MATH 901 LECTURE NOTES, FALL 2021

Contents

1. Category Theory

1.1. **Categories.**

Lecture of August 23, 2021

1.1.1. *Definition of category.*

Definition 1.1. A *category* $\mathscr{C}$ consists of the following data:

(1) a collection of *objects*, denoted $\mathrm{Ob}(\mathscr{C})$,
(2) for each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* (also known as *arrows*) from A to B,
(3) for each triple of objects $A, B, C \in \mathrm{Ob}(\mathscr{C})$, a function

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)$$

written as $(\alpha, \beta) \mapsto \beta \circ \alpha$ that we call the *composition rule*.

These data are required to satisfy the following axioms:

(1) (Disjointness) the Hom sets are disjoint: if $A \neq A'$ or $B \neq B'$, then

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \cap \mathrm{Hom}_{\mathscr{C}}(A', B') = \varnothing.$$

(2) (Identities) for every object A, there is an *identity morphism* $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.
(3) (Associativity) composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

Remark 1.2.
(1) The word "collection" as opposed to "set" is important here. The point is that there is no set of all sets, but by utilizing bigger collecting objects in set theory, we can sensibly talk about the collection of all sets. We'll sweep all of the set theory under the rug there, but it's worth keeping in mind that the objects of a category don't necessarily form a set. We did assume that the collections of morphisms between a pair of objects form a set, though not everyone does.
(2) The first axiom above guarantees that every morphism α in a category $\mathscr{C}$ has a well-defined *source* and *target* in $\mathrm{Ob}(\mathscr{C})$, namely, the unique A and B (respectively) such that $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

The name arrow dovetails with the common practice of depicting a morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ as

$$A \xrightarrow{\alpha} B.$$

The composition of $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ is $A \xrightarrow{\beta\circ\alpha} C$.

Optional Exercise 1.3. Prove that every element in a category has a unique identity morphism (i.e., a unique morphism that satisfies the hypothesis of axiom (2)).

1.1.2. *Examples of categories.* Many of our favorite objects from algebra naturally congregate in categories!

Example 1.4.
(1) There is a category **Set** where
- $\mathrm{Ob}(\mathbf{Set})$ is the collection of all sets
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the the set of functions from X to Y
- the composition rule is composition of functions

We observe that every set has an identity function, which behaves as an identity for composition, and that composition of functions is associative.

(2) There is a category **Grp** where
- $\mathrm{Ob}(\mathbf{Grp})$ is the collection of all groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Grp}}(X, Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

Note that the identity function on a group is a group homomorphism, and that a composition of two group homomorphisms is a group homomorphism.

(3) There is a category **Ab** where
- $\mathrm{Ob}(\mathbf{Ab})$ is the collection of all abelian groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Ab}}(X, Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

(4) In this class,
- A *semigroup* is a set S with an associative operation $\cdot$ that has an identity element; some may prefer the term *monoid*, but I don't.
- A *semigroup homomorphism* from semigroups $S \to T$ is a function that preserves the operation and maps the identity element to the identity element.

There is a category **Sgrp** where the objects are all semigroups and the morphisms are semigroup homomorphisms. (The composition rule is composition again.)

(5) In this class,
- A *ring* is a set R with two operations $+$ and $\cdot$ such that $(R, +)$ is abelian group, with identity 0, and $(R, \cdot)$ is a semigroup with identity 1, and such that the left and right distributive laws hold: $(r + s)t = rt + st$ and $t(r + s) = tr + ts$.
- A *ring homomorphism* is a function that preserves $+$ and $\cdot$ and sends 1 to 1.

There is a category **Ring** where the objects are all rings and the morphisms are ring homomorphisms.

(6) Let R be a ring. In this class,
- A *left R-module* is an abelian group $(M, +)$ equipped with a pairing $R \times M \to M$, written $(r, m) \mapsto rm$ or $(r, m) \mapsto r \cdot m$ such that
  (a) $r_1(r_2m) = (r_1r_2)m$,
  (b) $(r_1 + r_2)m = r_1m + r_2m$,
  (c) $r(m_1 + m_2) = rm_1 + rm_2$, and
  (d) $1m = m$.
- A *left module homomorphism* or *R-linear map* between left R-modules $\phi : M \to N$ is a homomorphism of abelian groups from $(M, +) \to (N, +)$ such that $\phi(rm) = r\phi(m)$.

There is a category $R-\mathbf{Mod}$ where the objects are all left R-modules and the morphisms are R-linear maps.

(7) There is a category **Fld** where the objects are all fields and the morphisms are all field homomorphisms.

(8) There is a category **Top** where the objects are all topological spaces and the morphisms are all continuous functions.

Remark 1.5. There are two special cases of the category of R-modules that are worth singling out:
- Every abelian group M is a $\mathbb{Z}$-module in a unique way, by setting
$$n \cdot m = \underbrace{m + \cdots + m}_{n-\text{times}} \quad \text{and} \quad -n \cdot m = -(\underbrace{m + \cdots + m}_{n-\text{times}}) \qquad \text{for } n \geqslant 0.$$
Thus, **Ab** is basically just $\mathbb{Z} - \mathbf{Mod}$.
- When $R = K$ happens to be a field, we are accustomed to calling K-modules *vector spaces*. Thus, we might write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

Lecture of August 25, 2021

Example 1.6. Here are some variations on the category $K - \mathbf{Vect}$.

(1) The collection of finite dimensional K-vector spaces with all linear transformations is a category; call it $K - \mathbf{vect}$.
(2) The collection of all n-dimensional K-vector spaces with all linear transformations is a category.
(3) The collection of all K-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.
(4) The collection of all K-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it's not closed under composition.
(5) The collection of all n-dimensional vector spaces with singular linear transformations is not a category, since it doesn't have identity maps.

Example 1.7. (1) There is a category $\mathbf{Set}_*$ of *pointed sets* where
- the objects are pairs (X, x) where X is a set and $x \in X$,
- for two pointed sets, the morphisms from (X, x) to (Y, y) are functions $f : X \to Y$ such that $f(x) = y$,
- usual composition.

(2) For a commutative ring A,
- A *commutative A-algebra* is a commutative ring R plus a homomorphism $\phi : A \to R$.

- Slightly more generally, an A-*algebra* is a ring R plus a homomorphism $\phi : A \to R$ such that $\phi(A)$ lies in the center of R: $r \cdot \phi(a) = \phi(a) \cdot r$ for any $a \in A$ and $r \in R$. (In the more general situation, A is still commutative but R may not be.)
- An A-*algebra homomorphism* between two A-algebras (R, ϕ) and (S, ψ) is a ring homomorphism $\alpha : R \to S$ such that $\alpha \circ \phi = \psi$.

The category of A-algebras is denoted $A - \mathbf{Alg}$, and the category of commutative A-algebras is $A - \mathbf{cAlg}$.

(3) Fix a field K, and define a category $\mathbf{Mat}_K$ as follows: the objects are the positive natural numbers $n \in \mathbb{N}_{>0}$, and $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is the set of $b \times a$ matrices with entries in K. To see this as a category, we need a composition rule. Given $B \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ and $A \in \mathrm{Hom}_{\mathscr{C}}(a, b)$, take the composition $A \circ B \in \mathrm{Hom}_{\mathscr{C}}(a, c)$ to be the product AB. Since matrix multiplication is associative, axiom (3) holds, and the $n \times n$ identity matrix serves as an identity morphism in the sense of axiom (2). Finally, if $A \in \mathrm{Hom}_{\mathscr{C}}(a, b) \cap \mathrm{Hom}_{\mathscr{C}}(a', b')$, then A is a $b \times a$ matrix and a $b' \times a'$ matrix, so $a = a'$ and $b = b'$. Notably, the morphisms in this category are not functions.

We can also make a bunch of categories in a hands-on way as follows:

Example 1.8. Let $(P, \leqslant)$ be a poset. We define a category $\mathbf{PO}(P)$ from P as follows. The objects of $\mathbf{PO}(P)$ are just the elements of P. For each pair $a, b \in P$ with $a \leqslant b$, form a symbol f_a^b. Then we set

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a, b) = \begin{cases} \{f_a^b\} & \text{if } a \leqslant b \\ \varnothing & \text{otherwise.} \end{cases}$$

There is only one possible composition rule:

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a, b) \times \mathrm{Hom}_{\mathbf{PO}(P)}(b, c) \longrightarrow \mathrm{Hom}_{\mathbf{PO}(P)}(a, c)$$

when $a \leqslant b$ and $b \leqslant c$ we also have $a \leqslant c$, and the unique pair on the left must map to the unique element on the right, so $f_b^c \circ f_a^b = f_a^c$; when either $a \nleqslant b$ or $b \nleqslant c$, there is nothing to compose!

Each morphism f_a^b is in only one Hom set (with source a and target b). Composition is associative since there is at most one function between one element sets. For any a, $f_a^a \in \mathrm{Hom}_{\mathbf{PO}(P)}(a, a)$ is the identity morphism.

For a specific example, we can think of $\mathbb{N}_{>0}$ as a category this way. Drawing all of the morphisms would be a mess, but any morphism is a composition of the ones depicted:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots .$$

Note that the objects of this category are exactly the same as in Example 1.7(3), but with much fewer morphisms!

Example 1.9. A category with one object is nothing but a semigroup.

1.1.3. *Constructions of categories.* Here are a few more basic constructions of categories:

Definition 1.10. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{\mathrm{op}}$ is the category with $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$, and $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$ for all $A, B \in \mathrm{Ob}(\mathscr{C})$.

That is, the opposite category is the "same category with the arrows reversed." To avoid confusion, we might write α^{op} for the morphism $B \xrightarrow{\alpha^{\mathrm{op}}} A$ in $\mathscr{C}^{\mathrm{op}}$ corresponding to $A \xrightarrow{\alpha} B$ in $\mathscr{C}$.

Definition 1.11. Given two categories $\mathscr{C}$ and $\mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ is the category with $\mathrm{Ob}(\mathscr{C} \times \mathscr{D})$ given by the collection of pairs (C, D) with $C \in \mathrm{Ob}(\mathscr{C})$ and $D \in \mathrm{Ob}(\mathscr{D})$, and $\mathrm{Hom}_{\mathscr{C} \times \mathscr{D}}((A, B), (C, D)) = \mathrm{Hom}_{\mathscr{C}}(A, C) \times \mathrm{Hom}_{\mathscr{D}}(B, D)$. We leave it to you to pin down the composition rule.

Definition 1.12. A category $\mathscr{D}$ is a *subcategory* of another category $\mathscr{C}$ provided

(1) every object of $\mathscr{D}$ is an object of $\mathscr{C}$
(2) for every $A, B \in \mathrm{Ob}(\mathscr{D})$, $\mathrm{Hom}_{\mathscr{D}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A, B)$, and
(3) for every $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ in $\mathscr{D}$, the composition of α and β in $\mathscr{D}$ equals the composition of α and β in $\mathscr{C}$.

If equality hold in (2) (for all A, B), we say that $\mathscr{D}$ is a *full subcategory* of $\mathscr{C}$.

Example 1.13. Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, since not every function between groups is a homomorphism, **Grp** is not a full subcategory of **Set**. Similarly, **Ab**, **Ring**, $R-$**Mod**, and **Top** are all subcategories of **Set**.

On the other hand, **Ab** is a full subcategory of **Grp**, and **Grp** is a full subcategory of **Sgrp**: a morphism of abelian groups is a morphism of groups that happens to be between abelian groups (and likewise for groups and semigroups)!

Lecture of August 27, 2021

1.2. **Basic notions with morphisms.**

Definition 1.14. A *diagram* in a category $\mathscr{C}$ is a directed multigraph whose vertices are objects in $\mathscr{C}$ and whose arrows/edges are morphisms in $\mathscr{C}$. A *commutative diagram* in $\mathscr{C}$ is a diagram in which for each pair of vertices A, B, any two paths from A to B compose to the same morphism.

Example 1.15. To say that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

commutes is to say that $\beta \circ \alpha = \delta \circ \gamma$ in $\mathrm{Hom}_{\mathscr{C}}(A, D)$.

Definition 1.16. Let $\mathscr{C}$ be any category and $A \xrightarrow{\alpha} B$ a morphism.

- α is an *isomorphism* if there exists $B \xrightarrow{\beta} A$ such that $\beta \circ \alpha = 1_A$ and $\alpha \circ \beta = 1_B$. Such an β is called the *inverse* of f.
- α has β as a *left inverse* if $\beta \circ \alpha = 1_A$. Similarly define *right inverse*.
- α is a *monomorphism* or is *monic* if for all arrows

$$\begin{tikzcd} C \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & A \arrow[r, "\alpha"] & B \end{tikzcd}$$

  if $\alpha\beta_1 = \alpha\beta_2$ then $\beta_1 = \beta_2$. That is, α can be cancelled from the left.
- α is an *epimorphism* or is *epic* if for all arrows

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & C \end{tikzcd}$$

  if $\beta_1\alpha = \beta_2\alpha$ then $\beta_1 = \beta_2$. That is, α can be cancelled from the right.

Remark 1.17. Note that α has a left inverse in $\mathscr{C}$ if and only if α^{op} has a right inverse in $\mathscr{C}^{\mathrm{op}}$, and that α is monic in $\mathscr{C}$ if and only if α^{op} is epic in $\mathscr{C}^{\mathrm{op}}$. We say that these are *dual* notions in category theory.

Lemma 1.18. *If α has a left inverse, then α is monic. Similarly for "right inverse" and "epic".*

Proof. If $\beta \circ \alpha = 1_A$ and γ_1, γ_2 are two morphisms from $C \to A$ such that $\alpha \circ \gamma_1 = \alpha \circ \gamma_2$, then

$$\gamma_1 = (\beta \circ \alpha) \circ \gamma_1 = \beta \circ (\alpha \circ \gamma_1) = \beta \circ (\alpha \circ \gamma_2) = (\beta \circ \alpha) \circ \gamma_2 = \gamma_2.$$

Similarly for "right inverse" and "epic". □

Example 1.19. In **Set**, the monomorphisms and left-invertible morphisms agree, and these are the injective functions. The epimorphisms and right-invertible morphisms agree, and there are the surjective functions.

Optional Exercise 1.20. For any poset P, in $\mathbf{PO}(P)$, every morphism is both monic and epic, but no nonidentity morphism has a left or right-inverse.

1.3. **Category-theoretic constructions of objects.** A property or construction is *category theoretic* if can be described just in terms of the data of the category rather than aspects of a particular category.

Example 1.21. Can we identify $\varnothing$ in **Set** without looking at the objects' and morphisms' names? We can: for every set S, there is a unique function $f : \varnothing \to S$; $\varnothing$ is the only set with this property.

Definition 1.22.
1. An object X in a category $\mathscr{C}$ is *initial* if there for every $Y \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $X \to Y$.
2. An object X in a category $\mathscr{C}$ is *terminal* if there for every $Y \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $Y \to X$.

