DEFINITION 20.2: A **subsequence** of a given sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence of the form $\{a_{n_k}\}_{k=1}^{\infty}$ where $\{n_k\}_{k=1}^{\infty}$ is any strictly increasing sequence of natural numbers.

(1) **True or false; justify.**
   (a) The sequence $\left\{\frac{1}{3n+7}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.
   (b) The constant sequence $\left\{\frac{1}{2}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.
   (c) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are both subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.
   (d) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are the only two subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.
   (e) The sequence $\{\sin(\pi n)\}_{n=1}^{\infty}$ is a subsequence of $\{\sin(n)\}_{n=1}^{\infty}$.

(2) Explain how the following Corollary follows from Theorem 20.4:

   COROLLARY 20.6: Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.
   (a) If there is a subsequence of $\{a_n\}_{n=1}^{\infty}$ that diverges, then the sequence $\{a_n\}_{n=1}^{\infty}$ diverges.
   (b) If there are two subsequences of $\{a_n\}_{n=1}^{\infty}$ that converge to different values, then $\{a_n\}_{n=1}^{\infty}$ diverges.

(3) Use Corollary 20.6 to give a quick proof that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

(4) **Prove or disprove:**
   (a) Every subsequence of a bounded sequence is bounded.
   (b) Every subsequence of a divergent sequence is divergent.
   (c) Every subsequence of a sequence that diverges to $-\infty$ also diverges to $-\infty$.