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STRUCTURAL SHAPE OPTIMIZATION—A SURVEY*

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This paper is a survey of structural shape optimization with an emphasis on techniques dealing with shape optimization of the boundaries of two- and three-dimensional bodies. Attention is focused on the special problems of structural shape optimization which are due to a finite element model which must change during the optimization process. These problems include the requirement for sophisticated automated mesh generation techniques and careful choice of design variables. They also include special problems in obtaining sufficiently accurate sensitivity derivatives.

1. Introduction

The first two decades of numerical structural optimization were almost exclusively focused on so-called sizing design variables. These are variables such as plate thicknesses and bar cross-sectional areas, which do not require a change in the finite element model of the structure as they are changed. Structural optimization with sizing design variables is simpler than with shape design variables, which control the geometry of the structure and typically require a finite element model which changes in the course of the optimization (see, for example, Fig. 1). However, for many problems shape design is more effective than sizing design. A typical example is that of a stress concentration at a hole boundary in a panel. Sizing design would increase or decrease the thickness of the panel near the hole, while shape design would change the shape of the hole boundary. The growing interest in shape design reflects a realization of the effectiveness of shape changes for improving structural performance. It also reflects a growing sophistication in structural analysis and optimization tools, which permit tackling the more difficult problems of shape optimization.

The present paper surveys recent work in structural shape optimization. The paper is focused on the difficulties which are encountered in shape optimization and which are not present, or are easier to solve, in sizing design. These difficulties are mostly encountered in two broad areas. First, because of the continuously changing finite element model, it is difficult to ensure that

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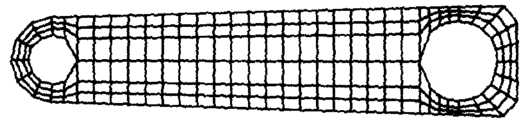
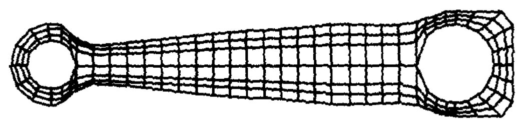


(a)



(b)

Fig. 1. Typical example of shape optimization (torque arm, [132]): (a) initial design; (b) final design.

the accuracy of the finite element analysis remains adequate throughout the design process. Second, it is more expensive to obtain good sensitivity derivatives with respect to shape design variables than with respect to sizing variables.

The present paper is limited mostly to design variables which control the boundary of two- and three-dimensional objects. It does not include a discussion of shape optimization of skeletal structures, because the problems in that area are quite different from those of boundary shape optimization. However, the interested reader is referred to a recent survey by Topping [120] on this topic.

Sections 2 and 3 of the paper deal with the problems created by the changing mesh. Section 2 deals with procedures for properly defining shape design variables so that the finite element mesh provides accurate stress calculation at the boundary. Section 3 deals with procedures for updating the finite element mesh as the boundary is changing. Section 4 deals with the problems of sensitivity analysis with respect to shape design variables, Section 5 lists the solution techniques used in solving the shape design problems, and Section 6 catalogs the shape design literature by area of application.

2. Selection of design variables

Shape optimization problems are often concerned with reducing stresses at a boundary by changing that boundary. However, if care is not exercised in the selection of the design variables, the accuracy of the finite element model can deteriorate as the optimization proceeds to the point where the results of the optimization are not useful. This often happens when the coordinates of the boundary nodes of the finite element model are used as design variables (a common practice in early work on shape optimization, e.g., [137]). The process is illustrated in Fig. 2, taken from [21]. Figure 2(a) shows the finite element model of the initial structure and Fig.

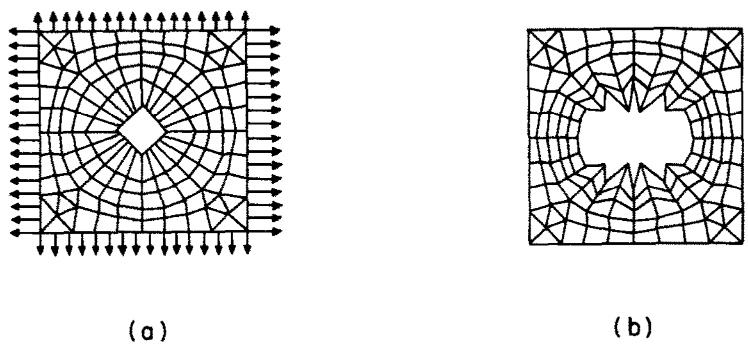


Fig. 2. Independent node movement technique for a hole in a plate [21]: (a) initial design; (b) final design.

