ASSIGNMENT #7: DUE FRIDAY, DECEMBER 13 AT 7PM

This problem set is to be turned in by Canvas. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

(1) Let $R \subset S$ be an inclusion of rings such that R is a direct summand of S. Show that the induced map on Spec is surjective.

(2) Let R be a ring, not necessarily Noetherian, and $S = R[X]$ a polynomial ring in one variable over R.
   (a) Show that for any prime ideal $\mathfrak{p}$ in R, any chain of prime ideals of S that all contract to $\mathfrak{p}$ has length at most one.
   (b) Show that if $\dim(R) = d$, then $d + 1 \leq \dim(S) \leq 2d + 1$.
   (c) Let $R = \mathbb{Q} + T\,\mathbb{R}[\![T]\!]$, i.e., the subring of $R' = \mathbb{R}[\![T]\!]$ consisting of all power series whose constant term is rational. Verify that R is a ring, that R has dimension one, and[1] that the dimension of $R[X]$ is three.

(3) Show that the $\mathbb{Z}$-module $\mathbb{Z}[1/2]/\mathbb{Z}$ is Artinian but not Noetherian.

(4) Let K be an algebraically closed field, $R = K[X_1, \dots, X_n]$, and

$$S = R[Y_1, \dots, Y_n] = K[X_1, \dots, X_n, Y_1, \dots, Y_n]$$

be polynomial rings. For $f \in R$, we will also write $f(\mathbf{X})$ for f, and we write $f(\mathbf{Y})$ for the element[2] of S obtained by replacing X-variables with Y-variables in f. For an ideal $I \subseteq R$, we write $I(\mathbf{Y}) = \{f(\mathbf{Y}) \mid f \in I\}$.

Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals in R.
   (a) Let $\{F_1, \dots, F_a\}$ be a set elements of R whose images generate a Noether normalization for $R/\mathfrak{p}$, and $\{G_1, \dots, G_b\}$ be a set of elements of R whose elements generate a Noether normalization for $R/\mathfrak{q}$. Show that the images of $\{F_1(\mathbf{X}), \dots, F_a(\mathbf{X}), G_1(\mathbf{Y}), \dots, G_b(\mathbf{Y})\}$ generate a Noether normalization[3] for $\dfrac{S}{\mathfrak{p}(\mathbf{X})S + \mathfrak{q}(\mathbf{Y})S}$. Deduce that

$$\dim\left(\frac{S}{\mathfrak{p}(\mathbf{X})S + \mathfrak{q}(\mathbf{Y})S}\right) = \dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}).$$

   (b) Show that

$$\frac{S}{\mathfrak{p}(\mathbf{X})S + \mathfrak{q}(\mathbf{Y})S + (X_1 - Y_1, \dots, X_n - Y_n)} \cong \frac{R}{\mathfrak{p} + \mathfrak{q}}.$$

   Deduce (this is the punchline) that $\operatorname{height}(\mathfrak{p} + \mathfrak{q}) \leq \operatorname{height}(\mathfrak{p}) + \operatorname{height}(\mathfrak{q})$.
   (c) Let $T = \dfrac{\mathbb{C}[X, Y, U, V]}{(XV - YU)}$, $\mathfrak{p} = (X, U)$, and $\mathfrak{q} = (Y, V)$. Show that $\operatorname{height}(\mathfrak{p}) = \operatorname{height}(\mathfrak{q}) = 1$ but $\operatorname{height}(\mathfrak{p} + \mathfrak{q}) = 3$.

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[1] Hint: Let $\mathfrak{p}$ be the prime ideal $T\mathbb{C}[\![T]\!] \subseteq R$ and let $\alpha : R[X] \to R'$ be the R-algebra homomorphism given by $\alpha(X) = e$. Show that $\ker(\alpha) \subsetneqq \mathfrak{p}S$.

[2] To be pedantic, let $\phi : R \to S$ be the K-algebra homomorphism given by $\phi(X_i) = Y_i$; then $f(Y) = \phi(f)$.

[3] Hint: Given an equation $\sum c_{\alpha,\beta} F^\alpha G^\beta \in \mathfrak{p}(\mathbf{X})S + \mathfrak{q}(\mathbf{Y})S$, evaluate the Y variables at $\beta \in Z(\mathfrak{q})$ to get an equation in R...