

FALL 2024 MATH 325 LECTURE NOTES AND WORKSHEETS

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1. MONDAY, AUGUST 26, 2024 §1.1

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \dots$. We will call these the *natural numbers* and write \mathbb{N} for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}.$$

(The symbol \mathbb{Z} is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as “two and a fourth”, but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as \mathbb{Q} (for “quotient”).

You might not have thought about it before, but an expression of the form $\frac{m}{n}$ is really an “equivalence class”: the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\dots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\dots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer “yes”, but the ancient Greeks believed for a time that every number was rational. Let’s convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c , must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number. Right? Now, let’s convince ourselves that there is no *rational number* with this property. In fact, I’ll make this a theorem.

Theorem 1.1. *There is no rational number whose square is 2.*

Preproof Discussion 1. *Before launching a formal proof, let’s philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I’d be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \implies \text{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement “there is a rational number whose square is 2”, the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction.

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of “rational number”, we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2’s.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a . But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. \square

Thus, if we would like to have a number corresponding to the length of the diagonal of a square with side length one, it must be a number that is real but not rational. Let’s record the common name for such a number.

Definition 1.2. A real number is *irrational* if it is not rational.

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.3 (Arithmetic and order properties of \mathbb{Q}). *The set of rational numbers form an “ordered field”. This means that the following ten properties hold:*

- (1) *There are operations $+$ and \cdot defined on \mathbb{Q} , so that if p, q are in \mathbb{Q} , then so are $p + q$ and $p \cdot q$.*
- (2) *Each of $+$ and \cdot is a commutative operation (i.e., $p + q = q + p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*
- (3) *Each of $+$ and \cdot is an associative operation (i.e., $(p + q) + r = p + (q + r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q , and r).*
- (4) *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0 + q = q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.*
- (5) *The distributive law holds: $p \cdot (q + r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.*
- (6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.*
- (7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.*
- (8) *There is a “total ordering” \leq on \mathbb{Q} . This means that*
 - (a) *For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
 - (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
 - (c) *For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
- (9) *The total ordering \leq is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.*
- (10) *The total ordering \leq is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

Which of the properties from Proposition 1.3 does \mathbb{N} satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for \mathbb{N} .

We expect everything from Proposition 1.3 to be true for the real numbers. We will build them into our definition. To define the real numbers \mathbb{R} , we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written \mathbb{R} , satisfies the following eleven properties:*

- (Axiom 1) *There are operations $+$ and \cdot defined on \mathbb{R} , so that if $x, y \in \mathbb{R}$, then so are $x + y$ and $x \cdot y$.*

- (Axiom 2) Each of $+$ and \cdot is a commutative operation.
- (Axiom 3) Each of $+$ and \cdot is an associative operation.
- (Axiom 4) The real number 0 is an identity element for addition and the real number 1 is an identity element for multiplication. This means that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.
- (Axiom 5) The distributive law holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.
- (Axiom 6) Every real number has an additive inverse: For any $x \in \mathbb{R}$, there is a number $-x$ satisfying $x + (-x) = 0$.
- (Axiom 7) Every nonzero real number has a multiplicative inverse: For any $x \in \mathbb{R}$ such that $x \neq 0$, there is a real number x^{-1} satisfying $x^{-1} \cdot x = 1$.
- (Axiom 8) There is a “total ordering” \leq on \mathbb{R} . This means that
- (a) For all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.
 - (b) If $x \leq y$ and $y \leq z$, then $x \leq z$.
 - (c) For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- (Axiom 9) The total ordering \leq is compatible with addition: If $x \leq y$ then $x + z \leq y + z$ for all z .
- (Axiom 10) The total ordering \leq is compatible with multiplication by nonnegative real numbers: If $x \leq y$ and $z \geq 0$ then $zx \leq zy$.
- (Axiom 11) The completeness axiom holds. (I will say what this means later.)

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

“Cancellation of Addition”: For real numbers, $x, y, z \in \mathbb{R}$, if $x + y = z + y$ then $x = z$.

