FALL 2024 MATH 325 LECTURE NOTES AND WORKSHEETS

Contents

1. Monday, August 26, 2024 §1.1

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \ldots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not havethought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\dots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\dots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number. Right? Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is 2.*

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \implies \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction.

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. □

Thus, if we would like to have a number corresponding to the length of the diagonal of a square with side length one, it must be a number that is real but not rational. Let's record the common name for such a number.

Definition 1.2. A real number is *irrational* if it is not rational.

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.3 (Arithmetic and order properties of $\mathbb{Q}$)**.** *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

1. *There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p + q$ and $p \cdot q$.*
2. *Each of $+$ and $\cdot$ is a commutative operation (i.e., $p + q = q + p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*
3. *Each of $+$ and $\cdot$ is an associative operation (i.e., $(p + q) + r = p + (q + r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q, and r).*
4. *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0 + q = q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.*
5. *The distributive law holds: $p \cdot (q + r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.*

(6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.*
(7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.*
(8) *There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that*
   (a) *For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
   (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
   (c) *For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
(9) *The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.*
(10) *The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

Which of the properties from Proposition 1.3 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$.

We expect everything from Proposition 1.3 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written $\mathbb{R}$, satisfies the following eleven properties:*

(Axiom 1) *There are operations $+$ and $\cdot$ defined on $\mathbb{R}$, so that if $x, y \in \mathbb{R}$, then so are $x + y$ and $x \cdot y$.*
(Axiom 2) *Each of $+$ and $\cdot$ is a commutative operation.*
(Axiom 3) *Each of $+$ and $\cdot$ is an associative operation.*
(Axiom 4) *The real number 0 is an identity element for addition and the real number 1 is an identity element for multiplication. This means that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.*
(Axiom 5) *The distributive law holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.*
(Axiom 6) *Every real number has an additive inverse: For any $x \in \mathbb{R}$, there is a number $-x$ satisfying $x + (-x) = 0$.*
(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any $x \in \mathbb{R}$ such that $x \neq 0$, there is a real number x^{-1} satisfying $x^{-1} \cdot x = 1$.*
(Axiom 8) *There is a "total ordering" $\leq$ on $\mathbb{R}$. This means that*
   (a) *For all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.*
   (b) *If $x \leq y$ and $y \leq z$, then $x \leq z$.*
   (c) *For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.*
(Axiom 9) *The total ordering $\leq$ is compatible with addition: If $x \leq y$ then $x + z \leq y + z$ for all z.*
(Axiom 10) *The total ordering $\leq$ is compatible with multiplication by nonnegative real numbers: If $x \leq y$ and $z \geq 0$ then $zx \leq zy$.*
(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

"Cancellation of Addition": For real numbers, $x, y, z \in \mathbb{R}$, if $x+y = z+y$ then $x = z$.

Let's prove this carefully, using just the list of axioms: Assume that $x + y = z + y$. Then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x + y) + (-y) = (z + y) + (-y)$. This can be rewritten as $x + (y + (-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

Here is the incredible fact: everything that we know about the real numbers follows from these axioms! In particular, all of the basic facts about arithmetic of real numbers follow from this, like "the product of two negative numbers is positive," as well as everything you encountered in Calculus, like the Mean Value Theorem.

2. Wednesday, August 27, 2024 §1.4 & 1.5

Making sense of if-then statements.

- The statement "If P then Q" is true whenever Q is true or P is false. Equivalently, the statement "If P then Q" is false whenever Q is false and P is true.
- The **converse** of the statement "If P then Q" is the statement "If Q then P".
- The **contrapositive** of the statement "If P then Q" is the statement "If not Q then not P".
- Any if-then statement is equivalent to its *contrapositive*, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Decide if original/contrapositive/converse true or false for real numbers a, b, but don't prove them yet.
  (a) If a is irrational, then $1/a$ is irrational.
  (b) If a and b are irrational, then ab is irrational.
  (c) If $a \geq 3$, then $a^2 \geq 9$.

