DIHEDRAL GROUPS

- A **isometry** of $\mathbb{R}^2$ is a bijective function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distances between pairs of points; these include rotations around a point, translations, and reflections over a line.
- Let $X \subseteq \mathbb{R}^2$. A **symmetry** of X is an isometry of $\mathbb{R}^2$ such that $f(X) = X$ as a set.
- The **dihedral group** D_n is the group of symmetries of a regular n-gon P_n in the plane, with composition of functions as the group operation.

THEOREM: The dihedral group D_n is indeed a group. It has exactly $2n$ elements consisting of:
- The identity map e,
- $n-1$ rotations $r, r^2, \dots, r^{n-1}$, where r is counterclockwise rotation by $2\pi/n$ (so r^i is counterclockwise rotation by $2\pi i/n$),
- n reflections. More precisely,
  - when n is odd, there are n distinct reflections over a line between a vertex and an opposite edge;
  - when n is even, there are $n/2$ distinct reflections between opposite pairs of vertices, and another $n/2$ distinct reflections between opposite pairs of edges.

(1) Orders of these elements?
(2) Proof

LEMMA: Let $v \in P_n$ be a vertex, and $s \in D_n$ the reflection through the axis containing s. Let $r \in D_n$ be counterclockwise rotation by $2\pi/n$. Then $srs^{-1} = r^{-1}$.

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(1) Every element of D_n can be written uniquely in the form
$$r^j \quad \text{for } j = 0, \dots, n-1, \qquad \text{or} \qquad r^j s \quad \text{for } j = 0, \dots, n-1.$$
(2) D_n is generated by r, s.
(3) D_n has the group presentation $\langle r, s \mid r^n = e, s^2 = e, srs^{-1} = r^{-1} \rangle$.

Symmetries of the circle.
Symmetries of a line.