MATH 901 LECTURE NOTES, FALL 2021

Contents

1. Category Theory

1.1. Categories.

Lecture of August 23, 2021

1.1.1. *Definition of category.*

Definition 1.1. A *category* $\mathscr{C}$ consists of the following data:

(1) a collection of *objects*, denoted $\mathrm{Ob}(\mathscr{C})$,
(2) for each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* (also known as *arrows*) from A to B,
(3) for each triple of objects $A, B, C \in \mathrm{Ob}(\mathscr{C})$, a function

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)$$

written as $(\alpha, \beta) \mapsto \beta \circ \alpha$ that we call the *composition rule*.

These data are required to satisfy the following axioms:

(1) (Disjointness) the Hom sets are disjoint: if $A \neq A'$ or $B \neq B'$, then

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \cap \mathrm{Hom}_{\mathscr{C}}(A', B') = \varnothing.$$

(2) (Identities) for every object A, there is an *identity morphism* $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.
(3) (Associativity) composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

Remark 1.2. (1) The word "collection" as opposed to "set" is important here. The point is that there is no set of all sets, but by utilizing bigger collecting objects in set theory, we can sensibly talk about the collection of all sets. We'll sweep all of the set theory under the rug there, but it's worth keeping in mind that the objects of a category don't necessarily form a set. We did assume that the collections of morphisms between a pair of objects form a set, though not everyone does.
(2) The first axiom above guarantees that every morphism α in a category $\mathscr{C}$ has a well-defined *source* and *target* in $\mathrm{Ob}(\mathscr{C})$, namely, the unique A and B (respectively) such that $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

The name arrow dovetails with the common practice of depicting a morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ as

$$A \xrightarrow{\alpha} B.$$

The composition of $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ is $A \xrightarrow{\beta\circ\alpha} C$.

Optional Exercise 1.3. Prove that every element in a category has a unique identity morphism (i.e., a unique morphism that satisfies the hypothesis of axiom (2)).

1.1.2. *Examples of categories.* Many of our favorite objects from algebra naturally congregate in categories!

Example 1.4.

(1) There is a category **Set** where
- Ob(**Set**) is the collection of all sets
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the the set of functions from X to Y
- the composition rule is composition of functions

We observe that every set has an identity function, which behaves as an identity for composition, and that composition of functions is associative.

(2) There is a category **Grp** where
- Ob(**Grp**) is the collection of all groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Grp}}(X, Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

Note that the identity function on a group is a group homomorphism, and that a composition of two group homomorphisms is a group homomorphism.

(3) There is a category **Ab** where
- Ob(**Ab**) is the collection of all abelian groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Ab}}(X, Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

(4) In this class,
- A *semigroup* is a set S with an associative operation $\cdot$ that has an identity element; some may prefer the term *monoid*, but I don't.
- A *semigroup homomorphism* from semigroups $S \to T$ is a function that preserves the operation and maps the identity element to the identity element.

There is a category **Sgrp** where the objects are all semigroups and the morphisms are semigroup homomorphisms. (The composition rule is composition again.)

(5) In this class,
- A *ring* is a set R with two operations $+$ and $\cdot$ such that $(R, +)$ is abelian group, with identity 0, and $(R, \cdot)$ is a semigroup with identity 1, and such that the left and right distributive laws hold: $(r + s)t = rt + st$ and $t(r + s) = tr + ts$.
- A *ring homomorphism* is a function that preserves $+$ and $\cdot$ and sends 1 to 1.

There is a category **Ring** where the objects are all rings and the morphisms are ring homomorphisms.