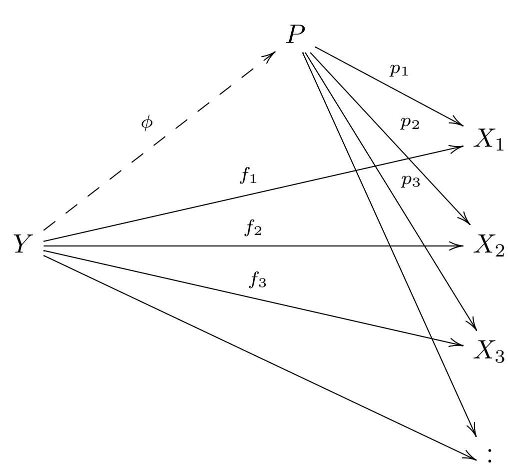
Example 1.23.
1. We just saw that $\varnothing$ is initial in **Set**. Any singleton is terminal.
2. A group with only one element $\{e\}$ is both initial and terminal in **Grp**.
3. $\mathbb{Z}$ is initial in **Ring**.

1.3.1. *Definitions of product and coproduct.*

Definition 1.24. Let $\mathscr{C}$ be a category, and $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of objects. A *product* of $\{X_\lambda\}_{\lambda \in \Lambda}$ is given by an object P and a family of morphisms $\{p_\lambda : P \to X_\lambda\}_{\lambda \in \Lambda}$ that is universal in the following sense:

Given an object Y and a family of morphisms $\{f_\lambda : Y \to X_\lambda\}_{\lambda \in \Lambda}$, there is a unique morphism $\phi : Y \to P$ such that $p_\lambda \circ \phi = f_\lambda$ for all λ.

Here is a diagram for the (first few) maps involved when $\Lambda = \mathbb{N}$ is countable:



We can also take a "big picture" view of this universal property:

$$\begin{tikzcd}
 & P \arrow[dr, squiggly, "\{p_\lambda\}"] & \\
Y \arrow[ur, dashed, "\phi"] \arrow[rr, squiggly, "\{f_\lambda\}"'] & & \{X_\lambda\},
\end{tikzcd}$$

where the squiggly arrows are again collections of maps instead of maps. The data of Y with a family of maps to each X_λ is the sort of thing a product might be, so we may think of it as a "product candidate." In this way, we can think of a product as a "terminal product candidate."

Lecture of August 30, 2021

Remark 1.25. Note that $(P, \{p_\lambda\}_{\lambda\in\Lambda})$ is a product of $\{X_\lambda\}_{\lambda\in\Lambda}$ if and only if the function

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(Y,P) & \longrightarrow & \times_{\lambda\in\Lambda} \mathrm{Hom}_{\mathscr{C}}(Y,X_\lambda) \\ \\ \phi & \longmapsto & (p_\lambda\circ\phi)_{\lambda\in\Lambda} \end{array}$$

is a bijection for all objects Y: the universal property says that everything in the target comes from a unique thing in the source.

Definition 1.26. Let $\mathscr{C}$ be a category, and $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects. A *coproduct* of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by an object C and a family of morphisms $\{i_\lambda : X_\lambda \to C\}_{\lambda\in\Lambda}$ that is universal in the following sense:

Given an object Y and a family of morphisms $\{f_\lambda : X_\lambda \to Y\}_{\lambda\in\Lambda}$, there is a unique morphism $\phi : C \to Y$ such that $\phi\circ i_\lambda = f_\lambda$ for all λ.

Here is a diagram for the (first few) maps involved when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & C \arrow[ddr, dashed, "\phi"] & \\
\vdots \arrow[ur] \arrow[drr] & & \\
X_3 \arrow[uur, "i_3"] \arrow[rr, "f_3"] & & Y \\
X_2 \arrow[uuur, "i_2"] \arrow[urr, "f_2"] & & \\
X_1 \arrow[uuuur, "i_1"] \arrow[uurr, "f_1"] & &
\end{tikzcd}$$

We can also take a "big picture" view of the universal property:

$$\begin{tikzcd}
 & C \arrow[dr, dashed, "\phi"] & \\
\{X_\lambda\} \arrow[ur, squiggly, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_\lambda\}"'] & & Y,
\end{tikzcd}$$

where the squiggly arrows are now collections of maps instead of maps. We can again think of the coproduct as the "initial coproduct candidate."

Remark 1.27. Note that $(C, \{i_\lambda\}_{\lambda\in\Lambda})$ is a coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ if and only if the function

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(C,Y) \arrow[r] & \bigtimes_{\lambda\in\Lambda} \mathrm{Hom}_{\mathscr{C}}(X_\lambda,Y) \\ \phi \arrow[r, mapsto] & (\phi\circ i_\lambda)_{\lambda\in\Lambda} \end{tikzcd}$$

is a bijection for all objects Y: the universal property says that everything in the target comes from a unique thing in the source.

Proposition 1.28. *If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda\in\Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a category $\mathscr{C}$, then there is a unique isomorphism $\alpha : P \xrightarrow{\sim} P'$ such that $p'_\lambda \circ \alpha = p_\lambda$ for all λ. The analogous statement holds for coproducts.*

Proof. We will just deal with products. The following picture is a rough guide:

$$\begin{tikzcd} P \arrow[r, dashed, "\alpha"] \arrow[rrd, squiggly, "\{p_\lambda\}"'] & P' \arrow[r, dashed, "\beta"] \arrow[rd, squiggly, "\{p'_\lambda\}"] & P \arrow[d, squiggly, "\{p_\lambda\}"] \\ & & \{X_\lambda\} \end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta : P' \to P$ such that $p_\lambda \circ \beta = p'_\lambda$. Switching roles, we obtain a unique map $\alpha : P \to P'$ such that $p'_\lambda \circ \alpha = p_\lambda$.

Consider the composition $\beta \circ \alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ for all λ. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ for all λ, so by the uniqueness property of products, $\beta \circ \alpha = 1_P$. We can again switch roles to see that $\alpha \circ \beta = 1_{P'}$. Thus α is an isomorphism. The uniqueness of α in the statement is part of the universal property. $\square$

Optional Exercise 1.29. Prove the analogous statement for coproducts.

We use the notation $\prod_{\lambda\in\Lambda} X_\lambda$ to denote the (object part of the) product of $\{X_\lambda\}$ and $\coprod_{\lambda\in\Lambda} X_\lambda$ to denote the (object part of the) coproduct of $\{X_\lambda\}$.

Observe that products and coproducts are dual notions in the same way as monic versus epic morphisms. The product of a family in $\mathscr{C}$ is the coproduct of the same family in $\mathscr{C}^{\mathrm{op}}$.

1.3.2. *Products in familiar categories.* The familiar notion of Cartesian product or direct product serves as a product in many of our favorite categories. Let's note first that given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in any of the categories $\mathbf{Set}, \mathbf{Sgrp}, \mathbf{Grp}, \mathbf{Ring}, R-\mathbf{Mod}, \mathbf{Top}$, the direct product $\bigtimes_{\lambda\in\Lambda} X_\lambda$ is an object of the same category:

- for sets, this is clear;
- for semigroups, groups, and rings, take the operation coordinate by coordinate: $(x_\lambda)_{\lambda\in\Lambda} \cdot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda}$;
- for modules, addition is coordinate by coordinate, and the action is the same on each coordinate: $r \cdot (x_\lambda)_{\lambda\in\Lambda} = (r \cdot x_\lambda)_{\lambda\in\Lambda}$;
- for topological spaces, use the product topology.

Note that this is not true for fields!

Proposition 1.30. *In each of the categories* **Set**, **Sgrp**, **Grp**, **Ring**, $R-$**Mod**, **Top**, *given a family* $\{X_\lambda\}_{\lambda\in\Lambda}$, *the direct product* $\times_{\lambda\in\Lambda} X_\lambda$ *along with the projection maps* $\pi_\lambda : \times_{\gamma\in\Lambda} X_\gamma \to X_\lambda$ *forms a product in the category.*

Proof. We observe that in each category, the direct product is an object, and the projection maps π_λ are morphisms in the category.

Let $\mathscr{C}$ be one of these categories, and suppose that we have morphisms $g_\lambda : Y \to X_\lambda$ for all λ in $\mathscr{C}$. We need to show there is a unique morphism $\phi : Y \to \times_{\lambda\in\Lambda} X_\lambda$ such that $\pi_\lambda \circ \phi = g_\lambda$ for all λ. The last condition is equivalent to $(\phi(y))_\lambda = (\pi_\lambda \circ \phi)(y) = g_\lambda(y)$ for all λ, which is equivalent to $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$, so if this is a valid morphism, it is unique. Thus, it suffices to show that the map $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$ is a morphism in $\mathscr{C}$, which is easy to see in each case. $\square$

1.3.3. *Coproducts in familiar categories.*

Example 1.31. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The product of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the cartesian product along with the projection maps. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the "disjoint union" with the various inclusion maps. By disjoint union, we simply mean union if the sets are disjoint; in general do something like replace X_λ with $X_\lambda \times \{\lambda\}$ to make them disjoint.

Proposition 1.32. *Let R be a ring, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left R-modules. A coproduct for the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is $(\bigoplus_{\lambda\in\Lambda} M_\lambda, \{\iota_\lambda\}_{\lambda\in\Lambda})$, where*

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \textit{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda$$

is the direct sum of the modules M_λ, and ι_λ is the inclusion map to the λ coordinate.

Lecture of September 1, 2021

Remark 1.33. If the index set Λ is finite, then the objects $\prod_{\lambda\in\Lambda} M_\lambda$ and $\bigoplus_{\lambda\in\Lambda} M_\lambda$ are identical, but the product and coproduct are not the same since one involves projection maps and the other involves inclusion maps.

Proof. Given R-module homomorphisms $g_\lambda : M_\lambda \to N$ for each λ, we need to show that there is a unique R-module homomorphism $\alpha : \bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ such that $\alpha \circ \iota_\lambda = g_\lambda$. We define

$$\alpha((m_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} g_\lambda(m_\lambda).$$

Note that since $(m_\lambda)_{\lambda\in\Lambda}$ is in the direct sum, at most finitely many m_λ are nonzero, so the sum on the right hand side is finite, and hence makes sense in N. We need to check that α is R-linear; indeed,

$$\alpha((m_\lambda) + (n_\lambda)) = \alpha((m_\lambda + n_\lambda)) = \sum g_\lambda(m_\lambda + n_\lambda) = \sum g_\lambda(m_\lambda) + \sum g_\lambda(n_\lambda) = \alpha((m_\lambda)) + \alpha((n_\lambda)),$$

and the check for scalar multiplication is similar. For uniqueness of α, note that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is generated by the elements $\iota_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha' \circ \iota_\lambda = g_\lambda$ for all λ, then $\alpha(\iota_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(\iota_\lambda(m_\lambda))$ so the maps must be equal. $\square$

Remark 1.34. For any indexing set Λ, $\coprod_{\lambda\in\Lambda} R$ is a free R-module. If $R = K$ happens to be a field, then $\prod_{\lambda\in\Lambda} K$ is free, since all vector spaces are free modules, but in general, $\prod_{\lambda\in\Lambda} R$ is not free for an infinite set Λ.

Remark 1.35.
- In **Top**, disjoint unions serve as coproducts.
- In **Sgrp** and **Grp**, coproducts exist, and are given as free products. You may see or have seen them in topology in the context of Van Kampen's theorem.
- In **Ring**, the story is more complicated. Let's note first that disjoint unions won't work, since they aren't rings. Direct sums of infinitely many rings don't have 1, so aren't rings, but even finite direct sums/products won't work, since the inclusion maps don't send 1 to 1. We will later on construct coproducts in **cRing** , the full subcategory of **Ring** consisting of commutative rings.

1.4. **Functors.**

Definition 1.36. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ is a mapping that assigns to each object $A \in \mathrm{Ob}(\mathscr{C})$ an object $F(A) \in \mathrm{Ob}(\mathscr{D})$ and to each morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ a morphism $F(\alpha) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ such that

(1) F preserves compositions, meaning $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ for all morphisms α, β in $\mathscr{C}$, and
(2) F preserves identity morphisms, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

A *contravariant functor* $F : \mathscr{C} \to \mathscr{D}$ is a mapping that assigns to each object $A \in \mathrm{Ob}(\mathscr{C})$ an object $F(A) \in \mathrm{Ob}(\mathscr{D})$ and to each morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ a morphism $F(\alpha) \in \mathrm{Hom}_{\mathscr{D}}(F(B), F(A))$ such that

(1) F preserves compositions, meaning $F(\alpha \circ \beta) = F(\beta) \circ F(\alpha)$ for all morphisms α, β in $\mathscr{C}$, and
(2) F preserves identity morphisms, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

Remark 1.37. One can also interpret a contravariant functor as a covariant functor from $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$, or as a covariant functor from $\mathscr{C} \to \mathscr{D}^{\mathrm{op}}$.

Example 1.38. Here are some examples of functors.

(1) Many of the categories we considered before are sets with extra structure, and the morphisms are functions that preserve the extra structure. The *forgetful functor* from such a category $\mathscr{C}$ to **Set**, is the covariant functor that forgets that extra structure are returns the underlying set of the object. For example the forgetful functor **Grp** $\to$ **Set** sends each group to its set of elements, and each homomorphism to its corresponding function of sets. Along similar lines, a ring is a group under addition with the bonus structure of multiplication, and we can talk about the forgetful functor from **Ring** to **Grp**, etc.
(2) The *identity functor* $1_{\mathscr{C}}$ on any category $\mathscr{C}$ sends each object to itself and each morphism to itself. It is covariant.
(3) There is a covariant functor $(-)^{\times 2}$: **Set** $\to$ **Set** that sends every set S to its cartesian square $S \times S$, and every function $f : S \to T$ to the function $(f, f) : S \times S \to T \times T$ that sends $(s_1, s_2) \mapsto (f(s_1), f(s_2))$. Let's check the axioms: given $g : S \to T$ and $f : T \to U$, we need to see that $(f, f) \circ (g, g) = (f \circ g, f \circ g)$, which is clear, and that $(1_S, 1_S)$ is the identity map on $S \times S$, which is also clear.
(4) Given a group G, the subgroup $G' \leqslant G$ generated by the set of commutators $\{ghg^{-1}h^{-1} \mid g, h \in G\}$ is a normal subgroup, and the quotient $G^{\mathrm{ab}} := G/G'$ is called the *abelianization* of G. The group G^{ab} is abelian. Given a group homomorphism $\phi : G \to H$, ϕ automatically takes commutators to commutators, so it induces a homomorphism $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. Put together, abelianization gives a covariant functor from **Grp** to **Ab**.
(5) Given any topological space X, the set of continuous functions from X to $\mathbb{R}$, $\mathrm{Cont}(X, \mathbb{R})$ is a ring with pointwise addition and multiplication. Given a continuous map $X \xrightarrow{\alpha} Y$, and a continuous

map $Y \xrightarrow{f} \mathbb{R}$, the composition $X \xrightarrow{\alpha\circ f} \mathbb{R}$ is a continuous function. In this way, we get a map from $\mathrm{Cont}(Y,\mathbb{R})$ to $\mathrm{Cont}(X,\mathbb{R})$. In fact, this map is a ring homomorphism. Put together, we obtain a contravariant functor from **Top** to **Ring**.

