Recommendation Systems
Part III

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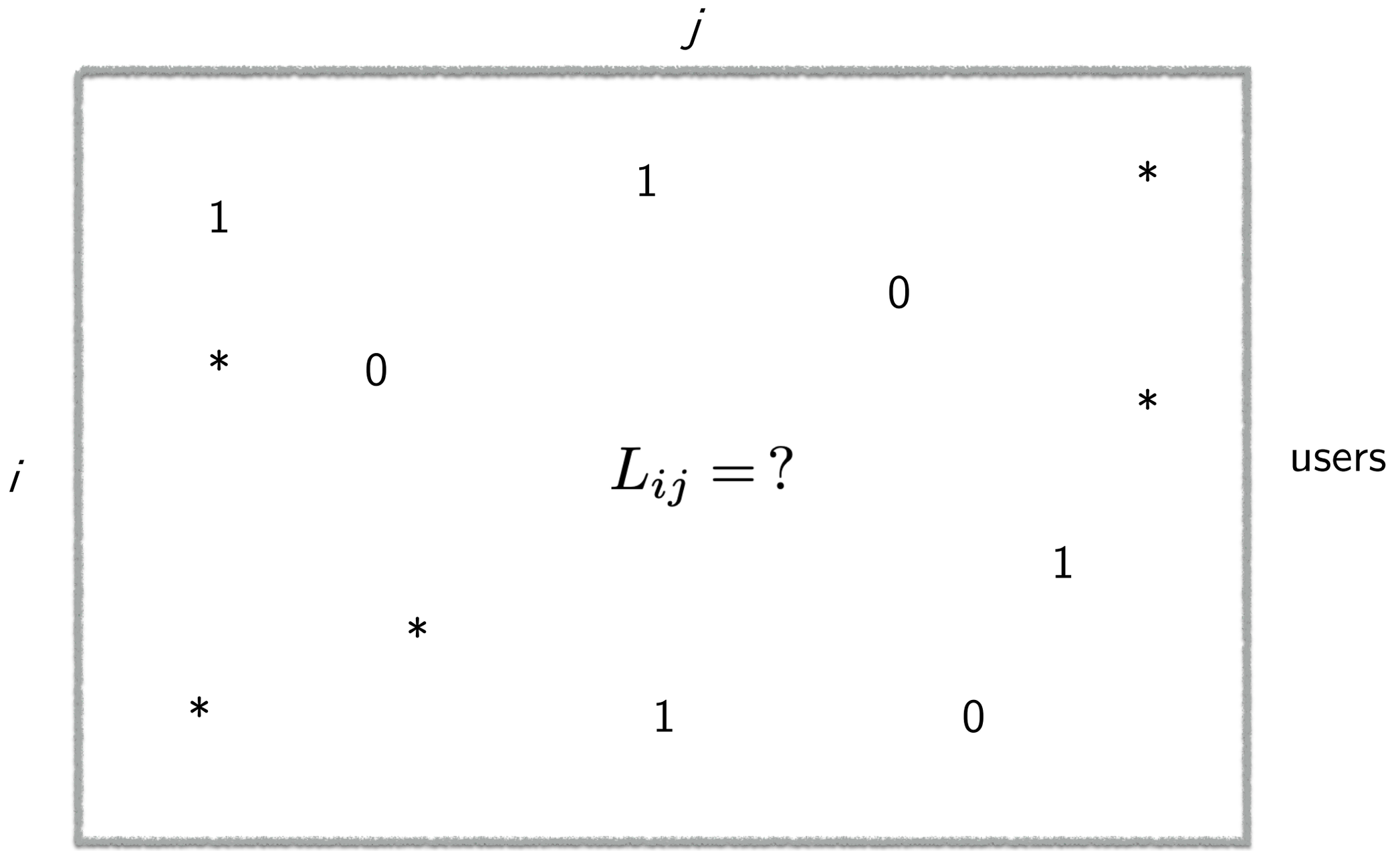
Recall — Part I

Introduction, simple methods

- Module 1: background
  - Recommendation systems: why and what?
  - Example datasets: Yelp & MovieLens
- Module 2: problem statement
  - Recommendation systems: a prediction problem
  - Model: estimating time varying tensor with side information
- Module 3: simple solutions
  - Solution I: averaging
  - Solution II: content-based

Recall: problem statement

We will start with simple problem statement: complete the matrix



Recommendation: Problem statement

We will start with simple problem statement: complete the matrix

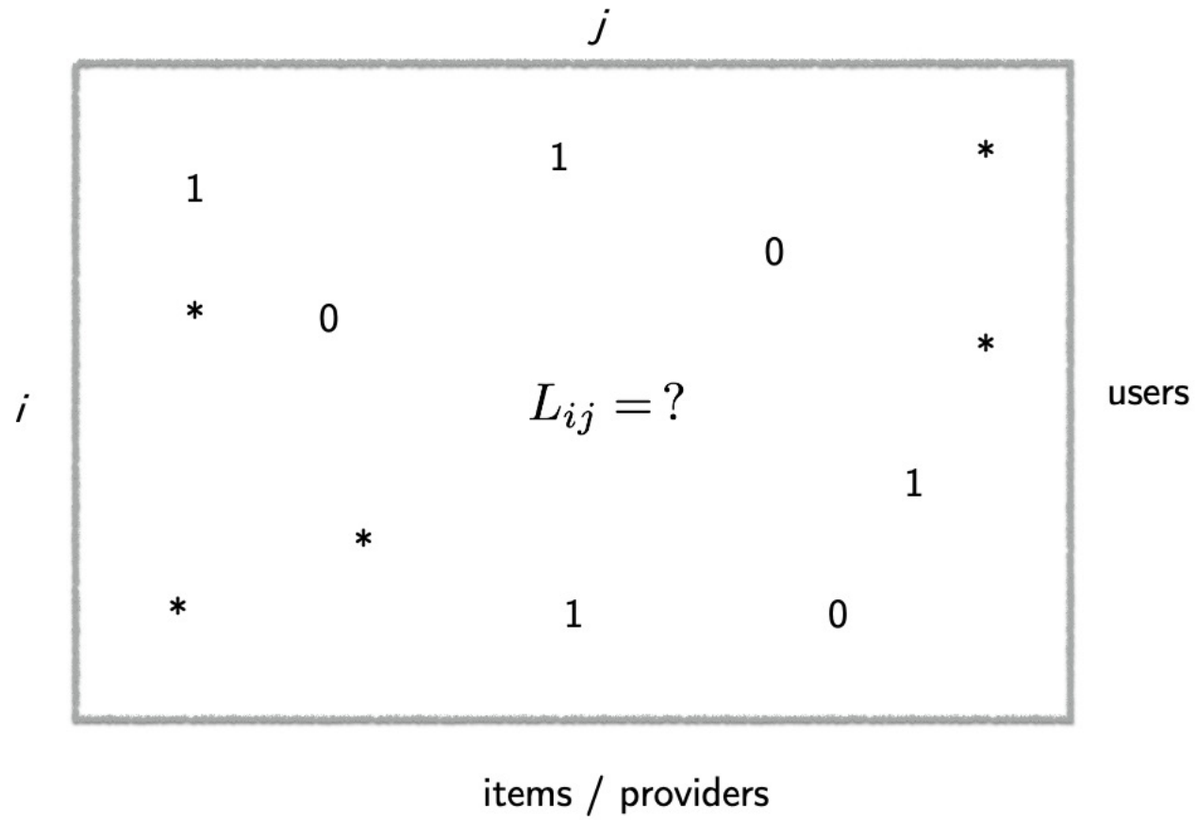
Observations: Y_{ij} over i in users, j in items

If (i, j) is observed

$$E[Y_{ij}] = L_{ij}$$

If (i, j) is *not* observed

$$Y_{ij} = \star \text{ or } ?$$



Goal: produce estimation $\hat{L}_{ij}$ for all i, j

so that $\hat{L}_{ij} \approx L_{ij}$ for all i, j

Recall: Solution 1 — Averaging

What if, instead we assume

All items or providers are identical

Then estimate: row average (+ correction for number of observations)

How to put these two simple estimators together?

$$2L_{ij} = L_{i\cdot} + \frac{1}{\sqrt{n_{i\cdot}}} + L_{\cdot j} + \frac{1}{\sqrt{n_{\cdot j}}}$$

where $L_{i\cdot}$ is average of observed entries in row i

$n_{i\cdot}$ is number of observed entries in row i

$L_{\cdot j}$ is average of observed entries in column j

$n_{\cdot j}$ is number of observed entries in column j

Recall: Solution 2 — Content-based Filtering

A little more involved assumption

Users and items have features

that are observed and can predict the likelihood

Let features of user i be x_i

MovieLens: demographics of users

Let features of item j be y_j

MovieLens: attributes of movies

Then, goal is to learn f where $L_{ij} = f(x_i, y_j)$

Outline — Part II

Solution evolves, Matrix estimation

- Module 1: clustering
  - Finding user and item clusters
  - Averaging *within* clusters
- Module 2: collaborative filtering aka personalized clustering
  - Finding users and items similar to a given user, item
  - Averaging *amongst* user-item specific *similar* users, items
- Module 3: singular value thresholding, optimization
  - It's a Matrix! find Singular Value Decomposition
  - Solve for least-squares

Clustering

How to cluster users (or items)?

Step 1. Compute similarity between each pair of N users [How?]

This gives N x N similarity matrix (that is symmetric)

Step 2. Obtain representation of each users in low-dim space [How?]

This assigns co-ordinates to each user in d dim space

Step 3. Perform k-means clusterings [How?]

Iteratively find k clusters till they make sense

Iterative collaborative filtering

Collaborative filtering

Use average over *similar* users (or items)

a user is *similar* when shared observations are *close enough*

A common challenge: not enough users with shared observations

Solution: friend of your friend is (almost) your friend

Find users *similar* to user of interest

Next find users *similar* to these users

And continue *iterating* this procedure to find more *similar* users

till *enough similar* users are found

Use the experiences of such users to obtain the estimate

Singular Value Thresholding

A natural algorithm: singular value thresholding

Step 1. Fill missing values in L by 0

[better way to fill missing values?]

Step 2. Compute SVD

$$L_{ij} = \sum_{k=1}^{\min(M,N)} s_k u_{ik} v_{jk}, \quad \text{for all } i,j$$

Step 3. Truncate SVD by keeping on top r components (+ normalize)

$$\widehat{L}_{ij} = \frac{1}{\hat{p}} \sum_{k=1}^{r} s_k u_{ik} v_{jk}, \quad \text{for all } i,j$$

where $\hat{p}$ is fraction of observed entries

[how to choose r?]

Optimization perspective

Singular value decomposition: optimization perspective

Let U, V be solution of

$$\text{minimize} \sum_{i,j}(X_{ij} - \sum_{k=1}^{r} u_{ik}v_{jk})^2$$

$$\text{over} \quad u_{ik} \in \mathbb{R},\ 1 \leq i \leq N,\ 1 \leq k \leq r$$

$$\text{over} \quad v_{jk} \in \mathbb{R}, 1 \leq j \leq M,\ 1 \leq k \leq r.$$

Then, U and V are (effectively) left/right singular vectors

And UV^T provides the best rank *r* approximate of X

Optimization perspective

This suggests the following algorithm

Find solution U, V of optimization problem

$$\text{minimize} \quad \sum_{(i,j):\text{obs}} \left(Y_{ij} - \sum_{k=1}^{r} u_{ik} v_{jk}\right)^2$$

$$\text{over} \quad u_{ik} \in \mathbb{R},\ 1 \le i \le N,\ 1 \le k \le r$$

$$\text{over} \quad v_{jk} \in \mathbb{R}, 1 \le j \le M,\ 1 \le k \le r.$$

For any i, j, produce estimate $\widehat{L}_{ij} = \sum_{k=1}^{r} u_{ik} v_{jk}$

Optimization perspective

The algorithm uses *only* observed entries

and does not require filling missing values as in for SVD

The optimization problem can be solved

via iteratively solving for U and V

also known as *Alternating Least Squares*

Food for thought:

Will it converge?

Optimization perspective

Exercise: Singular Value Thresholding meets Alternative Least Squares

Initialize by filling missing values with 0

Singular Value Thresholding to obtain an estimate.

Use outcome to fill missing values and then perform Alt. Least. Sqs.

