THEOREM 32.1 (BOUNDEDNESS THEOREM): Suppose f is continuous on the closed interval $[a, b]$ for some real numbers a, b with $a < b$. Then f is bounded on $[a, b]$ — that is, there are real numbers m and M so that $m \leq f(x) \leq M$ for all $x \in [a, b]$.

THEOREM 32.2 (EXTREME VALUE THEOREM): Assume f is continuous on the closed interval $[a, b]$ for some real numbers a and b with $a < b$. Then f attains a **minimum value** and a **maximum value** on $[a, b]$ — that is, there exists a number $r \in [a, b]$ such that $f(x) \leq f(r)$ for all $x \in [a, b]$ and there exists a number $s \in [a, b]$ such that $f(x) \geq f(s)$ for all $x \in [a, b]$.

(1) Let $f(x) = \sqrt{x^2 + 3}$.
  (a) Is $f(x)$ continuous on the closed interval $[-1, 2]$?
  (b) According to the theorems, does f attain a minimum value and/or a maximum value on $[-1, 2]$?
  (c) Find the minimum value and maximum value of f on $[-1, 2]$.

(2) In this problem we explore the necessity of the hypotheses in these theorems.
  (a) Draw a graph of a function on a closed interval $[a, b]$ that is *not continuous*, and is *not bounded* on $[a, b]$.
  (b) Draw a graph of a function that is continuous on an *open* interval (a, b), but is *not bounded* on (a, b).
  (c) Draw a graph of a function that is continuous on an *open* interval (a, b), and *is bounded* on (a, b), but for which the conclusion of the Extreme Value Theorem *fails*.

  Can you find formulas of functions that match each story?

(3) Prove or disprove: If f is continuous on a closed interval $[a, b]$, then f attains its minimum at a unique input value: that is, there is a unique $r \in [a, b]$ such that $f(r) \leq f(x)$ for all $x \in [a, b]$.

(4) Prove or disprove: If f is continuous on an open interval (a, b), then f either does not attain a minimum or does not attain a maximum on (a, b).

(5) True or false: A constant function attains a minimum value and a maximum value on any closed interval $[a, b]$.

(6) Explain why the Extreme Value Theorem actually implies the Boundedness Theorem. (The reason we state both is that we will have to first prove the Boundedness Theorem on the way to the Extreme Value Theorem.)

DEFINITION: Let I be an interval (either open or closed) and f a function defined on all of I.

- We say that f is **increasing** on I if for any $x, y \in I$ with $x < y$, we have $f(x) \leq f(y)$.
- We say that f is **decreasing** on I if for any $x, y \in I$ with $x < y$, we have $f(x) \geq f(y)$.
- We say that f is **monotone** on I if f is either increasing on I or decreasing on I.
- We say that f is **strictly increasing** on I if for any $x, y \in I$ with $x < y$, we have $f(x) < f(y)$.
- We say that f is **strictly decreasing** on I if for any $x, y \in I$ with $x < y$, we have $f(x) > f(y)$.
- We say that f is **strictly monotone** on I if f is either increasing on I or decreasing on I.

(7) Suppose that f is strictly monotone on (a, b). Show that f does not attain either a minimum or a maximum on (a, b).

(8) Let f be continuous on the closed interval $[-7, 3]$. Show that if f is one-to-one[1], then f must be strictly monotone.

(9) Prove or disprove: Let f be continuous on $\mathbb{R}$. Show that if f does *not* attain a minimum on the half-open interval $(0, 1]$ then there is some $a \in (0, 1)$ such that f is strictly increasing on $(0, 1)$.

[1]This means that $x \neq y$ implies $f(x) \neq f(y)$ for all x, y in the domain of f.