DEFINITION: Let G be a group. A nonempty subset H of G is a **subgroup** of G if H is a group under the the same operation as G (i.e., $h \cdot_H h' = h \cdot_G h'$ for $h, h' \in H$). We write $H \leq G$ to indicate that H is a subgroup of G.

Any group G has two **trivial subgroups** $\{e\}$ and G.

LEMMA 1: Let H be a subset of G.

- TWO STEP TEST: If H is nonempty, H is closed under multiplication[1] and H is closed under inverses[1], then H is a subgroup of G.
- ONE STEP TEST: If H is nonempty, and for all $x, y \in H$, $xy^{-1} \in H$, then H is a subgroup of G.

LEMMA 2 (GENERAL RECIPES FOR SUBGROUPS): Let G be a group.

(1) If $H \leq G$ and $K \leq H$, then $H \leq G$.
(2) If $\{H_\alpha\}_{\alpha \in J}$ is a collection of subgroups of G, then $\bigcap_{\alpha \in J} H_\alpha \leq G$.
(3) If $f : G \to H$ is a group homomorphism, then $\mathrm{im}(G) \leq H$.
(4) If $f : G \to H$ is a group homomorphism, and $K \leq G$, then $f(K) = \{f(k) \mid k \in K\} \leq H$.
(5) If $f : G \to H$ is a group homomorphism, and $K \leq G$, then $\ker(f) \leq G$.
(6) The center $Z(G)$ is a subgroup of G.

(1) Proving subsets are subgroups:
- **(a)** Choose a couple of parts of Lemma 2 and prove them; you can use Lemma 1.
- **(b)** Let $n \geq 3$ and consider the dihedral group D_n of symmetries of the n-gon.
  - (i) Is the set of all reflections in D_n a subgroup?
  - (ii) Is the set of all rotations in D_n a subgroup?
- **(c)** Let $n \in \mathbb{Z}_{\geq 1}$, and define $\mathrm{SL}_n(\mathbb{R})$ to be the set of $n \times n$ real matrices with determinant 1. Show[2] that $\mathrm{SL}_n(\mathbb{R}) \leq \mathrm{GL}_n(\mathbb{R})$. ($\mathrm{SL}_n(\mathbb{R})$ is called the **special linear group**.)
- **(d)** Let $n \in \mathbb{Z}_{\geq 1}$. Recall from linear algebra that an $n \times n$ matrix Q is *orthogonal* if $Q^T Q = I$, where ^{T} denotes transpose and I denotes the identity matrix. Define $\mathrm{O}_n(\mathbb{R})$ to be the set of $n \times n$ real orthogonal matrices. Show that $\mathrm{O}_n(\mathbb{R}) \leq \mathrm{GL}_n(\mathbb{R})$. ($\mathrm{O}_n(\mathbb{R})$ is called the **orthogonal group**.)
- **(e)** Define $\mathrm{SO}_n(\mathbb{R})$ to be the set of $n \times n$ real orthogonal matrices that have determinant 1. Show that $\mathrm{SO}_n(\mathbb{R}) \leq \mathrm{GL}_n(\mathbb{R})$. ($\mathrm{SO}_n(\mathbb{R})$ is called the **special orthogonal group**.)

(2) Prove or disprove: The union of two subgroups of a group is a subgroup.

(3) Prove Lemma 1.

[1] A subset $H \subseteq G$ is *closed under multiplication* if $x, y \in H \Rightarrow xy \in H$ and *closed under inverses* if $x \in H \Rightarrow x^{-1} \in H$.
[2] Hint: This becomes very quick with a proper use of Lemma 2.

DEFINITION: Let G be a group, and $S \subseteq G$ be a subset. The **subgroup of G generated by S** is the intersection of all subgroups of G that contain S:

$$\langle S \rangle := \bigcap_{\substack{H \leq G \\ S \subseteq H}} H$$

PROPOSITION: Let G be a group, and $S \subseteq G$ be a subset. Then

$$\langle S \rangle = \{x_1^{j_1} \cdots x_m^{j_m} \mid x_i \in S,\ j_i \in \mathbb{Z}\}.$$

(4) Explain why $\bigcap_{\substack{H \leq G \\ S \subseteq H}} H$ is a subgroup of G, and why it is the *unique smallest* subgroup of G that contains S.

(5) PROOF OF THE PROPOSITION: Let $K = \{x_1^{j_1} \cdots x_m^{j_m} \mid x_i \in S,\ j_i \in \mathbb{Z}\}$ as in the Proposition.
- (a) What concrete things do you need to show about K, S, and subgroups $H \leq G$ to prove this equality?
- (b) Complete the proof.

CAYLEY'S THEOREM: Let G be a finite group of order n. Then G is isomorphic to a subgroup of S_n.

(6) Prove[3] Cayley's Theorem.

[3] Hint: Let G act on G by left multiplication.