

WARMING UP WITH PROOF TECHNIQUES AND REAL NUMBERS

DEFINITION 2.1: A real number is *irrational* if it is not rational.

Making sense of if then statements and quantifier statements.

- The *converse* of the statement “If P then Q ” is the statement “If Q then P ”.
- The *contrapositive* of the statement “If P then Q ” is the statement “If not Q then not P ”.
- Any if then statement is equivalent to its contrapositive, but not necessarily to its converse!

- (1) For each of the following statements, write its contrapositive and its converse. Is the original/contrapositive/converse true or false for real numbers a, b ? Explain why (but don’t prove).
- (a) If a is irrational, then $1/a$ is irrational.
 - (b) If a and b are irrational, then ab is irrational.
 - (c) If $a > 3$, then $a^2 > 9$.

- The symbol for “for all” is \forall and the symbol for there exists is \exists .
- The negation of “For all $x \in S$, P ” is “There exists $x \in S$ such that not P ”.
- The negation of “There exists $x \in S$ such that P ” is “For all $x \in S$, not P ”.

- (2) Rewrite each statement with symbols in place of quantifiers, and write its negation. Is the original statement true or false? Explain why (but don’t prove them).
- (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
 - (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
 - (c) For all $x \in \mathbb{R}$ such that¹ $x \neq 0$, $x^2 > 0$.
 - (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
 - (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

¹In a statement of the form “For all $x \in S$ such that Q , P ”, the “such that Q ” part is part of the hypothesis: it is restricting the set S that we are “alling” over.

Proving if then statements and quantifier statements.

- The general outline of a direct proof of “If P then Q ” goes
 - (1) Assume P .
 - (2) Do some stuff.
 - (3) Conclude Q .
- Often it is easier to prove the contrapositive of an if then statement than the original, especially when the negation of the hypothesis or conclusion is something negative.
- The general outline of a proof of “For all $x \in S$, P ” goes
 - (1) Let $x \in S$ be arbitrary.
 - (2) Do some stuff.
 - (3) Conclude that P holds for x .
- To prove a there exists statement, you just need to give an example. To prove “There exists $x \in S$ such that P ” directly:
 - (1) Consider² x =[some specific element of S].
 - (2) Do some stuff.
 - (3) Conclude that P holds for x .

- (3) Let x and y be real numbers. Use the axioms of \mathbb{R} to prove³ that $x \geq y$ if and only if $-y \geq -x$.
- (4) Let x be a real number. Show that if x^2 is irrational, then x is irrational.
- (5) Let x be a real number. Use the axioms of \mathbb{R} and facts we have proven in class to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.
- (6) Prove that⁴ for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.
- (7) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.
- (8) Prove⁵ that (2d) is true and (2e) is false.
- (9) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.
- (10) Prove that $1 > 0$.
- (11) Let x, y be real numbers. Prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

²How you found this x is logically irrelevant to an existence proof, and should not be included.

³Hint: You may want to add something to both sides.

⁴Hint: Use (5).

⁵You can “work out of order here” and use (10) now.