

## REMINDER ON FUNCTIONS

Given any two sets  $S$  and  $T$ , a *function* from  $S$  to  $T$ , written  $f : S \rightarrow T$ , is a “rule”<sup>1</sup> that assigns to each element  $s \in S$  a unique element  $t \in T$ . The set  $S$  is called the *domain* of  $f$ . We will generally consider functions from some set of real numbers to  $\mathbb{R}$ . We often specify functions by formulas; when we do this we take the domain to be the set of all real numbers for which the formula evaluates to a unique real number. In particular,

$$f(x) = x + 1 \quad \text{and} \quad g(x) = \frac{x^2 - 1}{x - 1}$$

are *not* the same function, since their domains are different.

## LIMITS OF FUNCTIONS

**Definition 17.1:** Let  $S$  be a subset of  $\mathbb{R}$ . Let  $f : S \rightarrow \mathbb{R}$  be a function, and  $a$  and  $L$  be real numbers. We say that *the limit of  $f$  as  $x$  approaches  $a$  is  $L$*  provided:

for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $0 < |x - a| < \delta$ , then  $x$  is in the domain of  $f$  and  $|f(x) - L| < \varepsilon$ .

If this happens, we write  $\lim_{x \rightarrow a} f(x) = L$  to denote this.

### (1) UNPACKAGING PARTS OF THE DEFINITION.

- (a) Describe  $\{x \in \mathbb{R} \mid 0 < |x - 2| < 1\}$  as a union of two open intervals.
- (b) For a general  $a \in \mathbb{R}$  and  $\delta > 0$ , describe  $\{x \in \mathbb{R} \mid 0 < |x - a| < \delta\}$  as a union of two open intervals.
- (c) Focusing on the “domain” part of the definition, if the limit of  $f$  as  $x$  approaches  $a$  is  $L$ , then  $f$  must at least be defined \_\_\_\_\_ (where?).

### (2) THE $\varepsilon - \delta$ GAME.

- (a) Player 0 starts by graphing a function  $f$  (like a familiar one from calculus) and specifies an  $x$ -value  $a$  and a  $y$ -value  $L$  that (based on previous calculus knowledge) they think makes  $\lim_{x \rightarrow a} f(x) = L$  **true**. [The graph should be large.]
- (b) Player 1 chooses an  $\varepsilon$ . This is how close we would like our function to be to  $L$ . Thus,  $\varepsilon$  goes up and down from  $L$  (corresponding to  $|f(x) - L| < \varepsilon$ ). Draw horizontal dotted lines with  $y$ -values  $L - \varepsilon$  and  $L + \varepsilon$ . [The  $\varepsilon$  should be large enough for people to see and have room to work in the picture.]
- (c) Player 2 must find a  $\delta$  such that every  $x \in (a - \delta, a) \cup (a, a + \delta)$  is
  - in the domain of  $f$ , and
  - has an output  $f(x)$  within  $(L - \varepsilon, L + \varepsilon)$ .
 Draw vertical dotted lines for the  $x$ -values  $a - \delta$  and  $a + \delta$ . [Everyone in the team can assist player 2!]
- (d) Repeat with the same graph, players 1 & 2 switching roles (and a new  $\varepsilon$ ).

- (3) Draw the graph of  $g(x) = \frac{x^2 - 1}{x - 1}$ . Play the  $\varepsilon - \delta$  game with this function,  $a = 1$  and  $L = -3$ . What happens?

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<sup>1</sup>Here’s a real definition: a *function* from  $S$  to  $T$  is a subset  $G \subset S \times T$  of ordered pairs of elements of  $S$  and  $T$  with the property that for all  $s \in S$  there is a unique  $t \in T$  such that  $(s, t) \in G$ ; we write  $f(s)$  for this element  $t$ .

- (4) Consider the function  $g(x) = \frac{x^2 - 1}{x - 1}$ . It is true that  $\lim_{x \rightarrow 1} g(x) = 2$ .
- (a) I claim that for  $\varepsilon = 3$ , the choice  $\delta = 3$  “works” to make the rest of the definition true. Verify this.
  - (b) Find a  $\delta$  that “works” for  $\varepsilon = 1$ .
  - (c) Find a  $\delta$  that “works” for  $\varepsilon = 1/2$ .
  - (d) Find a  $\delta$  that “works” for  $\varepsilon > 0$ .
- (5) Consider the function  $g(x) = \frac{x^2 - 1}{x - 1}$ . It is not true that  $\lim_{x \rightarrow 1} g(x) = -3$ . I claim that for  $\varepsilon = 1$ , there is no choice of  $\delta > 0$  that “works” to make the rest of the definition true. Verify this.