GD method for the 1D linear advection equation

1. Problem overview

The linear advection equation is of the following form:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

where a is the advection speed and $u = u(x, t)$ is the unknown scalar function. This equation is hyperbolic.

For the initial conditions problem we have to specify the initial conditions $u(x, t) \mid_{t=0} = u_0(x)$, this can be solved analytically using the method of charcteristics. We will be solving this problem on an interval $I = \langle 0, L \rangle$, with the boundary condition on the left border (assuming $a > 0$) $u(x, t) \mid_{x=0} = \alpha(t)$.

2. Discontinous Galerkin Method

2.1. Spatial discretization

We split our domain (interval I) into N elements $I_j = \langle x_{j-\frac{2}{2}}, x_{j+\frac{1}{2}} \rangle$, $j = 1, 2, ..., N$, here x_j is the center of the element. We will use constant spatial step $h = \frac{L}{N}$, that means that $x_i = ih$. In each element we will be trying to approximate the solution with a polynomial of degree k.

We define a function space

$$V_h^k = \left\{ v \in L^2(I) : v \mid_{I_j} \in P_k\left(I_j\right), j = 1, 2, ..., N \right\},$$

where $P_k\left(I_j\right)$ is a space of polynomials of at most degree k on the interval I_j.

2.2. Weak formulation of the problem

To obtain the weak formulation we multiply the equation by a test function $v \in V_h^k$ and integrate over the interval I_j.

$$\int_{I_j} \frac{\partial u_h}{\partial t} v \,\mathrm{d}x + \int_{I_j} \frac{\partial u_h}{\partial x} v \,\mathrm{d}x = 0.$$

Using per-partes on the second integral we obtain

$$\int_{I_j} \frac{\partial u_h}{\partial t} v \,\mathrm{d}x - \int_{I_j} \frac{\partial v}{\partial x} u_h \,\mathrm{d}x + [a u_h v]_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} = 0$$

Now we have to define $u_h\left(x_{i+\frac{1}{2}}\right)$ to do that we define suitable numerical flux.

$$u_h\left(x_{x+\frac{1}{2}}\right) = \hat{u}_h\left(x_{x+\frac{1}{2}}\right)$$

One such numerical flux could be the well known upwind ($a > 0$)

$$\hat{u}\left(x_{x+\frac{1}{2}}\right) = u_h^-\left(x_{x+\frac{1}{2}}\right) = \lim_{x \to x_{j+\frac{1}{2}}} u_{h(x)}$$

The integral equation then becomes

$$\int_{I_j} \frac{\partial u_h}{\partial t} v \,\mathrm{d}x - \int_{I_j} \frac{\partial v}{\partial x} u_h \,\mathrm{d}x + \left[a\hat{u}_h\left(x_{j-\frac{1}{2}}\right)v\left(x_{j-\frac{1}{2}}\right) - a\hat{u}_h\left(x_{j+\frac{1}{2}}\right)v\left(x_{j+\frac{1}{2}}\right)\right] = 0$$

If we sum all the elements we get the following equation

$$\sum_{j=1}^{N}\left(\int_{I_j} \frac{\partial u_h}{\partial t} v \,\mathrm{d}x - \int_{I_j} \frac{\partial v}{\partial x} u_h \,\mathrm{d}x + \left[a\hat{u}_h\left(x_{j-\frac{1}{2}}\right)v\left(x_{j-\frac{1}{2}}\right) - a\hat{u}_h\left(x_{j+\frac{1}{2}}\right)v\left(x_{j+\frac{1}{2}}\right)\right]\right) = 0$$

2.3. Basis functions

When choosing the basis functions for the space V_h^k we have many options, for example

- Monomial basis
- Legendre polynomials
- Lagrange polynomials in Gauss points

Let us focus on the monomial basis.

2.3.1. Monomial basis

For easier computation we construct our basis on a so called refrence element $\hat{I} = \langle -1, 1\rangle$ and then we map it onto each element I_j using a affine transformation. Which is given by

$$x = x_j + \frac{h}{2}\xi,$$

where $\xi \in \hat{I}$ is a local variable on $\hat{I}$. The monomial basis then is

$$\hat{\varphi}^m(\xi) = \xi^m, \quad m = 1, 2, ...k$$

For computation itself we need to evaluate integrals of a product of two basis functions on I_j, we do that by substitution onto the reference interval

$$\int_{I_j} \varphi_j^m \varphi_j^n \,\mathrm{d}x = \int_{-1}^{1} \hat{\varphi}^m \hat{\varphi}^n \frac{h}{2} \,\mathrm{d}\xi =$$

$$= \frac{h}{2}\int_{-1}^{1} \xi^{m+n} \,\mathrm{d}\xi = \begin{cases} \frac{h}{m+n+1} & : \quad (m+n)\%2 = 0 \\ 0 & : \quad (m+n)\%2 \neq 0 \end{cases}$$

These form the mass matrix M.

