MIDTERM EXAM

Please turn in *four* of the following problems. If you intend to take an written algebra comprehensive exam, I recommend attempting the problems in a timed setting with no notes at first, and then continuing with the problems later.

(1) For each of the following, find an example with justification, or show that none exists:
  (a) A ring R such that $\mathbb{C} \subseteq R \subseteq \mathbb{C}[x, y]$ that is not Noetherian.
  (b) A ring R such that $\mathbb{C}[x^2, y^2] \subseteq R \subseteq \mathbb{C}[x, y]$ that is not Noetherian.

(2) Let R be a unique factorization domain and $I \subseteq R$ be an ideal. Show that I has height 1 if and only if I is principal.

(3) Let $(R, \mathfrak{m})$ be a local ring and M be a nonzero R-module. Assume that R is a domain. Show that if M is injective then M is not finitely generated.

(4) Give, with justification, ideals $I, J \subseteq \mathbb{C}[x, y]$ such that
  - $\sqrt{I} = \sqrt{J} = (y)$,
  - I has a unique minimal primary decomposition,
  - J has distinct minimal primary decompositions.

(5) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of modules. Show that $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(L) \cup \operatorname{Supp}_R(N)$.

(6) Let M be a finitely generated module over a Noetherian ring R. Show[1] that the following are equivalent:
  (a) M has finite length;
  (b) $\operatorname{Ass}_R(M) \subseteq \operatorname{Max}(R)$;
  (c) $\operatorname{Supp}_R(M) \subseteq \operatorname{Max}(R)$.

(7) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be ring homomorphisms. Show that $A \xrightarrow{\beta\alpha} C$ is integral if and only if $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ are both integral.

[1]Hint: For (3) $\Rightarrow$ (1), you may want to apply the previous problem to a prime filtration.