STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: INVARIANT FACTORS: Let G be a finitely generated abelian group. There exist integers $r \geqslant 0$, and $n_i \geqslant 2$, satisfying $n_1 \mid n_2 \mid \cdots \mid n_t$ such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_t.$$

Moreover, the list $r, n_1, \ldots, n_t$ is uniquely determined by G.

STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: ELEMENTARY DIVISORS: Let G be a finitely generated abelian group. Then there exist integers $r \geqslant 0$, not necessarily distinct positive prime integers $p_1, \ldots, p_s$, and integers $a_i \geqslant 1$ for $1 \leqslant i \leqslant s$ such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/p_1^{a_1} \times \cdots \times \mathbb{Z}/p_s^{a_s}.$$

Moreover, r and s are uniquely determined by G, and the list of prime powers $p_1^{a_1}, \ldots, p_s^{a_s}$ is unique up to the ordering.

(1) Converting between forms:

To convert a cyclic group $\mathbb{Z}/a$ to elementary divisor form, write each $a = p_1^{e_1} \cdots p_s^{e_s}$ as a product of prime powers, and use CRT get

$$\mathbb{Z}/a \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_s^{e_s}.$$

(a) Convert $\mathbb{Z}^2 \times \mathbb{Z}/50 \times \mathbb{Z}/60$ to elementary divisor form.

$\mathbb{Z}^2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/25$

To convert a group from elementary divisor form to invariant factor form,

- For each distinct prime p_j occurring, take the largest power E_j it has in an elementary divisor, and combine and combine $\prod_j \mathbb{Z}/p_j^{E_j} \cong \mathbb{Z}/(p_1^{E_1} \cdots p_\ell^{E_\ell})$ via CRT. If there's more than one copy of $\mathbb{Z}/p_j^{E_j}$, just take one of the copies and leave the rest.
- Repeat with the remaining factors.

(b) Convert $\mathbb{Z}^3 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/9 \times \mathbb{Z}/27 \times \mathbb{Z}/25$ to invariant factor form.

$\mathbb{Z}/3 \times \mathbb{Z}/36 \times \mathbb{Z}/2700$

(2) Which of the following groups are isomorphic or not?

- $\mathbb{Z}/5 \times \mathbb{Z}/12 \times \mathbb{Z}/36$
- $\mathbb{Z}/10 \times \mathbb{Z}/12 \times \mathbb{Z}/18$
- $\mathbb{Z}/30 \times \mathbb{Z}/54$

We use elementary divisor form to decide:

- $\mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/9$
- $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/9$
- $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \times \mathbb{Z}/5 \times \mathbb{Z}/27$

None are isomorphic. (We could also have used invariant factor form.)

(3) Classify all *abelian* groups of order 20 up to isomorphism. For each isomorphism class, give its expression in invariant factor form.

Note that $20 = 2^2 \times 5$. Thus, our possible elementary divisors are:

- $2, 2, 5$
- $4, 5$

This gives the two possibilities in invariant factor form:

- $\mathbb{Z}/2 \times \mathbb{Z}/10$
- $\mathbb{Z}/20$.

(4) Let $p < q$ be primes.

(a) Show that if p does not divide $q - 1$, then any group of order pq is isomorphic to C_{pq} by the following steps:

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for direct products.

By Sylow's Theorem, the number of q-Sylows n_q divides p. Since $p < q$, we have $n_q = 1$, so there is a unique, and hence normal, q-Sylow, Q. Also we have $n_p | q$ and $n_p \equiv 1 \mod p$. By the hypothesis, $q \not\equiv 1 \mod p$. It follows that $n_p = 1$, so there is a normal subgroup P of order p. We must have $P \cap Q = \{e\}$, since any element in the intersection has order dividing p and q, and $|PQ| = pq = |G|$, so $PQ = G$. Recognition Theorem for direct products, we have that $G \cong P \times Q$.

(b) Show from that if p does divide $q - 1$, then there are exactly two groups of order pq up to isomorphism by the following steps:

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for semidirect products.
- Use an Exercise from class about when two semidirect products are isomorphic.

We again have a unique q-Sylow Q for the same reasoning as above. now either $n_p = 1$ or $n_p = q$. If $n_p = 1$, then $G \cong P \times Q$ by the argument above. If $n_p = q$, then (using the same reasoning as above) the Recognition Theorem for semidirect products gives $G \cong Q \rtimes_\phi P$ for some ϕ. We claim that every nonabelian semidirect product $Q \rtimes_\phi P$ is isomorphic to each other. Indeed, since P is cyclic, by an exercise from class, if $\phi(P)$ and $\phi'(P)$ are conjugate in $\mathrm{Aut}(Q)$, then the corresponding semidirect products are isomorphic. However, $\mathrm{Aut}(Q)$ is cyclic, so it has a unique subgroup of order p. Since we assumed nonabelian, $\phi(P)$ is nontrivial, and hence has order p. This gives the uniqueness.

(5) Let p be a prime integer. Let G be a group of order p^2.

(a) Show[1] that G is abelian.

(b) Classify all groups of order p^2 up to isomorphism.

(6) Let p, q be primes such that $q = p + 2$ and $p \geq 5$. Show that any group of order p^2q^2 is either isomorphic to a cyclic group or a product of two cyclic groups.

[1]Hint: If not, what can you say about $Z(G)$ and $G/Z(G)$?