PELL'S EQUATION AND UNITS IN $\mathbb{Z}[\sqrt{D}]$

DEFINITION: The equation $x^2 - Dy^2 = 1$ for some fixed positive integer D that is not a perfect square, where the variables x, y range through integers is called a **Pell's equation**. We say that a solution (x_0, y_0) is a **positive solution** if x_0, y_0 are both positive integers. We say that one positive solution (x_0, y_0) is **smaller** than another positive solution (x_1, y_1) if $x_0 < x_1$; equivalently, $y_0 < y_1$.

(1) Warmup with Pell's equation:
  (a) Verify that $(9, 4)$ is a solution to Pell's equation with $D = 5$.
  (b) Fix some D. Show that if (x_0, y_0) is a solution to Pell's equation, then $(\pm x_0, \pm y_0)$ are solutions to Pell's equation with the same D.
  (c) What two trivial solutions does every Pell's equation have?
  (d) Explain how to recover all solutions from just the positive solutions.

(a) $9^2 - 5 \cdot 4^2 = 81 - 5 \cdot 16 = 1\checkmark$.
(b) $(\pm x_0)^2 - D(\pm y_0)^2 = x_0^2 - Dy_0^2 = 1$.
(c) $(\pm 1, 0)$.
(d) By throwing in $(\pm 1, 0)$ and taking $\pm$ each coordinate.

(2) By trial and error find the smallest positive solutions to Pell's equation with $D = 2$, $D = 3$, and $D = 5$.

For $D = 2$ we find $(3, 2)$. For $D = 3$ we find $(2, 1)$, For $D = 5$ we find $(9, 4)$.

(3) Suppose that D is a perfect square. Show that the equation $x^2 - Dy^2 = 1$ has no positive solutions.

If $D = d^2$ with $d > 0$, then $x^2 - Dy^2 = (x - dy)(x + dy)$. For any positive integers x, y, we have $x + dy > 1$, and $x - dy \in \mathbb{Z}$, so the product cannot be 1.

DEFINITION: Let D be a positive integer that is not a perfect square. We define the **quadratic ring** of D to be

$$\mathbb{Z}[\sqrt{D}] := \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}.$$

DEFINITION: For the quadratic ring $\mathbb{Z}[\sqrt{D}]$ we define the **norm** function

$$N : \mathbb{Z}[\sqrt{D}] \to \mathbb{Z} \qquad N(a + b\sqrt{D}) = a^2 - b^2 D.$$

Note that $N(a + b\sqrt{D}) = (a + b\sqrt{D})(a - b\sqrt{D})$.

LEMMA: For the quadratic ring $\mathbb{Z}[\sqrt{D}]$ the norm function satisfies the multiplicative property $N(\alpha\beta) = N(\alpha)N(\beta)$.

(4) Warmup with $\mathbb{Z}[\sqrt{D}]$:

(a) Show[1] that $\mathbb{Z}[\sqrt{D}]$ is a ring.
(b) Show that every element in $\mathbb{Z}[\sqrt{D}]$ has a unique expression in the form $a + b\sqrt{D}$.

(a) We check the conditions for a subring: Let $a + b\sqrt{D}, c + d\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$. Then,
- $1 = 1 + 0\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$
- $(a + b\sqrt{D}) - (c + d\sqrt{D}) = (a - c) + (b - d)\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$, and
- $(a + b\sqrt{D})(c + d\sqrt{D}) = (ac + bdD) + (ad + bc)\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$.

(b) If $a + b\sqrt{D} = c + d\sqrt{D}$ and $(a, b) \neq (c, d)$, then $a - c = (d - b)\sqrt{D}$. If $a \neq c$, then we must have $b \neq d$, so either way, $b \neq d$. Then $\sqrt{D} = \frac{a-c}{d-b}$, which contradicts that $\sqrt{D}$ is irrational. Thus, $a + b\sqrt{D} = c + d\sqrt{D}$ implies $(a, b) = (c, d)$.

(5) Norms, units, and Pell's equation:
(a) Prove the Lemma above.
(b) Show that an element of $\mathbb{Z}[\sqrt{D}]$ is a unit (has a multiplicative inverse) if and only if its norm is ± 1.
(c) Show that the set of units of $\mathbb{Z}[\sqrt{D}]$ forms a group under multiplication.
(d) Show that the set of elements $a + b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ such that (a, b) is a solution to the Pell's equation $x^2 - Dy^2 = 1$ forms a group under multiplication.

