ASSIGNMENT #3

(1) Let R be a commutative ring, and S be a multiplicatively closed subset. Let

$$F, G : R - \mathbf{Mod} \to S^{-1}R - \mathbf{Mod}$$

be the localization functor and the functor of extension of scalars $S^{-1}R \otimes_R -$, respectively. Show that F is naturally isomorphic to G.

(2) (a) Show that[1], for a commutative ring A, a commutative A-algebra R, and any ideal $I \subset A[x_1, \ldots, x_n]$, there is a ring isomorphism

$$R \otimes_A \frac{A[x_1, \ldots, x_n]}{I} \cong \frac{R[x_1, \ldots, x_n]}{IR[x_1, \ldots, x_n]}.$$

(b) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain.

(3) Let R be an integral domain. An element m of an R-module M is *torsion* if there is some $r \neq 0$ such that $rm = 0$. An R-module is *torsion* if every element is torsion.

(a) Show that there is a left exact functor $T : R - \mathbf{Mod} \to R - \mathbf{Mod}$ that on objects sends a module M to the submodule of M consisting of all its torsion elements.

(b) Let K be the fraction field of R. Show that for every R-module M, there is an isomorphism $T(M) \cong \ker(M \otimes_R R \xrightarrow{1_M \otimes i} M \otimes_R K)$, where i is the natural inclusion of R into K.

(4) (a) Prove that if A is a divisible abelian group and T is a torsion abelian group (i.e., a torsion $\mathbb{Z}$-module), then $A \otimes_{\mathbb{Z}} T = 0$.

(b) Prove[2] there does not exist a nonzero (unital) ring R such that the underlying abelian group $(R, +)$ is both torsion and divisible. (So, for example, there is no ring whose underlying abelian group is $\mathbb{Q}/\mathbb{Z}$.)

(5) Hom.

(a) Let $R = K[x]$ be a polynomial ring in one variable over a field K, and let $M = \mathrm{Hom}_K(R, K)$. Explicitly describe an element $m \in M$ such that $xm = m$ under the R-module action on M.

(b) Let $S = K[x, y]/(x^2, xy, y^2)$. This is a commutative ring that, as a K-vector space, has $\{1, x, y\}$ as a free basis. Explain how $N = \mathrm{Hom}_K(S, S)$ has two possible S-module structures, and show that these module structure are not isomorphic.

(c) Let $D = \mathbb{R}[\partial]$ be a polynomial ring in the indeterminate ∂. Explain why there is a D module action on the power series ring $\mathbb{R}[\![x]\!]$ given by $\partial \cdot f(x) = \dfrac{df(x)}{dx}$, and compute[3]

$$\mathrm{Hom}_D\left(\frac{D}{D(\partial - 1)}, \mathbb{R}[\![x]\!]\right).$$

[1] You can use that $R \otimes_A A[x_1, \ldots, x_n] \cong R[x_1, \ldots, x_n]$ via the map $r \otimes f(x) \mapsto rf(x)$.

[2] Hint: multiplication is biadditive.

[3] I.e., explicitly say what its elements are.