NORMAL SUBGROUPS

DEFINITION: A subgroup N of a group G is **normal** if $gNg^{-1} = N$ for all $g \in G$, where $gNg^{-1} = \{gng^{-1} \mid n \in N\}$. We write $N \trianglelefteq G$ to indicate that N is a normal subgroup of G.

LEMMA: Let N be a subgroup of a group G. The following are equivalent:

(a) N is a normal subgroup of G.
(b) For all $g \in G$, $gNg^{-1} \subseteq N$.
(c) For all $g \in G$, the *left coset* gN is equal to the *right coset* Ng.
(d) For all $g \in G$, $gN \subseteq Ng$.
(e) For all $g \in G$, $Ng \subseteq gN$.

(1) Examples of normal subgroups: Use the definition and/or the Lemma to show the following:

(a) If G is an abelian group and $H \leq G$, then $H \trianglelefteq G$.
(b) The center $Z(G)$ of a group G is a normal subgroup[1] of G.
(c) The[2] group $K = \{e, (12)(34), (13)(24), (14)(23)\} \leq S_4$ is normal.

[1]Recall that we have already shown that $Z(G) \leq G$.
[2]Hint: Recall from HW 1 that $\tau(i\ j)\tau^{-1} = (\tau(i)\ \tau(j))$.