1. August 23, 2022

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \ldots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not have thought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\ldots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\ldots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number.

Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is* 2.

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \Longrightarrow \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction.

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. $\square$

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.2 (Arithmetic and order properties of $\mathbb{Q}$)**.** *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

(1) *There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p+q$ and $p \cdot q$.*
(2) *Each of $+$ and $\cdot$ is a commutative operation (i.e., $p+q = q+p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*
(3) *Each of $+$ and $\cdot$ is an associative operation (i.e., $(p+q)+r = p+(q+r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q, and r).*
(4) *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0 + q = q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.*
(5) *The distributive law holds: $p \cdot (q + r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.*
(6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.*
(7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.*
(8) *There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that*
    (a) *For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
    (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
    (c) *For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
(9) *The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.*
(10) *The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

Which of the properties from Proposition 1.2 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$.

We expect everything from Proposition 1.2 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written* $\mathbb{R}$*, satisfies the following eleven properties:*

(Axiom 1) *There are operations* $+$ *and* $\cdot$ *defined on* $\mathbb{R}$*, so that if* $x, y \in \mathbb{R}$*, then so are* $x + y$ *and* $x \cdot y$*.*

(Axiom 2) *Each of* $+$ *and* $\cdot$ *is a commutative operation.*

(Axiom 3) *Each of* $+$ *and* $\cdot$ *is an associative operation.*

(Axiom 4) *The real number* 0 *is an identity element for addition and the real number* 1 *is an identity element for multiplication. This means that* $0 + x = x$ *and* $1 \cdot x = x$ *for all* $x \in \mathbb{R}$*.*

(Axiom 5) *The distributive law holds:* $x \cdot (y + z) = x \cdot y + x \cdot z$ *for all* $x, y, z \in \mathbb{R}$*.*

(Axiom 6) *Every real number has an additive inverse: For any* $x \in \mathbb{R}$*, there is a number* $-x$ *satisfying* $x + (-x) = 0$*.*

(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any* $x \in \mathbb{R}$ *such that* $x \neq 0$*, there is a real number* x^{-1} *satisfying* $x^{-1} \cdot x = 1$*.*

(Axiom 8) *There is a "total ordering"* $\leq$ *on* $\mathbb{R}$*. This means that*

  (a) *For all* $x, y \in \mathbb{R}$*, either* $x \leq y$ *or* $y \leq x$*.*

  (b) *If* $x \leq y$ *and* $y \leq z$*, then* $x \leq z$*.*

  (c) *For all* $x, y, z \in \mathbb{R}$*, if* $x \leq y$ *and* $y \leq z$*, then* $x \leq z$*.*

(Axiom 9) *The total ordering* $\leq$ *is compatible with addition: If* $x \leq y$ *then* $x + z \leq y + z$ *for all* z*.*

(Axiom 10) *The total ordering* $\leq$ *is compatible with multiplication by non-negative real numbers: If* $x \leq y$ *and* $z \geq 0$ *then* $zx \leq zy$*.*

(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

> "Cancellation of Addition": For real numbers, $x, y, z \in \mathbb{R}$, if $x + y = z + y$ then $x = z$.

Let's prove this carefully, using just the list of axioms: Assume that $x + y = z + y$. Then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x + y) + (-y) = (z + y) + (-y)$. This can be rewritten as $x + (y + (-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

For another example, we can deduce the following fact from the axioms:

$$r \cdot 0 = 0 \text{ for any real number } r.$$

Let's prove this carefully: Let r be any real number. We have $0 + 0 = 0$ (Axiom 4) and hence $r \cdot (0+0) = r \cdot 0$. But $r \cdot (0+0) = r \cdot 0 + r \cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

As I said, there are many other familiar properties of the real numbers that follow from these axioms, but I will not list them all. The great news is that all of these familiar properties follow from this short list of axioms. We will prove a couple, but for the most part, I'll rely on your innate knowledge that facts such as $r \cdot 0 = 0$ hold.

2. August 25, 2022

Definition 2.1. A real number is *irrational* if it is not rational.

Making sense of if then statements and quantifier statements.

- The *converse* of the statement "If P then Q" is the statement "If Q then P".
- The *contrapositive* of the statement "If P then Q" is the statement "If not Q then not P".
- Any if then statement is equivalent to its contrapositive, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Is the original/contrapositive/converse true or false for real numbers a, b? Explain why (but don't prove).
   (a) If a is irrational, then $1/a$ is irrational.
   (b) If a and b are irrational, then ab is irrational.
   (c) If $a > 3$, then $a^2 > 9$.

(1) true; contrapositive is "If $1/a$ is rational, a is rational" is true; converse is "if $1/a$ is irrational, then a is irrational" is true.
(2) false; contrapositive is "If ab is rational, either a or b is rational" is false; converse is "if ab is irrational then a and b are irrational" is false.
(3) true; contrapositive is "if $a^2 \leq 9$, then $a \leq 3$ is true; converse is "if $a^2 > 9$ then $a > 3$" is false.

- The symbol for "for all" is $\forall$ and the symbol for there exists is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that not P".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, not P".

