THEOREM 36.1 (CHAIN RULE): Suppose g is differentiable at s and f is differentiable at $g(s)$. Then $f \circ g$ is differentiable at s and

$$(f \circ g)'(s) = f'(g(s))g'(s).$$

(1) Use the chain rule to compute the derivative of $\sqrt{3x^2 + 13}$.

(2) Given that the derivative of $\sin(x)$ at $x = r$ is $\cos(r)$, and reusing any earlier computations, compute the derivative of the function

$$j(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at any real number $x = r$.

(3) Proof of Chain Rule:

(a) In the setting of the statement, set $r = g(s)$ and define

$$d(y) = \begin{cases} \dfrac{f(y) - f(r)}{y - r} & \text{if } y \neq r \\ f'(r) & \text{if } y = r. \end{cases}$$

Show that d is continuous at $y = r$.

(b) Show that if $x \neq s$, then

$$\frac{f(g(x)) - f(g(s))}{x - s} = d(g(x)) \cdot \frac{g(x) - g(s)}{x - s}.$$

Note: You might have to consider separately the cases with $g(x) \neq r$ and $g(x) = r$.

(c) Use the fact that a composition of continuous functions (at the suitable inputs) is continuous to compute $\lim_{x \to s} d(g(x))$.

(d) Compute $\lim_{x \to s}$ on both sides of (b). Use your computation to deduce the Chain Rule.