Bases and unique expression UMP for free modules

(1)

THEOREM 2: Let R be a ring. Let F be a free module with a basis B, and F' be a free module with a basis B'.

(1) If $|B| = |B'|$ (meaning there is a set bijection between B and B'), then $F \cong F'$.
(2) Let R be a commutative ring. If $F \cong F'$, then $|B| = |B'|$.

DEFINITION: Let R be a commutative ring, and F be a free module. The **rank** of F is the size of a basis B of F.

(1) Theorem 2:
- **(a)** What about the Definition above needs justification? Use Theorem 2 to justify it.
- **(b)** Prove part (1) of Theorem 2. (We will prove part (2) later as a consequence of the same result in the special case of vector spaces.)

(2) Let $R = M_\infty(\mathbb{R})$ be the ring of countably infinite matrices with real entries:

$$M_\infty(\mathbb{R}) = \{[a_{ij}]_{\substack{i=1,2,3,\dots \\ j=1,2,3,\dots}} \mid a_{ij} \neq 0 \text{ for at most finitely many pairs } (i,j)\}$$

with usual matrix addition and multiplication; you do not have to prove that this is a ring. Prove[1] that $R^1 \cong R^2$ as R-modules. What does this say about Theorem 2?

[1]Hint: Consider the map sending a matrix $[a_{ij}]$ to the pair of matrices $([a_{i,2j-1}], [a_{i,2j}])$ reconstituted from its odd columns and its even columns.