(6) Let R be a ring. In this class,
- A *left R-module* is an abelian group $(M, +)$ equipped with a pairing $R \times M \to M$, written $(r, m) \mapsto rm$ or $(r, m) \mapsto r \cdot m$ such that
  (a) $r_1(r_2m) = (r_1r_2)m$,
  (b) $(r_1 + r_2)m = r_1m + r_2m$,
  (c) $r(m_1 + m_2) = rm_1 + rm_2$, and
  (d) $1m = m$.
- A *left module homomorphism* or *R-linear map* between left R-modules $\phi : M \to N$ is a homomorphism of abelian groups from $(M, +) \to (N, +)$ such that $\phi(rm) = r\phi(m)$.

There is a category $R-\mathbf{Mod}$ where the objects are all left R-modules and the morphisms are R-linear maps.

(7) There is a category **Fld** where the objects are all fields and the morphisms are all field homomorphisms.

(8) There is a category **Top** where the objects are all topological spaces and the morphisms are all continuous functions.

Remark 1.5. There are two special cases of the category of R-modules that are worth singling out:

- Every abelian group M is a $\mathbb{Z}$-module in a unique way, by setting

$$n \cdot m = \underbrace{m + \cdots + m}_{n-\text{times}} \quad \text{and} \quad -n \cdot m = -(\underbrace{m + \cdots + m}_{n-\text{times}}) \qquad \text{for } n \geqslant 0.$$

  Thus, **Ab** is basically just $\mathbb{Z} - \mathbf{Mod}$.
- When $R = K$ happens to be a field, we are accustomed to calling K-modules *vector spaces*. Thus, we might write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

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Example 1.6. Here are some variations on the category $K - \mathbf{Vect}$.

(1) The collection of finite dimensional K-vector spaces with all linear transformations is a category; call it $K - \mathbf{vect}$.

(2) The collection of all n-dimensional K-vector spaces with all linear transformations is a category.

(3) The collection of all K-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.

(4) The collection of all K-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it's not closed under composition.

(5) The collection of all n-dimensional vector spaces with singular linear transformations is not a category, since it doesn't have identity maps.

Example 1.7. (1) There is a category $\mathbf{Set}_*$ of *pointed sets* where

- the objects are pairs (X, x) where X is a set and $x \in X$,
- for two pointed sets, the morphisms from (X, x) to (Y, y) are functions $f : X \to Y$ such that $f(x) = y$,
- usual composition.

(2) For a commutative ring A,

- A *commutative A-algebra* is a commutative ring R plus a homomorphism $\phi : A \to R$.
- Slightly more generally, an *A-algebra* is a ring R plus a homomorphism $\phi : A \to R$ such that $\phi(A)$ lies in the center of R: $r \cdot \phi(a) = \phi(a) \cdot r$ for any $a \in A$ and $r \in R$. (In the more general situation, A is still commutative but R may not be.)
- An *A-algebra homomorphism* between two A-algebras (R, ϕ) and (S, ψ) is a ring homomorphism $\alpha : R \to S$ such that $\alpha \circ \phi = \psi$.

The category of A-algebras is denoted $A - \mathbf{Alg}$, and the category of commutative A-algebras is $A - \mathbf{cAlg}$.

(3) Fix a field K, and define a category $\mathbf{Mat}_K$ as follows: the objects are the positive natural numbers $n \in \mathbb{N}_{>0}$, and $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is the set of $b \times a$ matrices with entries in K. To see this as a category, we need a composition rule. Given $B \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ and $A \in \mathrm{Hom}_{\mathscr{C}}(a, b)$, take the composition

$A \circ B \in \mathrm{Hom}_{\mathscr{C}}(a,c)$ to be the product AB. Since matrix multiplication is associative, axiom (3) holds, and the $n \times n$ identity matrix serves as an identity morphism in the sense of axiom (2). Finally, if $A \in \mathrm{Hom}_{\mathscr{C}}(a,b) \cap \mathrm{Hom}_{\mathscr{C}}(a',b')$, then A is a $b \times a$ matrix and a $b' \times a'$ matrix, so $a = a'$ and $b = b'$. Notably, the morphisms in this category are not functions.

