THE ARITHMETIC RANK OF DETERMINANTAL NULLCONES

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ABSTRACT. We compute the arithmetic rank of nullcone ideals arising from the classical actions of the symplectic group, the general linear group, and the orthogonal group. We use these arithmetic rank calculations to establish striking vanishing results on the local cohomology modules supported at these nullcone ideals. This is done by analyzing the integer torsion in the critical local cohomology modules. The vanishing theorems that we obtain are sharp in multiple ways.

CONTENTS

1. INTRODUCTION

Consider a polynomial ring S over a field $\mathbb{K}$, and a group G acting on S via degree-preserving $\mathbb{K}$-algebra automorphisms. By the *nullcone ideal* of the action, we mean the expansion of the homogeneous maximal ideal of the invariant ring S^G to the polynomial ring S. The notion arises at least as far back as Hilbert's proof of the finite generation of invariant rings [Hi], and has been studied extensively e.g., [He, HJPS, KS, KW, Lo2, PTW, Sc]. For classical invariant rings of characteristic zero, work of Kraft and Schwarz records precisely when the nullcone ideal is radical or prime [KS, Theorem 9.1]; the positive characteristic case is settled in [HJPS], where it is also determined precisely when the nullcone ideal is perfect, i.e., when it defines a Cohen–Macaulay ring — for each of the classical group actions, independent of the characteristic, it turns out that the minimal primes of the nullcone ideal are perfect. The F-regularity property is investigated in [PTW] and [Lo2].

Motivated by the Nullstellensatz, the *arithmetic rank* of an ideal is the least number of elements required to generate the ideal up to taking radicals. This is often a notoriously difficult invariant to compute, with some innocuous examples remaining a challenge for over sixty years, e.g., [Har]. Our paper begins with the observation that for the action of a linearly reductive group, the arithmetic rank of the nullcone ideal is readily determined:

Theorem 1.1. *Let S be a polynomial ring over a field $\mathbb{K}$, and let G be a linearly reductive group acting on S by degree-preserving $\mathbb{K}$-algebra automorphisms. Let S^G denote the ring of invariants, and $\mathfrak{m}_{S^G}$ the homogeneous maximal ideal of S^G. Then the nullcone ideal $\mathfrak{m}_{S^G}S$ has arithmetic rank $\dim S^G$.*

The proof is so elementary that we present it right away, though some notation and background is provided later in this section.

Proof. Set $R := S^G$ and $d := \dim R$. The homogeneous maximal ideal $\mathfrak{m}_R$ of R may be generated up to radical by d elements, namely by a homogeneous system of parameters for R. This gives us an upper bound for the arithmetic rank of the nullcone, namely

$$\operatorname{ara}(\mathfrak{m}_R S) \ \leqslant\ d.$$

Since the group G is assumed to be linearly reductive, the inclusion of rings $R \longrightarrow S$ is pure. It follows that the local cohomology module

$$H^d_{\mathfrak{m}_R}(R) \otimes_R S \ =\ H^d_{\mathfrak{m}_R}(S) \ =\ H^d_{\mathfrak{m}_R S}(S) \tag{1.1.1}$$

is nonzero. But then $\operatorname{ara}(\mathfrak{m}_R S) \geqslant d$, see [ILL$^+$, Proposition 9.12], or the discussion later in this section. □

Theorem 1.1 applies in the case of classical invariant rings of characteristic zero, i.e., when G is the general linear group, the symplectic group, the orthogonal group, or the special linear group, over a field of characteristic zero, and the action is as in Weyl's book: for the general linear group, consider a direct sum of copies of the standard representation and copies of the dual; in the other cases take copies of the standard representation. The invariant rings, respectively, are determinantal rings, rings defined by Pfaffians of alternating matrices, symmetric determinantal rings, and the Plücker coordinate rings of Grassmannians. It is the nullcones of these actions that have been studied extensively in [KS, HJPS, Lo2, PTW]. One of the main goals of the present paper is to determine the arithmetic rank of the corresponding nullcone ideals in the case of positive characteristic; the issue is that the classical groups are typically not linearly reductive in positive characteristic, and the inclusion $S^G \longrightarrow S$ is typically no longer pure. Indeed, the local cohomology obstruction (1.1.1) vanishes, and the lower bound on arithmetic rank is instead obtained using étale cohomology.

Sections 2, 3, and 4 summarize our results for Pfaffian nullcones, determinantal nullcones, and symmetric determinantal nullcones, respectively. In each case, we obtain the arithmetic rank of the nullcone ideal, and also study the critical local cohomology module. We prove vanishing theorems for local cohomology modules supported at nullcone ideals, that mirror corresponding results for determinantal ideals, [LSW, Theorem 1.1]; this involves analyzing the integer torsion in local cohomology. The singular and étale cohomology calculations required in these sections are performed in Sections 6, 7, and 8, for the respective cases of Pfaffian nullcones, determinantal nullcones, and symmetric determinantal nullcones. While preliminary remarks on singular and étale cohomology may be found in Section 5, we conclude this section with some definitions and notation:

The *local cohomological dimension* of an ideal $\mathfrak{a}$ in a Noetherian ring R is

$$\operatorname{lcd}\mathfrak{a} := \sup\{k \in \mathbb{Z} \mid H^k_{\mathfrak{a}}(R) \neq 0\}.$$

For $i > \operatorname{lcd}\mathfrak{a}$, it turns out that $H^i_{\mathfrak{a}}(M)$ vanishes for each R-module M, see for example [ILL$^+$, Theorem 9.6]. The *arithmetic rank* of $\mathfrak{a}$, denoted $\operatorname{ara}\mathfrak{a}$, is the least integer k with

$$\operatorname{rad}\mathfrak{a} \ =\ \operatorname{rad}(f_1, \ldots, f_k)R$$

for elements $f_i \in R$. Since $H^\bullet_{\mathfrak{a}}(R)$ may be computed using a Čech complex on $f_1, \dots, f_k$, it follows that $H^i_{\mathfrak{a}}(R) = 0$ for each $i > \operatorname{ara}\mathfrak{a}$. Hence $\operatorname{ara}\mathfrak{a} \geqslant \operatorname{lcd}\mathfrak{a}$, and indeed this is the local cohomology obstruction used in the proof of Theorem 1.1. The corresponding result for singular cohomology is [BS, Lemma 3]:

Lemma 1.2. *Let $W \subseteq \widetilde{W}$ be affine varieties over $\mathbb{C}$, such that $\widetilde{W} \smallsetminus W$ is nonsingular of pure dimension d. If there exist k polynomials $f_1, \dots, f_k$ with*

$$W \;=\; \widetilde{W} \cap \operatorname{Var}(f_1, \dots, f_k),$$

then

$$H^{d+i}_{\text{Sing}}(\widetilde{W} \smallsetminus W, \mathbb{C}) = 0 \quad \textit{for each } i \geqslant k.$$

We use the convention that the binomial coefficient $\binom{i}{j}$ is zero for $i < j$.

2. Local cohomology of Pfaffian nullcones

Let X be an $n \times n$ alternating matrix of indeterminates over a field $\mathbb{K}$, and $\operatorname{Pf}_{2t+2}(X)$ the ideal of $\mathbb{K}[X]$ generated by the Pfaffians of the size $2t+2$ principal submatrices of X; we refer to $\mathbb{K}[X]/\operatorname{Pf}_{2t+2}(X)$ as a *Pfaffian determinantal ring*. For an equivalent description, consider the $2t \times 2t$ alternating matrix

$$\Omega := \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}, \tag{2.0.1}$$

where the remaining entries are zero, and let Y be a $2t \times n$ matrix of indeterminates over $\mathbb{K}$. In this case, $Y^{\mathrm{tr}}\Omega Y$ is an $n \times n$ alternating matrix with rank at most $2t$, so the entrywise map provides a surjective ring homomorphism

$$\mathbb{K}[X]/\operatorname{Pf}_{2t+2}(X) \;\longrightarrow\; \mathbb{K}[Y^{\mathrm{tr}}\Omega Y],$$

that one verifies is an isomorphism via a dimension count. It follows that the subring $R := \mathbb{K}[Y^{\mathrm{tr}}\Omega Y]$ of $S := \mathbb{K}[Y]$ is isomorphic to the Pfaffian determinantal ring $\mathbb{K}[X]/\operatorname{Pf}_{2t+2}(X)$. The displayed isomorphism remains valid if the field $\mathbb{K}$ is replaced by $\mathbb{Z}$.

Consider the $\mathbb{K}$-linear action of the symplectic group $\operatorname{Sp}_{2t}(\mathbb{K})$ on S, where

$$M\colon Y \longmapsto MY \quad \text{ for } M \in \operatorname{Sp}_{2t}(\mathbb{K}). \tag{2.0.2}$$

When $\mathbb{K}$ is infinite, the invariant ring is precisely the subring R, see [We], [DP, §6], or [Has, Theorem 5.1], with the nullcone ideal being the ideal of S generated by the entries of the matrix $Y^{\mathrm{tr}}\Omega Y$.

We use $\mathfrak{P}$ or $\mathfrak{P}(Y)$, as needed, to denote the ideal of S generated by the entries of $Y^{\mathrm{tr}}\Omega Y$. By [HJPS, Theorem 6.8], the ideal $\mathfrak{P}$ is prime, $S/\mathfrak{P}$ is Cohen–Macaulay, and

$$\operatorname{ht}\mathfrak{P} = \begin{cases} \dbinom{n}{2} & \text{if } n \leqslant t+1, \\[2ex] nt - \dbinom{t+1}{2} & \text{if } n \geqslant t. \end{cases} \tag{2.0.3}$$

Note that $Y^{\mathrm{tr}}\Omega Y$ is an alternating matrix, so the ideal $\mathfrak{P}$ has $\binom{n}{2}$ minimal generators. In the case that $n \leqslant t+1$, it follows that $\mathfrak{P}$ is generated by a regular sequence of length $\binom{n}{2}$, which, of course, is then the arithmetic rank of $\mathfrak{P}$. More generally:

Theorem 2.1. *Let Y be a $2t \times n$ matrix of indeterminates over a field $\mathbb{K}$ of characteristic other than two. Then the arithmetic rank of the ideal $\mathfrak{P} := I_1(Y^{\mathrm{tr}}\Omega Y)$ in $\mathbb{K}[Y]$ is*

$$\binom{n}{2} - \binom{n-2t}{2}.$$

In particular, the following are equivalent:

(1) *the ideal $\mathfrak{P}$ is generated by a regular sequence;*
(2) *the ideal $\mathfrak{P}$ is a set theoretic complete intersection;*
(3) $n \leqslant t+1$.

Proof. Let X be an $n \times n$ alternating matrix of indeterminates. The subring $R := \mathbb{K}[Y^{\mathrm{tr}}\Omega Y]$ of $S := \mathbb{K}[Y]$ is isomorphic to the Pfaffian determinantal ring $\mathbb{K}[X]/\mathrm{Pf}_{2t+2}(X)$, that has dimension $c := \binom{n}{2} - \binom{n-2t}{2}$. If the field $\mathbb{K}$ has characteristic zero, c equals $\operatorname{ara}\mathfrak{P}$ by Theorem 1.1, using the $\mathrm{Sp}_{2t}(\mathbb{K})$ action on S as in (2.0.2). More generally, as in the proof of Theorem 1.1, the homogeneous maximal ideal of R is generated, up to radical, by c homogeneous elements, namely by a homogeneous system of parameters for R. These c elements, when viewed as elements of the ring S, generate an ideal that has radical $\mathfrak{P}$. Thus, independent of the characteristic of $\mathbb{K}$, one has

$$\operatorname{ara}\mathfrak{P} \;\leqslant\; c.$$

When $n \geqslant 2t$ and the characteristic of $\mathbb{K}$ is other than two, Theorem 6.1 yields $\operatorname{ara}\mathfrak{P} \geqslant c$. Assume next that $n < 2t$. Then, following our convention regarding binomial coefficients, $c = \binom{n}{2}$, and it remains to verify that this is a lower bound for $\operatorname{ara}\mathfrak{P}$. If $n \leqslant t$, then

$$\operatorname{ara}\mathfrak{P} \;\geqslant\; \operatorname{ht}\mathfrak{P} = \binom{n}{2}$$

by (2.0.3). Suppose for some n with $t < n < 2t$, the ideal $\mathfrak{P}$ is generated up to radical by fewer than $\binom{n}{2}$ elements. Consider the specialization

$$Y' := \begin{bmatrix} y_{11} & \cdots & y_{1t} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ y_{2t,1} & \cdots & y_{2t,t} & 0 & \cdots & 0 \end{bmatrix}$$

of Y, i.e., specialize the entries beyond the first t columns to 0, and let S' be the corresponding specialization of S, which we regard as a polynomial ring in $2t \times t$ indeterminates. Then the ideal $I_1(Y'^{\mathrm{tr}}\Omega Y')S'$ is generated up to radical by fewer than $\binom{n}{2}$ elements, contradicting what we have verified in the case of a $2t \times t$ matrix of indeterminates.

For the equivalences, note that (3) $\implies$ (1) follows from (2.0.3), while (1) $\implies$ (2) is immediate. Lastly, if $n \geqslant t+2$, we see that $\operatorname{ara}\mathfrak{P} > \operatorname{ht}\mathfrak{P}$, i.e., that

$$\binom{n}{2} - \binom{n-2t}{2} \;>\; nt - \binom{t+1}{2}$$

since

$$\binom{n}{2} - nt + \binom{t+1}{2} \;=\; \binom{n-t}{2} \;>\; \binom{n-2t}{2}. \qquad \square$$

Remark 2.2. Working over the integers, one continues to have an isomorphism

$$\mathbb{Z}[X]/\operatorname{Pf}_{2t+2}(X) \;\cong\; \mathbb{Z}[Y^{\mathrm{tr}}\Omega Y],$$

as in the proof of Theorem 2.1, given by mapping the entries of the alternating matrix of indeterminates X to the corresponding entries of $Y^{\mathrm{tr}}\Omega Y$. The displayed ring admits the structure of an algebra with a straightening law (ASL), see [DP, §6] or [Ba, §4], so one obtains the analogue of a homogeneous system of parameters for the ring $\mathbb{Z}[Y^{\mathrm{tr}}\Omega Y]$ in view of [BV, Lemma 5.9]. Using the lower bound from the proof of Theorem 2.1, the formula for arithmetic rank continues to hold when $\mathbb{K}$ is replaced by $\mathbb{Z}$, as recorded next:

Corollary 2.3. *Let Y be a $2t\times n$ matrix of indeterminates over a torsion-free $\mathbb{Z}$-algebra B. Set $\mathfrak{P} := I_1(Y^{\mathrm{tr}}\Omega Y)$ in $B[Y]$. Then*

$$\operatorname{ara}\mathfrak{P} \;=\; \operatorname{lcd}\mathfrak{P} \;=\; \binom{n}{2} - \binom{n-2t}{2}.$$

Proof. Note that $\operatorname{lcd}\mathfrak{P} \leqslant \operatorname{ara}\mathfrak{P} \leqslant \binom{n}{2} - \binom{n-2t}{2} =: c$, where the second inequality uses Remark 2.2. Since a torsion-free $\mathbb{Z}$-algebra is flat, base change gives $H^c_{\mathfrak{P}}(B) \neq 0$. □

The next theorem is the analogue of [LSW, Theorem 1.2] for the Pfaffian nullcone ideals considered in this section.

Theorem 2.4. *Let Y be a $2t\times n$ matrix of indeterminates; consider the ideal $\mathfrak{P} := I_1(Y^{\mathrm{tr}}\Omega Y)$ in the polynomial ring $\mathbb{Z}[Y]$. Then:*

(1) *For each integer k, local cohomology $H^k_{\mathfrak{P}}(\mathbb{Z}[Y])$ is a torsion-free $\mathbb{Z}$-module.*
(2) *If k differs from the height of $\mathfrak{P}$, then $H^k_{\mathfrak{P}}(\mathbb{Z}[Y])$ is a $\mathbb{Q}$-vector space.*
(3) *Set $c := \binom{n}{2} - \binom{n-2t}{2}$, which is the cohomological dimension of $\mathfrak{P}$. If $n \geqslant 2t+1$, then one has a degree-preserving isomorphism*

$$H^c_{\mathfrak{P}}(\mathbb{Z}[Y]) \;\cong\; H^{2tn}_{\mathfrak{m}}(\mathbb{Q}[Y]),$$

where $\mathfrak{m}$ is the homogeneous maximal ideal of $\mathbb{Q}[Y]$ under the standard grading.

The proof uses the following two results:

Lemma 2.5. [LSW, Corollary 2.18] *Let $S := \mathbb{Z}[y_1,\dots,y_d]$ be a polynomial ring with the $\mathbb{N}$-grading $[S]_0 = \mathbb{Z}$ and $\deg y_i = 1$ for each i. Let I be a homogeneous ideal, p a prime integer, and k a nonnegative integer. Suppose that the Frobenius action on*

$$\left[H^{d-k}_{(y_1,\dots,y_d)}(S/(I+pS))\right]_0$$

is nilpotent, and that the multiplication by p map

$$H^{k+1}_I(S)_{y_i} \xrightarrow{\;p\;} H^{k+1}_I(S)_{y_i}$$

is injective for each i. Then the multiplication by p map on $H^{k+1}_I(S)$ is injective.

Lemma 2.6. [PTW, Lemma 3.2] *Let $S := \mathbb{Z}[Y]$, for Y a $2t\times n$ matrix of indeterminates, and set $\mathfrak{P} := \mathfrak{P}(Y)$. Then there exists a $(2t-2)\times(n-1)$ matrix Y' with entries from $S_{y_{11}}$, and elements $f_2,\dots,f_n$ in $S_{y_{11}}$, such that:*

(1) *the elements $Y', y_{11},\dots,y_{1n}, y_{21},\dots,y_{2t,1}, f_2,\dots,f_n$ are algebraically independent;*
(2) *along with y_{11}^{-1}, the above elements generate $S_{y_{11}}$ as a $\mathbb{Z}$-algebra;*
(3) *the ideal $\mathfrak{P}S_{y_{11}}$ equals $\mathfrak{P}(Y')S_{y_{11}} + (f_2,\dots,f_n)S_{y_{11}}$.*

As long as it is suitably interpreted, the lemma remains valid if $t=1$ (in which case there is no Y') or if $n=1$ (in which case there is no Y' and no f_i).

Proof of Theorem 2.4. Multiplication by a prime p on S induces an exact sequence

$$\longrightarrow H^k_{\mathfrak{P}}(S/pS) \xrightarrow{\delta} H^{k+1}_{\mathfrak{P}}(S) \xrightarrow{p} H^{k+1}_{\mathfrak{P}}(S) \longrightarrow H^{k+1}_{\mathfrak{P}}(S/pS) \longrightarrow ,$$

and (1) is precisely the statement that, for each p, each connecting homomorphism δ is zero. The ring $S/(\mathfrak{P}+pS)$ is Cohen–Macaulay by [HJPS, Theorem 6.8], so

$$H^k_{\mathfrak{P}}(S/pS) \neq 0 \quad \text{if and only if } \ k = \operatorname{ht}\mathfrak{P}$$

by [PS, Proposition III.4.1]. Thus, in order to prove (1) and (2), it suffices to prove the injectivity of the map

$$H^{\operatorname{ht}\mathfrak{P}+1}_{\mathfrak{P}}(S) \xrightarrow{p} H^{\operatorname{ht}\mathfrak{P}+1}_{\mathfrak{P}}(S). \tag{2.6.1}$$

We proceed by induction on t. When $t=1$, the ideal $\mathfrak{P}$ coincides with $I_2(Y)$, so the injectivity of (2.6.1) is a special case of [LSW, Theorem 1.2]. Next note that the a-invariant of the ring $S/(\mathfrak{P}+pS)$ is negative since the ring is F-regular by [PTW, Theorem 3.6] or [Lo2, Proposition 4.7], and of positive dimension. By Lemma 2.5, it now suffices to show that the map (2.6.1) is injective upon inverting each indeterminate y_{ij}, without loss of generality, y_{11}. Using Lemma 2.6, $S_{y_{11}}$ is a free module over the subring

$$\mathbb{Z}[Y', f_2 \dots, f_n],$$

and $H^{\operatorname{ht}\mathfrak{P}+1}_{\mathfrak{P}}(S)_{y_{11}} = H^{\operatorname{ht}\mathfrak{P}+1}_{\mathfrak{P}}(S_{y_{11}})$ is a direct sum of copies of

$$H^{\operatorname{ht}\mathfrak{P}+1}_{\mathfrak{P}(Y')+(f_2,\dots,f_n)}(\mathbb{Z}[Y', f_2 \dots, f_n]) \ \cong\ H^{\operatorname{ht}\mathfrak{P}-n+2}_{\mathfrak{P}(Y')}(\mathbb{Z}[Y']) \otimes_{\mathbb{Z}} H^{n-1}_{(f_2,\dots,f_n)}(\mathbb{Z}[f_2,\dots,f_n]),$$

and hence a direct sum of copies of

$$H^{\operatorname{ht}\mathfrak{P}-n+2}_{\mathfrak{P}(Y')}(\mathbb{Z}[Y']). \tag{2.6.2}$$

It is readily verified using (2.0.3) that

$$\operatorname{ht}\mathfrak{P}-n+2 \ =\ \operatorname{ht}\mathfrak{P}(Y')+1,$$

so multiplication by p on the module (2.6.2) is injective by the inductive hypothesis.

It remains to prove (3). Since $n \geqslant 2t+1$, the equivalent conditions in Theorem 2.1 give $c > \operatorname{ht}\mathfrak{P}$, so $H^c_{\mathfrak{P}}(\mathbb{Z}[Y])$ is indeed a nonzero $\mathbb{Q}$-vector space. We change notation and work with $S := \mathbb{Q}[Y]$ for the remainder of the proof. We claim that the support of $H^c_{\mathfrak{P}}(S)$ is the homogeneous maximal ideal $\mathfrak{m}$ of S, for which it suffices, without loss of generality, to verify that $H^c_{\mathfrak{P}}(S)_{y_{11}}$ is zero. Using Lemma 2.6 as before, one sees that $H^c_{\mathfrak{P}}(S)_{y_{11}}$ is a direct sum of copies of

$$H^c_{\mathfrak{P}(Y')+(f_2,\dots,f_n)}(\mathbb{Q}[Y', f_2, \dots, f_n]) \ \cong\ H^{c-n+1}_{\mathfrak{P}(Y')}(\mathbb{Q}[Y']) \otimes_{\mathbb{Q}} H^{n-1}_{(f_2,\dots,f_n)}(\mathbb{Q}[f_2,\dots,f_n]),$$

where Y' is a matrix of indeterminates of size $(2t-2)\times(n-1)$. But

$$H^{c-n+1}_{\mathfrak{P}(Y')}(\mathbb{Q}[Y']) \ =\ 0,$$

since

$$\operatorname{ara}\mathfrak{P}(Y') \ =\ \binom{n-1}{2} - \binom{(n-1)-2(t-1)}{2} \ <\ c-n+1.$$

Set D to be the ring of $\mathbb{Q}$-linear differential operators on S. Recall that $H^c_{\mathfrak{P}}(S)$ is a holonomic D-module, [Ly1, Section 2] or [ILL+, Lecture 23]. Since it has support $\{\mathfrak{m}\}$, it is isomorphic, as a D-module, to a finite direct sum of copies of $H^{2tn}_{\mathfrak{m}}(S)$, as follows from [Ka, Proposition 4.3] or [Ly2, Lemma (c), page 208]. Moreover, this isomorphism is degree-preserving by [MZ, Theorem 1.1], see also [BBL+, Section 3.2]. It remains to check that $H^c_{\mathfrak{P}}(S)$ is isomorphic to *one* copy of $H^{2tn}_{\mathfrak{m}}(S)$, and that follows from Theorem 6.1. □

Remark 2.7. The requirement that $n \geqslant 2t+1$ in Theorem 2.4 (3) is needed: if $n = 2t$, one may see that the isomorphism $H^c_{\mathfrak{P}}(\mathbb{Z}[Y]) \cong H^{2tn}_{\mathfrak{m}}(\mathbb{Q}[Y])$ does not hold by verifying, for example, that $H^c_{\mathfrak{P}}(\mathbb{Z}[Y])_{y_{11}}$ is nonzero: by Lemma 2.6 it is a direct sum of copies of

$$H^{c-n+1}_{\mathfrak{P}(Y')}(\mathbb{Z}[Y']),$$

which is nonzero since $\operatorname{lcd}\mathfrak{P}(Y') = c-n+1$.

We next prove a vanishing theorem analogous to [LSW, Theorem 1.1]:

Theorem 2.8. *Let $M = (m_{ij})$ be a $2t \times n$ matrix with entries from a commutative Noetherian ring A, where $n \geqslant 2t+1$. Set $\mathfrak{p} := I_1(M^{\mathrm{tr}}\Omega M)$ and $c := \binom{n}{2} - \binom{n-2t}{2}$. Then:*

1. *The local cohomology module $H^c_{\mathfrak{p}}(A)$ is a $\mathbb{Q}$-vector space, and thus vanishes if the canonical homomorphism $\mathbb{Z} \longrightarrow A$ is not injective.*
2. *If $\dim A \otimes_{\mathbb{Z}} \mathbb{Q} < 2tn$, then $H^c_{\mathfrak{p}}(A) = 0$, i.e., $\operatorname{lcd}\mathfrak{p} < c$.*
3. *If the images of the matrix entries m_{ij} in the ring $A \otimes_{\mathbb{Z}} \mathbb{Q}$ are algebraically dependent over a field that is a subring of $A \otimes_{\mathbb{Z}} \mathbb{Q}$, then $H^c_{\mathfrak{p}}(A) = 0$.*

Proof. For Y a $2t \times n$ matrix of indeterminates and $\mathfrak{P} := I_1(Y^{\mathrm{tr}}\Omega Y)$, Theorem 2.4 (3) gives

$$H^c_{\mathfrak{P}}(\mathbb{Z}[Y]) \;\cong\; H^{2tn}_{\mathfrak{m}}(\mathbb{Q}[Y]).$$

Consider A as a $\mathbb{Z}[Y]$-algebra via $y_{ij} \longmapsto m_{ij}$, so that $\mathfrak{P}A$ equals $\mathfrak{p}$. By base change using the right-exactness of $A \otimes_{\mathbb{Z}[Y]} -$, see for example, [LSW, Lemma 3.3], one obtains

$$H^c_{\mathfrak{p}}(A) \;\cong\; H^{2tn}_{\mathfrak{m}A}(A \otimes_{\mathbb{Z}} \mathbb{Q}). \tag{2.8.1}$$

It follows that $H^c_{\mathfrak{p}}(A)$ is a $\mathbb{Q}$-vector space, which settles (1).

