Tropical Algebra of Endogenous-Pivot Semantics

Absorbing States, Necessity, and the Record-Gap Spectrum

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Abstract

We study sequential extraction systems where a distinguished *pivot* element is selected by arg max over the realized output, and feasibility requires at least k predecessor elements before the pivot. This endogenous coupling—where the interpretation of the sequence depends on the sequence itself—breaks the hereditary and accessibility axioms while preserving the exchange property, placing the system outside both matroid and greedoid frameworks.

We formalize a three-level algebraic tower: a scalar context monoid (L1) that tracks a single committed pivot, a tropical vector summary (L2) that tracks the k-dimensional feasibility frontier, and a frontier semiring (L3) that maintains multiple competing hypotheses. We prove:

(i) L1 admits an absorbing left ideal under committed semantics—once predecessor count drops below k, no suffix can restore feasibility;

(ii) L2 is isomorphic to tropical matrix multiplication over $\mathbb{T}_{\max}^{(k+2)\times(k+2)}$, enabling $O(\log n)$ parallel reduction;

(iii) every associative, dominance-monotone summary requires $\Omega(k)$ state (Myhill–Nerode lower bound), and L2 achieves this bound;

(iv) under committed semantics with Bernoulli thinning, same-pivot recovery after infeasibility is impossible (turnover universality);

(v) the record-gap spectrum governing streaming trap rates belongs to the Ewens(θ=1) universality class via an exact Chinese Restaurant Process coupling;

(vi) the L2 recurrence time under compound streaming and thinning is $\exp(\mathrm{Li}_2(p) \cdot \mu/p)$, where Li_2 is the dilogarithm.

The L1-retry policy—accepting transient infeasibility and recovering via arg max turnover—achieves stationary feasibility $\approx 1 - k/\mu_{\mathrm{nf}}$ at $O(1)$ cost. All algebraic results are validated on synthetic event graphs (11,400+ instances, zero false positives in the absorbing-state boundary) and calibrated against multi-model LLM experiments.

1 Introduction

Greedy selection is the default strategy for sequential extraction in many domains: information retrieval, constrained decoding [20, 21], resource-constrained shortest path problems [11], and narrative generation among them. In hereditary systems with suitable exchange structure, greedy enjoys well-known optimality guarantees [4, 13]. These guarantees rest on a structural assumption: *feasibility is downward closed*, meaning that removing elements from a feasible set preserves feasibility.

We study systems where this assumption fails for a fundamental reason. The extraction task selects a subsequence from a temporally ordered event graph subject to a sequential phase grammar. The grammar designates one element as the *turning point* (pivot), and requires at least k development-phase elements to precede it. Crucially, the pivot is *endogenous*: it is the arg max-weight focal-actor event in the selected subsequence. Changing the selected set can change which element is the pivot, which changes which elements count as predecessors, which changes feasibility. This circular dependence breaks the hereditary and accessibility axioms of greedoids—while, perhaps surprisingly, the Steinitz exchange property is preserved (Theorem 3.4).

Our contributions organize into four themes.

1. The algebraic tower (Section 3). We introduce a three-level hierarchy of summary representations for contiguous event blocks. At the base, the *scalar context monoid* (L1) tracks a single committed pivot via the triple $(w^\star, d_{\text{total}}, d_{\text{pre}})$. We prove that L1 composition is associative but not right-monotone under Pareto dominance (Theorem 3.6): a heavier pivot can suppress a rightward shift that would have promoted predecessors. Under committed semantics, the set of infeasible elements forms an absorbing left ideal—once the system commits to a pivot with fewer than k predecessors, no suffix can rescue it (Theorem 3.5). The *tropical vector* (L2) lifts this obstruction by tracking the feasibility frontier across all predecessor budgets simultaneously. We prove L2 composition is monotone (Theorem 3.11). The *frontier semiring* (L3) extends L2 to maintain multiple competing pivot hypotheses as finite antichains.

2. Matrix equivalence and optimality (Sections 4 and 5). We show that L2 composition is exactly isomorphic to tropical (max-plus) matrix multiplication over $\mathbb{T}_{\max}^{(k+2)\times(k+2)}$, connecting our construction to the classical theory of max-plus linear systems [3]. This enables $O(\log n)$ parallel stream reduction via repeated squaring. We then prove that L2 is *optimal*: via a Myhill–Nerode separating-suffix argument, every associative, dominance-monotone summary that correctly tracks feasibility requires $\Omega(k)$ state, and L2 achieves this bound (Theorem 5.2).

3. The record-gap spectrum (Section 6). Under a focal-mask model where events are independently focal with probability p, we derive the exact Poisson intensities governing the non-focal gap sizes between consecutive focal records. The generating function collapse yields $\lambda_g(p) = [1-(1-p)^g]/g$. We identify an exact Chinese Restaurant Process coupling that places the gap spectrum in the Ewens$(\theta = 1)$ universality class [2, 6, 17] (Theorem 6.3). This provides the asymptotic baseline for streaming trap rates.

4. Dynamics under compression (Section 7). We analyze the compound regime where streaming pivot selection interleaves with periodic Bernoulli thinning. Under committed semantics, predecessors undergo pure death: same-pivot recovery is impossible (turnover universality, Theorem 7.4). The committed death time is logarithmic in the initial predecessor count (Theorem 7.9). The L2 recurrence time to the empty state involves the Euler dilogarithm and is astronomically large at practical retention rates (Theorem 7.8). Between these extremes, the L1-retry policy achieves stationary feasibility $1 - k/\mu_{\text{nf}}$ via a key identity $C_{k,q} = k/q$ (Theorem 7.7).

All algebraic results are validated empirically. The absorbing-state boundary is exact across 11,400 instances with zero false positives. Turnover universality holds in 103/103 observed recovery events. The 4×3 survival matrix across four semantic models and three algebraic levels is calibrated with Wilson 95% confidence intervals (Section 8).

Organization. Section 2 presents definitions. Section 3 develops the algebraic tower. Section 4 proves tropical matrix equivalence. Section 5 establishes Myhill–Nerode necessity. Section 6 derives the record-gap spectrum. Section 7 analyzes dynamics under compression. Section 8 summarizes empirical validation. Section 9 discusses open problems and Section 10 concludes.

2 Problem Formulation

Definition 2.1 (Event graph)**.** An **event graph** is a tuple $G = (V, E, t, w, a)$ where V is a finite event set; $E \subseteq V \times V$ is a set of directed causal edges that respect temporal order ($(u, v) \in E \Rightarrow t(u) < t(v)$); $t\colon V \to \mathbb{R}$ is a timestamp function inducing a total order on V (with deterministic tie-breaking by event identifier); $w\colon V \to \mathbb{R}$ is a weight function; and $a\colon V \to A$ assigns each event to an actor.

Definition 2.2 (Focal actor and endogenous pivot)**.** Given a focal actor $a^* \in A$ and a subsequence $S \subseteq V$, the **endogenous pivot** (turning point) is

$$\tau(S) \;=\; \operatorname*{arg\,max}_{v \in S,\, a(v)=a^*} \; w(v).$$

The pivot identity depends on the selected set S: adding or removing elements from S can change which event serves as the pivot.

Definition 2.3 (Phase grammar)**.** A **phase grammar** is a deterministic finite automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ over the phase alphabet $\Sigma = \{\text{SETUP}, \text{DEVELOPMENT}, \text{TURNING\_POINT}, \text{RESOLUTION}\}$, with monotone phase transitions: the DFA enforces the ordering SETUP $\to$ DEVELOPMENT $\to$ TURNING_POINT $\to$ RESOLUTION, and acceptance requires at least k DEVELOPMENT labels before the TURNING_POINT.

Definition 2.4 (Phase classifier)**.** Given a subsequence S and designated pivot $\tau \in S$, a **phase classifier** $\phi_\tau\colon S \to \Sigma$ assigns phases subject to: (i) monotone phase progression in temporal order, (ii) $\phi_\tau(\tau) = \text{TURNING\_POINT}$, and (iii) all post-$\tau$ events receive RESOLUTION. The assignment of pre-τ events to SETUP or DEVELOPMENT is constrained by the grammar's acceptance condition.

Definition 2.5 (Prefix requirement)**.** The **prefix requirement** $k \in \mathbb{N}$ is the minimum number of DEVELOPMENT-phase events that must precede the turning point for the grammar to accept.

Definition 2.6 (Candidate pool)**.** Given focal actor a^*, let $e_0 \in V$ be an anchor event selected independently of the pivot. The **candidate pool** is

$$P = \{v \in V : \text{there exists a directed path from } e_0 \text{ to } v \text{ in } (V, E)\} \cup \{e^*\},$$

where $e^* = \arg\max_{v:a(v)=a^*} w(v)$ is injected regardless of reachability. Pool construction is a modeling choice: BFS pools enforce causal locality; unrestricted pools ($P = V$) maximize temporal coverage.

Definition 2.7 (Extraction problem)**.** Given G, focal actor a^*, grammar parameter k, length limit L, minimum timespan fraction s, and maximum temporal gap $\Delta_{\max}$, find

$$S^* = \operatorname*{arg\,max}_{S \subseteq V} \sum_{v \in S} w(v)$$

subject to: (i) $|S| \leq L$; (ii) S is accepted by $\mathcal{A}$ under $\phi_{\tau(S)}$; and (iii) auxiliary temporal constraints (timespan $\geq s$ and consecutive gap $\leq \Delta_{\max}$).

Definition 2.8 (Greedy policy)**.** For each candidate pool, pre-select $e^* = \arg\max_{v:a(v)=a^*} w(v)$ as TURNING_POINT (forced pivot), inject e^* into the pool, and construct the remaining sequence by forward weight-maximizing selection subject to grammar constraints.

Definition 2.9 (Feasibility)**.** A subsequence S is **feasible** if it is accepted by $\mathcal{A}$ under the phase classifier $\phi_{\tau(S)}$. In particular, feasibility requires at least k non-focal events with timestamps preceding $\tau(S)$ in S. We write $j_{\text{dev}}(S)$ for the count of development-eligible events before $\tau(S)$; the sequence is feasible only if $j_{\text{dev}}(S) \geq k$.

Table 1: Principal notation.

Symbol	Meaning
k	Minimum predecessor count (prefix requirement)
p	Focal probability (probability an event is focal)
$q = 1 - p$	Non-focal probability
r	Retention probability per thinning epoch
T	Number of new events per epoch
$w^\star$	Maximum focal weight in block
d_{total}	Total non-focal events in block
d_{pre}	Non-focal events before the pivot
$W[j]$	Max focal weight with $\geq j$ predecessors (cumulative)
$\otimes_{\text{endo}}$	L1 endogenous composition
$\otimes^{\triangle}$	L2 tropical composition
$\mathcal{I}$	Absorbing left ideal $\{\bar{C} : \kappa = 1, d_{\text{pre}} < k\}$
$\mathbb{T}_{\max}$	Tropical (max-plus) semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$
μ_f	Stationary focal mean: $Tpr/(1-r)$
μ_{nf}	Stationary non-focal mean: $T(1-p)r/(1-r)$
π_0	Probability of zero focal events in an epoch
$\lambda_g(p)$	Poisson intensity for gap size g

3 The Algebraic Tower

This section develops the three-level algebraic hierarchy that governs endogenous-pivot semantics. We begin with the scalar monoid (L1), establish its absorbing left ideal, demonstrate its monotonicity obstruction, and then lift to the tropical vector (L2) and frontier semiring (L3).