We can also make a bunch of categories in a hands-on way as follows:

Example 1.8. Let $(P, \leqslant)$ be a poset. We define a category $\mathbf{PO}(P)$ from P as follows. The objects of $\mathbf{PO}(P)$ are just the elements of P. For each pair $a, b \in P$ with $a \leqslant b$, form a symbol f_a^b. Then we set

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a,b) = \begin{cases} \{f_a^b\} & \text{if } a \leqslant b \\ \varnothing & \text{otherwise.} \end{cases}$$

There is only one possible composition rule:

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a,b) \times \mathrm{Hom}_{\mathbf{PO}(P)}(b,c) \longrightarrow \mathrm{Hom}_{\mathbf{PO}(P)}(a,c)$$

when $a \leqslant b$ and $b \leqslant c$ we also have $a \leqslant c$, and the unique pair on the left must map to the unique element on the right, so $f_b^c \circ f_a^b = f_a^c$; when either $a \nleqslant b$ or $b \nleqslant c$, there is nothing to compose!

Each morphism f_a^b is in only one Hom set (with source a and target b). Composition is associative since there is at most one function between one element sets. For any a, $f_a^a \in \mathrm{Hom}_{\mathbf{PO}(P)}(a,a)$ is the identity morphism.

For a specific example, we can think of $\mathbb{N}_{>0}$ as a category this way. Drawing all of the morphisms would be a mess, but any morphism is a composition of the ones depicted:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots.$$

Note that the objects of this category are exactly the same as in Example 1.7(3), but with much fewer morphisms!

Example 1.9. A category with one object is nothing but a semigroup.

1.1.3. *Constructions of categories.* Here are a few more basic constructions of categories:

Definition 1.10. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{\mathrm{op}}$ is the category with $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$, and $\mathrm{Hom}_{\mathscr{C}}(A,B) = \mathrm{Hom}_{\mathscr{C}}(B,A)$ for all $A, B \in \mathrm{Ob}(\mathscr{C})$.

That is, the opposite category is the "same category with the arrows reversed." To avoid confusion, we might write α^{op} for the morphism $B \xrightarrow{\alpha^{\mathrm{op}}} A$ in $\mathscr{C}^{\mathrm{op}}$ corresponding to $A \xrightarrow{\alpha} B$ in $\mathscr{C}$.

Definition 1.11. Given two categories $\mathscr{C}$ and $\mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ is the category with $\mathrm{Ob}(\mathscr{C} \times \mathscr{D})$ given by the collection of pairs (C, D) with $C \in \mathrm{Ob}(\mathscr{C})$ and $D \in \mathrm{Ob}(\mathscr{D})$, and $\mathrm{Hom}_{\mathscr{C}\times\mathscr{D}}((A,B),(C,D)) = \mathrm{Hom}_{\mathscr{C}}(A,C) \times \mathrm{Hom}_{\mathscr{D}}(B,D)$. We leave it to you to pin down the composition rule.

Definition 1.12. A category $\mathscr{D}$ is a *subcategory* of another category $\mathscr{C}$ provided

(1) every object of $\mathscr{D}$ is an object of $\mathscr{C}$
(2) for every $A, B \in \mathrm{Ob}(\mathscr{D})$, $\mathrm{Hom}_{\mathscr{D}}(A,B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A,B)$, and
(3) for every $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ in $\mathscr{D}$, the composition of α and β in $\mathscr{D}$ equals the composition of α and β in $\mathscr{C}$.

If equality hold in (2) (for all A, B), we say that $\mathscr{D}$ is a *full subcategory* of $\mathscr{C}$.

Example 1.13. Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, since not every function between groups is a homomorphism, **Grp** is not a full subcategory of **Set**. Similarly, **Ab**, **Ring**, $R-$**Mod**, and **Top** are all subcategories of **Set**.