Let’s prove this carefully, using just the list of axioms: Assume that $x + y = z + y$. Then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x + y) + (-y) = (z + y) + (-y)$. This can be rewritten as $x + (y + (-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

Here is the incredible fact: everything that we know about the real numbers follows from these axioms! In particular, all of the basic facts about arithmetic of real numbers follow from this, like “the product of two negative numbers is positive,” as well as everything you encountered in Calculus, like the Mean Value Theorem.

2. WEDNESDAY, AUGUST 27, 2024 §1.4 & 1.5

Making sense of if-then statements.

- The statement “If P then Q ” is true whenever Q is true or P is false. Equivalently, the statement “If P then Q ” is false whenever Q is false and P is true.
- The **converse** of the statement “If P then Q ” is the statement “If Q then P ”.
- The **contrapositive** of the statement “If P then Q ” is the statement “If not Q then not P ”.
- Any if-then statement is equivalent to its *contrapositive*, but not necessarily to its converse!

- (1) For each of the following statements, write its contrapositive and its converse. Decide if original/contrapositive/converse true or false for real numbers a, b , but don't prove them yet.
- (a) If a is irrational, then $1/a$ is irrational.
 - (b) If a and b are irrational, then ab is irrational.
 - (c) If $a \geq 3$, then $a^2 \geq 9$.

- (a) Contrapositive: If $1/a$ is rational, then a is rational. Converse: If $1/a$ is irrational, then a is irrational. Original, contrapositive, and converse all TRUE.
- (b) Contrapositive: If ab is rational, then a is rational or b is rational. Converse: if ab is irrational, then a and b are irrational. Original, contrapositive, and converse all FALSE.
- (c) Contrapositive: If $a^2 < 9$ then $a < 3$. Converse: $a^2 \geq 9$ then $a \geq 3$. Original and contrapositive are TRUE but converse is FALSE.

Proving if-then statements.

- The general outline of a direct proof of “If P then Q ” goes
 - (1) Assume P .
 - (2) Do some stuff.
 - (3) Conclude Q .
- Often it is easier to prove the contrapositive of an if-then statement than the original, especially when the conclusion is something negative. We sometimes call this an *indirect proof* or a *proof by contraposition*.

- (2) Consider the following proof of the claim “For real numbers x, y, z , if $x + y = z + y$, then $x = z$ ” from the axioms of \mathbb{R} . Match the parts of this proof with the general outline above. Which sentences are *assumptions* and which are

assertions? Is it clear *just from reading each sentence on its own* whether it is an assumption or an assertion? Is it clear *why* each assertion is true?

Proof. Suppose that $x + y = z + y$. Then adding $-y$ (which exists by Axiom 6) we get

$$(x + y) + (-y) = (z + y) + (-y).$$

This can be rewritten (by Axiom 3) as

$$x + (y + (-y)) = z + (y + (-y)),$$

and hence (by Axiom 6) as

$$x + 0 = z + 0,$$

which gives $x = z$ (by Axioms 4 and 2). \square

The first sentence is an assumption, while the others are assertions. It is clear that the first is an assumption since it says “Suppose”; another possibility is “Assume”.

- (3) Consider the following purported proof of the true fact “If $2x + 5 \geq 7$ then $x \geq 1$.” Is this a good proof? Is it a correct proof?

Proof.

$$x \geq 1.$$

Multiply both sides by two.

$$2x \geq 2.$$

Add five to both sides.

$$2x + 5 \geq 7.$$

\square

No. It is not clear what is an assumption and what is an assertion. From the order of the sentences, it looks like they are assuming the conclusion and deducing the hypothesis! This was OK in this example, but they could have done almost the exact same thing to “prove” the false statement “If $2x^2 + 5 \geq 7$ then $x \geq 1$.”

Proving if-then statements.

- (4) Prove or disprove each of the statements in (1). You might consider a proof by contraposition for some of these!

- (a) TRUE: We proceed by contraposition. Suppose that $1/a$ is rational, so $1/a = p/q$ for some integers p/q . Then $a = q/p$ is rational.
- (b) FALSE: Consider $a = \sqrt{2}$ and $b = \sqrt{2}$. Both a and b are irrational, but $ab = 2$ is rational.

(c) TRUE: If $a \geq 3$ then $a^2 = a \cdot a \geq a \cdot 3 \geq 3 \cdot 3 = 9$, where we used that $a \geq 3 \geq 0$ in the first inequality, and $3 \geq 0$ in the second.