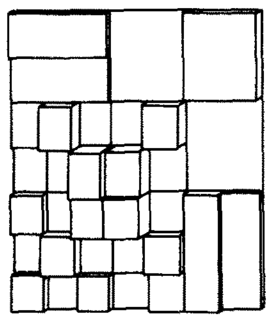


Fig. 3. Optimal thickness distribution in a quarter model of a plate [87].

2(b) a similar model for the final structure. Clearly, while the finite element model may provide adequate stresses for the initial design, it cannot be expected to be accurate for the final design. The problem also exists in some cases of sizing design as demonstrated by the final design of a plate obtained by Prasad and Haftka [87] shown in Fig. 3. The finite element mesh of 7×7 elements was quite adequate for the initial design which had uniform thickness, but is probably inadequate for the final design. The solution in both cases of sizing and shape design is to avoid a one-to-one correspondence between the finite element model and the design variables, and assure that there is an adequate number of elements for an accurate stress evaluation.

2.1. Polynomial representation of boundaries

Polynomials have been used to describe thickness distributions (e.g., [118]) with the polynomial coefficients being the design variables. This was, therefore, a natural choice for describing boundaries. Bhavikatti and Ramakrishnan [16] used a polynomial with coefficients taken as the design variables to characterize the shape. In the years following this publication, most researchers have used polynomials to describe the boundaries (for example, Pedersen and Laursen [85], Prasad and Emerson [88], etc.). A more general approach is to define the boundary as a linear combination of shape functions with the coefficients being the design variables. Thus, Kristensen and Madsen [68] defined the boundary using a linear combination of orthogonal functions added to the initial design by treating the coefficients of the functions as design parameters. Dems [38] also used a formulation describing the boundary using a set of prescribed shape functions and applied it to the simple case of piecewise linear boundaries.

2.2. *Spline representation of boundaries*

The use of high-order polynomials to describe the boundary can result in an oscillatory boundary shape. Splines eliminate this problem because they are composed of low-order polynomial pieces which are combined to maximize smoothness.

The cubic spline function, which has two continuous derivatives everywhere and possesses minimum mean curvature, is a natural choice for defining the moving boundary (e.g., [71, 126]). Yang and Choi [131] showed that the spline representation has better sensitivity accuracy than a piecewise linear representation of the boundary. Braibant et al. [21] used Bezier and B-spline blending functions to describe design element boundaries. The blending functions provide great flexibility for the geometrical description. With the B-spline formulation, boundary regularity requirements are automaticaly taken into account and also an analytical formulation of the sensitivity derivatives can be established.

2.3. *The design element concept*

One of the ways to achieve an adequate finite element model is to use the 'design element' concept introduced by Imam [61]. In this approach, the structure is divided into a few regions (see, for example, Fig. 4). These regions, or design elements, are described by a set of key nodes (or master nodes) that controls the geometry. Associated with the design element is a set of design variables that specify the location of the key nodes that are allowed to move in the design. Each design element consists of several finite elements. The design element boundary is described by using two-dimensional isoparametric finite element interpolation functions. Recently, Braibant et al. [21] and Braibant and Fleury [23, 24] used blending functions to represent the boundaries of the design element, while Wasserman [125] used interpolation functions based on position and tangent vectors.

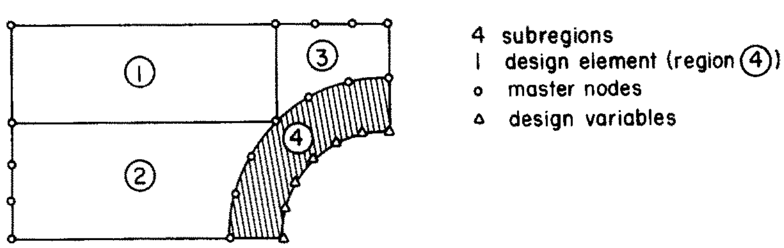


Fig. 4. The design element concept [21].