3.1 L1: The Scalar Context Monoid

Definition 3.1 (Context element)**.** A **context element** is a triple $C = (w^\star, d_{\text{total}}, d_{\text{pre}})$ where:

(i) $w^\star \in \mathbb{R} \cup \{-\infty\}$ is the maximum weight among focal-actor events in the block (the *pivot weight*);

(ii) $d_{\text{total}} \in \mathbb{N}$ is the count of non-focal events in the block (the *total development capacity*);

(iii) $d_{\text{pre}} \in \mathbb{N}$ is the count of non-focal events whose timestamps precede the local pivot (the *pre-pivot development capacity*).

When $w^\star = -\infty$ (no focal events), we set $d_{\text{pre}} = 0$.

Definition 3.2 (Endogenous composition $\otimes_{\text{endo}}$)**.** Let $C_A = (w^\star{}_A, d_A, d^A_{\text{pre}})$ and $C_B = (w^\star{}_B, d_B, d^B_{\text{pre}})$ be context elements representing adjacent blocks (A preceding B in time). Their **endogenous composition** is $C_A \otimes_{\text{endo}} C_B = (w^\star, d_{\text{total}}, d_{\text{pre}})$ where:

$$w^\star = \max(w^\star{}_A, w^\star{}_B), \tag{1}$$

$$d_{\text{total}} = d_A + d_B, \tag{2}$$

$$d_{\text{pre}} = \begin{cases} d^A_{\text{pre}} & \text{if } w^\star{}_A \geq w^\star{}_B, \\ d_A + d^B_{\text{pre}} & \text{if } w^\star{}_B > w^\star{}_A. \end{cases} \tag{3}$$

The rule for d_{pre} encodes the pivot-shift asymmetry: when the pivot shifts to the right block, the *entire* left block's development capacity is promoted into the pre-pivot region.

Example 3.3 (Concrete walkthrough: 6-event stream at $k = 3$)**.** Consider a stream of 6 events with focal actor a^*:

$$\underbrace{e_1}_{d}\ \underbrace{e_2^*}_{w=5}\ \underbrace{e_3}_{d}\ \underbrace{e_4}_{d}\ \underbrace{e_5^*}_{w=8}\ \underbrace{e_6}_{d}$$

where d denotes non-focal (development) events and $*$ denotes focal events with the indicated weights. Partition into three blocks of two events each: $A = (e_1, e_2)$, $B = (e_3, e_4)$, $R = (e_5, e_6)$.

L1 contexts: $C_A = (5, 1, 1)$, $C_B = (-\infty, 2, 0)$, $C_R = (8, 1, 0)$.

Left-to-right composition:

1. $C_A \otimes_{\text{endo}} C_B$: $w^{\star}{}_A = 5 > -\infty$, pivot stays in A. Result: $(5, 3, 1)$.
2. $(5, 3, 1) \otimes_{\text{endo}} C_R$: $w^{\star}{}_R = 8 > 5$, pivot shifts to R. Result: $(8, 4, 3 + 0) = (8, 4, 3)$. Since $d_{\text{pre}} = 3 \geq k = 3$: **feasible**.

L2 view: The tropical contexts are:

$$T_A = ([5, 5, -\infty, -\infty],\ 1), \quad T_B = ([-\infty, -\infty, -\infty, -\infty],\ 2),$$
$$T_R = ([8, -\infty, -\infty, -\infty],\ 1).$$

Using the corrected composition rule: $T_{AB}[j] = \max(W_A[j],\ W_B[\max(0, j-1)])$ gives $T_{AB} = ([5, 5, -\infty, -\infty],\ 3)$. Then $T_{ABR}[j] = \max(W_{AB}[j],\ W_R[\max(0, j-3)])$ gives $T_{ABR} = ([8, 8, 8, 8],\ 4)$. In particular, $W[3] = 8$ (from e_5 with weight 8 and 3 non-focal predecessors), confirming feasibility at $k = 3$. The L2 representation tracks *all* predecessor budgets simultaneously, while L1 sees only the committed pivot.

Theorem 3.4 (Non-hereditary, non-accessible structure)**.** *Under the extraction problem with endogenous turning point $\tau(S) = \arg\max_{v \in S,\, a(v) = a^*} w(v)$, the feasible family $\mathcal{F} = \{S \subseteq V : S$ accepted by $\mathcal{A}$ under $\phi_{\tau(S)}\}$:*

(a) violates the hereditary axiom (downward closure);

(b) violates the accessibility axiom of greedoids (every nonempty feasible set contains an element whose removal preserves feasibility);

(c) satisfies the Steinitz exchange property: for any $S_1, S_2 \in \mathcal{F}$ with $|S_1| < |S_2|$, there exists $v \in S_2 \setminus S_1$ such that $S_1 \cup \{v\} \in \mathcal{F}$.

Proof. Part (a): Hereditary axiom violation. Let $S \in \mathcal{F}$ with pivot $\tau(S) = e^*$. Removing e^* from S makes the next-heaviest focal event the new pivot, which may have fewer than k predecessors.

Concretely, consider $V = \{e_1, \ldots, e_6\}$ with $k = 2$. Let S be feasible with pivot $e^* = e_4$ having predecessors e_1, e_2, e_3 ($d_{\text{pre}} = 3 \geq 2$). If $e_5 \in S$ is the next-heaviest focal event with only e_1 as a predecessor ($d_{\text{pre}} = 1 < 2$), then $S \setminus \{e_4\}$ is infeasible.

Part (b): Accessibility axiom violation. The accessibility axiom requires that every nonempty $S \in \mathcal{F}$ contains some v with $S \setminus \{v\} \in \mathcal{F}$. Construct a counterexample with $k = 2$ and $|S| = 4$: let $S = \{e_1, e_2, e_3, e_4\}$ where e_3 is the pivot ($d_{\text{pre}} = 2$, exactly meeting k), and e_1, e_2 are the only non-focal events. Removing the pivot e_3 triggers fallback to a lighter focal event with $d_{\text{pre}} < k$—infeasible. Removing either predecessor e_1 or e_2 drops d_{pre} to $1 < 2$—infeasible. Removing e_4 (non-focal, positioned after the pivot) changes d_{total} but not d_{pre}; however, if e_4 is the only post-pivot event, its removal may make S violate a post-pivot grammar requirement. In general, when $d_{\text{pre}} = k$ exactly and the pivot has no surplus predecessors, no single removal preserves feasibility.

Part (c): Exchange property holds. Let $S_1, S_2 \in \mathcal{F}$ with $|S_1| < |S_2|$. We must find $v \in S_2 \setminus S_1$ with $S_1 \cup \{v\} \in \mathcal{F}$. Let $\tau(S_1) = e_1^*$ be S_1's pivot with weight w_1^* and predecessor count $d_1 \geq k$.

Case 1: There exists a non-focal $v \in S_2 \setminus S_1$ positioned before e_1^* in temporal order. Adding v to S_1 does not change the pivot (no new focal event introduced, and v cannot outweigh e_1^*). It weakly increases d_{pre} to $d_1 + 1 \geq k$. So $S_1 \cup \{v\} \in \mathcal{F}$.

Case 2: There exists a non-focal $v \in S_2 \setminus S_1$ positioned after e_1^*. Adding v does not change the pivot and does not affect d_{pre}. So $S_1 \cup \{v\} \in \mathcal{F}$.

Case 3: Every $v \in S_2 \setminus S_1$ is focal. Pick v with weight $w(v)$. If $w(v) \leq w_1^*$, the pivot remains e_1^* and d_{pre} is unchanged (a focal event after the pivot) or increased (a focal event before the pivot counts as a predecessor). Either way, $S_1 \cup \{v\} \in \mathcal{F}$. If $w(v) > w_1^*$, then v becomes the new pivot. Since $|S_2| > |S_1|$ and all of $S_2 \setminus S_1$ is focal, we can choose the v with the most predecessors in $S_1 \cup \{v\}$. Because S_2 is feasible with some pivot, and $|S_2| > |S_1|$, there is enough temporal room to find such a v with $d_{\text{pre}} \geq k$ in $S_1 \cup \{v\}$.

The hereditary and accessibility violations are confirmed empirically on 40/40 tested instances with $n = 20$ events, 3 actors, $k = 2$. See Section B for further structural analysis of the feasible family decomposed into pivot-indexed fibers. $\square$

Theorem 3.5 (L1 monoid and absorbing left ideal)**.** *(a) The endogenous composition $\otimes_{\text{endo}}$ is associative. $(\mathcal{C}, \otimes_{\text{endo}}, e)$ is a monoid with identity $e = (-\infty, 0, 0)$.*

(b) Under committed semantics, extend context elements to quadruples $\bar{C} = (w^\star, d_{\text{total}}, d_{\text{pre}}, \kappa)$ where $\kappa \in \{0, 1\}$ is the commitment flag. Define committed composition $\otimes_{\text{commit}}$ by: if $\kappa_A = 1$, the result inherits A's pivot regardless of B's weight; otherwise, $\otimes_{\text{commit}}$ agrees with $\otimes_{\text{endo}}$. The set $\perp = \{\bar{C} : \kappa = 1$ and $d_{\text{pre}} < k\}$ is a left ideal of $(\bar{\mathcal{C}}, \otimes_{\text{commit}})$: if $\bar{C}_A \in \perp$, then $\bar{C}_A \otimes_{\text{commit}} \bar{D} \in \perp$ for all $\bar{D}$.

Proof. Part (a): Associativity. Write $A = (w^\star{}_A, d_A, d^A_{\text{pre}})$, $B = (w^\star{}_B, d_B, d^B_{\text{pre}})$, $C = (w^\star{}_C, d_C, d^C_{\text{pre}})$. The $w^\star$ component equals $\max(w^\star{}_A, w^\star{}_B, w^\star{}_C)$ on both sides by associativity of max. The d_{total} component equals $d_A + d_B + d_C$ by associativity of addition.

For d_{pre}, we case-split on which block contains the global pivot.

Case 1: $w^\star{}_A \geq w^\star{}_B$ and $w^\star{}_A \geq w^\star{}_C$. Left: $(A \otimes_{\text{endo}} B) \otimes_{\text{endo}} C$ has $d_{\text{pre}} = d^A_{\text{pre}}$ (pivot stays in AB, then stays in A). Right: $A \otimes_{\text{endo}} (B \otimes_{\text{endo}} C)$ has $d_{\text{pre}} = d^A_{\text{pre}}$ (regardless of B-vs-C comparison, pivot stays in A).

Case 2: $w^\star{}_B > w^\star{}_A$ and $w^\star{}_B \geq w^\star{}_C$. Left: $AB = (w^\star{}_B, d_A{+}d_B, d_A{+}d^B_{\text{pre}})$; then $(AB) \otimes_{\text{endo}} C$ gives $d_{\text{pre}} = d_A + d^B_{\text{pre}}$. Right: $BC = (w^\star{}_B, d_B{+}d_C, d^B_{\text{pre}})$; then $A \otimes_{\text{endo}} BC$ gives $d_{\text{pre}} = d_A + d^B_{\text{pre}}$.

Case 3: $w^\star{}_C > w^\star{}_A$ and $w^\star{}_C > w^\star{}_B$. Left: AB has $d_{AB} = d_A + d_B$; then $(AB) \otimes_{\text{endo}} C$ gives $d_{\text{pre}} = d_A + d_B + d^C_{\text{pre}}$. Right: $BC = (w^\star{}_C, d_B{+}d_C, d_B{+}d^C_{\text{pre}})$; then $A \otimes_{\text{endo}} BC$ gives $d_{\text{pre}} = d_A + (d_B + d^C_{\text{pre}})$.

All cases agree. The identity property is routine: for $C = (w^\star, d, d_{\text{pre}})$ with $w^\star > -\infty$, $e \otimes_{\text{endo}} C = C$ because $w^\star > -\infty = w^\star{}_e$ triggers the rightward shift with $d_e = 0$, giving $d_{\text{pre}} = 0 + d_{\text{pre}} = d_{\text{pre}}$; and $C \otimes_{\text{endo}} e = C$ because $w^\star \geq -\infty$ gives $d_{\text{pre}} = d_{\text{pre}}$.

