BOUNDEDNESS THEOREM AND EXTREME VALUE THEOREM

Theorem (Boundedness Theorem): Suppose f is continuous on the closed interval $[a, b]$ for some real numbers a, b with $a < b$. Then f is bounded on $[a, b]$ — that is, there are real numbers m and M so that $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Theorem (Extreme Value Theorem): Assume f is continuous on the closed interval $[a, b]$ for some real numbers a and b with $a < b$. Then f attains a minimum and a maximum value on $[a, b]$ — that is, there exists a number $r \in [a, b]$ such that $f(x) \leq f(r)$ for all $x \in [a, b]$ and there exists a number $s \in [a, b]$ such that $f(x) \geq f(s)$ for all $x \in [a, b]$.

(1) Explain why the Extreme Value Theorem actually implies the Boundedness Theorem. (The reason we state both is that we have to prove the Boundedness Theorem on the way to the Extreme Value Theorem.)

(2) In this problem we explore the necessity of the hypotheses in these theorems.
  - (a) Draw a graph of a function on a closed interval $[a, b]$ that is *not continuous*, but is not bounded on $[a, b]$.
  - (b) Draw a graph of a function that is continuous on an *open* interval (a, b), but is not bounded on (a, b).

Lemma from homework: Let $a < b$ be real numbers and $[a, b]$ be a closed interval. Let $\{x_n\}_{n=1}$ be a sequence with $x_n \in [a, b]$ for all n, and assume that $\{x_n\}_{n=1}$ converges to r. Then,

- $r \in [a, b]$, and
- If f is continuous on the closed interval $[a, b]$, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(r)$.

(3) Proof of Boundedness Theorem:
  - (a) We argue by contradiction. What does it mean to suppose that the theorem is false? Assume it.
  - (b) Explain why there must be a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $f(x_n) > n$ for all $n \in \mathbb{N}$.
  - (c) Apply Bolzano-Weierstrass to the sequence $\{x_n\}_{n=1}^{\infty}$. What do you get?
  - (d) Now apply the Lemma from the homework. What do you get?

(4) Proof of Extreme Value Theorem:
  - (a) We will find a maximum value; finding a minimum value is similar (or follows from this part applied to $-f$).
  - (b) Let $R = \{f(x) \mid x \in [a, b]\}$. Explain why R has a supremum; call it ℓ.
  - (c) Explain why there must be a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $\ell - \frac{1}{n} < f(x_n) \leq \ell$ for all $n \in \mathbb{N}$.
  - (d) Apply Bolzano-Weierstrass to the sequence $\{x_n\}_{n=1}^{\infty}$. What do you get?
  - (e) Now apply the Lemma from the homework. What do you get?