FINAL EXAM

For this exam, you may use the lecture notes, your own notes, and the homework assignments, but you should not use other sources, including your classmates. However, if you plan to take a written algebra comp, I recommend that you first attempt the problems without notes, and then complete them with notes later. Turn in solutions to *four* of the problems below.

(1) Let G be a finite group. Show that G is abelian if and only if the group ring $\mathbb{C}[G]$ has no nonzero nilpotent elements.

(2) Let G be the group of order 12 with presentation $\langle a, b \,|\, a^3 = b^4 = e, bab^1 = a^{-1} \rangle$. Determine the number of isomorphism classes and the dimensions of the irreducible complex representations of G.

(3) Let K be a field and G be a finite group. There is a covariant additive functor $F : K[G] - \mathbf{Mod} \to K - \mathbf{Vect}$ that on objects sends V to $F(V) = V^G = \{v \in V \mid gv = v \text{ all } g \in G\}$, and sends a map $f : V \to W$ to its restriction from V^G to W^G.
   (a) Show that F is left exact.
   (b) Show that if the order of G is a unit in K, then $\mathbb{R}^i F(V) = 0$ for all $i > 0$ and all $K[G]$-modules V.

(4) (a) Let $P_\bullet$ be a complex of projective modules with $P_i = 0$ for all $i < 0$. Show that $P_\bullet$ is exact if and only if the identity map on $P_\bullet$ is nullhomotopic.
   (b) Let $E^\bullet$ be a complex of projective modules with $E^i = 0$ for all $i < 0$ Show that $P_\bullet$ is exact if and only if the identity map on $P_\bullet$ is nullhomotopic.
   (c) Show that the complex

$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \to \cdots$$

   of $\mathbb{Z}/4\mathbb{Z}$-modules is exact the identity map is not nullhomotopic. (Note that by an earlier homework problem, $\mathbb{Z}/4\mathbb{Z}$ is both projective and injective as an R-module.)

(5) Let R be a commutative ring, and M, N be R-modules. Show that for any $r \in R$, if $\mu_r : M \to M$ is the map of multiplication by r, then $\operatorname{Tor}_i^R(N, \mu_r)$ is the map of multiplication by r for every i.

(6) Let R be a principal ideal domain. Show that $\operatorname{Tor}_i^R(M, N) = 0$ and $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 1$ for all finitely generated modules M, N.

(7) Let R be a ring, and N be a right R-module. Show that if $\operatorname{Tor}_1^R(N, M) = 0$ for all R-modules M, then N is flat.