(a) Contrapositive: If $1/a$ is rational, then a is rational. Converse: If $1/a$ is irrational, then a is irrational. Original, contrapositive, and converse all TRUE.
(b) Contrapositive: If ab is rational, then a is rational or b is rational. Converse: if ab is irrational, then a and b are irrational. Original, contrapositive, and converse all FALSE.
(c) Contrapositive: If $a^2 < 9$ then $a < 3$. Converse: $a^2 \geq 9$ then $a \geq 3$. Original and contrapositive are TRUE but converse is FALSE.

Proving if-then statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if-then statement than the original, especially when the conclusion is something negative. We sometimes call this an *indirect proof* or a *proof by contraposition*.

(2) Consider the following proof of the claim "For real numbers x, y, z, if $x + y = z + y$, then $x = z$" from the axioms of $\mathbb{R}$. Match the parts of this proof with the general outline above. Which sentences are *assumptions* and which are

assertions? Is it clear *just from reading each sentence on its own* whether it is an assumption or an assertion? Is it clear *why* each assertion is true?

Proof. Suppose that $x + y = z + y$. Then adding $-y$ (which exists by Axiom 6) we get

$$(x + y) + (-y) = (z + y) + (-y).$$

This can be rewritten (by Axiom 3) as

$$x + (y + (-y)) = z + (y + (-y)),$$

and hence (by Axiom 6) as

$$x + 0 = z + 0,$$

which gives $x = z$ (by Axioms 4 and 2). □

> The first sentence is an assumption, while the others are assertions. It is clear that the first is an assumption since it says "Suppose"; another possibility is "Assume".

(3) Consider the following purported proof of the true fact "If $2x + 5 \geq 7$ then $x \geq 1$." Is this a good proof? Is it a correct proof?

Proof.

$$x \geq 1.$$

Multiply both sides by two.

$$2x \geq 2.$$

Add five to both sides.

$$2x + 5 \geq 7.$$

□

> No. It is not clear what is an assumption and what is an assertion. From the order of the sentences, it looks like they are assuming the conclusion and deducing the hypothesis! This was OK in this example, but they could have done almost the exact same thing to "prove" the false statement "If $2x^2 + 5 \geq 7$ then $x \geq 1$."

Proving if-then statements.

(4) Prove or disprove each of the statements in (1). You might consider a proof by contraposition for some of these!

> (a) TRUE: We proceed by contraposition. Suppose that $1/a$ is rational, so $1/a = p/q$ for some integers p/q. Then $a = q/p$ is rational.
> (b) FALSE: Consider $a = \sqrt{2}$ and $b = \sqrt{2}$. Both a and b are irrational, but $ab = 2$ is rational.

(c) TRUE: If $a \geq 3$ then $a^2 = a \cdot a \geq a \cdot 3 \geq 3 \cdot 3 = 9$, where we used that $a \geq 3 \geq 0$ in the first inequality, and $3 \geq 0$ in the second.

(5) Prove or disprove the *converse* of each of the statements in (1).

(a) TRUE: We proceed by contraposition. Suppose that a is rational, so $a = p/q$ for some integers p/q. Then $1/a = q/p$ is rational.
(b) FALSE: Consider $a = \sqrt{2}$ and $b = 1$. Then $ab = \sqrt{2}$ is irrational, but b is rational, so it is not true that a and b are irrational.
(c) FALSE: Consider $a = -3$; for this choice of a, $a \not\geq 3$ but $a^2 = 9 \geq 9$.

Using the axioms of $\mathbb{R}$ to prove basic arithmetic facts.

(6) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[1] that if $x \geq y$ then $-x \leq -y$.

Let $x \geq y$. Adding $(-x) + (-y)$ to both sides (which exists by Axiom 6), we obtain $-y = x + ((-x) + (-y)) \geq y + ((-x) + (-y)) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x+y)+(-x) \leq (x+y)+(-y) = x$ (by Axiom 9 and Axiom 5).