On the other hand, **Ab** is a full subcategory of **Grp**, and **Grp** is a full subcategory of **Sgrp**: a morphism of abelian groups is a morphism of groups that happens to be between abelian groups (and likewise for groups and semigroups)!

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Definition 1.14. Given a category $\mathscr{C}$ and an object $A \in \mathrm{Ob}(\mathscr{C})$, the *coslice category* $\mathscr{C}_A$ is the category with objects given by all arrows from A: namely morphisms of the form

$$A \xrightarrow{\alpha} X \quad \text{with} \quad X \in \mathrm{Ob}(\mathscr{C})$$

and morphisms

$$\mathrm{Hom}_{\mathscr{C}_A}(A \xrightarrow{\alpha} X, A \xrightarrow{\beta} Y) = \{X \xrightarrow{\gamma} Y \mid \gamma \circ \alpha = \beta\}.$$

Equivalently, we can think of such a morphism as a triangle:

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, "\gamma"] & & Y \end{tikzcd}$$

The composition rule comes from the composition rule on $\mathscr{C}$: if $X \xrightarrow{\gamma} Y$ and $Y \xrightarrow{\delta\circ\gamma} Z$.

This construction may be less new to you than you think!

Optional Exercise 1.15. Explain how $A-$**cAlg** and **Set**$_*$ are special cases of the coslice category construction. Can you describe the category $A-$**Alg** in terms of the terms above?

1.2. Basic notions with morphisms.

Definition 1.16. A *diagram* in a category $\mathscr{C}$ is a directed multigraph whose vertices are objects in $\mathscr{C}$ and whose arrows/edges are morphisms in $\mathscr{C}$. A *commutative diagram* in $\mathcal{C}$ is a diagram in which for each pair of vertices A, B, any two paths from A to B compose to the same morphism.

Example 1.17. To say that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

commutes is to say that $\beta \circ \alpha = \delta \circ \gamma$ in $\mathrm{Hom}_{\mathscr{C}}(A, D)$.

Definition 1.18. Let $\mathscr{C}$ be any category and $A \xrightarrow{\alpha} B$ a morphism.

- α is an *isomorphism* if there exists $B \xrightarrow{\beta} A$ such that $\beta \circ \alpha = 1_A$ and $\alpha \circ \beta = 1_B$. Such an β is called the *inverse* of f.
- α has β as a *left inverse* if $\beta \circ \alpha = 1_A$. Similarly define *right inverse*.
- α is a *monomorphism* or is *monic* if for all arrows

$$\begin{tikzcd} C \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & A \arrow[r, "\alpha"] & B \end{tikzcd}$$

if $\alpha\beta_1 = \alpha\beta_2$ then $\beta_1 = \beta_2$. That is, α can be cancelled from the left.

- α is an *epimorphism* or is *epic* if for all arrows

$$A \xrightarrow{\alpha} B \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} C$$

if $\beta_1\alpha = \beta_2\alpha$ then $\beta_1 = \beta_2$. That is, α can be cancelled from the right.

Remark 1.19. Note that α has a left inverse in $\mathscr{C}$ if and only if α^{op} has a right inverse in $\mathscr{C}^{\mathrm{op}}$, and that α is monic in $\mathscr{C}$ if and only if α^{op} is epic in $\mathscr{C}^{\mathrm{op}}$. We say that these are *dual* notions in category theory.

Lemma 1.20. *If α has a left inverse, then α is monic. Similarly for "right inverse" and "epic".*

Proof. If $\beta \circ \alpha = 1_A$ and γ_1, γ_2 are two morphisms from $C \to A$ such that $\alpha \circ \gamma_1 = \alpha \circ \gamma_2$, then

$$\gamma_1 = (\beta \circ \alpha) \circ \gamma_1 = \beta \circ (\alpha \circ \gamma_1) = \beta \circ (\alpha \circ \gamma_2) = (\beta \circ \alpha) \circ \gamma_2 = \gamma_2.$$

Similarly for "right inverse" and "epic". □

Example 1.21. In **Set**, the monomorphisms and left-invertible morphisms agree, and these are the injective functions. The epimorphisms and right-invertible morphisms agree, and there are the surjective functions.

