

### PROBLEM SET #3

- (1) Is  $\mathbb{Z}[\sqrt[3]{37}]$  a regular ring? What about  $\mathbb{Z}[\sqrt[3]{43}]$ ?
- (2) Let  $R$  be an  $A$ -algebra,  $f(x_1, \dots, x_n) \in A[x_1, \dots, x_n]$  a polynomial with coefficients in  $A$ , and  $r_1, \dots, r_n, s_1, \dots, s_n \in R$ .
  - (a) Prove the *chain rule* for the universal derivation:  $d_{R|A}(f(r_1, \dots, r_n)) = \sum_i \frac{df}{dx_i}(r_1, \dots, r_n) dr_i$ .
  - (b) Prove the *Taylor expansion* formula:  $f(r_1 + s_1, \dots, r_n + s_n) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{|\alpha|!} \frac{d^{|\alpha|} f}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}}(r_1, \dots, r_n) s_1^{\alpha_1} \dots s_n^{\alpha_n}$ .
- (3) Facts about  $p$ -bases/  $p$ -degree:
  - (a) Let  $K \subseteq L$  be an extension of fields of positive characteristic. Let  $T$  be a  $p$ -basis for  $L$  over  $K$ . Show that for any  $e$ , the set  $T^{[< p^e]}$  is a basis for  $L$  over  $K(L^{p^e})$ .
  - (b) Let  $K \subseteq L$  be a finite extension of fields of positive characteristic. Show that  $p \deg(K) = p \deg(L)$ .
  - (c) Let  $L = K(x_1, \dots, x_m)$  be a field of rational functions in  $m$  variables over  $K$ . Show that  $p \deg(L) = p \deg(K) + m$ .
- (4) Let  $k$  be a field of positive characteristic with a finite  $p$ -basis,  $R$  be a finitely generated  $k$ -algebra, and  $\mathfrak{p} \subseteq \mathfrak{q}$  be prime ideals of  $R$ . Show that
 
$$\dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = p \deg(\kappa(\mathfrak{p})) - p \deg(\kappa(\mathfrak{q})).$$
- (5) Let  $K$  be a field.
  - (a) Let  $R = K[x]$  be a polynomial ring in one variable and  $M = R^{\oplus \mathbb{N}}$  be a free  $R$ -module on a countable basis. Compute the  $(x)$ -adic completion of  $M$ .
  - (b) Let  $R = K[x_1, x_2, \dots]$  be a polynomial ring in countably many variables and  $\mathfrak{m} = (x_1, x_2, \dots)$ . Describe the elements of  $\hat{R}^{\mathfrak{m}}$ . Find an element in the maximal ideal of  $\hat{R}^{\mathfrak{m}}$  that is *not* an element of  $\mathfrak{m}\hat{R}^{\mathfrak{m}}$ .
- (6) Let  $K \subseteq L$  be an extension of fields.
  - (a) Suppose that  $L$  is a finitely generated over  $K$  as fields. Show that  $L$  is formally unramified over  $K$  if and only if the extension is separable algebraic.
  - (b) Show that the finite generation hypothesis is strictly necessary in part (1).