(5) Prove or disprove the *converse* of each of the statements in (1).

(a) TRUE: We proceed by contraposition. Suppose that a is rational, so $a = p/q$ for some integers p/q . Then $1/a = q/p$ is rational.
 (b) FALSE: Consider $a = \sqrt{2}$ and $b = 1$. Then $ab = \sqrt{2}$ is irrational, but b is rational, so it is not true that a and b are irrational.
 (c) FALSE: Consider $a = -3$; for this choice of a , $a \not\geq 3$ but $a^2 = 9 \geq 9$.

Using the axioms of \mathbb{R} to prove basic arithmetic facts.

(6) Let x and y be real numbers. Use the axioms of \mathbb{R} to prove¹ that if $x \geq y$ then $-x \leq -y$.

Let $x \geq y$. Adding $(-x) + (-y)$ to both sides (which exists by Axiom 6), we obtain $-y = x + ((-x) + (-y)) \geq y + ((-x) + (-y)) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x + y) + (-x) \leq (x + y) + (-y) = x$ (by Axiom 9 and Axiom 5).

(7) Let x and y be real numbers. Use the axioms of \mathbb{R} to prove that $x \geq y$ if and only if $-x \leq -y$.

We need to prove both directions. The first direction is the previous problem. Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x + y) + (-x) \leq (x + y) + (-y) = x$ (by Axiom 9 and Axiom 5).

(8) Let x, y be real numbers. Use the axioms of \mathbb{R} and facts we have already proven² to prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

¹Hint: You may want to add something to both sides.

²Be careful: are you using any facts that we have not already proven?

3. FRIDAY, AUGUST 29, 2024 §1.4 & 1.5

Making sense of quantifier statements.

- The symbol for “**for all**” is \forall and the symbol for “**there exists**” is \exists .
- The negation of “For all $x \in S$, P ” is “There exists $x \in S$ such that not P ”.
- The negation of “There exists $x \in S$ such that P ” is “For all $x \in S$, not P ”.

A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both)!

- (1) Match each informal story (i)–(iv) below with a precise quantifier statement (A)–(D).

Informal stories:

- (i) Every number past some point is red.
- (ii) There are arbitrarily big red numbers.
- (iii) All positive numbers are red.
- (iv) There are positive red number(s).

Precise statements:

- (A) For every $y > 0$, y is red.
- (B) There exists $y > 0$ such that y is red.
- (C) For every $x \in \mathbb{R}$, there is some $y > x$ such that y is red.
- (D) There exists $x \in \mathbb{R}$ such that for every $y > x$, y is red.

(i)=(D), (ii)=(C), (iii)=(A), (iv)=(B)

- (2) Draw a picture where (A) is false and (B) is true.

Answers may vary

- (3) Draw a picture where (C) is true and (D) is false.

Answers may vary

- (4) Suppose that (C) is true. Which of the following statements must also be true? Why?

- (a) There is some $y > 1000000000$ such that y is red.
- (b) For every $\mu \in \mathbb{R}$, there is some $\theta > \mu$ such that θ is red.
- (c) For every $x \in \mathbb{R}$, there is some $y > 2x$ such that y is red.

These are ALL TRUE.

The next problem is no longer about a spraypainting of the real number line.

(5) Rewrite each statement with symbols in place of quantifiers, and write its negation. Do you think the original statement is true or false (but don't prove them yet)?

- (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
- (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
- (c) For all $x \in \mathbb{R}$ such that³ $x \neq 0$, $x^2 > 0$.
- (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
- (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

- (a) $\exists x \in \mathbb{Q} : x^2 = 2$. NEGATION: For all $x \in \mathbb{Q}$, $x^2 \neq 2$.
- (b) $\forall x \in \mathbb{R}, x^2 > 0$. NEGATION: There exists $x \in \mathbb{R}$ such that $x^2 \leq 0$.
- (c) $\forall x \in \mathbb{R} : x \neq 0, x^2 > 0$. NEGATION: There exists $x \in \mathbb{R}$ such that $x \neq 0$ such that $x^2 \leq 0$.
- (d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$. NEGATION: There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x \geq y$.
- (e) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x < y$. NEGATION: For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $x \geq y$.