(a) Set $\alpha = a + b\sqrt{D}$, $\beta = c + d\sqrt{D}$. Then $\alpha\beta = (ac + bdD) + (ad + bc)\sqrt{D}$ so
$$\begin{aligned} N(\alpha\beta) &= (ac + bdD)^2 - (ad + bc)^2 D \\ &= a^2c^2 + 2abcdD + b^2d^2D^2 - a^2 + d^2D - 2abcdD - b^2c^2D \\ &= a^2c^2 + b^2d^2D^2 - a^2d^2D - b^2c^2D. \end{aligned}$$
On the other hand,
$$N(\alpha)N(\beta) = (a^2 - b^2D)(c^2 - d^2D) = a^2c^2 - a^2d^2D - b^2c^2D + b^2d^2D^2.$$
(b) If α is a unit so $\alpha\beta = 1$ for some β, then
$$1 = N(1) = N(\alpha\beta) = N(\alpha)N(\beta),$$
so $N(\alpha)$ is a unit in $\mathbb{Z}$, hence is ± 1. Conversely, if $\alpha = a + b\sqrt{D}$ and $N(\alpha) = \pm 1$, then $(a + b\sqrt{D})(a - b\sqrt{D}) = \pm 1$, so $(a + b\sqrt{D})(\pm(a - b\sqrt{D})) = 1$, and α is a unit.
(c) The product of two elements of norm 1 has norm 1, by the lemma. The element 1 has norm 1, which serves as the identity. By the previous part, an element of norm 1 has an inverse, which must have norm 1 by the lemma.

THEOREM: Let D be a positive integer that is not a perfect square. Consider the Pell's equation $x^2 - Dy^2 = 1$. Let (a, b) be the smallest positive solution (assuming that some positive solution exists). Then every positive solution (c, d) can be obtained by the rule
$$c + d\sqrt{D} = (a + b\sqrt{D})^k$$
for some positive integer k.

[1] Recall: to check that a subset of a ring is a subring, it suffices to show that it contains the multiplicative identity and is closed under subtraction and multiplication.

(7) Use the Theorem above and your work from (2) to give a formula for all solutions to each of the Pell's equations

- $x^2 - 2y^2 = 1$
- $x^2 - 3y^2 = 1$
- $x^2 - 5y^2 = 1$

Then, for each of these, find the smallest three solutions.

For $D = 2$, the solutions are the coefficients of $(3 + 2\sqrt{2})^k$. The first three solutions are $(3, 2)$, $(17, 12)$, and $(99, 70)$.

For $D = 3$, the solutions are the coefficients of $(2 + \sqrt{3})^k$. The first three solutions are $(2, 1)$, $(7, 4)$, and $(26, 15)$.

For $D = 5$, the solutions are the coefficients of $(9 + 4\sqrt{5})^k$. The first three solutions are $(9, 4)$, $(129, 72)$, and $(2025, 1164)$.

(8) Proof of Theorem: Assume that (a, b) is the smallest positive solution to the Pell's equation $x^2 - Dy^2 = 1$.

(a) Show that pair of the form (c, d) where $c + d\sqrt{D} = (a + b\sqrt{D})^k$ is a positive solution to the same Pell's equation.

(b) Suppose that $(c, d) \neq (a, b)$ is a positive solution to Pell's equation. Show that if

$$e + f\sqrt{D} := (c + d\sqrt{D})(a - b\sqrt{D}),$$

then (e, f) is a solution to Pell's equation.

(c) Show[2] that, for e, f as in the previous part, $e, f > 0$ and $e < c$.

(d) Complete the proof of the Theorem.

(a) From the lemma, $N((a + b\sqrt{D})^k) = N(a + b\sqrt{D})^k = 1$ for all k, so all of these are solutions.

(b) We have

$$N(e + f\sqrt{D}) = N(c + d\sqrt{D})N(a - b\sqrt{D}) = N(c + d\sqrt{D})N(a + b\sqrt{D}) = 1,$$

so it is a solution.

(c) From $a^2 - b^2D = 1 > 0$, we find that $a > b\sqrt{D}$, and similarly $c > d\sqrt{D}$. Then $ac > bdD$ so $e = ac - bdD > 0$. Since $0 < a < c$, we have

$$a^2d^2 = a^2(c^2 - 1) = a^2c^2 - a^2 > a^2c^2 - c^2 = (a^2 - 1)c^2 = b^2c^2,$$

so $ad > bc$, and $f = ad - bc > 0$. Finally, we have

$$c + d\sqrt{D} = (c + d\sqrt{D})(a - b\sqrt{D})(a + b\sqrt{D}) = (e + f\sqrt{D})(a + b\sqrt{D}),$$

so $c = ae + bfD > e$.

(d) If not, let $c + d\sqrt{D}$ be the smallest positive solution not of this form. Then $e + f\sqrt{D} := (c + d\sqrt{D})(a - b\sqrt{D})$ is also not a power of $a + b\sqrt{D}$, since if $e + f\sqrt{D} = (a + b\sqrt{D})^k$, then $c + d\sqrt{D} = (e + f\sqrt{D})(a + b\sqrt{D}) = (a + b\sqrt{D})^{k+1}$, a contradiction. But by the previous part, $e + f\sqrt{D}$ is a smaller positive solution; a contradiction.

[2] For $e > 0$, note that $a > b\sqrt{D}$ and $c > d\sqrt{D}$. For $f > 0$, you might start with $a^2(c^2 - 1) > (a^2 - 1)c^2$. For $e < c$, multiply the equation above by $a + b\sqrt{D}$.

(9) Use[3] your work from (7) to give a closed formula for all solutions to the same particular Pell's equations.

[3]Hint: The coefficients of $(m + n\sqrt{2})(3 + 2\sqrt{2})$ are the entries of $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}$.