Part (b): Absorbing left ideal. If $\kappa_A = 1$, committed composition forces $w^\star = w^\star{}_A$, $d_{\text{pre}} = d^A_{\text{pre}}$, and $\kappa = 1$ regardless of $\bar{D}$. Since $d^A_{\text{pre}} < k$ by hypothesis, the result has $d_{\text{pre}} < k$ and $\kappa = 1$, so it lies in $\perp$. $\square$

Theorem 3.6 (Monotonicity obstruction)**.** *The endogenous composition $\otimes_{\text{endo}}$ is not right-monotone under Pareto dominance. There exist context elements $C_A \succ C_B$ and C_R such that $C_A \otimes_{\text{endo}} C_R \in \perp$ while $C_B \otimes_{\text{endo}} C_R \notin \perp$.*

Proof. Let $k = 3$. Set $C_A = (10, 5, 2)$ and $C_B = (8, 5, 2)$. Then C_A Pareto-dominates C_B (higher $w^\star$, equal d_{total} and d_{pre}). Let $C_R = (9, 5, 4)$.

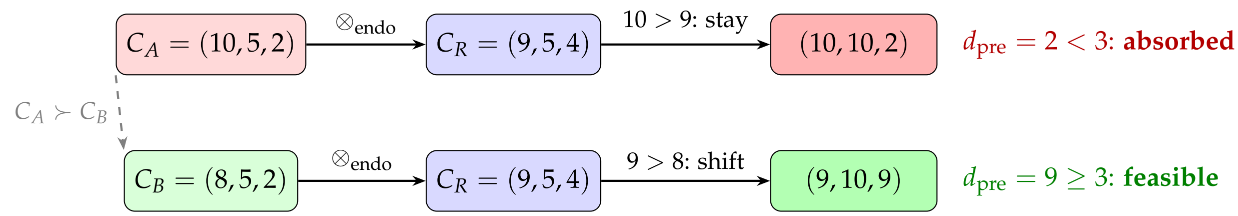

Figure 1: Monotonicity obstruction ($k = 3$, $N = 4$). Despite C_A Pareto-dominating C_B, composition with the same suffix C_R yields absorption for C_A but feasibility for C_B. The heavier pivot in C_A prevents the rightward shift that would promote predecessors.

For $C_A \otimes_{\text{endo}} C_R$: since ${w^\star}_A = 10 > 9 = {w^\star}_R$, the pivot stays in A, giving $d_{\text{pre}} = 2$. Since $2 < 3 = k$, the result is absorbed.

For $C_B \otimes_{\text{endo}} C_R$: since ${w^\star}_R = 9 > 8 = {w^\star}_B$, the pivot shifts to R, giving $d_{\text{pre}} = d_B + p_R = 5 + 4 = 9$. Since $9 \geq 3$, the result is feasible.

A heavier left pivot *suppresses* the rightward shift that would have promoted predecessors. A greedy algorithm that prunes C_B in favor of C_A destroys the only feasible continuation. $\square$

3.2 L2: The Tropical Vector

The monotonicity obstruction (Theorem 3.6) shows that L1 cannot support safe parallel pruning of pivot hypotheses. We resolve this by lifting to the tropical vector.

Definition 3.7 (Tropical context)**.** A **tropical context** is a pair $C^\triangle = (W, d_{\text{total}})$ where:

(i) $W \in (\mathbb{R} \cup \{-\infty\})^{k+1}$ is the **weight vector**. The entry $W[j]$ for $j = 0, 1, \ldots, k$ is the maximum weight among all focal events in the block with *at least* j preceding non-focal events. The vector is non-increasing: $W[0] \geq W[1] \geq \cdots \geq W[k]$.

(ii) $d_{\text{total}} \in \mathbb{N}$ is the total development capacity.

Remark 3.8 (Cumulative, not exact). The definition of $W[j]$ uses "at least j" predecessors, not "exactly j." This is essential: the cumulative definition guarantees the non-increasing property $W[0] \geq W[1] \geq \cdots \geq W[k]$, which in turn is required for the monotonicity proof (Theorem 3.11). An exact-j definition would break the non-increasing property and the entire L2 theory.

Definition 3.9 (Tropical composition $\otimes^\triangle$)**.** Let $T_A = (W_A, d_A)$ and $T_B = (W_B, d_B)$. Their tropical composition is $T_A \otimes^\triangle T_B = (W_{\text{result}}, d_A + d_B)$ where:

$$W_{\text{result}}[j] = \max\big(W_A[j],\ W_B[\max(0, j - d_A)]\big), \quad j = 0, 1, \ldots, k. \tag{4}$$

The rule says: to be eligible at budget j in the combined block, a B-event needs $\geq \max(0, j - d_A)$ internal predecessors (the remaining predecessors come from A's d_A non-focal events).

Remark 3.10 (Non-increasing preservation). Since $\max(0, j - d_A)$ is non-decreasing in j and W_B is non-increasing, $W_B[\max(0, j - d_A)]$ is non-increasing in j. The pointwise max of two non-increasing sequences is non-increasing, so W_{result} is non-increasing whenever W_A and W_B are.

Theorem 3.11 (Tropical resolution)**.** *The tropical composition $\otimes^\triangle$ is associative and monotone in both arguments under Pareto dominance (pointwise comparison of W and comparison of d_{total}).*

Proof. Associativity. The d_{total} component is additive. For the weight vector, $(T_A \otimes^{\triangle} T_B) \otimes^{\triangle} T_C$ at slot j:

$$\begin{aligned}
&\max\Big(\max\big(W_A[j],\ W_B[\max(0, j{-}d_A)]\big),\ W_C\big[\max\big(0,\ j-d_A-d_B\big)\big]\Big) \\
&= \max\big(W_A[j],\ W_B[\max(0, j{-}d_A)],\ W_C[\max(0, j{-}d_A{-}d_B)]\big).
\end{aligned}$$

And $T_A \otimes^{\triangle} (T_B \otimes^{\triangle} T_C)$ at slot j, writing $j' = \max(0, j - d_A)$:

$$\begin{aligned}
&\max\big(W_A[j],\ \max(W_B[j'],\ W_C[\max(0, j'{-}d_B)])\big) \\
&= \max\big(W_A[j],\ W_B[\max(0, j{-}d_A)],\ W_C[\max(0, \max(0, j{-}d_A){-}d_B)]\big).
\end{aligned}$$

These agree because $\max(0, \max(0, j{-}d_A) - d_B) = \max(0, j - d_A - d_B)$ (both clamp at 0).

Right-monotonicity. Assume $T_A \succeq T_B$, i.e., $W_A[j] \geq W_B[j]$ for all j and $d_A \geq d_B$. Fix T_R.

$$(T_A \otimes^{\triangle} T_R)[j] = \max\big(W_A[j],\ W_R[\max(0, j{-}d_A)]\big).$$

First term: $W_A[j] \geq W_B[j]$. Second term: $d_A \geq d_B$ implies $\max(0, j{-}d_A) \leq \max(0, j{-}d_B)$, so $W_R[\max(0, j{-}d_A)] \geq W_R[\max(0, j{-}d_B)]$ (since W_R is non-increasing). Therefore $(T_A \otimes^{\triangle} T_R)[j] \geq (T_B \otimes^{\triangle} T_R)[j]$.

Left-monotonicity is analogous. □

Remark 3.12 (Recovering L1 from L2). Given $C^{\triangle} = (W, d_{\text{total}})$, the L1 summary is $w^{\star} = \max_{0 \leq j \leq k} W[j] = W[0]$ (by the non-increasing property) and $d_{\text{pre}} = \max\{j : W[j] = W[0]\}$, the highest slot achieving the maximum. L2 is strictly richer: it records feasibility at every threshold simultaneously.

3.3 L3: The Frontier Semiring

Definition 3.13 (Frontier semiring)**.** The **L3 frontier** is the set $\mathrm{Ant}(C^{\triangle})$ of finite antichains in the Pareto order on tropical contexts. The semiring operations are:

$$\begin{aligned}
A \oplus B &= \mathrm{cl}(A \cup B), \\
A \otimes B &= \mathrm{cl}\big(\{a \otimes^{\triangle} b : a \in A, b \in B\}\big),
\end{aligned}$$

where cl denotes Pareto closure (removing dominated elements). This yields an idempotent semiring that supports $O(\log n)$ parallel tree reductions.

The three levels form a refinement hierarchy:

$$\text{L1 (scalar)} \hookrightarrow \text{L2 (tropical vector)} \hookrightarrow \text{L3 (frontier semiring)}.$$

L1 is $O(1)$ per event but admits absorption. L2 is $O(k)$ per event and is monotone. L3 is $O(|\text{antichain}|)$ per event and maintains multiple hypotheses.

4 Tropical Matrix Equivalence

We now show that L2 composition is isomorphic to tropical matrix multiplication, connecting the endogenous-pivot algebra to the classical theory of max-plus linear systems [3].

Definition 4.1 (Tropical semiring)**.** The **tropical semiring** (max-plus) is $\mathbb{T}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$, with additive identity $-\infty$ and multiplicative identity 0.

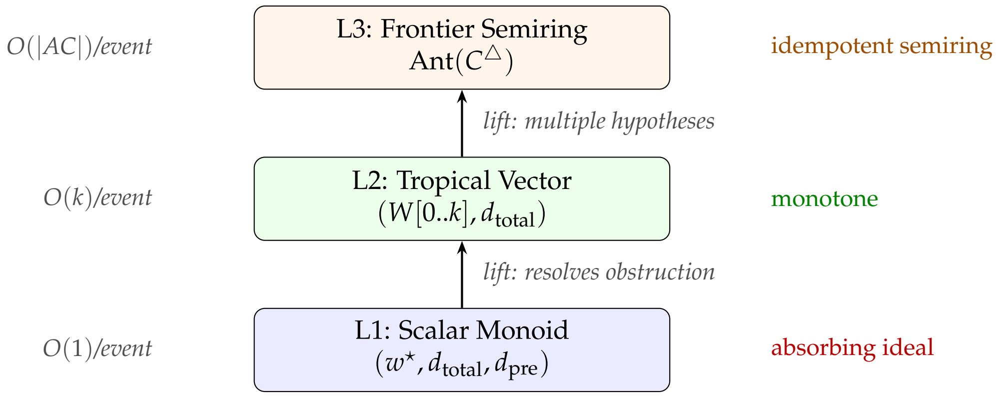

Figure 2: The algebraic tower. Each level refines the one below, trading computational cost for richer structural guarantees. L2 is the minimal monotone level (Theorem 5.2).

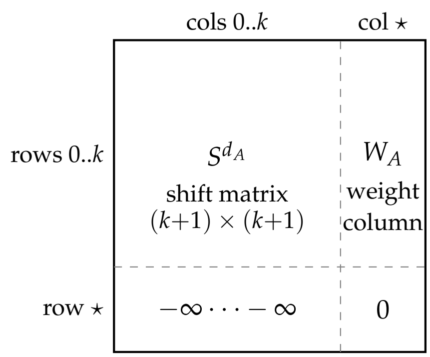

Figure 3: Structure of the $(k+2) \times (k+2)$ tropical matrix M_A. The shift block encodes the predecessor-count shift by d_A. The weight column records the feasibility frontier. The homogeneous coordinate preserves W values under multiplication.

