DEFINITION: Let R be a ring.

(i) An ideal I of R is a **maximal ideal** if I is proper and for any proper ideal J, $I \subseteq J$ implies $I = J$.
(ii) Let R be commutative. An ideal I of R is a **prime ideal** if I is proper and $ab \in I$ implies $a \in I$ or $b \in I$.

THEOREM: Let R be a commutative ring and I an ideal.

(i) The ideal I is maximal if and only if R/I is a field.
(ii) The ideal I is prime if and only if R/I is an integral domain.

(1)

THEOREM: Let R be a ring. Then R has a maximal ideal.

DEFINITION: Let $(P, \leq)$ be a partially ordered set.

(i) A **maximal element** of P is an element $x \in P$ such that $x \geq y$ for all $y \in P$.
(ii) A subset X of P is a **chain** if for all $x, y \in X$ either $x \leq y$ or $y \leq x$.
(iii) A **upper bound** for a subset X is an element x such that $x \geq y$ for all $y \in X$.

ZORN'S LEMMA: Let $(P, \leq)$ be a nonempty partially ordered set. If every chain $C \subseteq P$ has an upper bound $c \in C$, then P has a maximal element.

(2) Understanding Zorn's Lemma:

(a) The most common use of Zorn's Lemma occurs in the following situation: P is the collection of all subsets of some set Y ordered by inclusion ($A \leq B$ if and only if $A \subseteq B$), and S is some special family of subsets of Y. Rewrite[1] the statement of Zorn's Lemma in this context.

(b) Let $Y = \mathbb{Z}$ and P be the collection of all *finite* subsets of Y. Explain why there is no maximal element of P.

(3) Prove the Theorem.

(4) Prove that every proper ideal is contained in a maximal ideal.

(5) Prove that every prime ideal contains a minimal prime ideal.

[1] Meaning replace all $\leq$ with $\subseteq$ and unpackage the definitions of maximal element and upper bound.