Proving quantifier statements and using the axioms of \mathbb{R} .

- The general outline of a proof of “For all $x \in S$, P ” goes
 - (1) Let $x \in S$ be arbitrary.
 - (2) Do some stuff.
 - (3) Conclude that P holds for x .
- To prove a there exists statement, you just need to give an example. To prove “There exists $x \in S$ such that P ” directly:
 - (1) Consider² $x = [\text{some specific element of } S]$.
 - (2) Do some stuff.
 - (3) Conclude that P holds for x .

Note: explaining *how* you found your example “ x ” is *not* a logically necessary part of the proof.

(6) Circle the correct answer in each of the blanks below:

- To prove a “for all” statement, you need to give a
GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *disprove* a “for all” statement, you need to give a
GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To prove a “there exists” statement, you need to give a
GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *disprove* a “there exists” statement, you need to give a
GENERAL ARGUMENT / SPECIFIC EXAMPLE.

³In a statement of the form “For all $x \in S$ such that Q , P ”, the “such that Q ” part is part of the hypothesis: it is restricting the set S that we are “alling” over.

- If you want to *use* a “for all” statement that you know is true, you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE
- If you want to *use* a “there exists” statement that you know is true, you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE

- To prove a “for all” statement, you need to give a GENERAL ARGUMENT.
- To *disprove* a “for all” statement, you need to give a SPECIFIC EXAMPLE.
- To prove a “there exists” statement, you need to give a SPECIFIC EXAMPLE.
- To *disprove* a “there exists” statement, you need to give a GENERAL ARGUMENT.
- If you want to *use* a “for all” statement that you know is true, you CAN CHOOSE A SPECIFIC VALUE
- If you want to *use* a “there exists” statement that you know is true, you MUST USE A MYSTERY VALUE

- (7) Prove or disprove each of the statements in (5) using the axioms of \mathbb{R} and facts we have already proven.

- (a) FALSE: We showed that there is no rational number whose square is two.
- (b) FALSE: Take $x = 0$; then $x^2 = 0$ which is not strictly greater than zero.
- (c) TRUE.
- (d) TRUE: Let $x \in \mathbb{R}$ be given. Then $y = x + 1 > x$.
- (e) FALSE: Suppose that $x \in \mathbb{R}$. We claim that there is some $y \in \mathbb{R}$ such that $y \leq x$; just take x itself!

More practice with quantifier statements. Using the axioms of \mathbb{R} and statements that we’ve already proven (like cancellation of addition, or any problem on the list above the given one), prove the following:

- (8) Prove that there exists some $x \in \mathbb{R}$ such that $2x + 5 = 3$.

Take $x = -1$. Then $2x + 5 = 2(-1) + 5 = 3$.

- (9) Prove⁴ that for any real number r , we have $r \cdot 0 = 0$.

Let r be any real number. We have $0 + 0 = 0$ (Axiom 4) and hence $r \cdot (0 + 0) = r \cdot 0$. But $r \cdot (0 + 0) = r \cdot 0 + r \cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the

⁴Hint: You might find it useful to write $0 = 0 + 0$ and, in a later step, use cancellation of addition.

Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

- (10) Let x be a real number. Use the axioms of \mathbb{R} and facts we have already proven to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

Let x be a real number. Suppose that there exists a real number y such that $xy = 1$. By way of contradiction, suppose that $x = 0$. Then $1 = xy = 0y = 0$, which is a contradiction. Thus, we must have $x \neq 0$. Thus, if there exists a real number y such that $xy = 1$, then $x \neq 0$.

- (11) Prove that⁵ for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

⁵Hint: Use (10).

4. WEDNESDAY, SEPTEMBER 3, 2024 §1.8

I owe you a statement of the very important Completeness Axiom. Before we get there, I want to recall an axiom of \mathbb{N} that we haven't discussed yet. It pertains to minimum elements in sets. Let's be precise and define minimum element.

Definition 4.1. Let S be a set of real numbers. A **minimum** element of S is a real number x such that

- (1) $x \in S$, and
- (2) for all $y \in S$, $x \leq y$.