(6) Fix a field K. Given a vector space V, the collection V^* of linear transformations from V to K is again a K-vector space, the *dual vector space* of V. If $\phi : W \to V$ is a linear transformation and $\ell : V \to K$ is in V^* then $\ell \circ \phi : W \to K$ is in W^*, so there is a map $V^* \to W^*$. You can check that this together forms a functor $(-)^*$ that is contravariant.

(7) You may be familiar with the fundamental group of a pointed topological space; this is a group $\pi_1(X,x)$ assigned to a topological space and a point in it. The rule π_1 gives a functor from pointed topological spaces to groups.

(8) The unit group functor **Ring** $\to$ **Grp** sends each ring to its group of units. A homomorphism of rings restricts to a group homomorphism on the units: if $x \in R$ is a unit, so $xy = 1$, and $\phi : R \to S$ is a group homomorphism, then $1 = \phi(xy) = \phi(x)\phi(y)$, so $\phi(x)$ is a unit; ϕ preserves multiplication as well. This is covariant.

Lecture of September 3, 2021

It follows from the definition of covariant functor that if we apply a covariant functor F to a commutative diagram, we get another commutative diagram of the same shape, e.g.:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] \arrow[d, "F(\gamma)"'] & F(B) \arrow[d, "F(\beta)"] \\ F(C) \arrow[r, "F(\delta)"] & F(D). \end{tikzcd}$$

If we apply a contravariant functor G to a commutative diagram, we get a commutative diagram of the same shape with the arrows reversed, e.g.:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd} \quad \overset{G}{\rightsquigarrow} \quad \begin{tikzcd} G(A) & G(B) \arrow[l, "G(\alpha)"'] \\ G(C) \arrow[u, "G(\gamma)"] & G(D). \arrow[l, "G(\delta)"'] \arrow[u, "G(\beta)"'] \end{tikzcd}$$

Remark 1.39. A composition of two covariant functors, or of two contravariant functors, is a covariant functor. The composition of a covariant functor and a contravariant functor, or vice versa, is a contravariant functor.

1.5. **Natural transformations.**

Definition 1.40. Let F and G be covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A *natural transformation* η between F and G is a mapping that to each object A in $\mathscr{C}$ assigns a morphism $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes. We sometimes write $\eta : F \implies G$.

A *natural isomorphism* is a natural transformation η where each η_A is an isomorphism.

In short, a natural transformation is a rule to turn F of whatever into G of whatever in a reasonable way.

Optional Exercise 1.41. Let $F, G : \mathscr{C} \to \mathscr{D}$ be covariant functors. Show that a natural transformation $\eta : F \implies G$ is a natural isomorphism if and only if there is another natural transformation $\mu : G \implies F$ such that $\mu \circ \eta$ is the identity natural isomorphism on F and $\eta \circ \mu$ is the identity natural transformation on G.

We can make make a similar definition for contravariant functors.

Definition 1.42. Let F and G be contravariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A *natural transformation* between F and G is a mapping that to each object A in $\mathscr{C}$ assigns a morphism $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] & G(A) \\ F(B) \arrow[u, "F(f)"] \arrow[r, "\eta_B"'] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

commutes.

Example 1.43. Let's describe a natural transformation of functors $\eta : (-)^{\times 2} \implies 1_{\mathbf{Set}}$, namely we take

$$\eta_S : S \times S \to S \qquad (s_1, s_2) \mapsto s_1.$$

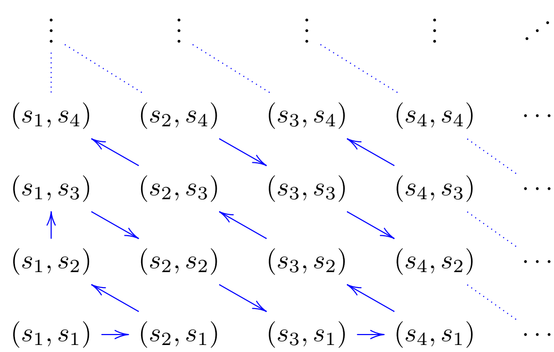
We need to check for every map $f : S \to T$ the commutativity of a diagram:

$$\begin{tikzcd} S \times S \arrow[r, "\eta_S"] \arrow[d, "{(f,f)}"'] & S \arrow[d, "f"] \\ T \times T \arrow[r, "\eta_T"'] & T. \end{tikzcd}$$

Going either down then left or right then down, (s_1, s_2) maps to $f(s_1)$, so this does commute, and we indeed have a natural transformation. This is not a natural isomorphism, since the map η_S is not always (almost never) an isomorphism of sets.

Example 1.44. Let $\mathscr{C}$ be the full subcategory of **Set** consisting of countable sets. For every $S \in \mathrm{Ob}(\mathscr{C})$, there is a $\mathscr{C}$-isomorphism, i.e., a bijection, $\eta_S : S \to S \times S$. Namely, we can take η_S as follows: enumerate S as $S = \{s_1, s_2, s_3, \dots\}$, and do the usual zigzag trick

$$\begin{aligned} s_1 &\mapsto (s_1, s_1) \\ s_2 &\mapsto (s_2, s_1) \\ s_3 &\mapsto (s_1, s_2) \\ s_4 &\mapsto (s_1, s_3) \\ s_5 &\mapsto (s_2, s_2) \\ s_6 &\mapsto (s_3, s_1) \\ &\vdots \end{aligned}$$



However, the bijections η_S do not form a natural bijection (in fact, if we just choose η_S like so for one set S, no matter what the other choices are, we can't get a natural transformation). Let $f : S \to S$ satisfy

$f(s_1) = s_2$ and $f(s_2) = s_1$. Then in the diagram

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[d, "f"'] & S \times S \arrow[d, "(f,f)"] \\ S \arrow[r, "\eta_S"'] & S \times S, \end{tikzcd}$$

we have $\eta_S(f(s_1)) = (s_2, s_1)$ while $(f, f)(\eta_S(s_1)) = (s_2, s_2)$, so the diagram does not commute.

Intuitively, we can blame the fact that our map η decided on a *choice* of enumeration of the set.

Example 1.45. Recall the contravariant functor $(-)^* : K - \mathbf{vect} \to K - \mathbf{vect}$; here we are restricting to finite dimensional vector spaces.

For every $V \in K - \mathbf{vect}$, there is an isomorphism $V \cong V^*$: if we fix a basis $\mathcal{B}$ for V, there is a dual basis for V^* (the $\mathcal{B}$-coordinate functions) of the same size, so they are isomorphic. However, there is no natural isomorphism $\eta : 1_{K-\mathbf{vect}} \implies (-)^*$, since $1_{K-\mathbf{vect}}$ is covariant and $(-)^*$ is contravariant. We will actually see a more compelling version of this nonnatruality statement in the homework.

Composing the dual functor with itself twice we get the covariant double-dual functor $(-)^{**} : K - \mathbf{vect} \to K - \mathbf{vect}$. We will show that there is a natural isomorphism $1_{K-\mathbf{vect}} \implies (-)^{**}$.

For every $v \in V$, there is a map $\mathrm{ev}_v : V^* \to V$ given by evaluation at v: $\mathrm{ev}_v(\ell) = \ell(v)$. So, $\mathrm{ev}_v \in V^{**}$. Since we have one for each v, there is a function $\mathrm{ev} : V \to V^{**}$ given by $\mathrm{ev}(v) = \mathrm{ev}_v$.

The map ev is a linear transformation:

$$\mathrm{ev}_{cv+w}(\ell) = \ell(cv + w) = c\ell(v) + \ell(w) = c\,\mathrm{ev}_v(\ell) + \mathrm{ev}_w(\ell).$$

It is injective, since any nonzero vector takes on a nonzero value for some linear functional. It is then a bijection since $\dim(V) = \dim(V^*) = \dim(V^{**})$.

We just need to check commutativity of the square:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "\phi"] & V^{**} \arrow[d, "\phi^{**}"] \\ W \arrow[r, "\mathrm{ev}"] & W^{**} \end{tikzcd}$$

This translates to

$$\mathrm{ev}_{\phi(v)} = (\mathrm{ev} \circ \phi)(v) \stackrel{?}{=} (\phi^{**} \circ \mathrm{ev})(v) = \phi^{**}(\mathrm{ev}_v)$$

in W^{**} for all $v \in V$. But, for all $\ell \in W^*$,

$$\mathrm{ev}_{\phi(v)}(\ell) = \ell(\phi(v)) = \phi^*(\ell)(v) = (\mathrm{ev}_v \circ \phi^*)(\ell) = \phi^{**}(\mathrm{ev}_v)(\ell),$$

so the equality holds.

In the homework, we will discuss some more examples from linear algebra. For example, for a pair of vector spaces $W \leqslant V$, there are isomorphisms $V \cong W \oplus V/W$, but no natural isomorphism of the sort. On the bright side, we will see that if V has an inner product, then V and V^* are naturally isomorphic in a suitable sense.

2. R-Modules

Lecture of September 8, 2021

2.0.1. *Left vs right vs both.* Recall that a left R-module is an abelian group M with an action map $R \times M \to M$ written $(r, m) \mapsto rm$ such that $r(sm) = (rs)m$, along with two distributive properties and the condition that 1 acts as the identity. A *right module* over R is defined similarly; we usually write the action as $(r, m) \mapsto mr$, and we have $(mr)s = m(rs)$, along with distributive and identity properties. The point is that when we act by a product $rs \in R$, we can think of it as an iterated action; in a left module, the left factor acts last while in a right module the right factor acts last.

Definition 2.1. If R is a ring, the *opposite ring* R^{op} is the ring with the same underlying set and same addition, but with multiplication given by $r \cdot_{R^{\mathrm{op}}} s = s \cdot_R r$.

A right R-module is exactly the same thing as a left R^{op}-module (except our convention for writing the action). In particular, if R is commutative, then a left R-module is exactly the same thing as a right R-module, and we will just say "module" in this case. By default, in general, when we say module, we will mean left R-module.

Example 2.2. Let R be a ring. The collection $M_n(R)$ of $n \times n$ matrices with entries in R forms a ring that in general is not commutative. The set R^n of column vectors of length n with entries in R is naturally a left $M_n(R)$-module. The collection of row vectors of length n with entries in R is naturally a right $M_n(R)$-module. We can also identify this latter action with a right module action on R^n by transposing any column vector into a row vector, acting, then transposing back:

$$v \cdot M = (v^T M)^T = M^T v.$$

We can think of R-module structures in a different way. To prepare, let's record a lemma.

Lemma 2.3. *If M is an abelian group, then $\mathrm{End}_{\mathbf{Ab}}(M) := \mathrm{Hom}_{\mathbf{Ab}}(M, M)$ forms a ring with pointwise addition and composition as multiplication. More generally, if M is a left R-module, then $\mathrm{End}_R(M) := \mathrm{Hom}_{R-\mathbf{Mod}}(M, M)$ forms a ring (with the aforementioned operations).*

Proof. The first statement is a special case of the first, since an abelian group is the same thing as a $\mathbb{Z}$-module, so we'll prove the second. Let $f, g \in \mathrm{End}_R(M)$. Since

$$(f+g)(rm+n) = f(rm+n) + g(rm+n) = rf(m) + f(n) + rg(m) + g(n) = r(f+g)(m) + (f+g)(n)$$

we see that $f + g \in \mathrm{End}_R(M)$. It's easy to see that $\mathrm{End}_R(M)$ is an abelian group under $+$. Associativity of multiplication is a special case of associativity of composition of functions. For distributive laws, we have

$$((f+g)h)(m) = (f+g)(h(m)) = f(h(m)) + g(h(m)) = (fh)(m) + (gh)(m)$$
$$(f(g+h))(m) = f(g(m)+h(m)) = f(g(m)) + f(h(m)) = (fg)(m) + (fh)(m);$$

for the latter distributive law, it was crucial that we are dealing with homomorphisms of abelian groups. We also have the identity map on M as a mutliplicative identity. $\square$

Optional Exercise 2.4. Show that there is a ring isomorphism $\mathrm{End}_R(R) \cong R^{\mathrm{op}}$.

Proposition 2.5. *Let R be a ring and $(M, +)$ an abelian group. There is a bijective correspondence*

$$\{R-\textit{module actions } R \times M \to M \,(\textit{with given } +)\} \longleftrightarrow \{\textit{ring homomorphisms } \rho : R \to \mathrm{End}_{\mathbb{Z}}(M)\}$$

$$\cdot \longmapsto \rho(r)(m) = r \cdot m$$

$$r \cdot m = \rho(r)(m) \longleftarrow\!\mapsfrom \rho.$$

Proof. We clearly have a bijection as long as the maps are well-defined.

Given an R-module action $\cdot$, one distributive property translates to the condition that $\rho(r)$ is $\mathbb{Z}$-linear; the identity condition means $\rho(1_R)$ is the identity function on M, which is the 1 element in $\mathrm{End}_{\mathbb{Z}}(M)$; the other distributive condition means ρ preserves addition; and the associativity condition means ρ preserves multiplication. Thus, ρ is a ring homomorphism. And conversely. □

It turns out that we often have a left module structure and a right module structure on something in a compatible way.

Definition 2.6. Let R and S be rings. An (R, S)-*bimodule* is an abelian group M equipped with a left R-module structure and a right S-module structure that commute with each other:

$$(r \cdot m) \cdot s = r \cdot (m \cdot s) \qquad \text{for all } m \in M, r \in R, s \in S.$$

Example 2.7. Here are some basic sources of bimodules:

(1) If R is a ring, then $M = R$ is an (R, R)-bimodule in the obvious way. More generally, if $\phi : A \to R$ is a ring homomorphism, then R is an (R, A)-bimodule by

$$s \cdot r \cdot a = sr\phi(a) \qquad \text{for } r, s \in R, a \in A;$$

equally well, R is an (A, R) or (A, A)-bimodule.

(2) If R is a commutative ring and M is any left module, then M is also a right module by the same action, and M is an (R, R)-bimodule with these structures. I.e., starting with an action $r \cdot m$, we set $m \cdot s$ to be $s \cdot m$, and

$$(r \cdot m) \cdot s = s \cdot (r \cdot m) = sr \cdot m = rs \cdot m = r \cdot (s \cdot m) = r \cdot (m \cdot s).$$

(3) Every left R-module is automatically an $(R, \mathbb{Z})$-bimodule in a unique way:

$$(r \cdot m) \cdot n = \underbrace{(r \cdot m) + \cdots + (r \cdot m)}_{n \text{ times}} = r \cdot (\underbrace{m + \cdots + m}_{n \text{ times}}) = r \cdot (m \cdot n) \qquad \text{for } n \in \mathbb{Z}_{\geqslant 0},$$

and similarly for $n \leqslant 0$. Likewise, every right R-module is automatically a $(\mathbb{Z}, R)$-bimodule.

Example 2.8. For a ring R, the set of column vectors of length n, R^n, is a $(M_n(R), R)$-bimodule. However, if we take the natural left action together with the right action $v \cdot M = M^T v$ discussed above, we do not get a bimodule structure, since $(M \cdot v) \cdot N = N^T M v$ generally differs from $M \cdot (v \cdot N) = M N^T v$.

Sometimes, when we want to keep track of various module and bimodule structures, we may write something like ${}_R M_S$ to indicate that M is an (R, S)-bimodule, or ${}_R M$ to indicate that M is a left R-module.