For (2), note that $H^{2tn}_{\mathfrak{m}A}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ vanishes if $\dim A \otimes_{\mathbb{Z}} \mathbb{Q} < 2tn$. But then, by (2.8.1),

$$H^c_{\mathfrak{p}}(A) \;=\; 0.$$

For (3), suppose the matrix entries m_{ij} are algebraically dependent over a field $\mathbb{F}$ contained in $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Take B to be the $\mathbb{F}$-subalgebra of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the images of m_{ij} and consider the ideal $I_1(M^{\mathrm{tr}}\Omega M)$ in B. Then $\dim B < 2tn$, so (2) gives $H^c_{I_1(M^{\mathrm{tr}}\Omega M)}(B) = 0$. But then

$$H^c_{\mathfrak{p}}(A) \;\cong\; H^c_{\mathfrak{p}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) \;\cong\; H^c_{I_1(M^{\mathrm{tr}}\Omega M)}(B) \otimes_B (A \otimes_{\mathbb{Z}} \mathbb{Q})$$

vanishes as well. □

Remark 2.9. The bound $\operatorname{lcd}\mathfrak{p} < c$ in Theorem 2.8 (2) is sharp: take $S := \mathbb{Q}[Y]$ for Y a $2t \times n$ matrix of indeterminates with $n \geqslant 2t+1$ and $A := S/y_{11}S$. Note that $\dim A < 2tn$. Set $\mathfrak{p} := \mathfrak{P}A$. Multiplication by y_{11} on S induces the exact sequence

$$\longrightarrow\; H^{c-1}_{\mathfrak{p}}(A) \;\longrightarrow\; H^c_{\mathfrak{P}}(S) \;\xrightarrow{\;y_{11}\;}\; H^c_{\mathfrak{P}}(S) \;\longrightarrow\; 0,$$

where the vanishing on the right is by Theorem 2.8 (2). But multiplication by y_{11} on $H^c_{\mathfrak{P}}(S)$ has a nonzero kernel by Theorem 2.4 (3), so $H^{c-1}_{\mathfrak{p}}(A)$ is nonzero, i.e., $\operatorname{lcd}\mathfrak{p} = c-1$.

The requirement $n \geqslant 2t+1$ in Theorem 2.8 is also optimal: Suppose instead that $n \leqslant 2t$. For indeterminates y_{ij} over $\mathbb{Q}$, set

$$M := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & y_{22} & y_{23} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & y_{2t,2} & y_{2t,3} & \cdots & y_{2t,n} \end{bmatrix}$$

and $A := \mathbb{Q}[M]$. Then $\dim A < 2tn$, and we claim that $H^c_{\mathfrak{p}}(A)$ in nonzero for $\mathfrak{p} := I_1(M^{\mathrm{tr}}\Omega M)$. Note that $c = \binom{n}{2}$ in this case, and that

$$\mathfrak{p} \;=\; \mathfrak{P}(M') + (y_{22}, \dots, y_{2n}),$$

where M' is the $(2t-2) \times (n-1)$ submatrix of M obtained by deleting the first column and the first two rows. But then $H^c_{\mathfrak{p}}(A)$ is a direct sum of copies of

$$H^{c-n+1}_{\mathfrak{P}(M')}(\mathbb{Q}[M']),$$

which is nonzero by Corollary 2.3.

3. Local cohomology of generic determinantal nullcones

Let X be an $m \times n$ matrix of indeterminates over a field K; we use $I_{t+1}(X)$ to denote the ideal of $K[X]$ generated by the size $t+1$ minors of X. The *determinantal ring* $\mathbb{K}[X]/I_{t+1}(X)$ is a subring of a polynomial ring as follows: Taking Y and Z to be matrices of indeterminates of sizes $m \times t$ and $t \times n$ respectively, the product matrix YZ has rank at most t, and the entrywise map provides an isomorphism

$$\mathbb{K}[X]/I_{t+1}(X) \;\longrightarrow\; \mathbb{K}[YZ].$$

It follows that the subring $R := \mathbb{K}[YZ]$ of $S := \mathbb{K}[Y,Z]$ is isomorphic to $\mathbb{K}[X]/I_{t+1}(X)$. This isomorphism remains valid if the field $\mathbb{K}$ is replaced by $\mathbb{Z}$.

Consider the $\mathbb{K}$-linear action of the general linear group $\mathrm{GL}_t(\mathbb{K})$ on S, where an element $M \in \mathrm{GL}_t(\mathbb{K})$ acts via

$$M\colon \begin{cases} Y & \longmapsto YM^{-1} \\ Z & \longmapsto MZ. \end{cases} \tag{3.0.1}$$

When $\mathbb{K}$ is infinite, the invariant ring is precisely the subring R, see [We], [DP, §3], or [Has, Theorem 4.1].

We use $\mathfrak{A}$ to denote the ideal of S generated by the entries of the product matrix YZ. Unlike the Pfaffian case, the ideal $\mathfrak{A}$ need not be prime or even equidimensional; its irreducible components correspond to varieties of complexes that have been studied extensively, beginning with Buchsbaum–Eisenbud [BE]. For the case at hand, consider a complex of $\mathbb{K}$-vector spaces

$$\mathbb{K}^m \xleftarrow{\;M\;} \mathbb{K}^t \xleftarrow{\;N\;} \mathbb{K}^n,$$

and regard the matrix entries of M,N as a point in affine space $\mathbb{A}^{mt+tn}_{\mathbb{K}}$. Note that

$$\operatorname{rank} M + \operatorname{rank} N \;\leqslant\; t.$$

Fixing nonnegative integers i,j with $i+j \leqslant t$, the corresponding *variety of complexes* is the algebraic set consisting of matrices M,N with $\operatorname{rank} M \leqslant i$, $\operatorname{rank} N \leqslant j$, and $MN = 0$. The defining ideal of this variety is

$$\mathfrak{p}_{i,j} := I_{i+1}(Y) + I_{j+1}(Z) + \mathfrak{A}.$$

The ring $S/\mathfrak{p}_{i,j}$ has rational singularities if $\mathbb{K}$ has characteristic zero, [Ke1, Ke2], and is F-regular if $\mathbb{K}$ has positive characteristic, [Lo1, Corollary 4.2], [PTW, Theorem 5.6]. The ideal $\mathfrak{A}$ equals the intersection of the $\mathfrak{p}_{i,j}$ with $i+j = t$. If $i \leqslant m$ and $j \leqslant n$, then

$$\operatorname{ht} \mathfrak{p}_{i,j} = (m-i)(t-i) + (n-j)(t-j) + ij, \tag{3.0.2}$$

see for example [Hu] or [DS]. Our first result in this section concerns the arithmetic rank of the ideal $\mathfrak{A}$:

Theorem 3.1. *Let Y and Z be matrices of indeterminates of sizes $m\times t$ and $t\times n$ respectively, over a field $\mathbb{K}$ of characteristic other than two. Then the arithmetic rank of the ideal $\mathfrak{A} := I_1(YZ)$ in $\mathbb{K}[Y,Z]$ is*

$$\operatorname{ara}\mathfrak{A} = \begin{cases} mt+nt-t^2 & \text{if } t<\min\{m,n\},\\ mn & \text{otherwise}.\end{cases}$$

The following are equivalent:

(1) *the ideal $\mathfrak{A}$ is generated by a regular sequence;*
(2) *the ideal $\mathfrak{A}$ is a set theoretic complete intersection;*
(3) $m+n\leqslant t+1$.

Proof. Let X be an $m\times n$ matrix of indeterminates. The subring $R:=\mathbb{K}[YZ]$ of $S:=\mathbb{K}[Y,Z]$ is isomorphic to the generic determinantal ring $\mathbb{K}[X]/I_{t+1}(X)$, that has dimension

$$c := \begin{cases} mt+nt-t^2 & \text{if } t<\min\{m,n\},\\ mn & \text{otherwise}.\end{cases}$$

If the field $\mathbb{K}$ has characteristic zero, the dimension c equals $\operatorname{ara}\mathfrak{A}$ by Theorem 1.1, using the $\mathrm{GL}_t(\mathbb{K})$ action on S as in (3.0.1). More generally, as in the proof of Theorem 1.1, the homogeneous maximal ideal of R is generated, up to radical, by a homogeneous system of parameters for R, and these c elements generate an ideal of S that has radical $\mathfrak{A}$. Thus, independent of the characteristic of $\mathbb{K}$, one has

$$\operatorname{ara}\mathfrak{A} \ \leqslant\ c.$$

Assume for the rest of the proof that the characteristic of $\mathbb{K}$ is not two. If $t\leqslant\min\{m,n\}$, Theorem 7.1 yields $\operatorname{ara}\mathfrak{A}\geqslant c$. For the remaining case, assume without loss of generality that $m\leqslant n$, and that $t>m$. We need to verify that $c=mn$ is a lower bound for $\operatorname{ara}\mathfrak{A}$. Suppose $\mathfrak{A}$ can be generated up to radical by fewer than mn elements. Consider the specializations

$$Y' := \begin{bmatrix} y_{11} & \cdots & y_{1m} & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & & \vdots\\ y_{m1} & \cdots & y_{mm} & 0 & \cdots & 0\end{bmatrix} \qquad\text{and}\qquad Z' := \begin{bmatrix} z_{11} & \cdots & z_{1n}\\ \vdots & & \vdots\\ z_{m1} & \cdots & z_{mn}\\ 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{bmatrix}$$

of Y and Z respectively, i.e., the entries of Y beyond the first m columns and the entries of Z beyond the first m rows are specialized to 0. Let S' denote the corresponding specialization of S. Then the ideal $I_1(Y'Z')S'$ is generated up to radical by fewer than mn elements, contradicting what we have verified earlier.

Among the equivalent conditions, (1) $\Longrightarrow$ (2) is immediate. For (2) $\Longrightarrow$ (3), since the ideal $\mathfrak{A}$ is a set theoretic complete intersection by assumption, minimal primes of $\mathfrak{A}$ must have the same height. If $t<n$, then $\mathfrak{p}_{0,t}$ is one of the minimal primes, so

$$\operatorname{ht}\mathfrak{A} \ =\ \operatorname{ht}\mathfrak{p}_{0,t} \ =\ mt \ < mt+nt-t^2 \ =\ \operatorname{ara}\mathfrak{A},$$

a contradiction. Similarly, one cannot have $t<m$. Thus, $t\geqslant m$ and $t\geqslant n$; it follows that $\operatorname{ara}\mathfrak{A}=mn$. If $t\leqslant m+n-2$, we obtain a contradiction: the ideal $\mathfrak{p}_{m-1,t-m+1}$ is a minimal prime of $\mathfrak{A}$, but

$$\operatorname{ht}\mathfrak{p}_{m-1,t-m+1} \ =\ mn+t-m-n+1 \ <\ mn =\ \operatorname{ara}\mathfrak{A}.$$

It remains to prove $(3) \implies (1)$. It suffices to prove that $S/\mathfrak{A}$ is a complete intersection ring after specializing the entries of $t+1-(m+n)$ columns of Y and $t+1-(m+n)$ rows of Z to zero, since this leaves the number of defining equations unchanged. Thus, we may assume that $m+n=t+1$, in which case

$$\mathfrak{A} \;=\; \mathfrak{p}_{m-1,n} \cap \mathfrak{p}_{m,n-1}.$$

Since $\operatorname{ht} \mathfrak{p}_{m-1,n} = mn = \operatorname{ht} \mathfrak{p}_{m,n-1}$, it follows that $\operatorname{ht} \mathfrak{A} = mn$, as desired. We remark that in this case the ideals $\mathfrak{p}_{m-1,n}$ and $\mathfrak{p}_{m,n-1}$ are *geometrically linked*, and a local cohomology sequence shows that the ring $S/\mathfrak{p}_{m-1,n-1}$ is Gorenstein. □

As in Remark 2.2, the theorem remains valid if the field $\mathbb{K}$ is replaced by $\mathbb{Z}$, since one has an isomorphism

$$\mathbb{Z}[X]/I_{t+1}(X) \;\cong\; \mathbb{Z}[YZ],$$

where the ring above has an ASL structure, [BV, Chapter 4]. Using this, one obtains:

Corollary 3.2. *Let Y and Z be $m \times t$ and $t \times n$ matrices of indeterminates over a torsion-free $\mathbb{Z}$-algebra B. Set $\mathfrak{A} := I_1(YZ)$ in $B[Y,Z]$. Then*

$$\operatorname{ara} \mathfrak{A} \;=\; \operatorname{lcd} \mathfrak{A} \;=\; \begin{cases} mt+nt-t^2 & \text{if } t < \min\{m,n\}, \\ mn & \text{otherwise}. \end{cases}$$

Corresponding to Theorem 2.4, for determinantal nullcones one has:

Theorem 3.3. *Let Y and Z be $m \times t$ and $t \times n$ matrices of indeterminates respectively. Consider the ideal $\mathfrak{A} := I_1(YZ)$ in $\mathbb{Z}[Y,Z]$. Then:*

1. *For each integer k, local cohomology $H^k_{\mathfrak{A}}(\mathbb{Z}[Y,Z])$ is a torsion-free $\mathbb{Z}$-module.*
2. *Suppose $1 < t < \min\{m,n\}$. Set $c := mt+nt-t^2$, which is the cohomological dimension of $\mathfrak{A}$. Then one has a degree-preserving isomorphism*

$$H^c_{\mathfrak{A}}(\mathbb{Z}[Y,Z]) \;\cong\; H^{mt+tn}_{\mathfrak{m}}(\mathbb{Q}[Y,Z]),$$

where $\mathfrak{m}$ is the homogeneous maximal ideal of $\mathbb{Q}[Y,Z]$ under the standard grading.

The following will be used in the proof:

Lemma 3.4. [PTW, Lemma 5.1] *Let Y and Z be matrices of indeterminates of sizes $m \times t$ and $t \times n$ respectively; set $S := \mathbb{Z}[Y,Z]$. Let Z' be the submatrix of Z obtained by deleting the first row. Then there exists an $(m-1) \times (t-1)$ matrix Y' with entries from $S_{y_{11}}$, and elements $f_1, \dots, f_n \in S_{y_{11}}$ such that:*

1. *The elements $Y', Z', y_{11}, \dots, y_{1t}, y_{21}, \dots, y_{m1}, f_1, \dots, f_n$ are algebraically independent;*
2. *Along with y_{11}^{-1}, the above elements generate $S_{y_{11}}$ as a $\mathbb{Z}$-algebra;*
3. *The ideal $I_1(YZ)S_{y_{11}}$ equals $I_1(Y'Z')S_{y_{11}} + (f_1, \dots, f_n)S_{y_{11}}$.*

Lemma 3.5. *Let Y and Z be $m \times t$ and $t \times n$ matrices of indeterminates respectively. Set $S := \mathbb{K}[Y,Z]$ for $\mathbb{K}$ a field of positive characteristic, and set $\mathfrak{A} := I_1(YZ)$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of S. If $t \geqslant 2$, then*

$$[H^\bullet_{\mathfrak{m}}(S/\mathfrak{A})]_0 \;=\; 0.$$

Proof. For ℓ an integer with $0 \leqslant \ell \leqslant t$, set

$$\mathfrak{a}_\ell := \bigcap_{i=0}^{\ell} \mathfrak{p}_{i,t-i}.$$

Suppose $\ell \leqslant t-1$. Up to taking radicals, the ideal

$$\mathfrak{a}_\ell + \mathfrak{p}_{\ell+1,t-\ell-1}$$

coincides with

$$\begin{aligned}(\mathfrak{p}_{0,t} + \mathfrak{p}_{\ell+1,t-\ell-1}) \cap (\mathfrak{p}_{1,t-1} + \mathfrak{p}_{\ell+1,t-\ell-1}) \cap \cdots \cap (\mathfrak{p}_{\ell,t-\ell} + \mathfrak{p}_{\ell+1,t-\ell-1}) \\ = \mathfrak{p}_{0,t-\ell-1} \cap \mathfrak{p}_{1,t-\ell-1} \cap \cdots \cap \mathfrak{p}_{\ell,t-\ell-1} = \mathfrak{p}_{\ell,t-\ell-1},\end{aligned}$$

since $\mathfrak{p}_{i_1,j_1} + \mathfrak{p}_{i_2,j_2} = \mathfrak{p}_{i,j}$ for $i := \min\{i_1, i_2\}$ and $j := \min\{j_1, j_2\}$. But

$$\mathfrak{p}_{\ell,t-\ell-1} \subseteq \mathfrak{a}_\ell + \mathfrak{p}_{\ell+1,t-\ell-1},$$

and $\mathfrak{p}_{\ell,t-\ell-1}$ is prime, so one has the equality

$$\mathfrak{p}_{\ell,t-\ell-1} = \mathfrak{a}_\ell + \mathfrak{p}_{\ell+1,t-\ell-1} \tag{3.5.1}$$

and hence an exact sequence

$$0 \longrightarrow S/\mathfrak{a}_{\ell+1} \longrightarrow S/\mathfrak{a}_\ell \oplus S/\mathfrak{p}_{\ell+1,t-\ell-1} \longrightarrow S/\mathfrak{p}_{\ell,t-\ell-1} \longrightarrow 0.$$

We examine the degree 0 strand of the induced local cohomology exact sequence

$$\longrightarrow H^k_\mathfrak{m}(S/\mathfrak{a}_{\ell+1}) \longrightarrow H^k_\mathfrak{m}(S/\mathfrak{a}_\ell) \oplus H^k_\mathfrak{m}(S/\mathfrak{p}_{\ell+1,t-\ell-1}) \longrightarrow H^k_\mathfrak{m}(S/\mathfrak{p}_{\ell,t-\ell-1}) \longrightarrow$$

as follows. The rings $S/\mathfrak{p}_{i,j}$ are F-regular for $i+j \leqslant t$, and those that occur above are of positive dimension, so

$$\left[H^k_\mathfrak{m}(S/\mathfrak{p}_{i,j})\right]_0 = 0,$$

bearing in mind that $S/\mathfrak{p}_{0,0}$ does not occur; it is here that we use $t \geqslant 2$. Hence for each k, ℓ, one has an isomorphism

$$\left[H^k_\mathfrak{m}(S/\mathfrak{a}_{\ell+1})\right]_0 \cong \left[H^k_\mathfrak{m}(S/\mathfrak{a}_\ell)\right]_0$$

and the desired result follows by induction on ℓ: for the base case of the induction, note that $S/\mathfrak{a}_0 \cong \mathbb{K}[Z]$ is a polynomial ring. □

Lemma 3.6. *Let Y and Z be $m \times t$ and $t \times n$ matrices of indeterminates respectively, over a field $\mathbb{K}$ of positive characteristic. If $1 < t < \min\{m,n\}$, then*

$$\operatorname{lcd} I_1(YZ) < mt + nt - t^2.$$

Proof. Set $S := \mathbb{K}[Y,Z]$, and recall that each $S/\mathfrak{p}_{i,j}$ is a Cohen–Macaulay ring of positive prime characteristic, so $\operatorname{lcd} \mathfrak{p}_{i,j} = \operatorname{ht} \mathfrak{p}_{i,j}$ by [PS, Proposition III.4.1]. Set $\mathfrak{a}_\ell := \bigcap_{i=0}^{\ell} \mathfrak{p}_{i,t-i}$ as in the proof of Lemma 3.5. We use induction on ℓ to prove that

$$\operatorname{lcd} \mathfrak{a}_\ell < mt + nt - t^2$$

for each $0 \leqslant \ell \leqslant t$. When $\ell = 0$, this is simply the verification that

$$\operatorname{ht} \mathfrak{p}_{0,t} = mt < mt + nt - t^2.$$

For the inductive step, recall that

$$\mathfrak{p}_{\ell,t-\ell-1} = \mathfrak{a}_\ell + \mathfrak{p}_{\ell+1,t-\ell-1}$$

by (3.5.1), so one has a Mayer–Vietoris sequence

$$\longrightarrow H^k_{\mathfrak{a}_\ell}(S) \oplus H^k_{\mathfrak{p}_{\ell+1,t-\ell-1}}(S) \longrightarrow H^k_{\mathfrak{a}_{\ell+1}}(S) \longrightarrow H^{k+1}_{\mathfrak{p}_{\ell,t-\ell-1}}(S) \longrightarrow$$

which gives

$$\operatorname{lcd} \mathfrak{a}_{\ell+1} \leqslant \max\left\{\operatorname{lcd} \mathfrak{a}_\ell,\ \operatorname{lcd} \mathfrak{p}_{\ell+1,t-\ell-1},\ \operatorname{lcd} \mathfrak{p}_{\ell,t-\ell-1} - 1\right\}.$$

It now suffices to verify that

$$\operatorname{ht}\mathfrak{p}_{\ell+1,t-\ell-1} \;<\; mt+nt-t^2 \qquad\text{and}\qquad \operatorname{ht}\mathfrak{p}_{\ell,t-\ell-1}-1 \;<\; mt+nt-t^2$$

for $0\leqslant \ell\leqslant t-1$. In view of (3.0.2), these simplify respectively as

$$0 \;<\; (n-t)(t-\ell-1)+(m-\ell-1)(\ell+1) \qquad\text{and}\qquad 0 \;<\; (n-t)(t-\ell-1)+\ell(m-\ell-1),$$

each of which holds since $1<t<\min\{m,n\}$. $\square$

Proof of Theorem 3.3. Set $S:=\mathbb{Z}[Y,Z]$, and let p be a prime integer. We induce on t to prove the injectivity of the map

$$H^{k+1}_{\mathfrak{A}}(S) \xrightarrow{\;p\;} H^{k+1}_{\mathfrak{A}}(S) \tag{3.6.1}$$

for each integer k. In the case $t=1$ one has $\mathfrak{A}=I_1(Y)\cap I_1(Z)$, and the Mayer–Vietoris sequence provides a commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H^k_{\mathfrak{m}}(S) & \longrightarrow & H^k_{I_1(Y)}(S)\oplus H^k_{I_1(Z)}(S) & \longrightarrow & H^k_{\mathfrak{A}}(S) & \longrightarrow & H^{k+1}_{\mathfrak{m}}(S) & \longrightarrow\\
& \big\downarrow p & & \big\downarrow p & & \big\downarrow p & & \big\downarrow p & \\
\longrightarrow & H^k_{\mathfrak{m}}(S) & \longrightarrow & H^k_{I_1(Y)}(S)\oplus H^k_{I_1(Z)}(S) & \longrightarrow & H^k_{\mathfrak{A}}(S) & \longrightarrow & H^{k+1}_{\mathfrak{m}}(S) & \longrightarrow
\end{array}$$

where $\mathfrak{m}:=I_1(Y)+I_1(Z)$. Since p is injective on $H^\bullet_{I_1(Y)}(S)$, $H^\bullet_{I_1(Z)}(S)$, and $H^\bullet_{\mathfrak{m}}(S)$, the claim follows from a routine diagram chase.

Next, suppose that $t\geqslant 2$. Since $[H^\bullet_{\mathfrak{m}}(S/(\mathfrak{A}+pS))]_0=0$ by Lemma 3.5, it suffices in view of Lemma 2.5 to show that the map (3.6.1) is injective after inverting any y_{ij} or z_{ij}, without loss of generality, y_{11}. In the notation of Lemma 3.4, the ring $S_{y_{11}}$ is a free module over the polynomial subring

$$\mathbb{Z}[Y',Z',f_1,\dots,f_n],$$

where Y' and Z' are size $(m-1)\times(t-1)$ and $(t-1)\times n$ respectively, and

$$\mathfrak{A}S_{y_{11}} \;=\; I_1(Y'Z')S_{y_{11}}+(f_1,\dots,f_n)S_{y_{11}}.$$

Therefore $H^{k+1}_{\mathfrak{A}}(S)_{y_{11}}=H^{k+1}_{\mathfrak{A}}(S_{y_{11}})$ is a direct sum of copies of

$$H^{k+1-n}_{I_1(Y'Z')}(\mathbb{Z}[Y',Z'])\otimes_{\mathbb{Z}} H^n_{(f_1,\dots,f_n)}(\mathbb{Z}[f_1,\dots,f_n]),$$

and hence a direct sum of copies of

$$H^{k+1-n}_{I_1(Y'Z')}(\mathbb{Z}[Y',Z']).$$

By the inductive hypothesis, multiplication by p is injective on the module displayed above, and hence on $H^{k+1}_{\mathfrak{A}}(S)_{y_{11}}$. This completes the proof of (1).

Next, assume that $1<t<\min\{m,n\}$. We first prove that $H^c_{\mathfrak{A}}(S)$ is p-divisible. Since c is the cohomological dimension of the ideal $\mathfrak{A}$, one has an exact sequence

$$\longrightarrow\; H^c_{\mathfrak{A}}(S) \xrightarrow{\;p\;} H^c_{\mathfrak{A}}(S) \longrightarrow H^c_{\mathfrak{A}}(S/pS) \longrightarrow 0,$$

and it suffices to verify that $H^c_{\mathfrak{A}}(S/pS)=0$. This holds by Lemma 3.6, and it follows that

$$H^c_{\mathfrak{A}}(\mathbb{Z}[Y,Z]) \;=\; H^c_{\mathfrak{A}}(\mathbb{Q}[Y,Z]).$$

For the rest of the proof, we change notation and work with $S:=\mathbb{Q}[Y,Z]$. We next claim that $H^c_{\mathfrak{A}}(S)$ has support $\{\mathfrak{m}\}$. For this it suffices, without loss of generality, to verify that $H^c_{\mathfrak{A}}(S)_{y_{11}}$ vanishes. As with the proof of (1), using Lemma 3.4, one sees that $H^c_{\mathfrak{A}}(S)_{y_{11}}$ is a direct sum of copies of

$$H^{c-n}_{I_1(Y'Z')}(\mathbb{Q}[Y',Z'])\otimes_{\mathbb{Q}} H^n_{(f_1,\dots,f_n)}(\mathbb{Q}[f_1,\dots,f_n]),$$

where Y' and Z' are matrices of indeterminates of sizes $(m-1)\times(t-1)$ and $(t-1)\times n$ respectively. But

$$H^{c-n}_{I_1(Y'Z')}(\mathbb{Q}[Y',Z']) \;=\; 0,$$

since

$$\operatorname{ara} I_1(Y'Z') \;=\; (m-1)(t-1)+n(t-1)-(t-1)^2 \;=\; c-n-m+t \;<\; c-n.$$

Set D to be the ring of $\mathbb{Q}$-linear differential operators on S. As in the proof of Theorem 2.4 (3), $H^c_{\mathfrak{A}}(\mathbb{Q}[Y,Z])$ is a holonomic D-module with support $\{\mathfrak{m}\}$, hence a direct sum of copies of $H^{mt+nt}_{\mathfrak{m}}(S)$. That it is exactly one copy follows from Theorem 7.1. $\square$

Remark 3.7. The requirement that $1<t<\min\{m,n\}$ in Theorem 3.3 (2) is indeed essential: If $t=1$, then $H^c_{\mathfrak{A}}(\mathbb{Z}[Y,Z]) = H^{m+n-1}_{\mathfrak{A}}(\mathbb{Z}[Y,Z])$ is not a $\mathbb{Q}$-vector space since in this case $\mathfrak{A} = I_1(Y)\cap I_1(Z)$, and a Mayer–Vietoris sequence shows that the cokernel of

$$H^{m+n-1}_{\mathfrak{A}}(S) \;\xrightarrow{\;p\;}\; H^{m+n-1}_{\mathfrak{A}}(S)$$

is nonzero for p any prime integer. If $t=m$, then $c=mn$, and the isomorphism in Theorem 3.3 (2) does not hold since $H^{mn}_{\mathfrak{A}}(\mathbb{Z}[Y,Z])_{y_{11}}$ is nonzero, being a direct sum of copies of

$$H^{mn-n}_{I_1(Y'Z')}(\mathbb{Z}[Y',Z'])$$

by Lemma 3.4, where Y' and Z' are size $(m-1)\times(m-1)$ and $(m-1)\times n$ respectively; note that $\operatorname{lcd} I_1(Y'Z') = mn-n$ by Corollary 3.2.

