PROBLEM SET #1

(1) * Basic rules with derivations:
   (a) Prove the generalized product rule for derivations: if $\partial : R \to M$ is a derivation, then $\partial(a_1 \cdots a_n) = \sum_{j=1}^{n} (\prod_{j \neq i} a_i)\partial(a_j)$.
   (b) Prove the power rule for derivations: if $\partial : R \to M$ is a derivation, then $\partial(r^n) = nr^{n-1}\partial(r)$.
   (c) Show that if R is a ring of characteristic p, then the subring $R^p := \{r^p \mid r \in R\}$ is in the kernel of every derivation.

(2) * Let A be a ring and $S = A[x_1, \dots, x_n]$ be a polynomial ring.
   (a) Let R be an $\mathbb{N}$-graded A-algebra such that A lives in degree zero. Show that there is a derivation on R such that for every homogeneous element f of degree d, $\partial(f) = d \cdot f$. This derivation is called the *Euler operator* associated to the grading.

   Proof. The rule above describes a well-defined function on R. We need to check that it is A-linear and satisfies the product rule. Let $r = \sum_i r_i$ and $s = \sum_i s_i$ be elements of R expressed as (finite) sums of homogeneous pieces with degree $r_i = i$ and $a \in A$. Then

   - $\partial(r+s) = \partial(\sum_i r_i + \sum_i s_i) = \partial(\sum_i (r_i + s_i)) = \sum_i i(r_i + s_i) = \sum_i ir_i + \sum_i is_i = \partial(r) + \partial(s)$.
   - $\partial(ar) = \partial(a \sum_i r_i) = \partial(\sum_i ar_i) = \sum_i iar_i = a \sum_i ir_i = a\partial(r)$.
   - $\partial(rs) = \partial(\sum_k \sum_{i+j=k} r_i s_j) = \sum_k k(\sum_{i+j=k} r_i s_j) = \sum_{i,j} ir_i s_j + r_i js_j = s\partial(r) + r\partial(s)$.

   $\square$

   (b) Let S be, as above,[1] a polynomial ring over A endowed with the $\mathbb{N}$-grading by the rule $\deg(x_i) = n_i$. Express the Euler operator of the grading as an S-linear combination of the partial derivatives.

   Proof. Take $\partial = \sum_i n_i x_i \frac{d}{dx_i}$. To check that this agrees with the Euler operator, by A-linearity it suffices to check on any monomial $x_1^{a_1} \cdots x_n^{a_n}$: we get

$$\partial(x_1^{a_1} \cdots x_n^{a_n}) = \sum_i n_i a_i x_1^{a_1} \cdots x_n^{a_n}$$

   and $\sum_i n_i a_i$ is just the degree of $x_1^{a_1} \cdots x_n^{a_n}$. $\square$

(3) Let A be a ring and $R = A[x_1, \dots, x_n]$ be a polynomial ring.
   (a) Give an explicit formula for the Lie algebra bracket on $\mathrm{Der}_{R|A}(R)$.
   (b) Does $\mathrm{Der}_{R|A}(R)$ have any nontrivial proper Lie ideals (i.e., A-submodules B such that $[d, b] \in B$ for all $b \in B$ and $d \in \mathrm{Der}_{R|A}(R)$)?

   Proof. It is possible in general. For a fun example, over $A = \mathbb{F}_2$, we can take $\mathbb{F}_2[x^2]\frac{d}{dx}$ as a Lie ideal of $\mathrm{Der}_{\mathbb{F}_2[x]|\mathbb{F}_2}(\mathbb{F}_2[x])$. Indeed, note that for any $f \in \mathbb{F}_2[x]$, $\frac{d}{dx}(f) \in \mathbb{F}_2[x^2]$, since any even

[1] For infinitely many variables, we will get the same formula with a formal sum, but this is not an S-linear combination of partial derivatives. Oops!

power of x picks up a coefficient of two in the derivative. Then given $f \in \mathbb{F}_2[x^2]$ and $g \in \mathbb{F}_2[x^2]$ we have

$$[f\frac{d}{dx}, g\frac{d}{dx}] = (f\frac{d}{dx}(g) - g\frac{d}{dx}(f))\frac{d}{dx} = g\frac{d}{dx}(f)\frac{d}{dx} \in \mathbb{F}_2[x^2]\frac{d}{dx}.$$

However, over a field of characteristic zero, this is false. □

(4) Let R be a ring of characteristic $p > 0$ and $\partial : R \to R$ be a derivation. Show that ∂^p, i.e., the p-fold self composition of ∂, is a derivation on R.

(5) Let $R = \mathcal{C}^\infty(\mathbb{R}^n)$ be the ring of smooth functions on $\mathbb{R}^n$, and $\mathfrak{m}$ be the maximal ideal consisting of functions that vanish at some point $x_0 \in \mathbb{R}^n$.

(a) * Show that $\mathfrak{m}^t$ consists of the functions $f \in R$ such that $\frac{d^{a_1}}{dx_1^{a_1}} \cdots \frac{d^{a_n}}{dx_n^{a_n}}(f)|_{x=x_0} = 0$ for all $a_1, \ldots, a_n$ with $0 \leqslant a_1 + \cdots + a_n < t$.

Proof. Let $J_n = \{f \in R \mid \frac{d^{a_1}}{dx_1^{a_1}} \cdots \frac{d^{a_n}}{dx_n^{a_n}}(f)|_{x=x_0} = 0 \ \forall a_1, \ldots, a_n : 0 \leqslant a_1 + \cdots + a_n < t\}$. We'll write d^a for an n-tuple a as shorthand for the iterated derivative above.

