

Problem Set 4

Due Thursday, February 11

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly*. As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let $R = \mathbb{Z}[X]$ and let $f : R^2 \rightarrow R$ be the linear transformation $f\left(\begin{bmatrix} p \\ q \end{bmatrix}\right) = 2p + Xq$. Show that there do not exist basis B for R^2 and C for R and $r \in R$ for which $[f]_B^C = [r \ 0]$.

Problem 2. Let K be a field, and G be a finite subgroup of K^\times . Show that G is cyclic.

Note that $(\mathbb{Z}/p)^\times$ is cyclic, as a consequence of this.

Problem 3. Prove¹ that if R is a PID, F is a free module, and $M \subseteq F$ is a submodule (not necessarily finitely generated) then M is free.

¹Hint:

- Fix a basis B for F . Let $\mathcal{P} = \{A \subseteq B \mid RA \cap M \text{ is free}\}$. Show that \mathcal{P} has a maximal element.
- Let $A \in \mathcal{P}$ be a maximal element, and suppose that there is some $b \in B \setminus A$. Consider the map $\pi : R(A \cup \{b\}) \cap M \rightarrow R$ that maps an element to its b -coefficient.