# §4.14: NOETHER NORMALIZATION AND ZARISKI'S LEMMA

NOETHER NORMALIZATION: Let $K$ be a field, and $R$ be a finitely-generated $K$-algebra. Then there exists a finite[1] set of elements $f_1, \dots, f_m \in R$ that are algebraically independent over $K$ such that $K[f_1, \dots, f_m] \subseteq R$ is module-finite; equivalently, there is a module-finite injective $K$-algebra map from a polynomial ring $K[X_1, \dots, X_m] \hookrightarrow R$. Such a ring $S$ is called a **Noether normalization** for $R$.

LEMMA: Let $A$ be a ring, and $F \in R := A[X_1, \dots, X_n]$ be a nonzero polynomial. Then there exists an $A$-algebra automorphism $\phi$ of $R$ such that $\phi(F)$, viewed as a polynomial in $X_n$ with coefficients in $A[X_1, \dots, X_{n-1}]$, has top degree term $aX_n^t$ for some $a \in A \smallsetminus 0$ and $t \geq 0$.

- If $A = K$ is an infinite field, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + \lambda_i X_n$ for some $\lambda_1, \dots, \lambda_{n-1} \in K$.
- In general, if the top degree of $F$ (with respect to the standard grading) is $D$, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + X_n^{D^{n-i}}$ for $i < n$.

ZARISKI'S LEMMA: An algebra-finite extension of fields is module-finite.

USEFUL VARIATIONS ON NOETHER NORMALIZATION:

- NN FOR DOMAINS: Let $A \subseteq R$ be a module-finite inclusion of domains[2]. Then there exists $a \in A \smallsetminus 0$ and $f_1, \dots, f_m \in R[1/a]$ that are algebraically independent over $A[1/a]$ such that $A[1/a][f_1, \dots, f_m] \subseteq R[1/a]$ is module-finite.
- GRADED NN: Let $K$ be an infinite field, and $R$ be a standard graded $K$-algebra. Then there exist algebraically independent elements $L_1, \dots, L_m \in R_1$ such that $K[L_1, \dots, L_m] \subseteq R$ is module-finite.
- NN FOR POWER SERIES: Let $K$ be an infinite field, and $R = K[\![X_1, \dots, X_n]\!]/I$. Then there exists a module-finite injection $K[\![Y_1, \dots, Y_m]\!] \hookrightarrow R$ for some power series ring in $m$ variables.

**(1)** Examples of Noether normalizations: Let $K$ be a field.

**(a)** Show that $K[x, y]$ is a Noether normalization of $R = \dfrac{K[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$, where $x, y$ are the classes of $X$ and $Y$ in $R$, respectively.

**(b)** Show that $K[x]$ is *not* a Noether normalization of $R = \dfrac{K[X, Y]}{(XY)}$. Then show that $K[x + y] \subseteq R$ *is* a Noether normalization.

**(c)** Show that $K[X^4, Y^4]$ is a Noether normalization for $R = K[X^4, X^3Y, XY^3, Y^4]$.

**(2)** Use Noether Normalization[3] to prove Zariski's Lemma.

[1] Possibly empty!

[2] The assumption that $R$ is a domain is actually not necessary, but can't quite state the general statement yet. We assume that $R$ is a domain so that there is fraction field of $R$ in which to take $R[1/a]$.

[3] and a suitable fact about integral extensions…

**(3)** Proof of Noether Normalization (using the Lemma): Proceed by induction on the number of generators of $R$ as a $K$-algebra; write $R = K[r_1, \dots, r_n]$.

- **(a)** Deal with the base case $n = 0$.
- **(b)** For the inductive step, first do the case that $r_1, \dots, r_n$ are algebraically independent over $K$.
- **(c)** Let $\alpha : K[X_1, \dots, X_n] \to R$ be the $K$-algebra homomorphism such that $\alpha(X_i) = r_i$, and let $\phi$ be a $K$-algebra automorphism of $K[X_1, \dots, X_n]$. Let $r'_i = \alpha(\phi(X_i))$ for each $i$. Explain[4] why $R = K[r'_1, \dots, r'_n]$, and for any $K$-algebra relation $F$ on $r_1, \dots, r_n$, the polynomial $\phi^{-1}(F)$ is a $K$-algebra relation on $r'_1, \dots, r'_n$.
- **(d)** Use the Lemma to find a $K$-subalgebra $R'$ of $R$ with $n-1$ generators such that the inclusion $R' \subseteq R$ is module-finite.
- **(e)** Conclude the proof.

(4) Proof of the "general case" of the Lemma:

- (a) Where do "base $D$ expansions" fit in this picture?
- (b) Consider the automorphism $\phi$ from the general case of the Lemma. Show that for a monomial, we have $\phi(aX_1^{d_1} \cdots X_n^{d_n})$ is a polynomial with unique highest degree term $aX_n^{d_1 D^{n-1} + d_2 D^{n-2} + \cdots + d_n}$.
- (c) Can two monomials $\mu, \nu$ in $F$, have $\phi(\mu)$ and $\phi(\nu)$ with the same highest degree term?
- (d) Complete the proof.

(5) Variations on NN.

- (a) Adapt the proof of NN to show Graded NN.
- (b) Adapt the proof of NN to show NN for domains.
- *(c)* Adapt the proof of NN to show NN for power series.

[4] Say $\alpha'$ is the $K$-algebra map given by $\alpha'(X_i) = r'_i$. Observe that $\alpha' = \alpha \circ \phi$. Why is this surjective?