First we show that $\mathfrak{m}^t \subseteq J_n$. We proceed by induction on t with $t = 1$ immediate from the definitions. Supposing the inclusion for a given t, take $f \in \mathfrak{m}^{t+1}$ and write $f = \sum g_i h_i$ with $g_i \in \mathfrak{m}^t$ and $h_i \in \mathfrak{m}$. Then each $g_i \in J_t$ by the induction hypothesis. Since $f \in \mathfrak{m}^t \in J_t$, we have $d^a(f)|_{x_0} = 0$ for all $|a| < t$. Given some a with $|a| = t+1$, we can write $d^a = d^b \frac{d}{d_{x_j}}$ for some j and some b with $|b| = t$. Then

$$d^a(f) = \sum_i d^a(g_i h_i) = \sum_i d^b \frac{d}{d_{x_j}}(g_i h_i) = \sum_i d^b\big(h_i \frac{d}{d_{x_j}}(g_i)\big) + \sum_i d^b\big(g_i \frac{d}{d_{x_j}}(h_i)\big).$$

We have $g_i \frac{d}{d_{x_j}}(h_i) \in \mathfrak{m}^t \subseteq J_t$ so the second sum evaluates to zero at x_0. Since $\frac{d}{d_{x_j}}(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t-1}$, we have $h_i \frac{d}{d_{x_j}}(g_i) \in \mathfrak{m}^t$, so the first sum evaluates to 0 at x_0 as well. Thus, $f \in J_{t+1}$, as required.

For the other containment, we will apply Taylor's Theorem for multivariate functions[2]. Recall that this this says that f agrees with a polynomial (in $x_i - (x_0)_i$) whose coefficients are determined by the iterated partial derivatives of f at x_0, plus some error term. Beware that in general a smooth function is not equal to its Taylor series, so we will need to consider the polynomial plus remainder version. Applying this, if $f \in J_t$, we can write

$$f = \sum_{|a|=t} \frac{t}{a_1! \cdots a_n!} \widetilde{x_1}^{a_1} \cdots \widetilde{x_n}^{a_n} \int_0^1 (1-s)^t d^a(f)|_{x_0 + s(x - x_0)} \mathrm{ds},$$

where $\widetilde{x_i} := x_i - (x_0)_i$. What is important to observe about this expression is that each

$$j_a(x) := \frac{t}{a_1! \cdots a_n!} \int_0^1 (1-s)^t d^a(f)|_{x_0 + s(x-x_0)} \mathrm{ds}$$

is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$: we omit the details, but the point is essentially that smoothness lets us differentiate under the integral sign. Thus, we have

$$f = \sum_{|a|=t} j_a \widetilde{x_1}^{a_1} \cdots \widetilde{x_n}^{a_n}$$

with $j_a \in R$ and $\widetilde{x_i} \in \mathfrak{m}$ for each i, so $f \in \mathfrak{m}^t$. □

(b) Show that $\mathrm{Der}_{R|\mathbb{R}}(R/\mathfrak{m}) \cong (\mathfrak{m}/\mathfrak{m}^2)^* \cong \mathbb{R}^n$ as vector spaces.

[2]cf., Folland's *Advanced Calculus*, Theorem 2.68

As a moral, we conclude that $\mathrm{Der}_{R|\mathbb{R}}(R/\mathfrak{m})$ serves as a model for the tangent space of $\mathbb{R}^n$ at x_0 constructed from the ring of smooth functions.

(6) * Let R be an A-algebra and I an ideal. Show that if the identity map on I/I^2 is in the image of $\mathrm{Der}_{R|A}(I/I^2) \xrightarrow{\text{res}} \mathrm{Hom}_R(I/I^2, I/I^2)$, then there is an A-algebra right inverse to the quotient map $\pi : R/I^2 \to R/I$. Conclude that the following are equivalent:
- $\mathrm{Der}_{R|A}(M) \xrightarrow{\text{res}} \mathrm{Hom}_R(I/I^2, M)$ is surjective for all R/I-modules M;
- $\mathrm{Der}_{R|A}(I/I^2) \xrightarrow{\text{res}} \mathrm{Hom}_R(I/I^2, I/I^2)$ is surjective;
- The quotient map $R/I^2 \to R/I$ has an A-algebra right inverse.

Proof. Suppose that $\partial : R \to I/I^2$ is a derivation whose restriction to I/I^2 (after factoring through R/I^2 as usual) is the identity map. Viewing ∂ as a derivation on R/I^2 by abuse of notation, note that $K := \ker(\partial)$ is a subring of R/I^2 containing A. Let $i : K \to R/I^2$ be the inclusion map. We claim that $K \cong R/I$ as A-algebras.

Since $-\partial$ is a derivation, the map $1 - \partial : R/I^2 \to R/I^2$ is a ring homomorphism, and $(1-\partial) \circ i$ is the identity on K (because K is the kernel of ∂). In particular, $1 - \partial$ is surjective. We just need to see that the kernel of $1 - \partial$ is I/I^2. We have $I//I^2$ is contained in the kernel, since for $a \in I/I^2$, $(1-\partial)(a) = a - \partial(a) = 0$; on the other hand if $r \in \ker(1-\partial)$, then $r \in \mathrm{im}(\partial)$, so $r \in I/I^2$. This completes the proof.

For the equivalences, the first implies the second since I/I^2 is an R/I-module, the second implies the third by what we just showed, and the third implies the first by a theorem from class. □

(7) Let R be a ring and M an R-module. Recall that $R \rtimes M$ denotes the Nagata idealization of M: the ring with additive structure $R \oplus M$ and multiplication $(r, m)(s, n) = (rs, rn + sm)$. Show that $\alpha : R \to M$ is a derivation if and only if $(1, \alpha) : R \to R \rtimes M$ $(r \mapsto (r, \alpha(r)))$ is a ring homomorphism.