2.1. **Kernels, images, and exact sequences.** To every homomorphism $\phi : M \to N$ in $R - \mathbf{Mod}$, the kernel $\ker(\phi) \subseteq M$ and image $\mathrm{im}(\phi) \subseteq N$ are in $R - \mathbf{Mod}$, and the inclusion maps are homomorphisms of R-modules. It is surprisingly convenient to keep track of and compare these data in terms of exact sequences.

Definition 2.9. A sequence of R-modules and R-module maps of the form

$$\cdots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

(possible infinite, possibly not) is a *chain complex*, or just *complex* for short, if $d_i \circ d_{i+1} = 0$ for all i or, equivalently, $\mathrm{im}(d_{i+1}) \subseteq \ker(d_i)$ for all i.

A chain complex is *exact* at M_i if $\mathrm{im}(d_{i+1}) = \ker(d_i)$; it is exact if it is exact at every module that has a map in and a map out.

Lecture of September 10, 2021

Example 2.10. Let A be a $b \times a$ matrix and B be a $c \times b$ matrix of real numbers. The sequence of maps

$$\mathbb{R}^a \xrightarrow{A\cdot} \mathbb{R}^b \xrightarrow{B\cdot} \mathbb{R}^c$$

is a complex if and only if $BA = 0$; equivalently, the columns of A are in the solution space (nullspace) of B. It is exact if and only if the columns of A span the solution space of B.

Remark 2.11.
- A sequence of the form $M \xrightarrow{g} N \to 0$ is exact if and only if g is surjective.
- A sequence of the form $0 \to M \xrightarrow{f} N$ is exact if and only if f is injective.
- A sequence of the form $0 \to M \xrightarrow{h} N \to 0$ is exact if and only if h is an isomorphism.
- A sequence of the form $0 \to M \to 0$ is exact if and only if $M = 0$.

Definition 2.12.
- A *left exact sequence* is an exact sequence of the form

$$0 \to M' \xrightarrow{i} M \xrightarrow{g} M''$$

  This means i is injective and $M' \cong \operatorname{im}(i) = \ker(g)$.
- A *right exact sequence* is an exact sequence of the form

$$M' \xrightarrow{f} M \xrightarrow{p} M'' \to 0$$

  This means p is onto and $\operatorname{im}(f) = \ker(p)$, so, $M'' \cong M/\ker(p) = M/\operatorname{im}(f)$. We denote $M/\operatorname{im}(f) = \operatorname{coker}(f)$ and call it the *cokernel* of f. Thus in a right exact sequence as above, $M'' \cong \operatorname{coker}(f)$.
- A *short exact sequence* (*SES*) is an exact sequence of the form

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

  Note that in a short exact sequence $M' \cong \ker(p)$ and $M'' \cong \operatorname{coker}(i)$. In particular, $M'' \cong M/M'$. We also say that M is an *extension* of M' and M'' if it fits in a short exact sequence as above.

Example 2.13. Let A be a $b \times a$ matrix and B be a $c \times b$ matrix of real numbers. The sequence of maps

$$0 \to \mathbb{R}^a \xrightarrow{A\cdot} \mathbb{R}^b \xrightarrow{B\cdot} \mathbb{R}^c$$

is a left exact sequence if and only if the columns of A form a basis for the null space of B.

2.1.1. *Presentations.* Recall that a set of elements B in a module M is a *free basis* if every element of M can be written as a (finite) R-linear combination of elements of B in a unique way, and a module M is a *free module* if it admits a free basis (which almost never is unique, by the way). As mentioned before, a free module is isomorphic to a direct sum of copies of the ring (considered as a module), which we may write as R^n or $R^{\oplus\Gamma}$ for some index Γ; such a free module has as a *standard basis* $\{e_\lambda\}_{\lambda\in\Lambda}$ consisting the elements that have a 1 in the λ coordinate and 0 in each of the others. Free modules are also characterized by a universal property:

If F free with basis B, then for any module M, and any function $f : B \to M$, there is a unique module homomorphism $\phi : F \to M$ such that the diagram commutes:

$$\begin{tikzcd} & F \arrow[dr, dashed, "\phi"] & \\ B \arrow[ur, hook, "\subseteq"] \arrow[rr, "f"] & & M \end{tikzcd}$$

i.e., any homomorphism is uniquely and freely specified by its values on the basis.

Note that a set of elements $\{m_\lambda\}_{\lambda\in\Lambda} \subseteq M$ generates M if and only if the homomorphism

$$\begin{aligned} R^\Lambda &\longrightarrow M \\ e_\lambda &\longmapsto m_\lambda \end{aligned}$$

is surjective. The kernel of such a map consists of the set of Λ-tuples (r_λ) such that $\sum_{\lambda\in\Lambda} r_\lambda m_\lambda = 0$; this is called the module of *relations* on the elements $\{m_\lambda\}$.

Definition 2.14. A *presentation* of a module M consists of a set of elements $\{m_\lambda\}$ that generates M, and a set of relations on $\{m_\lambda\}$ that generates the whole module of relations on $\{m_\lambda\}$.

We can express the data of a presentation in terms of a right exact sequence. Namely, if $\{m_\lambda\}_{\lambda\in\Lambda}$ is a generating set of M, and $\{(r_\lambda)_\gamma\}_{\gamma\in\Gamma}$ generates the module of relations on our generating set, then

$$R^{\oplus\Gamma} \to R^{\oplus\Lambda} \to M \to 0$$

is a right exact sequence, where the standard basis of $R^{\oplus\Lambda}$ maps to $\{m_\lambda\}$ and the standard basis of $R^{\oplus\Gamma}$ maps to $\{(r_\lambda)_\gamma\}$. Conversely, a right exact sequence of the form

$$R^{\oplus\Gamma} \to R^{\oplus\Lambda} \to M \to 0$$

is equivalent to the data of a presentation.

2.1.2. *Split exact sequences.* Given modules M' and M'', we have the "trivial" SES

$$0 \to M' \xrightarrow{\iota} M' \oplus M'' \xrightarrow{\pi} M'' \to 0$$

where ι is the canonical inclusion and π is the canonical projection. The following result gives equivalent conditions for when a SES is equivalent to a split one.

Theorem 2.15 (The splitting theorem)**.** *Given a SES of left R-modules*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0,$$

the following are equivalent:

(1) *There is a commutative diagram where each vertical arrow is an isomorphism*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & M \arrow[r, "p"] \arrow[d, "\theta"] & M'' \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "\iota"] & M' \oplus M'' \arrow[r, "\pi"] & M'' \arrow[r] & 0. \end{tikzcd}$$

(2) *There is an isomorphism $\theta : M \xrightarrow{\cong} M' \oplus M''$ such that $\theta \circ i = \iota$ and $\pi \circ \theta = p$.*
(3) *The map i has a left inverse q in $R - \mathbf{Mod}$.*
(4) *The map p has a right inverse j in $R - \mathbf{Mod}$.*
(5) *There are maps $q : M \to M'$ and $j : M'' \to M$ such that $q \circ i = \mathrm{id}_{M'}$, $p \circ j = \mathrm{id}_{M''}$, and $i \circ q + j \circ p = \mathrm{id}_M$.*

If these equivalent conditions hold, we call the SES a split exact sequence*.*

Proof. (1) $\Leftrightarrow$ (2) follows by definition of commutative diagram.

(1) $\Rightarrow$ (5): The main idea is that there are obvious splitting maps for the bottom SES. Define π' to be the canonical projection $\pi' : M' \oplus M'' \to M', (m', m'') \mapsto m'$ and ι'' to be the inclusion $\iota'' : M'' \to M' \oplus M'', m'' \mapsto (0, m'')$. Notice that $\pi' \circ \iota = \mathrm{id}_{M'}$ and $\pi \circ \iota'' = \mathrm{id}_{M''}$ and $i \circ \pi' + \iota'' \circ p = \mathrm{id}_{M'\oplus M''}$.

Lecture of September 13, 2021

We can use this to set $q = \pi' \circ \theta$ and $j = \theta^{-1} \circ \iota''$ and check

$$\begin{aligned} q \circ i &= \pi' \circ \theta \circ i = \pi' \circ \iota = \mathrm{id}_{M'} \\ p \circ j &= p \circ \theta^{-1} \circ \iota'' = \pi \circ \iota'' = \mathrm{id}_{M''} \end{aligned}$$

$$\begin{aligned} i \circ q + j \circ p &= i \circ \pi' \circ \theta + \theta^{-1} \circ \iota'' \circ p = \theta^{-1} \circ (\theta \circ i \circ \pi' + \iota'' \circ p \circ \theta^{-1}) \circ \theta \\ &= \theta^{-1} \circ (\iota \circ \pi' + \iota'' \circ \pi) \circ \theta = \theta^{-1} \circ \mathrm{id}_{M' \oplus M''} \circ \theta = \mathrm{id}_M. \end{aligned}$$

(5) $\Rightarrow$ (3, 4) is clear.

(3) $\Rightarrow$ (2): Given such a q, define $\theta(m) = (q(m), p(m))$. It is clear $\theta \circ i = \iota$ and $\pi \circ \theta = p$. We will now show that θ is injective: if $\theta(m) = 0$ then $p(m) = 0$ so $m \in \mathrm{im}(i)$ therefore $m = i(m')$ for some $m' \in M'$. But now $0 = q(m) = q(i(m')) = m'$ so $m' = 0$ and thus $m = 0$.

We next show that θ is surjective: $(m', m'') \in M' \oplus M''$. Since p is onto, then there exists some $u \in M$ so thar $p(u) = m''$. Let $m = i(m') + u - i(q(u))$. Then

$$\begin{aligned} \theta(m) &= (q(i(m')) + q(u) - q(i(q(u))), p(i(m')) + p(u) - p(i(q(u)))) \\ &= (m' + q(u) - q(u), m'' + 0 - 0) = (m', m''). \end{aligned}$$

Therefore θ is bijective, so it is an isomorphism.

The proof that (4) $\Rightarrow$ (2) is similar, and omitted. $\square$

We can also use splittings to show exactness.

Proposition 2.16. *Given a complex of R-modules of the form*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0,$$

if there are maps $q : M \to M'$ and $j : M'' \to M$ such that $q \circ i = \mathrm{id}_{M'}$, $p \circ j = \mathrm{id}_{M''}$, and $i \circ q + j \circ p = \mathrm{id}_M$, then the complex is exact, and hence split exact.

Proof. Since i has a left inverse, it is injective, and since p has a right inverse, it is surjective. To show exactness in the middle, let $m \in \ker(p)$. Then

$$m = (i \circ q)(m) + (j \circ p)(m) = i(q(m)) \in \mathrm{im}(i). \qquad \square$$

Remark 2.17. The proof in the previous example actually shows that, for any ring R, a SES whose right-most term is free is split exact.

Example 2.18. Here is an example of a non-split exact sequence: Take R to be any (commutative) integral domain and $r \in R$ any non-zero, non-unit element. Then, using that R is a domain, the sequence

$$0 \to R \xrightarrow{r} R \to R/r \to 0$$

is exact (where the second map is the canonical surjection). But it cannot by split exact: If it were, then we would have an isomorphism $R \cong R \oplus R/r$ of modules and so in particular there would be an ideal I of R isomorphic as a module to R/r. But then $rI = 0$ and since R is a domain, this could only happen if $I = 0$, which would mean r is a unit.

For example

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

is an exact, but not split exact, sequence of $\mathbb{Z}$-modules.

Example 2.19. Suppose $R = k$ is a field. Then every short exact sequence of R-modules

$$0 \to W \to V \to V/W \to 0$$

splits.

2.2. **Homomorphisms of R-modules.**

2.2.1. *Structure of* $\mathrm{Hom}_R(M, N)$. In general, we write $\mathrm{Hom}_R(M, N)$ for the set $\mathrm{Hom}_{R-\mathbf{Mod}}(M, N)$ of R-module morphisms between two left R-modules M and N.

It turns out that the set of homomorphisms between two R-modules has additional structure.

Proposition 2.20. *Let R be a ring, and M, N be two left R-modules.*

(1) $\mathrm{Hom}_R(M, N)$ *is an abelian group by pointwise addition, i.e.,*

$$(\alpha + \beta)(m) := \alpha(m) + \beta(m) \qquad \alpha, \beta \in \mathrm{Hom}_R(M, N),\ m \in M.$$

(2) *If R is commutative, then* $\mathrm{Hom}_R(M, N)$ *is an R-module via the action*

$$(r\alpha)(m) := r\alpha(m) = \alpha(rm) \qquad \alpha \in \mathrm{Hom}_R(M, N),\ r \in R,\ m \in M.$$

(3) *More generally,*
- *if M is a (R, S)-bimodule, then* $\mathrm{Hom}_R(M, N)$ *is a left S-module by the action $(s\alpha)(m) = \alpha(ms)$;*
- *if N is a (R, T)-bimodule, then* $\mathrm{Hom}_R(M, N)$ *is a right T-module by the action $(\alpha t)(m) = \alpha(m)t$;*
- *if M is a (R, S)-bimodule and N is a (R, T)-bimodule, then* $\mathrm{Hom}_R(M, N)$ *is a (S, T)-bimodule by the previous two actions.*

Proof. (1) is easy to check, and similar to what we checked with module endomorphisms.

Let's consider (2): The first thing to note is that $r\alpha(m) = \alpha(rm)$ by linearity of α. Let us check that the map $r\alpha$ defined this way is an R-module morphism:

$$(r\alpha)(m + m') = r\alpha(m + m') = r(\alpha(m) + \alpha(m')) = r\alpha(m) + r\alpha(m') = r\alpha(m) + r\alpha(m')$$

$$(r\alpha)(sm) = r\alpha(sm) = rs\alpha(m) = sr\alpha(m) = s(r\alpha(m));$$

note that commutativity of R is essential here.

The distributive rules are straightforward, and $((rs)\alpha)(m) = rs\alpha(m) = (r(s\alpha))(m)$, so $(rs)\alpha = r(s\alpha)$.

For (3), let's just focus on the first case. To see $s\alpha$ is R-linear, addition is similar to above, and

$$(s\alpha)(rm) = \alpha(rms) = r\alpha(ms) = r(s\alpha)(m).$$

Let's check the associativity property for the action: given $s, s' \in S$,

$$(ss')\alpha(m) = \alpha(mss') = s'\alpha(ms) = (s(s'\alpha))(m).$$

The other axioms are straightforward. □

The bonus module structures in case (3) are often useful, even for commutative rings. However, for many statements below we will just focus on cases (1) and (2) above for clarity.

Example 2.21. Let K be a field. Since K is commutative, $\mathrm{Hom}_K(K, K[x])$ and $\mathrm{Hom}_K(K[x], K)$ are K-vector spaces. The polynomial ring $K[x]$ is a $(K, K[x])$-bimodule. This gives $\mathrm{Hom}_K(K, K[x])$ a $K[x]$-module structure by postmultiplication: e.g., if α is the K-linear map such that $\alpha(1) = f(x)$, and $g(x) \in K[x]$, then $g(x)\alpha$ is the map that sends 1 to $f(x)g(x)$. Likewise, $\mathrm{Hom}_K(K[x], K)$ a $K[x]$-module structure by premultiplication: e.g., if α is the K-linear map such that $\alpha(x^i) = \gamma_i \in K$, then $x\alpha$ is the map that sends x^i to γ_{i+1}.