(7) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove that $x \geq y$ if and only if $-x \leq -y$.

We need to prove both directions. The first direction is the previous problem. Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x + y) + (-x) \leq (x + y) + (-y) = x$ (by Axiom 9 and Axiom 5).

(8) Let x, y be real numbers. Use the axioms of $\mathbb{R}$ and facts we have already proven[2] to prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

[1] Hint: You may want to add something to both sides.

[2] Be careful: are you using any facts that we have not already proven?

3. Friday, August 29, 2024 §1.4 & 1.5

Making sense of quantifier statements.

- The symbol for "**for all**" is $\forall$ and the symbol for "**there exists**" is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that not P".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, not P".

A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both)!

(1) Match each informal story (i)–(iv) below with a precise quantifier statement (A)–(D).

Informal stories:

(i) Every number past some point is red.
(ii) There are arbitrarily big red numbers.
(iii) All positive numbers are red.
(iv) There are positive red number(s).

Precise statements:

(A) For every $y > 0$, y is red.
(B) There exists $y > 0$ such that y is red.
(C) For every $x \in \mathbb{R}$, there is some $y > x$ such that y is red.
(D) There exists $x \in \mathbb{R}$ such that for every $y > x$, y is red.

(i)=(D), (ii)=(C), (iii)=(A), (iv)=(B)

(2) Draw a picture where (A) is false and (B) is true.

Answers may vary

(3) Draw a picture where (C) is true and (D) is false.

Answers may vary

(4) Suppose that (C) is true. Which of the following statements must also be true? Why?
(a) There is some $y > 1000000000$ such that y is red.
(b) For every $\mu \in \mathbb{R}$, there is some $\theta > \mu$ such that θ is red.
(c) For every $x \in \mathbb{R}$, there is some $y > 2x$ such that y is red.

These are ALL TRUE.

The next problem is no longer about a spraypainting of the real number line.

(5) Rewrite each statement with symbols in place of quantifiers, and write its negation. Do you think the original statement is true or false (but don't prove them yet)?.
  (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
  (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
  (c) For all $x \in \mathbb{R}$ such that[3] $x \neq 0$, $x^2 > 0$.
  (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
  (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

(a) $\exists x \in \mathbb{Q} : x^2 = 2$. NEGATION: For all $x \in \mathbb{Q}$, $x^2 \neq 2$.
(b) $\forall x \in \mathbb{R}, x^2 > 0$. NEGATION: There exists $x \in \mathbb{R}$ such that $x^2 \leq 0$.
(c) $\forall x \in \mathbb{R} : x \neq 0, x^2 > 0$. NEGATION: There exists $x \in \mathbb{R}$ such that $x \neq 0$ such that $x^2 \leq 0$.
(d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$. NEGATION: There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x \geq y$.
(e) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x < y$. NEGATION: For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $x \geq y$.

Proving quantifier statements and using the axioms of $\mathbb{R}$.

- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

Note: explaining *how* you found your example"x" is *not* a logically necessary part of the proof.

(6) Circle the correct answer in each of the blanks below:
  - To prove a "for all" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
  - To *dis*prove a "for all" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
  - To prove a "there exists" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
  - To *dis*prove a "there exists" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.

[3]In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

- If you want to *use* a "for all" statement that you know is true,
  you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE
- If you want to *use* a "there exists" statement that you know is true,
  you CAN CHOOSE A SPECIFIC VALUE / MUST USE A MYSTERY VALUE

> - To prove a "for all" statement, you need to give a GENERAL ARGUMENT.
> - To *dis*prove a "for all" statement, you need to give a SPECIFIC EXAMPLE.
> - To prove a "there exists" statement, you need to give a SPECIFIC EXAMPLE.
> - To *dis*prove a "there exists" statement, you need to give a GENERAL ARGUMENT.
>
> - If you want to *use* a "for all" statement that you know is true, you CAN CHOOSE A SPECIFIC VALUE
> - If you want to *use* a "there exists" statement that you know is true, you MUST USE A MYSTERY VALUE

(7) Prove or disprove each of the statements in (5) using the axioms of $\mathbb{R}$ and facts we have already proven.

