WARMING UP WITH IF-THEN STATEMENTS AND REAL NUMBERS: §1.4 AND §1.5

Making sense of if-then statements.

- The statement "If P then Q" is true whenever Q is true or P is false. Equivalently, the statement "If P then Q" is false whenever Q is false and P is true.
- The **converse** of the statement "If P then Q" is the statement "If Q then P".
- The **contrapositive** of the statement "If P then Q" is the statement "If not Q then not P".
- Any if-then statement is equivalent to its *contrapositive*, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Decide if original/contrapositive/converse true or false for real numbers a, b, but don't prove them yet.
  (a) If a is irrational, then $1/a$ is irrational.
  (b) If a and b are irrational, then ab is irrational.
  (c) If $a \geq 3$, then $a^2 \geq 9$.

Proving if-then statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if-then statement than the original, especially when the conclusion is something negative. We sometimes call this an *indirect proof* or a *proof by contraposition*.

(2) Consider the following proof of the claim "For real numbers x, y, z, if $x+y = z+y$, then $x = z$" from the axioms of $\mathbb{R}$. Match the parts of this proof with the general outline above. Which sentences are *assumptions* and which are *assertions*? Is it clear *just from reading each sentence on its own* whether it is an assumption or an assertion? Is it clear *why* each assertion is true?

Proof. Suppose that $x + y = z + y$. Then adding $-y$ (which exists by Axiom 6) we get

$$(x + y) + (-y) = (z + y) + (-y).$$

This can be rewritten (by Axiom 3) as

$$x + (y + (-y)) = z + (y + (-y)),$$

and hence (by Axiom 6) as

$$x + 0 = z + 0,$$

which gives $x = z$ (by Axioms 4 and 2). $\square$

(3) Consider the following purported proof of the true fact "If $2x + 5 \geq 7$ then $x \geq 1$." Is this a good proof? Is it a correct proof?

Proof.

$$x \geq 1.$$

Multiply both sides by two.

$$2x \geq 2.$$

Add five to both sides.

$$2x + 5 \geq 7.$$

$\square$

Proving if-then statements.

(4) Prove or disprove each of the statements in (1). You might consider a proof by contraposition for some of these!

(5) Prove or disprove the *converse* of each of the statements in (1).

Using the axioms of $\mathbb{R}$ to prove basic arithmetic facts.

(6) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[1] that if $x \geq y$ then $-x \leq -y$.

(7) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove that $x \geq y$ if and only if $-x \leq -y$.

(8) Let x, y be real numbers. Use the axioms of $\mathbb{R}$ and facts we have already proven[2] to prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

(9) Use[3] the axioms of $\mathbb{R}$ and facts we have already proven to prove that $1 > 0$.

[1]Hint: You may want to add something to both sides.
[2]Be careful: are you using any facts that we have not already proven?
[3]Hint: Try a proof by contradiction.