Contents

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This class is, as its name makes clear, is all about differential equations. Let's start with an example that is probably similar to something you've seen in Calculus.

Example 1.1. The equation

$$\frac{dy}{dx} = 7y$$

is a differential equation. The unknown in this equation, y, stands for a function. What makes this equation a differential equation is that the equation relates the mystery function and its derivative.

Let's see if we can guess a solution. This equation might remind us of a curious calculus coincidence. If the 7 wasn't there, we would be looking for a function whose derivative is equal to itself; e^x would work.

Let's try $y = 7e^x$ for our original equation. To test it, we plug it in:

$$y = 7e^x \rightsquigarrow y' = (7e^x)' = 7e^x \neq 7y = 49e^x.$$

How about putting the 7 somewhere else:

$$y = e^{7x} \rightsquigarrow y' = (e^{7x})' = e^{7x}(7x)' = 7e^{7x} = 7y.$$

So e^{7x} is a solution!

Could there be any others?

$$y = 5e^{7x} \rightsquigarrow y' = (5e^{7x})' = 5e^{7x}(7x)' = 7(5e^{7x}) = 7y.$$

In general, $y(x) = Ce^{7x}$ is a solution for any constant C.

Of course, at the end of the day, nothing was special about 7. If we replaced 7 by any real number a, for the same reason, we would find that for the differential equation

$$y' = ay$$

the *general solution* is

$$y(x) = Ce^{ax}.$$

Guessing, while successful here, is not going to be our preferred method in the class. Let's savor this victory, and be prepared to collect many methods for solving differential equations as we progress through the course.

Types of differential equations. There are many different ways of throwing together functions and derivatives in an equation, so we'll need some terminology to orient ourselves.

Definition 1.2. An *ordinary differential equation (ODE)* is a differential equation involving only one independent variable; i.e., derivatives with respect to just one variable.

For example,

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} = -y + \cos(ty)$$

is an ordinary differential equation.

In general an ODE is an equation of the form

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$

for some function F where $y = y(t)$: an equation relating the function y with its derivative(s).

Definition 1.3. A *partial differential equation (PDE)* is a differential equation involving multiple independent variable; i.e., derivatives with respect to different variables.

For example,

$$\frac{\partial u}{\partial t} - 5\frac{\partial u}{\partial x} = 0$$

and

$$\frac{\partial^2 z}{\partial x \partial y} - z^2 = xy$$

are PDEs. A solution of the first PDE would be a function $u(x, t)$ that depends two independent variables x and t.

The "ordinary" vs "partial" refers to what type of derivatives see.

This is a class about ODEs. Almost all of the rest of the differential equations we see this semester will be ordinary!

Definition 1.4. The *order* of a differential equation is the highest order derivative that occurs in the equation.

For example,

$$yy'' + y''' + \frac{1}{y} = 5x$$

is a third order ODE, due to the y''' term and

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} = -y + \cos(ty)$$

is a second order ODE.

Definition 1.5. A *linear* ODE is any ODE of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t).$$

For example,

$$5ty'' + \ln(t)y' + y = \cos(t)$$

is a second order linear ODE, but

$$yy' + 5y = 7$$

and

$$(y')^3 - ty^2 = 3e^t$$

are first order nonlinear ODEs.

We will be especially interested in linear ODEs in this course!

Discussion Questions.

(1) Is the differential equation $y' = y^{2/3}$ ordinary? linear? What is its order?

> Ordinary yes, linear no, order 1.

(2) Which of the following is a solution to the differential equation $y' = y^{2/3}$:
   (a) $y = 8t^2$
   (b) $y = e^{2t/3}$
   (c) $y = \frac{1}{27}t^3$
   (d) $y = 0$ (constant function 0)

> (a) No: $y' = 16t \neq y^{2/3} = 4t^{4/3}$.
> (b) No: $y' = 2/3e^{2t/3} \neq y^{2/3} = (e^{2t/3})^{2/3} = e^{4t/9}$.
> (c) Yes: $y' = \frac{1}{9}t^2 = y^{2/3}$.
> (d) Yes: $y' = 0 = y^{2/3}$.

(3) There is a solution to $xy'' = (4x-4)y$ of the form $y = xe^{ax}$ for some real number a. Find a.

> By the product rule,
> $$y' = (ax+1)e^{ax} \quad \text{and} \quad y'' = (a^2x + 2a)e^{ax},$$
> so
> $$xy'' - (4x-4)y = (a^2x^2 + 2ax)e^{ax} - (4x-4)xe^{ax}.$$

If this is zero, we must have

$$a^2x^2 + 2ax = 4x^2 - 4x$$

as functions of x, so $a = -2$.

(4*) If f, g are solutions to $y^{(3)} + 2e^x y^{(2)} - y = \cos(x)$, show that $\frac{f+g}{2}$ is too.

(5*) Using only calculus, justify the claim we made earlier that $y = Ce^{ax}$ is the general solution to $y' = ay$ for any $a \in \mathbb{R}$. That is, explain why there aren't any other solutions (exponential or otherwise).

Initial value problems. In our first example, we saw that there are many solutions to the differential equation $y' = 7y$. To pin one down, we might specify a value for our function at a point. The system

$$\begin{cases} y' = 7y \\ y(2) = 4 \end{cases}$$

is a example of an *initial value problem.* Geometrically, $y(2) = 4$ corresponds to the condition that the graph of our solution passes through $(2, 4)$.