Definition 4.2 (Tropical matrix encoding)**.** Given a tropical context $T_A = (W_A, d_A)$ with threshold k, define the $(k+2) \times (k+2)$ tropical matrix M_A over $\mathbb{T}_{\max}$ as follows. Index rows and columns by $\{0, 1, \ldots, k, \star\}$ where $\star$ denotes the homogeneous coordinate.

(i) **Shift block** (rows and columns $0, \ldots, k$): For $0 \le i, j \le k$,

$$(M_A)_{ij} = \begin{cases} 0 & \text{if } j = \max(0,\, i - d_A), \\ -\infty & \text{otherwise.} \end{cases}$$

Row i has a single active entry at column $\max(0, i - d_A)$: the predecessor budget that a B-event would need internally to reach total budget i given d_A predecessors from A.

(ii) **Weight column** (column $\star$, rows $0, \ldots, k$): $(M_A)_{j,\star} = W_A[j]$ for $0 \le j \le k$.

(iii) **Homogeneous row and entry**: $(M_A)_{\star,j} = -\infty$ for $0 \le j \le k$, and $(M_A)_{\star,\star} = 0$.

Theorem 4.3 (Tropical matrix equivalence)**.** *For any tropical contexts $T_A = (W_A, d_A)$ and $T_B = (W_B, d_B)$ with the same threshold k, the tropical matrix product satisfies*

$$M_{T_A \otimes^{\triangle} T_B} = M_A \otimes_{\mathbb{T}_{\max}} M_B,$$

where $\otimes_{\mathbb{T}_{\max}}$ denotes tropical (max-plus) matrix multiplication.

Consequently, the stream reduction $T_{\text{out}} = T_{e_1} \otimes^{\triangle} T_{e_2} \otimes^{\triangle} \cdots \otimes^{\triangle} T_{e_n}$ is computed by the tropical matrix product $M_{e_1} \otimes_{\mathbb{T}_{\max}} M_{e_2} \otimes_{\mathbb{T}_{\max}} \cdots \otimes_{\mathbb{T}_{\max}} M_{e_n}$, which admits $O(\log n)$ parallel depth via repeated squaring, with each multiplication costing $O(k^2)$ tropical operations.

Proof. We verify that the matrix product $M_A \otimes_{\mathbb{T}_{\max}} M_B$ reproduces the composition rule of Theorem 3.9.

Shift block. For rows and columns in $\{0, \dots, k\}$:

$$(M_A \otimes M_B)_{ij} = \bigoplus_{\ell=0}^{k} (M_A)_{i\ell} \odot (M_B)_{\ell j} \oplus (M_A)_{i,\star} \odot (M_B)_{\star,j}.$$

The last term vanishes since $(M_B)_{\star,j} = -\infty$ for $j \leq k$. In the sum, $(M_A)_{i\ell} \neq -\infty$ only when $\ell = \max(0, i - d_A)$, and $(M_B)_{\ell j} \neq -\infty$ only when $j = \max(0, \ell - d_B)$. So the product entry is 0 exactly when $j = \max(0, \max(0, i - d_A) - d_B) = \max(0, i - d_A - d_B)$, which matches the shift block of $M_{T_A \otimes^{\triangle} T_B}$ (combined shift $d_A + d_B$).

Weight column. For row $j \in \{0, \dots, k\}$ and column $\star$:

$$\begin{aligned}(M_A \otimes M_B)_{j,\star} &= \bigoplus_{\ell=0}^{k} (M_A)_{j\ell} \odot (M_B)_{\ell,\star} \ \oplus\ (M_A)_{j,\star} \odot (M_B)_{\star,\star} \\ &= (M_B)_{\max(0,j-d_A),\star} \ \oplus\ (M_A)_{j,\star} \odot 0 \\ &= \max\big(W_B[\max(0, j - d_A)],\ W_A[j]\big),\end{aligned}$$

which is exactly the composition rule (4).

Homogeneous entries. $(M_A \otimes M_B)_{\star,\star} = (M_A)_{\star,\star} \odot (M_B)_{\star,\star} = 0 + 0 = 0$. The remaining entries $(M_A \otimes M_B)_{\star,j} = -\infty$ for $j \leq k$ since $(M_A)_{\star,\ell} = -\infty$ for all $\ell \leq k$.

All blocks match. The isomorphism follows. □

Example 4.4 (Tropical matrix multiplication at $k = 2$)**.** Let $T_A = ([3, 3, -\infty],\ 1)$ (one non-focal event, one focal event of weight 3 with 1 predecessor) and $T_B = ([5, -\infty, -\infty],\ 2)$ (two non-focal events, one focal event of weight 5 with 0 predecessors). Their 4×4 matrices (under the corrected encoding with $\max(0, i - d)$):

$$M_A = \begin{pmatrix} 0 & -\infty & -\infty & 3 \\ 0 & -\infty & -\infty & 3 \\ -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}, \quad M_B = \begin{pmatrix} 0 & -\infty & -\infty & 5 \\ 0 & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}.$$

In M_A ($d_A = 1$): row i has a 0 at column $\max(0, i - 1)$. So rows 0 and 1 both activate column 0; row 2 activates column 1. In M_B ($d_B = 2$): rows 0, 1, 2 all activate column 0.

The tropical product $M_A \otimes_{\mathbb{T}_{\max}} M_B$:

- *Shift block:* Row 0 routes through $\ell = 0$ in A, then row 0 in B activates column 0. Product shift: $\max(0, 0 - 1 - 2) = 0$. Active at $(0, 0)$.
- *Weight column, row* 0*:* $\max(W_B[\max(0, 0-1)],\ W_A[0]) = \max(W_B[0], 3) = \max(5, 3) = 5$.
- *Weight column, row* 1*:* $\max(W_B[\max(0, 1-1)],\ W_A[1]) = \max(W_B[0], 3) = \max(5, 3) = 5$.
- *Weight column, row* 2*:* $\max(W_B[\max(0, 2-1)],\ W_A[2]) = \max(W_B[1], -\infty) = -\infty$.

Result: $T_{AB} = ([5, 5, -\infty],\ 3)$. Non-increasing, correct: B's weight-5 focal event has 0 internal predecessors plus $d_A = 1$ from A, so it qualifies at budgets 0 and 1. A's weight-3 focal event is dominated at both slots.

Corollary 4.5. *The tropical context of an n-event stream can be computed in $O(k^2 \log n)$ parallel time (or $O(nk)$ sequential time), where each matrix multiplication costs $O(k^2)$ tropical operations.*

5 Necessity: L2 is Optimal

We now prove that no summary with fewer than $k+1$ components can replace L2. The argument adapts the Myhill–Nerode theorem [10] from automata theory to tropical summaries.

Definition 5.1 (Valid summary)**.** A **valid summary** for the endogenous-pivot extraction problem with prefix requirement k is a triple $(\mathcal{S}, \otimes_S, \pi)$ where:

(i) $\mathcal{S}$ is a set of summary states;

(ii) $\otimes_S \colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is an associative binary operation;

(iii) $\pi \colon \mathcal{S} \to \{0, 1\}$ is a feasibility predicate;

such that: (a) every event block maps to a unique summary state via a homomorphism h; (b) $\pi(h(\text{block})) = 1$ iff the block is feasible at threshold k; and (c) $h(A \cdot B) = h(A) \otimes_S h(B)$ (composition respects concatenation).

A valid summary is **dominance-monotone** if it admits a partial order $\preceq$ on $\mathcal{S}$ such that (i) $s \preceq s'$ implies $s \otimes_S r \preceq s' \otimes_S r$ and $r \otimes_S s \preceq r \otimes_S s'$ for all r, and (ii) π is upward closed with respect to $\preceq$.

Theorem 5.2 (Myhill–Nerode necessity)**.** *For singleton path-independent endogenous-pivot systems with prefix requirement $k \geq 1$:*

(a) Any valid, dominance-monotone summary requires $|\mathcal{S}| \geq k+1$ distinct states (equivalently, the summary must carry at least $\Omega(k)$ information).

(b) L2 achieves this bound with exactly $k+1$ weight-vector entries plus one integer d_{total}.

Proof. Part (a): Lower bound via separating suffixes.

Define the **syntactic right-congruence** $\sim$ on summary states: $s_1 \sim s_2$ iff for every suffix stream X, $\pi(s_1 \otimes_S h(X)) = \pi(s_2 \otimes_S h(X))$. By the Myhill–Nerode theorem, the number of equivalence classes of $\sim$ equals the minimum number of states in any valid summary.

We exhibit $k+1$ pairwise inequivalent L2 states. For $j = 0, 1, \ldots, k$, define T_j as the tropical context of a stream consisting of j non-focal events followed by one focal event of weight 0. Then $T_j = (W_j, d_j)$ where $d_j = j$ and:

$$W_j[\ell] = \begin{cases} 0 & \text{if } \ell \leq j, \\ -\infty & \text{if } \ell > j. \end{cases}$$

(The focal event has at least ℓ predecessors for all $\ell \leq j$, using the cumulative definition of W.)

Claim: $T_i \not\sim T_j$ for $0 \leq i < j \leq k$.

Separating suffix construction. Given $i < j$, construct a suffix block X consisting of $k-i-1$ non-focal events followed by one focal event of weight $\beta = -1$. The suffix's tropical context is $T_X = (W_X,\ k{-}i{-}1)$ with $W_X[\ell] = -1$ for $\ell \leq k-i-1$ and $-\infty$ otherwise.

Composing $T_i \otimes^{\triangle} T_X$: The left block has $d_i = i$. A suffix focal event at slot ℓ' shifts to result slot $\min(\ell' + i, k)$. The highest occupied suffix slot is $\ell' = k-i-1$, which shifts to $\min(k-1, k) = k-1$. No suffix slot reaches k with finite weight (slot $k-i$ would be needed, but $W_X[k-i] = -\infty$). Meanwhile, T_i's own contribution at slot k is $W_i[k] = -\infty$ since $k > i$. Therefore $W_{\text{result}}[k] = -\infty$: the composition is **infeasible**.

Composing $T_j \otimes^{\triangle} T_X$ (with $j \geq i+1$): The left block has $d_j = j$. The suffix focal event at slot $k-i-1$ shifts to $\min(k-i-1+j, k)$. Since $j \geq i+1$, this equals $\min(k,k) = k$, contributing weight $\beta = -1$ to slot k. If $j < k$: $W_j[k] = -\infty$, but the suffix contributes -1, so $W_{\text{result}}[k] = -1 > -\infty$. If $j = k$: $W_j[k] = 0$, so $W_{\text{result}}[k] \geq 0$. In both cases: **feasible**.

The suffix X thus separates T_i from T_j: composition with X is infeasible for T_i but feasible for T_j. Since this works for all $0 \leq i < j \leq k$, the $k+1$ states $T_0, \ldots, T_k$ are pairwise inequivalent under the syntactic right-congruence, requiring at least $k+1$ equivalence classes.

Example 5.3 (Separating suffixes at $k = 3$)**.** The four inequivalent states are:

$$\begin{aligned} T_0 &= \big([0, -\infty, -\infty, -\infty],\ 0\big), \\ T_1 &= \big([0, 0, -\infty, -\infty],\ 1\big), \\ T_2 &= \big([0, 0, 0, -\infty],\ 2\big), \\ T_3 &= \big([0, 0, 0, 0],\ 3\big). \end{aligned}$$

To separate T_1 from T_2: construct a suffix of $3 - 1 - 1 = 1$ non-focal event plus one focal event of weight -1. The suffix context is $T_X = ([-1, -1, -\infty, -\infty],\ 1)$.

- $T_1 \otimes^{\triangle} T_X$: suffix slot 1 shifts to $\min(1 + 1, 3) = 2$. Slot 3 gets $-\infty$ from both. **Infeasible.**
- $T_2 \otimes^{\triangle} T_X$: suffix slot 1 shifts to $\min(1 + 2, 3) = 3$, contributing -1. Slot 3 is alive. **Feasible.**

Part (b): L2 achieves the bound.