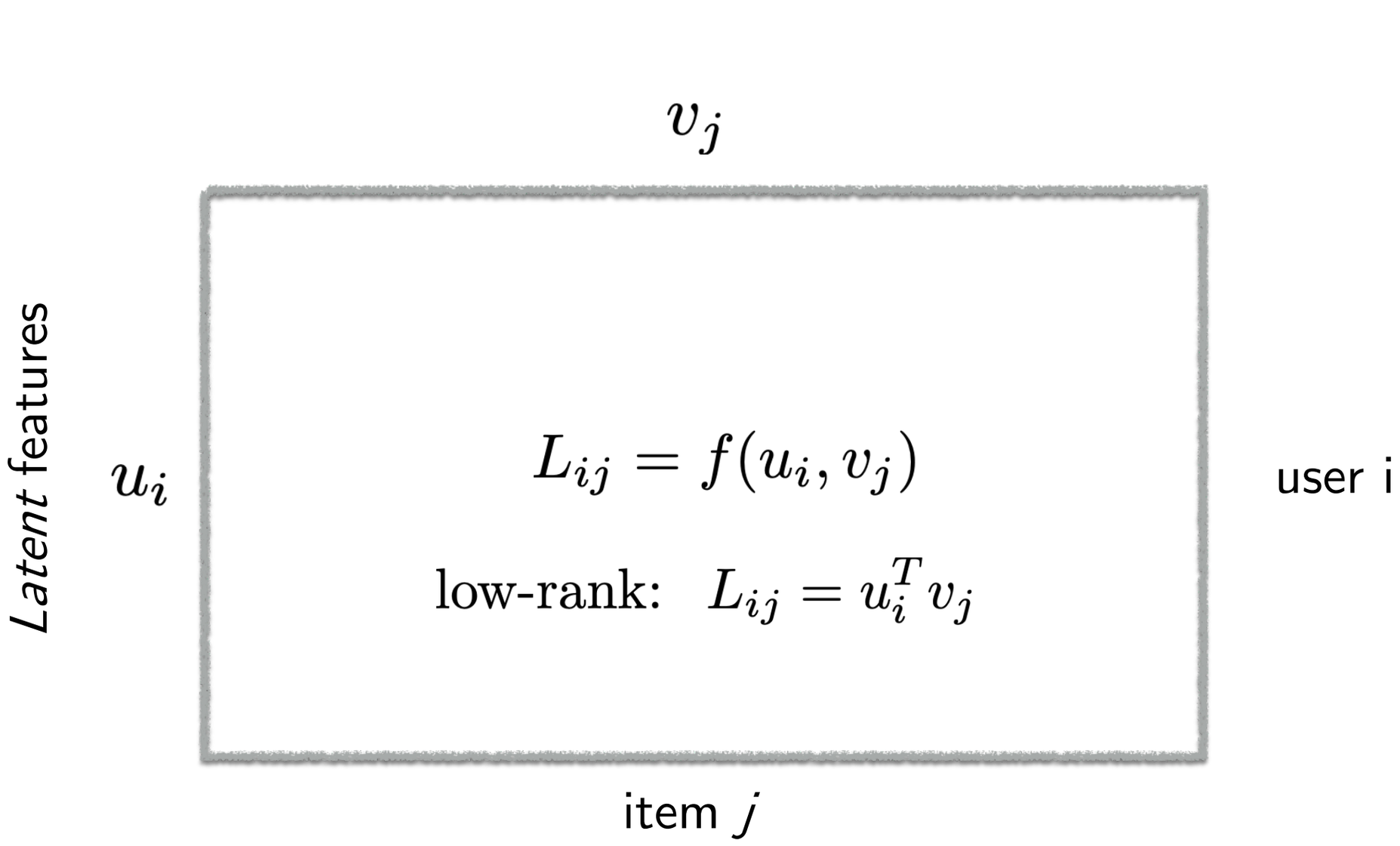
Iterate.

What are the advantages?

Use MovieLens and/or Yelp data to answer.

Matrix Estimation: Generic Model

Prediction problem: complete the matrix



Problem reduces to learning "factorization" of the matrix

either through similarities or algebraic approaches

Appendix: Matrix Estimation with Neural Networks

Singular Value Thresholding

bi-linear function of latent features

"Generalized" Singular Value Thresholding

generic "activation" function of latent features

multiple layers

this provides neural network implementation

Exercise: compare performance with other methods

Outline — Part III

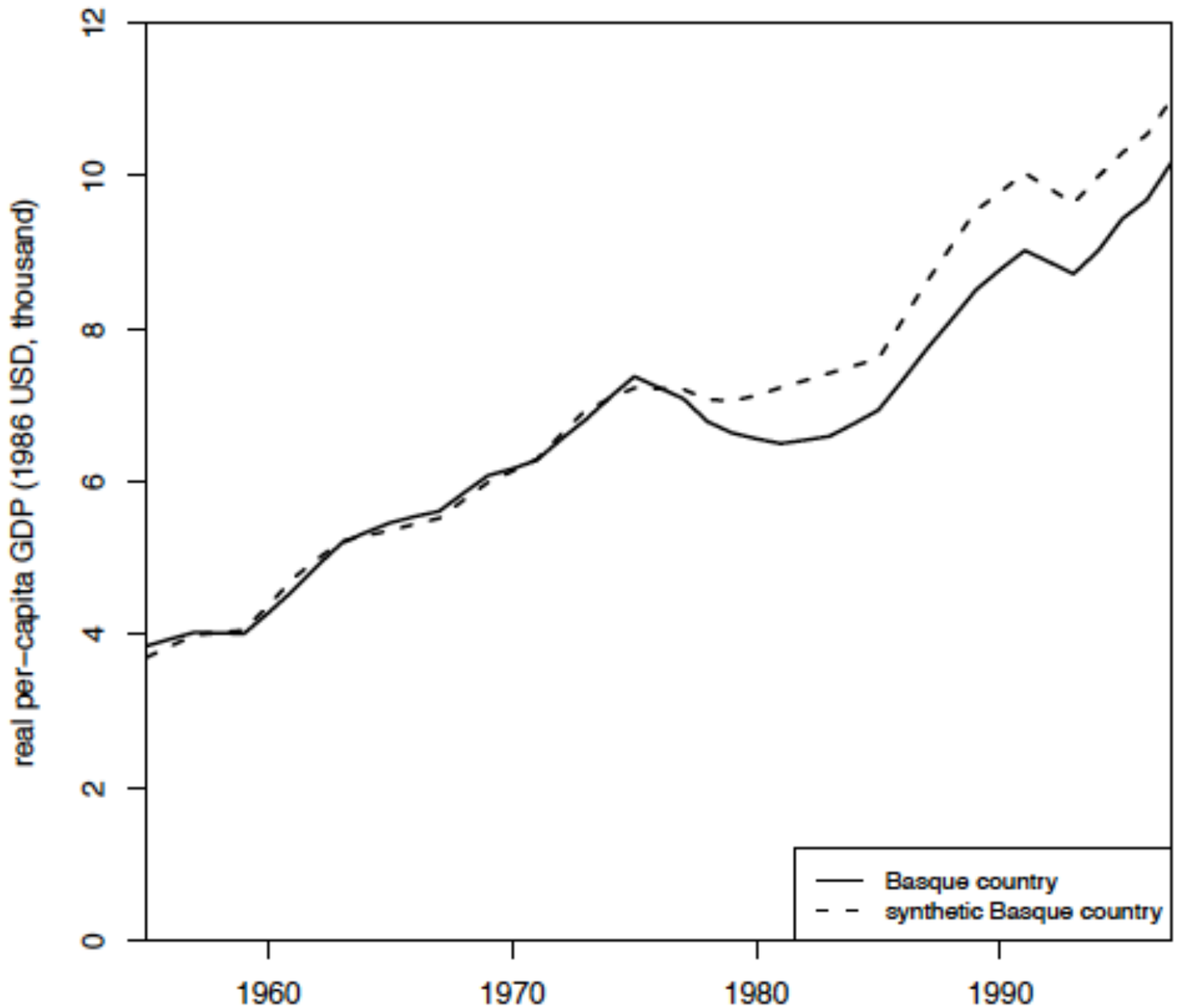
Getting things together, some more and connections

- Module 1: Matrix estimation meets content-based
  - Model
  - Estimation algorithm
- Module 2: Matrix estimation across time
  - Model
  - Estimation algorithm
- Module 3: Everything together
  - Model
  - Estimation algorithm

Some more applications

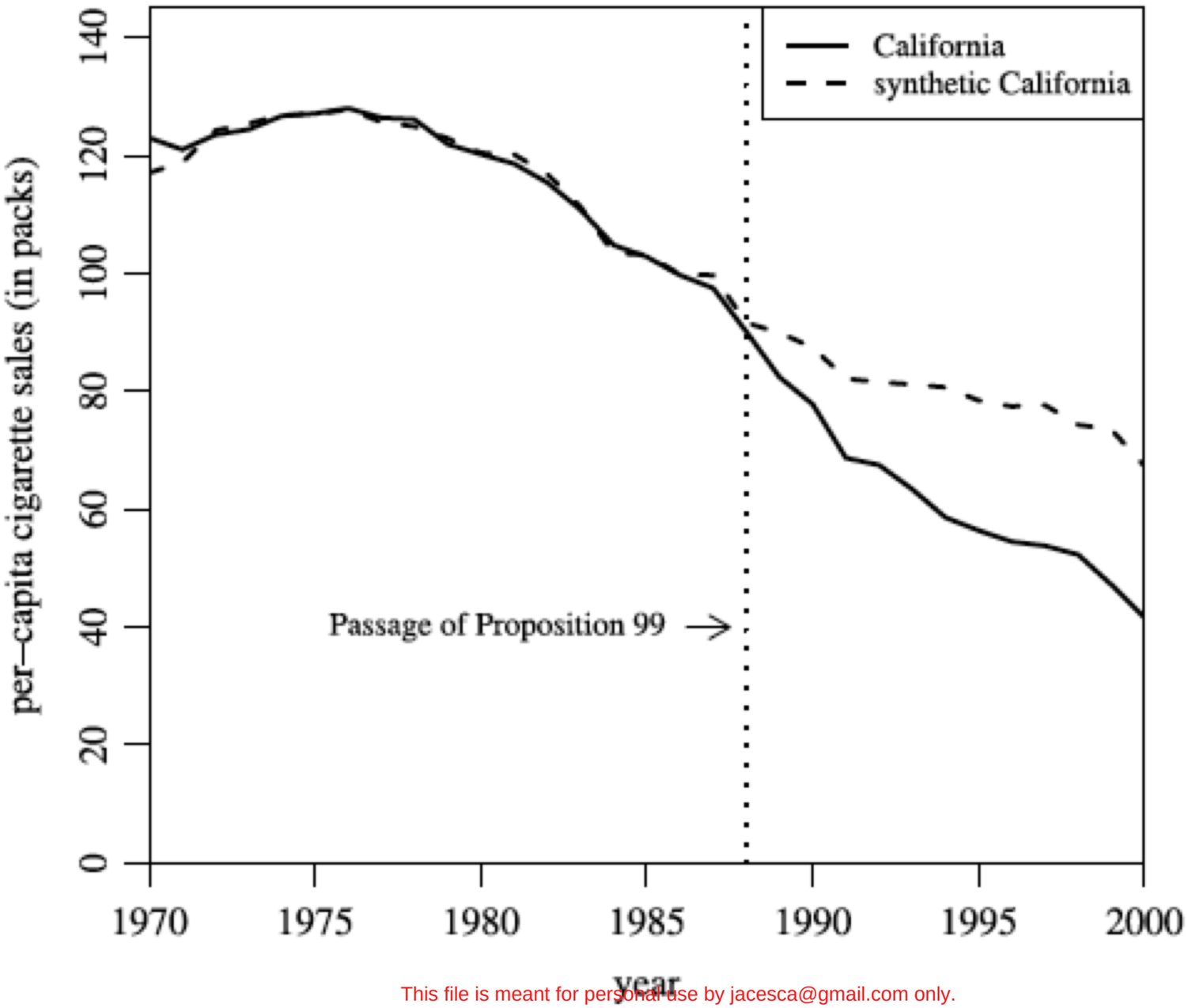
A Question

Did Terrorism have impact on Economy (of Basque Country)?

