List of problems for the first mid-term exam

The first mid-term for this class will consist of a subset of the problems in this list, possibly with some choices.

(1) Let G be a group (not necessarily finite) and H a nonempty subset of G that is closed under multiplication (i.e., if $x, y \in H$ then $xy \in H$). Suppose that for all $g \in G$ we have $g^2 \in H$. Prove the following:
  (a) H is a subgroup of G.
  (b) H is a normal group of G.
  (c) G/H is abelian.

(2) Fix a prime number p, and let A denote the abelian group of all complex roots of unity whose orders are powers of p; that is,

$$A = \{z \in \mathbb{C} \mid z^{p^n} = 1 \text{ for some integer } n \geq 1\}.$$

Prove the following statements.
  (a) Every non-trivial subgroup of A contains the group of p-th roots of unity.
  (b) Every proper subgroup of A is cyclic.
  (c) If B and C are subgroups of A, then either $B \subseteq C$ or $C \subseteq B$.
  (d) For each $n \geq 0$, there exists a unique subgroup of A with p^n elements.

(3) Let K be a subgroup of a group G.
  (a) Prove that K is a normal subgroup of G if and only if there is a group H and a homomorphism $f : G \to H$ such that $K = \ker(f)$.
  (b) Let G be the group of all 2×2 matrices with entries from $\mathbb{Z}$ having determinant 1. Let p be a prime number and take K to be the subset of G consisting of all matrices that are congruent to I_2 modulo p — that is, K consists of all matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a \equiv 1 \pmod{p}$, $d \equiv 1 \pmod{p}$, $b \equiv 0 \pmod{0}$, and $c \equiv 0 \pmod{p}$. Prove that K is a normal subgroup of G.

(4) Let G be a group and H and K subgroups. Recall that HK is the subset of G defined as $HK = \{hk \mid h \in H, k \in K\}$.
  (a) Prove $HK = KH$ if and only if HK is a subgroup of G.
  (b) Give an example (with justification) where HK is not a subgroup.

(5) Let G be a group and H a subgroup of G. The centralizer of H in G is the set of elements of G that commute with each element of H:

$$C_G(H) =: \{g \in G \mid gh = hg \text{ for all } h \in H\ \}.$$

Prove that if H is normal in G, then $C_G(H)$ is a normal subgroup of G and that $G/C_G(H)$ is isomorphic to a subgroup of the automorphism group of H.

(6) Let G be a group acting on a set A, and let H be a subgroup of G satisfying the condition that the induced action of H on A is transitive (that is, for all $a, b \in A$ there is an $h \in H$ with $ha = b$). Let $t \in A$, and let $\mathrm{Stab}_G(t)$ be the stabilizer of t in G. Show that $G = H\,\mathrm{Stab}_G(t)$.

(7) Let G be a finite group.
  (a) If N is a normal subgroup of G and $\#N = 2$, prove that N is contained in the center $Z(G)$ of G.
  (b) Suppose that $\#Z(G)$ is odd and that G contains a non-trivial simple subgroup H with $[G : H] = 2$. (Note: H is simple provided it has no normal subgroups other than $\{e\}$ and H itself.) Prove that H is the only non-trivial proper normal subgroup of G.

(8) Let G be a group. A subgroup H of G is called a *characteristic subgroup* of G if $\alpha(H) = H$ for every automorphism α of G. Show that if H is a characteristic subgroup of N and N is a normal subgroup of G, then H is a normal subgroup of G.

(9) Let H and K be groups. Recall that for any group G, an automorphism of G is an isomorphism from G to G, and $\mathrm{Aut}(G)$ denotes the group (under composition) of all automorphisms of G.
  (a) Show that the direct product group $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(H \times K)$.
  (b) Give an example, with justification, of groups H and K for which $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is not isomorphic to $\mathrm{Aut}(H \times K)$.

(10) Let G be a group of order p^n, for some prime p, acting on a finite set X.
  (a) Suppose p does not divide $\#X$. Prove that there exists some element of X that is fixed by all elements of G.
  (b) Suppose G acts faithfully on X. (Recall this means that if $g \cdot x = x$ for all $x \in X$, then $g = e_G$.) Prove that $\#X \geq n \cdot p$.

(11) Let $G = A_7$ and S be the set of all elements of G of order 7. Prove that S is not a conjugacy class of G.

(12) Let p be a prime number, G be a finite p-group (i.e., $|G| = p^n$ for some n), Z be the center of G, and $N \neq \{e\}$ a non-trivial, normal subgroup of G. Prove that $N \cap Z \neq \{e\}$. *Tip*: Use that G acts on N by conjugation.