# Problem Set 6.5
Not to be turned in for credit

**Problem 1.** (a) Let $G$ be a group of order 15 acting on a set $S$ with 7 elements. Prove that there exists at least one fixed point, i.e., there exists $s \in S$ such that $g \cdot s = s$ for all $g \in G$.

(b) Give an example of an action of $C_{15}$ on a set with 8 elements with no fixed points. Justify.

**Problem 2.** Let $G$ be a group acting on a set $A$, and let $H$ be a subgroup of $G$ satisfying the condition that the induced action of $H$ on $A$ is transitive (that is, for all $a, b \in A$ there is an $h \in H$ with $ha = b$). Let $t \in A$, and let $\operatorname{Stab}_G(t)$ be the stabilizer of $t$ in $G$. Show that $G = H \operatorname{Stab}_G(t)$.

**Problem 3.** Let $G$ be a finite group and let $H$ be a proper subgroup of $G$ with $[G : H] = h$.

1. Prove that $H$ has at most $h$ distinct conjugate sets $gHg^{-1}$ for $g \in G$.

2. Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$.

**Problem 4.** Let $G$ be a group of order $p^n$, for some prime $p$, acting on a finite set $X$.

(a) Suppose $p$ does not divide $\#X$. Prove that there exists some element of $X$ that is fixed by all elements of $G$.

(b) Suppose $G$ acts faithfully on $X$. (Recall this means that if $g \cdot x = x$ for all $x \in X$, then $g = e_G$.) Prove that $\#X \geq n \cdot p$.