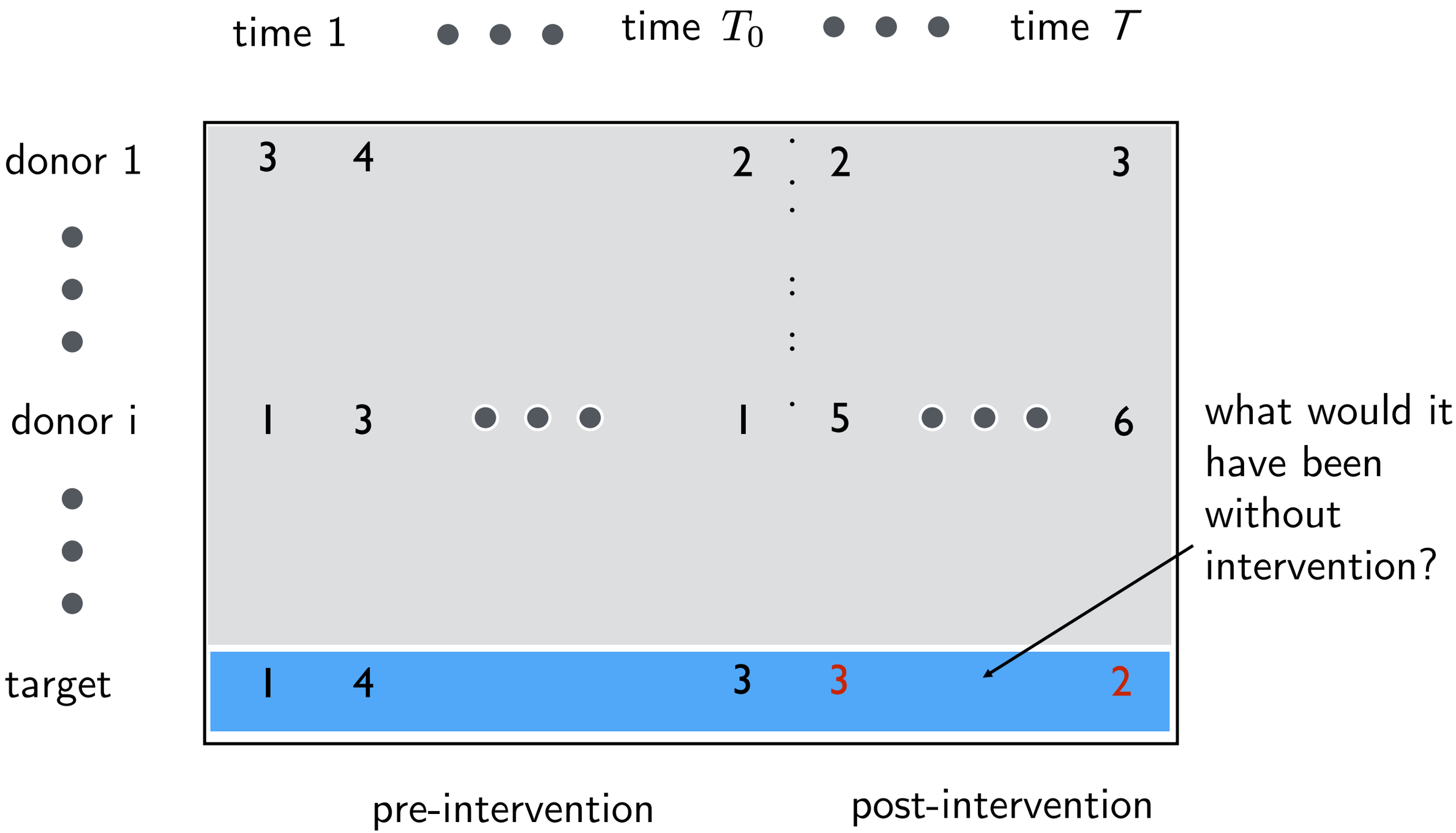


A Question

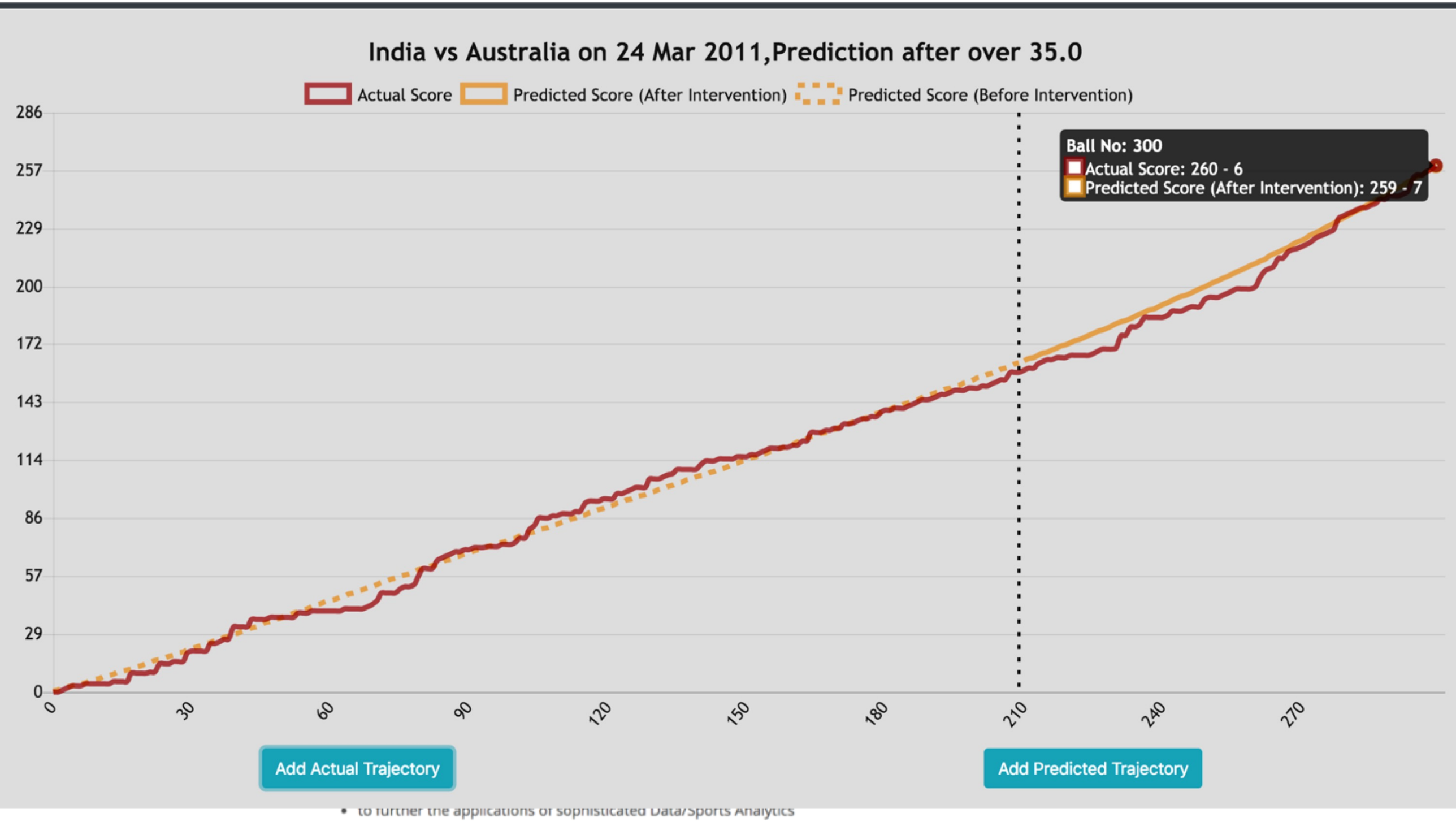
Did California tobacco control program (Prop 99) work?



How? Synthetic controls aka Matrix Estimation (!!)

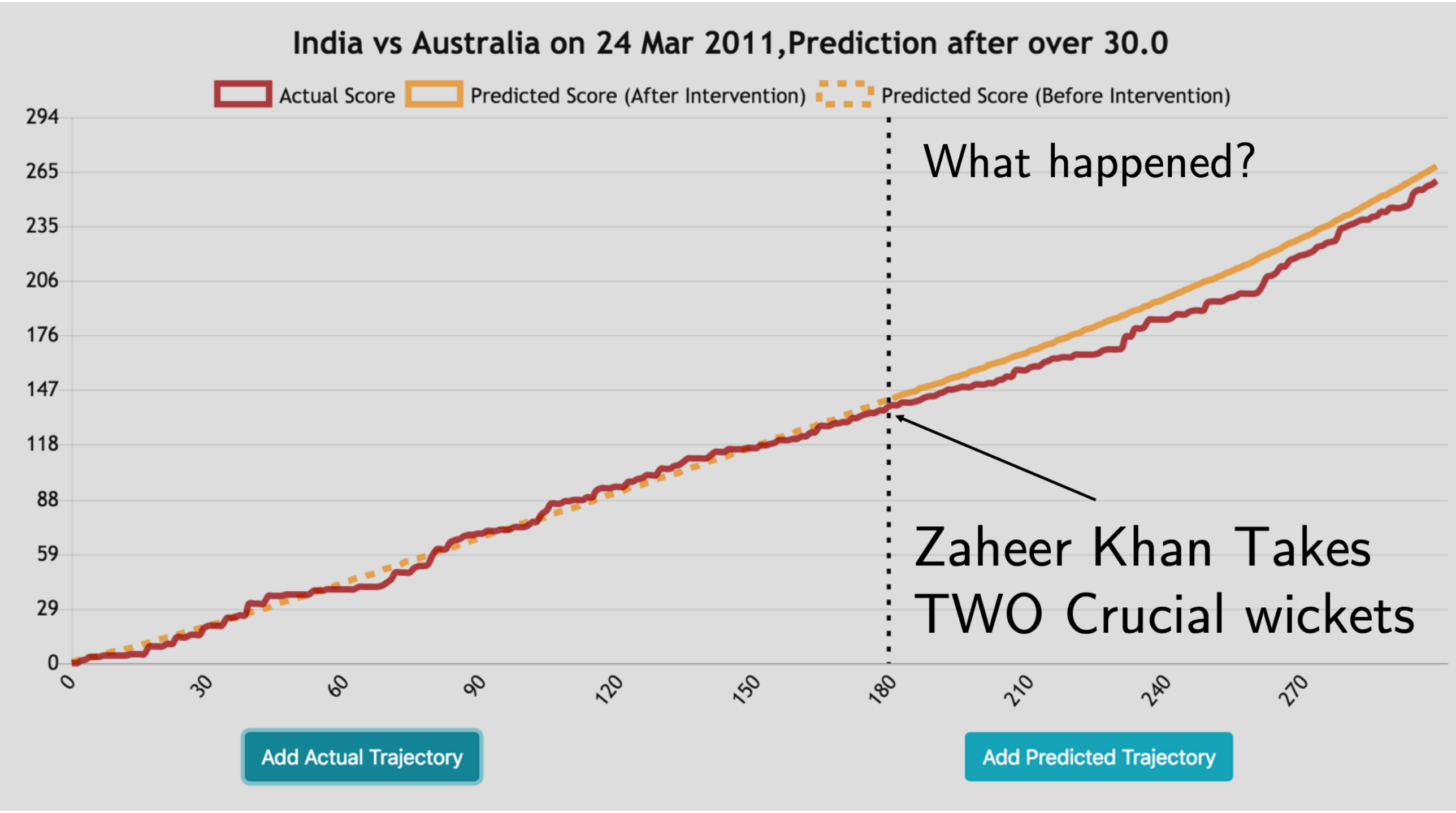


Another Example: Forecasting Cricket Trajectory



- to further the applications of sophisticated Data/Sports Analytics
- to contribute to academic literature on Sports Analytics
- to propose novel and practical solutions to (real) problems in the sport

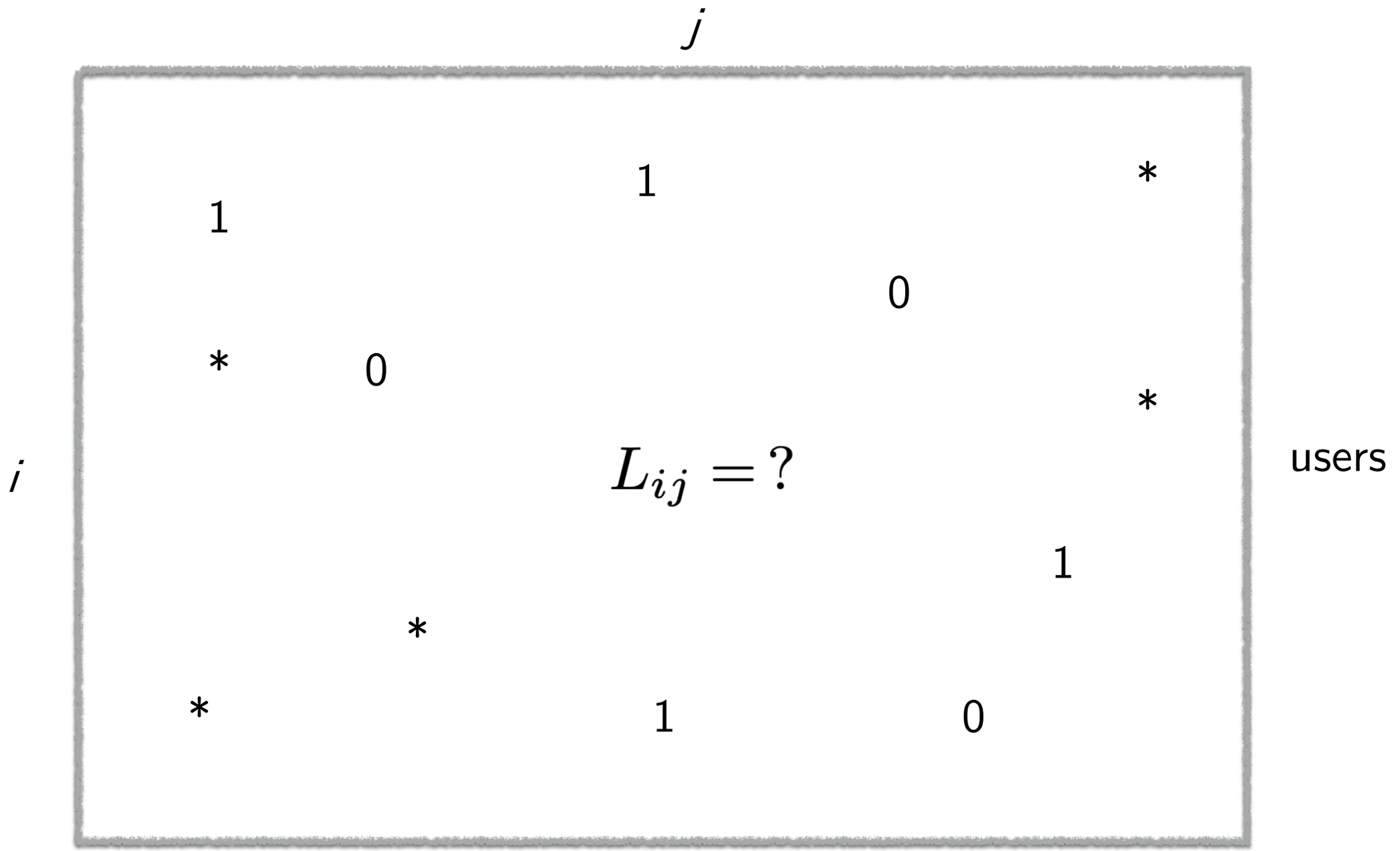
Another Example: Forecasting Cricket Trajectory



