

DEFINITION: Let F be a field, V an F -vector space of dimension n , and $\phi : V \rightarrow V$ a linear transformation.

- The **characteristic polynomial** of ϕ is the polynomial $c_\phi(x) := \det(xI_n - [\phi]_B^B)$ for some/any basis B of V .
- The **minimal polynomial** of ϕ is the monic generator $m_\phi(x)$ of the ideal $\text{ann}_{F[x]}(V_\phi)$. Equivalently, $m_\phi(x)$ is the monic polynomial of smallest degree such that $m_\phi(\phi) = 0$.

We write $c_A(x) := c_{t_A}(x)$ and $m_A(x) := m_{t_A}(x)$ for a matrix A .

PROPOSITION: Let F be a field, V an F -vector space of dimension n , and $\phi : V \rightarrow V$ a linear transformation. Let $g_1 | \dots | g_k$ be the invariant factors of ϕ .

- (1) $m_\phi(x) = g_k$.
- (2) $c_\phi(x) = g_1 \cdots g_k$.
- (3) $\deg(g_1) + \dots + \deg(g_k) = n$.

COROLLARY (CAYLEY-HAMILTON): $m_\phi(x) | c_\phi(x)$.

THEOREM: Let F be a field, V an F -vector space of dimension n , and $\phi : V \rightarrow V$ a linear transformation. For $\lambda \in F$, λ is an eigenvalue of $\phi \iff m_\phi(\lambda) = 0 \iff c_\phi(\lambda) = 0$.

(1) Let $A = \begin{bmatrix} -11 & -4 & -2 \\ 18 & 7 & 3 \\ 18 & 6 & 4 \end{bmatrix} \in \text{Mat}_3(\mathbb{Q})$. The SNF of $xI - A \in \text{Mat}_3(\mathbb{Q}[x])$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2+x-2 \end{bmatrix}$.

Use the results above to answer the following:

- (a)** What is the minimal polynomial of A ?
- (b)** What is the characteristic polynomial of A ?
- (c)** What is the RCF of A ?
- (d)** What are the eigenvalues of A ?

- (2)** Let F be a field, and $A \in \text{Mat}_n(F)$ be a nilpotent matrix, meaning that $A^t = 0$ for some $t \geq 1$.
 - (a)** Prove that $A^n = 0$.
 - (b)** If $n = 4$, what are the possible lists of invariant factors?
 - (c)** For $n = 4$, give a complete and nonredundant list of representatives of similarity classes of nilpotent matrices.

- (3)** Let $f(x) = (x^2 - 1)(x^4 - 1) \in \mathbb{Q}[x]$.
 - (a)** If $A \in \text{Mat}_6(\mathbb{Q})$ has characteristic polynomial $c_A(x) = f(x)$, then what are the possible lists of invariant factors of A ?
 - (b)** Give a complete and nonredundant list of representatives of similarity classes of rational matrices with characteristic polynomial f .

- (4)** Let F be a field.
 - (a)** Let A and B be two 3×3 matrices with entries in F . Prove A and B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial.
 - (b)** Show, by way of an example with justification, that the statement in part (a) would become false if 3×3 were replaced by 4×4 .

- (5)** Prove¹ the Proposition and the Theorem.

- (6)** Give a complete and nonredundant list of representatives for the conjugacy classes of $\text{GL}_3(\mathbb{Z}/2)$.

¹Hint: You did most of the work for (1) in a homework problem.