3. Finite element mesh generation

The problem of finite element mesh generation in shape optimization is due to the fact that in most cases the definition of a finite element mesh is a manual rather than an automated process. That is, the analyst uses judgement and experience-based intuition to select the mesh. Often the mesh is changed based on the result of a trial analysis, which reveals regions where the mesh needs to be refined. This manual approach is not adequate for shape optimization problems, because the analyst needs to define the mesh for a series of structures, without knowing their shape.

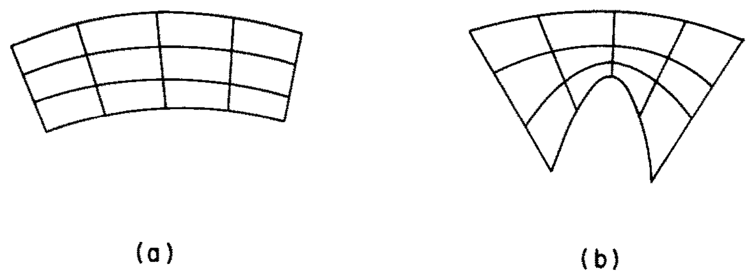


Fig. 5. Mesh deformation based on the modification rule: (a) initial design; (b) final design.

There are two basic solutions to the problem of adapting the mesh to the changing boundary. The first is to use simple modification rules for deforming the initial mesh. Often such simple modification rules run into trouble. This is illustrated by the simple mesh in Fig. 5. The 3×4 mesh for the initial design is adequate for stress calculation. The mesh is deformed so as to preserve a 3×4 uniform mesh. However, the final design has a fairly sharp corner and the mesh is not adequate for accurate stress calculation. Therefore, when a simple modification rule is used, it is often necessary to stop the optimization process and remesh manually. The second approach is based on the use of sophisticated automated mesh generation techniques, which generate a mesh and adaptively improve it based on the calculated response.

3.1. *Manual mesh refinement*

When some of the elements distort badly, the finite element model becomes incapable of evaluating high stresses in the valleys of the wavy boundary. Once the mesh is not capable of accurately modelling the problem, a refined mesh should be created. Oda and Yamazaki [82] regenerated the mesh after a number of optimization iterations. Yang et al. [132] manually remeshed the optimum shape design and then restarted the optimization procedure, with that design, for a final correction by the optimizer.

3.2. *Automated mesh generation*

An important tool of completely automated mesh generation is adaptive mesh refinement. With this concept, information from an analysis with a trial mesh is used to identify regions of the finite element mesh which need further refinement. This refinement can take either the form of adding additional elements in the area to be refined or of increasing the order of the finite element. The finite element mesh points are relocated whenever the boundary shape changes, and thus individual element distortion due to shape change during the optimization process can be kept to a minimum.

Luchi et al. [71] used automatic mesh generation at each step of optimization in designing a gas turbine disc. Queau and Trompette [89] used an automated mesh generator for several two-dimensional examples. The idea of regional (local) automatic mesh generation during the process of design has been presented by Botkin [19] for plate and shell components modelled by triangular flat plate elements. The techniques of 3-D shape optimization for solid components have been reported by Imam [61] and demonstrated on simple cantilever beams modelled by 3-D solid finite elements. Application of these techniques to the engine main bearing cap (Fig. 6) has been reported by Imam [62]. An integrated shape design program was developed by

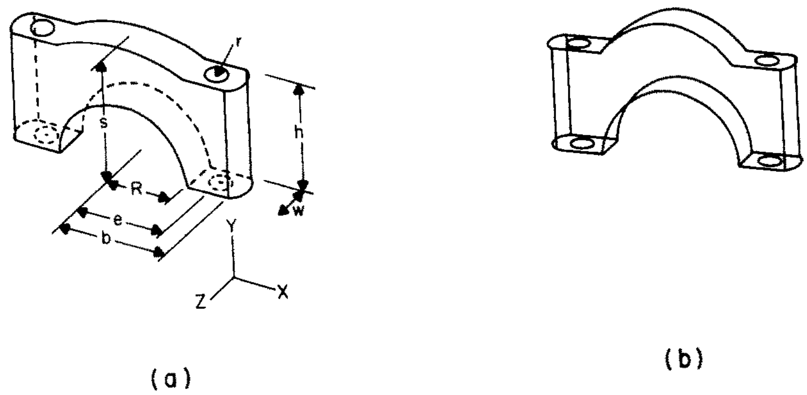


Fig. 6. Typical example of 3-D shape design bearing cap, [62]: (a) initial design; (b) final design.