(2) Rewrite each statement with symbols in place of quantifiers, and write its negation. Is the original statement true or false? Explain why (but don't prove them).
  (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
  (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
  (c) For all $x \in \mathbb{R}$ such that[1] $x \neq 0$, $x^2 > 0$.
  (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
  (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

(1) $\exists x \in \mathbb{Q} : x^2 = 2$ is false. Negation: $\forall x \in \mathbb{Q}, x^2 \neq 2$.
(2) $\forall x \in \mathbb{R}$, $x^2 > 0$ is false. Negation: $\exists x \in \mathbb{R} : x^2 \leq 0$.
(3) $\forall x \in \mathbb{R} : x \neq 0, x^2 > 0$ is true. Negation: $\exists x \in \mathbb{R} : x \neq 0, x^2 \leq 0$.
(4) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$ is true. Negation: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x \geq y$.
(5) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x < y$ is false. Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x \geq y$.

Proving if then statements and quantifier statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if then statement than the original, especially when the negation of the hypothesis or conclusion is something negative.
- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

[1] In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

(3) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[3] that $x \geq y$ if and only if $-y \geq -x$.
(4) Let x be a real number. Show that if x^2 is irrational, then x is irrational.
(5) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have proven in class to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.
(6) Prove that[4] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.
(7) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.
(8) Prove[5] that (2d) is true and (2e) is false.
(9) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.
(10) Prove that $1 > 0$.
(11) Let x, y be real numbers. Prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

(3) Let $x \geq y$. Adding $(-x) + (-y)$ to both sides (which exists by Axiom 6), we obtain $-y = x + ((-x) + (-y)) \geq y + ((-x)+(-y)) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x + y) + (-x) \leq (x + y) + (-y) = x$ (by Axiom 9 and Axiom 5).
(6) Let x and y be nonzero real numbers. By Axiom 7, there are element $x^{-1}, y^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$ and $yy^{-1} = 1$. Then $xy \cdot (x^{-1}y^{-1}) = (xx^{-1})(yy^{-1}) = 1$, using Axioms 2 and 3 in the first equality and Axiom 5 in the second. By the previous fact (applies to xy) we conclude that $xy \neq 0$.
(7) Consider $x = 1$. Let $y \in \mathbb{R}$. By Axiom 4, we have $xy = 1y = y$. Thus, for all $y \in \mathbb{R}$, we have $xy = x$.
(8) (2d): Let $x \in \mathbb{R}$. Consider $y = x + 1$. Since $1 > 0$ we have $y = x + 1 > x + 0 = x$. Thus, for each $x \in \mathbb{R}$, we have some y such that $x < y$. (2e): We claim this is false. Suppose, for the sake of contradiction that this was true, and let x

[2] How you found this x is logically irrelevant to an existence proof, and should not be included.
[3] Hint: You may want to add something to both sides.
[4] Hint: Use (5).
[5] You can "work out of order here" and *use* (10) now.

be as in the statement. Then for any $y \in \mathbb{R}$, we have $x < y$. But, for $y = x$, the inequality $x < y$ is false. This is a contradiction, so the statement must be false.

(10) First we establish two lemmas.

Lemma: For real numbers $x \in \mathbb{R}$ we have $-(-x) = x$.

Proof: We have

$$(-x) + (-(-x)) = 0$$

so

$$\begin{aligned} -(-x) &= 0 + -(-x) = (x + (-x)) + (-(-x)) \\ &= x + ((-x) + (-(-x))) = x. \quad \square \end{aligned}$$

Lemma: For real numbers $x, y \in \mathbb{R}$ we have $(-x)y = -(xy)$.

Proof: We have that

$$0 = 0y = (x + (-x))y = xy + (-x)y.$$

Adding $-(xy)$ to both sides we get

$$\begin{aligned} -(xy) &= -(xy) + (xy + (-x)y) \\ &= (-(xy) + (-x)y) + (-x)y \\ &= 0 + (-x)y = (-x)y. \quad \square \end{aligned}$$

We proceed with the proof. We either have $1 \geq 0$ or $1 \leq 0$. Suppose that $1 \leq 0$. Then $-1 \geq 0$, so

$$(-1)(-1) \geq (-1)0 = 0.$$

But

$$(-1)(-1) = -(1(-1)) = -(-1) = 1,$$

so $1 \geq 0$, contradicting the hypothesis.

3. August 30, 2022

I owe you a statement of the very important Completeness Axiom. Before we get there, I want to recall an axiom of $\mathbb{N}$ that we haven't discussed yet. It pertains to minimum elements in sets. Let's be precise and define minimum element.

Definition 3.1. Let S be a set of real numbers. A *minimum* element of S is a real number x such that

(1) $x \in S$, and
(2) for all $y \in S$, $x \leq y$.

In this case, we write $x = \min(S)$.

The definition of *maximum* is the same except with the opposite inequality.

Axiom 3.2 (Well-ordering axiom)**.** *Every nonempty subset of* $\mathbb{N}$ *has a minimum element.*

Example 3.3. If S is the set of even multiples of 7, then S has 14 as its minimum.

We generally like to say *the* minimum, rather than *a* minimum. To justify this, let's prove the following.

Proposition 3.4. *Let S be a set of real numbers. If S has a minimum, then the minimum is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P, then x and y must be the same thing.

