Problem Set #1

(1) Let $M = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ and let $G = \mathbb{Z}/4 = \langle g \rangle$. Consider the natural action of G on $V = K^2$ and the induced linear action on $S = \mathbb{C}[x,y]$. Find some nonzero elements of S^G. Can you find a generating set? (Hint: Compare to Example 1.1).

(2) Let $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$. Consider the natural action of G and H on $V = K^2$ and the induced linear action on $S = \mathbb{C}[x,y]$.
   (a) Compute the groups $H = \langle M \rangle$ and $G = \langle M, N \rangle$.
   (b) Use Molien's formula to find the Hilbert series of S^H and S^G. Compute both of them up to the t^4 term.
   (c) Find algebraically independent G-invariants of degrees 2 and 4. Explain why they must generate S^G.
   (d) Use the previous parts to determine the smallest degree of an element f that is H-invariant but not G-invariant, and find such an element f.
   (e) Observe something interesting about f^2. Can you find a generating set for S^H?

(3) Let G be a finite group. Given a homomorphism $G \hookrightarrow \mathfrak{S}_n$, for any field K one obtains a linear action of G on $K[x_1, \dots, x_n]$ by $g(x_i) := x_{g(i)}$, which we will call a permutation action. Show that, for such an action, S^G has a K-vector space basis given by orbit sums of monomials, i.e., elements of the form $\sum_{m' \in G \cdot m} m'$ where m is a monomial of S. Deduce that, in this setting, the Hilbert function of S^G is independent of K.

(4) Let $\mathfrak{A}_n$ be the alternating group on n letters, and let $\mathfrak{A}_n$ act by permuting the variables. Let K be a field of characteristic two.
   (a) Show that if K has characteristic two, then the discriminant $\Delta = \prod_{i<j}(x_i - x_j)$ is an element of $S^{\mathfrak{S}_n}$ and deduce that $S^{\mathfrak{A}_n} \neq K[e_1, \dots, e_n, \Delta]$.
   (b) Show that $\mu = \mathrm{Tr}^{\mathfrak{A}_n}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}) \in S^{\mathfrak{A}_n} \smallsetminus S^{\mathfrak{S}_n}$.
   (c) Show that $S^{\mathfrak{A}_n} = K[e_1, \dots, e_n, \mu]$.

(5) Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{F}_p)$ and $G = \langle M \rangle \cong \mathbb{Z}/p$. Consider the natural action of G on $V = K^2$ and the induced linear action on $S = K[x,y]$.
   (a) Explain why Molien's Theorem does not directly apply.
   (b) Show that $\mathbb{F}_p[x_1, N(x_2)] \subseteq S^G$, and explain why $\mathbb{F}_p[x_1, N(x_2)]$ is isomorphic to a polynomial ring in two variables. In particular, $\mathbb{F}_p[x_1, N(x_2)]$ is normal.
   (c) Show that $\mathbb{F}_p(x_1, N(x_2)) = \mathbb{F}_p(x_1, x_2)^G$.
   (d) Show that $\mathbb{F}_p[x_1, N(x_2)] \subseteq \mathbb{F}_p[x_1, x_2]$ is integral. Deduce that $S^G = \mathbb{F}_p[x_1, N(x_2)]$.

(6) Let $K = \mathbb{F}_2$, and let $G = \mathbb{Z}/2$ act on $S = K[x_1, x_2, x_3, y_1, y_2, y_3]$ by swapping x_i with y_i for each i. In this problem, we will show that S^G is not generated by elements of degree ≤ 2.

(a) Let $A = K[S^G_{\leq 2}]$ be the subalgebra of S^G generated by elements of degree at most 2. Show that A is generated by $\{x_i + y_i, x_i y_i, x_i y_j + x_j y_i \mid 1 \leq i < j \leq 3\}$.

(b) Let $I \subseteq S$ be the ideal generated by $\{x_i^2, x_i y_i, y_i^2 \mid i = 1, 2, 3\}$ and let $\overline{A}$ be the image of A in S/I. Compute the graded pieces $\overline{A_1}$ and $\overline{A_2}$ and find four linearly independent elements in $\overline{A_3}$.

(c) Show that the vector space $\overline{A_1} \cdot \overline{A_2}$ has $\mathbb{F}_2$-dimension at most three, and deduce the result.

(7) Let G be a finite group acting linearly on S. Show that the map $\pi : \operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(S^G)$ induced by the inclusion map is surjective and $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$ if and only if $G \cdot \mathfrak{p} = G \cdot \mathfrak{q}$. In particular, when $K = \overline{K}$, the maximal ideals of S^G correspond naturally to the G-orbits in V.

(8) Let G be a finite group of order m acting linearly on S. Let $A = K[S^G_{\leq m}]$ be the subalgebra of S^G generated by elements of degree at most m; in the modular case, this may be a proper subalgebra. Let $K = \overline{K}$. Show that the maximal ideals of A correspond naturally to the G-orbits in V.