Some old qualifying exam questions

Here are some old qualifying exam problems you are already ready to solve.

Problem 1. Let R be a commutative integral domain and M an R-module. A subset S of M is called a *maximal linearly independent* subset of M if S is linearly independent and any subset of M properly containing S is linearly dependent.

(a) Let T be a linearly independent subset of M. Prove that T is contained in some maximal linearly independent subset of M.

(b) Let T be a linearly independent subset of M and let N be the R-submodule of M generated by T. Prove that T is a maximal linearly independent subset if and only if M/N is torsion. (Recall that an R-module P is called "torsion" if for each $p \in P$, there is a $r \in R$ such that $r \neq 0$ and $rp = 0$.)

Problem 2. Let R be a nonzero commutative ring and suppose $p : R^m \to R^n$ is a surjective homomorphism of R-modules for some nonnegative integers m and n.

(a) Prove that if R is a field, then $m \geq n$.

(b) Prove that if R is any commutative ring, then $m \geq n$.

Problem 3. Let $A = \begin{bmatrix} 6 & 4 & 2 \\ 2 & 7 & 4 \end{bmatrix} \in \mathrm{Mat}_{2\times 3}(\mathbb{Z})$.

(a) Find invertible matrices P and Q such that $B := PAQ$ is diagonal (i.e., $b_{i,j} = 0$ for all $i \neq j$).

(b) Find all integer solutions of $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Problem 4. Let F be a field, V an F-vector space, and W a subspace of V. A subspace U of V is called a *complement* of W in V if $U \cap W = \{0\}$ and $U + W = V$.

(a) Prove that for every V and W as above, W has at least one complement in V.

(b) Prove that if U is a complement of W in V and V is finite dimensional, then $\dim_F(V) = \dim_F(W) + \dim_F(U)$.

Problem 5. Let R be a Euclidean domain, A an $m \times n$ matrix with elements from R, and A^T the transpose matrix of A. Let $\mathrm{coker}(A)$ denote the quotient of R^m by the submodule generated by the columns of A. The *torsion* submodule of an R-module M is the submodule

$$\{m \in M \mid rm = 0 \text{ for some } r \neq 0\}.$$

(It is indeed a submodule and you need not prove this.)

(a) Prove that the torsion submodules of $\mathrm{coker}(A)$ and $\mathrm{coker}(A^T)$ are isomorphic.

(b) Prove that the modules $\mathrm{coker}(A)$ and $\mathrm{coker}(A^T)$ are isomorphic if and only if $m = n$.

Problem 6. Let I be an ideal in a commutative ring R, let M and N be R-modules and let $f : M \to N$ be an R-module homomorphism.

(a) Prove there is a unique R-module homomorphism $f : M/IM \to N/IN$ such that $f \circ p = q \circ f$, where $p : M \to M/IM$ and $q : N \to N/IN$ are the canonical quotient maps.

(b) Prove that if $I^2 = 0$ and f is surjective, then so is f. (Recall that I^2 is the ideal generated by all elements of the form ab, where $a, b \in I$.)