Module 1: matrix est & content-based

Recommendation: Problem statement

Prediction problem: complete the matrix



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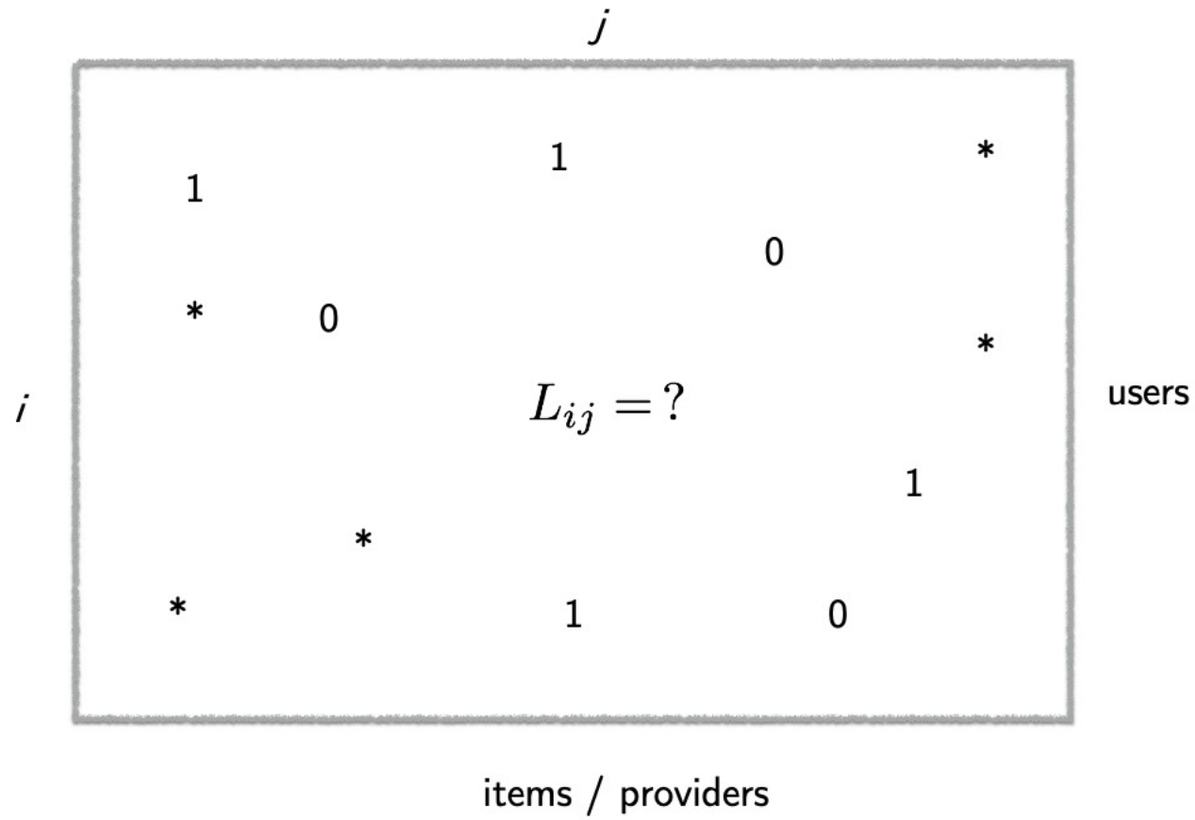


Goal: produce estimation $\hat{L}_{ij}$ for all i, j

so that $\hat{L}_{ij} \approx L_{ij}$ for all i, j

Content-based: Model

Prediction problem: complete the matrix

Observed features

y_j

Observed features

x_i

$$L_{ij} = f(x_i, y_j)$$

$$\text{e.g. } L_{ij} = \alpha x_i + \beta y_j + \gamma$$

$$\text{or } L_{ij} = \frac{\exp(\alpha x_i + \beta y_j + \gamma)}{1 + \exp(\alpha x_i + \beta y_j + \gamma)}$$

user i

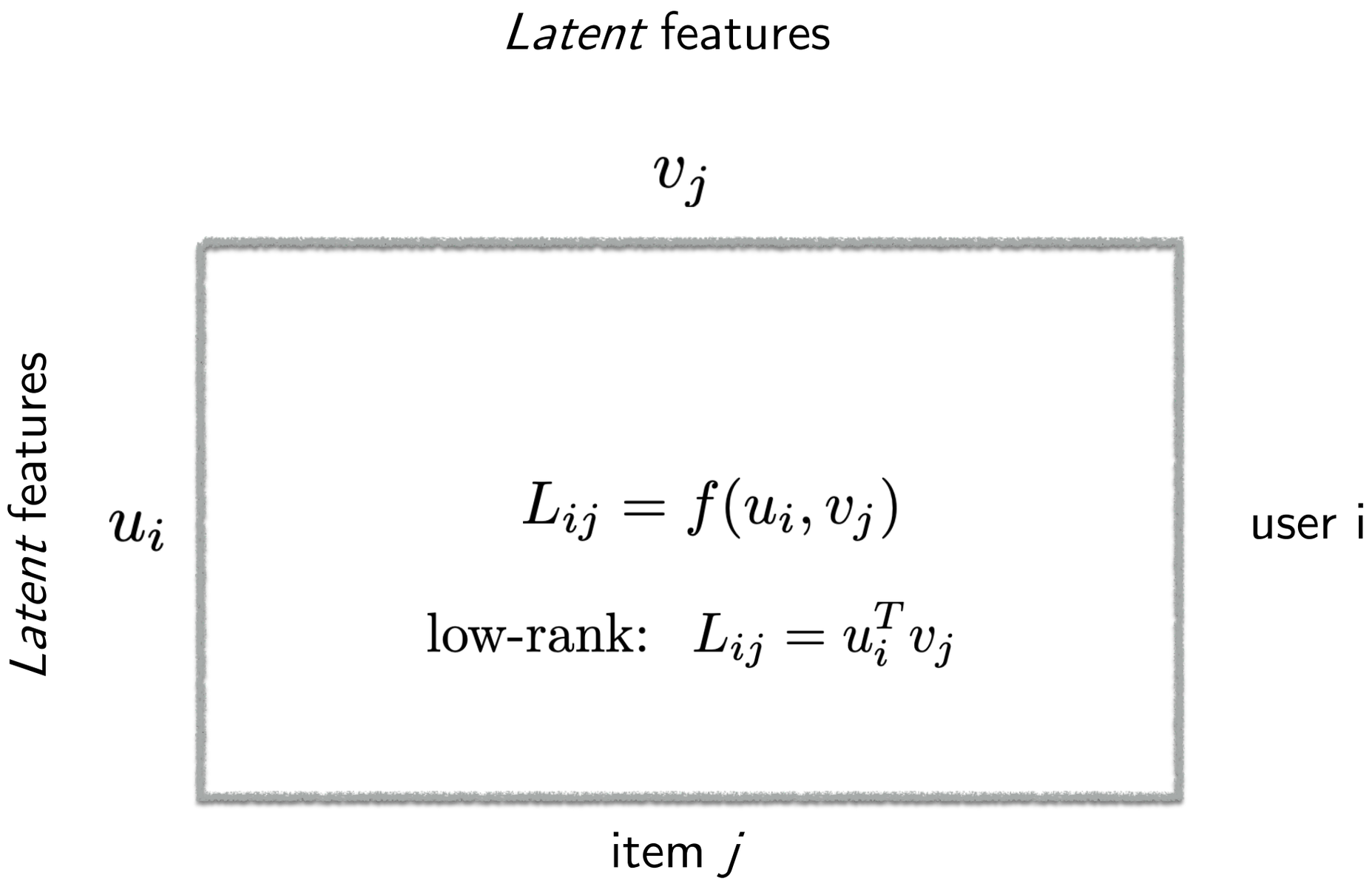
item *j*

Problem reduces to learning model *f*

That is, traditional supervised learning (regression or classification)

Matrix Estimation: Model

Prediction problem: complete the matrix



Problem reduces to learning "factorization" of the matrix

either through similarities or algebraic approaches

Matrix Estimation meets Content-based: Model

Prediction problem: complete the matrix

item features

latent: v_j, obs'ed: y_j

user features

latent: u_i

obs'ed: x_i

$$L_{ij} = f_{\text{obs}}(x_i, y_j) + f_{\text{latent}}(u_i, v_j)$$

$$L_{ij} = \alpha x_i + \beta y_j + u_i^T v_j + \gamma$$

user i

item j

Problem reduces to learning model f

That is, traditional supervised learning (regression or classification)

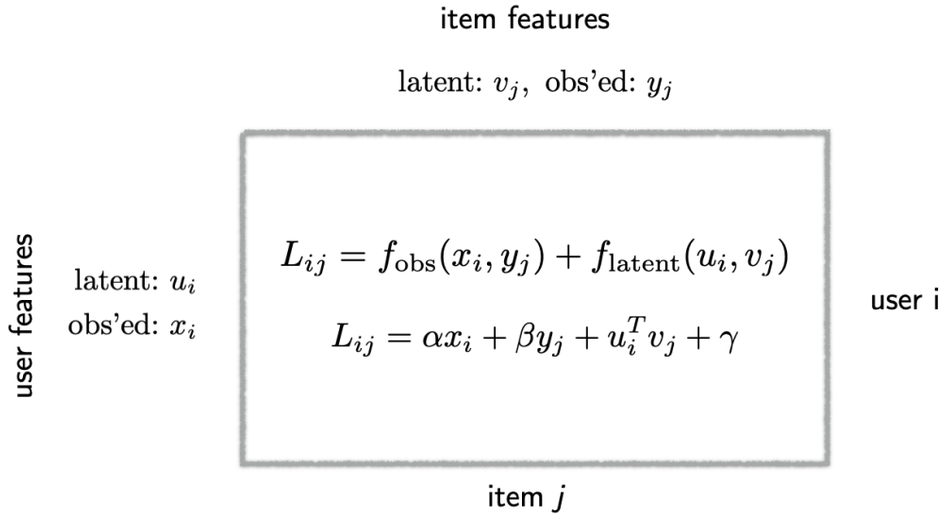
Matrix Estimation meets Content-based: Algorithm

Prediction problem: complete the matrix

Step 1. Content-based supervised learning

Learn the regressor (or classifier) f_{obs}

$$L_{ij}^{\text{obs}} = f_{\text{obs}}(x_i, y_j)$$



Step 2. Matrix estimation

Compute "difference" matrix

$$L_{ij}^{\text{diff}} = L_{ij} - L_{ij}^{\text{obs}}, \quad \text{over obs. entries}$$

Use matrix estimation on L^{diff} to produce L^{ME}

Step 3. Combine estimates:

$$\widehat{L}_{ij} = L_{ij}^{\text{obs}} + L_{ij}^{\text{ME}}$$

Module 2: Matrix Estimation Across Time

Matrix Estimation over Time: Model

Prediction problem: complete the matrix over time, indexed by t

Latent features

$v_j(t)$

Latent features

$u_i(t)$

$$L_{ij}(t) = f(u_i(t), v_j(t))$$

$$\text{low-rank:} \quad L_{ij}(t) = u_i^T(t) v_j(t)$$

$$u_i(\cdot),\ v_j(\cdot) \text{ time-series}$$

user i

item *j*

Problem reduces estimating time varying matrix where

Latent factors are time varying, observations are partial and noisy

Matrix Estimation over Time: Model

Prediction problem: complete the matrix over time, indexed by t

Multiple time series:

$$L_{ij}(t),\ i \in [N],\ j \in [M]$$

They obey latent structure:

$$L_{ij}(t) = u_i(t)^T v_j(t),\ \ u_i(t),\ v_j(t) \in \mathbb{R}^d$$

where each component of $u_i,\ v_j$ is a structured time-series

Observations:

Partial, noisy observations of the $L_{ij}(t),\ i \in [N],\ j \in [M]$

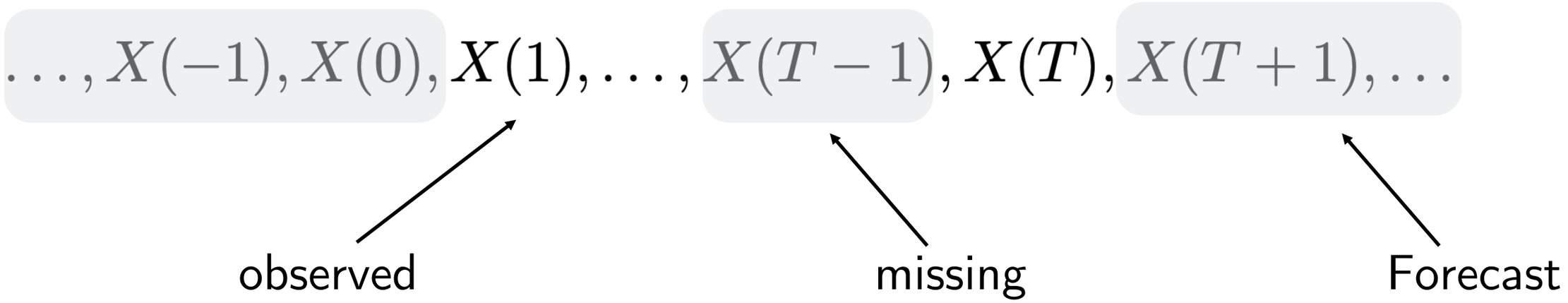
NOT $u_i(t),\ v_j(t) \in \mathbb{R}^d$

A digression: time series imputation, forecasting

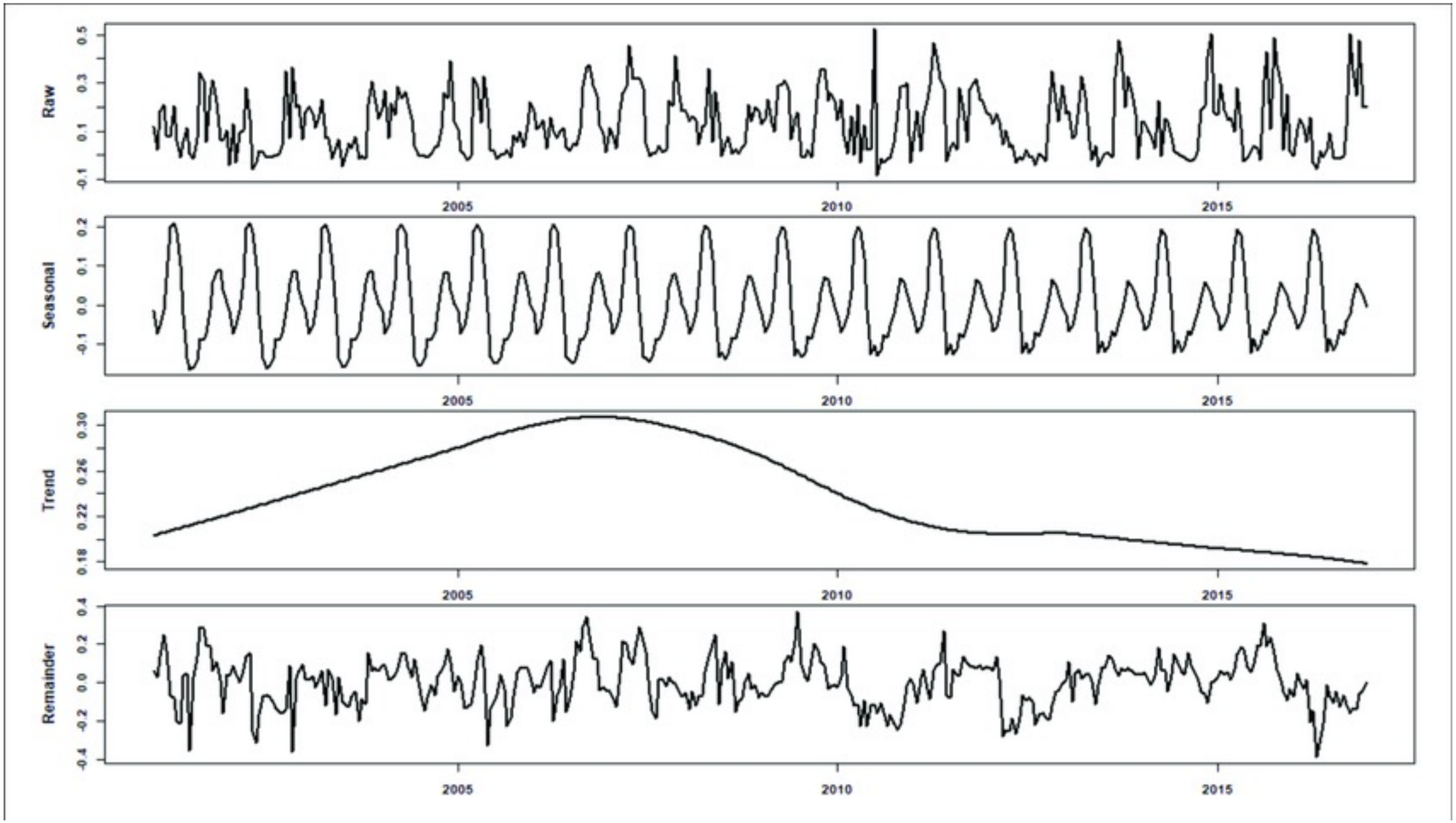
Ground truth of interest:

$$\ldots, X(-1), X(0), X(1), \ldots, X(T-1), X(T), X(T+1), \ldots$$

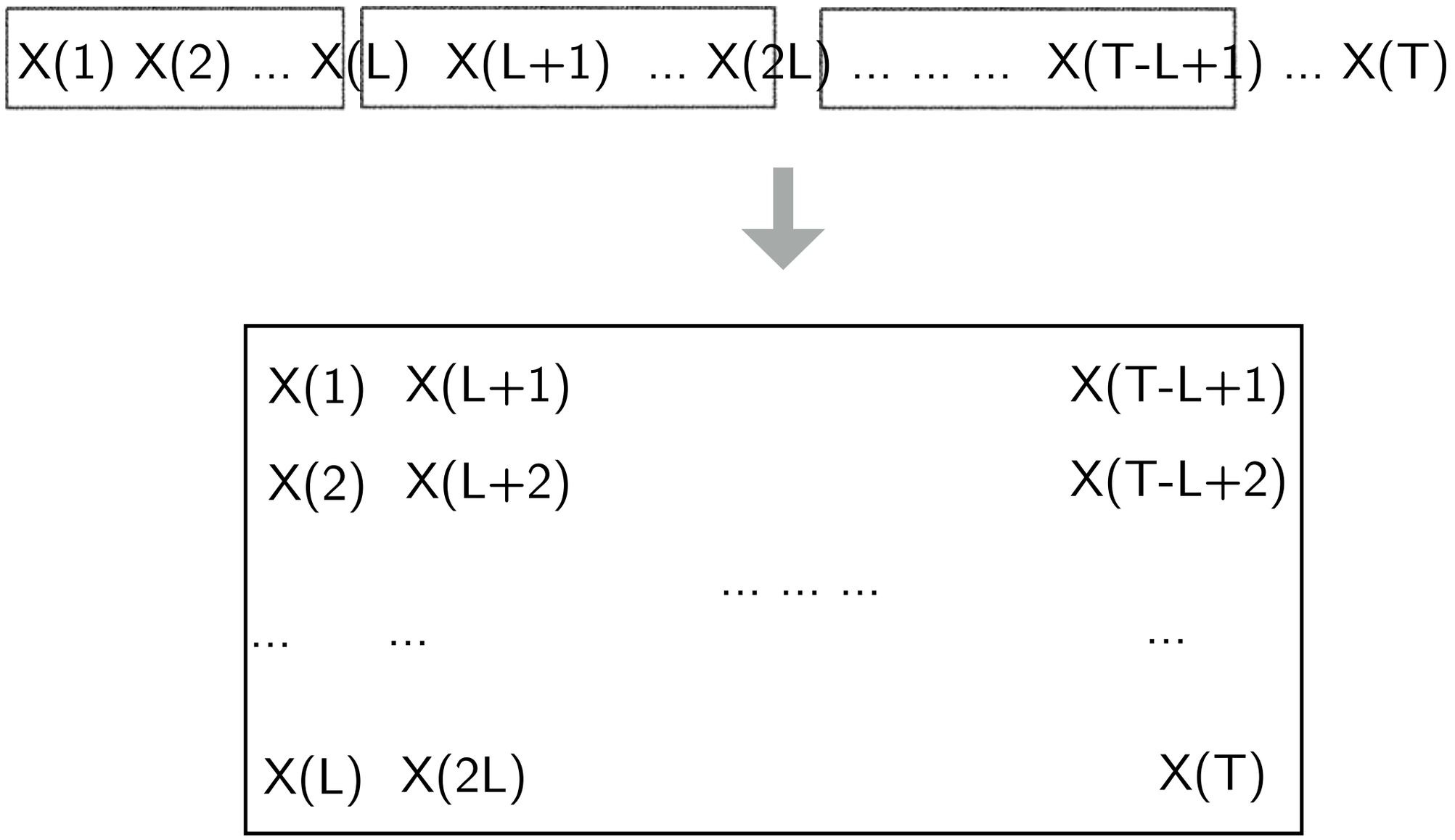
Observations:



Characteristics of time series data



Page Matrix



$$P = [P_{ij} = X(i + (j-1)L)] \in \mathbb{R}^{L \times \frac{T}{L}}$$

Imputation of time series data

Transform to Matrix, **Do Matrix Estimation**, Undo Transformation

X(1) X(2) ... X(L) X(L+1) ... X(2L) ... ... X(T-L+1) ... X(T)

X(1)	X(L+1)		X(T-L+1)
X(2)	X(L+2)		X(T-L+2)
		... ...	
...	...		...
X(L)	X(2L)		X(T)

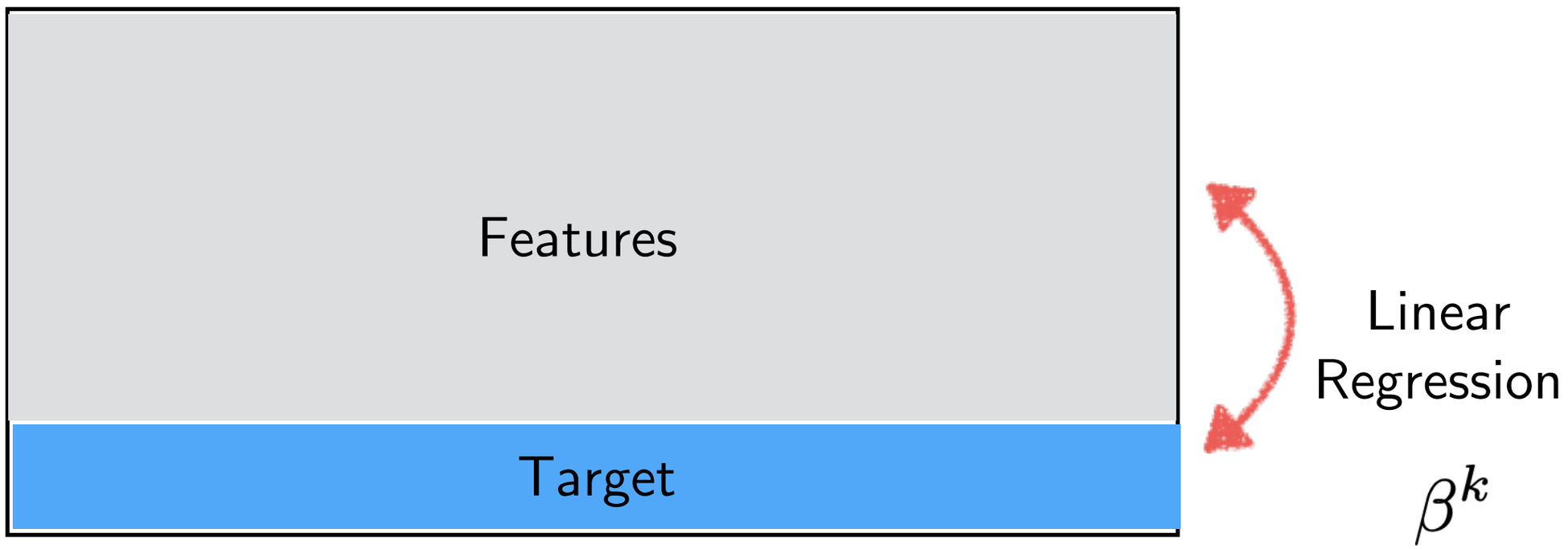
That's it!