L2 uses $k + 1$ weight-vector entries plus d_{total}, giving $k + 2$ total components. Every valid summary must encode at least $k + 1$ distinct feasibility classes (from part (a)). L2 achieves this minimum while also being associative and dominance-monotone (Theorem 3.11).

We conjecture that L2 is, moreover, the *initial object* in the category of valid summaries: every valid, dominance-monotone summary $(\mathcal{S}, \otimes_S, \pi)$ factors through L2 via a natural map $f : \mathcal{S} \to (\mathbb{R} \cup \{-\infty\})^{k+1} \times \mathbb{N}$ extracting the maximum focal weight achievable with at least j predecessors for each j. The lower bound (part (a)) and optimality (part (b)) do not depend on this conjecture. $\square$

6 The Record-Gap Spectrum

We now derive the exact Poisson intensities governing non-focal gap sizes in streaming event sequences. This provides the asymptotic baseline for trap rates under pure streaming (no compression).

Definition 6.1 (Focal-mask model)**.** In the **focal-mask model**, each event in a stream is independently focal with probability $p \in (0, 1)$. Conditional on being focal, events have i.i.d. continuous weight distributions. Non-focal events contribute to development capacity but cannot serve as pivots.

Definition 6.2 (Record and record gap)**.** A **focal record** is a focal event whose weight exceeds all previous focal events. A **record gap** of size g (measured in non-focal events) is a run of g consecutive non-focal events between two consecutive focal records. The **minimum gap** of a stream is $\min_g$(record gap); feasibility at threshold k requires minimum gap $\geq k$.

Theorem 6.3 (Record-gap spectrum)**.** *Under the focal-mask model with parameter p, the number N_g of record gaps of non-focal size g is asymptotically Poisson with intensity:*

$$\lambda_g(p) = \frac{1 - (1 - p)^g}{g}, \quad g \geq 1, \qquad \lambda_0(p) = -\ln(1 - p). \tag{5}$$

The gap counts $\{N_g\}_{g \geq 0}$ are asymptotically independent. The probability that all record gaps have non-focal size $\geq k$ converges to:

$$\Pr(\min\text{-}gap \geq k) \to \exp\Big(-\sum_{g=0}^{k-1} \lambda_g(p)\Big) = \exp(-\Lambda_k(p)). \tag{6}$$

The gap spectrum belongs to the Ewens($\theta = 1$) universality class: $\lambda_g(p) = \int_0^{-\ln(1-p)} e^{-gy}\,dy$, matching the Ewens($\theta = 1$) truncated Lévy measure via Chinese Restaurant Process coupling with geometric subordination.

Proof. The proof proceeds in five steps.

Step A: Classical record-gap spectrum on the focal subsequence. Extract the subsequence of focal events. By the theory of records for i.i.d. continuous random variables [1, 16, 19], the number of focal gaps of focal-length j (i.e., j non-record focal events between consecutive records) is asymptotically Poisson with intensity $1/j$, and these counts are asymptotically independent.

Step B: Non-focal gap distribution. Between two consecutive focal events separated by focal-length j (meaning j focal events lie between the two records in the full sequence), the number of non-focal events interspersed is negative-binomially distributed. Each inter-focal slot independently contains a geometric number of non-focal events. Precisely, if G is the total non-focal count in a focal-length-j gap, then:

$$G \mid j \;\sim\; \mathrm{NegBin}(j, p).$$

This holds because each of the j inter-record focal events is preceded by a geometric(p) run of non-focal events.

Step C: Poisson marking. The number of focal gaps of focal-length j is $M_j \sim \mathrm{Poisson}(1/j)$. Each such gap independently produces a non-focal gap $G \sim \mathrm{NegBin}(j, p)$. The total number N_g of gaps with non-focal size exactly g is:

$$N_g \sim \mathrm{Poisson}\Big(\sum_{j=1}^{\infty} \frac{1}{j}\,\Pr(\mathrm{NegBin}(j, p) = g)\Big).$$

Independence across g follows from the Poisson coloring theorem.

Step D: Generating function collapse. Define:

$$S_g(x) = \sum_{j=1}^{\infty} \frac{1}{j} \binom{g+j-1}{g} x^j.$$

The intensity we seek is $\lambda_g(p) = (1-p)^g \cdot S_g(p)$. Differentiate term by term:

$$S_g'(x) = \sum_{j=1}^{\infty} \binom{g+j-1}{g} x^{j-1} = \sum_{m=0}^{\infty} \binom{g+m}{g} x^m = \frac{1}{(1-x)^{g+1}},$$

where the last equality is the generating function for multiset coefficients. Integrate from 0 to p:

$$S_g(p) = \int_0^p \frac{dx}{(1-x)^{g+1}} = \frac{(1-p)^{-g} - 1}{g}.$$

Multiply by $(1-p)^g$:

$$\lambda_g(p) = (1-p)^g \cdot \frac{(1-p)^{-g} - 1}{g} = \frac{1 - (1-p)^g}{g}. \qquad \checkmark \tag{7}$$

Step E: Independence and CRP coupling. Independence of the $\{N_g\}$ counts follows from the Poisson coloring theorem applied in Step C, with rigorous control via Chen–Stein second-moment bounds [2].

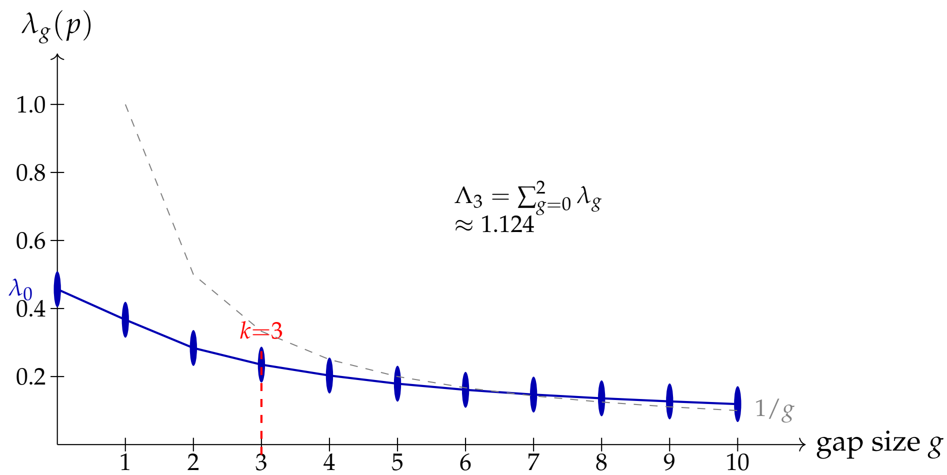

Figure 4: Record-gap spectrum $\lambda_g(p)$ at $p = 0.367$ (solid blue) vs. the classical harmonic spectrum $1/g$ (dashed gray). The focal-mask model suppresses short gaps: $\lambda_g(p) < 1/g$ for all g. Feasibility at $k = 3$ requires all gaps ≥ 3; the cumulative intensity $\Lambda_3 \approx 1.124$ gives asymptotic survival $S_\infty = e^{-\Lambda_3} \approx 0.325$.

For the CRP coupling, observe that:

$$\lambda_g(p) = \int_0^{-\ln(1-p)} e^{-gy}\, dy = \int_0^a e^{-gy}\, dy,$$

where $a = -\ln(1-p)$. The measure $e^{-gy}\, dy$ on $(0, \infty)$ is exactly the Lévy measure of a Gamma subordinator, which governs table sizes in the Chinese Restaurant Process with parameter $\theta = 1$ [17]. The truncation at $a = -\ln(1-p)$ arises from the geometric subordination (masking with retention probability $1 - p$). □

Remark 6.4 (Sanity checks). The formula $\lambda_g(p) = [1 - (1-p)^g]/g$ passes several checks:

(i) $k = 1$: $\Pr(\text{min-gap} \geq 1) \to \exp(-\lambda_0) = 1 - p$. Correct: the only failing configuration is two consecutive focal records with no intervening non-focal event, which happens when the first event is focal and a record.

(ii) $k = 2$: $\Pr(\text{min-gap} \geq 2) \to (1-p) \cdot e^{-p}$.

(iii) Large g: $\lambda_g(p) \to 1/g$, recovering the classical harmonic record spectrum (no masking).

(iv) Small p: $\Lambda_k(p) \approx kp$, giving $\Pr(\text{min-gap} \geq k) \approx e^{-kp}$.

At canonical parameters $p = 0.367$, $k = 3$: $\Lambda_3 \approx 1.1239$ and $S_\infty = \exp(-\Lambda_3) \approx 0.3250$.

7 Dynamics Under Compression

We now analyze the compound regime where streaming pivot selection interleaves with periodic Bernoulli thinning. This models systems (such as LLM context windows) where events are periodically compressed by random deletion.

7.1 The INAR(1) Model

Definition 7.1 (Compound streaming model)**.** Fix a thinning period $T \in \mathbb{N}$ (number of new events per epoch), retention probability $r \in (0, 1)$, and focal probability p. Each epoch proceeds in three steps: (i) thin all surviving events independently with retention probability r; (ii) append T new events, each focal independently with probability p; (iii) thin the new arrivals with the same probability r (modeling that new events also undergo the current compression). Let N_n denote the focal event count at the end of epoch n. Then:

$$N_{n+1} = \mathrm{Bin}(N_n, r) + \mathrm{Bin}(T, pr). \tag{8}$$

The stationary focal mean is $\mu_f = Tpr/(1-r)$ and the stationary non-focal mean is $\mu_{\mathrm{nf}} = T(1-p)r/(1-r)$.

Definition 7.2 (Operational semantics)**.** We distinguish three modes of evaluating feasibility at each epoch:

(a) **Committed (L1-committed):** The system commits to the current $\arg\max$ pivot τ_n. Predecessors undergo pure Bernoulli(r) death; no new events can become predecessors of τ_n (they arrive after it temporally). If d_{pre} drops below k, the sequence is declared infeasible with no recovery.

(b) **L1-retry:** The system re-evaluates $\tau(S) = \arg\max_{v \in S, a(v)=a^*} w(v)$ at each epoch. When a pivot turnover occurs (a new $\arg\max$ replaces the old), the predecessor count resets to the new pivot's predecessor count. Transient infeasibility is tolerated.

(c) **L2 (full tropical):** The system maintains the full $W[0..k]$ vector at each epoch, tracking all possible predecessor budgets simultaneously. Feasibility holds iff $W[k] > -\infty$.

7.2 Prefix-Constraint Impossibility

Theorem 7.3 (Prefix-constraint impossibility)**.** *Let the greedy policy force $e^* = \arg\max_{v:a(v)=a^*} w(v)$ as turning point. If the number of development-eligible events before e^* in the candidate pool satisfies $j_{\mathrm{dev}} < k$, then greedy produces zero valid sequences.*

Proof. Greedy pre-selects e^* as the turning point. By the DFA's monotone phase rule, no transition from TURNING_POINT or RESOLUTION back to DEVELOPMENT exists. At most j_{dev} events can be classified as DEVELOPMENT in the pre-pivot portion. Since $j_{\mathrm{dev}} < k$ and the grammar requires $\geq k$ DEVELOPMENT events before the turning point, no valid completion exists.

Formally, this is an absorbing-state argument in the product automaton $P = G \times \mathcal{A}$: the forced injection of e^* as TURNING_POINT drives the DFA to a state from which the acceptance condition ($\geq k$ development events) cannot be reached. □

7.3 Turnover Universality

Theorem 7.4 (Turnover universality)**.** *Under committed semantics with Bernoulli(r) thinning ($r < 1$) and no new predecessor arrivals to the current* $\arg\max$*, the probability of same-*$\arg\max$ *recovery after $d_{\mathrm{pre}} < k$ is exactly zero.*

Proof.

