RATIONAL CANONICAL FORM

DEFINITION: Let F be a field. For a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x],$$

the **companion matrix** of f is

$$C(f) = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ & & & -a_1 \\ & I_{n-1} & & \vdots \\ & & & -a_{n-1} \end{bmatrix}.$$

RATIONAL CANONICAL FORM: Let F be a field. Let V be a finite dimensional vector space, and $\phi : V \to V$ a linear transformation. Then there is a basis B for V such that

$$[\phi]_B^B = \begin{bmatrix} C(g_1) & 0 & \cdots & 0 \\ 0 & C(g_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(g_s) \end{bmatrix}$$

where $g_1 | \cdots | g_s$ are the invariant factors[1] of ϕ. This matrix is called the **rational canonical form** of ϕ, or $\mathrm{RCF}(\phi)$.

THEOREM: Let F be a field and $A, B \in \mathrm{Mat}_{n\times n}(F)$. The following are equivalent:

(1) A and B are similar matrices.
(2) A and B have the same rational canonical form.
(3) A and B have the same invariant factors.

(1) Computing some RCFs:
  (a) Let

(2) Classifying matrices up to similarity:
(3) Uniqueness of rational canonical form:

[1] The **invariant factors** of ϕ are the (monic) invariant factors of the $F[x]$-module V_ϕ.