DEFINITION: Let F be a field, V an F-vector space of dimension n, and $\phi : V \to V$ a linear transformation.

- The **characteristic polynomial** of ϕ is the polynomial $c_\phi(x) := \det(xI_n - [\phi]_B^B)$ for a basis B of V.
- The **minimal polynomial** of ϕ is the monic generator $m_\phi(x)$ of the ideal $\mathrm{ann}_{F[x]}(V_\phi)$. Equivalently, $m_\phi(x)$ is the monic polynomial of smallest degree such that $m_\phi(\phi) = 0$.

We write $c_A(x) := c_{t_A}(x)$ and $m_A(x) := m_{t_A}(x)$ for a matrix A.

PROPOSITION: Let F be a field, V an F-vector space of dimension n, and $\phi : V \to V$ a linear transformation. Let $g_1 \mid \cdots \mid g_k$ be the invariant factors of ϕ. Then,

(1) $m_\phi(x) = g_k$.
(2) $c_\phi(x) = g_1 \cdots g_k$.

COROLLARY (CAYLEY-HAMILTON): $m_\phi(x) \mid c_\phi(x)$.

THEOREM: Let F be a field, V an F-vector space of dimension n, and $\phi : V \to V$ a linear transformation. For $\lambda \in F$, λ is an eigenvalue of $\phi \iff m_\phi(\lambda) = 0 \iff c_\phi(\lambda) = 0$.

(1) Let $A = \begin{bmatrix} -11 & -4 & -2 \\ 18 & 7 & 3 \\ 18 & 6 & 4 \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{Q})$. The SNF of $xI - A \in \mathrm{Mat}_3(\mathbb{Q}[x])$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2+x-2 \end{bmatrix}$.

(a) What is the minimal polynomial of A?
(b) What is the characteristic polynomial of A?
(c) What is the RCF of A?
(d) What are the eigenvalues of A?

(2) Let F be a field, and $A \in \mathrm{Mat}_n(F)$ be a nilpotent matrix, meaning that $A^t = 0$ for some $t \geq 1$.

(a) What can you deduce about $m_A(x)$ from the fact that $A^t = 0$?
(b) Prove that $A^n = 0$.
(c) If $n = 4$, what are the possible lists of invariant factors?
(d) Give a complete nonredudant list of representatives of similarity classes of 4×4 nilpotent matrices.

(3) Let $f(x) = (x^2 - 1)(x^4 - 1) \in \mathbb{Q}[x]$.

(a) If $A \in \mathrm{Mat}_6(\mathbb{Q})$ has characteristic polynomial $c_A(x) = f(x)$, then what are the possible lists of invariant factors of A?
(b) Give a complete nonredudant list of representatives of similarity classes of rational matrices with characteristic polynomial f.

(4) Proofs:

(a) Show that the characteristic polynomial is well-defined (i.e., independent of choice of B).
(b) Prove[1] the Proposition and deduce Cayley-Hamilton as a Corollary.
(c) Prove the Theorem.

(5) Find the minimal and characteristic polynomials of rotation by $\pi/3$ counterclockwise in $\mathbb{R}^2$.

(6) Let F be a field.

(a) Let A and B be two 3×3 matrices with entries in F. Prove A and B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial.
(b) Show, by way of an example with justification, that the statement in part (a) would become false if 3×3 were replaced by 4×4.

(7) Give a complete nonredundant list of representatives for the conjugacy classes of $\mathrm{GL}_3(\mathbb{Z}/2)$.

[1]Hint: You did most of the work for (1) in a homework problem.