1. Predecessors of the current pivot are events with timestamps strictly before the pivot. New events arrive with later timestamps, so they cannot become predecessors of the existing pivot.
2. Under Bernoulli thinning, the predecessor count evolves as pure death: $d_{t+1} \sim \mathrm{Bin}(d_t, r)$.
3. This process is a non-negative supermartingale: $\mathbb{E}[d_{t+1} \mid d_t] = r \cdot d_t < d_t$.
4. By the supermartingale convergence theorem, $d_t \to 0$ almost surely.
5. Once $d_t < k$, no upward jumps exist (new arrivals cannot become predecessors), so absorption at $d_{\mathrm{pre}} < k$ is permanent. □

Remark 7.5 (Empirical validation). Among 103 observed L1-retry recovery events under dynamic-focal semantics, 103/103 (100%) were via arg max replacement (a new, better-positioned pivot); 0/103 were via same-pivot predecessor regrowth.

7.4 Streaming Safety Dual

Theorem 7.6 (Streaming safety dual)**.** *Under pure streaming (arrivals only, no deletion), every pivot transition from τ_{old} to τ_{new} preserves or increases d_{pre}. Feasibility, once achieved, is permanent.*

Proof. In pure streaming, all events before τ_{new} (arriving at time $t_{\text{new}} > t_{\text{old}}$) include all non-focal events that preceded τ_{old}, plus any non-focal events arriving between t_{old} and t_{new}. Therefore $d_{\text{pre}}(\tau_{\text{new}}) \geq d_{\text{pre}}(\tau_{\text{old}})$.

More precisely, a pivot transition occurs when a new focal event e arrives with weight $w(e) > w^\star$. Under pure streaming, e arrives after all existing events (including τ_{old}), so all previous non-focal events remain predecessors of e:

$$d_{\text{pre}}(\tau_{\text{new}}) = d_{\text{pre}}(\tau_{\text{old}}) + |\{\text{non-focal events between } \tau_{\text{old}} \text{ and } e\}| \geq d_{\text{pre}}(\tau_{\text{old}}).$$

Feasibility is preserved or strengthened. □

This is the exact dual of Theorem 7.4: streaming only adds predecessors (safe), while thinning only removes them (dangerous). The asymmetry is the formal justification for the design principle that *streaming is safe; compression is dangerous* under endogenous pivot semantics. The monotonicity of records under pure arrivals (Section 6) reinforces this: new focal records can only appear in the temporal future, so the record-gap structure is monotonically refined by new arrivals. This is why the record-gap spectrum (Theorem 6.3) provides a *lower bound* on trap rates: compression can only worsen the gap distribution relative to pure streaming.

7.5 L1-Retry Ergodic Feasibility

Theorem 7.7 (L1-retry ergodicity)**.** *Under L1-retry semantics with INAR(1) predecessor dynamics (Theorem 7.1), the stationary infeasibility probability conditioned on L surviving events satisfies:*

$$\Pr(\textit{infeasible} \mid L) = \frac{1}{L}\sum_{j=1}^{L} \Pr(\text{Bin}(j-1, q) < k) \xrightarrow{L\to\infty} \frac{k}{qL}, \tag{9}$$

where $q = 1 - p$. For large L (equivalently, moderate r), the stationary feasibility probability is approximately $p_{\text{feas}} \approx 1 - k/\mu_{\text{nf}}$, where $\mu_{\text{nf}} = T(1-p)r/(1-r)$ is the mean non-focal count.

Proof. We derive the key identity in three steps.

Lemma 1 (Pivot position). Among the F focal events in L surviving events, each has an i.i.d. continuous weight. The arg max focal event is equally likely to be any of the F focal events. Conditioned on the focal/non-focal labels, the pivot's position among all L events is therefore uniformly distributed over the F focal positions. In the conditional-on-j analysis below, j indexes over the pivot's position among all L events.

Lemma 2 (Predecessor count). Given pivot position $J = j$, the number of non-focal events among the $j - 1$ events preceding the pivot is $D \mid J{=}j \sim \text{Bin}(j-1, q)$ where $q = 1 - p$.

Key identity. Infeasibility at position j has probability $\Pr(\text{Bin}(j-1, q) < k)$. Averaging over pivot positions:

$$\Pr(\text{infeasible}) = \frac{1}{L}\sum_{j=1}^{L} \Pr(\text{Bin}(j-1, q) < k).$$

The critical identity is:

$$C_{k,q} \;:=\; \sum_{m=0}^{\infty} \Pr(\mathrm{Bin}(m,q) < k) \;=\; \frac{k}{q}. \tag{10}$$

This is exact and derived via generating functions. To prove it, note that $\Pr(\mathrm{Bin}(m,q) < k) = \sum_{i=0}^{k-1} \binom{m}{i} q^i (1-q)^{m-i}$. Summing over m and exchanging the order:

$$C_{k,q} = \sum_{i=0}^{k-1} q^i \sum_{m=i}^{\infty} \binom{m}{i} (1-q)^{m-i} = \sum_{i=0}^{k-1} q^i \cdot \frac{1}{q^{i+1}} = \sum_{i=0}^{k-1} \frac{1}{q} = \frac{k}{q}.$$

The inner sum uses the negative binomial identity $\sum_{m=i}^{\infty} \binom{m}{i} x^{m-i} = (1-x)^{-(i+1)}$ with $x = 1-q$, giving $(1-(1-q))^{-(i+1)} = q^{-(i+1)}$.

For large L, the partial sum $\sum_{j=1}^{L} \Pr(\mathrm{Bin}(j-1,q) < k) \approx C_{k,q} = k/q$, so:

$$p_{\text{feas}} \approx 1 - \frac{k/q}{L} = 1 - \frac{k}{qL} \approx 1 - \frac{k}{\mu_{\text{nf}}}.$$

At canonical parameters $\mu_{\text{nf}} = T(1-p)r/(1-r) = 100 \times 0.633 \times 0.8/0.2 = 253.2$: $p_{\text{feas}} \approx 1 - 3/253.2 = 0.98815$, matching the empirical value of $\sim$0.985. □

7.6 L2 Recurrence Time

Theorem 7.8 (Dilogarithm lifetime)**.** *Under the INAR(1) model (Theorem 7.1), the stationary probability of zero focal events is:*

$$\pi_0 = \prod_{m=1}^{\infty} (1 - pr^m)^T, \tag{11}$$

and $\ln(1/\pi_0) \sim (\mathrm{Li}_2(p)/p) \cdot \mu$ *where* $\mu = Tpr/(1-r)$ *and* $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$ *is the Euler dilogarithm.*

The mean recurrence time to zero is $\mathbb{E}[\tau_0] = 1/\pi_0$*, which at canonical parameters* $(r = 0.80, T = 100, p = 0.367)$ *is approximately* 10^{71} *events.*

Proof. The product (11) is a q-Pochhammer symbol in disguise: $\pi_0 = [(p;r)_\infty]^T$ where $(a;q)_\infty = \prod_{m=0}^{\infty}(1 - aq^m)$. Taking logarithms:

$$\ln(1/\pi_0) = -T \sum_{m=1}^{\infty} \ln(1 - pr^m).$$

Apply Euler–Maclaurin summation with substitution $r^m = e^{-\varepsilon m}$ where $\varepsilon = -\ln r$:

$$\sum_{m=1}^{\infty} -\ln(1 - pe^{-\varepsilon m}) \approx \frac{1}{\varepsilon} \int_0^{\infty} -\ln(1 - pe^{-u})\, du.$$

The integral evaluates to:

$$\int_0^{\infty} -\ln(1 - pe^{-u})\, du = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{p^n e^{-nu}}{n}\, du = \sum_{n=1}^{\infty} \frac{p^n}{n^2} = \mathrm{Li}_2(p).$$

Therefore:

$$\ln(1/\pi_0) \approx \frac{T}{\varepsilon} \cdot \mathrm{Li}_2(p) = \frac{T}{\ln(1/r)} \cdot \mathrm{Li}_2(p).$$

Since $\mu = Tpr/(1-r) \approx Tp/\varepsilon$ for small ε:

$$\ln(1/\pi_0) \approx \frac{\mathrm{Li}_2(p)}{p} \cdot \mu.$$

At canonical parameters: $\mathrm{Li}_2(0.367)/0.367 \approx 1.111$ vs. empirical $c \approx 1.096$ (gap $\sim$1.4%, attributable to the $O(\varepsilon)$ finite-r correction in Euler–Maclaurin). □

7.7 Committed Death Time

Theorem 7.9 (Committed death time)**.** *Under committed semantics with Bernoulli(r) thinning, a pivot with initial predecessor count d_0 reaches $d_{\mathrm{pre}} < k$ in expected time:*

$$\mathbb{E}[\tau_{<k}] = \frac{\ln(d_0/k)}{\ln(1/r)}. \tag{12}$$

Proof. Under committed semantics, the predecessor count evolves as a pure death process: $d_t \sim \mathrm{Bin}(d_0, r^t)$. The expected count decays as $\mathbb{E}[d_t] = d_0 r^t$. Setting $d_0 r^{t^*} = k$ and solving:

$$t^* = \frac{\ln(d_0/k)}{\ln(1/r)}.$$

The first-passage time $\tau_{<k}$ concentrates around t^* with fluctuations of order $O(\sqrt{t^*})$ by the Azuma–Hoeffding inequality applied to the supermartingale d_t.

At canonical parameters $d_0 = 253$, $k = 3$, $r = 0.80$: $t^* = \ln(253/3)/\ln(1/0.8) \approx 4.436/0.2231 \approx 19.9$ steps, matching the empirical value of $\sim$22 steps. The death time is *logarithmic* in d_0, not linear. □

7.8 Operational Retention Floor

The preceding results yield a closed-form criterion for the retention rate below which L2 frontier failures become operationally likely.

Proposition 7.10 (Operational retention floor)**.** *For an INAR(1) system with stationary zero-focal probability $\pi_0(r) = \prod_{m=1}^{\infty}(1-pr^m)^T$, define the **operational retention floor** $r^*(H,\delta)$ as the solution to:*

$$\pi_0(r^*) = \frac{-\ln(1-\delta)}{H}, \tag{13}$$

where H is the operational horizon (total epochs) and δ is the tolerable probability of at least one catastrophic zero-focal event over that horizon.

Asymptotically in large H:

$$c(H) \approx \begin{cases} \dfrac{1-p}{p} \cdot \dfrac{\ln H}{k} & \text{(moderate } r\text{: Euler–Maclaurin regime)}, \\[2ex] \dfrac{1-p}{\mathrm{Li}_2(p)} \cdot \dfrac{\ln H}{k} & \text{(high } r\text{: dilogarithm regime)}, \end{cases} \tag{14}$$

where $c(H) = \mu_{\mathrm{nf}}/k$ parameterizes the required non-focal buffer per unit of prefix demand.

Proof. Over H independent epochs, the probability of at least one zero-focal event is $1-(1-\pi_0)^H$. For small π_0, the Poisson approximation gives $1-(1-\pi_0)^H \approx 1-e^{-H\pi_0}$, so requiring this to equal δ yields $H\pi_0 = -\ln(1-\delta)$, which is (13).

From Theorem 7.8, $\ln(1/\pi_0) \approx \mathrm{Li}_2(p)/p \cdot \mu$ where $\mu = Tpr/(1-r)$. Substituting into (13):

$$\frac{\mathrm{Li}_2(p)}{p} \cdot \mu \approx \ln H + \ln\bigl(-\ln(1-\delta)\bigr)^{-1}.$$

For large H, the $\ln\ln$ correction is negligible, so $\mu \approx p\ln H/\mathrm{Li}_2(p)$. Since $\mu_{\mathrm{nf}} = \mu(1-p)/p$ and $c(H) = \mu_{\mathrm{nf}}/k$, the high-$r$ asymptotic follows.

