ASSIGNMENT #7: DUE FRIDAY, DECEMBER 13 AT 7PM

This problem set is to be turned in by Canvas. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

(1) Let R be a ring, not necessarily Noetherian, and $S = R[X]$ a polynomial ring in one variable over R.
   (a) Show that for any prime ideal $\mathfrak{p}$ in R, any chain of prime ideals of S that all contract to $\mathfrak{p}$ has length at most one.
   (b) Show that if $\dim(R) = d$, then $d + 1 \leq \dim(S) \leq 2d + 1$.
   (c) Let $R = \mathbb{Q} + T\,\mathbb{R}[\![T]\!]$, i.e., the subring of $R' = \mathbb{R}[\![T]\!]$ consisting of all power series whose constant term is rational. Verify that R is a ring, that R has dimension one, and[1] that the dimension of $R[X]$ is three.

(2) Let K be a field, and $R \subseteq S$ be a module-finite inclusion of domains[2] that are algebra-finite over K. Show that for any $\mathfrak{q} \in \operatorname{Spec}(S)$, the height of $\mathfrak{q}$ equals the height of $\mathfrak{q} \cap R$.

(3) Artinian modules:
   (a) Show that the $\mathbb{Z}$-module $\mathbb{Z}[1/2]/\mathbb{Z}$ is Artinian but not Noetherian.
   (b) Let R be a Noetherian ring and M be an R-module. Show that if M is Artinian, then $\operatorname{Supp}(M) \subseteq \operatorname{Max}(R)$.
   (c) Let K be a field, and R be a $\mathbb{N}$-graded ring with $R_0 = K$ and R finitely generated over K. Let M be a $\mathbb{Z}$-graded R-module. Show that if M is Artinian, then
      - there is some $k \in \mathbb{Z}$ such that $M_t = 0$ for all $t \geq k$, and
      - for $n = \max\{t \mid M_t \neq 0\}$, M_n is a finite-dimensional K-vectorspace.

[1] Hint: Let $\mathfrak{p}$ be the prime ideal $T\mathbb{C}[\![T]\!] \subseteq R$ and let $\alpha : R[X] \to R'$ be the R-algebra homomorphism given by $\alpha(X) = e$. Show that $\ker(\alpha)$
$subsetneqq \mathfrak{p} S$.

[2] Warning: we aren't assuming that R is normal.