The definition of continuous on a closed interval $[a, b]$ is actually a bit different: we shouldn't necessarily ask that f be continuous at a, since to know that would have to use something about f on input values outside of our interval!

Definition 22.1: Given a function $f(x)$ and real numbers $a < b$, we say f is *continuous on the closed interval* $[a, b]$ provided

(1) for every $r \in (a, b)$, f is continuous at r in the sense defined already,
(2) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $a \leq x < a + \delta$, then $f(x)$ is defined and $|f(x) - f(a)| < \varepsilon$.
(3) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $b - \delta < x \leq b$, then $|f(x) - f(b)| < \varepsilon$.

(1) Explain why if f is continuous at x for every $x \in [a, b]$, then f is continuous on the closed interval $[a, b]$. In particular, if f is continuous on any open interval containing $[a, b]$, then f is continuous on $[a, b]$. Conclude that every polynomial is continuous on every closed interval.

(2) Show that the function $f(x) = \sqrt{1 - x^2}$ is continuous on the closed interval $[-1, 1]$:
- For showing condition (1), I recommend using a Theorem about compositions of functions.
- For conditions (2) and (3), show that $\delta = \min\{\varepsilon^2/\sqrt{2}, 2\}$ works[1].

Is this function continuous on any open interval containing $[-1, 1]$?

Theorem 22.2. (Intermediate Value Theorem): Let $a < b$ and $f(x)$ be a function that is continuous on the closed interval $[a, b]$. If y is any real number between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ such that $f(c) = y$. More precisely, if $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$, then there is some $c \in [a, b]$ such that $f(c) = y$.

(3) Draw a picture of this theorem as follows:
- Mark some a and b on the x-axis.
- Graph a function f that is continuous on $[a, b]$.
- Mark $f(a)$ and $f(b)$ on the y-axis.
- Pick some y in between $f(a)$ and $f(b)$, and make a horizontal line for this y-value.
- Does it intersect the graph of f?

Repeat with at least one graph that is increasing, at least one graph that is decreasing, and at least one graph that is neither increasing nor decreasing.

(4) Give a counterexample to the statement of the Intermediate Value Theorem without the hypothesis that f is continuous on $[a, b]$.

(5) Prove or disprove: There is a real number $x \in [0, 2]$ such that $x^3 - 3x = 1$.

[1]Hint: Write $\sqrt{1 - x^2} = \sqrt{1 - x}\sqrt{1 + x}$.

(6) Prove or disprove: There[2] are at least two real numbers $x \in [0, 2]$ such that $x^3 - 3x = -1$.

(7) True or false: If $f(x)$ is continuous on $[a, b]$, and y is *not* in between $f(a)$ and $f(b)$, then there is no $c \in [a, b]$ such that $f(c) = y$.

(8) **Proof of the Intermediate Value Theorem:**
   (a) Let's assume that $f(a) \leq f(b)$ to get started. Explain why the cases $y = f(a)$ and $y = f(b)$ are easy. Hence, we assume that $f(a) < y < f(b)$.
   (b) Let $S = \{x \in [a, b] \mid f(r) < y \text{ for all } a \leq r \leq x\}$. In short, S is the set of x-values in the interval where the graph of f hasn't crossed y yet. Explain why S has a supremum, and let $c = \sup(S)$.
   (c) Show that $c > a$. [Hint: Apply part (2) of definition of continuous on $[a, b]$ with $\varepsilon = y - f(a)$, and show that a is not an upper bound for S.]
   (d) The argument that $c < b$ is similar (so come back to it later if you want). Thus, $c \in (a, b)$, so we know that f is continuous at c.
   (e) Suppose that $f(c) < y$, and obtain a contradiction. [Hint: Apply continuous at c with $\varepsilon = y - f(c)$, and show that c is not an upper bound for S.]
   (f) Suppose that $f(c) > y$, and obtain a contradiction. [Hint: Apply continuous at c with $\varepsilon = f(c) - y$, and find a smaller upper bound for S.]
   (g) This concludes the case when $f(a) \leq f(b)$. If $f(a) \geq f(b)$, what can you say about $g(x) = -f(x)$? Can we apply the case we just did?

We say that a function is *increasing* on an interval I if for any $x, y \in I$, $x < y$ implies $f(x) < f(y)$.
We say that a function is *decreasing* on an interval I if for any $x, y \in I$, $x < y$ implies $f(x) > f(y)$.
We say that a function is *monotone* on an interval I if it is either increasing on I or decreasing on I.
We say that a function is *one-to-one* on an interval I if for any $x, y \in I$, $x \neq y$ implies $f(x) \neq f(y)$.

(9) Show that if f is continuous and one-to one on an interval (a, b), then f is monotone.

[2] Draw a graph of this function before you declare victory on this problem.