Proof of Proposition 3.4. Let S be a set of real numbers, and let x and y be two minima of S. Applying part (1) of the definition of minimum to y, we have $y \in S$. Applying part (2) of the definition of minimum to x and the fact that $y \in S$, we get that $x \leq y$. Switching roles, we get that $y \leq x$. Thus $x = y$.

We conclude that if a minimum exists, it is necessarily unique. □

The previous proposition plus the Well-Ordering Axiom together imply that every nonempty subset of $\mathbb{N}$ has exactly one minimum element. A similar proof shows that if a maximum exists, it is necessarily unique. Could a set fail to have a maximum or a minimum? Yes!

Example 3.5.
1. The empty set $\varnothing$ has no minimum and no maximum element. (There is no $s \in \varnothing$!)
2. The set of natural numbers $\mathbb{N}$ has 1 as a minimum, but has no maximum. (Suppose there was: if $n = \max(\mathbb{N})$ was the maximum, then $n < n + 1 \in \mathbb{N}$ gives a contradiction.)
3. The open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has no minimum and no maximum. (Exercise later.)

Definition 3.6. Let S be any subset of $\mathbb{R}$. A real number b is called an *upper bound* of S provided that for every $s \in S$, we have $s \leq b$.

For example, the number 1 is an upper bound for the interval $(0, 1)$. The number 182 is also an upper bound of this set and so is π. It is pretty clear that 1 is the "best" (i.e., smallest) upper bound for this set, in the sense that every other upper bound of $(0,1)$ must be at least as big as 1. Let's make this official:

Proposition 3.7. *If b is an upper bound of the set $(0,1)$, then $b \geq 1$.*

I will prove this claim using just the axioms of the real numbers (in fact, I will only use the first 10 axioms):

Proof. Suppose b is an upper bound of the set $(0,1)$. By way of contradiction, suppose $b < 1$. (Our goal is to derive a contradiction from this.)

Consider the number $y = \frac{b+1}{2}$ (the average of b and 1). I will argue that $b < y$ and $b \geq y$, which is not possible.

Since we are assuming $b < 1$, we have $\frac{b}{2} < \frac{1}{2}$ and hence

$$b = \frac{2b}{2} = \frac{b}{2} + \frac{b}{2} < \frac{b}{2} + \frac{1}{2} = \frac{b+1}{2} = y.$$

So, $b < y$.

Similarly,

$$1 = \frac{1+1}{2} > \frac{b+1}{2} = y$$

so that

$$y < 1.$$

Since $\frac{1}{2} \in S$ and b is an upper bound of S, we have $\frac{1}{2} \leq b$. Since we already know that $b < y$, it follows that $\frac{1}{2} < y$ and hence $0 < y$. We have proven that $y \in (0,1)$. But, remember that b is an upper bound of $(0,1)$, and so we get $y \leq b$ by definition.

To summarize: given an upper bound b of $(0,1)$, starting with the assumption that $b < 1$, we have deduced the existence of a number y such that both $b < y$ and $y \leq b$ hold. As this is not possible, it must be that $b < 1$ is false, and hence $b \geq 1$. □

This claim proves the (intuitively obvious) fact that 1 is "least upper bound" of the set $(0,1)$. The notion of "least upper bound" will be an extremely important one in this class.

Definition 3.8. A subset S of $\mathbb{R}$ is called *bounded above* if there exists at least one upper bound for S. That is, S is bounded above provided there is a real number b such that for all $s \in S$ we have $s \leq b$.

For example, $(0,1)$ is bounded above, by for example 50.

The subset $\mathbb{N}$ of $\mathbb{R}$ is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the

Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of $\mathbb{R}$ that is bounded above.

Example 3.9. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 3.10. Suppose S is subset of $\mathbb{R}$ that is bounded above. A *supremum* (also known as a *least upper bound*) of S is a number ℓ such that

(1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
(2) if b is any upper bound of S, then $\ell \leq b$.

In this case we write $\sup(S) = \ell$.

Example 3.11. 1 is a supremum of $(0, 1)$. Indeed, it is clearly an upper bound, and in the "Claim" above, we proved that if b is any upper bound of $(0, 1)$ then $b \geq 1$. Note that this example shows that a supremum of S does not necessarily belong to S.

Example 3.12. I claim 1 is a supremum of

$$(0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}.$$

It is by definition an upper bound. If b is any upper bound of $(0, 1]$ then, since $1 \in (0, 1]$, by definition we have $1 \leq b$. So 1 is the supremum of $(0, 1]$.

Observation 3.13. *Let S be a set of real numbers. Suppose that $b \in S$ and that b is an upper bound for S. Then*

(1) *b is the maximum of S, and*
(2) *b is a supremum of S.*

The subset $\mathbb{N}$ does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly: the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term "supremum", I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom)**.** *Every nonempty, bounded-above subset of* $\mathbb{R}$ *has a supremum.*

4. September 1, 2022

(1) Write, in simplified form, the negation of the statement "b is an upper bound for S".

> There exists some $x \in S$ such that $x > b$.

(2) Write, in simplified form, the negation of the statement "S is bounded above".

> For every $b \in \mathbb{R}$, there exists $x \in S$ such that $x > b$.

(3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
   (a) If $x > \ell$, what is the most concrete thing you can say about x and S?
   (b) If $x < \ell$, what is the most concrete thing you can say about x and S?