We record the analogue of Theorem 2.8 for determinantal nullcones:

Theorem 3.8. *Let M and N be $m\times t$ and $t\times n$ matrices with entries from a commutative Noetherian ring A, where $1<t<\min\{m,n\}$. Set $\mathfrak{a} := I_1(MN)$ and $c := mt+nt-t^2$. Then:*

1. *The local cohomology module $H^c_{\mathfrak{a}}(A)$ is a $\mathbb{Q}$-vector space, and thus vanishes if the canonical homomorphism $\mathbb{Z}\longrightarrow A$ is not injective.*
2. *If $\dim A\otimes_{\mathbb{Z}}\mathbb{Q} < mt+nt$, then $H^c_{\mathfrak{a}}(A)=0$, i.e., $\operatorname{lcd}\mathfrak{a}<c$.*
3. *If the images of the matrix entries in the ring $A\otimes_{\mathbb{Z}}\mathbb{Q}$ are algebraically dependent over a field that is a subring of $A\otimes_{\mathbb{Z}}\mathbb{Q}$, then $H^c_{\mathfrak{a}}(A)=0$.*

Proof. For matrices of indeterminates Y and Z, set $S:=\mathbb{Z}[Y,Z]$, and regard A as an S-algebra via $y_{ij}\longmapsto m_{ij}$ and $z_{ij}\longmapsto n_{ij}$, so that $\mathfrak{a}$ equals $I_1(YZ)A$. By Theorem 3.3 (2)

$$H^c_{I_1(YZ)}(\mathbb{Z}[Y,Z]) \;\cong\; H^{mt+tn}_{\mathfrak{m}}(\mathbb{Q}[Y,Z]),$$

with $\mathfrak{m}$ the homogeneous maximal ideal of $\mathbb{Q}[Y,Z]$ and base change along $S\longrightarrow A$ gives

$$H^c_{\mathfrak{a}}(A) \;\cong\; H^{mt+nt}_{\mathfrak{m}A}(A\otimes_{\mathbb{Z}}\mathbb{Q}),$$

from which the assertions follow as in the proof of Theorem 2.8. $\square$

Remark 3.9. The bound $\operatorname{lcd}\mathfrak{a}<c$ in Theorem 3.8 (2) is sharp: take $S:=\mathbb{Q}[Y,Z]$ where Y and Z are $m\times t$ and $t\times n$ matrices of indeterminates respectively, and $t<\min\{m,n\}$. Set $\mathfrak{A}:=I_1(YZ)$, $A:=S/y_{11}S$, and $\mathfrak{a}:=\mathfrak{A}A$. Note that $\dim A<mt+nt$. Let c be as in the above theorem. Multiplication by y_{11} on S induces the exact sequence

$$\longrightarrow\; H^{c-1}_{\mathfrak{a}}(A) \;\longrightarrow\; H^c_{\mathfrak{A}}(S) \;\xrightarrow{\;y_{11}\;}\; H^c_{\mathfrak{A}}(S) \;\longrightarrow\; 0,$$

where the vanishing on the right is by Theorem 3.8 (2). Multiplication by y_{11} on $H^c_{\mathfrak{A}}(S)$ has a nonzero kernel by Theorem 3.3 (2), so $H^{c-1}_{\mathfrak{a}}(A)$ is nonzero, i.e., $\operatorname{lcd}\mathfrak{a}=c-1$.

The requirement $1 < t < \min\{m,n\}$ in Theorem 3.8 is needed: Suppose instead that one has $t = m$. For indeterminates y_{ij} and z_{ij} over $\mathbb{Q}$, set

$$M := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & y_{22} & \cdots & y_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & y_{m2} & \cdots & y_{mm} \end{bmatrix}, \qquad N := \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix},$$

and $A := \mathbb{Q}[M,N]$. Then $\dim A < m^2 + mn$ and $c = mn$, and we shall see that $H^{mn}_{I_1(MN)}(A)$ is nonzero. Note that

$$I_1(MN) \;=\; I_1(M'N') + (z_{11}, \ldots, z_{1n}),$$

where M' is the $(m-1)\times(m-1)$ submatrix of M obtained by deleting the first row and the first column, and N' is the submatrix of N obtained by deleting the first row. The nonvanishing of $H^{mn}_{I_1(MN)}(A)$ is now a consequence of the nonvanishing of $H^{mn-n}_{I_1(M'N')}(A)$, that in turn holds by Corollary 3.2.

4. Local cohomology of symmetric determinantal nullcones

Let X be a symmetric $n\times n$ matrix of indeterminates over a field K, and $I_{t+1}(X)$ the ideal generated by the size $t+1$ minors of X. The *symmetric determinantal ring* $\mathbb{K}[X]/I_{t+1}(X)$ is a subring of a polynomial ring: take Y to be a $t\times n$ matrix of indeterminates, in which case the product matrix $Y^{\mathrm{tr}}Y$ has rank at most t, and the entrywise map yields an isomorphism

$$\mathbb{K}[X]/I_{t+1}(X) \;\longrightarrow\; \mathbb{K}[Y^{\mathrm{tr}}Y].$$

Hence the subring $R := \mathbb{K}[Y^{\mathrm{tr}}Y]$ of $S := \mathbb{K}[Y]$ is isomorphic to $\mathbb{K}[X]/I_{t+1}(X)$, and the isomorphism remains valid if the field $\mathbb{K}$ is replaced by $\mathbb{Z}$.

The orthogonal group $\mathrm{O}_t(\mathbb{K})$ acts $\mathbb{K}$-linearly on S via

$$M\colon Y \longmapsto MY \qquad \text{for } M \in \mathrm{O}_t(\mathbb{K}). \tag{4.0.1}$$

When the field $\mathbb{K}$ is infinite, of characteristic other than 2, the invariant ring is precisely the subring R, see [We] or [DP, §5].

We use $\mathfrak{B}$ to denote the ideal of S generated by the entries of $Y^{\mathrm{tr}}Y$. Since $Y^{\mathrm{tr}}Y$ is symmetric, the ideal $\mathfrak{B}$ has $\binom{n+1}{2}$ minimal generators; in the case $n \leqslant (t+1)/2$, this is also the height of $\mathfrak{B}$, see [HJPS, Theorem 7.1]. Suppose next that $n > (t+1)/2$. Then

$$\operatorname{ht}\mathfrak{B} = \begin{cases} ns - \dbinom{s}{2} & \text{if } t = 2s, \\[2ex] ns + n - \dbinom{s+1}{2} & \text{if } t = 2s+1. \end{cases}$$

If t is odd or if the field $\mathbb{K}$ has characteristic two, then $\operatorname{rad}\mathfrak{B}$ is prime and $S/(\operatorname{rad}\mathfrak{B})$ is Cohen–Macaulay; if t is even and $\mathbb{K}$ contains a primitive fourth root of unity, then $\mathfrak{B}$ has minimal primes $\mathfrak{P}$ and $\mathfrak{Q}$ of the same height, with each of $S/\mathfrak{P}$, $S/\mathfrak{Q}$, $S/(\mathfrak{P}+\mathfrak{Q})$ being a Cohen–Macaulay integral domain; moreover,

$$\operatorname{ht}(\mathfrak{P}+\mathfrak{Q}) \;=\; ns+n+1-\binom{s+1}{2},$$

see Theorems 7.2, 7.12, and 7.13 of [HJPS]. Regarding the arithmetic rank of $\mathfrak{B}$, we prove:

Theorem 4.1. *Let Y be a $t \times n$ matrix of indeterminates over a field $\mathbb{K}$ of characteristic other than two. Then the arithmetic rank of the ideal $\mathfrak{B} := I_1(Y^{\mathrm{tr}}Y)$ in $\mathbb{K}[Y]$ is*

$$\binom{n+1}{2} - \binom{n+1-t}{2}.$$

The ideal $\mathfrak{B}$ is a set theoretic complete intersection if and only if $t = 1$ or $n \leqslant (t+1)/2$.

Proof. Let X be a symmetric $n \times n$ matrix of indeterminates. The subring $R := \mathbb{K}[Y^{\mathrm{tr}}Y]$ of $S := \mathbb{K}[Y]$ is isomorphic to $\mathbb{K}[X]/I_{t+1}(X)$ that has dimension

$$c := \binom{n+1}{2} - \binom{n+1-t}{2}.$$

If $\mathbb{K}$ has characteristic zero, then c equals $\operatorname{ara}\mathfrak{B}$ by Theorem 1.1 in view of the $\mathrm{O}_t(\mathbb{K})$ action (4.0.1). More generally, a homogeneous system of parameters for R generates $\mathfrak{B}$ up to radical, so

$$\operatorname{ara}\mathfrak{B} \;\leqslant\; c$$

independent of the characteristic.

Assume next that $\mathbb{K}$ has characteristic other than two. If $n \geqslant t$, the reverse inequality follows from Theorem 8.1. If $t > n$, we need to verify that $c = \binom{n+1}{2}$ is a lower bound for $\operatorname{ara}\mathfrak{B}$. If the arithmetic rank were less than c, considering the specialization

$$Y' := \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

yields a contradiction.

For the equivalent statements, we have already observed that the ideal $\mathfrak{B}$ is generated by a regular sequence if $n \leqslant (t+1)/2$, whereas, if $t = 1$, then $\mathfrak{B}$ is the square of the homogeneous maximal ideal of $\mathbb{K}[Y]$. It only remains to verify that $\operatorname{ara}\mathfrak{B} > \operatorname{ht}\mathfrak{B}$ outside of these cases.

If $t = 2s+1$, the required verification is

$$\binom{n+1}{2} - \binom{n-2s}{2} \;>\; ns + n - \binom{s+1}{2},$$

which holds since

$$\binom{n+1}{2} - ns - n + \binom{s+1}{2} \;=\; \binom{n-s}{2} \;>\; \binom{n-2s}{2}$$

as long as $n \geqslant s+2$ and $s \geqslant 1$.

If $t = 2s$, we need to check that

$$\binom{n+1}{2} - \binom{n+1-2s}{2} \;>\; ns - \binom{s}{2},$$

equivalently,

$$\binom{n+1}{2} - ns + \binom{s}{2} \;=\; \binom{n-s+1}{2} \;>\; \binom{n+1-2s}{2},$$

which holds if $n \geqslant s+1$. $\square$

As with Pfaffian rings and determinantal rings, e.g., Remark 2.2, the ring

$$\mathbb{Z}[X]/I_{t+1}(X) \;\cong\; \mathbb{Z}[Y^{\mathrm{tr}}Y],$$

has an ASL structure, see [DP, §5] or [Ba, §3], so one obtains the analogue of a homogeneous system of parameters in $\mathbb{Z}[Y^{\mathrm{tr}}Y]$. Using this:

Corollary 4.2. *Let Y be a $t \times n$ matrix of indeterminates over a torsion-free $\mathbb{Z}$-algebra B. Set $\mathfrak{B} := I_1(Y^{\mathrm{tr}}Y)$ in $B[Y]$. Then*

$$\operatorname{ara}\mathfrak{B} \;=\; \operatorname{lcd}\mathfrak{B} \;=\; \binom{n+1}{2} - \binom{n+1-t}{2}.$$

We next record some preliminary results towards studying $H^c_{\mathfrak{B}}(\mathbb{Z}[Y])$.

Lemma 4.3. *Let Y be a $t \times n$ matrix of indeterminates over a field $\mathbb{K}$ of positive characteristic. Set $s := \lfloor t/2 \rfloor$ and take $\mathfrak{B}$ to be the ideal $I_1(Y^{\mathrm{tr}}Y)$ in $S := \mathbb{K}[Y]$.*

If $n \geqslant s$, equivalently $n \geqslant (t-1)/2$, then

$$\operatorname{lcd}\mathfrak{B} \;\leqslant\; ns + n - \binom{s+1}{2}$$

with equality holding if the characteristic of $\mathbb{K}$ is odd, and also when t is odd.

Proof. If $\mathbb{K}$ has characteristic 2, by [HJPS, Theorem 7.2] $\operatorname{rad}\mathfrak{B}$ is a perfect ideal with

$$\operatorname{ht}\mathfrak{B} = \begin{cases} ns - \dbinom{s}{2} & \text{if } t = 2s, \\ ns + n - \dbinom{s+1}{2} & \text{if } t = 2s+1, \end{cases}$$

so the assertion follows by [PS, Proposition III.4.1]. When $\mathbb{K}$ has odd characteristic and t is odd, $\operatorname{rad}\mathfrak{B}$ is once again a perfect ideal with the height as above, [HJPS, Theorem 7.12].

The remaining case is when $\mathbb{K}$ has odd characteristic and t is even. After a flat base change, we may assume that $\mathbb{K}$ contains a primitive fourth root of unity, in which case $\mathfrak{B}$ has minimal primes $\mathfrak{P}$ and $\mathfrak{Q}$, with each of $\mathfrak{P}$, $\mathfrak{Q}$, and $\mathfrak{P}+\mathfrak{Q}$ being a perfect ideal by [HJPS, Theorem 7.13]. The result now follows from the Mayer–Vietoris sequence

$$\longrightarrow\; H^k_{\mathfrak{P}}(S) \oplus H^k_{\mathfrak{Q}}(S) \;\longrightarrow\; H^k_{\mathfrak{B}}(S) \;\longrightarrow\; H^{k+1}_{\mathfrak{P}+\mathfrak{Q}}(S) \;\longrightarrow$$

using the formulae for the heights of $\mathfrak{P}$, $\mathfrak{Q}$, and $\mathfrak{P}+\mathfrak{Q}$ from [HJPS, Theorem 7.13]. $\square$

Lemma 4.4. *Let Y be a $t \times n$ matrix of indeterminates over an infinite field $\mathbb{K}$ of characteristic other than 2. Set $\mathfrak{B}$ to be the ideal $I_1(Y^{\mathrm{tr}}Y)$ in $S := \mathbb{K}[Y]$. If $t \geqslant 2$, then*

$$\operatorname{ara}\mathfrak{B}S_{y_{11}} \;\leqslant\; \binom{n+1}{2} - \binom{n+2-t}{2}.$$

Proof. The assertion is immediate if $n \leqslant t-1$, since the upper bound asserted in that case is simply $\binom{n+1}{2}$, which is the number of generators of $\mathfrak{B}$. We assume $n \geqslant t$ henceforth.

The ideal $I_1(Y^{\mathrm{tr}}Y)S_{y_{11}}$ is unaffected by elementary column operations on Y so, after renaming variables, we may take

$$Y := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ y_{t1} & y_{t2} & y_{t3} & \cdots & y_{tn} \end{bmatrix}$$

and work in affine space with coordinates y_{ij} as above. After a change of notation, take S to be the polynomial ring in the $(t-1)\times n$ indeterminates above, and set $\mathfrak{B} := I_1(Y^{\mathrm{tr}}Y)S$. The task at hand is to prove that the arithmetic rank of $\mathfrak{B}$ is bounded above by $\binom{n+1}{2} - \binom{n+2-t}{2}$.

If $t=2$, then $\mathfrak{B}$ agrees up to radical with $(1+y_{21}^2,\ y_{22},\ y_{23},\ \dots,\ y_{2n})$, so the inequality holds. Assume $t \geqslant 3$. Let Z be a $(t-2)\times(n-1)$ matrix of indeterminates over $\mathbb{K}$. By Theorem 4.1, the ideal generated by the entries of $Z^{\mathrm{tr}}Z$ has arithmetic rank

$$\ell := \binom{n}{2} - \binom{n+2-t}{2}$$

and, in fact, the proof shows that ℓ general $\mathbb{K}$-linear combinations of the entries of $Z^{\mathrm{tr}}Z$ generate $I_1(Z^{\mathrm{tr}}Z)$ up to radical. With this in mind, let $\mathfrak{C}$ denote the ideal of S generated by the entries of the first row of $Y^{\mathrm{tr}}Y$, along with ℓ general linear combinations of the entries of the bottom right $(n-1)\times(n-1)$ submatrix of $Y^{\mathrm{tr}}Y$. Note that $\mathfrak{C}$ has

$$n+\ell \;=\; \binom{n+1}{2} - \binom{n+2-t}{2}$$

generators, so it suffices to prove that $\mathfrak{B}$ and $\mathfrak{C}$ agree up to radical; for this, we replace the field $\mathbb{K}$ by an algebraic closure, and use the Nullstellensatz as follows:

Consider a specialization

$$M := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_1 & m_2 & \cdots & m_n \end{bmatrix}$$

of Y that belongs to the algebraic set defined by $\mathfrak{C}$, where each m_i is a size $t-1$ column vector. The condition that the first row of $M^{\mathrm{tr}}M$ is zero (as enforced on M by the first n generators of the ideal $\mathfrak{C}$) reads

$$1+m_1^{\mathrm{tr}}m_1 = 0, \quad m_2^{\mathrm{tr}}m_1 = 0, \quad \dots, \quad m_n^{\mathrm{tr}}m_1 = 0.$$

Setting $i := \sqrt{-1}$ in $\mathbb{K}$, the vector im_1 has norm $(im_1)^{\mathrm{tr}}(im_1) = 1$, and may be taken to be the first column of a matrix

$$A := \begin{bmatrix} im_1 & a_2 & \cdots & a_{t-1} \end{bmatrix}$$

in $\mathrm{O}_{t-1}(\mathbb{K})$. Set

$$B := \begin{bmatrix} 1 & 0 \\ 0 & A^{\mathrm{tr}} \end{bmatrix}.$$

Since B is an orthogonal matrix, $\widetilde{M} := BM$ satisfies $\widetilde{M}^{\mathrm{tr}}\widetilde{M} = M^{\mathrm{tr}}M$. It suffices to verify that $\widetilde{M}$ belongs to the algebraic set defined by $\mathfrak{B}$, i.e., that $\widetilde{M}^{\mathrm{tr}}\widetilde{M}$ is zero, under the assumption that $\widetilde{M}$ belongs to the algebraic set defined by $\mathfrak{C}$.

The matrix $\widetilde{M}$ has the form

$$\widetilde{M} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & im_1^{\mathrm{tr}} \\ 0 & a_2^{\mathrm{tr}} \\ \vdots & \vdots \\ 0 & a_{t-1}^{\mathrm{tr}} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_1 & m_2 & \cdots & m_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ im_1^{\mathrm{tr}}m_1 & im_1^{\mathrm{tr}}m_2 & \cdots & im_1^{\mathrm{tr}}m_n \\ a_2^{\mathrm{tr}}m_1 & a_2^{\mathrm{tr}}m_2 & \cdots & a_2^{\mathrm{tr}}m_n \\ \vdots & \vdots & & \vdots \\ a_{t-1}^{\mathrm{tr}}m_1 & a_{t-1}^{\mathrm{tr}}m_2 & \cdots & a_{t-1}^{\mathrm{tr}}m_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -i & 0 & \cdots & 0 \\ 0 & a_2^{\mathrm{tr}}m_2 & \cdots & a_2^{\mathrm{tr}}m_n \\ \vdots & \vdots & & \vdots \\ 0 & a_{t-1}^{\mathrm{tr}}m_2 & \cdots & a_{t-1}^{\mathrm{tr}}m_n \end{bmatrix}.$$

Setting N to be the bottom right $(t-2)\times(n-1)$ submatrix of $\widetilde{M}$, it follows that

$$\widetilde{M}^{\mathrm{tr}}\widetilde{M} \;=\; \begin{bmatrix} 0 & 0 \\ 0 & N^{\mathrm{tr}}N \end{bmatrix}.$$

Since the ℓ general linear combinations that constitute the second set of generators for $\mathfrak{C}$ vanish on $\widetilde{M}$, it follows that $N^{\mathrm{tr}}N$ is zero, and hence that $\widetilde{M}^{\mathrm{tr}}\widetilde{M}$ is zero. □

Remark 4.5. In the case that Y is an $n\times n$ matrix of indeterminates over a field of characteristic other than 2, the preceding lemma says that

$$\operatorname{ara}\mathfrak{B}S_{y_{11}} \;\leqslant\; \binom{n+1}{2} - 1.$$

We point out that this inequality holds as well when $S := \mathbb{Z}[Y]$, as can be seen as follows:

After column operations that do not affect the ideal $\mathfrak{B}S_{y_{11}}$, and renaming variables, we may take Y to be the block matrix

$$Y := \begin{bmatrix} y_{11} & 0 \\ v & A \end{bmatrix}$$

with A a square matrix of size $n-1$. But then $\det Y = y_{11}\det A$, so $\det(Y^{\mathrm{tr}}Y) = y_{11}^2\det(A^{\mathrm{tr}}A)$ is a polynomial relation between the $\binom{n+1}{2}$ entries of the symmetric matrix $Y^{\mathrm{tr}}Y$. Using this, the ideal $I_1(Y^{\mathrm{tr}}Y)$ of $\mathbb{Z}[Y^{\mathrm{tr}}Y]$ can be generated up to radical by fewer than $\binom{n+1}{2}$ elements.

Corresponding to Theorems 2.4 and 3.3, we prove next:

Theorem 4.6. *Let Y be a $t\times n$ matrix of indeterminates, and consider $\mathfrak{B} := I_1(Y^{\mathrm{tr}}Y)$ in the polynomial ring $\mathbb{Z}[Y]$. Set $s := \lfloor t/2\rfloor$.*

(1) *If $n \geqslant s$, then $H^k_{\mathfrak{B}}(\mathbb{Z}[Y])$ is a $\mathbb{Q}$-vector space for each $k \geqslant ns+n+2-\binom{s+1}{2}$.*
(2) *Set $c := \binom{n+1}{2} - \binom{n+1-t}{2}$, which is the cohomological dimension of $\mathfrak{B}$. If $n \geqslant t \geqslant 3$, then one has a degree-preserving isomorphism*

$$H^c_{\mathfrak{B}}(\mathbb{Z}[Y]) \;\cong\; H^{tn}_{\mathfrak{m}}(\mathbb{Q}[Y]),$$

with $\mathfrak{m}$ the homogeneous maximal ideal of $\mathbb{Q}[Y]$ under the standard grading.

Proof. Set $S := \mathbb{Z}[Y]$. For each prime integer p, one has an exact sequence

$$\longrightarrow H^{k-1}_{\mathfrak{B}}(S/pS) \longrightarrow H^k_{\mathfrak{B}}(S) \xrightarrow{\;p\;} H^k_{\mathfrak{B}}(S) \longrightarrow H^k_{\mathfrak{B}}(S/pS) \longrightarrow$$

which gives (1) in light of Lemma 4.3.

Towards (2), it follows that $H^c_{\mathfrak{B}}(S)$ is a $\mathbb{Q}$-vector space whenever

$$\binom{n+1}{2} - \binom{n+1-t}{2} \;\geqslant\; ns+n+2-\binom{s+1}{2}.$$

Rearranging terms, this is

$$\binom{n-s}{2} \;\geqslant\; \binom{n+1-t}{2}+2,$$

which holds if $t \geqslant 3$ and $n \geqslant s+3$. We verify that $H^c_{\mathfrak{B}}(S)$ is also a $\mathbb{Q}$-vector space in the two cases $n=t=3$ and $n=t=4$. In each case, $H^c_{\mathfrak{B}}(S/pS)=0$ for each prime p by Lemma 4.3, and we need to check that $H^c_{\mathfrak{B}}(S)$ has no p-torsion. Using Lemma 2.5, it suffices to check that $H^c_{\mathfrak{B}}(S)_{y_{ij}}$ has no p-torsion, and that

$$\left[H^{tn-c+1}_{\mathfrak{m}}(S/\operatorname{rad}(\mathfrak{B}+pS))\right]_0 \;=\; 0.$$

In the case $n=t=3$, one has $c=6$, and $H^6_{\mathfrak{B}}(S)_{y_{ij}}$ vanishes since $\operatorname{ara}\mathfrak{B}S_{y_{ij}} \leqslant 5$ by Remark 4.5. As discussed at the beginning of this section, the ring $S/\operatorname{rad}(\mathfrak{B}+pS)$ is Cohen–Macaulay in this case; its a-invariant is -3, as can be seen, for example, by working modulo the system of parameters $y_{11},\ y_{12}-y_{21},\ y_{23}-y_{32},\ y_{33}$. It follows that

$$[H^4_{\mathfrak{m}}(S/\operatorname{rad}(\mathfrak{B}+pS))]_0 \;=\; 0.$$

For $n=t=4$, one may check that $\operatorname{ara}\mathfrak{B}S_{y_{ij}} \leqslant 9$ while $c=10$, so $H^{10}_{\mathfrak{B}}(S)_{y_{ij}}$ vanishes; we claim that $[H^7_{\mathfrak{m}}(S/\operatorname{rad}(\mathfrak{B}+pS))]_0$ vanishes as well. When $p=2$, this holds since the ring $S/\operatorname{rad}(\mathfrak{B}+pS)$ is Cohen–Macaulay, of dimension 9. When p is odd, after enlarging the field, $(\mathfrak{B}+pS)$ has minimal primes $\mathfrak{P}$ and $\mathfrak{Q}$ yielding an exact sequence

$$\longrightarrow\; H^6_{\mathfrak{m}}(S/(\mathfrak{P}+\mathfrak{Q})) \;\longrightarrow\; H^7_{\mathfrak{m}}(S/(\mathfrak{B}+pS)) \;\longrightarrow\; H^7_{\mathfrak{m}}(S/\mathfrak{P})\oplus H^7_{\mathfrak{m}}(S/\mathfrak{Q}) \;\longrightarrow\; .$$

The rings $S/\mathfrak{P}$ and $S/\mathfrak{Q}$ are Cohen–Macaulay of dimension 9, while $S/(\mathfrak{P}+\mathfrak{Q})$ is Cohen–Macaulay of dimension 6; working modulo a system of parameters, one checks that its a-invariant is -4, so the degree 0 strand of the displayed exact sequence indeed vanishes.

