IF THEN STATEMENTS

If P and Q are *statements* (sentences that are either true or false) then "If P then Q" is another statement. "If P then Q" is true when P is false or Q is true; it is false when P is true and Q is false. We might write $P \Rightarrow Q$ as shorthand for "If P then Q", but avoid this shorthand in proofs and theorem statements. There are many different ways to word an if then statement! Basically anything with a hypothesis and a conclusion is an if then statement; the hypothesis is the "if" part (in the role of P above) and the conclusion is the "then" part (in the role of Q above).

Proving an if then statement directly. The general outline of a (direct) proof of "If P then Q" goes

(1) Assume P.
(2) Do some stuff.
(3) Conclude Q.

Warning: in proving "If P then Q" we should never assume Q or just assert Q; we need to earn it, given P.

Applying an if then statement. If we know or can assume that "If P then Q" is true, and we also know that P is true, then we know that Q is true.

CONVERSES

The *converse* of the statement "If P then Q" is the statement "If Q then P". In symbols, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. An if then statement can be true and its converse be false! These are different, independent statements!

CONTRAPOSITIVES

The *contrapositive* of the statement "If P then Q" is the statement "If not Q then not P". In short, the contrapositive of $P \Rightarrow Q$ is $(\text{not}\, Q) \Rightarrow (\text{not}\, P)$. The contrapositive of a statement is logically equivalent[1] to the original statement!

Proving an if then statement by contraposition. Another way to prove an if then statement is by proving its contrapositive. The general outline of a proof of "If P then Q" *by contraposition* goes

(0) "We prove the contrapositive" or "We proceed by contraposition" (to orient the reader)
(1) Assume not Q.
(2) Do some stuff.
(3) Conclude not P.

IF-AND-ONLY-IF STATEMENTS

If P and Q are statements, then "P if and only if Q" means "If P then Q" and "If Q then P". The proof of such a statement generally has two parts: a proof of "If P then Q" (either directly or by contraposition) and a proof of "If Q then P" (either directly or by contraposition).

[1] Proof: "If not Q then not P" is false exactly when "not Q" is false and P is true, which is equivalent to P is true and Q is false, which happens exactly when "If P then Q" is false.

Quantifiers refers either *for all* or *there exists* quantifiers.

FOR ALL STATEMENTS

A *for all statement* is a statement of the form "For all $x \in S$, P" where S is a set and P is a statement (that might depend on x). It is true if every element of the set S makes the statement P true. In the statement "For all $x \in S$, P", the x is a *dummy variable*, which means it's just a temporary name given to help explain the statement; we need to use a letter x that doesn't mean anything yet, and after we've finished this sentence, the letter x no longer means anything! The symbol $\forall$ is shorthand for "for all".

We sometimes also write statements like "For all $x \in S$ such that Q, P" where S is a set and P and Q are statements (that might depend on x). It is true if every element of the set S that makes the statement Q true also makes the statement P true. The "such that" clause is restricting which elements of S we are "alling" over.

Proving a for all statement directly. The general outline of a proof of "For all $x \in S$, P" goes

(1) Let $x \in S$ be arbitrary.
(2) Do some stuff.
(3) Conclude that P holds for x.

Applying a for all statement. If we know or can assume that "For all $x \in S$, P" is true, and we have some element $y \in S$, then we can conclude that P holds for y.

THERE EXISTS STATEMENTS

A *there exists statement* is a statement of the form "There exists $x \in S$ such that P" where S is a set and P is a statement (that might depend on P). Again, the x in this statement is a dummy variable. The symbol $\exists$ is shorthand for "there exists".

Proving a there exists statement directly. To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:

(1) Consider $x =$[some specific element of S].
(2) Do some stuff.
(3) Conclude that P holds for x.

Applying a there exists statement. If we know or can assume that "There exists $x \in S$ such that P" is true, then you can take and use an element of S for which P is true. I.e., you can say "Let $x \in S$ be such that P."

NEGATIONS OF QUANTIFIER STATEMENTS

To negate a statement with a quantifier: *Switch the quantifier, and negate the condition.*

Negation of a for all statement. The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that $\text{not } P$". Switch the quantifier, and negate the condition. Thus, to *dis*prove a for all statement, we give a counterexample.

Negation of a there exists statement. The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, $\text{not } P$". Switch the quantifier, and negate the condition. Thus, to *dis*prove a for all statement, we give a prove a for all statement.