DECIMAL EXPANSIONS

In this worksheet, we are going to define decimal expansions and prove the basic properties about them. To simplify things, we are going to only deal with numbers between 0 and 1 (since we get all the the rest by adding integers and taking negatives). Along the way we will use induction and convergence of sequences in an important way. Before we define infinite decimal expansions, let's review finite decimal expansions.

(1) If $d \in \{0, 1, \dots, 9\}$ (i.e., d is an integer between 0 and 9), what does the decimal number $0.d$ mean? Express it as a rational number.

(2) If $d_1, d_2, \dots, d_n \in \{0, 1, \dots, 9\}$ (i.e., $d_1, \dots, d_n$ are a bunch of integers between 0 and 9, which may or may not have repeats), convince yourself that the decimal number $0.d_1 d_2 \cdots d_n$ in the way that we commonly use it is shorthand for

$$0.d_1 d_2 \cdots d_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.$$

Let's say that a sequence of the form $\{d_n\}_{n=1}^{\infty}$ is a *digit sequence* if $d_n \in \{0, 1, \dots, 9\}$ for all n. (That is a digit sequence is just a sequence of integers between 0 and 9.) Given a digit sequence $\{d_n\}_{n=1}^{\infty}$, define another sequence $\{D_n\}_{n=1}^{\infty}$ by the rule

$$\begin{aligned} D_1 &= \frac{d_1}{10^1} \\ D_2 &= \frac{d_1}{10^1} + \frac{d_2}{10^2} \\ &\vdots \\ D_n &= \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \\ &\vdots \end{aligned}$$

For example, for the digit sequence $2, 2, 2, \dots$, the corresponding $\{D_n\}_{n=1}^{\infty}$ sequence is

$$\frac{2}{10}, \ \frac{2}{10} + \frac{2}{100}, \ \frac{2}{10} + \frac{2}{100} + \frac{2}{1000}, \ \dots$$

We say that a digit sequence $\{d_n\}_{n=1}^{\infty}$ *represents a real number* r if the sequence $\{D_n\}_{n=1}^{\infty}$ converges to r, and in this case we write

$$0.d_1 d_2 d_3 d_4 \cdots = r.$$

In order to prepare for proving things about decimal expansions, we need a fact about geometric series.

(1) Let x and a be real numbers.

(a) Prove that for every $n \in \mathbb{N}$,

$$(1 - x)(1 + x + x^2 + x^3 + \cdots + x^n) = 1 - x^{n+1}.$$

(b) If $x \neq 1$, use (a) to show that for every $n \in \mathbb{N}$,

$$a + ax + ax^2 + \cdots + ax^n = a\frac{1 - x^{n+1}}{1 - x}.$$

(2) Use the definition (and perhaps the previous problem), but not our previous expectations about decimal expansions, to answer the following.
   (a) What number does the digit sequence $2, 3, 0, 0, 0, 0, 0, \ldots$ represent?
   (b) What number does the digit sequence $5, 0, 0, 0, 0, 0, 0, \ldots$ represent?
   (c) What number does the digit sequence $9, 9, 9, 9, 9, 9, 9, \ldots$ represent?
   (d) What number does the digit sequence $4, 9, 9, 9, 9, 9, 9, \ldots$ represent?

(3) Let $\{d_n\}_{n=1}^{\infty}$ be any digit sequence. Prove[1] that this sequence represents some real number: i.e., that the corresponding sequence $\{D_n\}_{n=1}^{\infty}$ is convergent.
   [Thus, every decimal expansion $0.d_1d_2d_3\cdots$ always gives us a real number.]

(4) In this problem, we will show that every real number $r \in [0, 1]$ is represented by some digit sequence.
   (a) Show that we can recursively define a digit sequence $\{d_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, in the corresponding sequence $\{D_n\}_{n=1}^{\infty}$, we have $0 \leq 10^n(r - D_n) \leq 1$.
   (b) Given a sequence as in part (a), show that $\{D_n\}_{n=1}^{\infty}$ converges to r.
   [Thus, every number can be written as a decimal expansion $0.d_1d_2d_3\cdots$.]

(5) Now we analyze uniqueness of decimal expansions. We will find it useful to use the following corollary of the proof of the Monotone Convergence Theorem: If $\{a_n\}_{n=1}^{\infty}$ is a bounded increasing sequence, $\{a_n\}_{n=1}^{\infty}$ converges to $\sup(\{a_n \mid n \in \mathbb{N}\})$.
   (a) Let $\{d_n\}_{n=1}^{\infty}$ be any digit sequence and $\{D_n\}_{n=1}^{\infty}$ be the corresponding sequence. Suppose that

[1] Hint: Use the monotone convergence theorem. You might want to use that $d_i \leq 9$ for all i and the previous problem!