THEOREM 39.1 (ROLLE'S THEOREM): Let f be continuous on the closed interval $[a, b]$ and differentiable at every point of (a, b). If $f(a) = f(b)$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

THEOREM 39.2 (MEAN VALUE THEOREM): Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

DEFINITION 39.3: Let f be a function, and $S \subseteq \mathbb{R}$ be a set of real numbers contained in domain of f. We say that

- f is **increasing** on S if for any $a, b \in S$ with $a < b$ we have $f(a) \leq f(b)$;
- f is **decreasing** on S if for any $a, b \in S$ with $a < b$ we have $f(a) \geq f(b)$;
- f is **constant** on S if for any $a, b \in S$ with $a < b$ we have $f(a) = f(b)$;
- f is **strictly increasing** on S if for any $a, b \in S$ with $a < b$ we have $f(a) < f(b)$;
- f is **strictly decreasing** on S if for any $a, b \in S$ with $a < b$ we have $f(a) > f(b)$.

COROLLARY 39.4: Suppose I is an open interval (that is, $I = (a, b)$, (a, ∞), $(-\infty, b)$, or (∞, ∞)) and f is differentiable on all of I.

(1) $f'(x) \geq 0$ for all $x \in I$ if and only if f is increasing on all of I.
(2) $f'(x) \leq 0$ for all $x \in I$ if and only if f is decreasing on all of I.
(3) $f'(x) = 0$ for all $x \in I$ if and only if f is a constant function on I.

(1) In this problem, we prove Rolle's Theorem.
  (a) First, assume that f is constant on $[a, b]$, and prove the Theorem in this case.
  (b) Explain why f has a minimum value and a maximum value on $[a, b]$.
  (c) Explain why, in the case that f is not constant, either the minimum or maximum value for f occurs in (a, b), and conclude the proof.

(2) Prove the Mean Value Theorem.
  - Suggestion: Let $\ell(x) = \left(\frac{f(b) - f(a)}{b - a}\right) x$, and show that $f(x) - \ell(x)$ satisfies the hypotheses of Rolle's Theorem.

(3) In this problem, we prove Corollary 39.4.
  (a) For the $(\Rightarrow)$ direction of (1), let $a, b \in I$ with $a < b$. Explain why the Mean Value Theorem applies to f on $[a, b]$, and apply it.
  (b) For the $(\Leftarrow)$ direction of (1), prove the contrapositive using a result from last time.
  (c) Prove the rest of the Corollary.

(4) Prove or disprove: If $J = (-\infty, 0) \cup (0, \infty)$ and that $f'(x) = 0$ for all $x \in J$, then f is constant on J.

(5) Prove or disprove: If f is differentiable on $\mathbb{R}$ and $f'(r) > 0$, then there is some $\delta > 0$ such that f is increasing on $(r - \delta, r + \delta)$.

(6) "THE FIRST DERIVATIVE TEST" Suppose that f is continuous on the closed interval $[a, b]$ and that there finitely many points $r_1 < r_2 < \cdots < r_{t-1}$ in (a, b) where either f' is zero or undefined. Set $r_0 = a$ and $r_t = b$.

For $i = 1, \ldots, t-1$, show that f attains a local maximum at r_i if and only if $f'(x) > 0$ for $x \in (r_{i-1}, r_i)$ and $f'(x) < 0$ for $x \in (r_i, r_{i+1})$. Likewise, f attains a local minimum at r_i if and only if $f'(x) < 0$ for $x \in (r_{i-1}, r_i)$ and $f'(x) > 0$ for $x \in (r_i, r_{i+1})$.

(7) "THE SECOND DERIVATIVE TEST" Suppose that f is continuous on the closed interval $[a, b]$ and that there finitely many points $r_1 < r_2 < \cdots < r_{t-1}$ in (a, b) where either f' is zero or undefined.

For $i = 1, \ldots, t-1$, show that f attains a local maximum at r_i if $f''(r_i) < 0$. Likewise, f attains a local minimum at r_i if $f''(r_i) > 0$. Give counterexamples to the converses.