

Making sense of quantifier statements.

- The symbol for “**for all**” is \forall and the symbol for “**there exists**” is \exists .
- The negation of “For all $x \in S$, P ” is “There exists $x \in S$ such that not P ”.
- The negation of “There exists $x \in S$ such that P ” is “For all $x \in S$, not P ”.

A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both)!

- (1) Match each informal story (i)–(iv) below with a precise quantifier statement (A)–(D).

Informal stories:

Precise statements:

- | | |
|---|--|
| (i) Every number past some point is red. | (A) For every $y > 0$, y is red. |
| (ii) There are arbitrarily big red numbers. | (B) There exists $y > 0$ such that y is red. |
| (iii) All positive numbers are red. | (C) For every $x \in \mathbb{R}$, there is some $y > x$ such that y is red. |
| (iv) There are positive red number(s). | (D) There exists $x \in \mathbb{R}$ such that for every $y > x$, y is red. |

- (2) Draw a picture where (A) is false and (B) is true.

- (3) Draw a picture where (C) is true and (D) is false.

- (4) Suppose that (C) is true. Which of the following statements must also be true? Why?

- (a) There is some $y > 1000000000$ such that y is red.
- (b) For every $\mu \in \mathbb{R}$, there is some $\theta > \mu$ such that θ is red.
- (c) For every $x \in \mathbb{R}$, there is some $y > 2x$ such that y is red.

The next problem is no longer about a spraypainting of the real number line.

- (5) Rewrite each statement with symbols in place of quantifiers, and write its negation. Do you think the original statement is true or false (but don’t prove them yet)?.

- (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
- (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
- (c) For all $x \in \mathbb{R}$ such that¹ $x \neq 0$, $x^2 > 0$.
- (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
- (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

¹In a statement of the form “For all $x \in S$ such that Q , P ”, the “such that Q ” part is part of the hypothesis: it is restricting the set S that we are “alling” over.

Proving quantifier statements and using the axioms of \mathbb{R} .

- The general outline of a proof of “For all $x \in S$, P ” goes
 - (1) Let $x \in S$ be arbitrary.
 - (2) Do some stuff.
 - (3) Conclude that P holds for x .
- To prove a there exists statement, you just need to give an example. To prove “There exists $x \in S$ such that P ” directly:
 - (1) Consider² x = [some specific element of S].
 - (2) Do some stuff.
 - (3) Conclude that P holds for x .

Note: explaining *how* you found your example “ x ” is *not* a logically necessary part of the proof.

(6) Circle the correct answer in each of the blanks below:

- To prove a “for all” statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *disprove* a “for all” statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To prove a “there exists” statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *disprove* a “there exists” statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.

- If you want to *use* a “for all” statement that you know is true,
you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE
- If you want to *use* a “there exists” statement that you know is true,
you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE

(7) Prove or disprove each of the statements in (??) (skip (c) for now) using the axioms of \mathbb{R} and facts we have already proven.

More practice with quantifier statements.

- (8) Prove that there exists some $x \in \mathbb{R}$ such that $2x + 5 = 3$.
- (9) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.
- (10) Let x be a real number. Use the axioms of \mathbb{R} and facts we have already proven to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.
- (11) Prove that² for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.
- (12) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.

²Hint: Use (??).