Making sense of quantifier statements.

- The symbol for "**for all**" is $\forall$ and the symbol for "**there exists**" is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that $\text{not } P$".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, $\text{not } P$".

A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both)!

(1) Match each informal story (i)–(iv) below with a precise quantifier statement (A)–(D).

Informal stories:	*Precise statements:*
(i) Every number past some point is red.	(A) For every $y > 0$, y is red.
(ii) There are arbitrarily big red numbers.	(B) There exists $y > 0$ such that y is red.
(iii) All positive numbers are red.	(C) For every $x \in \mathbb{R}$, there is some $y > x$ such that y is red.
(iv) There are positive red number(s).	(D) There exists $x \in \mathbb{R}$ such that for every $y > x$, y is red.

(2) Draw a picture where (A) is false and (B) is true.

(3) Draw a picture where (C) is true and (D) is false.

(4) Suppose that (C) is true. Which of the following statements must also be true? Why?
  (a) There is some $y > 1000000000$ such that y is red.
  (b) For every $\mu \in \mathbb{R}$, there is some $\theta > \mu$ such that θ is red.
  (c) For every $x \in \mathbb{R}$, there is some $y > 2x$ such that y is red.

The next problem is no longer about a spraypainting of the real number line.

(5) Rewrite each statement with symbols in place of quantifiers, and write its negation. Do you think the original statement is true or false (but don't prove them yet)?.
  (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
  (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
  (c) For all $x \in \mathbb{R}$ such that[1] $x \neq 0$, $x^2 > 0$.
  (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
  (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

[1] In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

Proving quantifier statements and using the axioms of $\mathbb{R}$.

- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

Note: explaining *how* you found your example"x" is *not* a logically necessary part of the proof.

(6) Circle the correct answer in each of the blanks below:
- To prove a "for all" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *dis*prove a "for all" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To prove a "there exists" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *dis*prove a "there exists" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.

- If you want to *use* a "for all" statement that you know is true,
  you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE
- If you want to *use* a "there exists" statement that you know is true,
  you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE

(7) Prove or disprove each of the statements in (5) (skip (c) for now) using the axioms of $\mathbb{R}$ and facts we have already proven.

More practice with quantifier statements.

(8) Prove that there exists some $x \in \mathbb{R}$ such that $2x + 5 = 3$.

(9) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.

(10) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have already proven to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

(11) Prove that[2] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

(12) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.

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[2]Hint: Use (10).