So far, we have established that $H^c_{\mathfrak{B}}(S)$ is a $\mathbb{Q}$-vector space under the hypotheses of (2). We now change notation and work with $S:=\mathbb{Q}[Y]$, and show that $H^c_{\mathfrak{B}}(S)$ has support $\{\mathfrak{m}\}$. For this, it suffices to verify that $H^c_{\mathfrak{B}}(S)_{y_{ij}}$ vanishes. This follows from Lemma 4.4 since

$$\operatorname{ara}\mathfrak{B}S_{y_{ij}} \;\leqslant\; \binom{n+1}{2}-\binom{n+2-t}{2} \;<\; \binom{n+1}{2}-\binom{n+1-t}{2} \;=\; c$$

whenever $n \geqslant t$. The familiar D-module argument implies that $H^c_{\mathfrak{B}}(S)$ is a direct sum of copies of $H^{tn}_{\mathfrak{m}}(S)$, while the fact that it is exactly one copy follows from Theorem 8.1. □

Remark 4.7. The hypothesis $t \geqslant 3$ in Theorem 4.6 (2) is essential: if $t=1$, then

$$H^c_{\mathfrak{B}}(\mathbb{Z}[Y]) \;=\; H^n_{(y_{11},\dots,y_{1n})}(\mathbb{Z}[Y])$$

is not a $\mathbb{Q}$-vector space; if $t=2$ then $c=2n-1$, and we claim that multiplication by an odd prime p is not surjective on $H^{2n-1}_{\mathfrak{B}}(\mathbb{Z}[Y])$. For this, consider the exact sequence

$$\longrightarrow\; H^{2n-1}_{\mathfrak{B}}(S) \;\xrightarrow{\;p\;}\; H^{2n-1}_{\mathfrak{B}}(S) \;\longrightarrow\; H^{2n-1}_{\mathfrak{B}}(S/pS) \;\longrightarrow\; 0,$$

where $H^{2n-1}_{\mathfrak{B}}(S/pS)$ is nonzero by Lemma 4.3.

Lastly, we have the analogue of Theorem 2.8 and Theorem 3.8:

Theorem 4.8. *Let M be a $t\times n$ matrix with entries from a commutative Noetherian ring A, where $n \geqslant t \geqslant 3$. Set $\mathfrak{b} := I_1(M^{\mathrm{tr}}M)$ and $c := \binom{n+1}{2}-\binom{n+1-t}{2}$. Then:*

(1) *The local cohomology module $H^c_{\mathfrak{b}}(A)$ is a $\mathbb{Q}$-vector space, and thus vanishes if the canonical homomorphism $\mathbb{Z} \longrightarrow A$ is not injective.*
(2) *If $\dim A \otimes_{\mathbb{Z}} \mathbb{Q} < tn$, then $H^c_{\mathfrak{b}}(A) = 0$, i.e., $\operatorname{lcd} \mathfrak{b} < c$.*
(3) *If the images of the matrix entries in the ring $A \otimes_{\mathbb{Z}} \mathbb{Q}$ are algebraically dependent over a field that is a subring of $A \otimes_{\mathbb{Z}} \mathbb{Q}$, then $H^c_{\mathfrak{b}}(A) = 0$.*

Proof. Set $S := \mathbb{Z}[Y]$ for Y a $t \times n$ matrix of indeterminates, and regard A as an S-algebra via $y_{ij} \longmapsto m_{ij}$, so that $\mathfrak{b}$ equals $I_1(Y^{\mathrm{tr}}Y)A$. Using Theorem 4.6 (2) and base change along the map $S \longrightarrow A$, one has

$$H^c_{\mathfrak{b}}(A) \;\cong\; H^{tn}_{\mathfrak{m}A}(A \otimes_{\mathbb{Z}} \mathbb{Q}),$$

from which the assertions follow as in the earlier cases. □

Remark 4.9. It follows from Remark 4.7 that (1) fails if $t \leqslant 2$. We observe next that (2) fails if $t \geqslant 3$ but $n < t$. For indeterminates y_{ij} over $\mathbb{Q}$, set

$$M := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ y_{21} & y_{22} & \cdots & y_{2n} \\ 0 & y_{32} & \cdots & y_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & y_{t2} & \cdots & y_{tn} \end{bmatrix}$$

and $A := \mathbb{Q}[M]$, in which case $\dim A < tn$. Set $\mathfrak{b} := I_1(M^{\mathrm{tr}}M)$, and note that $c = \binom{n+1}{2}$ since $n < t$. We claim that $H^c_{\mathfrak{b}}(A)$ is nonzero.

Setting M' to be the $(t-2) \times (n-1)$ submatrix of M obtained by deleting the first column and the first two rows, one has

$$\mathfrak{b} \;=\; I_1(M'^{\mathrm{tr}}M') + (1 + y_{21}^2,\ y_{21}y_{22},\ \ldots,\ y_{21}y_{2n}),$$

and this agrees up to radical with

$$I_1(M'^{\mathrm{tr}}M') + (1 + y_{21}^2,\ y_{22},\ \ldots,\ y_{2n}).$$

It follows that $H^c_{\mathfrak{b}}(A)$ is a direct sum of copies of

$$H^{c-n}_{I_1(M'^{\mathrm{tr}}M')}(\mathbb{Q}[M']),$$

and this is nonzero by Corollary 4.2.

5. Preliminaries on cohomology

5.1. **Notation and terminology.** For our main results on arithmetic rank and structure of local cohomology modules, we will require computations of singular cohomology of complex varieties and étale cohomology of varieties over fields of positive characteristic. We denote by

$$H^i_{\mathrm{Sing}}(X, G) \qquad \text{and} \qquad H^i_{c,\mathrm{Sing}}(X, G)$$

the singular cohomology and the compactly supported singular cohomology, respectively, of a complex quasiprojective variety X in the analytic (Euclidean) topology, with coefficients in an Abelian group G, or more generally, a local system of Abelian groups $\mathscr{G}$. We will implicitly use the identification of singular cohomology and sheaf cohomology with local coefficients for paracompact locally contractible spaces (e.g., quasiprojective complex varieties), [Bre, Theorem III.1.1] and [Di, §2.5]. We refer the reader to [Hat] and [Di] as general sources on these notions, though we will review in the next subsection the basic facts that we apply in the sequel.

Similarly, for an Abelian group G and a quasiprojective variety X over an algebraically closed field, we denote by

$$H^i_{c,\text{ét}}(X,G) \quad \text{and} \quad H^i_{\text{ét}}(X,G)$$

the étale cohomology groups and compactly supported cohomology groups, respectively, with coefficients in the constant sheaf on X given by G. More generally, we use the analogous notation with a local system of Abelian groups $\mathscr{G}$ on X. We refer the reader to [Mi] as a general source on these, and we will review in Subsection 5.3 the basic facts that we will require in the sequel.

We will be particularly interested in the first nonvanishing compact singular or étale cohomology groups of various spaces. We will say that the *singular compact dimension* of a space X with coefficient group G is

$$\operatorname{cptdim}(X,G) := \inf\{i \geqslant 0 \mid H^i_{c,\text{Sing}}(X,G) \neq 0\},$$

with *critical cohomology group* equal to $H^{\operatorname{cptdim}(X)}_{c,\text{Sing}}(X,G)$. We will use corresponding terminology and notation in the étale setting.

In many of our arguments, the computations of singular cohomology and étale cohomology are formally identical, and in these cases, we may drop the subscripts Sing or ét. In particular, for shorthand we will refer to *Setting* (AN) to mean that

- the ground field $\mathbb{K}$ is equal to $\mathbb{C}$, and the varieties under consideration are quasiprojective complex varieties;
- for a given such variety X, we set $H^i(X) := H^i_{\text{Sing}}(X,\mathbb{Q})$ and $H^i_c(X) := H^i_{c,\text{Sing}}(X,\mathbb{Q})$;
- for a given such variety X, we use $\operatorname{cptdim}(X)$ for the singular compact dimension, taking coefficients in $\mathbb{Q}$. The *rank* of the critical cohomology group refers to the rank of $H^{\operatorname{cptdim}(X)}_{c,\text{Sing}}(X,\mathbb{Q})$ as a $\mathbb{Q}$-vector space.

Likewise we will refer to *Setting* (ET) to mean that

- the ground field $\mathbb{K}$ is algebraically closed of odd characteristic, and the varieties under consideration are quasiprojective $\mathbb{K}$-varieties;
- for a given such variety X, we set $H^i(X) := H^i_{\text{ét}}(X,\mathbb{Z}/2)$ and $H^i_c(X) := H^i_{c,\text{ét}}(X,\mathbb{Z}/2)$;
- for a given such variety X, we use $\operatorname{cptdim}(X)$ for the étale compact dimension with coefficients in $\mathbb{Z}/2$. The *rank* of the critical cohomology group refers to the rank of $H^{\operatorname{cptdim}(X)}_{c,\text{ét}}(X,\mathbb{Z}/2)$ as a $\mathbb{Z}/2$-vector space.

5.2. **Singular cohomology.** We review some general facts about singular cohomology that will be applied below. We will tailor our discussion somewhat to the settings used in the sequel rather than giving the most general statements.

Poincaré duality. [Di, Corollary 3.3.12] Let X be an n-dimensional real connected manifold, not necessarily compact, that is oriented with respect to a field $\mathbb{L}$. Let $\mathscr{L}$ be a locally constant sheaf of finite dimensional $\mathbb{L}$-vector spaces on X, and let $\mathscr{L}^*$ denote the $\mathbb{L}$-dual sheaf. Then, for each i, there is an isomorphism

$$H^i_{c,\text{Sing}}(X,\mathscr{L}) \;\cong\; H^{n-i}_{\text{Sing}}(X,\mathscr{L}^*).$$

These isomorphisms hold in particular when X is a complex manifold and $\mathbb{L}$ equals $\mathbb{Q}$, since then X is oriented by [Di, Example 3.2.10]; a particular case arises when $\mathscr{L}$ is the constant sheaf with stalk $\mathbb{Q}$, in which case $\mathscr{L}^* = \mathscr{L}$.

Mayer–Vietoris sequence. [Iv, p. 103 and 185] Let X be a topological space with open subsets U,V such that $U\cup V=X$, and let $\mathscr{F}$ be a sheaf on X. Then there are long exact sequences of sheaf cohomology

$$\longrightarrow H^i(X,\mathscr{F})\longrightarrow H^i(U,\mathscr{F})\oplus H^i(V,\mathscr{F})\longrightarrow H^i(U\cap V,\mathscr{F})\longrightarrow H^{i+1}(X,\mathscr{F})\longrightarrow$$

and of compact supported cohomology

$$\longrightarrow H^i_c(U\cap V,\mathscr{F})\longrightarrow H^i_c(U,\mathscr{F})\oplus H^i_c(V,\mathscr{F})\longrightarrow H^i_c(X,\mathscr{F})\longrightarrow H^{i+1}_c(U\cap V,\mathscr{F})\longrightarrow .$$

In particular, if X is a complex quasi-projective variety and $\mathscr{L}$ is a local system of $\mathbb{Q}$-vector spaces, we have the long exact sequences of singular cohomology

$$\begin{aligned}\longrightarrow H^i_{\mathrm{Sing}}(X,\mathscr{L})\longrightarrow H^i_{\mathrm{Sing}}(U,\mathscr{L})\oplus H^i_{\mathrm{Sing}}(V,\mathscr{L})\longrightarrow H^i_{\mathrm{Sing}}(U\cap V,\mathscr{L})&\\ \longrightarrow H^{i+1}_{\mathrm{Sing}}(X,\mathscr{L})\longrightarrow&\end{aligned}$$

and of compactly supported singular cohomology

$$\begin{aligned}\longrightarrow H^i_{c,\mathrm{Sing}}(U\cap V,\mathscr{L})\longrightarrow H^i_{c,\mathrm{Sing}}(U,\mathscr{L})\oplus H^i_{c,\mathrm{Sing}}(V,\mathscr{L})\longrightarrow H^i_{c,\mathrm{Sing}}(X,\mathscr{L})&\\ \longrightarrow H^{i+1}_{c,\mathrm{Sing}}(U\cap V,\mathscr{L})\longrightarrow .&\end{aligned}$$

Long exact sequence of a subspace. [Iv, p. 185] Let X be a topological space, and consider a triple $U\subseteq X\supseteq Z$ where $Z\subseteq X$ is a closed subspace and $U=X\smallsetminus Z$. Then, for any Abelian group G, there is a long exact sequence of compactly supported singular cohomology

$$\longrightarrow H^i_{c,\mathrm{Sing}}(U,G)\longrightarrow H^i_{c,\mathrm{Sing}}(X,G)\longrightarrow H^i_{c,\mathrm{Sing}}(Z,G)\longrightarrow H^{i+1}_{c,\mathrm{Sing}}(U,G)\longrightarrow .$$

Affine vanishing. If X is a smooth complex affine variety of algebraic dimension d, then X is homotopy equivalent to a CW complex Y of dimension d [AF]. It follows from this that for every locally constant sheaf $\mathscr{L}$ of $\mathbb{Q}$-vector spaces on X, one has

$$H^i_{\mathrm{Sing}}(X,\mathscr{L})=0\qquad\text{for all } i>d.$$

Indeed, if $f\colon Y\longrightarrow X$ is the homotopy equivalence, then the pullback $f^{-1}(\mathscr{L})$ is a locally constant sheaf of $\mathbb{Q}$-vector spaces on Y, and $H^i(Y,f^{-1}(\mathscr{L}))\cong H^i(X,\mathscr{L})$ for all i by [Di, Remark 2.5.12], while $H^i(Y,f^{-1}(\mathscr{L}))=0$ for $i>d$ by [Di, Proposition 2.5.4].

If X is a smooth complex variety of algebraic dimension d that admits an open cover by t affines, it follows from the vanishing above and the Mayer–Vietoris sequence that

$$H^i_{\mathrm{Sing}}(X,\mathscr{L})=0\qquad\text{for all } i>d+t-1.$$

Combining this with Poincaré duality, under the same hypotheses we have

$$H^i_{c,\mathrm{Sing}}(X,\mathscr{L})=0\qquad\text{for all } i<d-t+1.$$

Leray–Serre spectral sequence. We say that

$$F\longrightarrow E\longrightarrow B$$

is a *locally trivial fiber bundle in the analytic topology* if there is a surjective map $E\xrightarrow{\pi}B$ and an open cover $U_1,\dots,U_t$ of B in the analytic topology such that

$$\pi|_{\pi^{-1}(U_i)}\colon \pi^{-1}(U_i)\longrightarrow U_i$$

is isomorphic to a projection $U_i\times F\longrightarrow U_i$. Given such a locally trivial fiber bundle and a coefficient group G, there is a *Leray–Serre spectral sequence*

$$H^i_{c,\mathrm{Sing}}(B,\mathscr{G})\Longrightarrow H^{i+j}_{c,\mathrm{Sing}}(E,G),$$

where $\mathscr{G}$ is a locally constant sheaf on B with stalk $H^j_{c,\mathrm{Sing}}(F,G)$. The monodromy action on $\mathscr{G}$ is induced by the monodromy action on F, i.e., the action of an element $[\gamma] \in \pi_1(B)$ on $\mathscr{G}$ is the map on cohomology induced by the automorphism of F given by lifting γ to E.

5.3. **Étale cohomology.** We discuss related statements for étale cohomology. Throughout this subsection, $\mathbb{K}$ is an algebraically closed field, and ℓ a prime integer invertible in $\mathbb{K}$.

Poincaré duality. [Mi, VI.11.1] Let X be a quasiprojective variety over $\mathbb{K}$ and $\mathscr{L}$ be a locally constant constructible sheaf of $\mathbb{Z}/\ell$-modules on X. Then there is a perfect pairing

$$H^i_{c,\text{ét}}(X,\mathscr{L}) \times H^{2d-i}_{\text{ét}}(X,\mathscr{L}^*) \longrightarrow \mathbb{Z}/\ell, \tag{5.0.1}$$

where $\mathscr{L}^* = \operatorname{Hom}(\mathscr{L},\mu_\ell^{\otimes d}) \cong \operatorname{Hom}(\mathscr{L},\mathbb{Z}/\ell)$.

Mayer–Vietoris sequences. Let X be a quasiprojective variety over $\mathbb{K}$, let U and V be Zariski open subsets of X with $U \cup V = X$, and let $\mathscr{F}$ be a sheaf of $\mathbb{Z}/\ell$-modules on X. Then there is a Mayer–Vietoris sequence from [Mi, III.2.24]

$$\longrightarrow H^i_{\text{ét}}(X,\mathscr{F}) \longrightarrow H^i_{\text{ét}}(U,\mathscr{F}) \oplus H^i_{\text{ét}}(V,\mathscr{F}) \longrightarrow H^i_{\text{ét}}(U\cap V,\mathscr{F}) \longrightarrow H^{i+1}_{\text{ét}}(X,\mathscr{F}) \longrightarrow .$$

and a Mayer–Vietoris sequence with compact supports

$$\begin{aligned}\longrightarrow H^i_{c,\text{ét}}(U\cap V,\mathscr{F}) \longrightarrow H^i_{c,\text{ét}}(U,\mathscr{F}) \oplus H^i_{c,\text{ét}}(V,\mathscr{F}) \longrightarrow H^i_{c,\text{ét}}(X,\mathscr{F})&\\ \longrightarrow H^{i+1}_{c,\text{ét}}(U\cap V,\mathscr{F}) \longrightarrow .&\end{aligned}$$

In lieu of a reference, we give a brief argument for the Mayer–Vietoris sequence with compact supports. Write $u : U \longrightarrow X$, $v : V \longrightarrow X$, and $w : U\cap V \longrightarrow X$ for the inclusion maps. Then the sheaf $u_!u^{-1}(\mathscr{F})$ is the sheafification of the presheaf with sections on W given by

$$\begin{cases}\Gamma(\mathscr{F},W) & \text{if } W \subseteq U\\ 0 & \text{if } W \not\subseteq U,\end{cases}$$

and likewise for $v_!v^{-1}(\mathscr{F})$ and $w_!w^{-1}(\mathscr{F})$. We then have a sequence morphisms of sheaves

$$0 \longrightarrow w_!w^{-1}\mathscr{F} \longrightarrow u_!u^{-1}\mathscr{F} \oplus v_!v^{-1}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow 0. \tag{5.0.2}$$

For any geometric point $x \in X$, one has

$$(u_!u^{-1}\mathscr{F})_x \cong \begin{cases}\mathscr{F}_x & \text{if } x \in U\\ 0 & \text{if } x \notin U,\end{cases}$$

and likewise for v,w. Thus, (5.0.2) is a short exact sequence of sheaves, and the Mayer–Vietoris sequence is the long exact sequence obtained from applying $H^\bullet_{c,\text{ét}}(X,-)$, using the isomorphisms $H^i_{c,\text{ét}}(X,u_!\mathscr{F}) \cong H^i_{c,\text{ét}}(U,\mathscr{F})$ and the analogues for v,w.

Long exact sequence of a subspace. Let X be a variety over $\mathbb{K}$, and $\mathscr{F}$ a sheaf of $\mathbb{Z}/\ell$-modules on X. Consider a triple $U \subseteq X \supseteq Z$ where $Z \subseteq X$ is closed and $U = X \smallsetminus Z$. Then there is a long exact sequence [Mi, p. 94]

$$\longrightarrow H^i_{c,\text{ét}}(U,\mathscr{F}) \longrightarrow H^i_{c,\text{ét}}(X,\mathscr{F}) \longrightarrow H^i_{c,\text{ét}}(Z,\mathscr{F}) \longrightarrow H^{i+1}_{c,\text{ét}}(U,\mathscr{F}) \longrightarrow . \tag{5.0.3}$$

Affine vanishing. Let X be an affine variety of dimension d over an algebraically closed field $\mathbb{K}$. According to [Mi, VI.7.2], for any ℓ-torsion sheaf $\mathscr{F}$ with ℓ invertible in $\mathbb{K}$,

$$H^i_{\text{ét}}(X,\mathscr{F}) = 0 \quad \text{for all } i > d.$$

It follows from this and the Mayer–Vietoris long exact sequence that if X is a variety of dimension d over an algebraically closed field $\mathbb{K}$ and admits an open cover by t affines, then

$$H^i_{\text{ét}}(X,\mathscr{F}) = 0 \quad \text{for all } i > d+t-1.$$

By Poincaré duality, it follows that for a locally constant invertible $\mathbb{Z}/\ell$ sheaf $\mathscr{L}$ that

$$H^i_{c,\text{ét}}(X,\mathscr{L}) = 0 \quad \text{for all } i < d-t+1.$$

Leray–Serre spectral sequence for compact supports. We say that

$$F \longrightarrow E \longrightarrow B$$

is a locally trivial fiber bundle in the étale topology if there is a surjective map $\pi : E \longrightarrow B$ and an open cover $U_1,\dots,U_t$ of B on the étale site such that the map $U_i \times_B E \longrightarrow U_i$ obtained by base change is isomorphic to the projection map $U_i \times F \longrightarrow U_i$.

In this setting, suppose that $\pi : E \longrightarrow B$ is a morphism of quasiprojective varieties over an algebraically closed field $\mathbb{K}$; then the functor $\pi_!$ exists [Mi, VI.3.3(e)]. Let $\rho : B \longrightarrow \operatorname{Spec}(\mathbb{K})$ be the constant map. By [Mi, VI.3.2(c)] there is a spectral sequence of the form

$$R^i\rho_! \circ R^j\pi_!(\mathbb{Z}/\ell) \Longrightarrow R^{i+j}(\rho\pi)_!(\mathbb{Z}/\ell),$$

which in this setting takes the form

$$H^i_{c,\text{ét}}(B, R^j\pi_!(\mathbb{Z}/\ell)) \Longrightarrow H^{i+j}_{c,\text{ét}}(E,\mathbb{Z}/\ell). \tag{5.0.4}$$

In a fibration, for each j, the sheaf $R^j\pi_!(\mathbb{Z}/\ell)$ is a locally constant constructible sheaf by [Mi, VI.3.2(d)], with stalk $H^j_{c,\text{ét}}(F,\mathbb{Z}/\ell)$. Indeed, let $\{U_i \longrightarrow B\}$ be an open cover of B such that

$$\begin{tikzcd} E \arrow[r, "\pi"] & B \\ U_i \times F \arrow[u] \arrow[r, "p"] & U_i \arrow[u] \end{tikzcd} \tag{5.0.5}$$

commutes, where p is the projection map. From the Cartesian diagram

$$\begin{tikzcd} U_i \times F \arrow[r, "p"] \arrow[d] & U_i \arrow[d] \\ F \arrow[r] & \operatorname{Spec}(\mathbb{K}) \end{tikzcd}$$

and the fact that $R^i\pi_!$ commutes with base change (cf. [Mi, VI.3.2(e)]), we obtain that $R^j p_!(\mathbb{Z}/\ell)$ is the constant sheaf $H^j_{c,\text{ét}}(F,\mathbb{Z}/\ell)$ on $U_i \times F$. Applying the same fact to the Cartesian square (5.0.5), we obtain that

$$R^j\pi_!(\mathbb{Z}/\ell)|_{U_i\times F} \cong R^j p_!(\mathbb{Z}/\ell).$$

This shows the claim.

5.4. **Additional Lemmas.** For convenience, we state the main consequence of the Leray–Serre spectral sequence that we apply below.

Lemma 5.1. *Let F, E, B be varieties over a field $\mathbb{K}$. We assume that Setting* (AN) *or* (ET) *holds. Furthermore, we make the following assumptions.*

1. *In Setting* (AN)*, $F \longrightarrow E \longrightarrow B$ is a locally trivial fiber bundle in the analytic topology; in Setting* (ET)*, $F \longrightarrow E \longrightarrow B$ is a locally trivial fiber bundle in the étale topology.*
2. *In either Setting* (AN) *or* (ET)*, the critical cohomology group of the fiber F has rank one.*
3. *In either Setting* (AN) *or* (ET)*, one of the following holds:*
   (a) *the base B is simply connected (in the analytic topology in Setting* (AN)*; in the étale topology in Setting* (ET)*) with critical cohomology group of rank one, or*
   (b) *the base B is smooth of dimension b as an algebraic variety, is covered by t affines and* $\operatorname{cptdim}(B) = b - t + 1$ *(for example, B could be smooth and affine with* $\operatorname{cptdim}(B) = b$*) with critical cohomology group of rank one.*
4. *In Setting* (AN)*, the monodromy action of $\pi_1(B)$ on the critical cohomology group of F is trivial (which is automatic when B is simply connected).*

Then $\operatorname{cptdim}(E) = \operatorname{cptdim}(F) + \operatorname{cptdim}(B)$ *with critical cohomology group of rank one.*

Proof. In both settings, the Lemma is a consequence of the Leray–Serre spectral sequence. We first consider Setting (AN). If B is simply connected, the spectral sequence takes the form
$$H^i_{c,\mathrm{Sing}}(B, H^j_{c,\mathrm{Sing}}(F,\mathbb{Q})) \Longrightarrow H^{i+j}_{c,\mathrm{Sing}}(E,\mathbb{Q}),$$
where $H^j_{c,\mathrm{Sing}}(\mathbb{Q})$ denotes the constant system of $\mathbb{Q}$-vector spaces on B, with corresponding stalk. Thus, all terms on the E_2 page with $j < \operatorname{cptdim}(F)$ or $i < \operatorname{cptdim}(B)$ vanish, and the term with $i = \operatorname{cptdim}(B)$ and $j = \operatorname{cptdim}(F)$ is one copy of $\mathbb{Q}$, and the conclusion follows.

Now, assume that B is smooth of dimension b as an algebraic variety, is covered by t affines with $\operatorname{cptdim}(B) = b - t + 1$, and the monodromy action of $\pi_1(B)$ on the critical cohomology group of F is trivial. By affine vanishing, we have $H^i(B, \mathscr{L}) = 0$ for any local system of $\mathbb{Q}$-vector spaces and any $i < b - t + 1 = \operatorname{cptdim}(B)$, so again all terms on the E_2 page with $j < \operatorname{cptdim}(F)$ or $i < \operatorname{cptdim}(B)$ vanish. The hypothesis on the monodromy action implies that the term with $i = \operatorname{cptdim}(B)$ and $j = \operatorname{cptdim}(F)$ is one copy of $\mathbb{Q}$, and the conclusion follows.

In Setting (ET), the argument is similar, using the analogous Leray–Serre spectral sequence and affine vanishing. The only difference in the argument is that $\mathbb{Z}/2\mathbb{Z}$ has only the trivial group automorphism, hence the monodromy action on $\mathbb{Z}/2\mathbb{Z}$ is necessarily trivial. □

Lemma 5.2 (Filtrations and cohomology)**.** *We assume Setting* (AN) *or* (ET) *holds. Suppose $V_t \sqcup V_{t-1} \sqcup \cdots \sqcup V_0$ is a partition of the topological space Y such that*

1. *V_i is open in $V_i \cup V_{i-1} \cup \cdots \cup V_0$ for $i = 1, \ldots, t$;*
2. $\operatorname{cptdim}(V_{i+1}) - 1 > \operatorname{cptdim}(V_i)$ *for $i = 1, \ldots, t-1$;*
3. $\operatorname{cptdim}(Y) > \operatorname{cptdim}(V_t) + 1$.

Then $\operatorname{cptdim}(V_0) = \operatorname{cptdim}(V_1) - 1$ *and the critical cohomology groups of V_0 and V_1 are isomorphic.*

Proof. For convenience, let us write
$$V_{\leqslant i} = V_i \cup V_{i-1} \cup \cdots \cup V_0, \quad d_i = \operatorname{cptdim}(V_i), \quad \text{and} \quad d = \operatorname{cptdim}(Y).$$

In particular, we have $V_{\leqslant t} = Y$ and V_i is open in $V_{\leqslant i}$ with closed complement $V_{\leqslant i-1}$ for $i = 1,\dots,t$.

