Math 445 — Problem Set #6
Due: Friday, November 3 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Use the methods from class to give a formula[1] for all solutions of the Pell's equation

$$x^2 - 13y^2 = 1.$$

We use the continued fraction algorithm:

$$\sqrt{13} = 3 + (\sqrt{13}-3) = 3 + \cfrac{1}{\left(\frac{1}{\sqrt{13}-3}\right)} = 3 + \cfrac{1}{\frac{\sqrt{13}+3}{4}} = 3 + \cfrac{1}{1+\frac{\sqrt{13}-1}{4}} = 3 + \cfrac{1}{1+\cfrac{1}{\left(\frac{4}{\sqrt{13}-1}\right)}}$$

$$= 3 + \cfrac{1}{1+\cfrac{1}{\left(\frac{4(\sqrt{13}+1)}{12}\right)}} = 3 + \cfrac{1}{1+\cfrac{1}{\left(\frac{\sqrt{13}+1}{3}\right)}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\frac{\sqrt{13}-2}{3}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{3}{\sqrt{13}-2}\right)}}}$$

$$= 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{3(\sqrt{13}+2)}{9}\right)}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{\sqrt{13}+2}{3}\right)}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{\sqrt{13}-1}{3}}}}$$

$$= 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{3}{\sqrt{13}-1}\right)}}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{3(\sqrt{13}+1)}{12}\right)}}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{\sqrt{13}+1}{4}\right)}}}}$$

$$= 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{\sqrt{13}-3}{4}}}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{4}{\sqrt{13}-3}\right)}}}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\left(\frac{4(\sqrt{13}+3)}{4}\right)}}}}}$$

$$= 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\sqrt{13}+3}}}}} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+(\sqrt{13}-3)}}}}}$$

and since $\sqrt{13}-3$ appears as a remainder in the first step, the continued fraction must start repeating. That is, $\sqrt{13} = [3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \dots]$.

We use this to generate a list of convergents $\frac{a}{b}$:

$$\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \frac{137}{38}, \frac{256}{71}, \frac{393}{109}, \frac{649}{180}, \dots$$

and for each, we test whether $a^2 - 13b^2 = 1$. The first solution we get is $(649, 180)$.

Now, by the theorem, every solution (x_k, y_k) arises of the form

$$(x_k, y_k) = (649 + 180\sqrt{13})^k.$$

(2) Closed formulas for solutions to Pell's equations.

[1] As in class, in terms of coefficients powers of some $a + b\sqrt{D}$.

(a) Explain why the kth positive solution (x_k, y_k) of the Pell's equation $x^2 - 2y^2 = 1$ satisfies the equation

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) Diagonalize the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and use this to give a closed expression for (x_k, y_k) in terms of k. Your formulas should be in terms of particular linear combinations of powers of two numbers.

(c) Use[2] your formulas from the previous part to show that

$$x_k = \left\lceil \frac{(3+2\sqrt{2})^k}{2} \right\rceil \qquad \text{and} \qquad y_k = \left\lfloor \frac{(3+2\sqrt{2})^k}{2\sqrt{2}} \right\rfloor.$$

Use this to quickly write down the first seven positive solutions to the Pell's equation $x^2 - 2y^2 = 1$.

(d) Repeat the steps above with the appropriate numbers for the Pell's equation $x^2 - 5y^2 = 1$.

(3) Not solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.

(a) Show that if $D \equiv 0 \pmod 4$ or $D \equiv 3 \pmod 4$, then the equation $x^2 - Dy^2 = -1$ has no integer solutions.

(b) Show that if $q \equiv 3 \pmod 4$ is prime and $q \mid D$, then the equation $x^2 - Dy^2 = -1$ has no integer solutions.

> (a) We know that $x^2, y^2 \equiv 0, 1 \pmod 4$. If $D \equiv 0 \pmod 4$, then $x^2 - Dy^2$ is congruent to 0 or 1 modulo 4, whereas -1 is congruent to 3 mod 4 so there can be no solutions. If $D \equiv 3 \pmod 4$, then $x^2 - Dy^2 \equiv x^2 + y^2 \pmod 4$, and the possible values are $0, 1, 2 \pmod 4$, so there again can be no solution.
>
> (b) Suppose that $q \equiv 3 \pmod 4$ and $q \mid D$. Given a solution $x^2 - Dy^2 = -1$, we obtain $x^2 \equiv -1 \pmod q$. By QR part -1, this has no solutions since -1 is not a quadratic residue in this case. Thus, $x^2 - Dy^2 = -1$ has no solutions.

(4) Solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.

(a) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, then $\frac{c}{d}$ is a convergent in the continued fraction expansion of $\sqrt{D}$.

(b) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, (a, b) is a positive integer solution to $x^2 - Dy^2 = 1$, and

$$e + f\sqrt{D} = (a + b\sqrt{D})(c + d\sqrt{D}),$$

then (e, f) is another positive integer solution to $x^2 - Dy^2 = -1$.