Optional Exercise 1.22. For any poset P, in $\mathbf{PO}(P)$, every morphism is both monic and epic, but no nonidentity morphism has a left or right-inverse.

1.3. **Products and coproducts.** A property or construction is *category theoretic* if can be described just in terms of the data of the category rather than aspects of a particular category.

Example 1.23. Can we identify $\varnothing$ in **Set** without looking at the objects' and morphisms' names? We can: for every set S, there is a unique function $f : \varnothing \to S$; $\varnothing$ is the only set with this property.

Definition 1.24.
(1) An object X in a category $\mathscr{C}$ is *initial* if there for every $Y \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $X \to Y$.
(2) An object X in a category $\mathscr{C}$ is *terminal* if there for every $Y \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $Y \to X$.

Example 1.25.
(1) We just saw that $\varnothing$ is initial in **Set**. Any singleton is terminal.
(2) A group with only one element $\{e\}$ is both initial and terminal in **Grp**.
(3) $\mathbb{Z}$ is initial in **Ring**.

1.3.1. *Definitions of product and coproduct.*

Definition 1.26. Let $\mathscr{C}$ be a category, and $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects. A *product* of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by an object P and a family of morphisms $\{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda}$ that is universal in the following sense:

Given an object Y and a family of morphisms $\{f_\lambda : Y \to X_\lambda\}_{\lambda\in\Lambda}$, there is a unique morphism $\alpha : Y \to P$ such that $p_\lambda \circ \alpha = f_\lambda$ for all λ.

Here is a diagram for the (first few) maps involved when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & P \arrow[dr, "p_1"] \arrow[ddr, "p_2"] \arrow[dddr, "p_3"] \arrow[ddddr] & \\
 & & X_1 \\
Y \arrow[uur, dashed, "\alpha"] \arrow[urr, "f_1"] \arrow[rr, "f_2"] \arrow[drr, "f_3"] \arrow[ddrr] & & X_2 \\
 & & X_3 \\
 & & \vdots
\end{tikzcd}$$

We can also take a "big picture" view of this universal property:

$$\begin{tikzcd}
 & P \arrow[dr, squiggly, "\alpha"] & \\
Y \arrow[ur, dashed, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_\lambda\}"'] & & \{X_\lambda\},
\end{tikzcd}$$

where the squiggly arrows are again collections of maps instead of maps. The data of Y with a family of maps to each X_λ is the sort of thing a product might be, so we may think of it as a "product candidate." In this way, we can think of a product as a "terminal product candidate."

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Remark 1.27. Note that $(P, \{p_\lambda\}_{\lambda\in\Lambda})$ is a product of $\{X_\lambda\}_{\lambda\in\Lambda}$ if and only if the function

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{C}}(Y, P) &\longrightarrow \bigtimes_{\lambda\in\Lambda} \operatorname{Hom}_{\mathscr{C}}(Y, X_\lambda) \\
\alpha &\longmapsto (p_\lambda \circ \alpha)_{\lambda\in\Lambda}
\end{aligned}$$

is a bijection: the universal property says that everything in the target comes from a unique thing in the source.

Definition 1.28. Let $\mathscr{C}$ be a category, and $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects. A *coproduct* of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by an object C and a family of morphisms $\{i_\lambda : X_\lambda \to Y\}_{\lambda\in\Lambda}$ that is universal in the following sense:

Given an object Y and a family of morphisms $\{f_\lambda : X_\lambda \to Y\}_{\lambda\in\Lambda}$, there is a unique morphism $\alpha : C \to Y$ such that $\alpha \circ i_\lambda = f_\lambda$ for all λ.

