Tacoma Narrows Bridge

On July 1, 1940, a bridge spanning the Tacoma Narrows opened to great celebration. It dramatically shortened the trip from Seattle to the Kitsap Peninsula. It was an elegant suspension bridge, a mile long (third longest in the US at the time) but just 39 feet across. Through the summer and early fall, drivers noticed that it tended to oscillate vertically, quite dramatically.

During the first fall storm, on November 7, 1940, with steady winds above 40 mph, the bridge began to exhibit different behavior. It *twisted*, part of one edge rising while the opposing edge fell, and then the reverse. At 10:00 AM the bridge was closed. The torsional oscillations continued to grow in amplitude, until, at just after 11:00, the central span of the bridge collapsed and fell into the water below. One car was lost.

Why did this collapse occur? Were the earlier oscillations a warning sign? Many differential equations textbooks announce that this is an example of *resonance*: the gusts of wind just happened to match the natural frequency of the bridge.



Models: Let the vertical deflection (positive direction downward) of the slice of the roadbed denoted by $y(t)$, where t represents time, and $y = 0$ represents the equilibrium position of the road.

1. Here are three simplified second order differential equations that model the situations of the Tacoma Bridge. Assume that at the initial time, the displacement of the bridge $y(0) = 0$, and the velocity of the bridge $y'(0) = 0.1$, so that the roadbed starts in the equilibrium position with a small downward velocity. For each of the following equations, find the solution to the IVP with these initial conditions. Graph the solutions, and describe the short-term and long-term behavior of the solutions.

   (a) **Without force.** $\dfrac{d^2y}{dt^2} + 4y = 0.$

   (b) **With periodic forcing.** $\dfrac{d^2y}{dt^2} + 4y = \cos(t).$

   (c) **In resonance.** $\dfrac{d^2y}{dt^2} + 4y = \cos(2t).$

2. Perfect coincidences are rare in nature: it's very unlikely for the wind frequency to exactly equal the natural frequency of the bridge.

   (a) Before solving, what do you expect to happen to the solution of the IVP with same initial conditions and equation $\dfrac{d^2y}{dt^2} + 4y = \cos(1.9t)$?

   (b) Graph the solution to the IVP described in the last part, and describe the short-term and long-term behavior of the solutions.

   (c) Repeat the last two questions for $\dfrac{d^2y}{dt^2} + 4y = \cos(1.99t)$.

   (d) What happens in the limit with the same IVP with $\dfrac{d^2y}{dt^2} + 4y = \cos(\alpha t)$ as $\alpha \to 2$?