1. August 23, 2022

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \ldots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not have thought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\ldots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\ldots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number.

Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is* 2*.*

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \Longrightarrow \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction. (Some mathematicians would actually not consider this to be a proof by contradiction. For some, a proof by contradiction refers to when the truth of a statement P is established by assuming the statement "not P" and deducing from that a falsity.)

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. $\square$

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.2 (Arithmetic and order properties of $\mathbb{Q}$)**.** *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

(1) *There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p + q$ and $p \cdot q$.*
(2) *Each of $+$ and $\cdot$ is a commutative operation (i.e., $p + q = q + p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*
(3) *Each of $+$ and $\cdot$ is an associative operation (i.e., $(p + q) + r = p + (q + r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q, and r).*
(4) *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0 + q = q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.*
(5) *The distributive law holds: $p \cdot (q + r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.*
(6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.*
(7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.*
(8) *There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that*
    (a) *For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
    (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
    (c) *For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
(9) *The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.*
(10) *The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

Which of the properties from Proposition 1.2 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$.

We expect everything from Proposition 1.2 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written $\mathbb{R}$, satisfies the following eleven properties:*

(Axiom 1) *There are operations $+$ and $\cdot$ defined on $\mathbb{R}$, so that if $x, y \in \mathbb{R}$, then so are $x + y$ and $x \cdot y$.*
(Axiom 2) *Each of $+$ and $\cdot$ is a commutative operation.*
(Axiom 3) *Each of $+$ and $\cdot$ is an associative operation.*
(Axiom 4) *The real number 0 is an identity element for addition and the real number 1 is an identity element for multiplication. This means that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.*
(Axiom 5) *The distributive law holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.*
(Axiom 6) *Every real number has an additive inverse: For any $x \in \mathbb{R}$, there is a number $-x$ satisfying $x + (-x) = 0$.*
(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any $x \in \mathbb{R}$ such that $x \neq 0$, there is a real number x^{-1} satisfying $x^{-1} \cdot x = 1$.*
(Axiom 8) *There is a "total ordering" $\leq$ on $\mathbb{R}$. This means that*
- (a) *For all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.*
- (b) *If $x \leq y$ and $y \leq z$, then $x \leq z$.*
- (c) *For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.*

(Axiom 9) *The total ordering $\leq$ is compatible with addition: If $x \leq y$ then $x + z \leq y + z$ for all z.*
(Axiom 10) *The total ordering $\leq$ is compatible with multiplication by non-negative real numbers: If $x \leq y$ and $z \geq 0$ then $zx \leq zy$.*
(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

> "Cancellation of Addition": For real numbers, $x, y, z \in \mathbb{R}$, if $x + y = z + y$ then $x = z$.

Let's prove this carefully, using just the list of axioms: Assume that $x + y = z + y$. Then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x + y) + (-y) = (z + y) + (-y)$. This can be rewritten as

$x + (y + (-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

For another example, we can deduce the following fact from the axioms:

$$r \cdot 0 = 0 \text{ for any real number } r.$$

Let's prove this carefully: Let r be any real number. We have $0+0 = 0$ (Axiom 4) and hence $r\cdot(0+0) = r\cdot 0$. But $r\cdot(0+0) = r\cdot 0 + r\cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

As I said, there are many other familiar properties of the real numbers that follow from these axioms, but I will not list them all. The great news is that all of these familiar properties follow from this short list of axioms. We will prove a couple, but for the most part, I'll rely on your innate knowledge that facts such as $r \cdot 0 = 0$ hold.

I owe you a description of the very important Completeness Axiom, and it will take a bit of time to do so. Before we get to this, it will be helpful to review set notation, and some basics of proof-writing.

2. August 25, 2022

Definition 2.1. A real number is *irrational* if it is not rational.

Making sense of if then statements and quantifier statements.

- The *converse* of the statement "If P then Q" is the statement "If Q then P".
- The *contrapositive* of the statement "If P then Q" is the statement "If not Q then not P".
- Any if then statement is equivalent to its contrapositive, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Is the original/contrapositive/converse true or false for real numbers a, b? Explain why (but don't prove).
  (a) If a is irrational, then $1/a$ is irrational.
  (b) If a and b are irrational, then ab is irrational.
  (c) If $a > 3$, then $a^2 > 9$.

(1) true; contrapositive is "If $1/a$ is rational, a is rational" is true; converse is "if $1/a$ is irrational, then a is irrational" is true.

(2) false; contrapositive is "If ab is rational, either a or b is rational" is false; converse is "if ab is irrational then a and b are irrational" is false.
(3) true; contrapositive is "if $a^2 \leq 9$, then $a \leq 3$ is true; converse is "if $a^2 > 9$ then $a > 3$" is false.