Forecasting of time series data

Transform to Matrix, **Do Matrix Estimation**, Regression, Prediction

Matrix Est

$\mathcal{X}$ → M

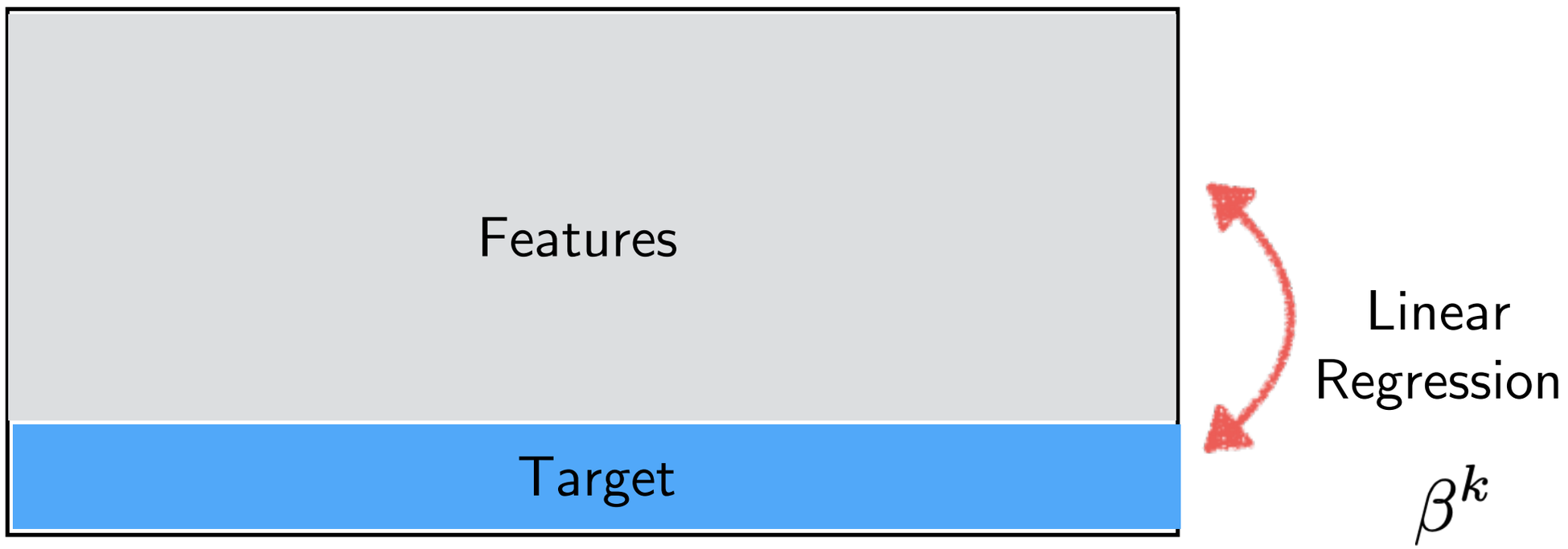


Forecasting of time series data

Transform to Matrix, Do Matrix Estimation, **Regression**, Prediction

Matrix Est

$\mathcal{X}$ → M



Forecasting

Transform to Matrix, Do Matrix Estimation, Regression, **Prediction**

[X(T-L+2) ... X(T)] → $V(T+1)$

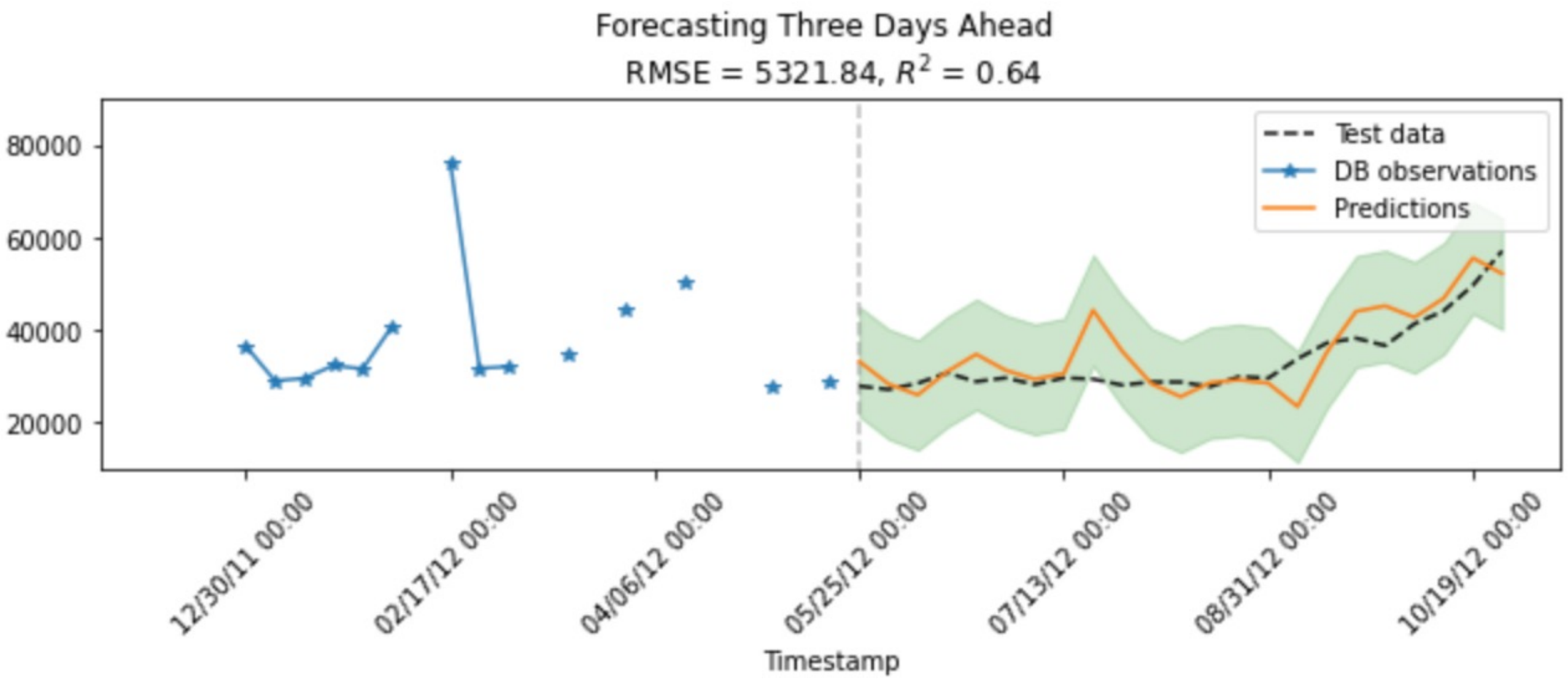
Project on Coln Space of M^k

Inner product with β^k

$\hat{X}(T+1)$

where $k = \left(T+1 \mod L\right)+1$

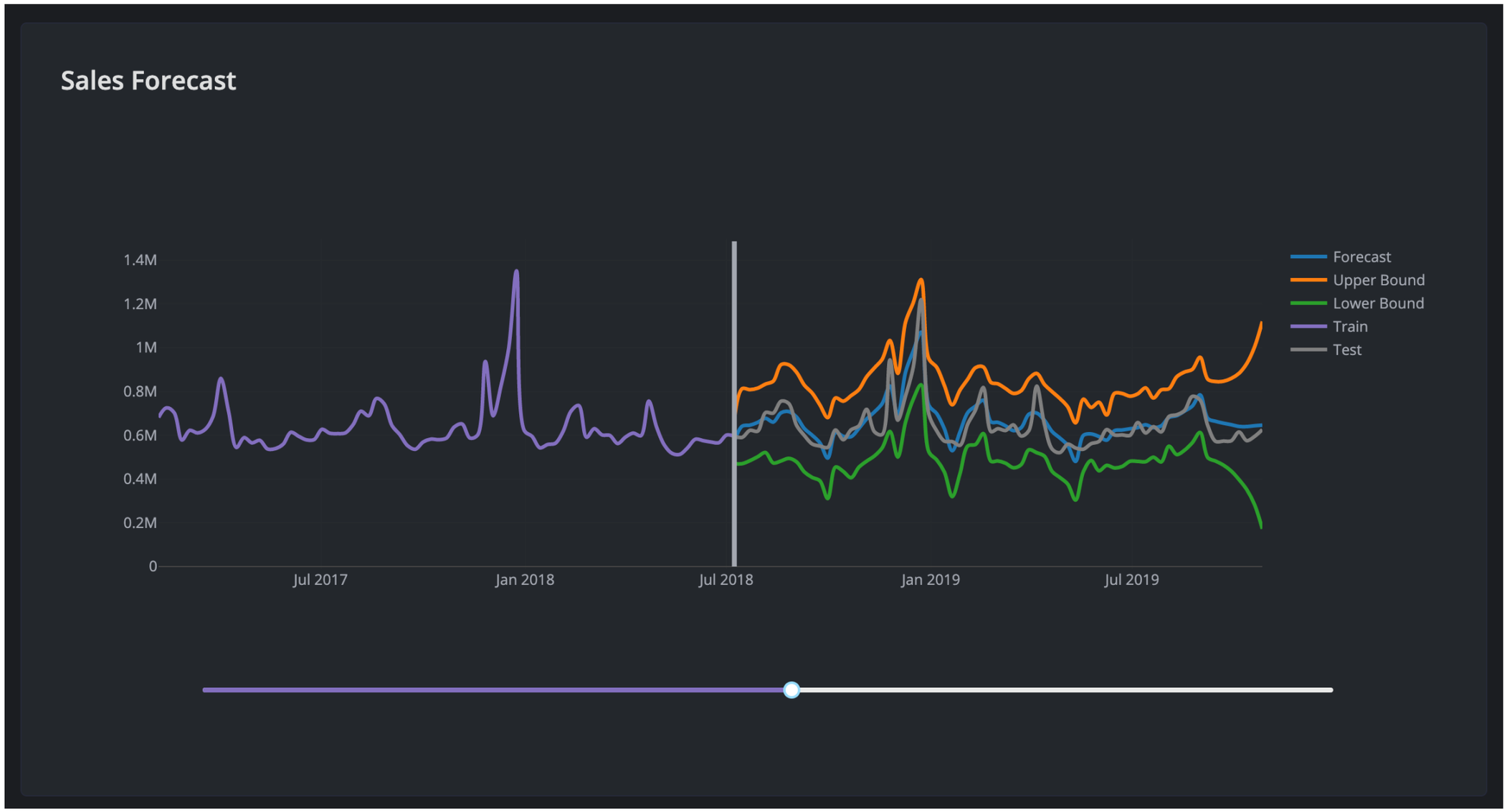
Does it really work?



Data URL:

https://www.kaggle.com/c/walmart-recruiting-store-sales-forecasting

Does it really work?



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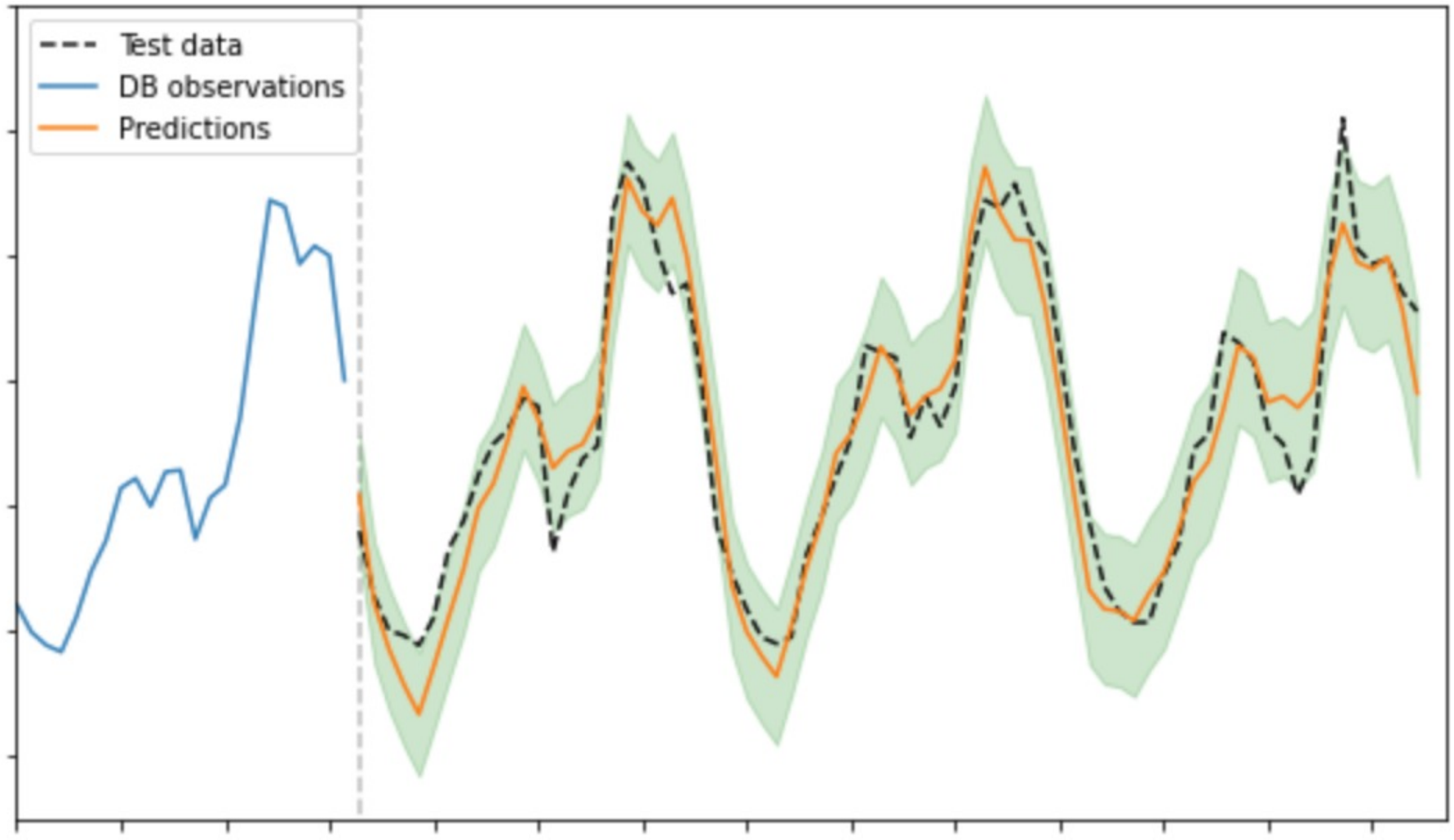
Does it really work?

Forecasting electricity consumption

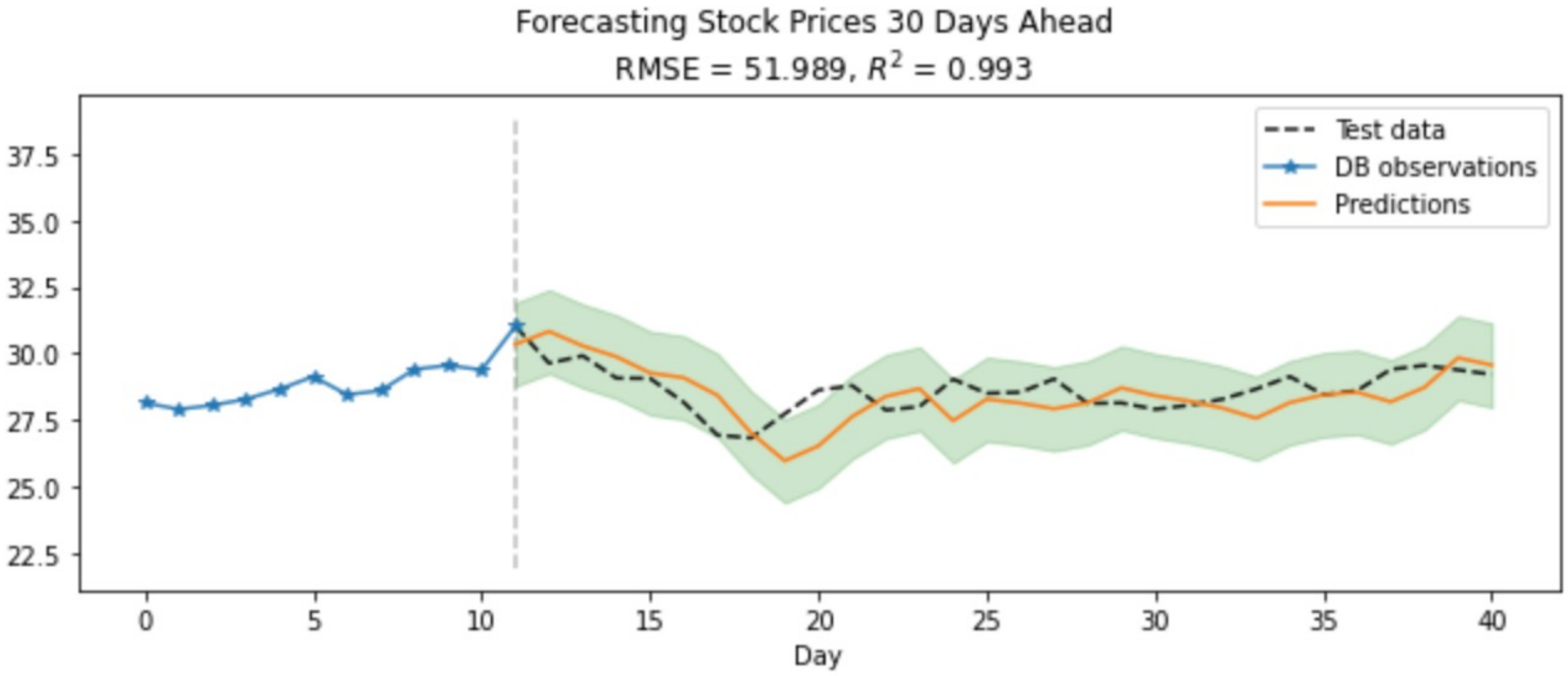


For more, see:

tspdb.mit.edu/#demo

Does it really work?