Lecture of September 15, 2021

2.2.2. *Hom as functors.*

Definition 2.22 (Covariant Hom)**.** Let R be a ring and M be an R-module. There is a covariant functor

$$\mathrm{Hom}_R(M,-) : R-\mathbf{Mod} \to \mathbf{Ab}$$

that maps each module A to $\mathrm{Hom}_R(M,A)$, and each morphism $A \xrightarrow{f} B$ to the homomorphism $\mathrm{Hom}_R(M,f)$ $=: f_*$ of "postcomposition by f":

$$\begin{array}{ccc} \mathrm{Hom}_R(M,A) & \xrightarrow{f_*} & \mathrm{Hom}_R(M,B) \\ g & \longmapsto & f \circ g \\ M \xrightarrow{g} A & & M \xrightarrow{g} A \xrightarrow{f} B. \end{array}$$

If R is commutative, then we consider $\mathrm{Hom}_R(M,-)$ as a functor from $R-\mathbf{Mod} \to R-\mathbf{Mod}$ by the same rule.

There are some things to check to verify that this is a functor.

Proof. We need to check that f_* is a valid morphism in $\mathbf{Ab}$, or in $R-\mathbf{Mod}$ in the commutative case. Given $g, h \in \mathrm{Hom}_R(A,B)$, we have

$$f_*(g+h)(m) = f((g+h)(m)) = f(g(m)+h(m)) = f(g(m)) + f(h(m)) = f_*(g)(m) + f_*(h)(m).$$

If R is commutative,

$$f_*(rg)(m) = f(g(rm)) = rf(g(m)) = (rf_*)(g)(m).$$

We also need to see that these satisfy the functor axioms. We have $(1_A)_*(g) = 1_A \circ g = g$, so $(1_A)_*$ is the identity map on $\mathrm{Hom}_R(M,A)$. Given $A \xrightarrow{g} B \xrightarrow{f} C$, and $h \in \mathrm{Hom}_R(M,A)$,

$$(fg)_*(h) = f \circ g \circ h = f \circ (g_*(h)) = f^*(g^*(h)) = (f_* \circ g_*)(h). \qquad \square$$

Remark 2.23*.* If M is an (R,S)-bimodule, then consider $\mathrm{Hom}_R(M,-)$ as a functor from $R-\mathbf{Mod} \to S-\mathbf{Mod}$ by the same rule.

Definition 2.24 (Contravariant Hom)**.** Let R be a ring and M be an R-module. There is a contravariant functor

$$\mathrm{Hom}_R(-,M) : R-\mathbf{Mod} \to \mathbf{Ab}$$

that maps each module A to $\mathrm{Hom}_R(A,M)$, and each morphism $A \xrightarrow{f} B$ to the homomorphism $\mathrm{Hom}_R(f,M)$ $=: f^*$ of "precomposition by f":

$$\begin{array}{ccc} \mathrm{Hom}_R(M,B) & \xrightarrow{f_*} & \mathrm{Hom}_R(M,A) \\ g & \longmapsto & g \circ f \\ B \xrightarrow{g} M & & A \xrightarrow{f} B \xrightarrow{g} M. \end{array}$$

If R is commutative, then we consider $\mathrm{Hom}_R(-,M)$ as a functor from $R-\mathbf{Mod} \to R-\mathbf{Mod}$ by the same rule.

There are some things to check to verify that this is a functor.

Proof. We need to check that f^* is a valid morphism in **Ab**, or in $R-\mathbf{Mod}$ in the commutative case. Given $g, h \in \mathrm{Hom}_R(A, B)$, we have

$$f^*(g+h)(m) = (g+h)(f(m)) = g(f(m)) + h(f(m)) = f^*(g)(m) + f^*(h)(m).$$

If R is commutative,

$$f^*(rg)(m) = rg(f(m)) = (rf^*(g))(m).$$

We also need to see that these satisfy the functor axioms. We have $(1_A)^*(g) = g \circ 1_A = g$, so $(1_A)^*$ is the identity map on $\mathrm{Hom}_R(A, M)$. Given $A \xrightarrow{g} B \xrightarrow{f} C$, and $h \in \mathrm{Hom}_R(C, M)$,

$$(fg)^*(h) = h \circ f \circ g = f^*(h) \circ g = g^*(f^*(h)) = (g^* \circ f^*)(h). \qquad \square$$

Remark 2.25. If M is an (R, S)-bimodule, then consider $\mathrm{Hom}_R(M, -)$ as a functor from $R - \mathbf{Mod} \to S^{\mathrm{op}} - \mathbf{Mod}$ by the same rule.

2.2.3. *Examples of Hom.*

Example 2.26. Let R be a ring. Then, by the universal property of free modules, since $\{1\}$ is a free basis for R as an R-module, the map

$$\begin{tikzcd} \mathrm{Hom}_R(R, M) \arrow[r, "\psi_M"] & M \\ \phi \arrow[r, mapsto] & \phi(1) \end{tikzcd}$$

is a bijection. Moreover, this is an isomorphism of abelian groups in general, and of R-modules in the commutative case:

$$\psi_M(\alpha + \beta) = (\alpha + \beta)(1) = \alpha(1) + \beta(1) = \psi_M(\alpha) + \psi_M(\beta)$$
$$\psi_M(r\alpha) = (r\alpha)(1) = r\alpha(1) = r\psi_M(\alpha).$$

Even better, in the commutative case, the collection of isomorphisms ψ_M form a natural isomorphism $\psi : \mathrm{Hom}_R(R, -) \Rightarrow 1_{R-\mathbf{Mod}}$. For this, we need to check that, given $\beta : M \to N$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_R(R, M) \arrow[r, "\beta_*"] \arrow[d, "\psi_M"] & \mathrm{Hom}_R(R, N) \arrow[d, "\psi_N"] \\ M \arrow[r, "\beta"] & N. \end{tikzcd}$$

Along either path, we get $\alpha \mapsto \beta(\alpha(1))$, so this is indeed the case.

Lecture of September 17, 2021

Example 2.27. Similarly, if $F = R^{\oplus\Lambda}$ is a free module, then $\mathrm{Hom}_R(R^{\oplus\Lambda}, M) \cong M^{\oplus\Lambda}$, where $M^{\times\Lambda} = \prod_{\lambda\in\Lambda} M$ by the map that sends a morphism to its tuple of values on the standard basis: as abelian groups, and as R-modules in the commutative case.

We can interpret the right-hand side as the values of a functor: set $F(M) = M^{\times\Lambda}$, and for $f : M \to N$, set $F(f)$ to be the map given by f on each coordinate. Interpreted like so, the isomorphisms again form a natural isomorphism.

Proposition 2.28. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules, and N be an R-module. There are isomorphisms of abelian groups*

$$\operatorname{Hom}_R(\bigoplus_{\lambda\in\Lambda} M_\lambda, N) \cong \prod_{\lambda\in\Lambda} \operatorname{Hom}_R(M_\lambda, N)$$
$$\operatorname{Hom}_R(N, \prod_{\lambda\in\Lambda} M_\lambda) \cong \prod_{\lambda\in\Lambda} \operatorname{Hom}_R(N, M_\lambda)$$

Moreover, these are isomorphisms of R-modules if R is commutative.

Proof. Since $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is the coproduct of $\{M_\lambda\}_{\lambda\in\Lambda}$ in $R-\mathbf{Mod}$, we have a bijection for every R-module N

$$\begin{aligned}\operatorname{Hom}_R(\textstyle\bigoplus_{\lambda\in\Lambda} M_\lambda, N) &\longrightarrow \textstyle\prod_{\lambda\in\Lambda} \operatorname{Hom}_R(M_\lambda, N)\\ \phi &\longmapsto (\phi\circ\iota_\lambda).\end{aligned}$$

We only have to observe that these maps preserve the abelian group and/or R-module structures. Similarly, since $\prod_{\lambda\in\Lambda} M_\lambda$ is the product of $\{M_\lambda\}_{\lambda\in\Lambda}$ in $R-\mathbf{Mod}$, we have a bijection for every R-module N

$$\begin{aligned}\operatorname{Hom}_R(N, \textstyle\prod_{\lambda\in\Lambda} M_\lambda) &\longrightarrow \textstyle\prod_{\lambda\in\Lambda} \operatorname{Hom}_R(N, M_\lambda)\\ \phi &\longmapsto (\pi_\lambda\circ\phi),\end{aligned}$$

and one verifies the additivity / linearity of this map. □

Example 2.29. As an important special case of the previous example, if R is commutative, and $R^{\oplus\Gamma}$ and $R^{\oplus\Lambda}$ are free modules, then every R-linear homomorphism $\alpha : R^{\oplus\Gamma} \to R^{\oplus\Lambda}$ is given by left multiplication by the (possibly infinite) $\Lambda\times\Gamma$ matrix where the γ column is the Λ-tuple $(\alpha(e_\gamma))_\lambda$.

Optional Exercise 2.30. Show that when R is not necessarily commutative, if we give $\operatorname{Hom}_R(R^{\oplus\Lambda}, M)$ the R-module structure via the (R,R)-bimodule structure on $R^{\oplus\Lambda}$, the isomorphisms $\operatorname{Hom}_R(R^{\oplus\Lambda}, M) \cong M^{\times\Lambda}$ are natural isomorphisms of R-modules.

Example 2.31. Let R be a commutative ring, and consider the module R/I for some ideal I. For every module M, there is an isomorphism $\operatorname{Hom}_R(R/I, M) \cong \operatorname{ann}_M(I)$, where $\operatorname{ann}_M(I)$ is the set of elements $m\in M$ such that $Im = 0$.

Indeed, every R-module homomorphism from R/I is determined by the image of 1, so the map $\operatorname{Hom}_R(R/I, M) \to M$ of evaluation at 1 is injective. The image consists of the set of elements $m \in M$ for which the map $r \mapsto rm$ is well-defined; this is the collection of elements that satisfy $Im = 0$.

Again, we can think of the right hand side as a functor $F : R-\mathbf{Mod} \to R-\mathbf{Mod}$ where on objects $F(M) = \operatorname{ann}_M(I)$, and on morphisms $M \xrightarrow{\alpha} N$ maps to the restriction of α to $\operatorname{ann}_M(I)$. This is a natural isomorphism again.

Example 2.32. For a field K, the functor $\operatorname{Hom}_K(-, K)$ is exactly the "vector space dual" functor $(-)^*$.

2.3. Exact functors and left exactness of Hom.

Definition 2.33. Let R, S be rings. A covariant functor $F : R-\mathbf{Mod} \to S-\mathbf{Mod}$ is *additive* if the function

$$\begin{aligned}\operatorname{Hom}_R(M, N) &\longrightarrow \operatorname{Hom}_S(F(M), F(N))\\ f &\longmapsto F(f)\end{aligned}$$

is a homomorphism of abelian groups. Likewise, a contravariant functor $G : R - \mathbf{Mod} \to S - \mathbf{Mod}$ is additive if the function

$$\begin{aligned} \operatorname{Hom}_R(M, N) &\longrightarrow \operatorname{Hom}_S(F(N), F(M)) \\ f &\longmapsto F(f) \end{aligned}$$

is a homomorphism of abelian groups.

Additive functors preserve a number of basic properties, e.g., zero morphisms go to zero morphisms, and the zero module maps to the zero module (since it's characterized by the fact that its identity map is its zero map).

Optional Exercise 2.34. The covariant and contravariant Hom functors are additive functors.

Definition 2.35. Let $F : R - \mathbf{Mod} \to S - \mathbf{Mod}$ be an additive covariant functor.

- F is *right exact* if whenever
$$M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$
is exact, then so is
$$F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'') \to 0.$$
(Recall $F(0) = 0$ since F is additive.)
- F is *left exact* if whenever
$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M''$$
is exact, then so is
$$0 \to F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'').$$
- F is *exact* if it is both left and right exact.

Remark 2.36. An exact functor takes any SES to a SES.

Definition 2.37. Let $G : R - \mathbf{Mod} \to S - \mathbf{Mod}$ be an additive contravariant functor.

- G is *right exact* if whenever
$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M''$$
is exact, then so is
$$G(M'') \xrightarrow{G(p)} G(M) \xrightarrow{G(i)} G(M') \to 0.$$
- G is *left exact* if whenever
$$M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$
is exact, then so is
$$0 \to G(M'') \xrightarrow{G(p)} G(M) \xrightarrow{G(i)} G(M').$$
- G is *exact* if it is additive and both left and right exact.

Optional Exercise 2.38. The definitions above all stay unchanged if for each condition we start with a short exact sequence. For example, a covariant additive functor F is left exact if for every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules,

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact.

Remark 2.39. If $F, G : R-\mathbf{Mod} \to S-\mathbf{Mod}$ are naturally isomorphic additive functors, then F is exact if and only if G is exact. Indeed, given a short exact sequence

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F(M') \arrow[r, "F(i)"] \arrow[d, "\theta'"] & F(M) \arrow[r, "F(p)"] \arrow[d, "\theta"] & F(M'') \arrow[r] \arrow[d, "\theta''"] & 0 \\ 0 \arrow[r] & G(M') \arrow[r, "G(i)"] & G(M) \arrow[r, "G(p)"] & G(M'') \arrow[r] & 0 \end{tikzcd}$$

where $\theta', \theta, \theta''$ isomorphisms. Then if the top row is exact, $G(i) = \theta F(i)(\theta')^{-1}$ is injective, $G(p) = \theta'' F(p)\theta^{-1}$ is surjective, and

$$x \in \ker G(p) = \ker(\theta'' F(p)\theta^{-1}) \iff \theta^{-1}(x) \in \ker F(p) \iff \theta^{-1}(x) \in \operatorname{im} F(i) \iff x \in \operatorname{im}(\theta F(i)(\theta')^{-1}) = \operatorname{im} G(i).$$

Similarly for "left exact" or "right exact".

Theorem 2.40. *Let M be an R-module.*

(1) *The functor* $\operatorname{Hom}_R(M, -)$ *is left exact.*
(2) *The functor* $\operatorname{Hom}_R(-, M)$ *is left exact.*

Lecture of September 20, 2021

Proof. (1) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C$$

be exact. We need to show that

$$0 \to \operatorname{Hom}_R(M, A) \xrightarrow{i_*} \operatorname{Hom}_R(M, B) \xrightarrow{p_*} \operatorname{Hom}_R(M, C)$$

is exact.

- i_* is injective: Let $f \in \operatorname{Hom}_R(M, A)$ be nonzero, so $f(m) \neq 0$ for some $m \in M$. Then $i_*(f)(m) = i(f(m)) \neq 0$ since i is injective, so $i_*(f) \in \operatorname{Hom}_R(M, B)$ is nonzero.
- $\operatorname{im}(i_*) \subseteq \ker(p_*)$: Let $g \in \operatorname{Hom}_R(M, B)$ be in the image of i_*, so we can write $g = i_*(f)$ for some $f \in \operatorname{Hom}_R(M, A)$. We have $p_*(i_*(f)) = p \circ i \circ f = 0$.
- $\ker(p_*) \subseteq \operatorname{im}(i_*)$: Let $g \in \operatorname{Hom}_R(M, B)$ be in the kernel of p_*, so $p \circ g = 0$. Then, for every $m \in M$, $g(m) \in \ker(p) = \operatorname{im}(i)$. As i is injective, i induces an isomorphism from i to the image of A in B, so there is an R-module homomorphism $q : \operatorname{im}(A) \to A$ such that $i \circ q = 1_{\operatorname{im}(A)}$. Thus, we obtain an R-module map $f := q \circ g : M \to A$ such that $i_*(f) = i \circ q \circ g = g$.