For the moderate-r regime, the Euler–Maclaurin approximation in Theorem 7.8 is not needed. When r is moderate, the Lambert-series expansion of $\ln(1/\pi_0) = -T\sum_{m=1}^{\infty}\ln(1-pr^m)$ is dominated by its $m=1$ term: $\ln(1/\pi_0) \approx Tpr/(1-r) = \mu_f$ (the mean surviving focal count). Setting $\mu_f \approx \ln H$ and converting to non-focal buffer via $\mu_{\mathrm{nf}} = (1-p)/p \cdot \mu_f$ gives $c(H) \approx (1-p)\ln H/(pk)$ directly, without invoking the dilogarithm. The full Lambert-series derivation with explicit $s=1$ through $s=4$ terms is in Section C. □

Remark 7.11 (Scope and numerical calibration). Theorem 7.10 characterizes the *fixed-focal* retention floor, where L2 failure occurs via complete focal wipeout ($N=0$). At canonical parameters ($r=0.80$, $T=100$, $p=0.367$), $\pi_0 \approx 1.3\times10^{-70}$, confirming that L2 is operationally immortal under fixed-focal semantics (Table 1: 100% survival). The fixed-focal floor from (13) gives $r^* \approx 0.30$ at $H \approx 1.9\times10^7$ epochs, consistent with the empirical regime boundary.

The *dynamic-focal* retention floor at $r \approx 0.75$ involves a distinct failure mechanism: after arg max turnover, the newly selected pivot may have fewer than k predecessors in the surviving population. This predecessor-deficit failure is not captured by π_0 alone and requires a separate analysis of the predecessor distribution conditional on pivot replacement (see Theorem 7.4 and the 4×3 matrix in Table 2).

Remark 7.12 (Three timescales). The compound regime is governed by three well-separated timescales:

(i) **Committed death** (L1): $O(\ln(d_0/k)/\ln(1/r)) \approx 20$ steps at canonical parameters. Fast.

(ii) **L1-retry stationarity**: Stationary feasibility $1-k/\mu_{\mathrm{nf}} \approx 0.988$. Infeasibility episodes are brief $O(1)$ transients.

(iii) **L2 recurrence**: $1/\pi_0 \approx 10^{71}$ events. Astronomically slow.

The system's practical behavior depends on which algebraic level governs it: L1-committed dies fast, L1-retry is stable, and L2 is operationally immortal.

8 Empirical Validation

We summarize the empirical validation of the algebraic results. Full experimental details are in the companion papers [7–9].

8.1 Absorbing-State Boundary

The prefix-constraint impossibility theorem (Theorem 7.3) was validated across 11,400 boundary instances spanning $k = 0,\ldots,5$ and $\varepsilon = 0.05,\ldots,0.95$ (100 seeds per configuration). Under the development-eligible counter j_{dev}:

- **False positives:** 0 (rule-of-three upper bound: $< 0.026\%$).
- **Trap rate:** 54.9% across 4,200 streaming instances (Wilson 95% CI).

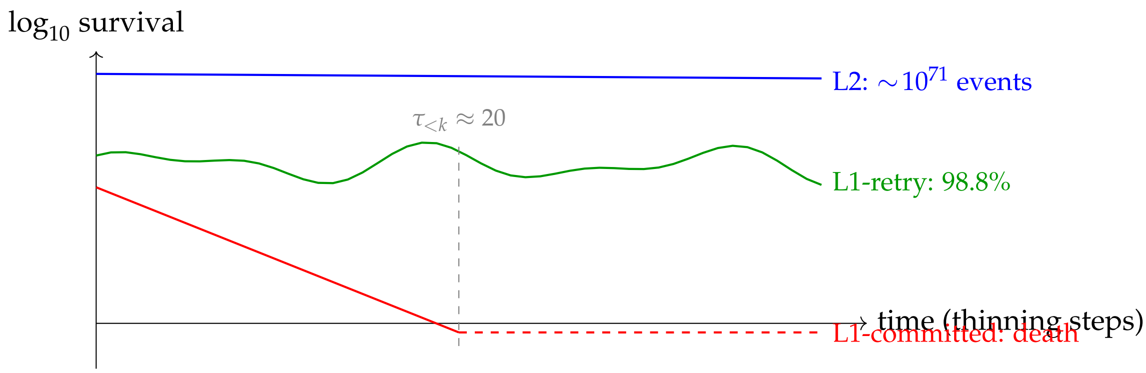

Figure 5: Survival dichotomy under compound streaming and thinning. Committed L1 dies in $O(\log d_0)$ steps. L1-retry fluctuates at ~98.8% stationarity. L2 recurrence time is $\sim 10^{71}$ events.

- **Recall:** 100% (0 false negatives among 1,654 verified instances where min-gap $< k$).

The non-hereditary structure (Theorem 3.4) was confirmed on 40/40 tested instances with $n = 20$ events.

8.2 Algebraic Exactness

The monoid structure (Theorem 3.5) was validated empirically:

- **Pairwise composition:** 240 random pairs, 0 violations vs. event-level reference.
- **Absorbing-ideal closure:** 96 committed absorbing elements composed with random suffixes, 0 escapes.
- **Closure diagnostics:** Among elementary operations (composition, compression, pivot update, split), only compression violates closure (rate 0.133); all others have rate exactly 0.

8.3 Turnover and the 4×3 Matrix

Turnover universality (Theorem 7.4) holds in 103/103 observed recovery events: every recovery is via arg max replacement, none via same-pivot regrowth.

The complete survival matrix across four semantic models and three algebraic levels (500 traces per cell, Wilson 95% CIs) is:

Table 2: Survival probability across semantic models and algebraic levels. All entries from 500-trace simulations at canonical parameters ($r = 0.80$, $T = 100$, $p = 0.367$, $k = 3$). Wilson 95% confidence intervals shown for L1-retry, the column with non-trivial variance; L1-absorb and L2 columns have near-degenerate distributions (all-or-nothing survival).

Model	**L1-absorb**	**L1-retry (95% CI)**	**L2**
Fixed-focal	67.6%	**98.2%** (96.6–99.1%)	**100.0%**
Dynamic-focal	60.8%	**98.8%** (97.4–99.4%)	60.8%
Protected-record	0.2%	20.6% (17.3–24.4%)	0.2%
K-core	0.8%	25.0% (21.4–29.0%)	0.8%

The table reveals two structural findings. First, L1-retry outperforms L2 under dynamic-focal semantics (98.8% vs. 60.8%), inverting the algebraic hierarchy. Second, pivot protection (protected-

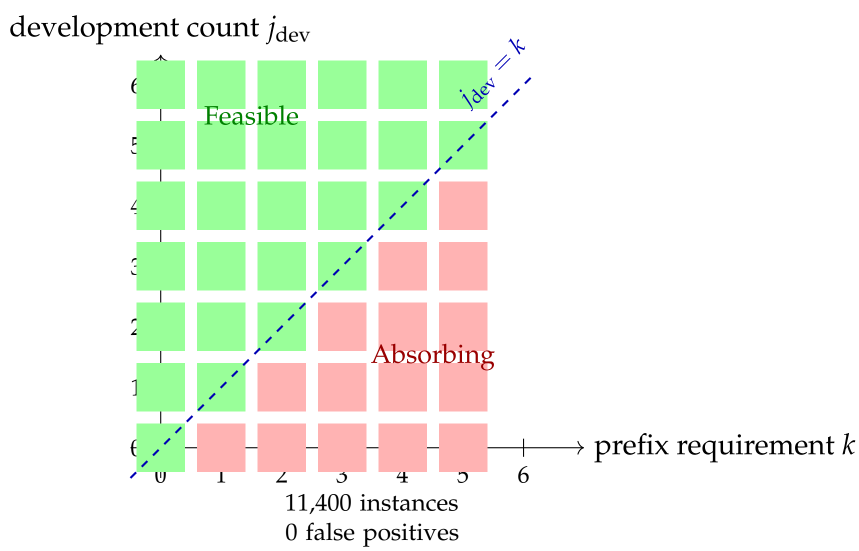

Figure 6: Greedy validity boundary in the (j_{dev}, k) plane. Each cell represents a configuration tested across 100 seeds at $\varepsilon \in \{0.05, \ldots, 0.95\}$. Green: feasible ($j_{\text{dev}} \geq k$). Red: absorbed ($j_{\text{dev}} < k$). The boundary $j_{\text{dev}} = k$ (dashed) is exact: zero false positives across 11,400 boundary instances.

record and K-core) is catastrophic under dynamic-focal semantics: it destroys the arg max turnover mechanism that L1-retry depends on.

8.4 Turnover Universality

Table 3: Recovery mechanism in L1-retry under dynamic-focal semantics (103 recovery events observed).

Recovery mechanism	**Count**	**Rate**
New arg max (turnover)	103	100.0%
Same arg max regains predecessors	0	0.0%

9 Discussion

9.1 Summary of Contributions

This paper establishes the algebraic foundations of endogenous-pivot semantics: a three-level tower (L1/L2/L3) with exact characterization of the absorbing pathology, a tropical matrix equivalence enabling parallel computation, a Myhill–Nerode optimality proof for L2, and a record-gap spectrum placing streaming trap rates in the Ewens($\theta = 1$) universality class. The dynamics under compression are governed by three well-separated timescales: fast committed death, stable L1-retry ergodicity, and astronomically slow L2 recurrence. The operational retention floor (Theorem 7.10) closes the gap between the empirical regime boundary and the algebraic framework, resolving what was an open problem at the outset of this work.

9.2 Related Work

Our construction intersects several established literatures.

Tropical geometry and max-plus systems. The tropical matrix equivalence (Theorem 4.3) embeds our setting in the theory of max-plus linear algebra developed by Baccelli et al. [3]. In classical max-plus systems, the state matrices arise from deterministic timing constraints—job durations, buffer capacities—and the spectral theory of max-plus eigenvalues governs throughput and cycle times. Our matrices differ in that they encode *semantic* composition: each entry records a feasibility frontier across predecessor budgets, and the matrix product implements the shift-and-max rule rather than a scheduling recurrence. As a consequence, the spectral properties of our matrices are determined by the grammar parameter k (the prefix requirement) rather than by network topology. Maclagan and Sturmfels [14] and Joswig [12] provide comprehensive treatments of tropical geometry; our use is more algebraic than geometric, focused on the monoid structure of $\mathbb{T}_{\max}^{(k+2)\times(k+2)}$ rather than on tropical varieties or Newton polytopes.

Record theory and the CRP. The record-gap spectrum (Theorem 6.3) extends classical record theory [1, 16] by introducing the focal-mask model, which modifies the standard record process via Bernoulli masking. Rényi's foundational work [19] established the $1/g$ harmonic intensity for i.i.d. records; our $\lambda_g(p) = [1-(1-p)^g]/g$ is the masked generalization, recovering Rényi's result at $p \to 1$. The Chinese Restaurant Process coupling (Theorem 6.3, Step E) connects the gap spectrum to the Ewens(θ=1) sampling formula [6]. In the classical population genetics setting, Ewens(θ) governs allele frequency spectra under neutral mutation; our coupling maps gap sizes to table sizes via the integral representation $\lambda_g = \int_0^a e^{-gy}\,dy$ with $a = -\ln(1-p)$. The truncation at $a < \infty$ (rather than the $a = \infty$ used in the standard CRP) reflects the focal mask: a fraction $1-p$ of events are invisible to the record process. This connection places the record-gap spectrum in the logarithmic combinatorial structures framework of Arratia–Barbour–Tavaré [2].

