DEFINITION: Let R be a commutative ring. Let V be a free R-module with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R-module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Given an R-module homomorphism $T\colon V \to W$, for each $j = 1, \dots n$, write

(♣) $$T(b_j) = r_{1,j}c_1 + \cdots + r_{n,j}c_n$$

for some elements $r_{i,j} \in R$. The matrix

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{bmatrix}$$

is the **matrix representing T in the bases $\mathcal{B}$ and $\mathcal{C}$.**

LEMMA: Let R be a commutative ring. Consider the standard bases $\mathcal{E}$ on R^n and $\mathcal{E}'$ on R^m. For any linear transformation $T : R^n \to R^m$, we have

$$T(v) = [T]_{\mathcal{E}}^{\mathcal{E}'} \cdot v, \quad \text{where the RHS is usual matrix-times-vector multiplication.}$$

(1) Warming up with the definition:

(a) If R is a field F, translate everything[1] in the definition into linear algebra terms.

(b) Use the equation (♣) to explain as concretely as possible what the j-th column of $[T]_{\mathcal{B}}^{\mathcal{C}}$ means in terms of T, $\mathcal{B}$, and $\mathcal{C}$.

(c) Explain why the entries $r_{i,j}$ are well-defined.

(d) *Just using your answer for part **(b)** and not looking at the formula*, describe the dimensions of the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ in terms of the rank of V and the rank of W.

(e) Let V be the $\mathbb{R}$-vector space of polynomials in $\mathbb{R}[x]$ of degree at most 3 along with the zero polynomial. The derivative map $\frac{d}{dx}$ is a linear transformation from V to V. Choose a basis $\mathcal{B}$ for V and compute the matrix $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{B}}$.

(f) Find another basis $\mathcal{C}$ for V such that $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(2) Prove the Lemma above.

PROPOSITION: Let R be a commutative ring. Let V be a free R-module with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R-module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Then the map

$$\begin{array}{ccc} \mathrm{Hom}_R(V, W) & \longrightarrow & \mathrm{Mat}_{m \times n}(R) \\ T & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{C}} \end{array}$$

is bijective. Moreover, this is an isomorphism of R-modules.
When $V = W$ and $\mathcal{B} = \mathcal{C}$, the same map

$$\begin{array}{ccc} \mathrm{End}_R(V) & \longrightarrow & \mathrm{Mat}_{n \times n}(R) \\ T & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{B}} \end{array}$$

is an isomorphism of rings.

(3) Prove that the map $T \mapsto [T]_{\mathcal{B}}^{\mathcal{C}}$ in the Proposition is bijective.

(4) Suppose that V is a free module with a countably infinite basis $\mathcal{B} = \{b_1, b_2, b_3, \dots\}$, and W is free with a countably infinite basis $\mathcal{C} = \{c_1, c_2, c_3, \dots\}$. What is the correct analogue of the Proposition in this case?

(5) Prove the Proposition.

[1] You can do this aloud instead of rewriting everything.