> (a) $x \notin S$.
> (b) There exists some $y \in S$ such that $y > x$.

(4) Let S be a set of real numbers, and let $T = \{2s \mid s \in S\}$. Prove that if S is bounded above, then T is bounded above.

> Assume that S is bounded above. Then there is some upper bound b for S, so for every $s \in S$, we have $b \geq s$. We claim that $2b$ is an upper bound for T. Indeed, if $t \in T$, then we can write $t = 2s$ for some $s \in S$, and $s \leq b$ implies $t = 2s \leq 2b$. Thus, T is bounded above.

(5) Let S be a set of real numbers. Show that if S has a supremum, then it is unique.

> Suppose both x and y are both suprema of the same subset S of $\mathbb{R}$. Then, since y is an upper bound of S and x is a supremum of S, by part (2) of the definition of "supremum" we have $y \geq x$. Likewise, since x is an upper bound of S and y is a supremum of S, we have $x \geq y$ by definition. Since $x \leq y$ and $y \leq x$, we conclude $x = y$.

(6) Let S be a set of real numbers, and let $T = \left\{ \frac{s}{2} \mid s \in S \right\}$. Directly prove that if S is unbounded above, then T is unbounded above.

> Assume that S is unbounded above. To show that T is unbounded above, let b be a real number. Since S is unbounded above, $2b$ is not an upper bound for S, so there is some $s \in S$ with $s > 2b$. Then $\frac{s}{2} > b$. By definition of T, we have $\frac{s}{2} \in T$, so b is not an upper bound of T. We conclude that T is unbounded above.

5. September 6, 2022

Let us now explore consequences of the completeness axiom. We know that there is no rational number whose square is 2; now we show that there is indeed a real number whose square is two.

Proposition 5.1. *There is a positive real number whose square is* 2*.*

Proof. Define S to be the subset

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

S is nonempty since, for example, $1 \in S$, and it is bounded above, since, for example, 2 is an upper bound for S, as we showed earlier. So, by the Completeness Axiom, S has a least upper bound, and we know it is unique from the proposition above. Let us call it ℓ. I will prove $\ell^2 = 2$.

We know one of $\ell^2 > 2$, $\ell^2 < 2$ or $\ell^2 = 2$ must hold. We prove $\ell^2 = 2$ by showing that both $\ell^2 > 2$ and $\ell^2 < 2$ are impossible.

We start by observing that $1 \leq \ell \leq 2$. The inequality $1 \leq \ell$ holds since $1 \in S$ and ℓ is an upper bound of S, and the inequality $\ell \leq 2$ holds since 2 is an upper bound of S and ℓ is the least upper bound of S.

Suppose $\ell^2 < 2$. We show this leads to a contradiction by showing that ℓ is not an upper bound of S in this case. We will do this by constructing a number that is ever so slightly bigger than ℓ and belongs to S. Let $\varepsilon = 2 - \ell^2$. Then $0 < \varepsilon \leq 1$ (since $\ell^2 < 2$ and $\ell^2 \geq 1$). We will now show that $\ell + \varepsilon/5$ is in S: We have

$$(\ell + \varepsilon/5)^2 = \ell^2 + \frac{2}{5}\ell\varepsilon + \frac{\varepsilon^2}{25} = \ell^2 + \varepsilon(\frac{2\ell}{5} + \frac{\varepsilon}{25}).$$

Now, using $\ell \leq 2$ and $0 < \varepsilon \leq 1$, we deduce

$$0 < \frac{2\ell}{5} + \frac{\varepsilon}{25} \leq \frac{4}{5} + \frac{\varepsilon}{25} < 1.$$

Putting these equations and inequalities together yields

$$(\ell + \frac{\varepsilon}{5})^2 < \ell^2 + \varepsilon = 2.$$

So, $\ell + \frac{\varepsilon}{5} \in S$ and yet $\ell + \frac{\varepsilon}{5} > \ell$, contradicting the fact that l is an upper bound of S. We conclude $\ell^2 < 2$ is not possible.

Assume now that $\ell^2 > 2$. Our strategy will be to construct a number ever so slightly smaller than ℓ, which therefore cannot be an upper bound of S, and use this to arrive at a contradiction. Let $\delta = \ell^2 - 2$. Then $0 < \delta \leq 2$ (since $\ell \leq 2$ and hence $\ell^2 - 2 \leq 2$). Since $\delta > 0$, we have $\ell - \frac{\delta}{5} < \ell$. Since ℓ is the least upper bound of S, $\ell - \frac{\delta}{5}$ must not be an upper bound of S. By definition, this means that there is $r \in S$ such that $\ell - \frac{\delta}{5} < r$. Since $\delta \leq 2$ and $\ell \geq 1$, it follows that $\ell - \frac{\delta}{5}$ is positive and hence so is r. We may thus square both sides of $\ell - \frac{\delta}{5} < r$ to obtain

$$(\ell - \frac{\delta}{5})^2 < r^2.$$

Now

$$(\ell - \frac{\delta}{5})^2 = \ell^2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = \delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25}$$

since $\ell^2 = \delta + 2$. Moreover,

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = 2 + \delta(1 - \frac{2\ell}{5} + \frac{\delta}{25}) \geq 2 + \delta(1 - \frac{4}{5} + \frac{\delta}{25})$$

since $\ell \leq 2$. We deduce that

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} \geq 2 + \delta(\frac{1}{5}) \geq 2.$$

Putting these inequalities together gives $r^2 > 2$, contrary to the fact that $r \in S$. We conclude that $\ell^2 > 2$ is also not possible.