We show by descending induction for $i = t,\dots,1$, that $\operatorname{cptdim}(V_{\leqslant i-1}) + 1 = \operatorname{cptdim}(V_i)$, and that the corresponding critical cohomology groups are isomorphic. For $i = t$ we have $\operatorname{cptdim}(V_t) + 1 < \operatorname{cptdim}(Y)$ from the hypothesis, whence in the long exact sequence for the triple $V_t \subseteq Y \supseteq V_{\leqslant t-1}$ one finds that $H_c^{d_t-1}(V_{\leqslant t-1}) \cong H_c^{d_t}(V_t)$ and all lower cohomology groups of $V_{\leqslant t-1}$ vanish. For $i = t-1,\dots,1$, the already established equality $\operatorname{cptdim}(V_{\leqslant i}) = d_{i+1} - 1$ implies with the hypothesis that $\operatorname{cptdim}(V_{\leqslant i}) = d_{i+1} - 1 > d_i$. In the long exact sequence for the triple $V_i \subseteq V_{\leqslant i} \supseteq V_{\leqslant i-1}$, it then follows that $H_c^{d_i-1}(V_{\leqslant i-1}) \cong H_c^{d_i}(V_i)$ and the lower cohomology groups vanish, completing the induction.

The statement of the Lemma is the assertion of the claim in the case $i = 1$. $\square$

6. Topology of Pfaffian nullcones

Let $\mathbb{K}$ be a field, and let $Y_{2t\times n}$ denote an $2t \times n$ matrix of indeterminates over $\mathbb{K}$. Let Ω denote the $2t \times 2t$ alternating matrix as in (2.0.1). The purpose of this section is to prove the following.

Theorem 6.1. *Let $\mathbb{K}$ be a field. Let $n \geqslant 2t > 0$ be natural numbers, and set*

$$X^0_{2t\times n} := \operatorname{Var}(Y^{\mathrm{tr}}\Omega Y) \subseteq \mathbb{K}^{2t\times n},$$

with Ω as in (2.0.1). Then

(1) *For $\mathbb{K} = \mathbb{C}$, we have*

$$H^i_{\mathrm{Sing}}(\mathbb{C}^{2t\times n} \smallsetminus X^0_{2t\times n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 4tn - \binom{2t+1}{2} - 1, \\ 0 & \text{if } i > 4tn - \binom{2t+1}{2} - 1. \end{cases}$$

(2) *For $\mathbb{K}$ an algebraically closed field of characteristic other than two, we have*

$$H^i_{\text{ét}}(\mathbb{K}^{2t\times n} \smallsetminus X^0_{2t\times n}, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = 4tn - \binom{2t+1}{2} - 1, \\ 0 & \text{if } i > 4tn - \binom{2t+1}{2} - 1. \end{cases}$$

We first study the cohomology of some auxiliary spaces defined by linear algebraic conditions. We will utilize the notation of Subsection 5.1. For a field $\mathbb{K}$ and integers $t \geqslant k > 0$ we define

$$\operatorname{Sp}(2t,2k) := \left\{ A \in \mathbb{K}^{2t\times 2k} \mid A^{\mathrm{tr}}\Omega A = \Omega \right\}.$$

Lemma 6.2. *Let $t \geqslant k > 0$ be integers. Then in either Setting* (AN) *or* (ET)*, $\operatorname{Sp}(2t,2k)$ is a smooth affine variety, and*

$$\operatorname{cptdim}(\operatorname{Sp}(2t,2k)) = \dim(\operatorname{Sp}(2t,2k)) = 4tk - \binom{2k}{2}$$

with critical cohomology group of rank one.

Furthermore, in Setting (AN)*, the space $\operatorname{Sp}(2t,2k)$ is simply connected.*

Proof. It is clear from the construction that $\operatorname{Sp}(2t,2k)$ is affine. Since the group $\operatorname{Sp}(2t)$ acts transitively on $\operatorname{Sp}(2t,2k)$ by left multiplication, $\operatorname{Sp}(2t,2k)$ is smooth as well.

For the other claims, we proceed by induction on k. For the base case $k = 1$, induce on $t \geqslant 1$ using the locally trivial fiber bundle

$$\mathbb{K}^{2t-1} \longrightarrow \operatorname{Sp}(2t,2) \longrightarrow \mathbb{K}^{2t} \smallsetminus \{0\} \tag{6.2.1}$$

given by mapping an element of $\mathrm{Sp}(2t,2)$ to its first column; cf. Lemma A.2. Note that $\mathbb{K}^{2t} \smallsetminus \{0\}$ is smooth of dimension $2t$, covered by $2t$ affines, and has compact dimension 1 with critical cohomology group of rank 1. Moreover, in Setting (AN), the space $\mathbb{K}^{2t} \smallsetminus \{0\}$ is homotopy equivalent to the sphere $\mathbb{S}^{4t-1}$ and, since $2t \geqslant 2$, simply connected. Thus, Lemma 5.1 applies and we have

$$\dim(\mathrm{Sp}(2t,2)) = \dim(\mathbb{K}^{2t-1}) + \dim(\mathbb{K}^{2t} \smallsetminus \{0\}) = 2t-1+2t = 4t-1, \text{ and}$$
$$\mathrm{cptdim}(\mathrm{Sp}(2t,2)) = \mathrm{cptdim}(\mathbb{K}^{2t-1}) + \mathrm{cptdim}(\mathbb{K}^{2t} \smallsetminus \{0\}) = 4t-2+1 = 4t-1,$$

with critical cohomology group of rank one.

Furthermore, the long exact homotopy sequence yields

$$\longrightarrow \pi_1(\mathbb{C}^{2t-1}) \longrightarrow \pi_1(\mathrm{Sp}(2t,2)) \longrightarrow \pi_1(\mathbb{C}^{2t} \smallsetminus \{0\}) \longrightarrow$$

which implies that $\mathrm{Sp}(2t,2)$ is simply connected in Setting (AN).

Now, we consider the locally trivial fiber bundle

$$\mathrm{Sp}(2t-2,2k-2) \longrightarrow \mathrm{Sp}(2t,2k) \longrightarrow \mathrm{Sp}(2t,2) \tag{6.2.2}$$

given by projection to the first column pair; cf. Lemma A.3. By the case established above and the induction hypothesis on k, the hypotheses of Lemma 5.1 hold in both Settings, so we obtain

$$\begin{aligned}\dim(\mathrm{Sp}(2t,2k)) &= \dim(\mathrm{Sp}(2t-2,2k-2)) + \dim(\mathrm{Sp}(2t,2)) \\ &= (2t-2)(2k-2) - \binom{2k-2}{2} + 4t - 1 = 4tk - \binom{2k}{2}.\end{aligned}$$

and likewise for compact dimension, with critical cohomology group of rank one. The long exact homotopy sequence gives that $\mathrm{Sp}(2t,2k)$ is simply connected along the same lines as above. □

Let

$$\mathrm{Alt}(2k) := \left\{ Y \in \mathbb{K}^{2k\times 2k} \mid Y \text{ is alternating and invertible} \right\},$$

and more generally

$$\mathrm{Alt}^{2k}_{n\times n} := \left\{ Y \in \mathbb{K}^{n\times n} \mid Y \text{ is alternating and } \mathrm{rank}(Y) = 2k \right\}.$$

Lemma 6.3. *Let $k > 0$ be an integer. Then in either Setting* (AN) *or* (ET)*, $\mathrm{Alt}(2k)$ is a smooth affine variety and*

$$\mathrm{cptdim}(\mathrm{Alt}(2k)) = \dim(\mathrm{Alt}(2k)) = \binom{2k}{2},$$

with critical cohomology group of rank one.

Proof. We note that $\mathrm{Alt}(2k)$ is the complement of a hypersurface in the $\binom{2k}{2}$ dimensional affine space of alternating matrices, and thus is smooth and affine of the claimed dimension.

For the assertion on cohomology, we proceed by induction on k. For $k = 1$, we note that $\mathrm{Alt}(2) \cong \mathbb{K}^\times$. For the inductive step, assuming the hypothesis for $\mathrm{Alt}(2k-2)$, first note that by the Künneth formula, we have

$$\mathrm{cptdim}(\mathrm{Alt}(2k-2) \times \mathbb{K}^{2k-2}) = \mathrm{cptdim}(\mathrm{Alt}(2k-2) + \mathrm{cptdim}(\mathbb{K}^{2k-2}) = \binom{2k-2}{2} + 4k - 4$$

with critical cohomology group of rank one.

Now, by [Ba, Lemma 1.3; Proof of Proposition 4.2] there is a locally trivial fiber bundle

$$\mathrm{Alt}(2k-2) \times \mathbb{K}^{2k-2} \longrightarrow \mathrm{Alt}(2k) \xrightarrow{\pi} \mathbb{K}^{2k-1} \smallsetminus \{0\}. \tag{6.3.1}$$

Since $\mathbb{K}^{2k-1} \smallsetminus \{0\}$ is smooth and covered by $2k-1$ affines, the conclusion follows from Lemma 5.1. □

Lemma 6.4. *Let $n > 2k > 0$ be integers. Then in either Setting* (AN) *or* (ET)*, $\mathrm{Alt}^{2k}_{n\times n}$ is a smooth variety with $\mathrm{cptdim}(\mathrm{Alt}^{2k}_{n\times n}) = \binom{2k}{2}$, and in Setting* (AN)*, the space $\mathrm{Alt}^{2k}_{n\times n}$ is simply connected.*

Proof. By [Ba], we have a locally trivial fiber bundle

$$\mathrm{Alt}(2k) \longrightarrow \mathrm{Alt}^{2k}_{n\times n} \longrightarrow \mathrm{Gr}(n-2k,n), \tag{6.4.1}$$

where $\mathrm{Gr}(n-2k,n)$ is the Grassmannian parameterizing the $(n-2k)$-dimensional subspaces of $\mathbb{K}^n$. The base is simply connected of compact dimension zero. The claim on cohomology follows from Lemma 6.3 and Lemma 5.1.

We examine the fundamental groups of $\mathrm{Alt}(2k)$ in Setting (AN). For $k=1$, we have $\mathrm{Alt}(2) \cong \mathbb{C}^\times$, so the fundamental group is infinite cyclic generated by the loop

$$\left\{ \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \mid \lambda \in \mathbb{C}^\times \right\}.$$

Now, for $k > 1$ there is a locally trivial fiber bundle

$$\mathrm{Alt}(2k-2) \times \mathbb{C}^{2k-2} \longrightarrow \mathrm{Alt}(2k) \xrightarrow{\pi} \mathbb{C}^{2k-1} \smallsetminus \{0\},$$

and since $\pi_2(\mathbb{C}^{2k-1} \smallsetminus \{0\}) = \pi_1(\mathbb{C}^{2k-1} \smallsetminus \{0\})$ is trivial by the Hurewicz theorem, the long exact sequence of homotopy groups yields that the inclusion map induces an isomorphism $\pi_1(\mathrm{Alt}(2k-2)) \cong \pi_1(\mathrm{Alt}(2k))$. In particular, each fundamental group $\pi_1(\mathrm{Alt}(2k))$ is generated by the loop

$$\left\{ \begin{bmatrix} L & \mathbf{0}_{2\times 2k-2} \\ \mathbf{0}_{2k-2\times 2} & \Omega_{2k-2} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} \mid u \in \mathbb{C}^\times \right\}.$$

Similarly, since Grassmannians are simply connected, the locally trivial fiber bundle

$$\mathrm{Alt}(2k) \longrightarrow \mathrm{Alt}^{2k}_{n\times n} \longrightarrow \mathrm{Gr}(n-2k,n) \tag{6.4.2}$$

and the long exact sequence yields a surjection

$$\pi_1(\mathrm{Alt}(2k)) \longrightarrow \pi_1(\mathrm{Alt}^{2k}_{n\times n}) \longrightarrow 0$$

induced by the inclusion map. To show that $\mathrm{Alt}^{2k}_{n\times n}$ is simply connected, it suffices to show that the image of the generating loop of $\mathrm{Alt}(2k)$ can be contracted in $\mathrm{Alt}^{2k}_{n\times n}$. Namely, write

$$\varepsilon(-) = \exp(2\pi\sqrt{-1}(-)),$$

and consider the loop

$$\lambda = \left\{ \begin{bmatrix} L & \mathbf{0}_{2\times(2k-2)} & \mathbf{0}_{2\times(n-2k)} \\ \mathbf{0}_{(2k-2)\times 2} & \Omega_{2k-2} & \mathbf{0}_{(2k-2)\times(n-2k)} \\ \mathbf{0}_{(n-2k)\times 2} & \mathbf{0}_{(n-2k)\times(2k-2)} & \mathbf{0}_{(2k-2)\times 2} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \varepsilon(t) \\ -\varepsilon(t) & 0 \end{bmatrix} \mid t \in [0,1] \right\}.$$

Setting $E_{2\times(2k-2)}$ to be the matrix with 1 in the top left corner, and 0 elsewhere, one can continuously deform λ to

$$\lambda' = \left\{ \begin{bmatrix} L & \mathbf{0}_{2\times(2k-2)} & E \\ \mathbf{0}_{(2k-2)\times 2} & \Omega_{2k-2} & \mathbf{0}_{(2k-2)\times(n-2k)} \\ -E^{\mathrm{tr}} & \mathbf{0}_{(n-2k)\times(2k-2)} & \mathbf{0}_{(2k-2)\times 2} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \varepsilon(t) \\ -\varepsilon(t) & 0 \end{bmatrix} \mid t \in [0,1] \right\}$$

in $\mathrm{Alt}^{2k}_{n\times n}$. But this deforms in $\mathrm{Alt}^{2k}_{n\times n}$ to the constant loop

$$\begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times(2k-2)} & E \\ \mathbf{0}_{(2k-2)\times 2} & \Omega_{2k-2} & \mathbf{0}_{(2k-2)\times(n-2k)} \\ -E^{\mathrm{tr}} & \mathbf{0}_{(n-2k)\times(2k-2)} & \mathbf{0}_{(2k-2)\times 2} \end{bmatrix},$$

and so λ', and hence λ, is contractible in $\mathrm{Alt}^{2k}_{n\times n}$. □

Lemma 6.5. *Let $t>0$ be an integer. In either Setting* (AN) *or* (ET)*, we have*

$$\mathrm{cptdim}(\mathrm{GL}_t)=t^2,$$

with critical cohomology group of rank one.

Proof. In either Setting (AN) or (ET) we have that $H^i(\mathrm{GL})$ vanishes for $i>t^2$ and has rank one for $i=t^2$ by [BS, Lemma 4 and 3.1]. Since GL_t is smooth, the claim for compact cohomology follows from Poincaré duality. □

The following auxiliary spaces will be used in our main calculation of cohomology:

(6.5.1) $$X^{2k}_{2t\times n} := \{Y\in\mathbb{K}^{2t\times n}\mid Y^{\mathrm{tr}}\Omega_{2t}Y \text{ has rank exactly } 2k\},$$

(6.5.2) $$G^{2k}_{2t\times n} := \left\{Y\in\mathbb{K}^{2t\times n}\mid Y^{\mathrm{tr}}\Omega_{2t}Y=\begin{bmatrix} A & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \text{ with } A\in\mathrm{Alt}(2k)\right\},$$

(6.5.3) $$F^{2k}_{2t\times n} := \left\{Y\in\mathbb{K}^{2t\times n}\mid Y^{\mathrm{tr}}\Omega_{2t}Y=\begin{bmatrix} \Omega_{2k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\}.$$

Theorem 6.6. *Let $\mathbb{K}$ be a field. Let n,t,k be natural numbers with $n\geqslant 2t\geqslant 2k\geqslant 0$. Then for the varieties $X^{2k}_{2t\times n}$ as defined in* (6.5.1)*, the following hold.*

(1) *For $\mathbb{K}=\mathbb{C}$, we have*

$$H^i_{c,\mathrm{Sing}}(X^{2k}_{2t\times n},\mathbb{Q})=\begin{cases}\mathbb{Q} & \text{if } i=\binom{2t+1}{2}+\binom{2k}{2},\\ 0 & \text{if } i<\binom{2t+1}{2}+\binom{2k}{2}.\end{cases}$$

(2) *For $\mathbb{K}$ an algebraically closed field of characteristic other than two, we have*

$$H^i_{c,\acute{e}t}(X^{2k}_{2t\times n},\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2 & \text{if } i=\binom{2t+1}{2}+\binom{2k}{2},\\ 0 & \text{if } i<\binom{2t+1}{2}+\binom{2k}{2}.\end{cases}$$

Proof. First we consider the case where $t=k$ in both Scenarios (AN) and (ET). We claim that

$$X^{2t}_{2t\times n}=\{Y_{2t\times n}\mid \mathrm{rank}(Y)=2t\}.$$

Indeed, if $\mathrm{rank}(Y)<2t$, then $\mathrm{rank}(Y^{\mathrm{tr}}\Omega Y)<2t$, and conversely, if $\mathrm{rank}(Y)=2t$, then multiplication by Y is surjective, and multiplication by Y^{tr} is injective, so $\mathrm{rank}(Y^{\mathrm{tr}}\Omega Y)=2t$. The claim then follows from [BS, Lemmas 2 and 2′].

Now we consider the case where $k=0$ and $t=1$. Note that $X^0_{2\times n}$ is closed in $\mathbb{K}^{2\times n}$ with open complement $X^2_{2\times n}$, which was handled in the previous case. From the long exact sequence for an open subset, we obtain that

$$\mathrm{cptdim}(X^0_{2\times n})=\mathrm{cptdim}(X^2_{2\times n})-1=3$$

with critical cohomology group of rank one.

Now we proceed by induction on t: fix $t > 1$ and assume that the claim holds for smaller values of t. We have already established the result for $t = k$; we now fix some k with $0 < k < t$. Consider the locally trivial fiber bundle

$$X^0_{(2t-2k)\times(n-2k)} \longrightarrow F^{2k}_{2t\times n} \longrightarrow \mathrm{Sp}(2t, 2k) \tag{6.6.1}$$

from Lemma A.6.3. By the induction hypothesis on t, and by Lemma 6.2, the hypotheses of Lemma 5.1 apply, and we deduce that

$$\begin{aligned}\mathrm{cptdim}(F^{2k}_{2t\times n}) &= \mathrm{cptdim}(X^0_{(2t-2k)\times(n-2k)}) + \mathrm{cptdim}(\mathrm{Sp}(2t, 2k))\\ &= \binom{2t-2k+1}{2} + 4tk - \binom{2k}{2} = \binom{2t+1}{2}\end{aligned}$$

with critical cohomology group of rank one.

At this step, we proceed slightly differently in the two settings. In Setting (AN), we now consider the locally trivial fiber bundle

$$F^{2k}_{2t\times n} \longrightarrow X^{2k}_{2t\times n} \longrightarrow \mathrm{Alt}^{2k}_{n\times n} \tag{6.6.2}$$

from Lemma A.6.4. Since the base $\mathrm{Alt}^{2k}_{n\times n}$ is simply connected, we can apply Lemma 5.1 and Lemma 6.4 to conclude that

$$\mathrm{cptdim}(X^{2k}_{2t\times n}) = \mathrm{cptdim}(F^{2k}_{2t\times n}) + \mathrm{cptdim}(\mathrm{Alt}^{2k}_{n\times n}) = \binom{2t+1}{2} + \binom{2k}{2}$$

with critical cohomology group of rank one. This completes the case $t > 1$ and $0 < k < t$ in Setting (AN).

In Setting (ET), we consider the locally trivial fiber bundle

$$F^{2k}_{2t\times n} \longrightarrow G^{2k}_{2t\times n} \longrightarrow \mathrm{Alt}(2k) \tag{6.6.3}$$

from Lemma A.6.2. By Lemma 6.3, the hypotheses of Lemma 5.1 apply, and so

$$\mathrm{cptdim}(G^{2k}_{2t\times n}) = \mathrm{cptdim}(F^{2k}_{2t\times n}) + \mathrm{cptdim}(\mathrm{Alt}(2k)) = \binom{2t+1}{2} + \binom{2k}{2}$$

with critical cohomology group of rank one.

Then we consider the locally trivial fiber bundle

$$G^{2k}_{2t\times n} \longrightarrow X^{2k}_{2t\times n} \longrightarrow \mathrm{Gr}(n-2k, n) \tag{6.6.4}$$

from Lemma A.6.1. Since $\mathrm{Gr}(n-2k, n)$ is simply connected with compact dimension zero and critical cohomology group of rank one, we apply Lemma 5.1 to obtain

$$\mathrm{cptdim}(X^{2k}_{2t\times n}) = \mathrm{cptdim}(G^{2k}_{2t\times n}) = \binom{2t+1}{2} + \binom{2k}{2}$$

with critical cohomology group of rank one, completing the case $t > 1$ and $0 < k < t$ in Setting (ET).

Finally, we deal with the case $k = 0$ in the induction on t. For this, we apply Lemma 5.2 to the filtration

$$\mathbb{K}^{2t\times n} = X^{2t}_{2t\times n} \sqcup X^{2t-2}_{2t\times n} \sqcup \cdots \sqcup X^{2}_{2t\times n} \sqcup X^{0}_{2t\times n}$$

to conclude that

$$\mathrm{cptdim}(X^0_{2t\times n}) = \mathrm{cptdim}(X^2_{2t\times n}) - 1 = \binom{2t+1}{2} + \binom{2}{2} - 1 = \binom{2t+1}{2}$$

with critical cohomology group of rank one. This completes the induction on t, and the proof of the Theorem. □

Proof of Theorem 6.1. From the long exact sequence in compactly supported cohomology associated to the subspace $X^0_{2t\times n} \subseteq \mathbb{K}^{2t\times n}$ and the previous Theorem, we obtain that for $\mathbb{K} = \mathbb{C}$,

$$H_{c,\mathrm{Sing}}^{\binom{2t+1}{2}+1}(\mathbb{C}^{2t\times n} \smallsetminus X^0_{2t\times n}, \mathbb{Q}) = \mathbb{Q}$$

and the lower cohomology groups vanish. Since $\mathbb{C}^{2t\times n} \smallsetminus X^0_{2t\times n}$ is a complex manifold of real dimension $4tn$ (with boundary), Poincaré duality applies to give

$$H_{\mathrm{Sing}}^{4tn-\binom{2t+1}{2}-1}(\mathbb{C}^{2t\times n} \smallsetminus X^0_{2t\times n}, \mathbb{Q}) = \mathbb{Q}$$

and the higher cohomology groups vanish. Similarly, over an algebraically closed field of characteristic other than two, we obtain that

$$H_{c,\text{ét}}^{\binom{2t+1}{2}+1}(\mathbb{K}^{2t\times n} \smallsetminus X^0_{2t\times n}, \mathbb{Z}/2) = \mathbb{Z}/2$$

and the lower cohomology groups vanish; we may again apply Poincaré duality and obtain

$$H_{\text{ét}}^{4tn-\binom{2t+1}{2}-1}(\mathbb{K}^{2t\times n} \smallsetminus X^0_{2t\times n}, \mathbb{Z}/2) = \mathbb{Z}/2$$

and the higher cohomology groups vanish. □

7. Topology of generic determinantal nullcones

For any field $\mathbb{K}$, we denote by $Y_{m\times t}$ and $Z_{t\times n}$ matrices of indeterminates of size $m\times t$ and $t\times n$ over $\mathbb{K}$. The purpose of this section is to prove the following.

Theorem 7.1. *Let $\mathbb{K}$ be a field. Let $m,n \geqslant t > 0$ be integers, and set*

$$X^0_{m,t,n} := \mathrm{Var}(YZ) \subseteq \mathbb{K}^{m\times t} \times \mathbb{K}^{t\times n}.$$

Then

(1) *For $\mathbb{K} = \mathbb{C}$, we have*

$$H^i_{\mathrm{Sing}}((\mathbb{C}^{m\times t} \times \mathbb{C}^{t\times n}) \smallsetminus X^0_{m,t,n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 2mt+2nt-t^2-1, \\ 0 & \text{if } i > 2mt+2nt-t^2-1. \end{cases}$$

(2) *For $\mathbb{K}$ an algebraically closed field of characteristic other than two, we have*

$$H^i_{\text{ét}}((\mathbb{K}^{m\times t} \times \mathbb{K}^{t\times n}) \smallsetminus X^0_{m,t,n}, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = 2mt+2nt-t^2-1, \\ 0 & \text{if } i > 2mt+2nt-t^2-1. \end{cases}$$

Theorem 7.1 will follow eventually as a consequence of Theorem 7.4 below. To this end, for integers $t \geqslant k > 0$, we examine the following auxiliary spaces, again utilizing the notation of Subsection 5.1:

$$\mathrm{GL}(t,k) := \left\{ A \in \mathbb{K}^{t\times k} \text{ with } \mathrm{rank}(A) = k \right\}, \tag{7.1.1}$$

$$\mathrm{P}(t,k) := \left\{ (A,B) \in \mathbb{K}^{k\times t} \times \mathbb{K}^{t\times k} \text{ with } AB = \mathbf{1}_k \right\}. \tag{7.1.2}$$

Lemma 7.2. *Let $\mathbb{K}$ be a field and $t \geqslant k > 0$ be integers. In Settings* (AN) *and* (ET), $\mathrm{GL}(t,k)$ *is a smooth variety with*

$$\dim(\mathrm{GL}(t,k)) = tk \quad \textit{and} \quad \mathrm{cptdim}(\mathrm{GL}(t,k)) = k^2,$$

with critical cohomology group of rank one. Moreover, $\mathrm{GL}(t,k)$ *is covered by $tk-k^2+1$ affine open sets. Furthermore, in Setting* (AN), *if $t > k$, then the space* $\mathrm{GL}(t,k)$ *is simply connected.*

Proof. The claim on dimension is immediate from the fact that $\mathrm{GL}(t,k)$ is an open subset of $\mathbb{K}^{t\times k}$. The claim on cohomology follows from [BS, Lemma 2 and Lemma 2′] and Poincaré duality. The claim on affine covering is [BS, Theorem 1(a)].

For $t \geqslant k > 1$, there is a fiber bundle (A.8.1) that is locally trivial in the Zariski topology. In Setting (AN), the fiber is homeomorphic to the product of $\mathbb{C}^{k-1}$ with $\mathbb{R}$ and a $2(m-k)+1$-sphere. Since $t \geqslant 2$, $\mathrm{GL}(t,1)$ is simply connected, and by the homotopy sequence to this fibration, so is inductively any $\mathrm{GL}(t,k)$ with $t > ik$. □

Lemma 7.3. *Let $\mathbb{K}$ be a field and $t \geqslant k > 0$ be integers. In either Setting* (AN) *or* (ET)*, $\mathrm{P}(t,k)$ is a smooth affine variety and*

$$\operatorname{cptdim}(\mathrm{P}(t,k)) = \dim(\mathrm{P}(t,k)) = 2tk - k^2$$

with critical cohomology group of rank one. Furthermore, in Setting (AN)*, the space $\mathrm{P}(t,k)$ is simply connected.*

Proof. By definition, $\mathrm{P}(t,k)$ is affine, and smoothness follows from the transitive $\mathrm{GL}(t)$-action given by $A\cdot(Y,Z) = (YA^{-1}, AZ)$.

