INTERMEDIATE VALUE THEOREM: Suppose f is a function, that $a < b$ are real numbers, and that f is continuous on the closed interval $[a, b]$. If y is any number between $f(a)$ and $f(b)$ (i.e., $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$), then there is a $c \in [a, b]$ such that $f(c) = y$

BOUNDEDNESS THEOREM: Suppose f is continuous on the closed interval $[a, b]$ for some real numbers a, b with $a < b$. Then f is bounded on $[a, b]$ — that is, there are real numbers m and M so that $m \leq f(x) \leq M$ for all $x \in [a, b]$.

EXTREME VALUE THEOREM: Assume f is continuous on the closed interval $[a, b]$ for some real numbers a and b with $a < b$. Then f attains a minimum value and a maximum value on $[a, b]$ — that is, there exists a number $r \in [a, b]$ such that $f(x) \leq f(r)$ for all $x \in [a, b]$ and there exists a number $s \in [a, b]$ such that $f(x) \geq f(s)$ for all $x \in [a, b]$.

LEMMA 33.1: Assume f is continuous on $[a, b]$ and that $\{x_n\}_{n=1}^{\infty}$ is any sequence such that $a \leq x_n \leq b$ for all n. If $\{x_n\}_{n=1}^{\infty}$ converges to some number r, then

(1) $r \in [a, b]$ and
(2) The sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(r)$.

(1) PROOF OF BOUNDEDNESS THEOREM:

We will argue that f is bounded above on $[a, b]$: i.e., there exists an M such that $f(x) \leq M$ for all $x \in [a, b]$; showing that f is bounded below is similar (or follows from this part applied to $-f$).

- (a) We argue by contradiction. What does it mean to suppose that the theorem is false? Assume it.
- (b) Explain why there must be a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $f(x_n) > n$ for all $n \in \mathbb{N}$.
- (c) Apply Bolzano-Weierstrass to the sequence $\{x_n\}_{n=1}^{\infty}$. What do you get?
- (d) Now apply the Lemma. What do you get?

(2) PROOF OF EXTREME VALUE THEOREM:

We will find a maximum value; finding a minimum value is similar (or follows from this part applied to $-f$).

- (a) Let $R = \{f(x) \mid x \in [a, b]\}$. Explain why R has a supremum; call it ℓ.
- (b) Explain why there must be a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $\ell - \frac{1}{n} < f(x_n) \leq \ell$ for all $n \in \mathbb{N}$.
- (c) Apply Bolzano-Weierstrass to the sequence $\{x_n\}_{n=1}^{\infty}$. What do you get?
- (d) Now apply the Lemma from the homework. What do you get?

(3) Prove Lemma 33.1.

(4) PROOF OF THE INTERMEDIATE VALUE THEOREM:

(a) Let's assume that $f(a) \leq f(b)$ to get started. Explain why the cases $y = f(a)$ and $y = f(b)$ are easy. Hence, we assume that $f(a) < y < f(b)$.

(b) Let $S = \{x \in [a, b] \mid f(r) < y \text{ for all } a \leq r \leq x\}$. In short, S is the set of x-values in the interval where the graph of f hasn't crossed y yet. Explain why S has a supremum, and let $c = \sup(S)$.

(c) Show that $c > a$. [Hint: Apply part (2) of definition of continuous on $[a, b]$ with $\varepsilon = y - f(a)$, and show that a is not an upper bound for S.]

(d) The argument that $c < b$ is similar (so come back to it later if you want). Thus, $c \in (a, b)$, so we know that f is continuous at c.

(e) Suppose that $f(c) < y$, and obtain a contradiction. [Hint: Apply continuous at c with $\varepsilon = y - f(c)$, and show that c is not an upper bound for S.]

(f) Suppose that $f(c) > y$, and obtain a contradiction. [Hint: Apply continuous at c with $\varepsilon = f(c) - y$, and find a smaller upper bound for S.]

(g) This concludes the case when $f(a) \leq f(b)$. If $f(a) \geq f(b)$, what can you say about $g(x) = -f(x)$? Can we apply the case we just did?