> (a) FALSE: We showed that there is no rational number whose square is two.
> (b) FALSE: Take $x = 0$; then $x^2 = 0$ which is not strictly greater than zero.
> (c) TRUE.
> (d) TRUE: Let $x \in \mathbb{R}$ be given. Then $y = x + 1 > x$.
> (e) FALSE: Suppose that $x \in \mathbb{R}$. We claim that there is some $y \in \mathbb{R}$ such that $y \leq x$; just take x itself!

More practice with quantifier statements. Using the axioms of $\mathbb{R}$ and statements that we've already proven (like cancellation of addition, or any problem on the list above the given one), prove the following:

(8) Prove that there exists some $x \in \mathbb{R}$ such that $2x + 5 = 3$.

> Take $x = -1$. Then $2x + 5 = 2(-1) + 5 = 3$.

(9) Prove[4] that for any real number r, we have $r \cdot 0 = 0$.

> Let r be any real number. We have $0 + 0 = 0$ (Axiom 4) and hence $r \cdot (0+0) = r \cdot 0$. But $r \cdot (0+0) = r \cdot 0 + r \cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the

[4]Hint: You might find it useful to write $0 = 0 + 0$ and, in a later step, use cancellation of addition.

Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

(10) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have already proven to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

Let x be a real number. Suppose that there exists a real number y such that $xy = 1$. By way of contradiction, suppose that $x = 0$. Then $1 = xy = 0y = 0$, which is a contradiction. Thus, we must have $x \neq 0$. Thus, if there exists a real number y such that $xy = 1$, then $x \neq 0$.

(11) Prove that[5] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

[5]Hint: Use (10).

4. Wednesday, September 3, 2024 §1.8

I owe you a statement of the very important Completeness Axiom. Before we get there, I want to recall an axiom of $\mathbb{N}$ that we haven't discussed yet. It pertains to minimum elements in sets. Let's be precise and define minimum element.

Definition 4.1. Let S be a set of real numbers. A **minimum** element of S is a real number x such that

(1) $x \in S$, and
(2) for all $y \in S$, $x \leq y$.

In this case, we write $x = \min(S)$.

The definition of **maximum** is the same except with the opposite inequality.

Axiom 4.2 (Well-ordering axiom)**.** *Every nonempty subset of* $\mathbb{N}$ *has a minimum element.*

Example 4.3. If S is the set of even multiples of 7, then S has 14 as its minimum.

We generally like to say *the* minimum, rather than *a* minimum. To justify this, let's prove the following.

Proposition 4.4. *Let S be a set of real numbers. If S has a minimum, then the minimum is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P, then x and y must be the same thing.

Proof of Proposition 4.4. Let S be a set of real numbers, and let x and y be two minima of S. Applying part (1) of the definition of minimum to y, we have $y \in S$. Applying part (2) of the definition of minimum to x and the fact that $y \in S$, we get that $x \leq y$. Switching roles, we get that $y \leq x$. Thus $x = y$.

We conclude that if a minimum exists, it is necessarily unique. □

The previous proposition plus the Well-Ordering Axiom together imply that every nonempty subset of $\mathbb{N}$ has exactly one minimum element. A similar proof shows that if a maximum exists, it is necessarily unique. Could a set fail to have a maximum or a minimum? Yes!

Example 4.5.

(1) The empty set $\varnothing$ has no minimum and no maximum element. (There is no $s \in \varnothing$!)
(2) The set of natural numbers $\mathbb{N}$ has 1 as a minimum, but has no maximum. (Suppose there was: if $n = \max(\mathbb{N})$ was the maximum, then $n < n + 1 \in \mathbb{N}$ gives a contradiction.)
(3) The open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has no minimum and no maximum. (Exercise later.)