Financial data



For more, see:

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Does it really work?

	Mean Imputation (NRMSE)				Mean Forecasting (NRMSE)			
	Electricity	Traffic	Synthetic	Financial	Electricity	Traffic	Synthetic	Financial
mSSA	**0.391**	**0.494**	**0.253**	**0.283**	**0.483**	0.525	**0.196**	**0.358**
SSA	0.519	0.608	0.626	0.466	0.552	0.704	0.522	0.592
LSTM	NA	NA	NA	NA	0.551	**0.473**	0.444	1.203
DeepAR	NA	NA	NA	NA	0.484	0.474	0.331	0.395
TRMF	0.694	0.512	0.325	0.513	0.534	0.570	0.267	0.464
Prophet	NA	NA	NA	NA	0.582	0.617	1.005	1.296

Out-performs:

DeepAR from Amazon

Prophet from Facebook

LSTM, the *shiny* neural network approach

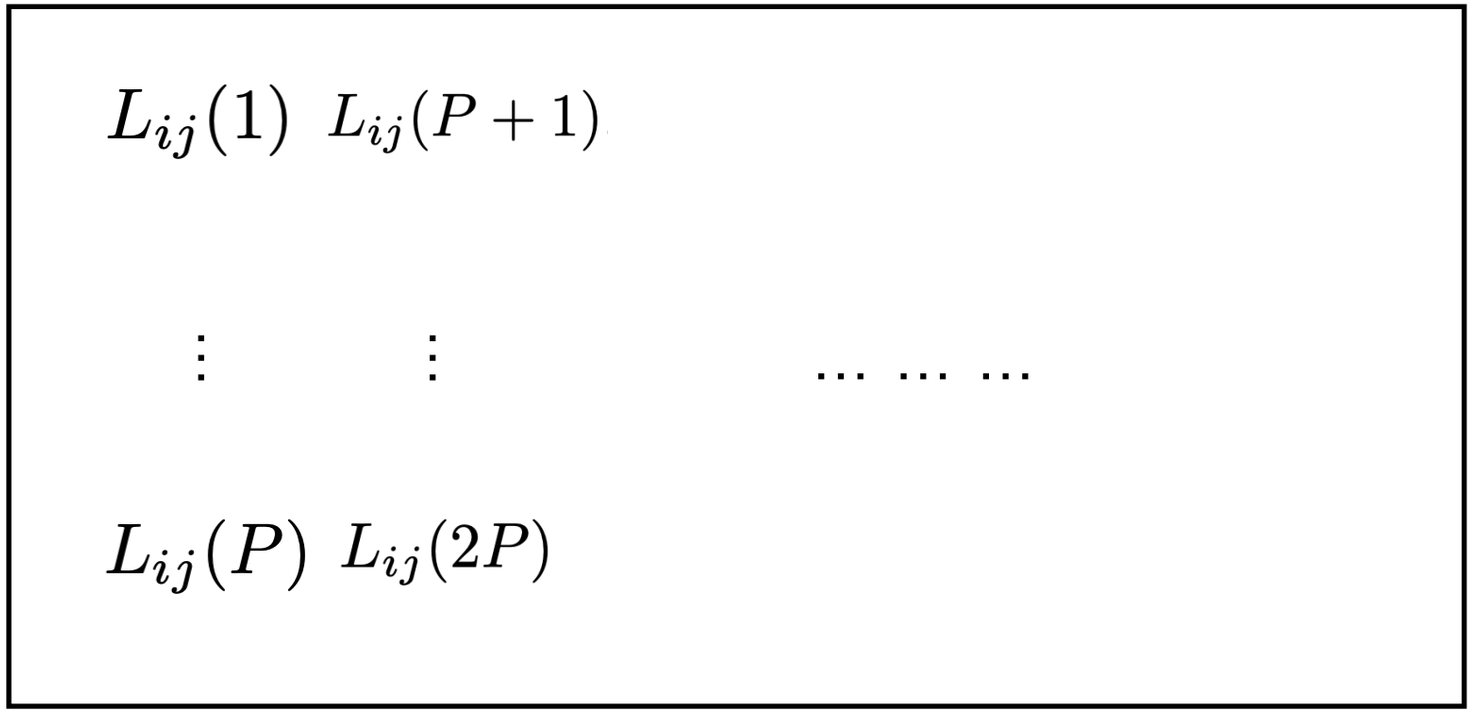
Easy to use implementation: tspdb.mit.edu

NeurIPS 2020 demo: https://www.powtoon.com/s/fkVh3axA4Jy/1/m

Back to Matrices Changing with Time

Step 1. Convert time series for each component into a (Page) matrix

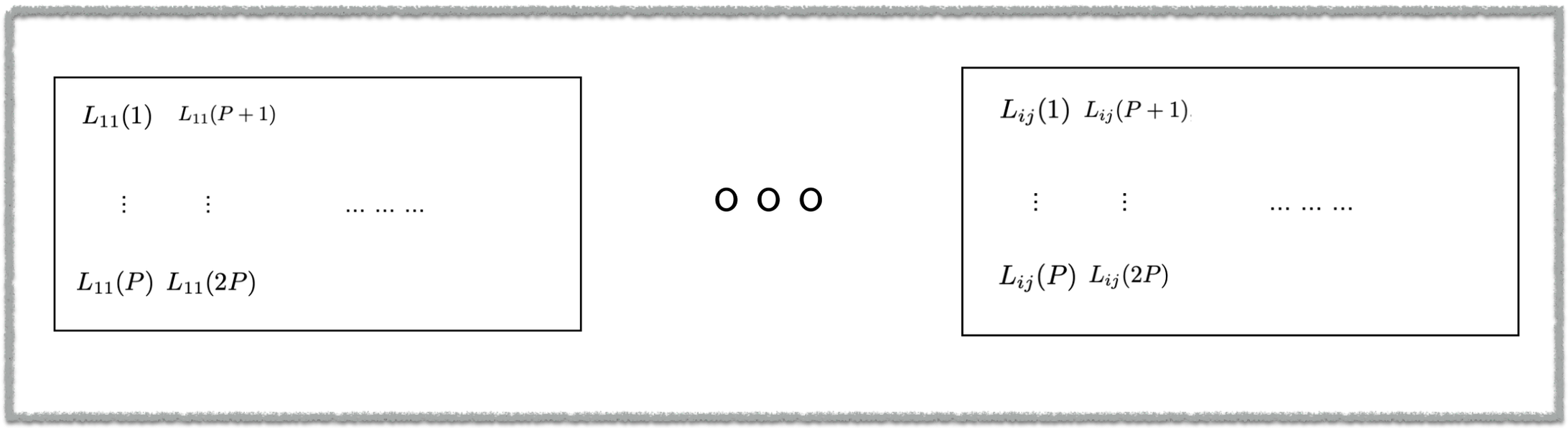
$$\boxed{L_{ij}(1), \cdots, L_{ij}(P)} \boxed{L_{ij}(P+1), \cdots, L_{ij}(2P)} \quad \cdots \cdots \cdots$$



Matrix Estimation over Time: Algorithm

Step 2. Concatenate matrices of all components along columns

observations over total of T time units



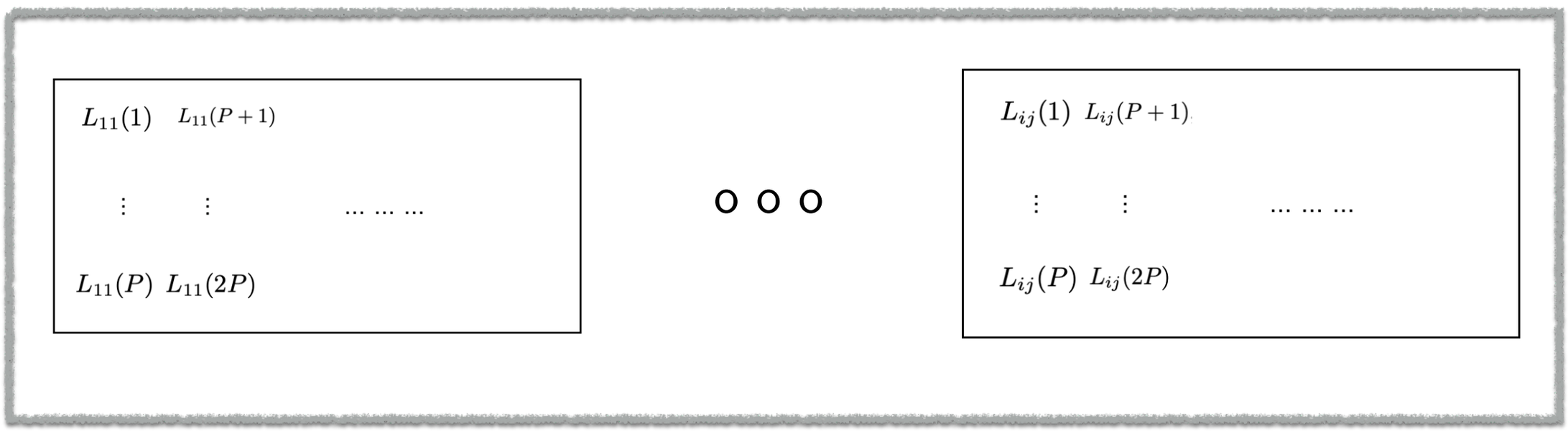
This combined matrix, say Z, has

P rows

T/P x N x M columns

Matrix Estimation over Time: Algorithm

Step 3. Perform matrix estimation over Z



Let estimated matrix from Z is $\widehat{Z}$

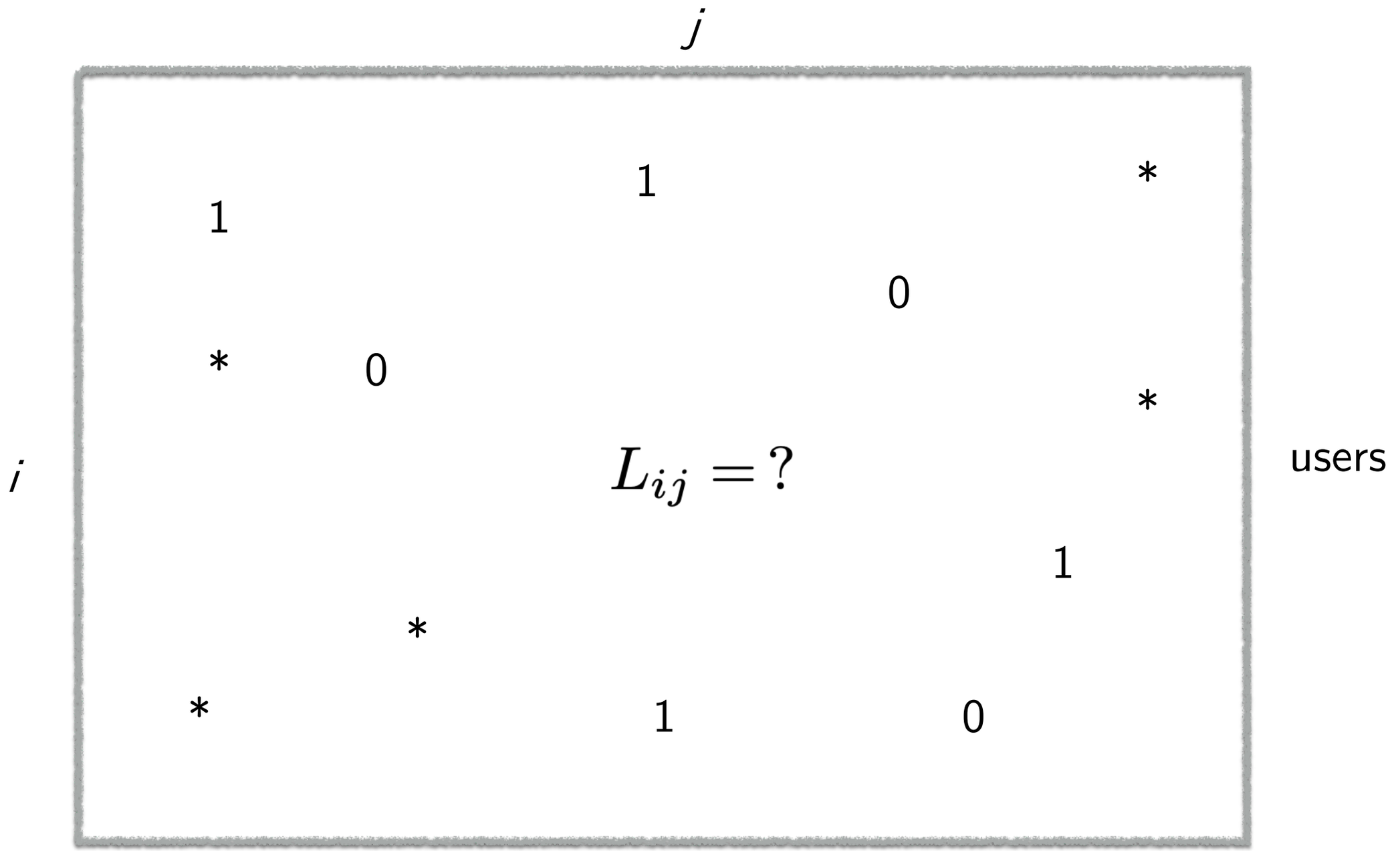
By reversing the map we can obtain

$$\widehat{L}_{ij}(t),\ t \in [T],\ i \in [N],\ j \in [M]$$

Module 3: Everything together

Recommendation: Problem statement

Prediction problem: complete the matrix



Recommendation: Model 1

Content-based

Observed features

y_j

Observed features

x_i

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$$\text{e.g. } L_{ij} = \alpha x_i + \beta y_j + \gamma$$