In this case, we write $x = \min(S)$.

The definition of **maximum** is the same except with the opposite inequality.

Axiom 4.2 (Well-ordering axiom). *Every nonempty subset of \mathbb{N} has a minimum element.*

Example 4.3. If S is the set of even multiples of 7, then S has 14 as its minimum.

We generally like to say *the* minimum, rather than *a* minimum. To justify this, let's prove the following.

Proposition 4.4. *Let S be a set of real numbers. If S has a minimum, then the minimum is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P , then x and y must be the same thing.

Proof of Proposition 4.4. Let S be a set of real numbers, and let x and y be two minima of S . Applying part (1) of the definition of minimum to y , we have $y \in S$. Applying part (2) of the definition of minimum to x and the fact that $y \in S$, we get that $x \leq y$. Switching roles, we get that $y \leq x$. Thus $x = y$.

We conclude that if a minimum exists, it is necessarily unique. \square

The previous proposition plus the Well-Ordering Axiom together imply that every nonempty subset of \mathbb{N} has exactly one minimum element. A similar proof shows that if a maximum exists, it is necessarily unique. Could a set fail to have a maximum or a minimum? Yes!

Example 4.5.

- (1) The empty set \emptyset has no minimum and no maximum element. (There is no $s \in \emptyset$!)
- (2) The set of natural numbers \mathbb{N} has 1 as a minimum, but has no maximum. (Suppose there was: if $n = \max(\mathbb{N})$ was the maximum, then $n < n + 1 \in \mathbb{N}$ gives a contradiction.)
- (3) The open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has no minimum and no maximum. (Exercise later.)

Definition 4.6. Let S be any subset of \mathbb{R} . A real number b is called an **upper bound** of S provided that for every $s \in S$, we have $s \leq b$.

Definition 4.7. A subset S of \mathbb{R} is called **bounded above** if there exists at least one upper bound for S . That is, S is bounded above provided there is a real number b such that for all $s \in S$ we have $s \leq b$.

For example, the open interval $(0, 1)$ is bounded above, by for example 50.

The subset \mathbb{N} of \mathbb{R} is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of \mathbb{R} that is bounded above.

Example 4.8. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S ? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 4.9. Suppose S is subset of \mathbb{R} that is bounded above. A **supremum** (also known as a **least upper bound**) of S is a number ℓ such that

- (1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
- (2) if b is any upper bound of S , then $\ell \leq b$.

In this case we write $\sup(S) = \ell$.

5. FRIDAY, SEPTEMBER 5, 2024 §1.8

Let S be a set of real numbers.

- A number b is an **upper bound** for S provided for all $x \in S$ we have $b \geq x$.
- The set S is **bounded above** provided there exists at least one upper bound for S .
- A number m is the **maximum** of S provided
 - (1) $m \in S$, and
 - (2) m is an upper bound of S .
- A number ℓ is a **supremum** of S provided
 - (1) ℓ is an upper bound of S , and
 - (2) for any upper bound b for S , we have $\ell \leq b$.

- (1) Write, in simplified form, the negation of the statement “ b is an upper bound for S ”.

There exists some $x \in S$ such that $x > b$.

- (2) Write, in simplified form, the negation of the statement “ S is bounded above”.

For every $b \in \mathbb{R}$, there exists $x \in S$ such that $x > b$.

- (3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
- (a) If $x > \ell$, what is the most concrete thing you can say about x and S ?
 - (b) If $x < \ell$, what is the most concrete thing⁶ you can say about x and S ?

- (a) $x \notin S$.
- (b) There exists some $y \in S$ such that $y > x$.

- (4) Let $S = \{x \in \mathbb{R} \mid x^3 + x < 5\}$. Use the definition of supremum to answer the following:
- (a) Is 1 the supremum of S ? Why or why not?
 - (b) Is 2 the supremum of S ? Why or why not?

- (a) No. 1 is not an upper bound for S , because $1.5^3 + 1.5 = 4.875 < 5$, so 1.5 is a larger element of S .
- (b) No. We claim that 1.75 is an upper bound for S , so 2 is not the smallest upper bound of S . Indeed, if $x \in S$ then $x \leq 1.75$; if $x > 1.75$ then $x^3 + x > 1.75^3 + 1.75 = 7.109375 > 5$ so $x \notin S$.

