

NORMAL SUBGROUPS

DEFINITION: A subgroup N of a group G is **normal** if $gNg^{-1} = N$ for all $g \in G$, where $gNg^{-1} = \{gng^{-1} \mid n \in N\}$. We write $N \trianglelefteq G$ to indicate that N is a normal subgroup of G .

LEMMA: Let N be a subgroup of a group G . The following are equivalent:

- (1) N is a normal subgroup of G .
- (2) For all $g \in G$, $gNg^{-1} \subseteq N$.
- (3) For all $g \in G$, the *left coset* gN is equal to the *right coset* Ng .
- (4) For all $g \in G$, $gN \subseteq Ng$.
- (5) For all $g \in G$, $Ng \subseteq gN$.

(1) Examples of normal subgroups: Use the definition and/or the Lemma to show the following:

- (a)** If G is an abelian group and $H \leq G$, then $H \trianglelefteq G$.
- (b)** The center $Z(G)$ of a group G is a normal subgroup¹ of G .
- (c)** The² group $K = \{e, (12)(34), (13)(24), (14)(23)\} \leq S_4$ is normal.
- (d)** Let $H = \{e, (12)(34)\} \leq K$, with K as above. Check that $H \trianglelefteq K$ and $K \trianglelefteq S_4$, but $H \not\trianglelefteq S_4$. Draw a moral from this example.
- (e)** Is the subgroup of all rotations a normal subgroup of D_n ?
- (f)** Is the subgroup generated by one reflection a normal subgroup of D_n ?

(2) Prove the Lemma.

(3) Let G be a group and $H \leq G$ a subgroup of index 2. Show that H must be normal.

RECALL:

- An equivalence relation \sim on a group is **compatible with multiplication** if $x \sim y$ implies $xz \sim yz$ and $zx \sim zy$ for all $x, y, z \in G$. If \sim is compatible with multiplication, then the equivalence classes of \sim obtain a well-defined group structure via the rule $[x][y] = [xy]$.
- For a subgroup H , we define an equivalence relation on G by $x \sim_H y$ if and only if $hx = y$ for some $h \in H$. The equivalence classes are the right cosets Hx .

THEOREM: Let G be a group. An equivalence relation \sim is compatible with multiplication if and only if $\sim = \sim_N$ for some $N \trianglelefteq G$.

COROLLARY: If G is a group and N is a normal subgroup, the collection of left cosets $\{gN \mid g \in G\}$ of N forms a group by the rule $gN \cdot hN = ghN$.

(4) Explain why the Corollary follows from the Theorem.

(5) Prove the (\Leftarrow) direction of the Theorem.

(6) Prove³ the (\Rightarrow) direction of the Theorem.

¹Recall that we have already shown that $Z(G) \leq G$.

²Hint: Recall from HW 1 that $\tau(ij)\tau^{-1} = (\tau(i)\tau(j))$.

³Hint: The main issue here is to find a candidate N . Think first about how you would reconstruct N from \sim_N .