

Math 445 — Problem Set #6
Due: Friday, October ?? by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like “I collaborated with Steven Smale on problems 1 and 3”. If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

- (1) Use the methods from class to give a formula¹ for all solutions of the Pell’s equation

$$x^2 - 13y^2 = 1.$$

- (2) Closed formulas for solutions to Pell’s equations.

- (a) Explain why the k th positive solution (x_k, y_k) of the Pell’s equation $x^2 - 2y^2 = 1$ satisfies the equation

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- (b) Diagonalize the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and use this to give a closed expression for (x_k, y_k) in terms of k . Your formulas should be in terms of particular linear combinations of powers of two numbers.
- (c) Use² your formulas from the previous part to show that

$$x_k = \left\lfloor \frac{(3 + \sqrt{2})^k}{2} \right\rfloor \quad \text{and} \quad y_k = \left\lfloor \frac{(3 + \sqrt{2})^k}{2\sqrt{2}} \right\rfloor.$$

Use this to quickly write down the first seven positive solutions to the Pell’s equation $x^2 - 2y^2 = 1$.

- (d) Repeat the steps above with the appropriate numbers for the Pell’s equation $x^2 - 5y^2 = 1$.
- (3) Not solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.
- (a) Show that if $D \equiv 0 \pmod{4}$ or $D \equiv 3 \pmod{4}$, then the equation $x^2 + Dy^2 = -1$ has no integer solutions.
- (b) Show that if $q \equiv 3 \pmod{4}$ is prime and $q \mid D$, then the equation $x^2 + Dy^2 = -1$ has no integer solutions.
- (4) Solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.
- (a) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, then $\frac{c}{d}$ is a convergent in the continued fraction expansion of \sqrt{D} .
- (b) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, (a, b) is a positive integer solution to $x^2 - Dy^2 = 1$, and

$$e + f\sqrt{D} = (a + b\sqrt{D})(c + d\sqrt{D}),$$

then (e, f) is another positive integer solution to $x^2 - Dy^2 = -1$.

- (c) Describe infinitely many solutions to the equation $x^2 - 13y^2 = -1$.

¹As in class, in terms of coefficients powers of some $a + b\sqrt{D}$.

²Recall that $\lfloor x \rfloor$ denotes the greatest integer n such that $n \leq x$ and $\lceil x \rceil$ denotes the smallest integer n such that $n \geq x$.

The remaining problem is only required for Math 845 students, though all are encouraged to think about it.

- (5) Let D be a positive integer that is not a perfect square. Suppose that $x^2 - Dy^2 = -1$ has a solution, and let (c, d) be the smallest positive integer solution. Let (a, b) be the smallest integer solution to the Pell's equation $x^2 - Dy^2 = 1$. Show that $(c + d\sqrt{D})^2 = a + b\sqrt{D}$, and use this to describe all solutions to $x^2 - Dy^2 = -1$ in terms of c and d .