Definition 4.6. Let S be any subset of $\mathbb{R}$. A real number b is called an **upper bound** of S provided that for every $s \in S$, we have $s \leq b$.

Definition 4.7. A subset S of $\mathbb{R}$ is called **bounded above** if there exists at least one upper bound for S. That is, S is bounded above provided there is a real number b such that for all $s \in S$ we have $s \leq b$.

For example, the open interval $(0, 1)$ is bounded above, by for example 50.

The subset $\mathbb{N}$ of $\mathbb{R}$ is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of $\mathbb{R}$ that is bounded above.

Example 4.8. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 4.9. Suppose S is subset of $\mathbb{R}$ that is bounded above. A **supremum** (also known as a **least upper bound**) of S is a number ℓ such that

(1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
(2) if b is any upper bound of S, then $\ell \leq b$.

In this case we write $\sup(S) = \ell$.

5. Friday, September 5, 2024 §1.8

Let S be a set of real numbers.

- A number b is an **upper bound** for S provided for all $x \in S$ we have $b \geq x$.
- The set S is **bounded above** provided there exists at least one upper bound for S.
- A number m is the **maximum** of S provided
  (1) $m \in S$, and
  (2) m is an upper bound of S.
- A number ℓ is a **supremum** of S provided
  (1) ℓ is an upper bound of S, and
  (2) for any upper bound b for S, we have $\ell \leq b$.

(1) Write, in simplified form, the negation of the statement "b is an upper bound for S".

There exists some $x \in S$ such that $x > b$.

(2) Write, in simplified form, the negation of the statement "S is bounded above".

For every $b \in \mathbb{R}$, there exists $x \in S$ such that $x > b$.

(3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
  (a) If $x > \ell$, what is the most concrete thing you can say about x and S?
  (b) If $x < \ell$, what is the most concrete thing[6] you can say about x and S?

(a) $x \notin S$.
(b) There exists some $y \in S$ such that $y > x$.

(4) Let $S = \{x \in \mathbb{R} \mid x^3 + x < 5\}$. Use the definition of supremum to answer the following:
  (a) Is 1 the supremum of S? Why or why not?
  (b) Is 2 the supremum of S? Why or why not?

(a) No. 1 is not an upper bound for S, because $1.5^3 + 1.5 = 4.875 < 5$, so 1.5 is a larger element of S.
(b) No. We claim that 1.75 is an upper bound for S, so 2 is not the smallest upper bound of S. Indeed, if $x \in S$ then $x \leq 1.75$; if $x > 1.75$ then $x^3 + x > 1.75^3 + 1.75 = 7.109375 > 5$ so $x \notin S$.

(5) Consider the open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

[6]Hint: Use one of the previous problems.

(a) Prove[7] that $(0,1)$ has no maximum element.
(b) Prove that $\sup(\,(0,1)\,) = 1$.

(a) To obtain a contradiction, suppose that $(0,1)$ has a maximum element; call it z. Since $z \in (0,1)$ so $0 < z < 1$. First, we claim that $y = \frac{z+1}{2} \in (0,1)$. Indeed, $z > 0$ implies $y > \frac{0+1}{2} = \frac{1}{2} > 0$, and $z < 1$ implies $y < \frac{1+1}{2} = 1$, so $y \in (0,1)$. Now we claim that y is larger than z: $y = \frac{z+1}{2} > \frac{z+z}{2} = z$. But this contradicts that z is an upper bound for $(0,1)$, so z cannot be the maximum. Since assuming the existence of a maximum element leads to a contradiction, we conclude that no maximum exists.

