GD method for the 1D linear advection equation

1. Problem overview

The linear advection equation is of the following form:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

where a is the advection speed and $u = u(x, t)$ is the unknown scalar function. This equation is hyperbolic.

For the initial conditions problem we have to specify the initial conditions $u(x, t) \mid_{t=0} = u_0(x)$, this can be solved analytically using the method of charcteristics. We will be solving this problem on an interval $I = \langle 0, L \rangle$, with the boundary condition on the left border (assuming $a > 0$) $u(x, t) \mid_{x=0} = \alpha(t)$.

2. Discontinous Galerkin Method

2.1. Spatial discretization

We split our domain (interval I) into N elements $I_j = \langle x_{j-\frac{2}{2}}, x_{j+\frac{1}{2}} \rangle$, $j = 1, 2, ..., N$, here x_j is the center of the element. We will use constant spatial step $h = \frac{L}{N}$, that means that $x_i = ih$. In each element we will be trying to approximate the solution with a polynomial of degree k.

We define a function space

$$V_h^k = \left\{ v \in L^2(I) : v \mid_{I_j} \in P_k\left(I_j\right), j = 1, 2, ..., N \right\},$$

where $P_k\left(I_j\right)$ is a space of polynomials of at most degree k on the interval I_j.

2.2. Weak formulation of the problem

To obtain the weak formulation we multiply the equation by a test function $v \in V_h^k$ and integrate over the interval I_j.

$$\int_{I_j} \frac{\partial u_h}{\partial t} v \,\mathrm{d}x + \int_{I_j} \frac{\partial u_h}{\partial x} v \,\mathrm{d}x = 0.$$

Using per-partes on the second integral we obtain

$$\int_{I_j} \frac{\partial u_h}{\partial t} v \,\mathrm{d}x - \int_{I_j} \frac{\partial v}{\partial x} u_h \,\mathrm{d}x + \left[a u_h v\right]_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} = 0$$

Now we have to define $u_h\left(x_{i+\frac{1}{2}}\right)$ to do that we define suitable numerical flux.

$$u_h\left(x_{x+\frac{1}{2}}\right) = \hat{u}_h\left(x_{x+\frac{1}{2}}\right)$$

One such numerical flux could be the well known upwind ($a > 0$)

$$\hat{u}\left(x_{x+\frac{1}{2}}\right) = u_h^-\left(x_{x+\frac{1}{2}}\right) = \lim_{x \to x_{j+\frac{1}{2}}} u_{h(x)}$$

The integral equation then becomes

$$\int_{I_j} \frac{\partial u_h}{\partial t} v \, \mathrm{d}x - \int_{I_j} \frac{\partial v}{\partial x} u_h \, \mathrm{d}x + \left[a\hat{u}_h \left(x_{j-\frac{1}{2}} \right) v \left(x_{j-\frac{1}{2}} \right) - a\hat{u}_h \left(x_{j+\frac{1}{2}} \right) v \left(x_{j+\frac{1}{2}} \right) \right] = 0$$

If we sum all the elements we get the following equation

$$\sum_{j=1}^{N} \left(\int_{I_j} \frac{\partial u_h}{\partial t} v \, \mathrm{d}x - \int_{I_j} \frac{\partial v}{\partial x} u_h \, \mathrm{d}x + \left[a\hat{u}_h \left(x_{j-\frac{1}{2}} \right) v \left(x_{j-\frac{1}{2}} \right) - a\hat{u}_h \left(x_{j+\frac{1}{2}} \right) v \left(x_{j+\frac{1}{2}} \right) \right] \right) = 0$$

2.3. Basis functions

When choosing the basis functions for the space V_h^k we have many options, for example

- Monomial basis
- Legendre polynomials
- Lagrange polynomials in Gauss points

Let us focus on the monomial basis.

2.3.1. Monomial basis

For easier computation we construct our basis on a so called refrence element $\hat{I} = \langle -1, 1 \rangle$ and then we map it onto each element I_j using a affine transformation. Which is given by

$$x = x_j + \frac{h}{2} \xi,$$

where $\xi \in \hat{I}$ is a local variable on $\hat{I}$. The monomial basis then is

$$\hat{\varphi}^m(\xi) = \xi^m, \quad m = 1, 2, ...k$$

For computation itself we need to evaluate integrals of a product of two basis functions on I_j, we do that by substitution onto the reference interval

$$\int_{I_j} \varphi_j^m \varphi_j^n \, \mathrm{d}x = \int_{-1}^{1} \hat{\varphi}^m \hat{\varphi}^n \frac{h}{2} \, \mathrm{d}\xi =$$

$$= \frac{h}{2} \int_{-1}^{1} \xi^{m+n} \, \mathrm{d}\xi = \begin{cases} \frac{h}{m+n+1} & : \quad (m+n)\%2 = 0 \\ 0 & : \quad (m+n)\%2 \neq 0 \end{cases}$$

These form the mass matrix M.

Next we will also need integrals involving a derivative of a basis function

$$\int_{I_j} \varphi_j^m$$

2.4. Time discretization