If $t = k$, then $\mathrm{P}(t,k)$ identifies with $\mathrm{GL}(k) = \mathrm{GL}(k,k)$ and then the claims follow from Lemma 7.2. Henceforth we assume that $t > k$, and proceed by induction on k.

For $k = 1$, we have

$$\mathrm{P}(t,1) = \mathrm{Var}(1 - \sum_{i=1}^{t} v_i w_i) \subseteq \mathbb{K}^t \times \mathbb{K}^t.$$

In Setting (AN), $\mathrm{P}(t,1)$ is carried into $\mathrm{Var}(1-\sum_1^t (v_i^2 + w_i^2))$ by a suitable coordinate change. Suppose $a, b \in \mathbb{R}^s$ are real and imaginary part of $x \in \mathrm{Var}(1-\sum_1^s x_i^2) \subseteq \mathbb{C}^s$. Abbreviate $\sqrt{\sum a_i^2}$ to $||a||$. Then $1 + ||b|| = ||a||$ and a is perpendicular to b. Thus, there is a diffeomorphism

$$\begin{aligned} \mathrm{Var}(1-\sum_1^s x_i^2) \ni x = a + b\sqrt{-1} \quad &\longmapsto \quad \Big(\frac{a}{\sqrt{1+||b||^2}}, b\Big), \\ (a\sqrt{1+||b||^2} + ||a||b\sqrt{-1}) \quad &\longleftarrow \quad (a,b) \in T\mathbb{S}^{s-1}. \end{aligned} \tag{7.3.1}$$

with the tangent bundle of the real $(s-1)$-sphere $\mathbb{S}^{s-1}$. Moreover, one has a locally trivial fiber bundle

$$\mathbb{R}^{s-1} \longrightarrow T\mathbb{S}^{s-1} \longrightarrow \mathbb{S}^{s-1}, \tag{7.3.2}$$

and if $s \geqslant 3$ then $\mathbb{S}^{s-1}$ is simply connected of compact dimension zero with critical cohomology group of rank one, and $\mathbb{R}^{s-1}$ is simply connected of compact dimension $s-1$ with critical cohomology group of rank one. Applying this to the diffeomorphism between $\mathrm{P}(t,1)$ and the tangent bundle of $\mathbb{S}^{2t-1}$, one concludes via the Leray–Serre spectral sequence that $\operatorname{cptdim}(\mathrm{P}(t,1)) = 2t-1$ with critical cohomology group of rank one. The corresponding conclusion holds in Setting (ET) by [De2, Table 3.7].

For $t > k > 1$, we consider the Zariski locally trivial fiber bundle

$$\mathrm{P}(t-k+1,1) \longrightarrow \mathrm{P}(t,k) \longrightarrow \mathrm{P}(t,k-1)$$

from Lemma A.7. Lemma 5.1 applies by the induction hypothesis on k in both settings, so

$$\operatorname{cptdim}(\mathrm{P}(t,k)) = \operatorname{cptdim}(\mathrm{P}(t,k-1)) + \operatorname{cptdim}(\mathrm{P}(t-k+1,1)) = 2tk - k^2.$$

The simply connectedness in Setting (AN) follows from the long exact sequence of homotopy groups. □

We now define some auxiliary spaces that will be used in our main calculation of cohomology:

$$X^k_{m,t,n} := \{(A,B) \in \mathbb{K}^{m\times t} \times \mathbb{K}^{t\times n} \mid AB \text{ has rank exactly } k\}, \tag{7.3.3}$$

$$G^k_{m,t,n} := \left\{(A,B) \in \mathbb{K}^{m\times t} \times \mathbb{K}^{t\times n} \mid \ker(AB) = \operatorname{im}\begin{bmatrix} \mathbf{0}_{k\times(n-k)} \\ \mathbf{1}_{n-k} \end{bmatrix}\right\}, \tag{7.3.4}$$

$$F^k_{m,t,n} := \left\{(A,B) \in \mathbb{K}^{m\times t} \times \mathbb{K}^{t\times n} \mid AB = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(m-k)\times k} & \mathbf{0}_{(m-k)\times(n-k)} \end{bmatrix}\right\}. \tag{7.3.5}$$

Theorem 7.4. *Let $\mathbb{K}$ be a field. Let $m,n \geqslant t \geqslant k \geqslant 0$ be natural numbers. Then*

(1) *For $\mathbb{K} = \mathbb{C}$, we have*

$$H^i_{c,\mathrm{Sing}}(X^k_{m,t,n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = t^2+k^2, \\ 0 & \text{if } i < t^2+k^2. \end{cases}$$

(2) *For $\mathbb{K}$ an algebraically closed field of characteristic other than two, we have*

$$H^i_{c,\text{ét}}(X^k_{m,t,n}, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = t^2+k^2, \\ 0 & \text{if } i < t^2+k^2. \end{cases}$$

Proof. First consider the case $t = k$ in both Settings (AN) and (ET). Then $(A,B) \in X^t_{m,t,n}$ if and only if A,B both have rank t, so

$$X^t_{m,t,n} \cong \mathrm{GL}(m,t) \times \mathrm{GL}(n,t),$$

and thus $\operatorname{cptdim}(X^t_{m,t,n}) = 2t^2 = t^2+k^2$ with critical cohomology group of rank one.

Now consider the case where $k = 0$ and $t = 1$. The space $X^0_{m,1,n}$ is the union of one copy each of $\mathbb{K}^m$ and $\mathbb{K}^n$, intersecting at a point. The Mayer–Vietoris sequence gives $\operatorname{cptdim}(X^0_{m,1,n}) = 1$ with critical cohomology group of rank one.

We now proceed by induction on t: fix $t > 1$ and assume the claim holds for all smaller values of t. Fix some k with $0 < k < t$. Consider the locally trivial fiber bundle

$$X^0_{(m-k),(t-k),(n-k)} \longrightarrow F^k_{m,t,n} \longrightarrow \mathrm{P}(t,k) \tag{7.4.1}$$

from Lemma A.9.3. By the induction hypothesis and Lemma 7.3, the hypotheses of Lemma 5.1 are in force, and we deduce that

$$\operatorname{cptdim}(F^k_{m,t,n}) = \operatorname{cptdim}(X^0_{(m-k),(t-k),(n-k)}) + \operatorname{cptdim}(\mathrm{P}(t,k)) = (t-k)^2 + 2tk - k^2 = t^2.$$

Next, consider the locally trivial fiber bundle

$$F^k_{m,t,n} \longrightarrow G^k_{m,t,n} \longrightarrow \mathrm{GL}(m,k) \tag{7.4.2}$$

from Lemma A.9.2. By Lemma 7.2, we can apply Lemma 5.1 and deduce that

$$\operatorname{cptdim}(G^k_{m,t,n}) = \operatorname{cptdim}(F^0_{m,t,n}) + \operatorname{cptdim}(\mathrm{GL}(m,k)) = t^2+k^2.$$

Then, we have the locally trivial fiber bundle

$$G^k_{m,t,n} \longrightarrow X^k_{m,t,n} \longrightarrow \mathrm{Gr}(n-k,n) \tag{7.4.3}$$

from Lemma A.9.1, and by Lemma 5.1 we deduce that

$$\operatorname{cptdim}(X^k_{m,t,n}) = \operatorname{cptdim}(G^k_{m,t,n}) + \operatorname{cptdim}(\mathrm{Gr}(n-k,n)) = t^2+k^2.$$

This completes the inductive step in the case $k > 0$. Finally, we apply Lemma 5.2 to complete the inductive step for $k = 0$ as in the proof of Theorem 6.6. $\square$

Proof of Theorem 7.1. This follows from Theorem 7.4 along the same lines as the proof of Theorem 6.1. □

8. Topology of symmetric determinantal nullcones

Let $Y_{t\times n}$ denote a $t\times n$ matrix of indeterminates over a field $\mathbb{K}$. The purpose of this section is to prove the following.

Theorem 8.1. *Let $\mathbb{K}$ be a field. Let $n \geqslant t > 0$ be natural numbers, and set*

$$X^0_{t\times n} := \operatorname{Var}(Y^{\mathrm{tr}}Y) \subseteq \mathbb{K}^{t\times n}.$$

Then

(1) *For $\mathbb{K} = \mathbb{C}$, we have*

$$H^i_{\mathrm{Sing}}(\mathbb{C}^{t\times n} \smallsetminus X^0_{t\times n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 2tn - \binom{t}{2} - 1, \\ 0 & \text{if } i > 2tn - \binom{t}{2} - 1. \end{cases}$$

(2) *For $\mathbb{K}$ an algebraically closed field of characteristic other than two, we have*

$$H^i_{\text{ét}}(\mathbb{K}^{t\times n} \smallsetminus X^0_{t\times n}, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = 2tn - \binom{t}{2} - 1, \\ 0 & \text{if } i > 2tn - \binom{t}{2} - 1. \end{cases}$$

Theorem 8.1 will follow eventually as a consequence of Theorem 8.5 below. For the proof, we examine some auxiliary spaces, again utilizing the notation of Subsection 5.1. For integers $t \geqslant k \geqslant 1$, define

$$\mathrm{O}(t,k) := \{Y \in \mathbb{K}^{t\times k} \mid Y^{\mathrm{tr}}Y = 1_k\} \tag{8.1.1}$$

and

$$\operatorname{Sym}(k) := \{Y \in \mathbb{K}^{k\times k} \mid Y \text{ is symmetric and invertible}\}. \tag{8.1.2}$$

We begin with studying $\mathrm{O}(t,k)$.

Lemma 8.2. *Choose an integer $t > 1$.*

(1) *In either Setting* (AN) *or* (ET)*, the variety* $\mathrm{O}(t,1)$ *is smooth affine and*

$$\operatorname{cptdim}(\mathrm{O}(t,1)) = \dim(\mathrm{O}(t,1)) = t-1,$$

with critical cohomology group of rank one.

(2) *In Setting* (AN)*, if $t > 2$, then* $\mathrm{O}(t,1)$ *is simply connected.*

(3) *In Setting* (AN)*, the negation map $\nu_t : \mathrm{O}(t,1) \longrightarrow \mathrm{O}(t,1)$ given by $\nu_t(x) = -x$ induces the identity map on $H^{t-1}_{c,\mathrm{Sing}}(\mathrm{O}(t,1))$.*

Proof. We first consider Setting (AN). As in (7.3.1), there is a diffeomorphism

$$\mathrm{O}(t,1) \cong T\mathbb{S}^{t-1}.$$

Using the locally trivial fiber bundle (7.3.2), the statements (1) and (2) follow readily.

The map ν_t corresponds, via (7.3.1), to the map $\overline{\nu}_t$ that sends $(a,b) \longmapsto (-a,-b)$ on $T\mathbb{S}^{t-1}$.

If $t = 2$, let

$$M_\varphi := \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}.$$

Then for any positive integer r, one has $\overline{\nu}_2(a,b) = (\overline{\nu}_{2,r})^r(a,b)$ where

$$\overline{\nu}_{2,r}(a,b) = (a^{\mathrm{tr}}M_{\pi/r}, b^{\mathrm{tr}}M_{\pi/r}).$$

Thus, the isomorphism on $H^{t-1}_{c,\mathrm{Sing}}(\mathrm{O}(t,1),\mathbb{Z})$ induced by $\overline{\nu}_2$ is the r-th power of the isomorphism induced by $\overline{\nu}_{2,r}$. But for $r \in \mathbb{N}$ large, the discreteness of $\mathbb{Z}$ forces the latter isomorphism to be the identity, and *a fortiori* so is the isomorphism induced by $\overline{\nu}_2$, for any coefficient group.

Now assume that $t > 2$ and choose $0 < \varepsilon \ll 1$. Let $U_1 := \{(a_1,\dots,a_t) \in \mathbb{S}^{t-1} \mid -\varepsilon < a_1\}$ and $U_2 := \{(a_1,\dots,a_t) \in \mathbb{S}^{t-1} \mid a_1 < \varepsilon\}$. Then set $U := U_1 \cap U_2$ and write a' and b' for the parts $(a_2,\dots,a_t)$ and $(b_2,\dots,b_t)$ of two points $a,b \in \mathbb{R}^t$ respectively. Observe that U is diffeomorphic with the cylinder $(-\varepsilon,\varepsilon)\times\mathbb{S}^{t-2}$, and that there is a corresponding diffeomorphism of tangent spaces given by

$$TU \ni (a,b) \longmapsto (a_1,b_1) \times \left(\frac{a'}{||a'||}, b' - \frac{b'\cdot a'}{||a'||}a'\right) \in T(-\varepsilon,\varepsilon)\times T\mathbb{S}^{t-2}.$$

There is a piece of the Mayer–Vietoris sequence of the form

$$H^{t-1}_{c,\mathrm{Sing}}(U_1 \sqcup U_2,\mathbb{Q}) \longrightarrow H^{t-1}_{c,\mathrm{Sing}}(T\mathbb{S}^{t-1},\mathbb{Q}) \xrightarrow{\delta} H^{t}_{c,\mathrm{Sing}}(U,\mathbb{Q}) \longrightarrow H^{t}_{c,\mathrm{Sing}}(U_1 \sqcup U_2,\mathbb{Q}).$$

Note that TU_1, TU_2 are diffeomorphic to $\mathbb{R}^{2t-2}$. Since $t \geqslant 3$, then also $2t-2 > t$, whence the two outer groups in the displayed sequence are zero. This allows to identify the inner groups via the connecting morphism δ.

The map $\overline{\nu}_t$ restricts to TU and corresponds on $T(-\varepsilon,\varepsilon)\times T\mathbb{S}^{t-2}$ to the negation map on base and fiber of $T(-\varepsilon,\varepsilon)$, and on $T\mathbb{S}^{t-2}$ to $\overline{\nu}_{t-1}$. Since the negation map on $T(-\varepsilon,\varepsilon)$ induces the trivial map on $H^2_{c,\mathrm{Sing}}(T(-\varepsilon,\varepsilon),\mathbb{Q})$, the lemma follows by induction.

Statement (1) in Setting (ET) follows from [De2, Table 3.7]. □

Proposition 8.3. *Choose integers $t > k > 0$.*

(1) *In either Setting* (AN) *or* (ET)*, the variety* $\mathrm{O}(t,k)$ *is smooth affine and*
$$\mathrm{cptdim}(\mathrm{O}(t,k)) = \dim(\mathrm{O}(t,k)) = tk - \binom{k+1}{2},$$
with critical cohomology group of rank one.

(2) *In Setting* (AN)*, if $t-k>1$, then* $\mathrm{O}(t,k)$ *is simply connected.*

(3) *In Setting* (AN)*, the map $\nu_{t,k} : \mathrm{O}(t,k) \longrightarrow \mathrm{O}(t,k)$ given by*
$$\nu_{t,k}([w_1,w_2,\dots,w_k]) = [-w_1,w_2,\dots,w_k]$$
induces the identity map on $H^{tk-\binom{k+1}{2}}_{c,\mathrm{Sing}}(\mathrm{O}(t,k))$.

Proof. The case $k=1$ is Lemma 8.2. For the general case, we proceed by induction on k, using the locally trivial fiber bundle

$$\mathrm{O}(t-1,k-1) \longrightarrow \mathrm{O}(t,k) \longrightarrow \mathrm{O}(t,1) \tag{8.3.1}$$

that arises from mapping an element of $\mathrm{O}(t,k)$ to its first column. Since now $t > k > 1$, the base $\mathrm{O}(t,1)$ in (8.3.1) is simply connected in Setting (AN) by Lemma 8.2, and so the hypotheses of Lemma 5.1 apply in both Settings (AN) and (ET). Thus, Claim (1) follows from Lemma 5.1, Lemma 8.2, and induction. Likewise, Claim (2) follows from the long exact homotopy sequence, Lemma 8.2, and induction.

For Claim (3), note that $\nu_{t,k}$ is compatible with the map $\pi : \mathrm{O}(t,k) \longrightarrow \mathrm{O}(t,1)$ in the sense that $\pi\circ\nu_{t,k} = \nu_t\circ\pi$, and $\nu_{t,k}$ restricts to the identity map on the fiber $\mathrm{O}(t-1,k-1)$. In particular, the restriction of $\nu_{t,k}$ induces the identity map on the critical cohomology group of the fiber $\mathrm{O}(t-1,k-1)$, while it also does so on the critical cohomology group of the base $\mathrm{O}(t,1)$ by Lemma 8.2. Claim (3) is then a consequence of the naturality of the Leray–Serre spectral sequence of (8.3.1). □

Our next goal is to compute the compactly-supported cohomology of $\mathrm{Sym}(k)$. We start with some preliminaries from linear algebra.

Recall that a square matrix U is *unitary* if the transpose of the conjugate is the inverse: $(\overline{U})^{\mathrm{tr}} = U^{-1}$; a matrix P is *Hermitian* if the conjugate is the transpose: $\overline{P} = P^{\mathrm{tr}}$. We will abbreviate $\overline{(-)^{\mathrm{tr}}}$ by $(-)^\star$ as is common; then P is Hermitian if $P^\star = P$, and U is unitary if $U^\star = U^{-1}$. A square matrix B is *normal* if $B^\star B = BB^\star$.

The Schur Decomposition Theorem states that any square complex matrix A can be written as UTU^{-1} for a unitary U and upper triangular T. If A is normal, then T must be normal and thus diagonal: normal matrices are unitarily diagonalizable.

The Polar Decomposition Theorem (PDT for short) [Hal, Section 2.5] states that any $k \times k$ complex matrix A can be written as products

$$P'U' = A = UP$$

in which U, U' are unitary $k \times k$ matrices and P, P' are Hermitian positive semi-definite $k \times k$ matrices. Moreover, if A is invertible, then P, P' can be chosen to be positive definite Hermitian, and the factorizations become unique. We make a few observations on this.

(1) Whenever $P'U' = A = UP$ with invertible matrices as in the PDT, then $U' = U$. Indeed, $UP = (UPU^{-1})U$, and UPU^{-1} is Hermitian and positive definite whenever P is. Namely,

$$\overline{UPU^{-1}} = \overline{U}\,\overline{P}\,\overline{U^{-1}} = U^{-\mathrm{tr}} P^{\mathrm{tr}} U^{\mathrm{tr}} = (UPU^{-1})^{\mathrm{tr}}$$

shows that it is Hermitian, and for a vector $x \in \mathbb{C}^k$ we have

$$x^\star (UPU^{-1}) x = x^\star UPU^\star x = y^\star P y$$

for $y = U^\star x$. The claim then follows from the uniqueness of polar decomposition.

(2) If $A = UP$ is a PDT factorization in which A is symmetric then so is U. Indeed, if $P'U = A = UP$ and $A = A^{\mathrm{tr}}$ then $P'U = A$ and $P^{\mathrm{tr}} U^{\mathrm{tr}} = A$ give the claim by uniqueness again. Note that from the previous item we have $P' = UPU^{-1}$ and so $UPU^{-1} = P^{\mathrm{tr}} (= \overline{P})$.

(3) Conversely, suppose P is Hermitian positive definite with $UPU^{-1} = P^{\mathrm{tr}} (= \overline{P})$ for some unitary symmetric U. Then UP is also symmetric, and invertible. Indeed,

$$(UP)^{\mathrm{tr}} = P^{\mathrm{tr}} U^{\mathrm{tr}} = UPU^{-1}U = UP.$$

(4) If B is normal, then it has all eigenvalues on the circle precisely when it is unitary. Indeed, if we diagonalize $B = UDU^{-1}$ with a unitary matrix then one unravels that

$$A^\star A = (UDU^{-1})^\star (UDU^{-1}) = UD^\star U^\star UDU^{-1} = UD^\star DU^{-1}$$

is the identity precisely if the diagonal elements λ_i of D satisfy $\overline{\lambda_i} = 1/\lambda_i$, which characterizes the points on the unit circle.

Now let

$$\mathrm{US}(k) := \{Y \in \mathrm{Sym}(k) \mid Y \text{ is unitary}\}.$$

By Lemma A.13, any unitary symmetric matrix has a Euclidean open neighborhood W where the squaring map has a section ρ. Let $\pi\colon \mathrm{Sym}(k) \longrightarrow \mathrm{US}(k)$ be the map that associates to a symmetric matrix A the unitary symmetric matrix U in the Polar Decomposition Theorem $A = UP$, with P the positive definite Hermitian factor for A. Then by Item (2) above for each $U \in W$, the set $\pi^{-1}(U)$ consists of positive definite Hermitian matrices P such that $\overline{P} = UPU^{-1}$. Denote this set by $\mathscr{P}_U$.

We claim that $\mathscr{P}_U$ is diffeomorphic to the space of positive definite real symmetric matrices, and the diffeomorphism is smooth in U. Indeed, write V for the symmetric unitary matrix $\rho(U)$ and consider the matrix $P' = VPV^{-1}$, which is Hermitian, positive definite by (1) above. Then we have

$$(P')^{\mathrm{tr}} = V^{-\mathrm{tr}}P^{\mathrm{tr}}V^{\mathrm{tr}} = \overline{V}(UPU^{-1})(\overline{V})^{-1} = \overline{V}V^2PV^{-2}(\overline{V})^{-1}.$$

As V is symmetric and unitary, $\overline{V}V^2 = V$ and it follows that $\overline{P'} = P'$. Conversely, if P' is positive definite real symmetric, then $P = \rho(U)^{-1}P'\rho(U)$ is positive definite symmetric by Item (1) and satisfies $\overline{P} = UPU^{-1}$. This proves the claim.

It follows that there is a locally trivial fiber bundle

$$\mathrm{PosSymR}(k) \longrightarrow \mathrm{Sym}(k) \longrightarrow \mathrm{US}(k) \tag{8.3.2}$$

with $\mathrm{PosSymR}(k)$ the positive definite real symmetric $k \times k$ matrices. The fiber is defined in the vector space of $k \times k$ symmetric real matrices by the open conditions of having positive principal minors. It is clearly nonempty, and it is convex as follows immediately from the characterization of positive definite symmetric matrices as those M such that $x^{\mathrm{tr}}Mx > 0$ for all $x \in \mathbb{R}^k$. It follows that $\mathrm{PosSymR}(k)$ is a contractible $\binom{k+1}{2}$-dimensional real manifold.

Proposition 8.4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic other than two and $k > 0$. In either Setting* (AN) *or* (ET)*, the variety* $\mathrm{Sym}(k)$ *is smooth affine and*

$$\mathrm{cptdim}(\mathrm{Sym}(k)) = \dim(\mathrm{Sym}(k)) = \binom{k+1}{2}$$

with critical cohomology group of rank one. Furthermore, in Setting (AN)*, the fundamental group of* $\mathrm{Sym}(k)$ *is free of rank one, with a generator represented by the loop* $\lambda_S\colon [0,1] \longrightarrow \mathrm{Sym}(k)$ *given by*

$$t \longmapsto \begin{bmatrix} \exp(2\pi t\sqrt{-1}) & 0 \\ 0 & I_{k-1} \end{bmatrix}.$$

Proof. In Setting (ET), this follows[1] from [Ba, Proposition 3.7] and Poincaré duality.

In Setting (AN), we use the locally trivial fiber bundle (8.3.2). Since $\mathrm{US}(k)$ is a connected compact manifold we have $\mathrm{cptdim}(\mathrm{US}(k)) = 0$ with critical cohomology group of rank one; we have $\mathrm{cptdim}(\mathrm{PosSymR}(k)) = \binom{k+1}{2}$ with critical cohomology group of rank one from the discussion above. The inclusion of $\mathrm{US}(k)$ into $\mathrm{Sym}(k)$ is a section of the projection in (8.3.2), and so the monodromy action on the fiber is trivial. Thus, the claim on the cohomology follows from Lemma 5.1.

In order to compute the fundamental group, let $\lambda\colon [0,1] \longrightarrow \mathrm{Sym}_{\mathbb{C}}(k)$ be any loop in $\mathrm{Sym}_{\mathbb{C}}(k)$. Denote $\lambda_{i,j}(t)$ the entries of $\lambda(t)$. Since $\mathbb{C}$ has real dimension 2, a generic change of coordinates makes sure that $\lambda_{1,1}(t) \neq 0$ for all t.

Conjugating $\lambda(t)$ by matrices $\begin{bmatrix} 1 & sv \\ sv^T & \mathbf{1}_{k-1} \end{bmatrix}$, with $0 \leqslant s \leqslant 1$ and appropriate $v \in \mathbb{C}^{k-1}$, gives a homotopy between λ and a loop λ' in which $\lambda_{i,1}(t) = 0$ for all $i > 1$. Assume λ is already in this form. Now change coordinates in the lower right $(k-1)$-minor, and iterate. Thus, λ can be assumed to be diagonal.

Let us write

$$\varepsilon(t) = \exp(2\pi t\sqrt{-1}).$$

[1] Note that it is claimed in [Ba] that $\mathbb{K}^t \smallsetminus \{0\}$ is simply connected, which is not the case in the étale topology. However, one can apply our Lemma 5.1 to rescue the result.

Since λ is a loop in the k-torus, which is contained in $\mathrm{Sym}(k)$, λ is homotopic in $\mathrm{Sym}(k)$ to a loop

$$\begin{bmatrix} \varepsilon(\ell_1 t) & 0 & \cdots & \cdots & 0 \\ 0 & \varepsilon(\ell_2 t) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \varepsilon(\ell_k t) \end{bmatrix}$$

with $\ell_1, \ldots \ell_k \in \mathbb{Z}$. It hence suffices to show that for $k = 2$ the loops $\begin{bmatrix} \varepsilon(t) & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon(t) \end{bmatrix}$ are homotopic.

For $0 \leqslant s, t \leqslant 1$, the matrices

$$\begin{bmatrix} (1-s) + s\varepsilon(t) & s(1-s)\varepsilon(t) \\ s(1-s)\varepsilon(t) & s + (1-s)\varepsilon(t) \end{bmatrix} \tag{8.4.1}$$

are symmetric and their determinant is

$$s(1-s) + \varepsilon(t)[(1-s)^2 + s^2] + (\varepsilon(t))^2[s(1-s) - s^2(1-s)^2].$$

The roots $x = 1/\varepsilon(t)$ of

$$x^2 s(1-s) + x[(1-s)^2 + s^2] + [s(1-s) - s^2(1-s)^2]$$

are

$$-\frac{(1-s)^2 + s^2}{2s(1-s)} \pm \sqrt{\left(\frac{(1-s)^2 + s^2}{2s(1-s)}\right)^2 - (1 - s(1-s))}.$$

For $0 < s < 1$, the square under the root is the square of a number that is the average of the positive number $(1-s)/2$ and its inverse, hence greater than 1. The second term under the root is at most 1. Thus, the roots $x = 1/\varepsilon(t)$ of the quadric are real (and negative in light of the term before the root).

