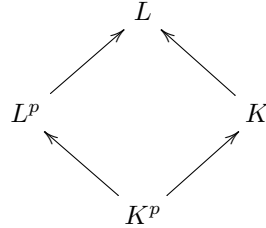


PROBLEM SET #3

- (1) Is $\mathbb{Z}[\sqrt[3]{37}]$ a regular ring? What about $\mathbb{Z}[\sqrt[3]{43}]$?
- (2) Let R be an A -algebra, $f(x_1, \dots, x_n) \in A[x_1, \dots, x_n]$ a polynomial with coefficients in A , and $r_1, \dots, r_n, s_1, \dots, s_n \in R$.
- (a) Prove the *chain rule* for the universal derivation: $d_{R|A}(f(r_1, \dots, r_n)) = \sum_i \frac{df}{dx_i}(r_1, \dots, r_n) dr_i$.
- (b) Prove the *Taylor expansion* formula: $f(r_1 + s_1, \dots, r_n + s_n) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{|\alpha|!} \frac{d^{|\alpha|} f}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}}(r_1, \dots, r_n) s_1^{\alpha_1} \dots s_n^{\alpha_n}$.
- (3) Facts about p -bases/ p -degree:
- (a) Let L be a field of positive characteristic. Let T be a p -basis for L . Show that for any e , the set $T^{[<p^e]}$ is a basis for L .
- (b) Let $K \subseteq L$ be a finite extension of fields of positive characteristic. Show that $p \deg(K) = p \deg(L)$.
- (c) Let $L = K(x_1, \dots, x_m)$ be a field of rational functions in m variables over K . Show that $p \deg(L) = p \deg(K) + m$.

Proof. (a) By induction on e , with $e = 1$ as the definition. If the claim is true for e , so $T^{[<p^e]}$ is a basis for L/L^{p^e} , taking p th powers we have that $(T^p)^{[<p^e]}$ is a basis for $L^p/L^{p^{e+1}}$. But $T^{[<p^{e+1}]} = (T^p)^{[<p^e]} T^{[<p]}$ (i.e., the first set is the set of products of the two sets on the right-hand side), so from field theory, the left hand side is a basis for $L/L^{p^{e+1}}$.

(b) Consider the diagram



From field theory $[L : K][K : K^p] = [L^p : K^p][L : L^p]$, and $[L^p : K^p] = [L : K]$, so $[K : K^p] = [L : L^p]$. Then $p \deg(K) = \log_p([K : K^p]) = \log_p([L : L^p]) = p \deg(L)$.

(c) Take a p -basis T for K . One checks that $T \cup \{x_1, \dots, x_m\}$ is a p -basis for L . □

- (4) Let k be a field of positive characteristic with a finite p -basis, R be a finitely generated k -algebra, and $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals of R . Show that

$$\dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = p \deg(\kappa(\mathfrak{p})) - p \deg(\kappa(\mathfrak{q})).$$

Proof. We will show that $\dim(R/\mathfrak{p}) = p \deg(\kappa(\mathfrak{p})) - p \deg(k)$; the formula above then follows. We can replace R by R/\mathfrak{p} and assume R is a domain and $\mathfrak{p} = 0$. Take a Noether normalization A for R . By part (2) of the previous problem, $p \deg(\kappa(\mathfrak{p})) = p \deg(\text{frac}(R)) = p \deg(\text{frac}(A))$. By part (3) of the previous problem, $p \deg(\text{frac}(A)) = \dim(A) + p \deg(k) = \dim(R) + p \deg(k)$. The conclusion then follows. □

- (5) Let K be a field.

- (a) Let $R = K[x]$ be a polynomial ring in one variable and $M = R^{\oplus \mathbb{N}}$ be a free R -module on a countable basis. Compute the (x) -adic completion of M .
- (b) Let $R = K[x_1, x_2, \dots]$ be a polynomial ring in countably many variables and $\mathfrak{m} = (x_1, x_2, \dots)$. Describe the elements of $\hat{R}^{\mathfrak{m}}$. Find an element in the maximal ideal of $\hat{R}^{\mathfrak{m}}$ that is *not* an element of $\mathfrak{m}\hat{R}^{\mathfrak{m}}$.

(6) Let $K \subseteq L$ be an extension of fields.

- (a) Suppose that L is a finitely generated over K as fields. Show that L is formally unramified over K if and only if the extension is separable algebraic.
- (b) Show that the finite generation hypothesis is strictly necessary in part (1).

Proof. (a) We just need to show that unramified implies separable algebraic. Any transcendental element is a p -independent set, which contradicts unramified, so unramified implies algebraic. Write $K \subseteq F \subseteq L$, with $F \subseteq L$ purely inseparable. By finite generation plus algebraic, this is finite. We can then choose some f with $f \in L \setminus F$ and $f^p \in F$ using finiteness. Then f is p -independent in L over F , contradicting unramified.

- (b) Take $K = \mathbb{F}_p(t)$ and $L = \bigcup_{e \in \mathbb{N}} \mathbb{F}_p(t^{1/p^e})$.

□