mathematical justifications. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 153–170). New York, NY: Routledge.

Knuth, E., Kalish, C., Ellis, A. B., Williams, C., & Felton, M. (2012). Adolescent reasoning in mathematical and non-mathematical domains: Exploring the paradox. In V. Reyna, S. Chapman, M. Dougherty, & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making* (pp. 183–210). Washington, DC: American Psychological Association.

Küchemann, D. (2010). Using patterns generically to see structure. *Pedagogies: An International Journal, 3*(5), 233–250.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from* (Vol. 6). New York: Basic Books.

Lappan, G., Phillips, E. E., Fey, J. T., & Friel, S. (2014). *A guide to Connected Mathematics 3: Understanding, implementing and teaching.* Boston, MA: Pearson.

Leron, U., & Zaslavsky, O. (2013). Generic proving: Reflections on scope and method. *For the Learning of Mathematics, 33*(3), 24–30.

Lin, F. L., Yang, K. L., & Chen, C. Y. (2004). The features and relationships of reasoning, proving and understanding proof in number patterns. *International Journal of Science and Mathematics Education, 2*, 227–256.

Lockwood, E., & Reed, Z. (2016). Students' meanings of a (potentially) powerful tool for generalizing in combinatorics. In T. Fukawa-Connelly, K. Keene, & M. Zandieh (Eds.), *Proceedings for the nineteenth special interest group of the MAA on research on undergraduate mathematics education*.

Lockwood, E., Ellis, A. B., & Lynch, A. G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education, 2*(2), 165–196.

Lynch, A. G., & Lockwood, E. (2019). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *Journal of Mathematical Behavior, 53*, 323–338.

MacGregor, M., & Stacey, K. (1993). Seeing a pattern and writing a ruleI. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.). *Proceedings of the 17th international conference for the psychology of mathematics education* (Vol. 1,, 181–188.

Martinez, M. V., & Pedemonte, B. (2014). Relationship between inductive arithmetic argumentation and deductive algebraic proof. *Educational Studies in Mathematics, 86*(1), 125–149.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277–289.

Mhlolo, M. K. (2016). Students' dichotomous experiences of the illuminating and illusionary nature of pattern recognition in mathematics. *African Journal of Research in Mathematics, Science and Technology Education, 20*(1), 45–56.

Moss, J., Beatty, R., McNab, S. L., & Eisenband, J. (2006). The potential of geometric sequences to foster young students' ability to generalize in mathematics. *Paper presented at the annual meeting of the American Educational Research Association*.

Piaget, J. (2001). In R. Campbell (Ed.), *Studies in reflecting abstraction.* Sussex: Psychology Press.

Piaget, J. (1976). *The grasp of consciousness: Action and concept in the young child.* Cambridge, MA: Harvard University Press.

Pytlak, M. (2015). Learning geometry through paper-based experiences. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the ninth congress of the European society for research in mathematics education* (pp. 571–577).

Radford, L. (1996). Some reflections on teaching algebra through generalization. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 107–111). Dordrecht, The Netherlands: Kluwer.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5–29.

Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM Mathematics Education, 40*, 65–82.

Rowland, T. (2002). Generic proofs in number theory. In S. R. Campbell, & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 157–184). Westport: Ablex Publishing.

Rowland, T. (1998). *Conviction, explanation and generic examples*. ISSN ISSN-0771-100X PUB DATE 1998-00-00 NOTE 366 p.; For volumes 1-3, see SE 062 271-273; for the 1998 (p. 78).

Schoenfeld, A. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 225–264). Hillsdale, NJ: Lawrence Erlbaum.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15*, 3–31.

Steffe, L. P. (1991). The learning paradox: A plausible counterexample. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26–44). New York: Springer-Verlag.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly, & R. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Strauss, A., & Corbin, C. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.

Stylianides, G., & Stylianides, J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314–352.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education*. Reston: VA National Council of Teachers of Mathematics.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(20), 5–24.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2011). Cognitive development of proof. In G. Hanna, & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 13–49). Dordrecht: Springer.

Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189–243). Hillsdale, NJ: Erlbaum.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

Tuomi-Gröhn, T., & Engeström, Y. (2003). *Between school and work: New perspectives on transfer and boundary crossing*. Amsterdam, the Netherlands: Pergamon.

Vlahović-Štetić, V., Pavlin-Bernardić, N., & Rajter, M. (2010). Illusion of linearity in geometry: Effect in multiple-choice problems. *Mathematical Thinking and Learning, 12*, 54–67.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb, & H. Bauersfeld (Eds.), *Emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163–201). Hillsdale, NJ: Erlbaum.

Waring, S. (2000). *Can you prove it? Developing concepts of proof in primary and secondary schools*. Leicester, UK: The Mathematical Association.

Yopp, D. A., & Ely, R. (2016). When does an argument use a generic example? *Educational Studies in Mathematics, 91*(1), 37–53.

Zazkis, R., & Liljedahl, P. (2002). Arithmetic sequence as a bridge among conceptual fields. *Canadian Journal of Science, Mathematics and Technology Education, 2*(1), 91–118.