(b) We need to show that 1 is an upper bound for $(0,1)$, and that any other upper bound b for $(0,1)$ satisfies $b \geq 1$. By definition of the interval $(0,1)$, the number 1 is an upper bound. Now suppose that b is an upper bound for $(0,1)$. We claim that $b \geq 1$. Suppose otherwise that $b < 1$ to obtain a contradiction. Consider $y = \frac{b+1}{2}$. Then by the same algebraic argument as the previous part, $y \in (0,1)$ and $y > b$, so y is not an upper bound for $(0,1)$. This contradicts the assumption that $b < 1$, so $b \geq 1$. Thus, every upper bound of $(0,1)$ is at least 1. This shows that 1 is the supremum of $(0,1)$).

(6) Let S be a set of real numbers, and let $T = \{2s \mid s \in S\}$. Prove that if S is bounded above, then T is bounded above.

Assume that S is bounded above. Then there is some upper bound b for S, so for every $s \in S$, we have $b \geq s$. We claim that $2b$ is an upper bound for T. Indeed, if $t \in T$, then we can write $t = 2s$ for some $s \in S$, and $s \leq b$ implies $t = 2s \leq 2b$. Thus, T is bounded above.

(7) Let S be a set of real numbers. Show that if S has a supremum, then it is unique.

Suppose both x and y are both suprema of the same subset S of $\mathbb{R}$. Then, since y is an upper bound of S and x is a supremum of S, by part (2) of the definition of "supremum" we have $y \geq x$. Likewise, since x is an upper bound of S and y is a supremum of S, we have $x \geq y$ by definition. Since $x \leq y$ and $y \leq x$, we conclude $x = y$.

[7] Hint: Try a proof by contradiction!

Can you think of a set with no supremum?

The subset $\mathbb{N}$ does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly: the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term "supremum", I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom)**.** *Every nonempty, bounded-above subset of* $\mathbb{R}$ *has a supremum.*

This is an incredibly powerful fact about the real numbers. To give a sense of what it does, recall that we showed that there is no rational number whose square is two; the completeness axiom is what allows us to show that there is a real number whose square is two.

Proposition 5.1. *There is a positive real number whose square is* 2*.*

In class, let's just discuss the outline of this proof rather than the algebraic details, so we can see the big picture. You can read the full proof in the lecture notes.

Proof outline.

Step 1: Define S to be the subset $S = \{x \in \mathbb{R} \mid x^2 < 2\}$.

Step 2: Show that S is nonempty and bounded above. Thus, by the Completeness Axiom, there is a real number $\ell = \sup(S)$.

Step 3: Show that $\ell^2 < 2$ leads to a contradiction. (Assuming $\ell^2 < 2$, we can find a number s slightly larger than ℓ that is an element of S, contradicting that ℓ is an upper bound of S.)

Step 4: Show that $\ell^2 > 2$ leads to a contradiction. (Assuming $\ell^2 > 2$, we can find a number b slightly smaller than ℓ that is an upper bound of S, contradicting that ℓ is the smallest upper bound of S.)

Step 5: There's no other possibilty: we must have $\ell^2 = 2$! $\square$

Here is the proof in full detail.

Proof. Define S to be the subset

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

S is nonempty since, for example, $1 \in S$, and it is bounded above, since, for example, 2 is an upper bound for S, as we showed earlier. So, by the Completeness Axiom, S has a least upper bound, and we know it is unique from the proposition above. Let us call it ℓ. I will prove $\ell^2 = 2$.

We know one of $\ell^2 > 2$, $\ell^2 < 2$ or $\ell^2 = 2$ must hold. We prove $\ell^2 = 2$ by showing that both $\ell^2 > 2$ and $\ell^2 < 2$ are impossible.

We start by observing that $1 \leq \ell \leq 2$. The inequality $1 \leq \ell$ holds since $1 \in S$ and ℓ is an upper bound of S, and the inequality $\ell \leq 2$ holds since 2 is an upper bound of S and ℓ is the least upper bound of S.

