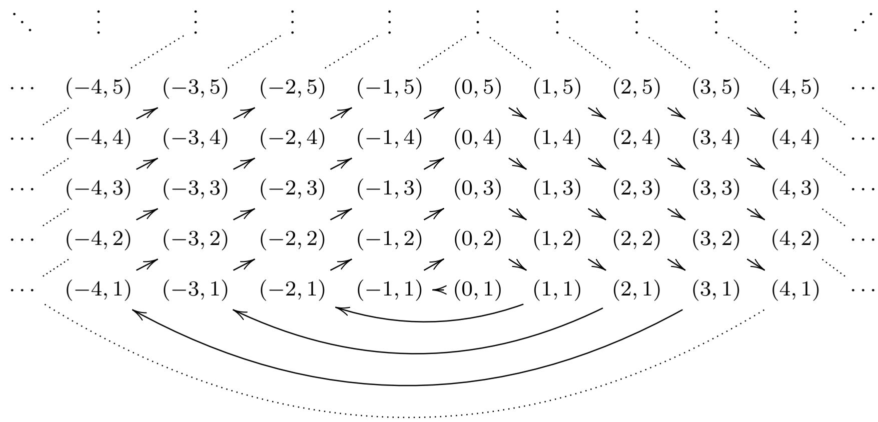
Copy the following picture onto the board:



This gives the list of points

$$(0,1),(-1,1),(0,2),(1,1),(-2,1),(-1,2),(0,3),(1,2),(2,1),(-3,1),\ldots$$

Now convert these to a list of rational numbers by changing (m,n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1},\frac{-1}{1},\frac{0}{2},\frac{1}{1},\frac{-2}{1},\frac{-1}{2},\frac{0}{3},\frac{1}{2},\frac{2}{1},\frac{-3}{1},\ldots$$

of rational numbers. Call this sequence $\{w_n\}_{n=1}^{\infty}$.

(1) True or false: Every rational number occurs in this sequence. That is, for every $q \in \mathbb{Q}$, there is some $n \in \mathbb{N}$ such that $w_n = q$.

(2) True or false: Every rational number occurs in this sequence infinitely many times. That is, for every $q \in \mathbb{Q}$, there are *infinitely many* natural numbers $n \in \mathbb{N}$ such that $w_n = q$.

(3) True or false: For every rational number $q \in \mathbb{Q}$, the constant sequence $\{q\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.

(4) True or false: For every real number $r \in \mathbb{R}$, the constant sequence $\{r\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.

(5) True or false: Every sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.

(6) True or false: For every real number $r \in \mathbb{R}$, there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ that converges to r.

THEOREM 21.1: There exists a sequence of rational numbers such that

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THEOREM 21.2 (CANTOR'S THEOREM): There is no sequence of real numbers such that every real number r occurs in the sequence.

By contradiction, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that every real number occurs as some a_n and write out the decimal expansions (where each $d_{i,j} \in \{0, 1, \ldots, 9\}$ is a digit).

$$\begin{aligned}
a_1 &= \text{(integer part)}.d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}d_{1,6}\cdots \\
a_2 &= \text{(integer part)}.d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}d_{2,6}\cdots \\
a_3 &= \text{(integer part)}.d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}d_{3,6}\cdots \\
a_4 &= \text{(integer part)}.d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}d_{4,6}\cdots \\
a_5 &= \text{(integer part)}.d_{5,1}d_{5,2}d_{5,3}d_{5,4}d_{5,5}d_{5,6}\cdots \\
a_6 &= \text{(integer part)}.d_{6,1}d_{6,2}d_{6,3}d_{6,4}d_{6,5}d_{6,6}\cdots \\
a_7 &= \text{(integer part)}.d_{7,1}d_{7,2}d_{7,3}d_{7,4}d_{7,5}d_{7,6}\cdots \\
\vdots \quad & \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}$$