Since $\ell^2 < 2$ and $\ell^2 > 2$ are impossible, we must have $\ell^2 = 2$. $\square$

The collection of rational numbers does not satisfy the completeness axiom and indeed it is precisely the completeness axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Example 5.2. Within the set $\mathbb{Q}$ the subset $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum. That is, no matter which rational number you pick that is an upper bound for S, you may always find an even smaller one that is also an upper bound of S.

It is precisely the completeness axiom that assures us that everything that ought to be a number (like the length of the diagonal of a square with side length 1) really is a number. It gives us that there are "no holes" in the real number line — the real numbers are *complete.*

For example, we can use it to prove that $\sqrt[8]{147}$ exists: Let $S = \{x \in \mathbb{R} \mid x^8 < 147\}$. Then S is nonempty (e.g., $0 \in S$) and bounded above (e.g., 50 is an upper bound) and so it must have a supremum ℓ. A proof similar to (but even messier than) the proof of Proposition 5.1 above shows that ℓ satisfies $\ell^8 = 147$.

The completeness axiom is also at the core of the Intermediate Value Theorem and many of the other major theorems we will cover in this class.

We also need the completeness axiom to understand the relationship between $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$.

Theorem 5.3. *If x is any real number, then there exists a natural number n such that $n > x$.*

This looks really stupid at first. How could it be false? But consider: there are examples of ordered fields, i.e. situations in which Axioms 1–10 hold, in which this Theorem is not true! So, its proof must rely on the Completeness Axiom.

Proof. Let x be any real number. By way of contradiction, suppose there is no natural number n such that $n > x$. That is, suppose that for all $n \in \mathbb{N}$, $n \leq x$. Then $\mathbb{N}$ is a bounded above (by x). Since it is also clearly nonempty, by the Completeness Axiom, $\mathbb{N}$ has a supremum, call it ℓ. Consider the number $y := \ell - 1$. Since $y < \ell$ and ℓ is the supremum of $\mathbb{N}$, y cannot be an upper bound of $\mathbb{N}$. So, there must be some $m \in \mathbb{N}$ such that such that $\ell - 1 < m$. But then by adding 1 to both sides of this inequality we get $\ell < m + 1$ and, since $m + 1 \in \mathbb{N}$, this contradicts that assumption that ℓ is the supremum of $\mathbb{N}$.

We conclude that, given any real number x, there must exist a natural number n such that $n > x$. $\square$

Corollary 5.4 (Archimedean Principle)**.** *If $a \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$, then for some natural number n we have $na > b$.*

"No matter how small a is and how large b is, if we add a to itself enough times, we can overtake b."

Proof. We apply Theorem 5.3 to the real number $x = \frac{b}{a}$. It gives that there is a natural number n such that $n > x = \frac{b}{a}$. Since $a > 0$, upon multiplying both sides by a we get $n \cdot a > b$. $\square$

Theorem 5.5 (Density of the Rational Numbers)**.** *Between any two distinct real numbers there is a rational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. We will prove this by consider two cases: $x \geq 0$ and $x < 0$.

Let us first assume $x \geq 0$. We apply the Archimedean Principle using $a = y - x$ and $b = 1$. (The Principle applies as $a > 0$ since $y > x$.) This gives us that there is a natural number $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1$$

and thus

$$0 < \frac{1}{n} < y - x.$$

Consider the set $S = \{p \in \mathbb{N} \mid p\frac{1}{n} > x\}$. Since $\frac{1}{n} > 0$, using the Archimedean principle again, there is at least one natural number $p \in S$. By the Well Ordering Axiom, there is a smallest natural number $m \in S$.

We claim that $\frac{m-1}{n} \leq x$. Indeed, if $m > 1$, then $m - 1 \in \mathbb{N} \setminus S$ (because $m - 1$ is less than the minimum), so $\frac{m-1}{n} \leq x$; if $m = 1$, then $m - 1 = 0$, so $\frac{m-1}{n} = 0 \leq x$.

So, we have

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

By adding $\frac{1}{n}$ to both sides of $\frac{m-1}{n} \leq x$ and using that $\frac{1}{n} < y - x$, we get

$$\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$$

and hence

$$x < \frac{m}{n} < y.$$

Since $\frac{m}{n}$ is clearly a rational number, this proves the result in this case (when $x > 0$).

We now consider the case $x < 0$. The idea here is to simply "shift" up to the case we've already proven. By Theorem 5.3, we can find a natural number j such that $j > -x$ and thus $0 < x + j < y + j$. Using the first case, which we have already proven, applied to the number $x+j$ (which is positive), there is a rational number q such that

$x + j < q < y + j$. We deduce that $x < q - j < y$, and, since $q - j$ is also rational, this proves the theorem in this case. □

6. September 8, 2022

(1) Let W be the set of real numbers x that satisfy the inequality $x^3 + x < 10$.
  (a) Write W mathematically in set notation.
  (b) Does W have a supremum? Why or why not?
  (c) Is $\sup(W) = 1$? Why or why not?
  (d) Is $\sup(W) = 4$? Why or why not?