(2) Let

$$A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be exact. We need to show that

$$0 \to \operatorname{Hom}_R(C, M) \xrightarrow{p^*} \operatorname{Hom}_R(B, M) \xrightarrow{i^*} \operatorname{Hom}_R(A, M)$$

is exact.

- p^* is injective: Let $f \in \operatorname{Hom}_R(C, M)$ be nonzero, so $f(c) \neq 0$ for some $m \in M$. Then, since p is surjective, there is some $b \in B$ such that $p(b) = c$, and hence $p^*(f)(b) = f(p(b)) = f(c) \neq 0$, so $p^*(f) \neq 0$.
- $\operatorname{im}(p^*) \subseteq \ker(i^*)$: Let $g \in \operatorname{Hom}_R(B, M)$ be in the image of p_*, so we can write $g = p^*(f)$ for some $f \in \operatorname{Hom}_R(M, C)$. We have $i^*(p^*(f)) = f \circ p \circ i = 0$.
- $\ker(i^*) \subseteq \operatorname{im}(p^*)$: Let $g \in \operatorname{Hom}_R(B, M)$ be in the kernel of i^*, so $g \circ i = 0$. Thus, as $g|_{\operatorname{im}(i)} = 0$, we can factor $g = \overline{g} \circ \pi$, where $\pi : B \to B/\operatorname{im}(i) = B/\ker(p)$ is the quotient map, and $\overline{g} : B/\operatorname{im}(i) \to M$. Note that, since p is surjective, writing $p = \overline{p} \circ \pi$, the map $\overline{p} : B/\operatorname{im}(i) = B/\ker(p) \to C$ is an isomorphism, so there is a map $j : C \to B/\ker(p)$ such that $j \circ \overline{p}$ is the identity on $B/\operatorname{im}(i)$, so $j \circ p = j \circ \overline{p} = \circ\pi = \pi$. Set $f = \overline{g} \circ j$. We then have $p^*(f) = \overline{g} \circ j \circ p = \overline{g} \circ \pi = g$. Thus $g \in \operatorname{im}(p^*)$. $\square$

Example 2.41. Neither Hom functor is exact. For example, consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

If we apply $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to this sequence, we get

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

This is exact up until $\mathbb{Z}/2\mathbb{Z}$ (which agrees with the left exactness), but not at $\mathbb{Z}/2\mathbb{Z}$. Likewise, apply $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ to get

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0.$$

This is again exact up to the last $\mathbb{Z}/2\mathbb{Z}$, but not there.

We can use left exactness to compute various Hom modules.

Example 2.42. Let R be commutative, and M be a finitely presented R-module with presentation

$$R^m \xrightarrow{A\cdot} R^n \to M \to 0.$$

Then $\operatorname{Hom}_R(M, R)$ sits in a left exact sequence

$$0 \to \operatorname{Hom}(M, R) \to \operatorname{Hom}_R(R^n, R) \xrightarrow{\operatorname{Hom}_R(A\cdot, R)} \operatorname{Hom}_R(R^m, R).$$

We have $\operatorname{Hom}_R(R^n, R)$ is free with free basis given by the coordinate functions $\{e_1^*, \dots, e_n^*\}$; likewise for $\operatorname{Hom}_R(R^m, R)$ with basis $\{\bar{e}_1^*, \dots, \bar{e}_m^*\}$ (we will write bars for basis elements in R^m). In these bases, to compute the jth column of the matrix, we have that e_j^* maps to $e_j^* A$, and to compute $e_j^* A$ in terms of the given basis (to find the entry in the ith row), we observe $(e_j^* A)(\bar{e}_i)$ is A_{ji}, so the map $\operatorname{Hom}_R(A\cdot, R)$ is given by $A^T\cdot$. We get a left exact sequence

$$0 \to \operatorname{Hom}(M, R) \to R^n \xrightarrow{A^T\cdot} R^m,$$

so $\operatorname{Hom}(M, R) \cong \ker(A^T)$.

2.4. **Tensor products.**

Lecture of September 22, 2021

2.4.1. *Definition of tensor product.*

Definition 2.43. For a ring R, a right R-module M, a left R-module N, and an abelian group A, a function

$$b : M \times N \to A$$

is called *R-balanced biadditive* if the following conditions hold:

(1) $b(m + m', n) = b(m, n) + b(m', n)$ for all $m, m' \in M$, $n \in N$,
(2) $b(m, n + n') = b(m, n) + b(m, n')$ for all $m \in M$, $n, n' \in N$, and
(3) $b(mr, n) = b(m, rn)$ for all $m \in M$, $n, \in N$, and $r \in R$.

Assume R is commutative and A is an R-module (not just an abelian group). Such a pairing b is called *R-bilinear* if we also have

(4) $b(mr, n) = b(m, rn) = rb(m, n)$ for all $m \in M$, $n, \in N$, and $r \in R$.

Conditions (1) and (2) alone are the biadditive part, and condition (3) is the balancedness. Condition (4) says that the biadditive map b is an R-linear function in either argument if we fix the other one.

Example 2.44. (1) If R is any ring, the map $f : R \times R \to R, f(r, s) = rs$ is R-balanced biadditive, and bilinear if R is commutative.
(2) For R commutative, an ideal I, and a left module M, the map $f : (R/I) \times M \to M/IM, f(\overline{r}, m) = \overline{rm}$ is R-bilinear.
(3) For K a field, $f : K^n \times K^n \to K$ given by the usual dot product is K-bilinear. Recall that we can view K^n as a right $M_n(K)$ module via $v \cdot A = A^T v$ and as a left $M_n(K)$ module via $A \cdot v = Av$. With these structures, f is $M_n(K)$-balanced biadditive. The balanced part is the least obvious one:

$$f(v \cdot A, w) = (A^T v) \cdot w = v^T Aw = v \cdot (Aw) = f(v, A \cdot w).$$

We now define tensor products using a universal property.

Definition 2.45. Let R be a (not necessarily commutative) ring, let M be a right R-module, let N be a left R-module.

An abelian group $M \otimes_R N$ together with an R-balanced biadditive map $h : M \times N \to M \otimes_R N$ is called a *tensor product* of M and N if it has the following universal property: for any abelian group A and R-balanced biadditive map $b : M \times N \to A$, there exists a unique abelian group homomorphism $\alpha : M \otimes_R N \to A$ such that $b = \alpha \circ h$.

$$\begin{tikzcd} & M \otimes_R N \arrow[dr, dashed, "\exists!\alpha"] & \\ M \times N \arrow[ur, "h"] \arrow[rr, "b"] & & A \end{tikzcd}$$

Lemma 2.46. *If (X, h), (Y, k) are two tensor products for M and N, then there is a unique isomorphism of abelian groups $\alpha : X \to Y$ such that $k = \alpha \circ h$.*

Proof. The following diagram is a rough guide for the argument:

$$\begin{tikzcd} & X \arrow[r, dashed, "\alpha"] & Y \arrow[r, dashed, "\beta"] & X. \\ M \times N \arrow[ur, "h"] \arrow[urr, "k"] \arrow[urrr, "h"'] & & & \end{tikzcd}$$

Applying the universal property of (X, h) with the R-balanced biadditive map k, we get a unique abelian group homomorphism α above that makes its triangle commute; in particular, the uniqueness statement is clear. Likewise, applying the universal property of (Y, k) with h, we get an abelian group homomorphism β that makes its triangle commute. Then, $\beta \circ \alpha$ is an abelian group homomorphism such that $(\beta \circ \alpha) \circ h = h$, and the identity map is another. By the uniqueness property of (X, h), $\beta \circ \alpha$ is the identity. A similar argument shows that $\alpha \circ \beta$ is the identity too, so α is an isomorphism. ☐

Theorem 2.47. *Let R be a (not necessarily commutative) ring, let M be a right R-module, let N be a left R-module. Then a tensor product $M \otimes_R N$ exists and is given by defining an abelian group $M \otimes_R N$ by generators and relations as follows:*

- *The generators are all expressions of the form $m \otimes n$ for $m \in M$ and $n \in N$.*
- *The relations are*
  1. *$(m + m') \otimes n = m \otimes n + m' \otimes n$ for all $m, m' \in M$ and $n \in N$,*
  2. *$m \otimes (n + n') = m \otimes n + m \otimes n'$ for all $m \in M$ and $n, n' \in N$, and*
  3. *$(mr) \otimes n = m \otimes (rn)$ for all $m \in M$, $n \in N$, and $r \in R$.*

Equivalently, $M \otimes_R N$ is the quotient

$$\frac{\bigoplus_{(m,n)\in M\times N} \mathbb{Z}\cdot(m\otimes n)}{(Y)}$$

where

$$Y = \{(m + m') \otimes n) - m \otimes n - m' \otimes n\} \cup \{m \otimes (n + n') - m \otimes n - m \otimes n'\} \cup \{(mr) \otimes n - m \otimes (rn)\}.$$

Further we define $h : M \times N \to M \otimes_R N$ to be the function $h(m, n) = m \otimes n$.

Then the pair $(M \otimes_R N, h)$ defined above is the tensor product of M and N.

Proof. It is immediate from the construction that h is R-balanced biadditive. Given a biadditive map $b : M \times N \to A$, define $\tilde{b} : \bigoplus_{(m,n)\in M\times N} \mathbb{Z}\cdot(m\otimes n) \to A$ to be the unique homomorphism of abelian groups sending the basis element $m \otimes n$ to $b(m, n)$. Since b is biadditive, we have

$$\tilde{b}((m + m') \otimes n - m \otimes n - m' \otimes n) = b(m + m', n) - b(m, n) - b(m', n) = 0,$$

$$\tilde{b}(m \otimes (n + n') - m \otimes n - m \otimes n) = b(m, n + n') - b(m, n) - b(m, n') = 0,$$

and

$$\tilde{b}((mr) \otimes n - m \otimes (rn)) = b(mr, n) - b(m, rn) = 0.$$

Thus $\tilde{b}(< Y >) = 0$ and so it induces a homomorphism of abelian groups

$$\alpha : M \otimes_R N \to A.$$

It is evident from the construction that $\alpha \circ h = b$. Since the image of B generates $M \otimes_A N$ as an abelian group, α is the unique homomorphism satisfying this equation.

If β is any abelian group homomorphism with $\beta \circ h = b$, we have $\beta(m \otimes n) = \beta(h(m, n)) = b(m, n) = \alpha(m \otimes n)$. Since the elements of the form $m \otimes n$ generate $M \otimes_R N$ as an abelian group, we must have $\beta = \alpha$. ☐

Note that the map induced by a biadditive map b sends $m \otimes n \mapsto b(m, n)$.

Remark 2.48. In this explicit construction, every element is a *sum* of *simple tensors* (elements of the form $m \otimes n$) but in general, not every element is itself a simple tensor.

Remark 2.49. While the construction of tensor products may feel easier to work with at first, it is important to keep in mind that it is hard to tell when two combinations of simple tensors are equal. In general, when we want to define a map from a tensor product, it is better to use the universal property, since we don't have to worry about well-definedness. However, to define a map into a tensor product, using the concrete description is often easier.

Optional Exercise 2.50. In $M \otimes_R N$ we have $0_M \otimes n = 0_{M \otimes_R N} = m \otimes 0_N$ for each $m \in M, n \in N$.

Lecture of September 24, 2021

Example 2.51. I claim $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/g\mathbb{Z}$ where $g = \gcd(m, n)$.

Proof. Define a function
$$b : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/g\mathbb{Z}$$
by $b(\bar{i}, \bar{j}) = \overline{ij}$. It is not hard to see that b is well-defined (exercise!) and $\mathbb{Z}$-balanced biadditive. By the universal property, it therefore induces a homomorphism of abelian groups
$$\alpha : \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/g\mathbb{Z}$$
such that $\alpha(\bar{i} \otimes \bar{j}) = \overline{ij}$.

Now define a homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ by sending 1 to $1 \otimes 1$. Notice that
$$\phi(g) = g \cdot (1 \otimes 1) = g \otimes 1 = 1 \otimes g.$$
Recall that $g = im + jn$ for some $i, j \in \mathbb{Z}$. So
$$g \otimes 1 = im \otimes 1 + 1 \otimes jn = 0 \otimes 1 + 1 \otimes 0 = 0 + 0 = 0.$$
So, ϕ induces a homomorphism
$$\beta = \overline{\phi} : \mathbb{Z}/g\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$
with $\beta(\bar{i}) = \bar{i} \otimes 1 = 1 \otimes \bar{i}$.

We have $\alpha(\beta(\bar{i})) = \alpha(\bar{i} \otimes 1) = \bar{i}$ so that $\alpha \circ \beta = \mathrm{id}$.

A typical element of $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ has the form $\sum_t \overline{i_t} \otimes \overline{j_t}$. We have
$$\beta(\alpha(\sum_t \overline{i_t} \otimes \overline{j_t})) = \sum_t \overline{i_t} \cdot \overline{j_t} \otimes 1 = \sum_t \overline{i_t} \otimes \overline{j_t}$$
and so $\beta \circ \alpha = \mathrm{id}$. □

2.4.2. *Module structure of tensor product.*

Proposition 2.52. (1) *If R is commutative, and M and N are R-modules, then*

(a) *$M \otimes_R N$ is an R-module via the action*
$$r \cdot (\sum_i m_i \otimes n_i) = \sum_i (rm_i) \otimes n_i = \sum_i m_i \otimes (rn_i).$$
(b) *The natural map $h : M \times N \to M \otimes_R N$ is R-bilinear.*

(c) *For any R-module A and R-bilinear map $b : M \times N \to A$, there is a unique R-module homomorphism $\alpha : M \otimes_R N \to A$ such that $b = \alpha \circ h$.*

(2) *If M is an (S, R)-bimodule, and N is an R-module, then consider $M \times N$ as an S-module by the action $s(m, n) = (sm, n)$. We have*

(a) *$M \otimes_R N$ is an S-module via the action*

$$s \cdot (\sum_i m_i \otimes n_i) = \sum_i (sm_i) \otimes n_i.$$

(b) *The natural map $h : M \times N \to M \otimes_R N$ is S-linear.*

(c) *For any S-module A and S-linear R-balanced biadditive map $b : M \times N \to A$, there is a unique S-module homomorphism $\alpha : M \otimes_R N \to A$ such that $b = \alpha \circ h$.*

(3) *If M is an R-module, and N is an (R, S)-bimodule, then consider $M \times N$ as a right S-module by the action $(m, n)s = (m, ns)$. We have*

(a) *$M \otimes_R N$ is a right S-module via the action*

$$s \cdot (\sum_i m_i \otimes n_i) = \sum_i m_i \otimes (n_i s).$$

(b) *The natural map $h : M \times N \to M \otimes_R N$ is right S-linear.*

(c) *For any right S-module A and right S-linear R-balanced biadditive map $b : M \times N \to A$, there is a unique right S-module homomorphism $\alpha : M \otimes_R N \to A$ such that $b = \alpha \circ h$.*

Proof. Let's consider case (2).