$$\text{or } L_{ij} = \frac{\exp(\alpha x_i + \beta y_j + \gamma)}{1 + \exp(\alpha x_i + \beta y_j + \gamma)}$$

user i

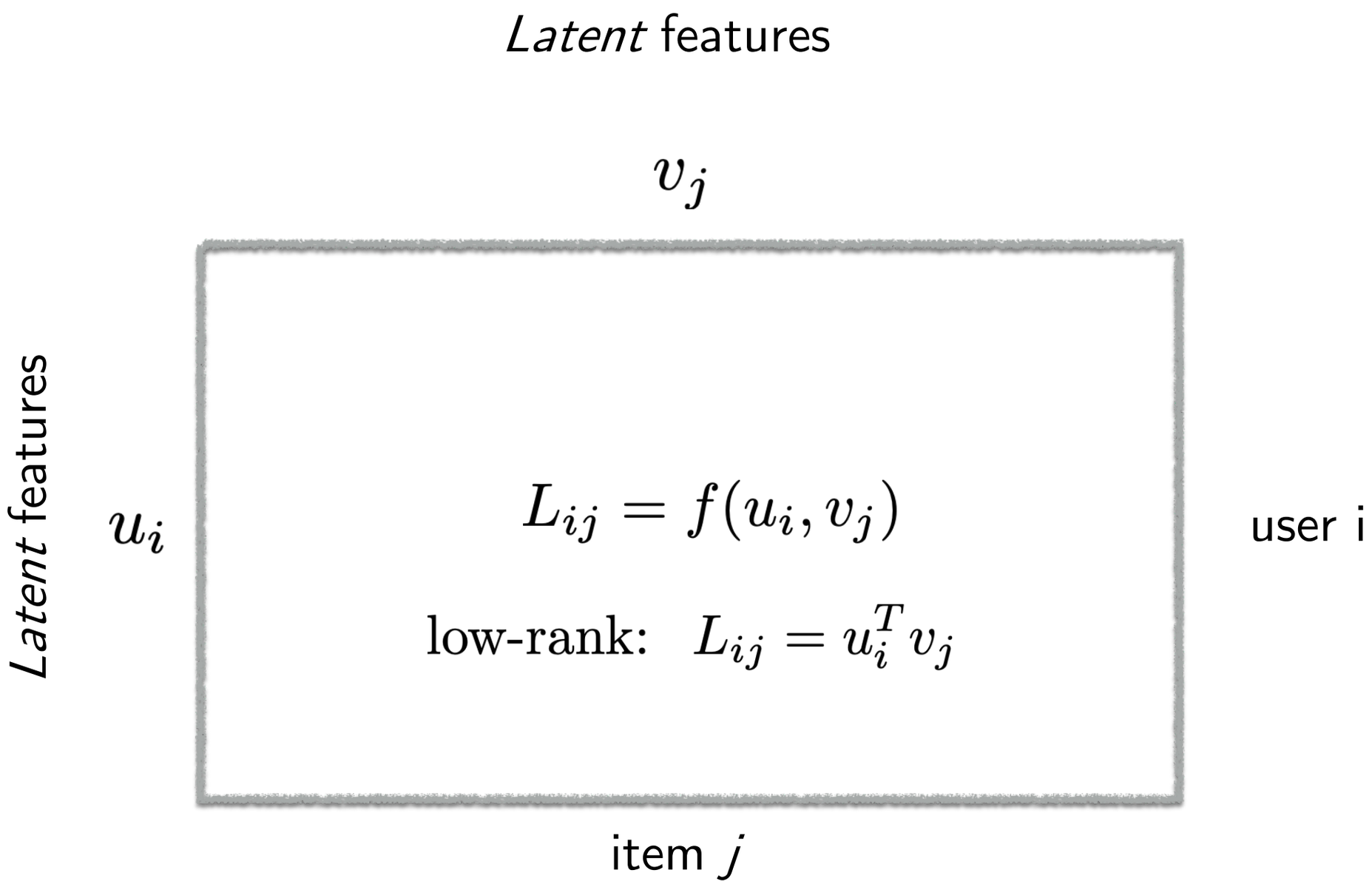
item *j*

Problem reduces to learning model *f*

That is, traditional supervised learning (regression or classification)

Recommendation: Model 2

Matrix estimation



Problem reduces to learning "factorization" of the matrix

either through similarities or algebraic approaches

Recommendation: Model 3

Matrix estimation over time

Latent features

$v_j(t)$

Latent features $u_i(t)$

$$L_{ij}(t) = f(u_i(t), v_j(t))$$

$$\text{low-rank:} \quad L_{ij}(t) = u_i^T(t) v_j(t)$$

$$u_i(\cdot),\ v_j(\cdot) \text{ time-series}$$

user i

item *j*

Problem reduces estimating time varying matrix where

Latent factors are time varying, observations are partial and noisy

Recommendation: Model 4

Multiple measurements

Recommendation: Model Everything

Put everything together

Users: N, Items: M

Time horizon: T

Measurements: K

Quantity of interest: user i, item j, measurement k at time t

$$L_{ijk}(t)$$

Problem statement:

Estimate the above using noisy, sparse observations

Recommendation: Model Everything

The model

$$L_{ijk}(t) = f^k_{\text{obs}}(x_i, y_j) + f^k_{\text{latent}}(u_i(t), v_j(t))$$

$$u_i(\cdot),\ v_j(\cdot) \text{ time-series}$$

A useful special instance

$$f^k_{\text{obs}}(x_i, y_j) = \alpha^k x_i + \beta^k y_j + \gamma^k$$

$$f^k_{\text{latent}}(u_i(t), v_j(t)) = \sum_{\ell=1}^{d} u_{i\ell}(t) v_{j\ell}(t) w_{k\ell}$$

Recommendation: Algorithm

Step 1. Content based learning

For each measurement k, learn via supervised learning

$$f^k_{\text{obs}}, \text{ that is } (\alpha^k, \beta^k, \gamma^k)$$

Step 2. Obtain difference (not learnt through content)

$$L^{\text{diff}}_{ijk}(t) = L_{ijk}(t) - L^{\text{obs}}_{ijk}$$

where $L^{\text{obs}}_{ijk} = f^k_{\text{obs}}(x_i, y_j)$

Recommendation: Algorithm

Step 3. Build stacked Page matrice across entries, slices of tensor

That is, for measurement k, create Page matrix with

P rows, T/P x N x M columns

$$\begin{matrix} L_{ijk}(1) & L_{ijk}(P+1) & \\ \vdots & \vdots & \dots \ \dots \ \dots \\ L_{ijk}(P) & L_{ijk}(2P) & \end{matrix}$$

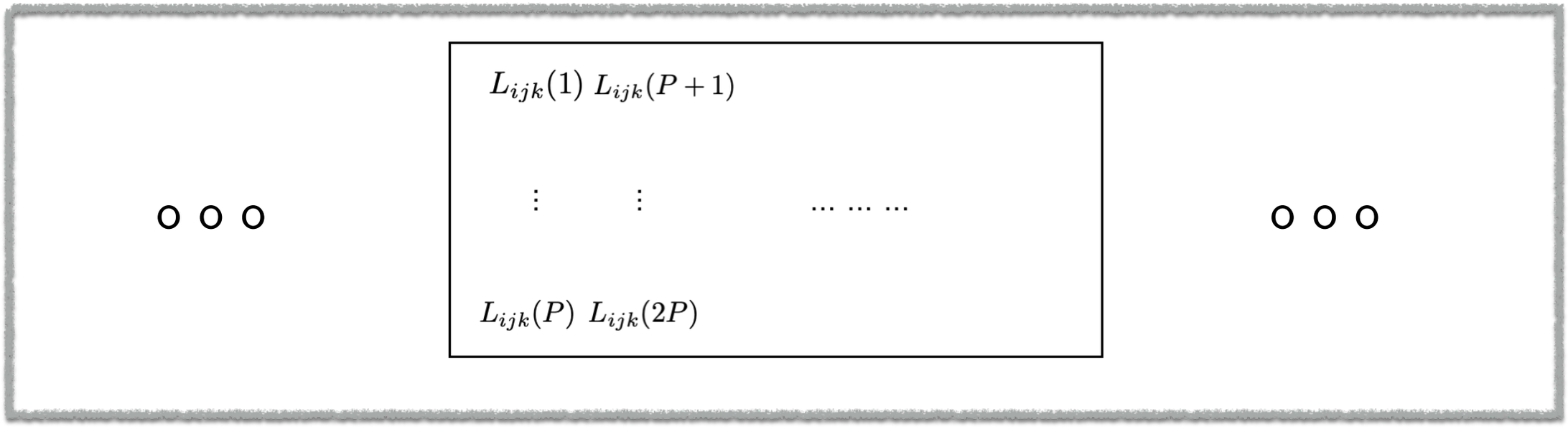
Recommendation: Algorithm

Step 3. Build stacked Page matrice across entries, slices of tensor

That is, for measurement k, create Page matrix with

P rows, T/P x N x M columns

Call it Z^k



Recommendation: Algorithm

Step 3. Build stacked Page matrice across entries, slices of tensor

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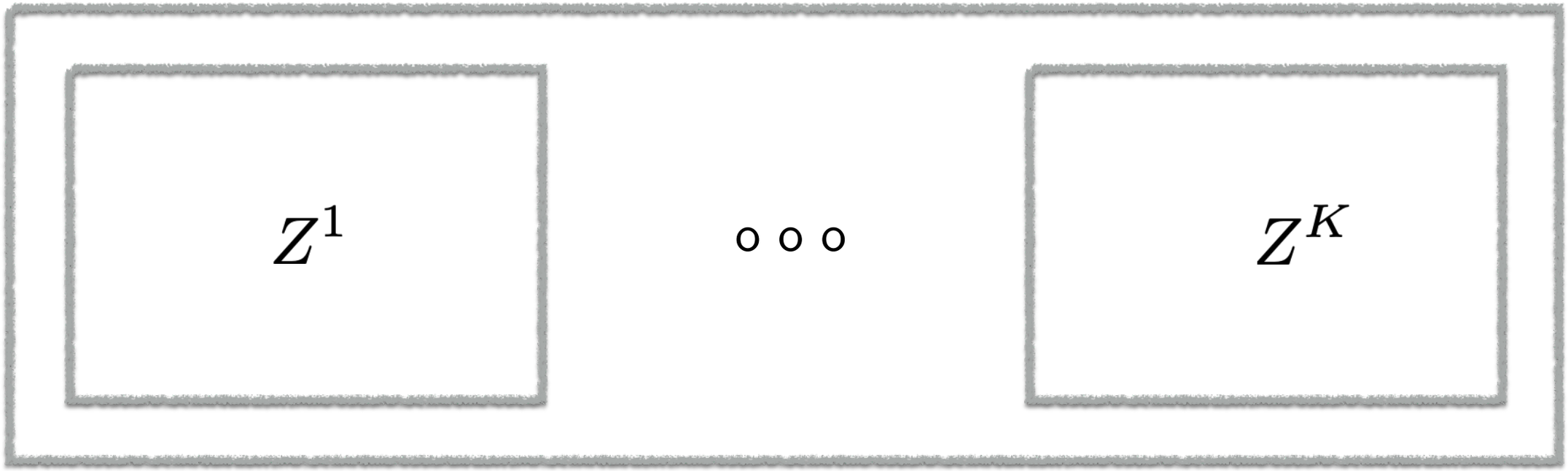

Step 4. Perform matrix estimation on it to obtain

$$\widehat{L}^{\text{diff}}_{ijk}(t)$$

Recommendation: Algorithm

Step 5. Final estimate

$$\widehat{L}_{ijk}(t) = \widehat{L}^{\text{diff}}_{ijk}(t) + L^{\text{obs}}_{ijk}$$

Some remarks:

- We *flattened* tensor in matrix to estimate to $\widehat{L}^{\text{diff}}_{ijk}(t)$
- In extremely sparse data regime, directly estimating tensor can help
- But, it comes at the cost of increased computation

In Summary — Part I

Introduction, simple methods

- Module 1: background
  - Recommendation systems: why and what?
  - Example datasets: Yelp & MovieLens
- Module 2: problem statement
  - Recommendation systems: a prediction problem
  - Model: estimating time varying tensor with side information
- Module 3: simple solutions
  - Solution I: averaging
  - Solution II: content-based

In Summary — Part II

Solution evolves, Matrix estimation

- Module 1: clustering
  - Finding user and item clusters
  - Averaging *within* clusters
- Module 2: collaborative filtering aka personalized clustering
  - Finding users and items similar to a given user, item
  - Averaging *amongst* user-item specific *similar* users, items
- Module 3: singular value thresholding, optimization
  - It's a Matrix! find Singular Value Decomposition
  - Solve for least-squares

In Summary — Part III

Getting things together, some more and connections

- Module 1: Matrix estimation meets content-based
  - Model
  - Estimation algorithm
- Module 2: Matrix estimation across time
  - Model
  - Estimation algorithm
- Module 3: Everything together
  - Model
  - Estimation algorithm

That's all, Folks!