Assignment #6: Due Friday, November 22 at 7pm

This problem set is to be turned in by Canvas. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

(1) Find[1] a minimal primary decomposition of the principal ideal (14) in $\mathbb{Z}[\sqrt{-13}]$ and explain why this minimal primary decomposition is unique.

(2) Let R be a Noetherian ring. Show that R is reduced if and only if every associated prime is minimal and $R_{\mathfrak{p}}$ is a field for every $\mathfrak{p} \in \operatorname{Min}(R)$.

(3) Let R be a ring, not necessarily Noetherian. Let I be an ideal such that $V(I) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$ is a finite set of maximal ideals.
  (a) Show that $Q_i = IR_{\mathfrak{m}_i} \cap R$ is primary.
  (b) Show[2] that I has a primary decomposition $I = Q_1 \cap \cdots \cap Q_t$.
  (c) Show that $R/I \cong R/Q_1 \times \cdots \times R/Q_t$.

(4) Let R be a domain.
  (a) Show that R is a UFD if and only if every prime of height one is principal.
  (b) Give an example of a height one ideal in a UFD that is not principal.
  (c) Give an example of a height one prime in a domain that is not principal.

(5) Use Macaulay2 to
  (a) In $\mathbb{Q}[X, Y, U, V]$, the ideal $(X^2 - U^2, XY - UV, Y^2 - V^2)$.

[1] Hint: $14 = 2 \cdot 7 = (1 + \sqrt{-13}) \cdot (1 - \sqrt{-13})$. You might also find it useful to think of $\mathbb{Z}[\sqrt{-13}]$ as $\mathbb{Z}[X]/(X^2 + 13)$.

[2] As a T-shirt once said, "Be wise—localize!"