Calculus 3 Notes

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June 10, 2024

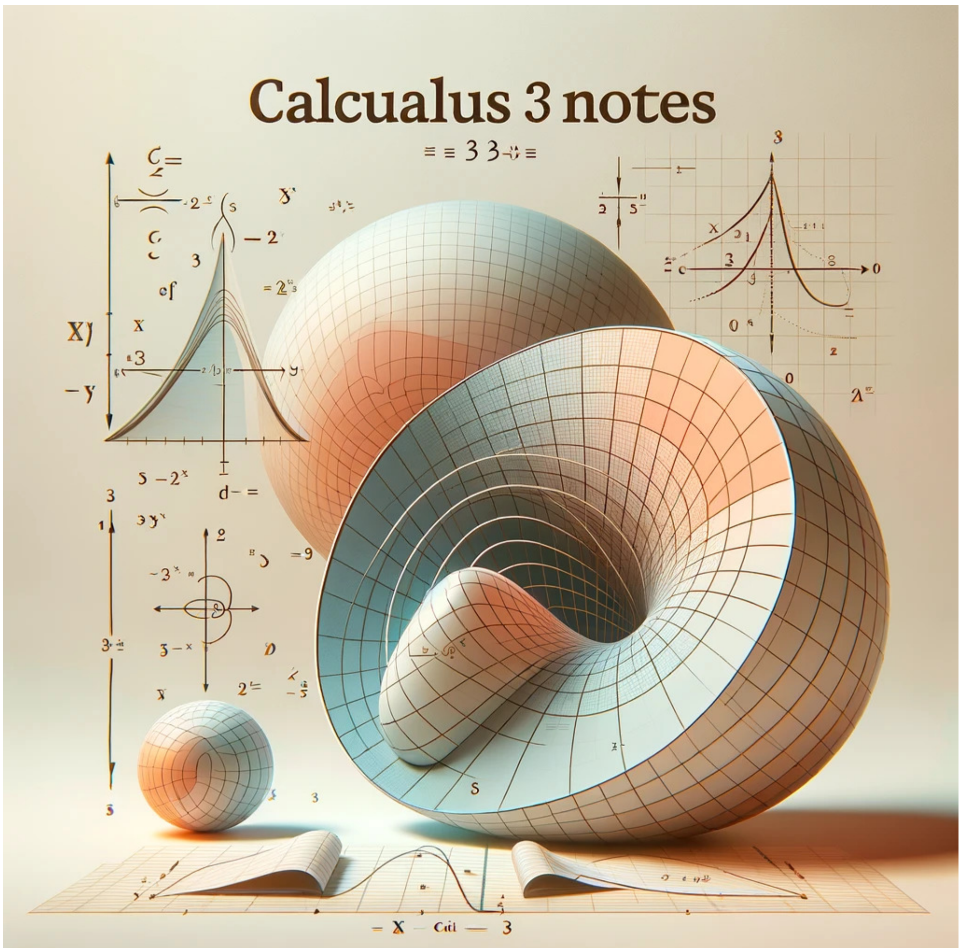

Figure 1: ChatGPT Visualization of this Topic

1 Vectors

1.1 Dot Product

The dot product of two vectors is defined as:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1}$$

Where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

1.2 Cross Product

The cross product of two vectors is defined as:

$$\vec{a} \times \vec{b} = \langle (a_2 b_3 - a_3 b_2), (a_3 b_1 - a_1 b_3), (a_1 b_2 - a_2 b_1) \rangle \tag{2}$$

Where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

The cross product $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$, therefore:

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0 \quad \text{and} \quad \vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

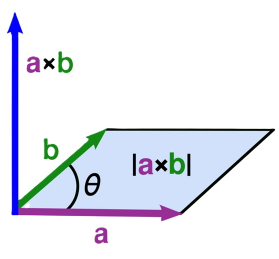

Figure 2: Cross Product in Vector Algebra

1.3 Vector Magnitude

The magnitude of a vector $\vec{a}$ is given by:

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{3}$$

2 Practical Calculations in 3D Space

2.1 Projection of a Vector

The projection of a vector $\vec{a}$ onto a vector $\vec{b}$ is given by:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\vec{b} \tag{4}$$

2.2 Find the Unit Vector

The unit vector of a vector $\vec{a}$ is given by:

$$\hat{a} = \frac{1}{\|\vec{a}\|} \cdot \vec{a} \tag{5}$$

2.3 Find the Angle Between Two Vectors

The angle between two vectors $\vec{a}$ and $\vec{b}$ is given by:

$$\theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) \tag{6}$$

2.4 Area of a Parallelogram

The area of a parallelogram spanned by two vectors $\vec{a}$ and $\vec{b}$ is given by:

$$\text{Area} = \|\vec{a} \times \vec{b}\| \tag{7}$$

2.5 Find distance of point to line

Given points P, Q, and R where P is the point, and Q and R define the line $\overleftrightarrow{QR}$, the distance from P to the line is given by:

$$\text{Distance} = \frac{\|\vec{QP} \times \vec{QR}\|}{\|\vec{QR}\|} = \frac{\text{Area}}{\text{Base}} = \text{Perpendicular Height} \tag{8}$$

2.6 Find the equation of a plane

Given a point P and a normal vector $\vec{n}$, the equation of a plane is given by:

$$\vec{n} \cdot \vec{PQ} = 0 \tag{9}$$

Where $\vec{PQ}$ is the vector from point P to any point Q on the plane.

$$\therefore \vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \tag{10}$$

Where (x_0, y_0, z_0) is the point P. Solve for the equation in the format $ax + by + cz + d = 0$.

2.7 Find distance of Point to Plane

Given a point P and a plane $ax + by + cz + d = 0$, the distance from the point to the plane is given by:

$$\text{Distance} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{11}$$

Where (x_0, y_0, z_0) is the point P.

3 Determining 3-Demensional Shapes

3.1 Breaking Down a 3D Shape into 2D Planes

We replace each value x, y, and z in a 3D shape with a constant k to determine the shape. Given the equation $x^2 - 2y^2 - z^2 = 1$, we can replace x, y, and z with k to get the following:

$$\text{(xy-plane:)} \quad x^2 - 2y^2 = 1 + k^2 \qquad \text{(hyperbolic)} \tag{12}$$

$$\text{(xz-plane:)} \quad x^2 - z^2 = 1 + 2k^2 \qquad \text{(hyperbolic)} \tag{13}$$

$$\text{(yz-plane:)} \quad 2y^2 + z^2 = k^2 - 1 \qquad \text{(ellipse, where } k^2 - 1 > 0\text{)} \tag{14}$$

Therefore we can conclude that this shape is an **elliptical hyperboloid of one sheet**. This is not to be mistaken with an **elliptical hyperboloid of two sheets**, exemplified by the equation $(x^2 - 2y^2 - z^2 = 1)$, where:

$$\text{(xy-plane:)} \quad x^2 + 2y^2 = 1 + k^2 \qquad \text{(ellipse)} \tag{15}$$

$$\text{(xz-plane:)} \quad x^2 - z^2 = 1 - 2k^2 \qquad \text{(hyperbolic)} \tag{16}$$

$$\text{(yz-plane:)} \quad 2y^2 - z^2 = 1 - k^2 \qquad \text{(hyperbolic)} \tag{17}$$