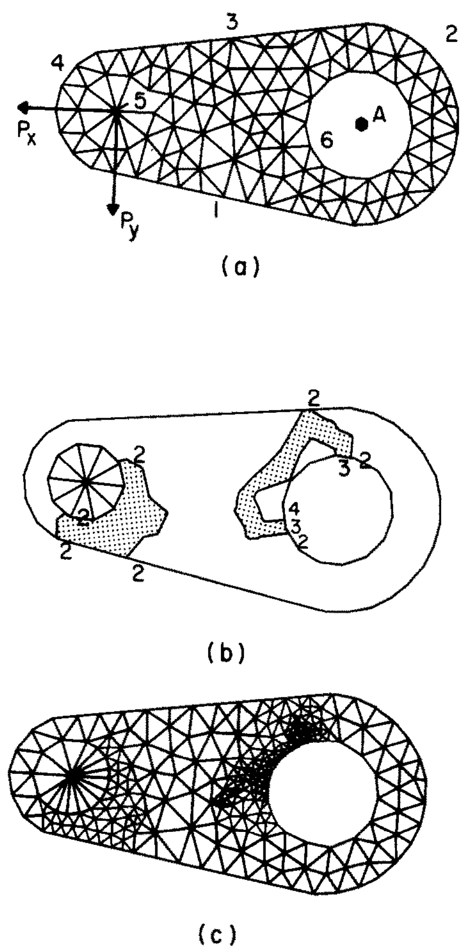


Fig. 7. Automated mesh refinement [15]: (a) initial uniform mesh; (b) strain energy density difference contours; (c) refined mesh.

Bennett and Botkin [14, 15] for 2-D problems. The program includes finite element analysis, automatic mesh generation, and structural optimization as an integrated package. The finite element model accuracy is improved by an adaptive mesh refinement scheme using the strain energy density gradients to identify regions which require mesh refinement (Fig. 7). This concept has been extended by Botkin and Bennett [20] to three-dimensional structures. The mesh generation is not an inexpensive part of optimization; it took approximately one third of the total CPU time in the applications reported by Botkin and Bennett [20]. However, subsequent improvements in the efficiency of the mesh generator reduced its cost to a few percent of the total.

3.3. *Optimum mesh refinement*

Because automated mesh generation must be an integral part of shape optimization, optimal mesh refinement is a closely related concept to optimum shape design. The two concepts of optimum mesh and optimum shape converge in the work of Kikuchi, Taylor, and their coworkers. Thus, earlier work on optimal grids (e.g., [42]) and optimal shape modification (e.g., [76]) led to the combination of the two [35, 65]. Of particular interest is the finding that a good adaptive mesh refinement strategy can avoid the jagged shape otherwise produced by using the coordinates of the finite element grid as design variables.

4. Sensitivity derivatives calculation

Design sensitivity analysis, that is, the calculation of quantitative information on how the response of a structure is affected by changes in the variables that define its shape, is a fundamental requirement for shape optimization. The first partial derivatives of structural response quantities with respect to shape design variables provide the essential information required to couple mathematical optimization and structural analysis procedures. There are two basic approaches to sensitivity derivatives calculation. The first is based on differentiation of the discretized (FE) system, and the second on differentiating the continuum equations.

4.1. *Differentiation of finite element model*

The finite element equilibrium equations representing a linear elastic model are written as

$$KU = F\,, \tag{1}$$

where K is the overall stiffness matrix, U is the displacement vector, and F is the external load vector.

The derivatives of the displacement U or of constraints based on U may be calculated by the finite differences approach. This is a popular approach, but it is quite costly when the number of design variables is large. The alternative is implicit differentiation of (1).

