

SUBGROUPS

DEFINITION: Let G be a group. A nonempty subset H of G is a **subgroup** of G if it is a group under the same operation as H (i.e., $h \cdot_H h' = h \cdot_G h'$ for $h, h' \in H$). We write $H \leq G$ to indicate that H is a subgroup of G .

Any group G has two **trivial subgroups** $\{e\}$ and G .

LEMMA 1: Let H be a subset of G .

- **TWO STEP TEST:** If H is nonempty, H is closed under multiplication¹ and H is closed under inverses¹, then H is a subgroup of G .
- **ONE STEP TEST:** If H is nonempty, and for all $x, y \in H$, $xy^{-1} \in H$, then H is a subgroup of G .

LEMMA 2 (GENERAL RECIPES FOR SUBGROUPS): Let G be a group.

- (1) If $H \leq G$ and $K \leq H$, then $H \leq G$.
- (2) If $\{H_\alpha\}_{\alpha \in J}$ is a collection of subgroups of G , then $\bigcap_{\alpha \in J} H_\alpha \leq G$.
- (3) If $f : G \rightarrow H$ is a group homomorphism, then $\text{im}(f) \leq H$.
- (4) If $f : G \rightarrow H$ is a group homomorphism, and $K \leq G$, then $f(K) = \{f(k) \mid k \in K\} \leq H$.
- (5) If $f : G \rightarrow H$ is a group homomorphism, and $K \leq G$, then $\ker(f) \leq G$.
- (6) The center $Z(G)$ is a subgroup of G .

(1) Proving subsets are subgroups:

- (a)** Choose a couple of parts of Lemma 2 and prove them; you can use Lemma 1.
- (b)** Let $n \in \mathbb{Z}_{\geq 1}$, and define $\text{SL}_n(\mathbb{R})$ to be the set of $n \times n$ real matrices with determinant 1. Show² that $\text{SL}_n(\mathbb{R}) \leq \text{GL}_n(\mathbb{R})$.
- (c)** Let $n \in \mathbb{Z}_{\geq 1}$. Recall from linear algebra that an $n \times n$ matrix Q is **orthogonal** if $Q^T Q = I$, where T denotes transpose and I denotes the identity matrix. Define $\text{O}_n(\mathbb{R})$ to be the set of $n \times n$ real orthogonal matrices. Show that $\text{O}_n(\mathbb{R}) \leq \text{GL}_n(\mathbb{R})$.
- (d)** Define $\text{SO}_n(\mathbb{R})$ to be the set of $n \times n$ real orthogonal matrices that have determinant 1. Show that $\text{SO}_n(\mathbb{R}) \leq \text{GL}_n(\mathbb{R})$.

(2) Prove or disprove: The union of two subgroups of a group is a subgroup.

(3) Prove Lemma 1.

DEFINITION: Let G be a group, and $S \subseteq G$ be a subset. The **subgroup of G generated by S** is the intersection of all subgroups of G that contain S :

$$\langle S \rangle := \bigcap_{\substack{H \leq G \\ S \subseteq H}} H$$

CAYLEY'S THEOREM:

¹A subset $H \subseteq G$ is *closed under multiplication* if $x, y \in H \Rightarrow xy \in H$ and *closed under inverses* if $x \in H \Rightarrow x^{-1} \in H$.

²Hint: This becomes very quick with a proper use of Lemma 2.