Here is a diagram for the (first few) maps involved when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & C \arrow[ddr, dashed, "\alpha"] & \\
\vdots \arrow[ur] \arrow[drr] & & \\
X_3 \arrow[uur, "i_3"] \arrow[rr, "f_3"] & & Y \\
X_2 \arrow[uuur, "i_2"] \arrow[urr, "f_2"] & & \\
X_1 \arrow[uuuur, "i_1"] \arrow[uurr, "f_1"] & &
\end{tikzcd}$$

We can also take a "big picture" view of the universal property:

$$\begin{tikzcd}
 & C \arrow[dr, dashed, "\alpha"] & \\
\{X_\lambda\} \arrow[ur, squiggly, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_\lambda\}"'] & & Y,
\end{tikzcd}$$

where the squiggly arrows are now collections of maps instead of maps. We can again think of the coproduct as the "initial coproduct candidate."

Remark 1.29. Note that $(C, \{i_\lambda\}_{\lambda\in\Lambda})$ is a coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ if and only if the function

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{C}}(C, Y) &\longrightarrow \textstyle\bigtimes_{\lambda\in\Lambda} \operatorname{Hom}_{\mathscr{C}}(X_\lambda, Y) \\
\alpha &\longmapsto (\alpha \circ i_\lambda)_{\lambda\in\Lambda}
\end{aligned}$$

is a bijection: the universal property says that everything in the target comes from a unique thing in the source.

Proposition 1.30. *If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda\in\Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a category $\mathscr{C}$, then there is a unique isomorphism $\alpha : P \xrightarrow{\sim} P'$ such that $p'_\lambda \circ \alpha = p_\lambda$ for all λ. The analogous statement holds for coproducts.*

Proof. We will just deal with products. The following picture is a rough guide:

$$\begin{tikzcd}
P \arrow[r, dashed, "\alpha"] \arrow[drrr, squiggly, "\{p_\lambda\}"'] & P' \arrow[r, dashed, "\beta"] \arrow[drr, squiggly, "\{p'_\lambda\}"] & P \arrow[dr, squiggly, "\{p_\lambda\}"] & \\
 & & & \{X_\lambda\}
\end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta : P' \to P$ such that $p_\lambda \circ \beta = p'_\lambda$. Switching roles, we obtain a unique map $\alpha : P \to P'$ such that $p'_\lambda \circ \alpha = p_\lambda$.

Consider the composition $\beta \circ \alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ for all λ. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ for all λ, so by the uniqueness property of products, $\beta \circ \alpha = 1_P$. We can again switch roles to see that $\alpha \circ \beta = 1_{P'}$. Thus α is an isomorphism. The uniqueness of α in the statement is part of the universal property. $\square$

Optional Exercise 1.31. Prove the analogous statement for coproducts.

Remark 1.32. If we drop the "uniqueness" clause in the definition of product, the notion we get is not well-defined up to isomorphism on objects. For example, let X be an object in **Set**. We claim that any set Z and surjective function $p : Z \to X$ satisfies the "existence" part of the "exists a unique" criterion in the definition of product for the family consisting of the single set X in **Set**. Indeed, given any set Y and function $f : Y \to X$, there is a map $\phi : Y \to Z$ given by taking $\phi(y)$ to be any preimage of $f(y)$ under p. Note that if p is not bijective, then there are different functions we can choose this way (as long as there is an element in the image of ϕ with multiple preimages under f), so Z is not a product unless p is a bijection.

We use the notation $\prod_{\lambda\in\Lambda} X_\lambda$ to denote the (object part of the) product of $\{X_\lambda\}$ and $\coprod_{\lambda\in\Lambda} X_\lambda$ to denote the (object part of the) coproduct of $\{X_\lambda\}$.

Observe that products and coproducts are dual notions in the same way as monic versus epic morphisms. The product of a family in $\mathscr{C}$ is the coproduct of the same family in $\mathscr{C}^{\mathrm{op}}$.