Differentiating (1) with respect to a design variable x, we obtain

$$K\,\mathrm{d}U/\mathrm{d}x = \mathrm{d}F/\mathrm{d}x - U\,\mathrm{d}K/\mathrm{d}x\,. \tag{2}$$

The derivatives of the displacement vector, $\mathrm{d}U/\mathrm{d}x$, can be computed by solving (2); this is the so-called direct method. If derivatives of a constraint function are needed, an adjoint variable method can be used.

Using (2) to calculate sensitivity derivatives has two disadvantages. First, even small changes of the boundary can change the entire finite element mesh and therefore the calculation of $\mathrm{d}K/\mathrm{d}x$ is quite costly. Second, changes in shape can lead to the distortion of the finite elements and reduced accuracy. Thus, the derivatives obtained from (2) have a spurious component, which reflects the changing accuracy of the solution when the mesh is distorted [14, 19]. However, computational experience to date did not show that this spurious component presents serious problems.

4.2. Continuum derivatives

There has been a substantial amount of work in obtaining sensitivity derivatives by differentiating the continuum equations, using the concept of material derivatives, and only then discretizing the problem. The reverse order of differentiation and discretization avoids spurious errors due to mesh distortion. Also, the derivatives can be expressed in terms of boundary integrals, which are cheaper to calculate than the term $\mathrm{d}K/\mathrm{d}x$, because they are localized to the changing boundaries. Several authors have proposed formulations using boundary integrals and the adjoint method [28, 30, 32–34, 37, 39, 55, 56, 58, 100–102, 115, 131, 135, 139]. Unfortunately, there are considerable numerical difficulties associated with the evaluation of boundary integrals (see [130–132]), especially for low-order elements which do not model well a curved boundary.

Braibant and Fleury [22–24] avoid these numerical difficulties by using domain instead of boundary integrals in a direct approach formulation. Such a domain integral approach was also used with the adjoint formulation in [29, 31, 59]. The domain integrals avoid the numerical difficulaties associated with boundary integrals, but they are about as expensive as the use of (2). The main advantage of the continuum approach seems to be the generality of its results. It is equally applicable to finite element, boundary element, or any other numerical or analytical solution technique.

5. Solution techniques for shape optimum design

The methods used for solving the shape design problems range from the calculus of variations to experimental techniques employing photoelastic models. However, most of the work is based on employing mathematical programming methods coupled with finite element analysis of the structure. For typical applications of mathematical programming techniques, see [21, 32, 52, 137]. The following discussion is focused mostly on techniques which have been specifically developed for shape optimization problems.

5.1. Calculus of variations and optimality criteria methods

One-dimensional shape optimization problems are often solved by the calculus of variations. For example, Curtis and Walpole [36] solved for the maximal torsional rigidity of axisymmetric hollow shafts. Similarly, Plaut and Olhoff [86] used the calculus of variations to obtain the optimal forms of shallow arches for vibration and stability requirements.

Optimality criteria can be either rigorous or intuitive. Rigorous optimality criteria are derived mathematically, for example, by using variational methods [37, 77]. Optimality criteria can also be stipulated intuitively. For example, it can be stipulated that the tangential stress or the strain energy density should be uniform along the boundary.

Banichuk [5, 6] formulated a problem of selecting the optimum shape of a cross-section for a shaft to maximize torsional stiffness, with a given amount of material available. He exploited the fact that the functional minimized by the warping function in a variational formulation of the boundary value problem is proportional to the torsional stiffness of that part. He then used variations of this functional with respect to both the warping function and boundary variation, using material derivatives, and obtained a necessary condition for optimum location of the boundary, which he solved analytically.

Dems and Mroz, in [37–40], used the principle of virtual work and a boundary perturbation analysis to derive optimality criteria for attaining a minimum of mean elastic compliance. The optimal design was obtained by iteratively solving the optimality conditions using a finite element representation of the equations. A similar approach was used by Na et al. [75–77]. Olhoff and Taylor [83] introduced the concept of shape remodelling, that is, optimal modification of a specified shape.

