

**STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: INVARIANT FACTORS:**

Let  $G$  be a finitely generated abelian group. There exist integers  $r \geq 0$ , and  $n_i \geq 2$ , satisfying  $n_1 \mid n_2 \mid \cdots \mid n_t$  such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_t.$$

Moreover, the list  $r, n_1, \dots, n_t$  is uniquely determined by  $G$ .

**STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: ELEMENTARY DIVISORS:**

Let  $G$  be a finitely generated abelian group. Then there exist integers  $r \geq 0$ , not necessarily distinct positive prime integers  $p_1, \dots, p_s$ , and integers  $a_i \geq 1$  for  $1 \leq i \leq s$  such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/p_1^{a_1} \times \cdots \times \mathbb{Z}/p_s^{a_s}.$$

Moreover,  $r$  and  $s$  are uniquely determined by  $G$ , and the list of prime powers  $p_1^{a_1}, \dots, p_s^{a_s}$  is unique up to the ordering.

**(1) Converting between forms:**

To convert a cyclic group  $\mathbb{Z}/a$  to elementary divisor form, write each  $a = p_1^{e_1} \cdots p_s^{e_s}$  as a product of prime powers, and use CRT get

$$\mathbb{Z}/a \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_s^{e_s}.$$

**(a) Convert  $\mathbb{Z}^2 \times \mathbb{Z}/50 \times \mathbb{Z}/60$  to elementary divisor form.**

To convert a group from elementary divisor form to invariant factor form,

- For each distinct prime  $p_j$  occurring, take the largest power  $E_j$  it has in an elementary divisor, and combine and combine  $\prod_j \mathbb{Z}/p_j^{E_j} \cong \mathbb{Z}/(p_1^{E_1} \cdots p_\ell^{E_\ell})$  via CRT. If there's more than one copy of  $\mathbb{Z}/p_j^{E_j}$ , just take one of the copies and leave the rest.
- Repeat with the remaining factors.

**(b) Convert  $\mathbb{Z}^3 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/9 \times \mathbb{Z}/27 \times \mathbb{Z}/25$  to invariant factor form.**

**(2) Which of the following groups are isomorphic or not?**

- $\mathbb{Z}/5 \times \mathbb{Z}/12 \times \mathbb{Z}/36$
- $\mathbb{Z}/10 \times \mathbb{Z}/12 \times \mathbb{Z}/18$
- $\mathbb{Z}/30 \times \mathbb{Z}/54$

**(3) Classify all abelian groups of order 20 up to isomorphism. For each isomorphism class, give its expression in invariant factor form.**

**(4) Let  $p < q$  be primes.**

**(a) Show that if  $p$  does not divide  $q - 1$ , then any group of order  $pq$  is isomorphic to  $C_{pq}$  by the following steps:**

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for direct products.

**(b) Show from that if  $p$  does divide  $q - 1$ , then there are exactly two groups of order  $pq$  up to isomorphism by the following steps:**

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for semidirect products.
- Use an Exercise from class about when two semidirect products are isomorphic.

- (5) Let  $p$  be a prime integer. Let  $G$  be a group of order  $p^2$ .
- (a) Show<sup>1</sup> that  $G$  is abelian.
  - (b) Classify all groups of order  $p^2$  up to isomorphism.
- (6) Let  $p, q$  be primes such that  $q = p + 2$  and  $p \geq 5$ . Show that any group of order  $p^2q^2$  is either isomorphic to a cyclic group or a product of two cyclic groups.

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<sup>1</sup>Hint: If not, what can you say about  $Z(G)$  and  $G/Z(G)$ ?