For (a), the first thing we need to show that the action of an element $s \in S$ on $M \otimes_R N$ is a well-defined function. To do this, consider the map $\mu_s : M \times N \to A$ given by the rule $\mu_s(m, n) = sm \otimes n$. We claim that this is R-balanced biadditive. Indeed,

$$\mu_s(m + m', n) = (s(m + m')) \otimes n = (sm + sm') \otimes n = sm \otimes n + sm' \otimes n = \mu_s(m, n) + \mu_s(m', n),$$

similarly $\mu_s(m, n + n') = \mu_s(m, n) + \mu_s(m, n')$, and

$$\mu_s(mr, n) = smr \otimes n = sm \otimes rn = \mu_s(m, rn).$$

Thus, we obtain a well-defined map $M \otimes_R N \to M \otimes_R N$ that sends $m \otimes n \mapsto sm \otimes n$, and the given formula follows. It is easy to check that this action satisfies the module axioms.

For (b), we already know this map is additive. To see that it is S-linear, we compute

$$h(s(m, n)) = h(sm, n) = sm \otimes n = s(m \otimes n) = sh(m, n).$$

For (c), we know that since f is R-balanced biadditive map there exists a unique additive map α such that $b = \alpha \circ h$. We just need to show that this map is S-linear:

$$\begin{aligned}\alpha(s(\sum_i m_i \otimes n_i)) &= \alpha(\sum_i sm_i \otimes n_i) = \sum_i \alpha(sm_i \otimes n_i) = \sum_i \alpha(h(sm_i, n_i)) \\ &= \sum_i b(sm_i, n_i) = s(\sum_i b(m_i, n_i)) = s(\sum_i \alpha(m_i \otimes n_i)) = s(\alpha(\sum_i m_i \otimes n_i)).\end{aligned}$$

Case (3) is quite analogous. Case (1) is a special case of (2): we consider M as an (R, R)-bimodule. The extra equality in (a) follows from $rm \otimes n = mr \otimes n = m \otimes rn$. For (b) and (c), we note that R-bilinear is equivalent to R-balanced biadditive plus R-linear with respect to the module structure given in case (2). $\square$

We can take tensor products of maps as well.

Lemma 2.53. *Let $f : M \to M'$ be a homomorphism of right R-modules and $g : N \to N'$ be a homomorphism of left R-modules. There exists a unique homomorphism of abelian groups $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ such that*

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$$

for all $m \in M$ and $n \in N$.

If R is commutative, this map is R-linear.

If M and M' are (S, R)-bimodules, and f is also S-linear, then this map is an S-module homomorphism.

Proof. The function

$$\begin{aligned} M \times N &\longrightarrow M' \otimes_R N' \\ (m, n) &\longmapsto f(m) \otimes g(n) \end{aligned}$$

is R-balanced biadditive (and bilinear when R is commutative), so the universal property of tensor products gives the desired R-module homomorphism, which is unique. In the bimodule case, S-linearity follows from observing that the function displayed above is S-linear on the first argument. □

Definition 2.54. Let R be a ring and M be a right R-module. There is an additive covariant functor

$$M \otimes_R - : R - \mathbf{Mod} \to \mathbf{Ab}$$

that on objects sends N to $M \otimes_R N$, and on morphisms sends $f : N \to N'$ to the map $1_M \otimes f$.

If R is commutative, we can consider $M \otimes_R -$ as a functor from $R - \mathbf{Mod} \to R - \mathbf{Mod}$.

If M is a (S, R)-bimodule, we can consider $M \otimes_R -$ as a functor from $R - \mathbf{Mod} \to S - \mathbf{Mod}$.

Proof. Well definedness of the maps comes from the lemma. Given $A \xrightarrow{g} B \xrightarrow{f} C$, we have

$$(1_M \otimes (fg))(m \otimes a) = m \otimes (fg)(a) = (1_M \otimes f)(1_M \otimes g)(m \otimes a),$$

so $(1_M \otimes (fg)) - (1_M \otimes f)(1_M \otimes g)$ vanishes on a generating set for $M \otimes_R A$, and hence is zero. Similarly for the identity property.

For additivity, we observe that

$$(1_M \otimes (f + g))(m \otimes n) = m \otimes (f + g)(n) = m \otimes f(n) + m \otimes g(n) = ((1_M \otimes f) + (1_M \otimes g))(m \otimes n),$$

and since simple tensors generate, we have $1_M \otimes (f + g) = 1_M \otimes f + 1_M \otimes g$. □

Remark 2.55. We can equally well discuss $- \otimes_R N : R^{\mathrm{op}} - \mathbf{Mod} \to \mathbf{Ab}$ (or other targets when we have more structure akin to above).

Lecture of September 27, 2021

The key to unlocking more examples of tensor will be to prove that it is right exact.

Theorem 2.56. *Let M be a right R-module. The functor $M \otimes_R - : R - \mathbf{Mod} \to \mathbf{Ab}$ is right exact.*

Proof. Let

$$A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be exact. We need to show that

$$M \otimes_R A \xrightarrow{1_M \otimes i} M \otimes_R B \xrightarrow{1_M \otimes p} M \otimes_R C \to 0$$

is exact.

- $1_M \otimes p$ is surjective: Given $\sum_i m_i \otimes c_i \in M \otimes_R C$, we can find $b_i \in B$ such that $p(b_i) = c_i$ for all i; then $(1_M \otimes p)(\sum_i m_i \otimes b_i) = \sum_i m_i \otimes c_i$.
- $\operatorname{im}(1_M \otimes i) \subseteq \ker(1_M \otimes p)$: We have $(1_M \otimes p)(1_M \otimes i) = 1_M \otimes (pi) = 1_M \otimes 0 = 0$.
- $\ker(1_M \otimes p) \subseteq \operatorname{im}(1_M \otimes i)$: From above, the map $1_M \otimes p$ induces a surjection $\alpha : (M \otimes_R B)/\operatorname{im}(1_M \otimes i) \to M \otimes_R C$ that maps $m \otimes b \mapsto m \otimes p(b)$. We will construct an inverse for this map.

Consider the map

$$\begin{aligned} \mu : M \times C &\longrightarrow (M \otimes_R B)/\operatorname{im}(1_M \otimes i) \\ (m,c) &\longmapsto m \otimes b \qquad\qquad \text{for some } b \text{ with } p(b) = c. \end{aligned}$$

To see this is well-defined, note that if $p(b) = p(b') = c$, then $p(b - b') = 0$, so $b - b' = i(a)$ for some $a \in A$, so

$$(m \otimes b) - (m \otimes b') = m \otimes (b - b') = m \otimes i(a) \in \operatorname{im}(1_M \otimes i).$$

We then check μ is R-balanced biadditive:

$$\mu(m + m', c) = (m + m') \otimes b = m \otimes b + m' \otimes b = \mu(m, c) + \mu(m', c).$$

If $p(b) = c$ and $p(b') = c'$, then $p(b + b') = c + c'$, so

$$\mu(m, c + c') = m \otimes (b + b') = m \otimes b + m \otimes b' = \mu(m, c) + \mu(m, c'),$$

and, if $p(b) = c$, then $p(rb) = rc$, so

$$\mu(mr, c) = mr \otimes b = m \otimes rb = \mu(m, rc).$$

Thus, μ induces an additive homomorphism $\beta : M \otimes_R C \to (M \otimes_R B)/\operatorname{im}(1_M \otimes i)$. By construction, we have $\beta \circ \alpha(m \otimes b) = m \otimes b$ for all simple tensors, and thus this is the identity map since simple tensors generate.

Since α has a left inverse, it follows that α is injective, so $\operatorname{im}(1_M \otimes i)$ is equal to the kernel of $1_M \otimes p$. $\square$

2.4.3. *Examples of tensors.*

Proposition 2.57. *Let R be a ring. There is a natural isomorphism between $R \otimes_R -$ and the identity functor on $R-$* **Mod**. *In particular, for every R-module M, there is an R-module isomorphism $R \otimes_R M \cong M$ for every (left) R-module M.*

Proof. Note that R is an (R, R)-bimodule, so $R \otimes_R M$ is again an R-module. Now,

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\longmapsto rm \end{aligned}$$

is biadditive (by distributive laws), R-balanced (by associativity module axiom), and R-linear, so it induces a homomorphism of R-modules $R \otimes_R M \xrightarrow{\varphi_M} M$. By construction, φ_M is surjective. Moreover, the map

$$\begin{aligned} M &\xrightarrow{f_M} R \otimes_R M \\ m &\longmapsto 1 \otimes m \end{aligned}$$

is a homomorphism of R-modules, since

$$f_M(a + b) = 1 \otimes (a + b) = 1 \otimes a + 1 \otimes b$$

$$f_M(ra) = 1 \otimes (ra) = r \otimes a = r(1 \otimes a) = r f_M(a).$$

For every $m \in M$, $\varphi_M f_M(m) = \varphi_M(1 \otimes m) = 1m = m$, and for every simple tensor, $f_M \varphi_M(r \otimes m) = f_M(rm) = 1 \otimes (rm) = r \otimes m$. This shows that φ_M is an isomorphism.

Finally, given any $f \in \mathrm{Hom}_R(M, N)$, since f is R-linear we conclude that the diagram

$$\begin{tikzcd} R \otimes_R M \arrow[r, "\varphi_M"] \arrow[d, "1\otimes f"'] & M \arrow[d, "f"] \\ R \otimes N \arrow[r, "\varphi_N"'] & N \end{tikzcd}$$

commutes, as $r \otimes m \mapsto rf(m)$ either way, so our isomorphism is natural. □

Proposition 2.58. *Let R be a ring, $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of right R-modules, N be a left R-module. There is an isomorphism*

$$\phi : \left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) \otimes_R N \xrightarrow{\cong} \bigoplus_{\lambda\in\Lambda} (M_\lambda \otimes_R N)$$

that sends $(m_i)_{i\in I} \otimes n$ to $(m_i \otimes n)_{i\in I}$. This is an isomorphism of abelian groups in general, of R-modules in the commutative case, of S-modules if each M_λ is an (S, R)-bimodule, and of right S-modules if N is an (R, S)-bimodule.

Proof. Define

$$b : \left(\bigoplus_{\lambda\in\Lambda} M\right) \times N \to \bigoplus_{\lambda\in\Lambda} (M_\lambda \otimes_R N)$$

by

$$b((m_\lambda), n) = (m_\lambda \otimes n).$$

The map b is R-balanced biadditive in general, and linear with respect to the specified action in each of the other cases. Thus, it induces a morphism ϕ of the specified type.

To show ϕ is an isomorphism, we construct an inverse. For each i we define a pairing

$$b_\lambda : M_\lambda \times N \to \left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) \otimes N$$

by $b_\lambda(x, n) = \iota_\lambda(x) \otimes n$, where $\iota_\lambda : M_\lambda \to (\bigoplus_{\lambda\in\Lambda} M_\lambda)$ is the canonical inclusion map. Then b_λ is R-balanced biadditive in general, and linear with respect to the specified action in each of the other cases and hence induces a morphism $\psi_i : M_\lambda \otimes_R N \to (\bigoplus_{\lambda\in\Lambda} M_\lambda) \otimes N$.

By the universal mapping property for coproducts the maps $\psi_\lambda, \lambda \in \Lambda$ determine a morphism

$$\psi : \bigoplus_{\lambda\in\Lambda} (M_\lambda \otimes_R N) \to \left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) \otimes N.$$

It is easy to see that both $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps by observing that they act as the identity on simple tensors. □

Remark 2.59. The same property holds on the right side of the tensor.

Example 2.60. If $F = R^{\oplus\Lambda}$ is a free module, and M is any R-module, then $R^{\oplus\Lambda} \otimes_R M \cong M^{\oplus\Lambda}$, and this isomorphism is natural in M.

Example 2.61. As a special case, $R^{\oplus\Gamma} \otimes_R R^{\oplus\Lambda}$ is a free module on the basis $\{e_\gamma \otimes e_\lambda \mid (\gamma, \lambda) \in \Gamma \times \Lambda\}$.

Even more concretely, if K is a field, $K^m \otimes_K K^n \cong K^{m\times n}$ is isomorphic to the collection of $m \times n$ matrices, by the isomorphism that takes $e_i \otimes e_j$ to the matrix that has a 1 in the i, j entry and zeroes elsewhere. This morphism then sends $(a_1, \ldots, a_m) \otimes (b_1, \ldots, b_n)$ to $[a_i b_j]$, the outer product of these matrices. Observe that the simple tensors correspond exactly to the matrices of rank at most one.

Remark 2.62. Let R be a ring, M be a right R-module, and N be a left R-module. We can compute $M \otimes_R N$ by taking a presentation of M

$$R^{\oplus \Gamma} \xrightarrow{\phi} R^{\oplus \Lambda} \to M \to 0$$

and tensoring with N to get

$$N^{\oplus \Gamma} \to N^{\oplus \Lambda} \to M \otimes_R N \to 0,$$

so $M \otimes_R N$ is isomorphic to the cokernel of the map $N^{\oplus \Gamma} \to N^{\oplus \Lambda}$ induced by ϕ. We can also compute $M \otimes_R N$ by taking a presentation of M

$$R^{\oplus \Xi} \xrightarrow{\psi} R^{\oplus \Omega} \to N \to 0$$

and tensoring with M to get

$$M^{\oplus \Xi} \to M^{\oplus \Omega} \to M \otimes_R N \to 0,$$

so $M \otimes_R N$ is isomorphic to the cokernel of the map $M^{\oplus \Gamma} \to M^{\oplus \Lambda}$ induced by ψ.

Example 2.63. Let R be a commutative ring, I an ideal, and M a module. There is an isomorphism $R/I \otimes_R M \cong M/IM$. Indeed, if $I = (\{f_\gamma\})$, then we have a presentation

$$R^{\oplus \Gamma} \xrightarrow{\cdot[\{f_\gamma\}]} R \to R/I \to 0,$$

so

$$M^{\oplus \Gamma} \xrightarrow{\cdot[\{f_\gamma\}]} M \to R/I \otimes_R M \to 0$$

is exact. The image of the first map is just IM, so we obtain the isomorphism.

Lecture of September 29, 2021

Tensor is not exact in general.

Example 2.64. Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and apply $- \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. We obtain the complex

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0,$$

which is exact at the last two $\mathbb{Z}/2\mathbb{Z}$'s but not at the first one.

The following properties are also useful properties for computing tensors.

Optional Exercise 2.65. Let R be commutative and M and N be R-modules. There is an isomorphism $M \otimes_R N \cong N \otimes_R M$.

Optional Exercise 2.66. Let R and S be rings. Let L be a right R-modules, M be an (R, S)-bimodule, and N be an S-module. Then $(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$.

An important case of the tensor functor is tensoring with a ring.