- The symbol for "for all" is $\forall$ and the symbol for there exists is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that not P".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, not P".

(2) Rewrite each statement with symbols in place of quantifiers, and write its negation. Is the original statement true or false? Explain why (but don't prove them).
   (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
   (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
   (c) For all $x \in \mathbb{R}$ such that[1] $x \neq 0$, $x^2 > 0$.
   (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
   (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

(1) $\exists x \in \mathbb{Q} : x^2 = 2$ is false. Negation: $\forall x \in \mathbb{Q}, x^2 \neq 2$.
(2) $\forall x \in \mathbb{R}$, $x^2 > 0$ is false. Negation: $\exists x \in \mathbb{R} : x^2 \leq 0$.
(3) $\forall x \in \mathbb{R} : x \neq 0, x^2 > 0$ is true. Negation: $\exists x \in \mathbb{R} : x \neq 0, x^2 \leq 0$.
(4) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$ is true. Negation: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x \geq y$.
(5) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x < y$ is false. Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x \geq y$.

Proving if then statements and quantifier statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.

[1]In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

- Often it is easier to prove the contrapositive of an if then statement than the original, especially when the negation of the hypothesis or conclusion is something negative.
- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

(3) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[3] that $x \geq y$ if and only if $-y \geq -x$.
(4) Let x be a real number. Show that if x^2 is irrational, then x is irrational.
(5) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have proven in class to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.
(6) Prove that[4] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.
(7) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.
(8) Prove[5] that (2d) is true and (2e) is false.
(9) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.
(10) Prove that $1 > 0$.
(11) Let x, y be real numbers. Prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

(3) Let $x \geq y$. Adding $(-x) + (-y)$ to both sides (which exists by Axiom 6), we obtain $-y = x + ((-x) + (-y)) \geq y +$

[2]How you found this x is logically irrelevant to an existence proof, and should not be included.
[3]Hint: You may want to add something to both sides.
[4]Hint: Use (5).
[5]You can "work out of order here" and *use* (10) now.

$((-x)+(-y)) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq -y$. Adding $x+y$ to both sides, we obtain $y = (x+y)+(-x) \leq (x+y)+(-y) = x$ (by Axiom 9 and Axiom 5).

(6) Let x and y be nonzero real numbers. By Axiom 7, there are element $x^{-1}, y^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$ and $yy^{-1} = 1$. Then $xy \cdot (x^{-1}y^{-1}) = (xx^{-1})(yy^{-1}) = 1$, using Axioms 2 and 3 in the first equality and Axiom 5 in the second. By the previous fact (applies to xy) we conclude that $xy \neq 0$.

(7) Consider $x = 1$. Let $y \in \mathbb{R}$. By Axiom 4, we have $xy = 1y = y$. Thus, for all $y \in \mathbb{R}$, we have $xy = x$.

(8) (2d): Let $x \in \mathbb{R}$. Consider $y = x+1$. Since $1 > 0$ we have $y = x+1 > x+0 = x$. Thus, for each $x \in \mathbb{R}$, we have some y such that $x < y$. (2e): We claim this is false. Suppose, for the sake of contradiction that this was true, and let x be as in the statement. Then for any $y \in \mathbb{R}$, we have $x < y$. But, for $y = x$, the inequality $x < y$ is false. This is a contradiction, so the statement must be false.

(10) First we establish two lemmas.

Lemma: For real numbers $x \in \mathbb{R}$ we have $-(-x) = x$.

Proof: We have

$$(-x) + (-(-x)) = 0$$

so

$$\begin{aligned} -(-x) &= 0 + -(-x) = (x + (-x)) + (-(-x)) \\ &= x + ((-x) + (-(-x))) = x. \quad \square \end{aligned}$$

Lemma: For real numbers $x, y \in \mathbb{R}$ we have $(-x)y = -(xy)$.

Proof: We have that

$$0 = 0y = (x + (-x))y = xy + (-x)y.$$

Adding $-(xy)$ to both sides we get

$$\begin{aligned} -(xy) &= -(xy) + (xy + (-x)y) \\ &= (-(xy) + (-x)y) + (-x)y \\ &= 0 + (-x)y = (-x)y. \quad \square \end{aligned}$$

We proceed with the proof. We either have $1 \geq 0$ or $1 \leq 0$. Suppose that $1 \leq 0$. Then $-1 \geq 0$, so

$$(-1)(-1) \geq (-1)0 = 0.$$

But

$$(-1)(-1) = -(1(-1)) = -(-1) = 1,$$

so $1 \geq 0$, contradicting the hypothesis.