Cherpanov [26] considered the problem of finding a hole shape in a planar solid to make the boundary tangential stress constant, under the assumption that this is a condition of optimality (see [9] for the rigorous optimality condition). He solved the problem analytically. Dhir [43–45] used a procedure based on first developing analytical boundary stress expressions as functions of the hole geometry, plate material, amount of reinforcement, and specified load. The integral of the square of the stress around the opening boundary was assumed to be a reasonable objective function and was minimized analytically. He concluded that a uniform tangential stress along the opening boundary would lead to the minimum stress concentration. Weck and Steinke [126] assumed that a uniform tangential stress is the optimality condition. They made assumptions relating the tangential stress to the curvature of the boundary to simplify the problem.

Two additional intuitive optimality criteria methods, the pattern transformation method and the photoelasticity technique, deserve particular attention because of their unique resizing approaches.

5.2. Pattern transformation methods

The pattern transformation method of Oda [79–82] is a technique of transforming the shape of the boundary based on the stress ratio in the boundary finite elements. In the first step, the stress ratio in the boundary finite elements is calculated. In the second step, the size of the boundary elements is scaled up or down based on their stress ratio. Umetani and Hirai [122, 123] also used the same stress ratio approach, whereas Seguchi and Tada [112, 119], and Hemada et al. [57], considered strain energy ratios instead of stress ratios for the shape modification.

5.3. The photoelasticity technique

One of the important problems in the design of plates and shells is to minimize the stress concentration due to the presence of holes and other structural discontinuities. Some recent investigations have attempted to optimize the shape of the holes experimentally. In [46, 47, 92, 93], a step-by-step procedure is used for modifying hole boundaries in a two-dimensional photoelastic model until the tensile and compressive boundary stresses were approximately constant. The hole shapes were changed by removing material from low stress regions to obtain uniform stress conditions as well as significant reductions in stress intensity factors. This technique, therefore, calls for a series of experiments for each particular problem. Schnack [107, 111] developed a procedure for the numerical equivalent of this experimental approach.

5.4. Boundary element method

Because the boundary element technique requires only the modelling of the boundary, it seems to be ideally suited for boundary shape optimization. The boundary element method overcomes the two major drawbacks of the finite element method, that are finite element mesh regeneration and difficulties in sensitivity derivatives calculation stemming from inaccurate boundary representation. Recently, Soares et al. [113, 114] formulated the problem of optimization of the geometry in terms of the boundary element method. Only the boundary of the structure was discretized and the optimization problem was solved by Pshenichny's linearization method. However, the boundary element method is still not as reliable as the finite element method. Recently, Eizadian and Trompette [48] reported erratic performance in the application of the method to two-dimensional shape optimization problems.

6. Applications

The previous sections summarized shape optimization work by type of approach. The present section gives a summary of the shape optimization literature by area of application. The information is summarized in Table 1.

Table 1
Shape optimization literature by area of application

Application	Reference(s)
Beams	[4, 27, 54, 61, 123, 138]
Bracket	[14, 22]
Columns	[1, 12, 50, 51, 64, 67, 119]
Connecting rod	[52, 88, 133–135]
Cross-sectional shape in torsion	[5–7, 10, 11, 25, 28, 36, 37, 39, 41, 53 58–60, 66, 69, 70, 74–76, 84, 85, 113, 114]
Dams	[80, 96, 99, 124, 125, 132, 137]
Disks	[2, 17, 40, 41, 71, 78, 81, 82, 116]
Engine main bearing cap	[62, 82]
Fillets	[16, 21, 52, 68, 77, 85, 89, 96, 104 109, 121, 132]
Plate with a hole	[2, 9, 10, 19, 21, 22, 37, 43–47, 49, 68, 95, 97, 103, 105, 117, 126]
Pressure vessel components	[72, 73, 79, 89, 96, 124]
Rotating turbine machinery	[137]
Shells	[3, 18–20, 63, 92–94, 126, 129, 136]
Stress concentration minimization	[2, 8, 16, 46, 49, 52, 68, 80, 85, 88, 92, 93 96, 104, 105, 107, 108, 110, 121, 127, 128]
Torque arm	[13, 14, 22, 23, 130, 132]
Turbine and compressor blades	[90, 91]

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