Now, $1/\varepsilon(t)$ is real and negative for $0 \leqslant t \leqslant 1$ only when $t = 1/2$. In that case, the matrices in question are $\begin{bmatrix} 1-2s & -s(1-s) \\ -s(1-s) & 2s-1 \end{bmatrix}$, with determinant

$$-(2s-1)^2 - s^2(1-s^2),$$

which is never zero for $0 < s < 1$. It follows that the symmetric matrices in (8.4.1) are invertible and provide a homotopy in $\mathrm{Sym}(k)$ from $\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon(t) \end{bmatrix}$ to $\begin{bmatrix} \varepsilon(t) & 0 \\ 0 & 1 \end{bmatrix}$.

Thus, the fundamental group is generated by

$$\widetilde{\lambda} = \begin{bmatrix} \varepsilon(t) & \mathbf{0}_{1\times(k-1)} \\ \mathbf{0}_{(k-1)\times 1} & \mathbf{1}_{(k-1)\times(k-1)} \end{bmatrix}.$$

This loop has no relations since it generates the fundamental group $\mathbb{Z}$ of the larger space of invertible square matrices. $\square$

The following auxiliary spaces will be used in the proof of the main cohomology calculation of this section:

$$X^k_{t\times n} \quad := \quad \{Y \in \mathbb{K}_{t\times n} \mid Y^{\mathrm{tr}}Y \text{ has rank exactly } k\}, \tag{8.4.2}$$

$$G^k_{t\times n} \quad := \quad \left\{Y \in \mathbb{K}_{t\times n} \mid Y^{\mathrm{tr}}Y = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ with } A \in \mathrm{Sym}(k)\right\}, \tag{8.4.3}$$

$$F^k_{t\times n} \quad := \quad \left\{Y \in \mathbb{K}_{t\times n} \mid Y^{\mathrm{tr}}Y = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}\right\}. \tag{8.4.4}$$

For $k = t \leqslant n$, the containment $Y \in X^k_{t\times n}$ requires that Y be of full rank t, and conversely this rank condition is sufficient to assure that $Y \in X^k_{t\times n}$. Observe that for $k < t$ the condition $Y \in X^k_{t\times n}$ is usually different from the condition $\mathrm{rank}(Y) = k$.

Theorem 8.5. *Let $\mathbb{K}$ be a field and $n \geqslant t \geqslant k \geqslant 0$ integers. Then*

(1) *For $\mathbb{K} = \mathbb{C}$, we have*

$$H^i_{c,\mathrm{Sing}}(X^k_{t\times n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = \binom{t}{2} + \binom{k+1}{2}, \\ 0 & \text{if } i < \binom{t}{2} + \binom{k+1}{2}. \end{cases}$$

(2) *For $\mathbb{K}$ an algebraically closed field of characteristic other than two, we have*

$$H^i_{c,\text{ét}}(X^k_{t\times n}, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = \binom{t}{2} + \binom{k+1}{2}, \\ 0 & \text{if } i > \binom{t}{2} + \binom{k+1}{2}. \end{cases}$$

Proof. We start with the case $t = k$ in Settings (AN) and (ET). In this case, $X^t_{t\times n}$ is simply the set of $t \times n$ matrices of full rank, and then the result follows from [BS, Lemmas 2 and 2′].

For the case $k = 0$ and $t = 1$, we note that $X^0_{1\times n}$ is simply the zero matrix. Now we proceed by induction on t: fix $t > 1$ and assume that the claim holds for smaller values of t. We have already established the result for $t = k$; we now fix some k with $0 < k < t$.

Consider the locally trivial fiber bundle

$$X^0_{(t-k)\times(n-k)} \longrightarrow F^k_{t\times n} \longrightarrow \mathrm{O}(t,k) \tag{8.5.1}$$

from Lemma A.14.3. If $k = t-1$, then $X^0_{(t-k)\times(n-k)} = \{0\}$ and so $F^k_{t\times n} \cong \mathrm{O}(t,k)$. By Proposition 8.3,

$$\mathrm{cptdim}(F^1_{t\times n}) = \mathrm{cptdim}(\mathrm{O}(t,t-1)) = \binom{t}{2}$$

with critical cohomology group of rank one. For $k < t-1$, by Proposition 8.3, the hypotheses of Lemma 5.1 apply, and we deduce from (8.5.1) that

$$\mathrm{cptdim}(F^k_{t\times n}) = \mathrm{cptdim}(X^0_{(t-k)\times(n-k)}) + \mathrm{cptdim}(\mathrm{O}(t,k)) = \binom{t-k}{2} + tk - \binom{k+1}{2} = \binom{t}{2}$$

with critical cohomology group of rank one.

We now consider the locally trivial fiber bundle

$$F^k_{t\times n} \longrightarrow G^k_{t\times n} \longrightarrow \mathrm{Sym}(k) \tag{8.5.2}$$

from Lemma A.14.2. Proposition 8.4 shows that the hypotheses of Lemma 5.1 hold in Setting (ET); in Setting (AN), we must also verify that the monodromy action is trivial. We take the representative loop λ_S for the generator of $\pi_1(\mathrm{Sym}(k))$ from Lemma 8.4. For a matrix $Y \in F^k_{t\times n}$, the loop λ_S lifts (uniformly in Y) to the map $\widetilde{\lambda} : [0,1] \longrightarrow G^k_{t\times n}$ given by $\widetilde{\lambda}(t) = Y\lambda_S(t/2)$. Thus, the monodromy action on the fiber $F^k_{t\times n}$ is given by

$$\begin{aligned} \nu_F : F^k_{t\times n} &\longrightarrow F^k_{t\times n}, \\ [w_1, w_2, \dots, w_k] &\longmapsto [-w_1, w_2, \dots, w_k]. \end{aligned}$$

This map is compatible with the projection $\pi : F^k_{t\times n} \longrightarrow \mathrm{O}(t,k)$ from (8.5.1) in the sense that $\pi \circ \nu_F = \nu_{t,k} \circ \pi$, with $\nu_{t,k}$ as in Proposition 8.3. Note that ν_F restricts to the identity map on $X^0_{(t-k)\times(n-k)}$; by Proposition 8.3, $\nu_{t,k}$ induces the identity map on the critical cohomology group of $\mathrm{O}(t,k)$. It follows from naturality of the Leray–Serre spectral sequence that ν_F induces the identity map on the critical cohomology group of $F^k_{t\times n}$. By Lemma 5.1, we deduce that

$$\operatorname{cptdim}(G^k_{t\times n}) = \operatorname{cptdim}(F^k_{t\times n}) + \operatorname{cptdim}(\mathrm{Sym}(k)) = \binom{t}{2} + \binom{k+1}{2}$$

with critical cohomology group of rank one.

We now consider the locally trivial fiber bundle

$$G^k_{t\times n} \longrightarrow X^k_{t\times n} \longrightarrow \mathrm{Gr}(n-k,n) \tag{8.5.3}$$

from Lemma A.14.1. Since $\mathrm{Gr}(n-k,n)$ is simply connected of compact dimension zero, we obtain that

$$\operatorname{cptdim}(X^k_{t\times n}) = \operatorname{cptdim}(G^k_{t\times n}) = \binom{t}{2} + \binom{k+1}{2}$$

with critical cohomology group of rank one.

The case $k = 0$ follows by the previous cases using Lemma 5.2 as in the proof of Theorem 6.6. This completes the induction on t, and the proof. □

Proof of Theorem 8.1. This follows from Theorem 8.5 along the same lines as the proof of Theorem 6.1. □

APPENDIX A. SOME LOCALLY TRIVIAL FIBER BUNDLES

The Appendix is devoted to justifying the locally trivial fiber bundles utilized above; the main results are Lemmas A.6, A.9, and A.14. In order to establish the local triviality of these fiber bundles, we collect a number of linear algebra lemmas for which we could not find a reference.

A.1. **The Alternating Case.**

In this subsection we shall utilize the symplectic product

$$\langle a,b\rangle := a^{\mathrm{tr}}\Omega_{2t}b,$$

with Ω_{2t} as in (2.0.1), for any pair of $(2t)$-vectors a,b and observe that $\langle a,a\rangle$ vanishes. Denote

$$a^{\perp} := \{b \in \mathbb{K}^{2t} \mid \langle a,b\rangle = 0\}.$$

Lemma A.1 (Alternating Gram–Schmidt)**.** *For $t > k > 0$ let*

$$\pi\colon\ \mathrm{Sp}(2t,2t) \longrightarrow \mathrm{Sp}(2t,2k)$$

denote the projection onto the first $2k$ columns. Then there exists a Zariski open subset $U \subseteq \mathrm{Sp}(2t,2k)$ such that the base change of the projection map

$$U \times_{\mathrm{Sp}(2t,2k)} \mathrm{Sp}(2t,2t) \xrightarrow{\pi\times U} U$$

admits a section.

Proof. Let R and S denote the coordinate rings of $\mathrm{Sp}(2t,2k)$ and $\mathrm{Sp}(2t,2t)$, respectively. Let $u_1,v_1,\dots,u_k,v_k \in R^{2t}$ denote the column vector pairs of coordinates of $\mathrm{Sp}(2t,2k)$, and let $u_1',v_1',\dots,u_t',v_t' \in S^{2t}$ denote the column vector pairs of coordinates of $\mathrm{Sp}(2t,2t)$. Let $e_1,f_1,\dots,e_t,f_t \in \mathbb{K}^{2t}$ denote the columns of Ω_{2t}.

Set $R_j = R$, $w_j = u_j$, and $z_j = v_j$ for $1 \leqslant j \leqslant k$. For $k < i \leqslant t$, we inductively define vectors $w_i, z_i \in R_{i-1}^{2t}$, elements $\ell_i \in R_{i-1}$, and localizations R_i of R_{i-1} as follows:

$$w_i := e_i - \sum_{j=1}^{i-1} \langle w_j/\ell_j, e_i\rangle z_j + \sum_{j=1}^{i-1} \langle z_j, e_i\rangle w_j/\ell_j,$$

$$z_i := f_i - \sum_{j=1}^{i-1} \langle w_j/\ell_j, f_i\rangle z_j + \sum_{j=1}^{i-1} \langle z_j, f_i\rangle w_j/\ell_j,$$

$$\ell_i := \langle w_i, z_i\rangle,$$

and $R_i = R_{i-1}[1/\ell_i]$.

Consider the $\mathbb{K}$-algebra homomorphism $\varphi\colon R \longrightarrow \mathbb{K}$ given by

$$\varphi(u_j) = e_j, \quad \varphi(v_j) = f_j, \quad 1 \leqslant j \leqslant k.$$

For $i \leqslant t$ we claim that φ extends to a homomorphism $\varphi_i\colon R_i \longrightarrow \mathbb{K}$ such that

$$\varphi_i(w_j) = e_j, \quad \varphi_i(z_j) = f_j, \quad \varphi_i(\ell_j) = 1, \quad 1 \leqslant j \leqslant i.$$

By induction on i, we can assume that a map $\varphi_{i-1}\colon R_{i-1} \longrightarrow \mathbb{K}$ like so exists; then

$$\begin{aligned}\varphi_{i-1}(w_i) &= \varphi_{i-1}(e_i) - \sum_{j=1}^{i-1} \varphi_{i-1}(\langle w_j/\ell_j, e_i\rangle)\varphi_{i-1}(z_j) + \sum_{j=1}^{i-1} \varphi_{i-1}(\langle z_j, e_i\rangle)\varphi_{i-1}(w_j/\ell_j)\\ &= e_i - (\langle e_j/1, e_i\rangle)\varphi_{i-1}(f_j) + \varphi_{i-1}(\langle f_j, e_i\rangle)\varphi_{i-1}(e_j/1)) = e_i,\end{aligned}$$

and similarly $\varphi_{i-1}(z_i) = f_i$. Then $\varphi_{i-1}(\ell_i) = \langle e_i, f_i\rangle = 1$, and φ_{i-1} extends to φ_i as claimed. In particular, ℓ_i and R_i are nonzero for each i.

The projection map $\pi\colon \mathrm{Sp}(2t,2t) \longrightarrow \mathrm{Sp}(2t,2k)$ corresponds to the map $\pi^*\colon R \longrightarrow S$ given by $\pi^*(u_i) = u_i'$ and $\pi^*(v_i) = v_i'$. Setting $U = \mathrm{Spec}(R_t) \subseteq \mathrm{Spec}(R) = \mathrm{Sp}(2t,2k)$, the map

$$U \times_{\mathrm{Sp}(2t,2k)} \mathrm{Sp}(2t,2t) \xrightarrow{\pi\times U} U$$

corresponds to the map

$$R_t \xrightarrow{R_t\otimes\pi^*} R_t \otimes_R S.$$

We claim that there is an R_t-algebra left-inverse to $R_t \otimes \pi^*$ given by $\psi\colon S \otimes_R R_t \longrightarrow R_t$ determined by

$$\psi(u_j') = \frac{1}{\ell_j} w_j, \quad \psi(v_j') = z_j, \quad k < j \leqslant t.$$

For this, it suffices to check that the images of u_j' and v_j' satisfy the defining relations of S over R; namely that

$$\left\langle \frac{1}{\ell_i} w_i, z_j \right\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad \left\langle \frac{1}{\ell_i} w_i, \frac{1}{\ell_j} w_j \right\rangle = 0\,, \quad \text{and} \quad \langle z_i, z_j\rangle = 0.$$

It is clear from the construction that $\left\langle \frac{1}{\ell_i} w_i, z_i \right\rangle = 1$ for each i. In order to show that

$$\langle w_i, w_j\rangle = \langle w_i, z_j\rangle = \langle z_i, z_j\rangle = 0$$

for $1 \leqslant j < i \leqslant t$, we proceed by induction on i: one has

$$\begin{aligned}\langle w_i, w_j\rangle &= \langle e_i, w_j\rangle - \left\langle \frac{w_j}{\ell_j}, e_i\right\rangle \langle z_j, w_j\rangle + \langle z_j, e_i\rangle \left\langle \frac{w_j}{\ell_j}, w_j\right\rangle \\ &\qquad - \sum_{a \neq j < i} \left\langle \frac{w_a}{\ell_a}, e_i\right\rangle \langle z_a, w_a\rangle + \sum_{a \neq j < i} \langle z_a, e_i\rangle \left\langle \frac{w_a}{\ell_a}, w_a\right\rangle \\ &= \langle e_i, w_j\rangle + \left\langle \frac{w_j}{\ell_j}, e_i\right\rangle (-\ell_j) = 0,\end{aligned}$$

where the terms in the sums on the second line all vanish by the induction hypothesis on i.

Checking the other two sets of relations is done in similar fashion. $\square$

Lemma A.2. *For $1 \leqslant t \in \mathbb{N}$ there is a Zariski locally trivial fiber bundle*

$$\mathbb{K}^{2t-1} \longrightarrow \mathrm{Sp}(2t,2) \longrightarrow \mathbb{K}^{2t} \smallsetminus \{0\} \tag{A.2.1}$$

given by mapping an element of $\mathrm{Sp}(2t,2)$ *to its first column.*

Proof. Let $e_1, f_1, \dots, e_t, f_t$ be the columns of Ω_{2t} from (2.0.1). Let u, v be the column vectors of coordinates of $\mathrm{Sp}(2t,2)$. Let U_i, U_i' be the open subsets of $\mathbb{K}^{2t} \smallsetminus \{0\}$ where $\langle u, f_i\rangle \neq 0$ and $\langle u, e_i\rangle \neq 0$, respectively. Then, identifying $\mathbb{K}^{2t-1} \cong e_i^{\perp}$ one has isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U_i) & \cong & U_i \times \mathbb{K}^{2t-1} \\ [u,v] & \longmapsto & (u, v + \langle v, e_i\rangle f_i) \\ \left[u, v + \dfrac{1 - \langle u, v\rangle}{\langle u, f_i\rangle} f_i\right] & \longleftarrow & (u,v); \end{array}$$

one has corresponding isomorphisms for the open sets where $\langle u, f_i\rangle \neq 0$. This shows that $\mathbb{K}^{2t} \smallsetminus \{0\} = \bigcup (U_i \cup U_i')$, and that (A.2.1) is a Zariski locally trivial fiber bundle. $\square$

Lemma A.3. *For $1 < t \in \mathbb{N}$, there is a Zariski locally trivial fiber bundle*

$$\mathrm{Sp}(2t-2, 2k-2) \longrightarrow \mathrm{Sp}(2t, 2k) \longrightarrow \mathrm{Sp}(2t,2) \tag{A.3.1}$$

given by mapping an element of $\mathrm{Sp}(2t,2k)$ *to its first column pair.*

Proof. Note that $\mathrm{Sp}(2t,2)$ is homogeneous under the action of $\mathrm{Sp}(2t,2t)$. By Lemma A.1, $\mathrm{Sp}(2t,2)$ is covered by Zariski open sets on which the projection from $\mathrm{Sp}(2t,2t)$ admits a section; hence, it suffices to show for any such $U \subseteq \mathrm{Sp}(2t,2)$ that the preimage of U in $\mathrm{Sp}(2t,2k)$ is isomorphic to the product $\mathrm{Sp}(2t-2,2k-2) \times U$, compatibly with the projection map in (A.3.1).

Let α be a section $U \longrightarrow \mathrm{Sp}(2t,2t)$. For $M \in \mathrm{Sp}(2t,2k)$ set $\beta(M) = \alpha(\pi(M))^{-1}M$. Then $\alpha(\pi(M))$ is a symplectic matrix with $\alpha(\pi(M))e_1, \alpha(\pi(M))f_1$ equal to the first column pair of M. Hence, $\beta(M) \in \mathrm{Sp}(2t,2k)$ has first column pair $\{e_1, f_1\}$ and thus every other column has zeroes in rows one and two. In particular, erasing the first two rows and columns matches $\beta(M)$ with a matrix $\beta'(M) \in \mathrm{Sp}(2t-2,2k-2)$.

Thus, one has isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times \mathrm{Sp}(2t-2,2k-2) \\ M & \longmapsto & (\pi(M), \beta'(M)) \\ \alpha(M)\begin{bmatrix} \Omega_2 & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix} & \longleftarrow & (M,N). \end{array}$$

This shows that (A.3.1) is a Zariski locally trivial fiber bundle. $\square$

Lemma A.4 (Jozefiak–Pragacz [JP]). *Let U denote the variety of $n \times n$ alternating matrices $A = [a_{ij}]$ over $\mathbb{K}$ with $a_{12} \neq 0$. There is a morphism $\alpha\colon U \longrightarrow \mathrm{GL}_n$ such that for each $A \in U$, the matrix $B := \alpha(A)^{\mathrm{tr}} A \alpha(A)$ is alternating, and block decomposes as*

$$B = \begin{bmatrix} \Omega_2 & \mathbf{0} \\ \mathbf{0} & A' \end{bmatrix}$$

with $\mathrm{Pf}_{2t}(A) = \mathrm{Pf}_{2t-2}(A')$.

Lemma A.5 (Alternating roots). *Consider the map*

$$\begin{aligned} \mathrm{GL}_{2t} &\xrightarrow{\mu} \mathrm{Alt}(2t), \\ M &\longmapsto M^{\mathrm{tr}} \Omega_{2t} M. \end{aligned}$$

For any $A \in \mathrm{Alt}(2t)$, there is some Zariski open set $U \subseteq \mathrm{Alt}(2t)$ containing A such that there is a section $\psi\colon U \longrightarrow \mathrm{GL}_{2t}$ of the map μ.

Proof. We proceed by induction on t. For $t = 1$, one can globally take

$$\psi\left(\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose we have for every invertible alternating $(2t-2) \times (2t-2)$ matrix a Zariski open neighborhood with a section as in the statement. Let $A = [a_{ij}]$ be an invertible alternating $2t \times 2t$ matrix. For some $i < j$, we have $a_{ij} \neq 0$. Let P be a permutation matrix swapping columns 1 and i, and swapping columns 2 and j; then there is a nonempty Zariski open neighborhood U_1 of A in $\mathrm{Alt}(2t)$ such that $P^{\mathrm{tr}} Y P$ has nonzero $(1,2)$-entry for all $Y \in U_1$. Let $\sigma\colon\ \mathrm{Alt}(2t) \longrightarrow \mathrm{Alt}(2t)$ be the map $\sigma(Y) = P^{\mathrm{tr}} Y P$. By Lemma A.4, there is a morphism $\alpha\colon \sigma(U_1) \longrightarrow \mathrm{GL}_{2t}$ such that the map $\beta\colon \sigma(U_1) \longrightarrow \mathrm{GL}_{2t}$ given by $\beta(Y) = \alpha(Y)^{\mathrm{tr}} Y \alpha(Y)$ has image in the space of matrices

$$\left\{ \begin{bmatrix} \Omega_2 & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix} \;\middle|\; M \in \mathrm{Alt}(2t-2) \right\};$$

we set $\beta'\colon \sigma(U_1) \longrightarrow \mathrm{Alt}(2t-2)$ to be the corresponding map. By induction hypothesis, there is some Zariski open neighborhood $U_2 \subseteq \mathrm{Alt}(2t-2)$ of $\beta'(\sigma(A))$ and a map $\psi'\colon U_2 \longrightarrow \mathrm{GL}_{2t-2}$ such that $\psi'(Y')^{\mathrm{tr}} \Omega_{2t-2} \psi'(Y') = Y'$ for all $Y' \in U_2$.

Set $U = U_1 \cap (\beta'\sigma)^{-1}(U_2)$ and define $\gamma\colon U \longrightarrow \mathrm{GL}_{2t}$ by $\gamma(Y) = \begin{bmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & \psi'(\beta'(\sigma(Y))) \end{bmatrix}$. Then $\psi(Y) = \gamma(Y)\alpha(\sigma(Y))^{-1} P^{-1}$ is a section of μ. □

Lemma A.6. *Let n, t, k be integers with $n \geqslant 2t > 2k \geqslant 0$. Each of the following is a Zariski locally trivial fiber bundle:*

$$G^{2k}_{2t\times n} \longrightarrow X^{2k}_{2t\times n} \longrightarrow \mathrm{Gr}(n-2k, n), \tag{A.6.1}$$

where $Y \in X^{2k}_{2t\times n}$ maps to $\ker(Y^{\mathrm{tr}} \Omega_{2t} Y) \in \mathrm{Gr}(n-2k, n)$;

$$F^{2k}_{2t\times n} \longrightarrow G^{2k}_{2t\times n} \longrightarrow \mathrm{Alt}(2k), \tag{A.6.2}$$

where an element of $G^{2k}_{2t\times n}$ maps to the top left $2k \times 2k$ submatrix of $Y^{\mathrm{tr}} \Omega_{2t} Y$;

$$X^0_{(2t-2k)\times(n-2k)} \longrightarrow F^{2k}_{2t\times n} \longrightarrow \mathrm{Sp}(2t, 2k), \tag{A.6.3}$$

where an element of $F^{2k}_{2t\times n}$ maps to its first k pairs of columns;

$$F^{2k}_{2t\times n} \longrightarrow X^{2k}_{2t\times n} \longrightarrow \mathrm{Alt}^{2k}_{n\times n}, \tag{A.6.4}$$

where $Y \in X^{2k}_{2t\times n}$ maps to $Y^{\mathrm{tr}} \Omega_{2t} Y$.

Proof. (A.6.1): Suppose $d \in \mathbb{N}$ satisfies $1 \leqslant d \leqslant n$. Choose a basis $\{e_1, \dots, e_n\}$ for $\mathbb{K}^n$ and denote K, K' the subspaces of $\mathbb{K}^n$ spanned by $\{e_1, \dots, e_d\}$ and $\{e_{d+1}, \dots, e_n\}$ respectively. Let $U \subseteq \mathrm{Gr}(n-d, n)$ be the collection of subspaces $W \subseteq \mathbb{K}^n$ such that $\dim(W) = n-d$ and $W \cap K = 0$. Let A be an $m \times n$ matrix of rank d, for some $m \geqslant d$. Then $W := \ker(A) \in U$ if and only if the submatrix of A consisting of the first d columns has full rank. In this case there is a unique matrix M_W of the form $\begin{bmatrix} \mathbf{1}_d & * \\ \mathbf{0}_{(n-d)\times d} & \mathbf{1}_{n-d} \end{bmatrix}$ such that AM_W has kernel K', or, equivalently, that $W = M_W K'$. The matrix M_W depends only on W, and the assignment $W \longmapsto M_W$ is a regular function from U to $\mathbb{K}^{n\times n}$.

Now set $d = 2k$ and let $(-)_{\leqslant d}$ be the operator that projects a matrix to its submatrix consisting of the leftmost d columns. Then the preimage $\pi^{-1}(U)$ in $X^{2k}_{2t\times n}$ is

$$\{Y \in \mathbb{K}^{2t\times n} \mid \mathrm{rank}(Y^{\mathrm{tr}}\Omega_{2t}Y) = 2k = \mathrm{rank}((Y^{\mathrm{tr}}\Omega_{2t}Y)_{\leqslant 2k})\}.$$

Then

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times G^{2k}_{2t\times n} \\ Y & \longmapsto & (W = \ker(Y^{\mathrm{tr}}\Omega_{2t}Y), YM_W) \\ Y'(M_W)^{-1} & \longleftarrow & (W, Y') \end{array}$$

is given by regular functions and hence an isomorphism. Indeed, for $W = \ker(Y^{\mathrm{tr}}\Omega_{2t}Y)$, the kernel of $(YM_W)^{\mathrm{tr}}\Omega_{2t}(YM_W) = (M_W)^{\mathrm{tr}}Y^{\mathrm{tr}}\Omega_{2t}YM_W$ is the kernel of $Y^{\mathrm{tr}}\Omega_{2t}YM_W$, and that is K' by definition of M_W. On the other hand, the assignment $Y'(M_W)^{-1} \longleftarrow (W, Y')$ clearly lands in $\pi^{-1}(U)$, and $\ker((Y'(M_W)^{-1})^{\mathrm{tr}}\Omega_{2t}Y'(M_W)^{-1})$ equals

$$\ker((Y')^{\mathrm{tr}}\Omega_{2t}Y'(M_W)^{-1}) = M_W \ker((Y')^{\mathrm{tr}}\Omega_{2t}Y') = M_W K' = W.$$

Similar computations apply on the open sets in $\mathrm{Gr}(n-d, n)$ defined by the nonvanishing of any other d-minor.