> (a) $W = \{x \in \mathbb{R} \mid x^3 + x < 10\}$.
> (b) Yes. It is nonempty, since $0 \in W$, and bounded above, e.g., by 3: if $x > 3$, then $x^3 + x > 3^3 + 3 = 30$, so $x \notin W$.
> (c) No: 1 is not an upper bound, because $1.5 \in W$.
> (d) No: 3 is an upper bound, and $3 < 4$.

(2) Use the Archimedean Principle to show that for any positive number $\varepsilon > 0$, there is a natural number n such that $0 < \varepsilon < \frac{1}{n}$.

(3) Prove that the supremum of the set $S = \left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\}$ is 1.

(4) Let S be a set of real numbers, and suppose that $\sup(S) = \ell$. Let $T = \{s + 7 \mid s \in S\}$. Prove that $\sup(T) = \ell + 7$.

> First, we show that $\ell + 7$ is an upper bound of T. Let $t \in T$. Then there is some $s \in S$ such that $t = s + 7$. Since $s \leq \ell$, we have $t = s + 7 < \ell + 7$, so $\ell + 7$ is indeed an upper bound. Next, let b be an upper bound for T. We claim that $b - 7$ is an upper bound for S. Indeed, if $s \in S$, then $s + 7 \in T$ so $s + 7 \leq b$, so $s \leq b - 7$. Then, by definition of supremum, we have $b - 7 \geq \ell$, eso $b \geq \ell + 7$.

(5) Prove the following:

Corollary 6.1 (Density of irrational numbers)**.** *For any real numbers x, y with $x < y$, there is some irrational number z such that $x < z < y$.*

Let $x < y$ be real numbers. Then we have $x - \sqrt{2} < y - \sqrt{2}$. By density of rationals, there is some rational number q such that $x - \sqrt{2} < q < y - \sqrt{2}$. Then $x < q + \sqrt{2} < y$. Since q is rational and $\sqrt{2}$ is irrational, $z = q + \sqrt{2}$ is irrational, and hence the number we seek.

(6) True or false & justify: There is a rational number x such that $|x^2 - 2| = 0$.

False: this would imply that x is a rational number whose square is 2.

(7) True or false & justify: There is a rational number x such that $|x^2 - 2| < \frac{1}{1000000}$.

True: By density of rational numbers, there is a rational number q such that $\sqrt{2} - \frac{1}{5000000} < q < \sqrt{2}$. Then

$$\begin{aligned} |x^2 - 2| &= |x - \sqrt{2}|\,|x + \sqrt{2}| \\ &< \frac{1}{5000000}\left(\frac{1}{5000000} + 4\right) \\ &< \frac{1}{5000000} \cdot 5 \\ &= \frac{1}{1000000}. \end{aligned}$$

7. September 13, 2022

We now turn our attention to the next major topic of this class: sequences of real numbers. We will spend the next few weeks developing their properties carefully and rigorously. Sequences form the foundation for much of what we will cover for the rest of the semester.

Definition 7.1. A *sequence* is an infinite list of real numbers indexed by $\mathbb{N}$:

$$a_1, a_2, a_3, \ldots.$$

(Equivalently, a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$: the value of the function at $n \in \mathbb{N}$ is written as a_n.)

We will usually write $\{a_n\}_{n=1}^{\infty}$ for a sequence.

Example 7.2. To describe sequences, we will typically give a formula for the n-th term, a_n, either an explicit one or a recursive one. On rare occasion we'll just list enough terms to make the pattern clear. Here are some examples:

(1) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is the sequence that starts

$$4, \frac{11}{2}, \frac{14}{3}, \frac{21}{4}, \frac{24}{5}, \dots.$$

(2) Let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. This gives the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

This is an example of a recursively defined sequence. It is the famed *Fibonacci sequence*.

(3) Let $\{c_n\}_{n=1}^{\infty}$ be the sequence whose n-th term is the n-th smallest positive prime integer:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \dots.$$

Note that here I have not really given an explicit formula for the terms of the sequence, but it is possible to describe an algorithm that lists every term of the sequence in order.

You have all probably seen an "intuitive" definition of the limit of a sequence before. For example, you probably believe that

$$5 + (-1)^n \frac{1}{n}$$

converges to 5. Let's give the rigorous definition.

Definition 7.3. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence and L a real number. We say $\{a_n\}_{n=1}^{\infty}$ *converges* to L provided the following condition is met:

> For every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

This is an extremely important definition for this class. Learn it by heart!

In symbols, the definition is

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided
> $\forall \varepsilon > 0, \exists N \in \mathbb{R} : \forall n \in \mathbb{N}$ s.t. $n > N,\ |a_n - L| < \varepsilon$.

It's a complicated definition — three quantifiers!

Here is what the definition is saying somewhat loosely: No matter how small a number ε you pick, so long as it is positive, if you go far enough out in the sequence, all of the terms from that point on will be within a distance of ε of the limiting value L.

