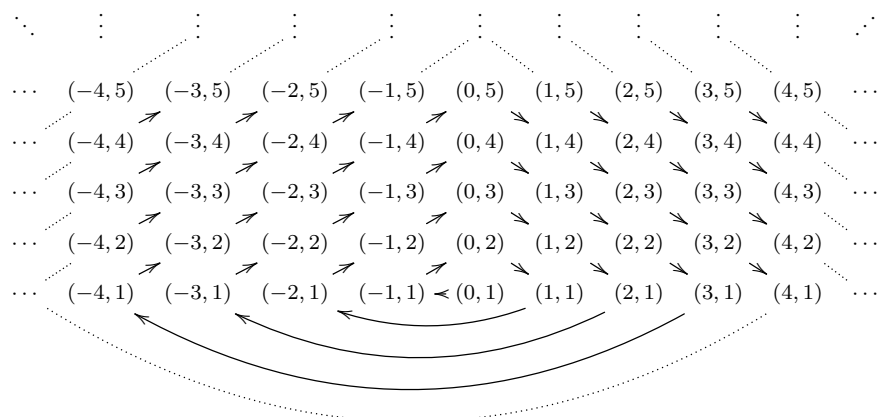


A CURIOUS SEQUENCE §2.3

Copy the following picture onto the board:



This gives the list of points

$$(0, 1), (-1, 1), (0, 2), (1, 1), (-2, 1), (-1, 2), (0, 3), (1, 2), (2, 1), (-3, 1), \dots$$

Now convert these to a list of rational numbers by changing (m, n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1}, \frac{-1}{1}, \frac{0}{2}, \frac{1}{1}, \frac{-2}{1}, \frac{-1}{2}, \frac{0}{3}, \frac{1}{2}, \frac{2}{1}, \frac{-3}{1}, \dots$$

of rational numbers. Call this sequence $\{w_n\}_{n=1}^{\infty}$.

- (1) True or false: Every rational number occurs in this sequence. That is, for every $q \in \mathbb{Q}$, there is some $n \in \mathbb{N}$ such that $w_n = q$.
- (2) True or false: Every rational number occurs in this sequence infinitely many times. That is, for every $q \in \mathbb{Q}$, there are *infinitely many* natural numbers $n \in \mathbb{N}$ such that $w_n = q$.
- (3) True or false: For every rational number $q \in \mathbb{Q}$, the constant sequence $\{q\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.
- (4) True or false: For every real number $r \in \mathbb{R}$, the constant sequence $\{r\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.
- (5) True or false: Every sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.
- (6) True or false: For every real number $r \in \mathbb{R}$, there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ that converges to r .

THEOREM 21.1: There exists a sequence of rational numbers such that

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A CURIOUS IMPOSSIBILITY §2.3

THEOREM 21.2 (CANTOR'S THEOREM): There is no sequence of real numbers such that every real number r occurs in the sequence.

By contradiction, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that every real number occurs as some a_n and write out the decimal expansions (where each $d_{i,j} \in \{0, 1, \dots, 9\}$ is a digit).

$$\begin{aligned}
 a_1 &= (\text{integer part}).d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}d_{1,6} \cdots \\
 a_2 &= (\text{integer part}).d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}d_{2,6} \cdots \\
 a_3 &= (\text{integer part}).d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}d_{3,6} \cdots \\
 a_4 &= (\text{integer part}).d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}d_{4,6} \cdots \\
 a_5 &= (\text{integer part}).d_{5,1}d_{5,2}d_{5,3}d_{5,4}d_{5,5}d_{5,6} \cdots \\
 a_6 &= (\text{integer part}).d_{6,1}d_{6,2}d_{6,3}d_{6,4}d_{6,5}d_{6,6} \cdots \\
 a_7 &= (\text{integer part}).d_{7,1}d_{7,2}d_{7,3}d_{7,4}d_{7,5}d_{7,6} \cdots \\
 &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots
 \end{aligned}$$