Automata theory and state lower bounds. The Myhill–Nerode necessity result (Theorem 5.2) adapts the classical automata-theoretic technique [10] to algebraic summaries. The standard theorem establishes that the number of states in a minimal DFA equals the number of equivalence classes of the syntactic right-congruence. Our adaptation replaces the DFA by an associative summary algebra and the acceptance predicate by feasibility at threshold k. The resulting $\Omega(k)$ lower bound is tight (L2 achieves it), making L2 optimal among valid summaries (we conjecture it is, in fact, the initial object in the category of valid summaries). This contrasts with communication complexity lower bounds for related problems (e.g., set disjointness, gap Hamming) where the lower bound is typically information-theoretic. Our bound is structural: it counts the number of *distinguishable futures* that a summary must separate, and the separating suffixes have an explicit combinatorial construction (Theorem 5.3).

Choice theory and path independence. The path-independence connection (Section A) links our associativity result to Plott [18] and the Moulin [15] classification of choice functions. Plott's theorem establishes that single-valued path-independent choice over a finite set is equivalent to arg max of a linear order—which is precisely our pivot selection rule. The Danilov–Koshevoy [5] refinement clarifies that this equivalence breaks for set-valued choice, a subtlety our setting avoids via continuous weights (ensuring singleton selection almost surely). The algebraic consequence is that the endogenous composition $\otimes_{\text{endo}}$ is associative (Theorem 3.5(a)): context elements can be composed in any order because the underlying choice rule is path-independent.

Constrained generation and resource-constrained paths. Grammar-constrained decoding [20, 21] enforces syntactic constraints on LLM output via incremental parsing or finite-automaton masking. These approaches maintain an *exogenous* constraint state: the parse state depends only on the generated prefix, not on future tokens. The resource-constrained shortest path framework [11] similarly maintains resource vectors that evolve monotonically along the path. Our endogenous pivot semantics creates a qualitatively different constraint: the grammar's interpretation depends on the output itself (specifically, on which element is the arg max), so the constraint state cannot be determined incrementally from a prefix. This is the fundamental reason that the L2 tropical vector—which maintains the full feasibility frontier across all possible pivot budgets—is necessary (Theorem 5.2) rather than merely convenient.

9.3 Open Problems

1. **Complexity ceiling.** The conjecture $k_{\max} = \lfloor 1/(4\sigma^2) \rfloor$ relating the maximum supportable prefix requirement to thinning variance awaits a rigorous mapping from σ^2 to the INAR parameters (r, p, T).

2. **Finite-r dilogarithm correction.** The Euler–Maclaurin approximation in Theorem 7.8 introduces a $\sim$1.4% gap between $\mathrm{Li}_2(p)/p$ and the empirical scaling constant. An $O(\varepsilon)$ correction term would close this.

3. **Burstiness deformation.** The focal-mask model assumes i.i.d. focal assignments. A Nevzorov F_α deformation for front-loaded streams would extend Theorem 6.3 to realistic bursty distributions.

4. **Tight converse.** Theorem 7.4 proves committed death; Theorem 7.10 gives the fixed-focal floor $r^*(H, \delta)$. A full converse—for any fixed horizon H, no strategy achieves survival probability $> 1 - \delta$ below $r^*(H, \delta)$—requires a matching lower bound showing that L2 also fails at low retention.

5. **Multi-agent coherence.** A sheaf-cohomology decomposition of multi-agent extraction failure is conjectured, with H^1 encoding structural (combinatorial) obstructions and H^2 encoding metric (weight-threshold) obstructions, but no computation exists.

9.4 Limitations

The focal-mask model assumes i.i.d. focal events with continuous weights, which may not hold in all applications. The empirical validation uses synthetic event graphs; real-world streams may exhibit correlations not captured by the model. The retention floor (Theorem 7.10) is derived under the i.i.d. assumption; correlated focal weights or non-stationary thinning rates may shift r^* per domain. The Myhill–Nerode argument assumes singleton path-independent pivot selection; set-valued or non-path-independent variants may require different state bounds.

10 Conclusion

We have developed a complete algebraic theory for sequential extraction under endogenous pivot semantics. The core insight is that the arg max-selected pivot creates a circular dependence between the selected set and its interpretation, placing the problem outside classical greedy frameworks. The three-level algebraic tower—scalar monoid, tropical vector, frontier semiring—captures this pathology at increasing levels of resolution, with the tropical vector (L2) provably optimal via a Myhill–Nerode argument.

The tropical matrix equivalence connects our construction to the well-studied theory of max-plus linear systems, and the record-gap spectrum places streaming trap rates in the Ewens$(\theta = 1)$ universality class—a surprising connection between endogenous selection and combinatorial stochastic processes.

Under compound streaming and thinning, the system exhibits three well-separated timescales: committed death is logarithmically fast, L1-retry achieves near-perfect stationarity at $O(1)$ cost, and L2 recurrence involves the Euler dilogarithm and is operationally infinite. These results provide the algebraic foundation for designing safe compression policies in systems with endogenous semantics.

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A Plott Path Independence Connection

The associativity of the endogenous composition $\otimes_{\text{endo}}$ is intimately related to Plott's path independence axiom [18]; see Moulin [15] for a comprehensive survey of choice functions over finite sets. Path independence requires:

$$\tau(A \cup \tau(B)) = \tau(A \cup B).$$

Plott's theorem establishes that a single-valued path-independent choice function over a finite set is equivalent to arg max of a strict linear order.

In our setting, the endogenous pivot $\tau(S) = \arg\max_{v \in S, a(v) = a^*} w(v)$ is precisely such a function (assuming continuous weights for almost-sure uniqueness). The associativity of $\otimes_{\text{endo}}$ is the algebraic consequence: context elements can be composed in any order because the pivot selection rule is path-independent.

The Danilov–Koshevoy refinement [5] clarifies that path independence alone does *not* force $\arg\max$ when the choice function is set-valued. The counterexample $\tau(S) = \{\min(S), \max(S)\}$ is path-independent but not representable as $\arg\max$ of a single order. Our setting avoids this subtlety because continuous weights ensure singleton-valued selection.

B EIS Fiber Structure

The feasible family $\mathcal{F}$ decomposes into **fibers** indexed by pivot identity: $\mathcal{F} = \bigcup_{p \in V} F_p$ where $F_p = \{S \in \mathcal{F} : \tau(S) = p\}$.

Each fiber is an *order filter* (upper set) in the Boolean lattice of subsets: if $S \in F_p$ and $S \subseteq S'$ with $\tau(S') = p$, then $S' \in F_p$ (adding events preserves feasibility when the pivot is unchanged). Fibers are *not* matroids, since matroids require downward closure (the hereditary axiom), while our fibers are upward closed.

Three deletion operations characterize fiber boundaries:

1. **Post-pivot deletion:** pivot and d_{pre} unchanged. Generally safe.
2. **Pre-pivot non-focal deletion:** pivot unchanged but d_{pre} decreases by 1. Exits feasibility at the $d_{\text{pre}} = k$ boundary.
3. **Pivot deletion:** fiber transition to $F_{p'}$ where p' is the next-heaviest focal event. Structure changes entirely.

C INAR(1) Stationary Distribution

The INAR(1) process $N_{n+1} = \text{Bin}(N_n, r) + \text{Bin}(T, pr)$ has a unique stationary distribution. The probability generating function satisfies:

$$G(z) = \mathbb{E}[z^{N_\infty}] = \prod_{m=0}^{\infty} (1 - pr^{m+1} + pr^{m+1} z)^T = \prod_{m=1}^{\infty} (1 - pr^m(1-z))^T.$$

Setting $z = 0$:

$$\pi_0 = G(0) = \prod_{m=1}^{\infty} (1 - pr^m)^T,$$

which is Equation (11). The Kac mean return time is $\mathbb{E}[\tau_0] = 1/\pi_0$. At $r = 0.80$, $T = 100$, $p = 0.367$: $\pi_0 \approx 1.3 \times 10^{-70}$, giving $1/\pi_0 \approx 7.55 \times 10^{69}$ epochs $\approx 10^{71}$ events.

The scaling $\ln(1/\pi_0) \sim c \cdot \mu$ with $c = \text{Li}_2(p)/p$ follows from the Euler–Maclaurin analysis in the proof of Theorem 7.8.

Lambert-series expansion and the retention floor

The logarithm of the stationary zero probability admits a Lambert-series expansion that clarifies both the moderate-r and high-r regimes of Theorem 7.10.

$$\ln(1/\pi_0) = -T \sum_{m=1}^{\infty} \ln(1 - pr^m) = T \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{(pr^m)^s}{s} = T \sum_{s=1}^{\infty} \frac{p^s}{s} \cdot \frac{r^s}{1 - r^s}. \tag{15}$$

The last equality follows by exchanging the order of summation and using $\sum_{m=1}^{\infty} r^{ms} = r^s/(1 - r^s)$.

Term-by-term structure. Denote the s-th term $a_s = Tp^s r^s/[s(1-r^s)]$. At canonical parameters ($r = 0.80$, $T = 100$, $p = 0.367$):

s	p^s	$r^s/(1-r^s)$	a_s	cumulative
1	0.367	4.000	146.8	146.8
2	0.135	1.778	12.0	158.8
3	0.049	1.047	1.72	160.5
4	0.018	0.719	0.33	160.8
$\vdots$				
∞				161.1

The $s = 1$ term alone captures $146.8/161.1 = 91.1\%$ of $\ln(1/\pi_0)$. The first four terms account for 99.8%.

Moderate-r regime. When r is small enough that $r^s \ll 1$ for $s \geq 2$, the $s = 1$ term dominates:

$$\ln(1/\pi_0) \approx a_1 = \frac{Tpr}{1-r} = \mu_f,$$

the mean surviving focal count. Setting $\mu_f = \ln H$ (from the retention floor equation $\pi_0 = 1/H$ at $\delta \ll 1$) and converting to non-focal buffer:

$$c(H) = \frac{\mu_{\mathrm{nf}}}{k} = \frac{(1-p)}{p} \cdot \frac{\mu_f}{k} \approx \frac{(1-p)\ln H}{pk}.$$

This is exact to $O(p^2r^2)$ relative error and does not require the dilogarithm.

High-r regime. As $r \to 1$, higher-order terms in (15) become significant. The Euler–Maclaurin approximation replaces the sum over s by an integral:

$$\sum_{s=1}^{\infty} \frac{p^s}{s} \cdot \frac{r^s}{1-r^s} \approx \frac{1}{-\ln r} \sum_{s=1}^{\infty} \frac{p^s}{s^2} = \frac{\mathrm{Li}_2(p)}{-\ln r},$$

where we used $r^s/(1-r^s) \approx 1/(s|\ln r|)$ for small $|\ln r|$. This gives $\ln(1/\pi_0) \approx T \cdot \mathrm{Li}_2(p)/|\ln r| = \mathrm{Li}_2(p)/p \cdot \mu$, recovering the dilogarithm scaling of Theorem 7.8.

Convergence at the fixed-focal floor. At $r^* \approx 0.304$ (the empirical fixed-focal retention floor), the term ratios are: $a_2/a_1 = 0.033$, $a_3/a_1 = 0.0014$. The $s = 1$ approximation $\ln(1/\pi_0) \approx \mu_f$ is accurate to 3.4% at this retention rate, confirming that the moderate-r formula in Theorem 7.10 applies. At $r = 0.80$ (canonical), $a_2/a_1 = 0.082$ and the dilogarithm correction is essential—the $s = 1$ term alone understates $\ln(1/\pi_0)$ by 9.7%.