

§2.9: NOETHERIAN RINGS

DEFINITION: A ring R is **Noetherian** if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some N such that $I_n = I_N$ for all $n \geq N$.

HILBERT BASIS THEOREM: If R is a Noetherian ring, then the polynomial ring $R[X]$ and power series ring $R[[X]]$ are also Noetherian.

We will return to the proof of Hilbert Basis Theorem after discussing Noetherian modules next time.

COROLLARY: Every finitely generated algebra over a field is Noetherian.

- (1) Equivalences for Noetherianity.
 - (a) Show¹ that R is Noetherian if and only if every ideal is finitely generated.
 - (b) Show² that R is Noetherian if and only if every nonempty collection of ideals has a maximal³ element.
- (2) Some Noetherian rings:
 - (a) Show that fields and PIDs are Noetherian.
 - (b) Show that if R is Noetherian and $I \subseteq R$, then R/I is Noetherian.
 - (c) Is⁴ every subring of a Noetherian ring Noetherian?
- (3) Use the Hilbert Basis Theorem to deduce the Corollary.
- (4) Some nonNoetherian rings:
 - (a) Let K be a field. Show that $K[X_1, X_2, \dots]$ is not Noetherian.
 - (b) Let K be a field. Show that $K[X, XY, XY^2, \dots]$ is not Noetherian.
 - (c) Show that $\mathcal{C}([0, 1], \mathbb{R})$ is not Noetherian.
- (5) Let R be a Noetherian ring. Show that for every ideal I , there is some n such that $\sqrt{I}^n \subseteq I$. In particular, there is some n such that for every nilpotent element z , $z^n = 0$.
- (6) Let R be Noetherian. Show that every element of R admits a decomposition into irreducibles.
- (7) Prove the principle of **Noetherian induction**: Let \mathcal{P} be a property of a ring. Suppose that “For every nonzero ideal I , \mathcal{P} is true for R/I implies that \mathcal{P} is true for R ”. Then \mathcal{P} is true for every Noetherian ring.
- (8)
 - (a) Suppose that every maximal ideal of R is finitely generated. Must R be Noetherian?
 - (b) Suppose that every ascending chain of prime ideals stabilizes. Must R be Noetherian?
 - (c) Suppose that every prime ideal of R is finitely generated. Must R be Noetherian?

¹For the backward direction, consider $\bigcup_{n \in \mathbb{N}} I_n$

²Hint: For the forward direction, show the contrapositive.

³This means that if \mathcal{S} is our collection of ideals, there is some $I \in \mathcal{S}$ such that no $J \in \mathcal{S}$ properly contains I . It does not mean that there is a maximal ideal in \mathcal{S} .

⁴Hint: Every domain has a fraction field, even the domain from (4a).