Example 7.4. To say that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5 gives us a different statement for every $\varepsilon > 0$. For example:

- Setting $\varepsilon = 3$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 3$. Namely, we can take $N = 0$, since for *every* term a_n of the sequence, $|a_n - 5| < 3$ holds true.
- Setting $\varepsilon = \frac{1}{3}$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < \frac{1}{3}$. We cannot take $N = 0$ anymore, since $1 > 0$ and $|a_1 - 5| = 1 > \frac{1}{3}$. However, we can take $N = 3$, since for $n > 3$, $|a_n - 5| = \frac{1}{n} < \frac{1}{3}$.
- Setting $\varepsilon = 1/1000000$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 1/1000000$. We need a bigger N; now $N = 1000000$ works.

In general, our choice of N may depend on ε, which is OK since our definition is of the form $\forall \varepsilon > 0, \exists N \ldots$ rather than $\exists N : \forall \varepsilon > 0 \ldots$.

Example 7.5. I claim the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5. I'll give a rigorous proof, along with some commentary and "scratch work" within the parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: Given this ε, our goal is to find N so that if $n > N$, then $|5 + (-1)^n \frac{1}{n} - 5| < \varepsilon$. The latter simplifies to $\frac{1}{n} < \varepsilon$, which in turn is equivalent to $\frac{1}{\varepsilon} < n$ since ε and n are both positive. So, it seems we've found the N that "works". Back to the formal proof....)

Let $N = \frac{1}{\varepsilon}$. Then $\frac{1}{N} = \varepsilon$, since ε is positive.

(Comment: We next show that this is the N that "works" in the definition. Since this involves proving something about every natural number that is bigger than N, we start by picking one.)

Pick any $n \in \mathbb{N}$ such that $n > N$. Then $\frac{1}{n} < \frac{1}{N}$ and hence

$$|a_n - 5| = |5 + (-1)^n \frac{1}{n} - 5| = |(-1)^n \frac{1}{n}| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

This proves that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. $\square$

Remark 7.6. A direct proof that a certain sequence converges to a certain number follows the general outline:

- Let $\varepsilon > 0$ be given. (or, if your prefer, "Pick $\varepsilon > 0$.")
- Let $N =$ [expression in terms of ε from scratch work].
- Let $n \in \mathbb{N}$ be such that $n > N$.

- [Argument that $|a_n - L| < \varepsilon$.]
- Thus $\{a_n\}_{n=1}^{\infty}$ converges to L.

Example 7.7. I claim that the sequence

$$\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$$

converges to $\frac{2}{5}$. Again I'll give a proof with commentary and scratch work in parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: We need n to be large enough so that

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This simplifies to $\left|\frac{-7}{25n+5}\right| < \varepsilon$ and thus to $\frac{7}{25n+5} < \varepsilon$, which we can rewritten as $\frac{7}{25\varepsilon} - \frac{1}{5} < n$.)

Let $N = \frac{7}{25\varepsilon} - \frac{1}{5}$. We solve this equation for ε: We get $\frac{7}{25\varepsilon} = \frac{5N+1}{5}$ and hence $\frac{25\varepsilon}{7} = \frac{5}{5N+1}$, which gives finally

$$\varepsilon = \frac{7}{25N+5}.$$

(Next we show this value of N works....)

Now pick any $n \in \mathbb{N}$ is such that $n > N$. Then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| = \left|\frac{10n-5-10n-2}{25n+5}\right| = \frac{7}{25n+5}.$$

Since $n > N$, $25n+5 > 25N+5$ and hence

$$\frac{7}{25n+5} < \frac{7}{25N+5} = \varepsilon.$$

We have proven that if $n \in \mathbb{N}$ and $n > N$, then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This proves $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$. $\square$

8. September 15, 2022

(1) Let c be a real number. Prove that the constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.

> Let $\varepsilon > 0$. Take $N = 0$ (or $N = 588$, or $N = -10000000$, or any other real number). For any natural number $n > N$, we have $|a_n - c| = 0 < \varepsilon$. Thus the sequence converges to c.

(2) Prove that the sequence $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 0.

(3) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose we know that $\{a_n\}_{n=1}^{\infty}$ converges to 1. Prove that there is a natural number $n \in \mathbb{N}$ such that $a_n > 0$.

> Take $\varepsilon = 1$. By definition of converges to 1, there is some N such that for all $n > N$, $|a_n - 1| < 1$, and in particular $a_n > 0$. So, take any natural number greater than n, and the conclusion follows.

(4) Prove or disprove: The sequence $\left\{\frac{n+1}{2n}\right\}_{n=1}^{\infty}$ converges to 0.

> Take $\varepsilon = 1/2$. We claim that there is no N such that for all $n > N$ we have $|a_n - 0| < 1/2$. Indeed, given N, take any n to be any natural number greater than N. Then $a_n = 1/2 + 1/2n > 1/2$, so $|a_n| > 1/2$. Thus, there is no N satisfying the desired property. This means that the sequence does not converge to 1/2.

(5) Prove or disprove: The sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = \begin{cases} 1 & \text{if } n = 10^m \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

converges to 0.

