Problem Set 5
Due Thursday, February 19

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} x & 1 & 0 \\ 1 & x & -3 \\ 0 & 0 & x-1 \end{bmatrix} \in \mathrm{Mat}_{3\times3}(R),$$

where $R = \mathbb{Q}[x]$.

(a) Determine the Smith normal form for A.

(b) Determine the representative in the isomorphism class of the module presented by A which is written in invariant factor form and in elementary divisor form.

Problem 2. Let R be a domain. An R-module M is **torsionfree** if for $r \in R$ and $m \in M$, we have $rm = 0$ implies $r = 0$ or $m = 0$.

(a) Show that if R is a PID and M is a finitely generated torsionfree module, then M is free.

(b) Give an example of a torsionfree module M over a PID R such that M is not free.

(c) Give an example of a finitely generated torsionfree module M over a domain R such that M is not free.

Problem 3. Let F be a field and consider a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ in $F[x]$ with $n \geqslant 1$.

(a) Show that the principal ideal $(f(x))$ is a subspace of the F-vector space $F[x]$.

(b) Show that the set $B = \{\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}\}$, where $\overline{x^i} = x^i + (f(x))$, is a basis for the quotient F-vector space $F[x]/(f(x))$.

(c) Consider the linear transformation $l_x : F[x]/(f(x)) \to F[x]/(f(x))$ defined by $l_x(v) = \overline{x}v$ for any $v \in F[x]/(f(x))$. Find the matrix representing l_x in the basis B from part (b).

Problem 4. Let K be a field, and G be a finite subgroup of $K^\times$. Show[1][2] that G is cyclic.

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[1]Hint: Consider the invariant factors of G, and let e be the largest one. Show that every element of G is a root of the polynomial $x^e - 1 \in K[x]$.

[2]Note that as a special case of this, $(\mathbb{Z}/p)^\times$ is cyclic.