(c) Describe infinitely many solutions to the equation $x^2 - 13y^2 = -1$.

> (a) We have that $1 = |c^2 - d^2 D| = |c + d\sqrt{D}| \cdot |c - d\sqrt{D}|$, so $\left|\frac{c}{d} + \sqrt{D}\right| \cdot \left|\frac{c}{d} - \sqrt{D}\right| = \frac{1}{d^2}$. Since $|\frac{c}{d} - \sqrt{D}| \le 1$ and $\sqrt{D} > 1$, we have $\left|\frac{c}{d} + \sqrt{D}\right| > 2$, so $\left|\frac{c}{d} - \sqrt{D}\right| < \frac{1}{2d^2}$. By the Theorem on Good Approximations and convergents, this implies that $\frac{c}{d}$ is a convergent.

[2]Recall that $\lfloor x \rfloor$ denotes the greatest integer n such that $n \le x$ and $\lceil x \rceil$ denotes the smallest integer n such that $n \ge x$.

(b) We have $N(e+f\sqrt{D}) = N(a+b\sqrt{D})N(c+d\sqrt{D})$. Since (a,b) is a solution to Pell's equation, $N(a+b\sqrt{D}) = 1$, and the given equation implies that $N(c+d\sqrt{D}) = -1$, so $N(e+f\sqrt{D}) = -1$, which means that (e,f) is a solution to the given equation.

(c) From the convergent $\frac{18}{5}$ computed above, we get the first solution $18^2 - 13 * 5^2 = -1$. Then any $(x_{,k}, y_k)$ such that $x + k + y_k\sqrt{D} = (649 + 180\sqrt{13})^k(18 + 13\sqrt{5})$ for $k \geq 0$ is a solution of the given equation.

The remaining problem is only required for Math 845 students, though all are encouraged to think about it.

(5) Let D be a positive integer that is not a perfect square. Suppose that $x^2 - Dy^2 = -1$ has a solution, and let (c,d) be the smallest positive integer solution. Let (a,b) be the smallest integer solution to the Pell's equation $x^2 - Dy^2 = 1$. Show that $(c + d\sqrt{D})^2 = a + b\sqrt{D}$, and use this to describe all solutions to $x^2 - Dy^2 = -1$ in terms of c and d.

Let (c,d) be the smallest positive integer solution to $x^2 - Dy^2 = -1$ and (a,b) be the smallest integer solution to the Pell's equation $x^2 - Dy^2 = 1$. Note first that $\alpha = a + b\sqrt{D}$ has $N(\alpha) = 1$ and $\gamma = c + d\sqrt{D}$ has $N(\gamma) = -1$. In particular, $N(\gamma^2) = 1$, so γ^2 is some positive solution to to the Pell's equation $x^2 - Dy^2 = 1$. Based on our results on Pell's equation, we must have $\gamma^2 = \alpha^k$ for some $k \geq 1$.

We consider the elements of $\mathbb{Z}[\sqrt{D}]$:

$$\alpha\gamma^{-1} = (a + b\sqrt{D})(d\sqrt{D} - c) = (bd\sqrt{D} - ac) + (ad - bc)\sqrt{D}$$
$$\alpha^{-1}\gamma = (a - b\sqrt{D})(c + d\sqrt{D}) = (ac - bd\sqrt{D}) + (ad - bc)\sqrt{D}.$$

Note that $ad - bc > 0$ since $a > b\sqrt{D}$ and $d\sqrt{D} > c$, so $ad\sqrt{D} > bc\sqrt{D}$.

We claim that $bd\sqrt{D} - ac > 0$. To see this, first, if equality holds, then $-1 = N(\alpha\gamma^{-1}) = -(ad - bc)^2 D$ is a contradiction. If $bd\sqrt{D} - ac < 0$, then $N(\alpha^{-1}\gamma) = -1$ and the coefficients of $\alpha^{-1}\gamma$ yield a positive integer solution to $x^2 - Dy^2 = -1$; say (e,f). But then $\gamma = (\alpha^{-1}\gamma)\alpha = (e+f\sqrt{D})(a+b\sqrt{D})$ is easily seen to have larger positive coefficients than (e,f), which contradicts minimality of (c,d). This establishes the claim.

Thus, the coefficients of $\alpha\gamma^{-1}$ yield a positive solution to $x^2 - Dy^2 = -1$; say (e,f); set $\varepsilon = e + f\sqrt{D}$. By an argument similar to above, we have (e,f) is less than (a,b). Then, the coefficients of ε^2 are less than those of α^2. Since $N(\varepsilon^2) = (-1)^2 = 1$, the coefficients of ε^2 are a solution to the Pell's equation $x^2 - Dy^2 = 1$, and since every positive solution comes from a power of α, we must have $\varepsilon^2 = \alpha$. That is: $(\alpha\gamma^{-1})^2 = \alpha$, so $\gamma^2 = \alpha$. This is what we wanted to show.