1.3.2. *Products in familiar categories.* The familiar notion of Cartesian product or direct product serves as a product in many of our favorite categories. Let's note first that given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in any of the categories $\mathbf{Set}, \mathbf{Sgrp}, \mathbf{Grp}, \mathbf{Ring}, R-\mathbf{Mod}, \mathbf{Top}$, the direct product $\times_{\lambda\in\Lambda} X_\lambda$ is an object of the same category:

- for sets, this is clear;
- for semigroups, groups, and rings, take the operation coordinate by coordinate: $(x_\lambda)_{\lambda\in\Lambda} \cdot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda}$;
- for modules, addition is coordinate by coordinate, and the action is the same on each coordinate: $r \cdot (x_\lambda)_{\lambda\in\Lambda} = (r \cdot x_\lambda)_{\lambda\in\Lambda}$;
- for topological spaces, use the product topology.

Note that this is not true for fields!

Proposition 1.33. *In each of the categories* $\mathbf{Set}, \mathbf{Sgrp}, \mathbf{Grp}, \mathbf{Ring}, R-\mathbf{Mod}, \mathbf{Top}$, *given a family* $\{X_\lambda\}_{\lambda\in\Lambda}$, *the direct product* $\times_{\lambda\in\Lambda} X_\lambda$ *along with the projection maps* $\pi_\lambda : \times_{\gamma\in\Lambda} X_\gamma \to X_\lambda$ *forms a product in the category.*

Proof. We observe that in each category, the direct product is an object, and the projection maps π_λ are morphisms in the category.

Let $\mathscr{C}$ be one of these categories, and suppose that we have morphisms $g_\lambda : Y \to X_\lambda$ for all λ in $\mathscr{C}$. We need to show there is a unique morphism $\alpha : Y \to \times_{\lambda\in\Lambda} X_\lambda$ such that $\pi_\lambda \circ \alpha = g_\lambda$ for all λ. The last condition is equivalent to $(\alpha(y))_\lambda = (p_\lambda \circ \alpha)(y) = g_\lambda(y)$ for all λ, which is equivalent to $\alpha(y) = (g_\lambda)_{\lambda\in\Lambda}$, so if this is a valid morphism, it is unique. Thus, it suffices to show that the map $\alpha(y) = (g_\lambda)_{\lambda\in\Lambda}$ is a morphism in $\mathscr{C}$, which is easy to see in each case. $\square$

Remark 1.34. We already saw that direct products are not products in the category of fields. In fact, there are no products in the category of fields in general. For example, suppose that P was a product of $\mathbb{F}_p$ and $\mathbb{F}_q$ in **Fld** for two primes $p \neq q$. Then $\mathrm{Hom}_{\mathbf{Fld}}(K, P)$ is bijective with $\mathrm{Hom}_{\mathbf{Fld}}(K, \mathbb{F}_p) \times \mathrm{Hom}_{\mathbf{Fld}}(K, \mathbb{F}_q)$ for all K, so in particular, $\mathrm{Hom}_{\mathbf{Fld}}(\mathbb{F}_p, P) \neq 0$ and $\mathrm{Hom}_{\mathbf{Fld}}(\mathbb{F}_q, P) \neq 0$, but no such field exists!

1.3.3. *Coproducts in familiar categories.*

Example 1.35. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The product of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the cartesian product along with the projection maps. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the "disjoint union" with the various inclusion maps. By disjoint union, we simply mean union if the sets are disjoint; in general do something like replace X_λ with $X_\lambda \times \{\lambda\}$ to make them disjoint.

Proposition 1.36. *Let R be a ring, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left R-modules. A coproduct for the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is $(\bigoplus_{\lambda\in\Lambda} M_\lambda, \{\iota_\lambda\}_{\lambda\in\Lambda})$, where*

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \textit{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda$$

is the direct sum of the modules M_λ, and ι_λ is the inclusion map to the λ coordinate.