- (5) Consider the open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

⁶Hint: Use one of the previous problems.

- (a) Prove⁷ that $(0, 1)$ has no maximum element.
 (b) Prove that $\sup((0, 1)) = 1$.

- (a) To obtain a contradiction, suppose that $(0, 1)$ has a maximum element; call it z . Since $z \in (0, 1)$ so $0 < z < 1$. First, we claim that $y = \frac{z+1}{2} \in (0, 1)$. Indeed, $z > 0$ implies $y > \frac{0+1}{2} = \frac{1}{2} > 0$, and $z < 1$ implies $y < \frac{1+1}{2} = 1$, so $y \in (0, 1)$. Now we claim that y is larger than z : $y = \frac{z+1}{2} > \frac{z+z}{2} = z$. But this contradicts that z is an upper bound for $(0, 1)$, so z cannot be the maximum. Since assuming the existence of a maximum element leads to a contradiction, we conclude that no maximum exists.
- (b) We need to show that 1 is an upper bound for $(0, 1)$, and that any other upper bound b for $(0, 1)$ satisfies $b \geq 1$. By definition of the interval $(0, 1)$, the number 1 is an upper bound. Now suppose that b is an upper bound for $(0, 1)$. We claim that $b \geq 1$. Suppose otherwise that $b < 1$ to obtain a contradiction. Consider $y = \frac{b+1}{2}$. Then by the same algebraic argument as the previous part, $y \in (0, 1)$ and $y > b$, so y is not an upper bound for $(0, 1)$. This contradicts the assumption that $b < 1$, so $b \geq 1$. Thus, every upper bound of $(0, 1)$ is at least 1. This shows that 1 is the supremum of $(0, 1)$.

- (6) Let S be a set of real numbers, and let $T = \{2s \mid s \in S\}$. Prove that if S is bounded above, then T is bounded above.

Assume that S is bounded above. Then there is some upper bound b for S , so for every $s \in S$, we have $b \geq s$. We claim that $2b$ is an upper bound for T . Indeed, if $t \in T$, then we can write $t = 2s$ for some $s \in S$, and $s \leq b$ implies $t = 2s \leq 2b$. Thus, T is bounded above.

- (7) Let S be a set of real numbers. Show that if S has a supremum, then it is unique.

Suppose both x and y are both suprema of the same subset S of \mathbb{R} . Then, since y is an upper bound of S and x is a supremum of S , by part (2) of the definition of “supremum” we have $y \geq x$. Likewise, since x is an upper bound of S and y is a supremum of S , we have $x \geq y$ by definition. Since $x \leq y$ and $y \leq x$, we conclude $x = y$.

⁷Hint: Try a proof by contradiction!

Can you think of a set with no supremum?

The subset \mathbb{N} does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly: the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term “supremum”, I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom). *Every nonempty, bounded-above subset of \mathbb{R} has a supremum.*

This is an incredibly powerful fact about the real numbers. To give a sense of what it does, recall that we showed that there is no rational number whose square is two; the completeness axiom is what allows us to show that there is a real number whose square is two.

Proposition 5.1. *There is a positive real number whose square is 2.*

In class, let’s just discuss the outline of this proof rather than the algebraic details, so we can see the big picture. You can read the full proof in the lecture notes.

Proof outline.

Step 1: Define S to be the subset $S = \{x \in \mathbb{R} \mid x^2 < 2\}$.

Step 2: Show that S is nonempty and bounded above. Thus, by the Completeness Axiom, there is a real number $\ell = \sup(S)$.

Step 3: Show that $\ell^2 < 2$ leads to a contradiction. (Assuming $\ell^2 < 2$, we can find a number s slightly larger than ℓ that is an element of S , contradicting that ℓ is an upper bound of S .)

Step 4: Show that $\ell^2 > 2$ leads to a contradiction. (Assuming $\ell^2 > 2$, we can find a number b slightly smaller than ℓ that is an upper bound of S , contradicting that ℓ is the smallest upper bound of S .)

Step 5: There’s no other possibility: we must have $\ell^2 = 2$! □

Here is the proof in full detail.

