DETERMINANTS

DEFINITION: Let R be a commutative ring, and $A \in \mathrm{Mat}_{n\times n}(R)$. The determinant of M is

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

THEOREM 1: Identify $\mathrm{Mat}_{n\times n}(R)$ with $\underbrace{R^n \times \cdots \times R^n}_{n \text{ times}}$ by considering a matrix as an n-tuple of columns. The determinant is the unique function

$$\det\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$$

that satisfies the following three properties:

- det is **multilinear**, meaning
$$\begin{aligned}\det(v_1, \dots, v_{i-1}, \boldsymbol{v} + \boldsymbol{w}, v_{i+1}, \dots, v_n) &= \det(v_1, \dots, v_{i-1}, \boldsymbol{v}, v_{i+1}, \dots, v_n) \\ &\quad + \det(v_1, \dots, v_{i-1}, \boldsymbol{w}, v_{i+1}, \dots, v_n) \\ \det(v_1, \dots, v_{i-1}, \boldsymbol{rv}, v_{i+1}, \dots, v_n) &= \boldsymbol{r} \det(v_1, \dots, v_{i-1}, \boldsymbol{v}, v_{i+1}, \dots, v_n)\end{aligned}$$
- det is **alternating**, meaning
$$\det(v_1, \dots, v_n) = 0 \qquad \text{if } v_i = v_j \text{ for some } i \neq j.$$
- $\det(e_1, \dots, e_n) = 1$.

LEMMA: Let $F\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$ be an alternating multilinear function. Then

$$F(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \mathrm{sgn}(\sigma) F(v_1, \dots, v_n).$$

(1)

THEOREM 2: Let R be a commutative ring and $A, B \in \mathrm{Mat}_{n\times n}(R)$. Then

$$\det(AB) = \det(A)\det(B).$$

PROPOSITION: Let R be a commutative ring. Let A be a square matrix, and B be a matrix obtained from A by an elementary column operation.

- For the operation "add $r \in R$ times column i to column j" we have $\det(B) = \det(A)$.
- For the operation "multiply column i by $u \in R^\times$" we have $\det(B) = u\det(A)$.
- For the operation "swap column i and column j" we have $\det(B) = -\det(A)$.

(3) Prove the Proposition.

(4) Proof of Theorem in the case $R = F$ is a field:

- **(a)** Use the Proposition (and the fact that over a field every invertible matrix is a product of elementary matrices) that if A is invertible, then $\det(A) \neq 0$.
- **(b)** Show[1] that if A is not invertible, then $\det(A) = 0$.
- **(c)**

[1] Hint: First show that the columns of A are linearly dependent, and express some v_i as a linear combination of the others.