§4.18: SPECTRUM OF A RING

DEFINITION: Let R be a ring, and $I \subseteq R$ an ideal of R.

- The **spectrum** of a ring R, denoted $\mathrm{Spec}(R)$, is the set of prime ideals of R.
- We set $V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$, the set of primes containing I.
- We set $D(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \not\subseteq \mathfrak{p}\}$, the set of primes *not* containing I.
- More generally, for any subset $S \subseteq R$, we define $V(S)$ and $D(S)$ analogously.

DEFINITION/PROPOSITION: The collection $\{V(I) \mid I \text{ an ideal of } R\}$ is the collection of closed subsets of a topology on R, called the **Zariski topology**; equivalently, the open sets are $D(I)$ for I an ideal of R.

DEFINITION: Let $\phi : R \to S$ be a ring homomorphism. Then the **induced map on Spec** corresponding to ϕ is the map $\phi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $\phi^*(\mathfrak{p}) := \phi^{-1}(\mathfrak{p})$.

LEMMA: Let $\mathfrak{p}$ be a prime ideal. Let I_λ, J be ideals.

(1) $\sum_\lambda I_\lambda \subseteq \mathfrak{p} \Longleftrightarrow I_\lambda \subseteq \mathfrak{p}$ for all λ.
(2) $IJ \subseteq \mathfrak{p} \Longleftrightarrow I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
(3) $I \cap J \subseteq \mathfrak{p} \Longleftrightarrow I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
(4) $I \subseteq \mathfrak{p} \Longleftrightarrow \sqrt{I} \subseteq \mathfrak{p}$

(1) The spectrum of some reasonably small rings.
- **(a)** Let $R = \mathbb{Z}$ be the ring of integers.
  - **(i)** What are the elements of $\mathrm{Spec}(R)$? Be careful not to forget (0)!
  - **(ii)** Draw a picture $\mathrm{Spec}(R)$ (with $\cdots$ since you can't list everything) with a line going up from $\mathfrak{p}$ to $\mathfrak{q}$ if $\mathfrak{p} \subset \mathfrak{q}$.
  - **(iii)** Describe the sets $V(I)$ and $D(I)$ for any ideal I.
- **(b)** Same questions for $R = K$ a field.
- **(c)** Same questions for the polynomial ring $R = \mathbb{C}[X]$.
- **(d)** Same questions[1] for the power series ring $R = K[\![X]\!]$ for a field K.

(2) More Spectra.
- **(a)** Let $R = \mathbb{C}[X, Y]$ be a polynomial ring in two variables. Find some maximal ideals, the zero ideal, and some primes that are neither. Draw a picture like the ones from the previous problem to illustrate some containments between these.
- **(b)** Let R be a ring and I be an ideal. Use the Second Isomorphism Theorem to give a natural bijection between $\mathrm{Spec}(R/I)$ and $V(I)$.
- **(c)** Let $R = \dfrac{\mathbb{C}[X,Y]}{(XY)}$. Let $x = [X]$ and $y = [Y]$.
  - **(i)** Use the definition of prime ideal to show that $\mathrm{Spec}(R) = V(x) \cup V(y)$.
  - **(ii)** Use the previous problem to completely describe $V(x)$ and $V(y)$.
  - **(iii)** Give a complete description/picture of $\mathrm{Spec}(R)$.

[1] Spoiler: The only primes are (0) and (X). To prove it, show/recall that any nonzero series f can be written as $f = X^n u$ for some unit $u \in K[\![X]\!]$.

(3) Let R be a ring.

(a) Show that for any subset S of R, $V(S) = V(I)$ where $I = (S)$.

(b) Translate the lemma to fill in the blanks:

$V(I)$ ___ $V(\sqrt{I})$	$D(I)$ ___ $D(\sqrt{I})$
$V(\sum_\lambda I_\lambda)$ _____ $V(I_\lambda)$	$D(\sum_\lambda I_\lambda)$ _____ $D(I_\lambda)$
$V(f_1, \ldots, f_n)$ __ $V(f_1)$ __ $\cdots$ __ $V(f_n)$	$D(f_1, \ldots, f_n)$ __ $D(f_1)$ __ $\cdots$ __ $D(f_n)$
$V(IJ)$___$V(I)$___$V(J)$	$D(IJ)$___$D(I)$___$D(J)$
$V(I \cap J)$___$V(I)$___$V(J)$	$D(I \cap J)$___$D(I)$___$D(J)$

(c) Use the above to verify that the Zariski topology indeed satisfies the axioms of a topology.

(4) The induced map on Spec: Let $\phi : R \to S$ be a ring homomorphism.

(a) Show that for any prime ideal $\mathfrak{q} \subseteq S$, the ideal $\phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$ is a prime ideal of R.

(b) Show that for any ideal $I \in R$, we have

$$(\phi^*)^{-1}(V(I)) = V(IS) \text{ and } (\phi^*)^{-1}(D(I)) = D(IS).$$

(c) Show that ϕ^* is continuous.

(d) If $\phi : R \to R/I$ is quotient map, describe ϕ^*.

(5) Let R and S be rings. Describe $\mathrm{Spec}(R \times S)$ in terms of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$.

(6) Properties of $\mathrm{Spec}(R)$.

(a) Show that for any ring R, the space $\mathrm{Spec}(R)$ is compact.

(b) Show that if $\mathrm{Spec}(R)$ is Hausdorff, then every prime of R is maximal.

(c) Show that $\mathrm{Spec}(R) \cong \mathrm{Spec}(R/\sqrt{0})$.

(7) Let K be a field, and $R = \dfrac{K[X_1, X_2, \ldots]}{(\{X_i - X_i X_j \mid 1 \leq i \leq j\})}$. Describe $\mathrm{Spec}(R)$ as a set and as a topological space.