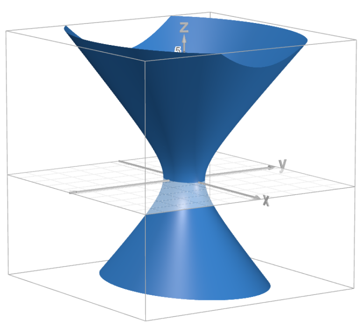

Figure 3: Elliptical Hyperboloid of One Sheet
ex. $x^2 + 2y^2 - z^2 = 1$

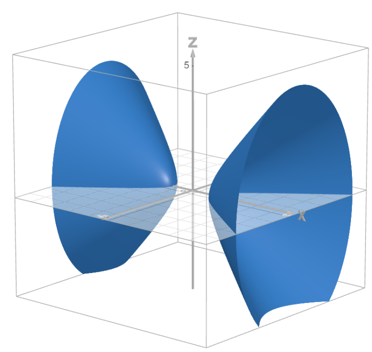

Figure 4: Elliptical Hyperboloid of Two Sheets
ex. $x^2 - 2y^2 - z^2 = 1$

3.2 Shapes of 2D Planes in 3D Space

1. $ax^2 + by^2 = C + ck^n$: Ellipse
2. $ax^2 + by^2 = C - ck^n$: Ellipse (Where $C - ck^n > 0$)
3. $ax^2 - by^2 = C \pm ck^n$: Hyperbolic
4. $ax^2 - by = C \pm ck^n$: Parabolic
5. $ax \pm by = ck^n$: Line

4 Arc Length

The arc length of a curve C defined by the vector function $\vec{r}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$ is given by:

$$L = \int_a^b \|\vec{r}'(t)\| \, dt \tag{18}$$

Where $\vec{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$$\therefore \|\vec{r}'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \tag{19}$$

4.1 Vector Velocity

The velocity of a particle moving along a curve C is given by the derivative of the vector function $\vec{r}(t)$:

$$\vec{v}(t) = \vec{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \tag{20}$$

$$\therefore \|\vec{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \|\vec{r}'(t)\| \tag{21}$$

5 Tangent and Normal Vectors

5.1 Tangent Vector

The tangent vector to a curve C at a point P is given by the derivative of the vector function $\vec{r}(t)$:

$$\vec{T}(t) = \frac{1}{\|\vec{r}'(t)\|} \cdot \vec{r}'(t) = \frac{1}{\|\vec{v}(t)\|} \cdot \vec{v}(t) \tag{22}$$

5.2 Normal Vector

The normal vector to a curve C at a point P is given by the derivative of the tangent vector $\vec{T}(t)$:

$$\vec{N}(t) = \frac{1}{\left\| \frac{d\vec{T}}{dt} \right\|} \cdot \frac{d\vec{T}}{dt} \tag{23}$$

5.3 Curvature

The curvature of a curve C at a point P is given by:

$$\kappa = \frac{1}{\frac{ds}{dt}} \cdot \left\| \frac{d\vec{T}}{dt} \right\| = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3} \tag{24}$$

Where $\vec{v}(t)$ is the velocity vector and $\vec{a}(t)$ is the acceleration vector.

6 Acceleration

6.1 Acceleration as Linear Components

Now that we have the unit tangent vector $\vec{T}(t)$ and the unit normal vector $\vec{N}(t)$, we can break down the acceleration vector $\vec{a}(t)$ into linear components:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt} \cdot \vec{T}\right) = \frac{d^2s}{dt^2} \cdot \vec{T} + \frac{ds}{dt} \cdot \frac{d\vec{T}}{dt} \tag{25}$$

$$\text{Since} \quad \frac{d\vec{T}}{dt} = \left\|\frac{d\vec{T}}{dt}\right\| \cdot \vec{N} \quad \text{and} \quad \left\|\frac{d\vec{T}}{dt}\right\| = \kappa \cdot \frac{ds}{dt} \tag{26}$$

$$\text{we have:} \tag{27}$$

$$\frac{d\vec{T}}{dt} = \frac{ds}{dt} \cdot \kappa \cdot \vec{N} \tag{28}$$

$$\therefore \vec{a}(t) = \frac{d^2s}{dt^2} \cdot \vec{T} + \left(\frac{ds}{dt}\right)^2 \cdot \kappa \cdot \vec{N} = a_T \cdot \vec{T} + a_N \cdot \vec{N} \tag{29}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{\vec{v}(t) \cdot \vec{a}(t)}{\|\vec{v}(t)\|} \tag{30}$$

$$a_N = \left(\frac{ds}{dt}\right)^2 \cdot \kappa \tag{31}$$

Since these two components of acceleration are orthogonal, the total acceleration is given by:

$$\|\vec{a}(t)\| = \sqrt{a_T^2 + a_N^2} \tag{32}$$