Next we will also need integrals involving a derivative of a basis function

$$\int_{I_j} \varphi_j^m \frac{\mathrm{d}\varphi_j^n}{\mathrm{d}x} \,\mathrm{d}x = \int_{\hat{I}} \hat{\varphi}^m \frac{\mathrm{d}\hat{\varphi}^n}{\mathrm{d}\xi} \frac{2}{h} \frac{h}{2} \,\mathrm{d}\xi =$$

$$= \int_{\hat{I}} \xi^m n \xi^{n-1} \,\mathrm{d}\xi = \begin{cases} 0 & : \quad (m+n)\%2 = 0 \\ \frac{2n}{m+n} & : \quad (m+n)\%2 \neq 0 \end{cases}$$

These integrals form the matrix C.

Next we need to know the values of the basis functions on the cell boundaries:

$$\varphi_j^m\left(x_{j-\frac{1}{2}}\right) = (-1)^m,$$

$$\varphi_j^m\left(x_{j+\frac{1}{2}}\right) = 1$$

those values will form the matrix B.

2.3.2. Legender polynomials

2.3.3. Lagrange polynomials

2.4. Setting up discretizied system

We expect the approximation on element I_j to be a linear combination of the basis functions:

$$u_h(x,t)\mid_{I_j} = \sum_{m=0}^{k} U_j^m(t)\varphi_j^m(x),$$

where U_j^m is a vector of coefficients of the approximate solution on element I_j.

We also expect the test function to be of the same form, their vector of coefficients is V_j^m

Now we can discretize our equation in the weak formulation:

1. $$\int_{I_j} \frac{\partial u_h}{\partial t} v\,\mathrm{d}x = \int_{I_j} \frac{\partial}{\partial t}\left(U_j^m \varphi_j^m\right) V_j^n \varphi_j^n\,\mathrm{d}x = \int_{I_j} \frac{\mathrm{d}U_j^m}{\mathrm{d}t} V_j^n \varphi_j^m \varphi_j^n\,\mathrm{d}x = \left(\frac{\mathrm{d}U_j}{\mathrm{d}t}\right)^T M V_j$$
2. $$\int_{I_j} -a u_h \frac{\partial v}{\partial x}\,\mathrm{d}x = -a \int_{I_j} U_j^m \varphi_j^m \frac{\partial V_j^n \varphi_j^n}{\partial x}\,\mathrm{d}x = -a \int_{I_j} U_j^m V_j^n \varphi_j^m \frac{\mathrm{d}\varphi_j^n}{\mathrm{d}x}\,\mathrm{d}x = -a U_j C V_j$$
3. $$a\hat{u}_h\left(x_{j+\frac{1}{2}}\right) v\left(x_{j+\frac{1}{2}}\right) - a\hat{u}_h\left(x_{j-\frac{1}{2}}\right) v\left(x_{j-\frac{1}{2}}\right) = B \begin{pmatrix} -a\hat{u}_h\left(x_{j-\frac{1}{2}}\right) \\ a\hat{u}_h\left(x_{j+\frac{1}{2}}\right) \end{pmatrix}$$

Thus we get the following matrix equation

$$\left(\frac{\mathrm{d}U_j}{\mathrm{d}t}\right)^T M V_j - a U_j C V_j + B \begin{pmatrix} -a\hat{u}_h\left(x_{j-\frac{1}{2}}\right) \\ a\hat{u}_h\left(x_{j+\frac{1}{2}}\right) \end{pmatrix} = 0$$

$$\left(\frac{\mathrm{d}U_j}{\mathrm{d}t}\right)^T M - a U_j C + B \begin{pmatrix} -a\hat{u}_h\left(x_{j-\frac{1}{2}}\right) \\ a\hat{u}_h\left(x_{j+\frac{1}{2}}\right) \end{pmatrix} = 0$$

2.5. Time discretization

To solve the equation for the coefficients $U_j(t)$ we apply RK2 method.

$$\left(\frac{\mathrm{d}U_j}{\mathrm{d}t}\right)^T = M^{-1} a U_j C - M^{-1} B \begin{pmatrix} -a\hat{u}_h\left(x_{j-\frac{1}{2}}\right) \\ a\hat{u}_h\left(x_{j+\frac{1}{2}}\right) \end{pmatrix} = F\left(U_j\right)$$

$$T_1 = U_j^i + \Delta t F\left(U_j\right)$$

$$T_2 = \left(3U_j^i + T_1 + \Delta t F(T_1)\right)\frac{1}{4}$$

$$U_j^{i+1} = \left(U_j^i + 2T_2 + 2\delta t F(T_2)\right)\frac{1}{3}$$