§3.13: Finiteness Theorem for Invariant rings

Hilbert's finiteness Theorem: Let K be a field of characteristic zero, and $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let G be a finite group acting on R by degree-preserving automorphisms. Then the invariant ring R^G is algebra-finite over K.

Theorem: Let R be an $\mathbb{N}$-graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is algebra-finite over R_0.

Definition: Let $R \subseteq S$ be an inclusion of rings. We say that R is a **direct summand** of S if there is an R-module homomorphism $\pi : S \to R$ such that $\pi|_R = \mathbb{1}_R$.

Proposition: A direct summand of a Noetherian ring is Noetherian.

Lemma: In the setting of Hilbert's finiteness Theorem,

(1) R^G is $\mathbb{N}$-graded with $(R^G)_0 = K$.
(2) R^G is a direct summand of R.

(1) Use the Lemma, Proposition, and Theorem to deduce Hilbert's finiteness Theorem.

(2) Proof of Theorem:
- (a) Explain the direction ($\Leftarrow$).
- (b) Show that R Noetherian implies R_0 is Noetherian.
- (c) Let $f_1, \dots, f_t$ be a homogeneous generating set for R_+, the ideal generated by positive degree elements of R. Show[1] by (strong) induction on d that every element of R_d is contained in $R_0[f_1, \dots, f_t]$.
- (d) Conclude the proof of the Theorem.

(3) Proof of Proposition:
- (a) Show that if R is a direct summand of S, and I is an ideal of R, then $IS \cap R = I$.
- (b) Complete the proof of the proposition.

(4) Proof of Lemma part (2): Consider $r \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot r$.

(5) Show that a direct summand of a normal ring is normal.

(6) Let $\mathcal{S}_3$ denote the symmetric group on 3 letters, and let $\mathcal{S}_3$ act on $R = \mathbb{C}[X_1, X_2, X_3]$ by permuting variables; i.e., σ is the $\mathbb{C}$-algebra homomorphism given by $\sigma \cdot X_i = X_{\sigma(i)}$. Find a $\mathbb{C}$-algebra generating set for $R^{\mathcal{S}_3}$. What about replacing 3 by n?

[1]Hint: Start by writing $h \in R_d$ as $h = \sum_i r_i f_i$ with $d = \deg(r_i) + \deg(f_i)$ for all i.