Proof. Define S to be the subset

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

S is nonempty since, for example, $1 \in S$, and it is bounded above, since, for example, 2 is an upper bound for S , as we showed earlier. So, by the Completeness Axiom, S has a least upper bound, and we know it is unique from the proposition above. Let us call it ℓ . I will prove $\ell^2 = 2$.

We know one of $\ell^2 > 2$, $\ell^2 < 2$ or $\ell^2 = 2$ must hold. We prove $\ell^2 = 2$ by showing that both $\ell^2 > 2$ and $\ell^2 < 2$ are impossible.

We start by observing that $1 \leq \ell \leq 2$. The inequality $1 \leq \ell$ holds since $1 \in S$ and ℓ is an upper bound of S , and the inequality $\ell \leq 2$ holds since 2 is an upper bound of S and ℓ is the least upper bound of S .

Suppose $\ell^2 < 2$. We show this leads to a contradiction by showing that ℓ is not an upper bound of S in this case. We will do this by constructing a number that is ever so slightly bigger than ℓ and belongs to S . Let $\varepsilon = 2 - \ell^2$. Then $0 < \varepsilon \leq 1$ (since $\ell^2 < 2$ and $\ell^2 \geq 1$). We will now show that $\ell + \varepsilon/5$ is in S : We have

$$(\ell + \varepsilon/5)^2 = \ell^2 + \frac{2}{5}\ell\varepsilon + \frac{\varepsilon^2}{25} = \ell^2 + \varepsilon\left(\frac{2\ell}{5} + \frac{\varepsilon}{25}\right).$$

Now, using $\ell \leq 2$ and $0 < \varepsilon \leq 1$, we deduce

$$0 < \frac{2\ell}{5} + \frac{\varepsilon}{25} \leq \frac{4}{5} + \frac{\varepsilon}{25} < 1.$$

Putting these equations and inequalities together yields

$$(\ell + \frac{\varepsilon}{5})^2 < \ell^2 + \varepsilon = 2.$$

So, $\ell + \frac{\varepsilon}{5} \in S$ and yet $\ell + \frac{\varepsilon}{5} > \ell$, contradicting the fact that ℓ is an upper bound of S . We conclude $\ell^2 < 2$ is not possible.

Assume now that $\ell^2 > 2$. Our strategy will be to construct a number ever so slightly smaller than ℓ , which therefore cannot be an upper bound of S , and use this to arrive at a contradiction. Let $\delta = \ell^2 - 2$. Then $0 < \delta \leq 2$ (since $\ell \leq 2$ and hence $\ell^2 - 2 \leq 2$). Since $\delta > 0$, we have $\ell - \frac{\delta}{5} < \ell$. Since ℓ is the least upper bound of S , $\ell - \frac{\delta}{5}$ must not be an upper bound of S . By definition, this means that there is $r \in S$ such that $\ell - \frac{\delta}{5} < r$. Since $\delta \leq 2$ and $\ell \geq 1$, it follows that $\ell - \frac{\delta}{5}$ is positive and hence so is r . We may thus square both sides of $\ell - \frac{\delta}{5} < r$ to obtain

$$(\ell - \frac{\delta}{5})^2 < r^2.$$

Now

$$(\ell - \frac{\delta}{5})^2 = \ell^2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = \delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25}$$

since $\ell^2 = \delta + 2$. Moreover,

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = 2 + \delta(1 - \frac{2\ell}{5} + \frac{\delta}{25}) \geq 2 + \delta(1 - \frac{4}{5} + \frac{\delta}{25})$$

since $\ell \leq 2$. We deduce that

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} \geq 2 + \delta(\frac{1}{5}) \geq 2.$$

Putting these inequalities together gives $r^2 > 2$, contrary to the fact that $r \in S$. We conclude that $\ell^2 > 2$ is also not possible.

Since $\ell^2 < 2$ and $\ell^2 > 2$ are impossible, we must have $\ell^2 = 2$. □

6. MONDAY, SEPTEMBER 9, 2024 §1.9

Recall the Completeness Axiom:

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

The collection of rational numbers does not satisfy the completeness axiom and indeed it is precisely the completeness axiom that differentiates \mathbb{R} from \mathbb{Q} .