(A.6.2): To see that this is Zariski locally trivial, by Lemma A.5, we can take a Zariski open cover of $\mathrm{Alt}(2k)$ by sets U for which there is a map $\psi\colon U \longrightarrow \mathrm{GL}_{2k}$ such that $\psi(A)^{\mathrm{tr}}\Omega_{2k}\psi(A) = A$. We then have isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times F^{2k}_{2t\times n} \\ Y\begin{bmatrix} \psi(A) \\ 0 \end{bmatrix} & \longleftarrow & (A, Y) \\ Y & \longmapsto & \left(\pi(Y), Y\begin{bmatrix} \psi(\pi(Y))^{-1} \\ 0 \end{bmatrix}\right). \end{array}$$

(A.6.3): By Lemma A.1, there is a covering of $\mathrm{Sp}(2t, 2k)$ by Zariski open sets U for which $U \times_{\mathrm{Sp}(2t,2k)} \mathrm{Sp}(2t, 2t) \longrightarrow U$ has a section. Fix such a U and a section α. Let

$$X' = \left\{ \begin{bmatrix} \Omega_{2k} & 0 \\ 0 & Y_{(2t-2k)\times(n-2k)} \end{bmatrix} \;\middle|\; Y \in X^0_{(2t-2k)\times(n-2k)} \right\} \cong X^0_{(2t-2k)\times(n-2k)}.$$

We then have isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times X' \\ \alpha(A)Y & \longleftarrow & (A, Y) \\ Y & \longmapsto & \pi(Y)\alpha(\pi(Y))^{-1}Y. \end{array}$$

(A.6.4): By [Ba, p. 77], there is a Zariski locally trivial fiber bundle

$$\mathrm{Alt}(2k) \longrightarrow \mathrm{Alt}^{2k}_{n\times n} \longrightarrow \mathrm{Gr}(n-2k, n)$$

given by mapping a matrix $Y \in \mathrm{Alt}^{2k}_{n\times n}$ to its kernel. Take a Zariski open cover $\{V_i\}$ of $\mathrm{Gr}(n-2k,n)$ on which this bundle and the bundle (A.6.1) both trivialize, and let U_i and T_i be the preimages of V_i in $\mathrm{Alt}^{2k}_{n\times n}$ and $X^{2k}_{2t\times n}$, respectively. We then have a commutative diagram of the form

$$\begin{tikzcd} T_i \arrow[r] \arrow[d, "\cong"] & U_i \arrow[r] \arrow[d, "\cong"] & V_i \\ G^{2k}_{2t\times n}\times V_i \arrow[r] & \mathrm{Alt}(2k)\times V_i \arrow[ur] & \end{tikzcd}$$

where the map along the bottom is the product of the projection in (A.6.2) with the identity on V_i. Since (A.6.2) is Zariski locally trivial, one can take an open cover of $\mathrm{Alt}(2k)$ on which the map $G^{2k}_{2t\times n} \longrightarrow \mathrm{Alt}(2k)$ decomposes as a product with fiber $F^{2k}_{2t\times n}$; taking the preimage of this cover in each U_i gives a cover of $\mathrm{Alt}^{2k}_{n\times n}$ on which (A.6.4) decomposes as a product. □

A.2. **The Generic Case.** For the following lemma, refer to display (7.1.2).

Lemma A.7. *For integers $t \geqslant k \geqslant 1$ there are Zariski locally trivial fiber bundles*

(A.7.1) $$\mathrm{P}(t-k+1,1) \longrightarrow \mathrm{P}(t,k) \longrightarrow \mathrm{P}(t,k-1)$$

given by projecting a pair (Y,Z) to the top $k-1$ rows of Y and the leftmost $k-1$ columns of Z.

Proof. There is a natural $\mathrm{GL}(t)$-action on $\mathrm{P}(k-1,t)$ sending $(M,(Y,Z))$ to $(YM, M^{-1}Z)$. This action commutes with the projection of the proposed bundle and allows to replace a given (Y,Z) with a pair of the form $((\mathbf{1}_{k-1}, \mathbf{0}_{k-1\times(t-k+1)}), (\mathbf{1}_{k-1}, Z_0)^{\mathrm{tr}})$ where $Z_0 \in \mathbb{K}^{(k-1)\times(t-k+1)}$. The fiber of $P(t,k) \longrightarrow P(t,k-1)$ over $((\mathbf{1}_{k-1}, \mathbf{0}_{(k-1)\times(t-k+1)}), (\mathbf{1}_{k-1}, Z_0)^T)$ can be identified with

$$\{(A,B) \in \mathbb{K}^{1\times(t-k+1)} \times \mathbb{K}^{(t-k+1)\times 1} \mid AB = \mathbf{1}\}$$

via

$$(A,B) \longmapsto \left(\begin{bmatrix} \mathbf{1}_{k-1} & \mathbf{0}_{(k-1)\times(t-k+1)} \\ -AZ_0^{\mathrm{tr}} & A \end{bmatrix}, \begin{bmatrix} \mathbf{1}_{k-1} & \mathbf{0}_{(k-1)\times 1} \\ Z_0^{\mathrm{tr}} & B \end{bmatrix} \right).$$

Zariski triviality follows. □

Lemma A.8. *For $t > k \geqslant 1$, there is a fiber bundle, locally trivial in the Zariski topology,*

(A.8.1) $$(\mathbb{K}^t \smallsetminus \mathbb{K}^k) \longrightarrow \mathrm{GL}(t,k+1) \xrightarrow{\pi} \mathrm{GL}(t,k),$$

that forgets the last column.

Proof. Let $S := \{s_1 < \cdots < s_k\} \in \binom{\{1,\dots,m\}}{k}$, and let U_S be the open subset of $\mathrm{GL}(m,k)$ consisting of matrices A where the k-submatrix $A(S)$ with rows in S is nonzero. These U_S are an open cover of $\mathrm{GL}(m,k)$. For each $A \in U_S$ there is a unique $N_A \in \mathrm{GL}(m)$ such that

- $n_{i,j} = 1$ if $i = j \notin S$;
- the submatrix of N with rows and columns in S is the inverse of $A(S)$; and
- all other entries are zero.

Let P_S be the $m\times m$ permutation matrix to the permutation that swaps i with s_i for $1 \leqslant i \leqslant k$ and fixes all $i > k$. The function $A \longrightarrow N_A$ is regular on U_S.

Denote $U'_S \subseteq \mathrm{GL}(t,k+1)$ the preimage of U_S under π in (A.8.1). Then under $A \longmapsto P_S N_A A$, U_S is isomorphic as variety to the set of matrices $\begin{bmatrix} \mathbf{1}_k \\ A_S \end{bmatrix}$ where $A_S \in \mathbb{K}^{(t-k)\times k}$. The same process, $B \longmapsto P_S N_{\pi(B)} B$, identifies $\pi^{-1}(U_S)$ with the matrices $\begin{bmatrix} \mathbf{1}_k & b_S \\ \pi(B)_S & b'_S \end{bmatrix}$, where $(b_S, b'_S) \in \mathbb{K}^k \oplus \mathbb{K}^{t-k}$ is independent of the other columns. Independence corresponds to $b'_S \neq b_S^{\mathrm{tr}} \pi(B)_S$ and hence the composition of maps

$$B \longmapsto P_S N_{\pi(B)} B = \begin{bmatrix} \mathbf{1}_k & b_S \\ \pi(B)_S & b'_S \end{bmatrix} \longmapsto (\pi(B)_S, b_S, b'_S - b_S^{\mathrm{tr}} \pi(B)_S)$$

gives an isomorphism of varieties between $\pi^{-1}(U_S)$ and $U_S \times \mathbb{K}^k \times (\mathbb{K}^{t-k} \smallsetminus \{0\})$. $\square$

Refer to (7.3.3)–(7.3.5) for notation in the following lemma.

Lemma A.9. *Let m,n,t,k be integers such that $m,n \geqslant t > k \geqslant 0$. The following are Zariski locally trivial fiber bundles:*

$$G^k_{m,t,n} \longrightarrow X^k_{m,t,n} \longrightarrow \mathrm{Gr}(n-k,n), \tag{A.9.1}$$

sending $X^k_{m,t,n} \ni (Y,Z) \longmapsto \ker(YZ)$;

$$F^k_{m,t,n} \longrightarrow G^k_{m,t,n} \longrightarrow \mathrm{GL}(m,k), \tag{A.9.2}$$

sending (Y,Z) to the leftmost k columns of YZ; and

$$X^0_{(m-k),(t-k),(n-k)} \longrightarrow F^k_{m,t,n} \longrightarrow \mathrm{P}(t,k), \tag{A.9.3}$$

sending $(Y,Z) \in F^k_{m,t,n}$ to the pair of matrices consisting of the top $k \times t$ part of Y and the leftmost $t \times k$ part of Z.

Proof. (A.9.1): One may follow along similar lines to (A.6.1); retaining the notation from there, with the sole change that $d = t$, we have the isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times G^k_{m,t,n} \\ Y & \longmapsto & (W = \ker(YZ), (Y, ZM_W)) \\ (Y, Z'(M_W)^{-1}) & \longleftarrow & (W, (Y, Z')). \end{array}$$

(A.9.2): We use the notation of the proof of Lemma A.8. Denote $\pi(Y,Z)$ the image of $(Y,Z) \in G^k_{m,t,n}$ in $\mathrm{GL}(m,k)$ and assume that it is in U_S. Then,

$$P_S N_{\pi(Y,Z)} YZ = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(m-k)\times k} & \mathbf{0}_{(m-k)\times(n-k)} \end{bmatrix}$$

and so $(P_S N_{\pi(Y,Z)} Y, Z) \in F^k_{m,t,n}$. Note that N and P can be reconstructed just from S and $\pi(Y,Z)$ for $(Y,Z) \in G^k_{m,t,n} \cap \pi^{-1}(U_S)$.

Over U_S we can now identify $\pi^{-1}(U_S)$ with $F^k_{m,t,n} \times U_S$ by sending (Y,Z) to $(P_S N_{\pi(Y,Z)} Y, Z)$ paired with $\pi(Y,Z)$. In reverse, from $((Y,Z),A) \in F^k_{m,t,n} \times \pi(G^k_{m,t,n})$, first recover N_A and P_S, and then map $((Y,Z),A)$ to $((N_A)^{-1} P_S Y, Z)$.

(A.9.3): Let $(A,B) \in \mathrm{P}(t,k)$. Suppose $\{s_1 < \cdots < s_k\} = S \in \binom{\{1,\dots,t\}}{k}$ is such that the submatrix $B(S)$ of B with rows in S is nonzero, denote the corresponding open set in $\mathrm{P}(t,k)$ by V_S, and denote P_S the $t \times t$ permutation matrix to the permutation that swaps i with s_i for $1 \leqslant i \leqslant k$, as in the proof of Lemma A.8. Denote M_B the invertible $t \times t$ matrix such that

- $m_{i,j} = 1$ if $i = j \notin S$;

- the submatrix of M_B in the rows and columns of S is the inverse of $B(S)$; all other entries are zero.

For $(Y,Z) \in F^k_{m,t,n}$, we write $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ where $Y_1 \in \mathbb{K}^{k\times t}$ and $Z = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix}$ where $Z_1 \in \mathbb{K}^{t\times k}$. Then on $\pi^{-1}(V_S)$ we have an morphism of varieties

$$(Y,Z) \longmapsto ((Y',Z'),Z_1) := ((Y(M_{Z_1})^{-1}P_S^{-1}, P_S M_{Z_1} Z), Z_1),$$

to the subset of $\mathrm{P}(t,k) \times \mathbb{K}^{t\times(n-k)}$ where $Z' = \begin{bmatrix} Z'_1, Z'_2 \end{bmatrix}$ is of the form $\begin{bmatrix} \mathbf{1}_k & D \\ C & E \end{bmatrix}$ (noting that $YZ = Y'Z'$). Observe that P_S, M_{Z_1} are regular functions in the entries of Z_1 and so this is in fact an isomorphism.

A vector $(b,b') \in \mathbb{K}^k \oplus \mathbb{K}^{t-k}$ satisfies $(b,b')^{\mathrm{tr}} Z'_1 = 0$ exactly if it is a linear combination of the vectors $v_i \in \mathbb{K}^k \times \mathbb{K}^{t-k}$ that have $\mathbb{K}^k$-component the i-th row of $-C$ and $\mathbb{K}^{t-k}$-component the i-th unit vector in $\mathbb{K}^{t-k}$, with $1 \leqslant i \leqslant t-k$. Reading v_i as a regular function in the entries of C, define the $t\times t$ matrix $M = M_{Y'_1,Z'_2}$ given by

- the top k rows of M are exactly Y'_1;
- the rows $k+1 \leqslant i \leqslant t$ of M are $v_1,\dots,v_{t-k}$ in that order.

Note that this matrix is full rank: the top and bottom are bases for two subspaces of $\mathbb{K}^t$ that meet in the origin, since any $y \in \ker(Z'_1)$ dots to zero with Z'_1 but a nonzero element of the row span of Y'_1 cannot do so. Note also that M is determined by Y'_1 and Z'_1, and hence also by the original Y_1 and Z_1.

It follows that the assignment

$$F^k_{m,t,n} \ni (Y,Z) \longmapsto ((Y',Z'),Z_1) \longmapsto (Y'',Z'') := ((Y'M^{-1}, MZ'), (Y_1,Z_1))$$

is a well-defined morphism with $Y'' = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_{k\times(t-k)} \\ B & C \end{bmatrix}$ and $Z'' = \begin{bmatrix} \mathbf{1}_k & D \\ \mathbf{0}_{(t-k)\times k} & E \end{bmatrix}$. Remark that $Y''Z'' = YZ$ and hence $B = 0$, whence $CE = 0$. Thus, $(Y,Z) \longmapsto ((Y_1,Z_1),(C,E))$ takes values in $\mathrm{P}(t,k) \times X^0_{(m-k),(t-k),(n-k)}$.

Since the matrices M_{Z_1}, P_S, M are regular functions in (Y_1,Z_1), the morphism is an isomorphism. It follows that on $\pi^{-1}(V_S)$, (A.9.3) is identified with the projection of $X^0_{(m-k),(t-k),(n-k)} \times \mathrm{P}(t,k)$ onto the second factor. □

A.3. **The Symmetric Case.** In this subsection we shall utilize the standard inner product

$$\langle a,b \rangle := a^{\mathrm{tr}} b,$$

for any pair of t-vectors a,b.

Refer to (8.1.1) in the next lemma.

Lemma A.10. *Let $t > k > 0$ be integers. Let $\pi\colon \mathrm{O}(t,t) \longrightarrow \mathrm{O}(t,k)$ denote the projection onto the first k columns.*

1. *In Setting* (AN)*, there exists a Euclidean open cover of* $\mathrm{O}(t,k)$ *such that on each open set U in the cover, the base change of the projection map*

$$U \times_{\mathrm{O}(t,k)} \mathrm{O}(t,t) \xrightarrow{\pi\times U} U$$

admits a section.

2. *In Setting* (ET)*, there exists an affine étale cover* $\mathrm{O}(t,k)$ *such that for each extension $U \longrightarrow \mathrm{O}(t,k)$ in the cover, the base change of the projection map*

$$U \times_{\mathrm{O}(t,k)} \mathrm{O}(t,t) \xrightarrow{\pi\times U} U$$

admits a section.

Proof. We begin with the étale case.

Utilizing the transitive $\mathrm{O}(t,t)$ action, for which π is equivariant, it suffices to find some nontrivial étale U for which the map $U \times_{\mathrm{O}(t,k)} \mathrm{O}(t,t) \xrightarrow{\pi \times U} U$ admits a section.

Let R denote the coordinate ring of $\mathrm{O}(t,k)$ and S denote the coordinate ring of $\mathrm{O}(t,t)$. Let $v_1, \dots, v_k \in R^t$ denote the vectors of coordinates of $\mathrm{O}(t,k)$ and let $v'_1, \dots, v'_t \in S^t$ denote the column vectors of coordinates of $\mathrm{O}(t,t)$. Let $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{K}^t$.

Set $R_i = R$ and $w_i = v_i$ for $1 \leqslant j \leqslant k$. For $k < i \leqslant t$, we inductively define vectors $w_i \in R^t_{i-1}$, elements $\ell_i \in R_{i-1}$, and R-algebras R_i as follows:

$$w_i := e_i - \sum_{j=1}^{i-1} \langle e_i, w_j \rangle w_j, \quad \ell_i := \langle w_i, w_i \rangle, \quad \text{and} \quad R_i := R_{i-1}[1/\ell_i][s_i]/(s_i^2 - \ell_i).$$

Consider the $\mathbb{K}$-algebra homomorphism $\varphi \colon R \longrightarrow \mathbb{K}$ given by

$$\varphi(v_j) = e_j, \quad 1 \leqslant j \leqslant k.$$

By induction on $i \leqslant t$, φ extends to a homomorphism $\varphi_i \colon R_i \longrightarrow \mathbb{K}$ such that

$$\varphi_i(w_j) = e_j, \quad \varphi_i(\ell_j) = 1, \quad \varphi_i(s_j) = 1, \quad 1 \leqslant j \leqslant i.$$

In particular, there is a nonzero homomorphism $\varphi_i \colon R_i \longrightarrow \mathbb{K}$ such that $\varphi_i(\ell_i) \neq 0$. But then $\ell_i \neq 0$ in R_i and thus R_t is a nonzero étale extension of R.

We claim that there is a well-defined R_t-algebra homomorphism $\psi \colon S \otimes_R R_t \longrightarrow R_t$ given by $\psi(w_i) = v_i/s_i$, where we interpret $s_i = 1$ for $i < k$. It suffices to check that the vectors v_i/s_i satisfy the defining relations

$$\langle w_i, w_j \rangle = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Both are easily verified inductively.

For the analytic case, one proceeds along similar lines, inductively defining w_i as above, and choosing U_i to be sufficiently small so that $\langle w_i, w_i \rangle$ admits a holomorphic square root. □

We require the following result of Micali and Villamayor.

Lemma A.11 ([MV]). *Let U denote the variety of $n \times n$ symmetric matrices $A = [a_{ij}]$ over $\mathbb{K}$ with $a_{11} \neq 0$. In Setting* (ET) *there is an étale morphism $\alpha \colon U \longrightarrow \mathrm{GL}_n$ such that for each $A \in U$, the matrix $B := \alpha(A)^{\mathrm{tr}} A \alpha(A)$ is symmetric, and block decomposes as*

$$B = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}$$

with $\mathrm{I}_t(A) = \mathrm{I}_{t-1}(A')$.

For the following lemma, refer to (8.1.2).

Lemma A.12 (Symmetric roots)**.** *Consider the map*

$$\begin{array}{ccc} \mathrm{GL}_t & \xrightarrow{\mu} & \mathrm{Sym}(t) \\ M & \longmapsto & M^{\mathrm{tr}} M. \end{array}$$

(1) *In Setting* (AN)*, there exists a Euclidean open subset $U \subseteq \mathrm{Sym}(t)$ containing A such that the base change of the projection map*

$$U \times_{\mathrm{Sym}(t)} \mathrm{GL}_t \xrightarrow{U \times \mu} U$$

admits a section.

(2) *In Setting* (ET)*, there exists an affine étale extension* $U \longrightarrow \mathrm{Sym}(t)$ *such that the base change of the projection map*

$$U \times_{\mathrm{Sym}(t)} \mathrm{GL}_t \xrightarrow{U\times\mu} U$$

admits a section.

Proof. We start with Setting (ET), where we proceed by induction on t. For $t = 1$, we take the étale cover

$$U = \mathrm{Spec}(\mathbb{C}[x,1/x][y]/(x-y^2)) \longrightarrow \mathrm{Sym}(1) = \mathrm{Spec}(\mathbb{C}[x,1/x])$$

given by adjoining a square root. In this case, $U \times_{\mathrm{Sym}(1)} \mathrm{GL}_1$ identifies with $\mathbb{K}^\times \sqcup \mathbb{K}^\times$, and the map from each component to $\mathrm{Sym}(1) \cong \mathbb{K}^\times$ identifies with the identity, and the claim follows.

Now let $t > 1$. On the Zariski open set $U_{1,1}$ of $\mathrm{Sym}(t) \ni A$ with $a_{1,1} \neq 0$, Lemma A.11 reduces the search for σ to the case of a $(t-1)\times(t-1)$ matrix. If $a_{1,1} = 0$, let $U_{1,k}$ be the Zariski open set where $a_{1,k} \neq 0$. Let E be the elementary operation that adds row k to row 1. Then EAE^{tr} is in $U_{1,1}$ (since $2 \neq 0$) and thus $E^{-1}\sigma E^{-\mathrm{tr}}$, with σ the section found over $U_{1,1}$, gives the required section over the Zariski open set $E^{-1}U_{1,1}E^{-\mathrm{tr}} \cap U_{1,k} \ni A$.

In the analytic setting, follow the étale construction and take a Euclidean neighborhood of A on which the constructed étale cover is a covering space. □

The following proof is a fleshed out version of [Is].

Lemma A.13 (Unitary symmetric square roots)**.** *The map from the set of unitary symmetric matrices to itself given by* $U \longmapsto U^2$ *has Euclidean local sections.*

Proof. We consider, for variables $r = \{r_1,\dots,r_k\}$, the rational function

$$f_r(z) := \sum_{i=1}^{k} \frac{(z-r_i^2+r_i)\cdot\prod_{j\neq i}(z-r_j^2)}{\prod_{j\neq i}(r_i^2-r_j^2)}.$$

Clearly, $f_r(z)$ has at worst poles at $r_i + r_{i'}$ and at $r_i - r_{i'}$, and our first claim on $f_r(z)$ is that only the former poles will occur. Indeed, the only summands where $r_i - r_{i'}$ is a pole are those of index i and i'. However,

$$\frac{(z-r_i^2+r_i)\cdot\prod\limits_{j\neq i}(z-r_j^2)}{\prod\limits_{j\neq i}(r_i^2-r_j^2)} + \frac{(z-r_{i'}^2+r_{i'})\cdot\prod\limits_{j\neq i'}(z-r_j^2)}{\prod\limits_{j\neq i'}(r_{i'}^2-r_j^2)}$$

can—up to the factor $r_i + r_{i'}$)—be interpreted (reading $r_{i'}$ as $r_i + \Delta r_i$) as the difference quotient of $g_{\overline{r}}(z)$ where

$$g_{\overline{r}}(z) = \frac{(z-r_i^2+r_i)\prod\limits_{i'\neq j\neq i}(z-r_j^2)}{\prod\limits_{i'\neq j\neq i}(r_i^2-r_j^2)}$$

in the variables z and $\overline{r} = \{r_1,\dots,r_{i'-1},r_{i'+1},\dots,r_k\}$. Since $g_{\overline{r}}(z)$ is differentiable, the claim follows.

We observe next, that $f_r(z)$ evaluates to r_i at r_i^2. Indeed, setting $z = r_i^2$ wipes out all terms except term i, which returns r_i.

Choose a unitary symmetric $k \times k$ matrix U_0 and denote its eigenvalues $\lambda_1, \ldots, \lambda_k$. Choose a ray R emanating from the origin in $\mathbb{C}$ and not containing any λ_i, and a section $\sqrt{-}$ of the square function on $\mathbb{C} \smallsetminus R$. Note that $\sqrt{a} + \sqrt{b} = 0$ is then impossible on $\mathbb{C} \smallsetminus R$. For $\mu \in (\mathbb{C} \smallsetminus R)^k$, let $f_{\sqrt{\mu}}(z)$ denote the function $f_r(z)$ from above, with parameters $\sqrt{\mu_1}, \ldots, \sqrt{\mu_k}$. Then the rational function $f_{\sqrt{\mu}}(z)$ has no poles on $(\mathbb{C} \smallsetminus R)^k \times \mathbb{C}$; this follows from the discussion on poles of $f_r(z)$ above, in light of the fact that roots cannot sum to zero on $\mathbb{C} \smallsetminus R$. In particular, for any fixed choice μ of the parameters, $f_{\sqrt{\mu}(z)}$ is a well-defined polynomial that varies analytically with μ.

We now consider for unitary symmetric U with eigenvalues $\mu \in (\mathbb{C} \smallsetminus R)^k$ the matrix $f_{\sqrt{\mu}}(U)$. As $f_{\sqrt{\mu}}(\mu_i) = \sqrt{\mu_i}$, $(f_{\sqrt{\mu}}(z))^2 - z$ is zero at each μ_i. If U is unitary with eigenvalues μ all in $\mathbb{C} \smallsetminus R$, then the minimal polynomial of U divides $(f_{\sqrt{\mu}}(z))^2 - z$ and thus $(f_{\sqrt{\mu}}(U))^2 = U$. In particular, eigenvalues of $f_{\sqrt{\mu}}(U)$ are, like those of U, on the unit circle.

Thus, for symmetric unitary U, $f_{\sqrt{\mu}}(U)$ is symmetric (as $f_{\sqrt{\mu}}$ is a polynomial and U symmetric), normal (as U is normal, and $f_{\sqrt{\mu}}$ is a polynomial), unitary (since it is normal and has its eigenvalues are on the unit circle). It follows that $f_{\sqrt{\mu}}(U)$ is an analytic section of the square function on unitary symmetric matrices with eigenvalues different from the intersection of R with the unit circle. □

Refer to (8.4.2)–(8.4.4) for notation in the following lemma.

Lemma A.14. *Let n, t, k be integers with $n \geqslant t > k \geqslant 0$. The following are Zariski locally trivial fiber bundles:*

(A.14.1) $$G^k_{t\times n} \longrightarrow X^k_{t\times n} \longrightarrow \mathrm{Gr}(n-k, n)$$

sending $Y \in X^k_{t\times n}$ to the kernel of $Y^{\mathrm{tr}}Y$;

(A.14.2) $$F^k_{t\times n} \longrightarrow G^k_{t\times n} \longrightarrow \mathrm{Sym}(k)$$

sending $Y \in G^k_{t\times n}$ with $Y^{\mathrm{tr}}Y = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ to $A \in \mathrm{Sym}(k)$; and

(A.14.3) $$X^0_{(t-k)\times(n-k)} \longrightarrow F^k_{t\times n} \longrightarrow \mathrm{O}(t, k),$$

sending $Y \in F^k_{t\times n}$ to the submatrix consisting of the leftmost k columns.

Proof. (A.14.1): This is entirely similar to the proof of (A.6.1), if one takes $d = k$ and replaces Ω_{2t} by the $t \times t$ identity matrix.

(A.14.2): By Lemma A.12, the projection is surjective, and there is an open cover of $\mathrm{Sym}(k)$ by sets U for which there is a map $\psi : U \longrightarrow \mathrm{GL}_k$ such that $\psi(A)^{\mathrm{tr}}\psi(A) = A$; the open sets U are Euclidean open in Setting (AN), and étale open in Setting (ET). We then have isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times F^k_{t\times n} \\ Y\begin{bmatrix} \psi(A) & 0 \\ 0 & \mathbf{1}_{n-k} \end{bmatrix} & \longleftarrow & (A, Y) \\ Y & \longmapsto & \left(\pi(Y),\ Y\begin{bmatrix} \psi(\pi(Y))^{-1} & 0 \\ 0 & \mathbf{1}_{n-k} \end{bmatrix}\right). \end{array}$$

(A.14.3): By Lemma A.10, there is an open cover of $\mathrm{O}(t, k)$ by sets U for which there is a section $\alpha : U \longrightarrow U \times_{\mathrm{O}(t,k)} \mathrm{O}(t, t)$ of the projection; the sets U are Euclidean open in

Setting (AN), and étale open in Setting (ET). Identifying

$$X' = \left\{ \begin{bmatrix} \mathbf{1}_k & 0 \\ 0 & Y_{(t-k)\times(n-k)} \end{bmatrix} \;\middle|\; Y \in X^0_{(t-k)\times(n-k)} \right\} \cong X^0_{(t-k)\times(n-k)},$$

we have isomorphisms

$$\begin{array}{rcl} \pi^{-1}(U) & \cong & U \times X' \\ \alpha(A)Y & \longleftarrow & (A,Y) \\ Y & \longmapsto & (\pi(Y), \alpha(\pi(Y))^{-1}Y)\,. \end{array}$$

This concludes the proof. □

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