Definition 2.67. Let $\phi : R \to S$ be a ring homomorphism. The functor

$$S \otimes_R - : R - \mathbf{Mod} \to S - \mathbf{Mod}$$

is called the functor of *extension of scalars* from R to S.

Observe that S is an (S, R)-bimodule, so this functor does indeed return S-modules. By the discussion above, extension of scalars turns an R-module into the S-module with the same presentation.

2.4.4. *Hom tensor adjointess.* The Hom and tensor functors are closely related.

Theorem 2.68. *Let R, S be rings, and A be an (R, S)-bimodule, B be an S-module, and C be and R-module. There is an isomorphism*

$$\mathrm{Hom}_R(A \otimes_S B, C) \cong \mathrm{Hom}_S(B, \mathrm{Hom}_R(A, C)).$$

Moreover, these isomorphisms are natural in each argument.

Proof. Take

$$\eta : \mathrm{Hom}_R(A \otimes_S B, C) \to \mathrm{Hom}_S(B, \mathrm{Hom}_R(A, C))$$

by the rule $\eta(\phi)(b)(a) = \phi(a \otimes b)$. We check

The map $\eta(\phi)(b)$ that sends $a \mapsto \phi(a \otimes b)$ for fixed b is R-linear: addition is fine and

$$\eta(\phi)(b)(ra) = \phi(ra \otimes b) = \phi(r(a \otimes b)) = r\phi(a \otimes b) = r\eta(\phi)(b)(a).$$

The map $\eta(\phi)$ that sends $b \mapsto \phi(- \otimes b)$ is S-linear: addition is fine and

$$\eta(\phi)(sb)(a) = \phi(a \otimes sb) = \phi(as \otimes b) = (s\eta(\phi)(b))(a).$$

Now take

$$\mu : \mathrm{Hom}_S(B, \mathrm{Hom}_R(A, C)) \to \mathrm{Hom}_R(A \otimes_S B, C)$$

by the rule $\mu(\psi)(a \otimes b) = \psi(b)(a)$. We need to check that $\mu(\psi)$ is well-defined and R-linear: to do this we check that the map $A \times B \to C$ given by $(a, b) \mapsto \psi(b)(a)$ is S-balanced biadditive and R-linear on the left factor (omitted).

We then see that μ and η are mutually inverse:

$$(\mu \circ \eta)(\phi)(a \otimes b) = \eta(\phi)(b)(a) = \phi(a \otimes b)$$

$$(\eta \circ \mu)(\psi)(b)(a) = \mu(\psi)(a \otimes b) = \psi(b)(a).$$

What do the naturality claims mean? First, this is a natural isomorphism as a functor of A:

$$\mathrm{Hom}_R(- \otimes_S B, C) \overset{\cong}{\Longrightarrow} \mathrm{Hom}_S(B, \mathrm{Hom}_R(-, C)),$$

and likewise for B and C. We won't write these out, but they are straghtforward. □

Hom-tensor adjunction has a nice consequence in terms of extension of scalars.

Definition 2.69. Let $\phi : R \to S$. There is a functor

$$\mathrm{Res}_\phi : S - \mathbf{Mod} \to R - \mathbf{Mod}$$

called the functor of *restriction of scalars* that maps an S-module M to the R-module that is the same abelian group as M with action $r \cdot m = \phi(r)m$, and is the identity mapping on morphisms.

When ϕ is injective, this restriction is literally just restricting the action. Evidently, this functor is exact, as it does nothing on the level of abelian groups, and exactness can be characterized there.

Proposition 2.70. *Let $\phi : R \to S$ be a ring homomorphism. Let M be an R-module and N be an S-module. There is an isomorphism*

$$\mathrm{Hom}_R(M, \mathrm{Res}_\phi(N)) \cong \mathrm{Hom}_S(S \otimes_R M, N).$$

These isomorphisms are natural in M and in N.

Proof. Consider S as an (S, R)-bimodule, where the right action is through ϕ. With this structure, $\mathrm{Hom}_S(S, N) \cong \mathrm{Res}_\phi(N)$. Thus, this follows from Hom-tensor adjunction. $\square$

Lecture of October 1, 2021

2.4.5. *Multilinear maps.* Let R be a commutative ring. Associativity of tensor implies that for any finite set of modules $M_1, \dots, M_n$, we can tensor them all together and it doesn't matter how we group them.

Observe that for any R-modules M and N, if M is generated by $m_1, \dots, m_a$ and N is generated by $n_1, \dots, n_b$, then $M \otimes_R N$ is generated by $\{m_i \otimes n_j \mid i = 1, \dots, a; j = 1, \dots, b\}$: we can write any element as a sum of simple tensors, and write each simple tensor $m \otimes n = (\sum_i r_i m_i) \otimes (\sum_j s_j b_j) = \sum_{i,j} r_i s_j (m_i \otimes n_j)$.

Likewise, by a straightforward induction on n, in $M_1 \otimes_R \cdots \otimes_R M_n$, every element is a sum of simple tensors, and an R-linear combination of simple tensors of generators of the modules M_i.

Definition 2.71. Let R be a commutative ring, and $M_1, \dots, M_n, N$ be R-modules. We say that a map $\psi : M_1 \times \cdots \times M_n \to N$ is *multilinear* or *R-multilinear* if it is R-linear in each argument: i.e., for each i,

$$\psi(m_1, \dots, rm_i + m_i', \dots, m_n) = r\psi(m_1, \dots, m_i, \dots, m_n) + \psi(m_1, \dots, m_i', \dots, m_n).$$

Note that when $n = 2$, this is just the notion of R-bilinear.

Proposition 2.72. *There is a multilinear map*

$$h : M_1 \times \cdots \times M_n \to M_1 \otimes_R \cdots \otimes_R M_n$$

that satisfies the following universal property: for any multilinear map $\psi : M_1 \times \cdots \times M_n \to N$, there is an R-linear map $\alpha : M_1 \otimes_R \cdots \otimes_R M_n \to N$ such that $\psi = \alpha \circ h$.

Proof. For the map h, we take $h(m_1, \dots, m_n) = m_1 \otimes \cdots \otimes m_n$. Then, if such a map α exists, we must have $\alpha(m_1 \otimes \cdots \otimes m_n) = \psi(m_1, \dots, m_n)$; since simple tensors generate, α is unique if it exists. For existence, we can proceed by induction on n. For any fixed $m_n \in M_n$, the map ψ is a multilinear map on the first $n-1$ arguments, so by the inductive hypothesis, we obtain an R-linear map $M_1 \otimes_R \cdots \otimes_R M_{n-1} \to N$ that sends $m_1 \otimes \cdots \otimes m_{n-1} \mapsto \psi(m_1, \dots, m_n)$. Since we have such a map for each m_n, we get a map $M_1 \otimes_R \cdots \otimes_R M_{n-1} \times M_n \to N$ that we can check to be bilinear, and this induces the desired map. $\square$

2.4.6. *Tensor products of rings.*

Proposition 2.73. *Let A be a commutative ring, and R and S be commutative A-algebras. Then the tensor product $R \otimes_A S$ is a commutative ring, where the multiplication on simple tensors is given by $(r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'$.*

Proof. We need to show that there is a well-defined map that corresponds to this formula for multiplication. Note that the map

$$\begin{aligned} R \times S \times R \times S &\longrightarrow R \otimes_A S \\ (r, s, r', s') &\longmapsto rr' \otimes ss' \end{aligned}$$

is multilinear over A. Thus, we get a well defined map

$$\begin{aligned} R \otimes_A S \otimes_A R \otimes_A S &\longrightarrow R \otimes_A S \\ r \otimes s \otimes r' \otimes s' &\longmapsto rr' \otimes ss'. \end{aligned}$$

Thinking of $R \otimes_A S \otimes_A R \otimes_A S = (R \otimes_A S) \otimes_A (R \otimes_A S)$ and precomposing with the natural map from product to tensor product, we get a well defined A-bilinear map

$$\begin{aligned}(R \otimes_A S) \times (R \otimes_A S) &\longrightarrow R \otimes_A S \\ (r \otimes s, r' \otimes s') &\longmapsto rr' \otimes ss'.\end{aligned}$$

The bilinearity of this map translates into the distributive laws. Commutativity and associativity of multiplication can be checked on simple tensors, since these generate, and for each these follow from the same properties in R and S. $1 \otimes 1$ is an evident multiplicative identity. □

Optional Exercise 2.74. If R and S are commutative rings, then $R \otimes_{\mathbb{Z}} S$ is the coproduct of R and S in the category of commutative rings. Moreover, if R and S are commutative A-algebras, then $R \otimes_A S$ is the coproduct of R and S in the category of commutative A-algebras.

Proposition 2.75. *If A is a commutative ring, and R is an A-algebra, then $A[x_1, \dots, x_n] \otimes_A R \cong R[x_1, \dots, x_n]$ as rings.*

Proof. Consider the map $A[x_1, \dots, x_n] \times R \to R[x_1, \dots, x_n]$ given by $(f(\mathbf{x}), r) \mapsto rf(\mathbf{x})$. This is A-bilinear, so we get an induced map on the tensor product. It is evidently additive, and also clearly multiplicative, so it is a ring homomorphism.

Consider the structure of $A[x_1, \dots, x_n]$ as an A-module. Every element is an A-linear combination of monomials in a unique way, so the monomials form a free basis. Similarly for $R[x_1, \dots, x_n]$. Thus, we have

$$A[x_1, \dots, x_n] \otimes_A R = (\bigoplus_{\alpha} Ax^{\alpha}) \otimes_A R \cong \bigoplus_{\alpha} Rx^{\alpha} = R[x_1, \dots, x_n],$$

where the middle isomorphism is the extension of scalars isomorphism that sends $x^{\alpha} \otimes 1$ to x^{α}, so this isomorphism is the same map considered above; hence our map is an isomorphism. □

Example 2.76. $A[x] \otimes_A A[x] = A[x, y]$.

Proposition 2.77. *If A is a commutative ring, R is an A-algebra, and $S = A[x_1, \dots, x_n]/I$ is an A-algebra, then*

$$R \otimes_A S \cong \frac{R[x_1, \dots, x_n]}{IR[x_1, \dots, x_n]}.$$

Lecture of October 4, 2021

2.5. Projective, injective, and flat modules.

2.5.1. *Projective modules.*

Definition 2.78. An R-module P is *projective* if given any surjective homomorphism of modules $p : N \twoheadrightarrow N''$ and a homomorphism $f : P \to N''$, there is a homomorphism $g : P \to N$ such that $p \circ g = h$. In other words, given the solid arrows in the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g"'] \arrow[d, "f"] & \\ N \arrow[r, "p"] & N'' \arrow[r] & 0 \end{tikzcd}$$

in which the bottom row is exact, there exists at least one dotted arrow that causes the triangle to commute.

Proposition 2.79. *Every free R-module is projective.*

Proof. Suppose P is free with basis B and let a diagram as in the definition be given. Since p is surjective, for each $b \in B$, we can find an element $n_b \in N$ such that $f(b) = p(n_b)$. Since B is a basis, the assignment $b \mapsto n_b$ extends uniquely to an R-module homomorphism $g : P \to N$. The triangle commutes since $p \circ g$ and f agree on B. $\square$

We will see soon that the converse is false.

Proposition 2.80. *For a ring R and module P, the following are equivalent:*

(1) *P is projective,*
(2) *the functor $\mathrm{Hom}_R(P, -)$ is exact,*
(3) *every short exact sequence of the form $0 \to N' \to N \to P \to 0$ is split,*
(4) *every surjective R-module homomorphism $p : N \twoheadrightarrow P$ has a right inverse, and*
(5) *P is a summand of a free R-module; i.e., there is an R-module Q such that $F = P \oplus Q$ is a free R-module.*

Proof. Since $\mathrm{Hom}_R(P, -)$ is left exact for any module P, $\mathrm{Hom}_R(P, -)$ is exact if and only if it preserves surjections. The definition of "projective" is just an unpackaging of the property that $\mathrm{Hom}_R(P, -)$ preserves surjections. The equivalence of (1) and (2) is thus essentially by definition.

The equivalence of (3) and (4) follows from the Splitting Theorem. Note that given an surjective map $p : N \twoheadrightarrow P$, we may form the short exact sequence $0 \to \ker(p) \to N \xrightarrow{p} P \to 0$.

Suppose (1) holds and $p : N \twoheadrightarrow P$ is onto. Applying the definition with $f = \mathrm{id}_P$ and $p = p$ gives an R-linear map g such that $p \circ f = \mathrm{id}_P$. So (1) $\Rightarrow$ (4).

To see (1) implies (4), let $p : N \twoheadrightarrow P$ be surjective, and consider the identity map on P. By (1), the identity map factors through p, so p has a right inverse.

Assume (3) holds. By choosing a generating set for P (e.g., all of P) we may find a surjection $p : F \twoheadrightarrow P$ with F a free R-module. This map splits by assumption, and thus $P \oplus \ker(p) \cong F$, so that (5) holds. So (3) $\Rightarrow$ (5).

Assume (5) holds. Say $F = P \oplus Q$ is free, and let a diagram as in the definition be given. Let $\pi : F \twoheadrightarrow P$ be the canonical surjection. Since F is projective (by the example above), there is a $h : F \to N$ so that $p \circ h = f \circ \pi$. Define $g : P \to N$ to be $h \circ \iota$ where $\iota : P \to F$ sends x to $(x, 0)$. Then $p(g(x)) = p(h(x, 0)) = f(\pi(x, 0)) = f(x)$. So P is projective (i.e. (1) holds). $\square$

Remark 2.81. The proof of (5) $\Rightarrow$ (1) shows more than advertised: it shows that if P is a summand of projective R-module, then P is projective.

Example 2.82. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let P be the ideal $(2, 1 + \sqrt{-5})$. We claim P is projective as an R-module, but not free.

It's not free since an ideal in an integral domain is free as a module if and only if it is principal (exercise). And you should have seen in 818 that this ideal is not principal.

To prove it is projective I will prove it is a summand of a free module. Let

$$\pi : R^2 \twoheadrightarrow P$$

be the map given by the row vector $[2, 1 + \sqrt{-5}]$; that is $\pi(x, y) = 2x + (1 + \sqrt{-5})y$, which is clearly onto. Define $j : P \to R^2$ to be the map

$$j(z) = (-z, 3z/(1 + \sqrt{-5})).$$

The target of j really is R^2 since for $z = 2\alpha + (1+\sqrt{-5})\beta$ we have

$$j(z) = (-z, (1-\sqrt{-5})\alpha + 3\beta) \in R^2,$$

using that $3 \cdot 2 = (1-\sqrt{-5})(1+\sqrt{-5})$. We have

$$\pi(j(z)) = -2z + 3z = z;$$

that is, p is a split surjection with splitting j. It follows that

$$R^2 \cong P \oplus \ker(\pi),$$

and hence P is projective.

Example 2.83. Let

$$R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$$

and let P be the kernel of the map

$$\pi : R^3 \xrightarrow{[x,y,z]} R.$$

π is in fact a split surjection, since $\pi \circ j = \mathrm{id}_R$ where $j(r) = (xr, yr, zr)$. This also follows because R is projective. So we have

$$R^3 \cong P \oplus R$$

and in particular this shows P is projective.

It's not free; can you prove it? Tip: Hairy Ball Theorem.

Index