Example 6.1. Within the set \mathbb{Q} the subset $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum. That is, no matter which rational number you pick that is an upper bound for S , you may always find an even smaller one that is also an upper bound of S .

It is precisely the completeness axiom that assures us that everything that ought to be a number (like the length of the diagonal of a square with side length 1) really is a number. It gives us that there are “no holes” in the real number line — the real numbers are *complete*.

The completeness axiom is also at the core of the Intermediate Value Theorem and many of the other major theorems we will cover in this class.

We also need the completeness axiom to understand the relationship between \mathbb{N} , \mathbb{Q} , and \mathbb{R} .

Theorem 6.2. *If x is any real number, then there exists a natural number n such that $n > x$.*

This looks really stupid at first. How could it be false? But consider: there are examples of ordered fields, i.e. situations in which Axioms 1–10 hold, in which this Theorem is not true! So, its proof must rely on the Completeness Axiom.

Proof. Let x be any real number. By way of contradiction, suppose there is no natural number n such that $n > x$. That is, suppose that for all $n \in \mathbb{N}$, $n \leq x$. Then \mathbb{N} is a bounded above (by x). Since it is also clearly nonempty, by the Completeness Axiom, \mathbb{N} has a supremum, call it ℓ . Consider the number $y := \ell - 1$. Since $y < \ell$ and ℓ is the supremum of \mathbb{N} , y cannot be an upper bound of \mathbb{N} . So, there must be some $m \in \mathbb{N}$ such that $\ell - 1 < m$. But then by adding 1 to both sides of this inequality we get $\ell < m + 1$ and, since $m + 1 \in \mathbb{N}$, this contradicts that assumption that ℓ is the supremum of \mathbb{N} .

We conclude that, given any real number x , there must exist a natural number n such that $n > x$. \square

Corollary 6.3 (Archimedean Principle). *If $a \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$, then for some natural number n we have $na > b$.*

“No matter how small a is and how large b is, if we add a to itself enough times, we can overtake b .”

Proof. We apply Theorem 6.2 to the real number $x = \frac{b}{a}$. It gives that there is a natural number n such that $n > x = \frac{b}{a}$. Since $a > 0$, upon multiplying both sides by a we get $n \cdot a > b$. \square

Theorem 6.4 (Density of the Rational Numbers). *Between any two distinct real numbers there is a rational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. We will prove this by consider two cases: $x \geq 0$ and $x < 0$.

Let us first assume $x \geq 0$. We apply the Archimedean Principle using $a = y - x$ and $b = 1$. (The Principle applies as $a > 0$ since $y > x$.) This gives us that there is a natural number $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1$$

and thus

$$0 < \frac{1}{n} < y - x.$$

Consider the set $S = \{p \in \mathbb{N} \mid p \frac{1}{n} > x\}$. Since $\frac{1}{n} > 0$, using the Archimedean principle again, there is at least one natural number $p \in S$. By the Well Ordering Axiom, there is a smallest natural number $m \in S$.

We claim that $\frac{m-1}{n} \leq x$. Indeed, if $m > 1$, then $m-1 \in \mathbb{N} \setminus S$ (because $m-1$ is less than the minimum), so $\frac{m-1}{n} \leq x$; if $m = 1$, then $m-1 = 0$, so $\frac{m-1}{n} = 0 \leq x$.

So, we have

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

By adding $\frac{1}{n}$ to both sides of $\frac{m-1}{n} \leq x$ and using that $\frac{1}{n} < y - x$, we get

$$\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$$

and hence

$$x < \frac{m}{n} < y.$$

Since $\frac{m}{n}$ is clearly a rational number, this proves the result in this case (when $x \geq 0$).

We now consider the case $x < 0$. The idea here is to simply “shift” up to the case we’ve already proven. By Theorem 6.2, we can find a natural number j such that $j > -x$ and thus $0 < x + j < y + j$. Using the first case, which we have already proven, applied to the number $x + j$ (which is positive), there is a rational number q such that $x + j < q < y + j$. We deduce that $x < q - j < y$, and, since $q - j$ is also rational, this proves the theorem in this case. \square

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