

DEFINITION: Let R be a commutative ring. Let V be a free R -module with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R -module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Given an R -module homomorphism $T: V \rightarrow W$, for each $j = 1, \dots, n$, write

$$(\clubsuit) \quad T(b_j) = r_{1,j}c_1 + \cdots + r_{m,j}c_m$$

for some elements $r_{i,j} \in R$. The matrix

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{bmatrix}$$

is the **matrix representing T in the bases \mathcal{B} and \mathcal{C}** .

(1) Warming up with the definition:

- (a) If R is a field F , translate everything¹ in the definition into linear algebra terms.
- (b) Use the equation (\clubsuit) to explain as concretely as possible what the j -th column of $[T]_{\mathcal{B}}^{\mathcal{C}}$ means in terms of T , \mathcal{B} , and \mathcal{C} .
- (c) Explain why the entries $r_{i,j}$ are well-defined.
- (d) Just using your answer for part (b) and not looking at the formula, describe the dimensions of the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ in terms of the rank of V and the rank of W .
- (e) Let V be the \mathbb{R} -vector space of polynomials in $\mathbb{R}[x]$ of degree at most 3 along with the zero polynomial. The derivative map $\frac{d}{dx}$ is a linear transformation from V to V . Choose a nice basis \mathcal{B} for V and compute the matrix $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{B}}$.
- (f) Find another² basis \mathcal{C} for V such that $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(2) Show that if $\mathcal{E} = \{e_1, \dots, e_n\}$ is the standard basis on R^n and $\mathcal{E}' = \{e_1, \dots, e_m\}$ is the standard basis on R^m , then $T(v) = [T]_{\mathcal{E}}^{\mathcal{E}'} \cdot v$, where the RHS is usual matrix-times-vector multiplication.

PROPOSITION: Let R be a commutative ring. Let V be a free R -module with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R -module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Then the map

$$\begin{aligned} \text{Hom}_R(V, W) &\longrightarrow \text{Mat}_{m \times n}(R) \\ T &\longmapsto [T]_{\mathcal{B}}^{\mathcal{C}} \end{aligned}$$

is bijective. Moreover, this is an isomorphism of R -modules.

When $V = W$ and $\mathcal{B} = \mathcal{C}$, the same map

$$\begin{aligned} \text{End}_R(V) &\longrightarrow \text{Mat}_{n \times n}(R) \\ T &\longmapsto [T]_{\mathcal{B}}^{\mathcal{B}} \end{aligned}$$

is an isomorphism of rings.

(3) Prove that the map $T \mapsto [T]_{\mathcal{B}}^{\mathcal{C}}$ in the Proposition is bijective.

(4) Suppose that V is a free module with a countably infinite basis $\mathcal{B} = \{b_1, b_2, b_3, \dots\}$, and W is free with a countably infinite basis $\mathcal{C} = \{c_1, c_2, c_3, \dots\}$. What is the analogue of the Proposition in this case?

(5) Prove the Proposition.

¹You can do this aloud instead of rewriting everything.

²You might have to reorder or change \mathcal{B} if you are unlucky.