PROPOSITION: Let R be a ring and F be a free module with basis B. Then every element of $f \in F$ admits a unique expression as a linear combination[1] of elements of B.

UNIVERSAL MAPPING PROPERTY FOR FREE MODULES: Let R be a ring and F be a free module with basis B. Let N be an arbitrary R-module. Then for any function $j : B \to N$, there is a unique R-module homomorphism $h : F \to N$ such that $h(b) = j(b)$ for all $b \in B$.

(1) Let R be a ring and $n \in \mathbb{Z}_{>0}$. The **standard free module of rank n** and its **standard basis** are, respectively,

$$R^n = \left\{ \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \;\middle|\; r_i \in R \right\} \quad \text{and} \quad \text{the set with elements } e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

We also write elements in the form $(r_1, \dots, r_n)$.

- **(a)** Let $R = \mathbb{Z}[x]$ and $M = R^3$. Give the unique expression of $v = (2x + 3, 1, x^4)$ as a linear combination of the standard basis.
- **(b)** Let $R = \mathbb{Z}[x]$, and $M = R^3$, and $N = \mathbb{Z}/5[x]$. Let $h : M \to N$ be the unique R-linear map such that $h(e_1) = [2]$, $h(e_2) = [0]$, and $h(e_3) = x$. Compute $h(v)$.

(2) Proving things.

- **(a)** Prove the Proposition above.
- **(b)** Prove the UMP for free modules above.

THEOREM: Let R be a ring. Let F be a free module with a basis B, and F' be a free module with a basis B'.

(1) If $|B| = |B'|$, meaning there is a set bijection between B and B', then $F \cong F'$.
(2) Let R be a commutative ring. If $F \cong F'$, then $|B| = |B'|$.

DEFINITION: Let R be a commutative ring, and F be a free module. The **rank** of F is the size of a basis B of F.

(3) Rank:

- **(a)** What about the Definition above needs justification? Use the Theorem to justify it.
- **(b)** Prove part (1) of Theorem 2. (We will prove part (2) later as a consequence of the same result in the special case of vector spaces.)

(4) Let $A = M_\infty(\mathbb{R})$ be the ring of countably infinite matrices with real entries:

$$M_\infty(\mathbb{R}) = \{[a_{ij}]_{\substack{i=1,2,3,\dots \\ j=1,2,3,\dots}} \mid a_{ij} \neq 0 \text{ for at most finitely many pairs } (i,j)\}$$

with usual matrix addition and multiplication; you do not have to prove that this is a ring. Prove[2] that $A^1 \cong A^2$ as A-modules. What does this say about the Theorem?

[1] Recall that a linear combination of B is a sum of the form $r_1 b_1 + \cdots + r_n b_n$ for some finite list of elements $b_1, \dots, b_n \in B$ and $r_1, \dots, r_n \in R$.

[2] Hint: Consider the map sending a matrix $[a_{ij}]$ to the pair of matrices $([a_{i,2j-1}], [a_{i,2j}])$ reconstituted from its odd columns and its even columns.