

DETERMINANTS

DEFINITION: Let R be a commutative ring, and $A \in \text{Mat}_{n \times n}(R)$. The **determinant** of A is

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n x_{i,\sigma(i)}.$$

THEOREM 1: Identify $\text{Mat}_{n \times n}(R)$ with $\underbrace{R^n \times \cdots \times R^n}_{n \text{ times}}$ by considering a matrix as an n -tuple of columns. The determinant is the unique function

$$\det: \underbrace{R^n \times \cdots \times R^n}_{n \text{ times}} \rightarrow R$$

that satisfies the following three properties:

- \det is **multilinear**, meaning

$$\begin{aligned} \det(v_1, \dots, v_{i-1}, \mathbf{v} + \mathbf{w}, v_{i+1}, \dots, v_n) &= \det(v_1, \dots, v_{i-1}, \mathbf{v}, v_{i+1}, \dots, v_n) \\ &\quad + \det(v_1, \dots, v_{i-1}, \mathbf{w}, v_{i+1}, \dots, v_n) \\ \det(v_1, \dots, v_{i-1}, \mathbf{r}\mathbf{v}, v_{i+1}, \dots, v_n) &= \mathbf{r} \det(v_1, \dots, v_{i-1}, \mathbf{v}, v_{i+1}, \dots, v_n) \end{aligned}$$

- \det is **alternating**, meaning

$$\det(v_1, \dots, v_n) = 0 \quad \text{if } v_i = v_j \text{ for some } i \neq j.$$

- $\det(e_1, \dots, e_n) = 1$.

(1) Working with Theorem 1:

- Use Theorem 1 to explain why the determinant of a diagonal matrix is the product of its diagonal entries.
- Use Theorem 1 to show that if some column of A is a linear combination of the other columns of A , then $\det(A) = 0$.
- Use part (a) to show that if $R = F$ is a field, and A is not invertible, then $\det(A) = 0$.
- Use Theorem 1 to show¹ that

$$\det(\mathbf{v}_2, \mathbf{v}_1, v_3, \dots, v_n) = -\det(\mathbf{v}_1, \mathbf{v}_2, v_3, \dots, v_n).$$

Likewise, the same holds for swapping any two entries.

(2) Uniqueness part of Theorem 1:

- Use 1(d) to show that for any $\sigma \in S_n$,

$$\det(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \text{sgn}(\sigma) \det(v_1, \dots, v_n).$$

- Explain the following claim: if $F: \underbrace{R^n \times \cdots \times R^n}_{n \text{ times}} \rightarrow R$ is multilinear, then F is completely determined by $F(e_{i_1}, \dots, e_{i_n})$ for $1 \leq i_1, \dots, i_n \leq n$.
- Explain the following claim: if $F: \underbrace{R^n \times \cdots \times R^n}_{n \text{ times}} \rightarrow R$ is multilinear and alternating, then F is completely determined by $F(e_1, \dots, e_n)$.

¹Hint: Consider $\det(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, v_3, \dots, v_n)$.

THEOREM 2: Let R be a commutative ring and $A, B \in \text{Mat}_{n \times n}(R)$. Then

$$\det(AB) = \det(A)\det(B).$$

PROPOSITION: Let R be a commutative ring. Let A be a square matrix, and B be a matrix obtained from A by an elementary column operation.

- For the operation “add $r \in R$ times column i to column j ” we have $\det(B) = \det(A)$.
- For the operation “multiply column i by $u \in R^\times$ ” we have $\det(B) = u\det(A)$.
- For the operation “swap column i and column j ” we have $\det(B) = -\det(A)$.

(3) Use Theorem 1 to prove the Proposition.

(4) Use the Proposition (and not the definition) to compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 7 & 13 \end{bmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{Q})$.

(5) Proof of Theorem 2 in the case $R = F$ is a field:

- (a)** Prove Theorem 2 in the case $B = E$ is an elementary matrix.
- (b)** Prove² Theorem 2 in the case A and B are both invertible matrices.
- (c)** Show that AB is invertible if and only if A and B are both invertible.
- (d)** Show³ that $\det(A) \in F^\times$ if and only if A is invertible.
- (e)** Complete the proof of Theorem 2 in the field case.

(6) Prove that $\det(A) = \det(A^T)$.

(7) Prove the Laplace expansion formula (along the first column): for $A \in \text{Mat}_{n \times n}(R)$,

$$\det(A) = \sum_{i=1}^n (-1)^{i+1} a_{i,1} \det(\widehat{A}_{i,1}),$$

where $\widehat{A}_{i,1}$ is the $(n - 1) \times (n - 1)$ matrix obtained from A by removing the i th row and first column.

²Hint: You can use the fact that over a field, every invertible matrix is a product of elementary matrices

³Hint: Use part (a) and 1(c).