PROBLEM SET #1

(1) (a) Prove the generalized product rule for derivations: if $\partial : R \to M$ is a derivation, then $\partial(a_1 \cdots a_n) = \sum_{i=1}^{n} (\prod_{j \neq i} a_i) \partial(a_j)$.
  (b) Prove the power rule for derivations: if $\partial : R \to M$ is a derivation, then $\partial(r^n) = nr^{n-1}\partial(r)$.
  (c) Show that if R is a ring of characteristic p, then the subring $R^p := \{r^p \mid r \in R\}$ is in the kernel of every derivation.

(2) Let A be a ring and $S = A[x_1, \ldots, x_n]$ be a polynomial ring.
  (a) Let R be an $\mathbb{N}$-graded A-algebra such that A lives in degree zero. Show that there is a derivation on R such that for every homogeneous element f of degree d, $\partial(f) = df$. This derivation is called the *Euler operator* associated to the grading.
  (b) Let $S = A[x_\lambda \mid \lambda \in \Lambda]$ be a polynomial ring over A endowed with the $\mathbb{N}$-grading by the rule $\deg(x_\lambda) = n_\lambda$. Express the Euler operator of the grading as an S-linear combination of the partial derivatives.

(3) Let A be a ring and $R = A[x_1, \ldots, x_n]$ be a polynomial ring.
  (a) Give an explicit formula for the Lie algebra bracket on $\mathrm{Der}_{R|A}(R)$.
  (b) Does $\mathrm{Der}_{R|A}(R)$ have any nontrivial proper Lie ideals (i.e., A-submodules B such that $[d, b] \in B$ for all $b \in B$ and $d \in \mathrm{Der}_{R|A}(R)$)?

(4) Let R be a ring of characteristic $p > 0$ and $\partial : R \to R$ be a derivation. Show that ∂^p, i.e., the p-fold self composition of ∂, is a derivation on R.