> Take $\varepsilon = 1/2$. We claim that there is no N such that for all $n > N$ we have $|a_n - 0| < 1/2$. Indeed, let N be any real number. Let m be a natural number larger than N, and $n = 10^m$. Then $n = 10^m > m > N$, and $a_n = 1$, so $|a_n - 0| = 1 > 1/2$. This shows the claim, and hence that the sequence does not converge to 0

Definition 8.1. A sequence $\{a_n\}_{n=1}^{\infty}$ is *convergent* if there is a real number L such that $\{a_n\}_{n=1}^{\infty}$ converges to L. Otherwise, it is said to be *divergent*.

(6) In this problem, we will prove that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.
- Proceed by contradiction and suppose it converges to L.
- Apply the definition of "converges to L" with $\varepsilon = \frac{1}{2}$, so we get some N.
- Take an odd integer n bigger than N: what does this say about L?
- Take an even integer n bigger than N: what does this say about L?
- Conclude the proof.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 5.3. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1 - L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. $\square$

9. September 20, 2022

Example 9.1. Let's prove the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent. This means that there is no L to which it converges.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 5.3. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1 - L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. $\square$

Proposition 9.2. *If a sequence converges, then there is a unique number to which it converges.*

Proof. Recall that to show something satisfying certain properties is unique, one assumes there are two such things and argues that they must be equal. So, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to L and that also converges to M. We will prove $L = M$.

By way of contradiction, suppose $L \neq M$. Then set $\varepsilon = \frac{|L-M|}{3}$. Since we are assuming $L \neq M$, we have $\varepsilon > 0$. According to the definition of convergence, since the sequence converges to L, there is a real number N_1 such that for $n \in \mathbb{N}$ such that $n > N_1$ we have

$$|a_n - L| < \varepsilon.$$

Also according to the definition, since the sequence converges to M, there is a real number N_2 such that for $n \in \mathbb{N}$ and $n > N_2$ we have

$$|a_n - M| < \varepsilon.$$

Pick n to be any natural number larger than $\max\{N_1, N_2\}$ (which exists by Theorem 5.3). For such an n, both $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$ hold. Using the triangle inequality and these two inequalities, we get

$$|L - M| \leq |L - a_n| + |M - a_n| < \varepsilon + \varepsilon.$$

But by the choice of ε, we have $\varepsilon + \varepsilon = \frac{2}{3}|L - M|$. That is, we have deduced that $|L - M| < \frac{2}{3}|L - M|$ which is impossible. We conclude that $L = M$. ☐

From now on, given a sequence $\{a_n\}_{n=1}^{\infty}$ and a real number L, will we use the short-hand notation

$$\lim_{n\to\infty} a_n = L$$

to mean that the given sequence converges to the given number. For example, we showed above that

$$\lim_{n\to\infty} \frac{2n-1}{5n+1} = \frac{2}{5}.$$

But, to be clear, the statement "$\lim_{n\to\infty} a_n = L$" signifies nothing more and nothing less than the statement "$\{a_n\}_{n=1}^{\infty}$ converges to L".

Here is some terminology we will need:

Definition 9.3. Suppose $\{a_n\}_{n=1}^{\infty}$ is any sequence.

(1) We say $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if there exists at least one real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$; we say $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if there exists at least one real number m such that $a_n \geq m$ for all $n \in \mathbb{N}$; and we say $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below.

(2) We say $\{a_n\}_{n=1}^{\infty}$ is *increasing* if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$; we say $\{a_n\}_{n=1}^{\infty}$ is *decreasing* if for all $n \in \mathbb{N}$, $a_n \geq a_{n+1}$; and we say $\{a_n\}_{n=1}^{\infty}$ is *monotone* if it is either decreasing or increasing.

(3) We say $\{a_n\}_{n=1}^{\infty}$ is *strictly increasing* if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$. I leave the definition of *strictly decreasing* and *strictly monotone* to your imaginations.

Remark 9.4. Be sure to interpret "monotone" correctly. It means

$$(\forall n \in \mathbb{N}, a_n \leq a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_n \geq a_{n+1});$$

it does *not* mean

$$\forall n \in \mathbb{N}, (a_n \leq a_{n+1}) \text{ or } (a_n \geq a_{n+1}).$$

Do you see the difference?

Proposition 9.5. *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *converges then it is bounded.*

Proof. Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number L. Applying the definition of "converges to L" using the particular value $\varepsilon = 1$ gives the following fact: There is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < 1$. The latter inequality is equivalent to $L - 1 < a_n < L + 1$ for all $n > N$.

Let m be any natural number such that $m > N$, and consider the finite list of numbers

$$a_1, a_2, \ldots, a_{m-1}, L + 1.$$

Let b be the largest element of this list. I claim the sequence is bounded above by b. For any $n \in \mathbb{N}$, if $1 \leq n \leq m - 1$, then $a_n \leq b$ since in this case a_n is a member of the above list and b is the largest element of this list. If $n \geq m$ then since $m > N$, we have $n > N$ and hence $a_n < L + 1$ from above. We also have $L + 1 \leq b$ (since $L + 1$ is in the list) and thus $a_n < b$. This proves $a_n \leq b$ for all n as claimed.

Now take p to be the smallest number in the list

$$a_1, a_2, \ldots, a_{m-1}, L - 1.$$

A similar argument shows that $a_n \geq p$ for all $n \in \mathbb{N}$. $\square$

Index