Suppose $\ell^2 < 2$. We show this leads to a contradiction by showing that ℓ is not an upper bound of S in this case. We will do this by constructing a number that is ever so slightly bigger than ℓ and belongs to S. Let $\varepsilon = 2 - \ell^2$. Then $0 < \varepsilon \leq 1$ (since $\ell^2 < 2$ and $\ell^2 \geq 1$). We will now show that $\ell + \varepsilon/5$ is in S: We have

$$(\ell + \varepsilon/5)^2 = \ell^2 + \frac{2}{5}\ell\varepsilon + \frac{\varepsilon^2}{25} = \ell^2 + \varepsilon(\frac{2\ell}{5} + \frac{\varepsilon}{25}).$$

Now, using $\ell \leq 2$ and $0 < \varepsilon \leq 1$, we deduce

$$0 < \frac{2\ell}{5} + \frac{\varepsilon}{25} \leq \frac{4}{5} + \frac{\varepsilon}{25} < 1.$$

Putting these equations and inequalities together yields

$$(\ell + \frac{\varepsilon}{5})^2 < \ell^2 + \varepsilon = 2.$$

So, $\ell + \frac{\varepsilon}{5} \in S$ and yet $\ell + \frac{\varepsilon}{5} > \ell$, contradicting the fact that l is an upper bound of S. We conclude $\ell^2 < 2$ is not possible.

Assume now that $\ell^2 > 2$. Our strategy will be to construct a number ever so slightly smaller than ℓ, which therefore cannot be an upper bound of S, and use this to arrive at a contradiction. Let $\delta = \ell^2 - 2$. Then $0 < \delta \leq 2$ (since $\ell \leq 2$ and hence $\ell^2 - 2 \leq 2$). Since $\delta > 0$, we have $\ell - \frac{\delta}{5} < \ell$. Since ℓ is the least upper bound of S, $\ell - \frac{\delta}{5}$ must not be an upper bound of S. By definition, this means that there is $r \in S$ such that $\ell - \frac{\delta}{5} < r$. Since $\delta \leq 2$ and $\ell \geq 1$, it follows that $\ell - \frac{\delta}{5}$ is positive and hence so is r. We may thus square both sides of $\ell - \frac{\delta}{5} < r$ to obtain

$$(\ell - \frac{\delta}{5})^2 < r^2.$$

Now

$$(\ell - \frac{\delta}{5})^2 = \ell^2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = \delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25}$$

since $\ell^2 = \delta + 2$. Moreover,

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = 2 + \delta(1 - \frac{2\ell}{5} + \frac{\delta}{25}) \geq 2 + \delta(1 - \frac{4}{5} + \frac{\delta}{25})$$

since $\ell \leq 2$. We deduce that

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} \geq 2 + \delta(\frac{1}{5}) \geq 2.$$

Putting these inequalities together gives $r^2 > 2$, contrary to the fact that $r \in S$. We conclude that $\ell^2 > 2$ is also not possible.

Since $\ell^2 < 2$ and $\ell^2 > 2$ are impossible, we must have $\ell^2 = 2$. $\square$

6. Monday, September 9, 2024 §1.9

Recall the Completeness Axiom:

Completeness Axiom: Every nonempty bounded above set of real numbers has a supremum.

The collection of rational numbers does not satisfy the completeness axiom and indeed it is precisely the completeness axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Example 6.1. Within the set $\mathbb{Q}$ the subset $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum. That is, no matter which rational number you pick that is an upper bound for S, you may always find an even smaller one that is also an upper bound of S.

It is precisely the completeness axiom that assures us that everything that ought to be a number (like the length of the diagonal of a square with side length 1) really is a number. It gives us that there are "no holes" in the real number line — the real numbers are *complete*.

The completeness axiom is also at the core of the Intermediate Value Theorem and many of the other major theorems we will cover in this class.

We also need the completeness axiom to understand the relationship between $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$.

Theorem 6.2. *If x is any real number, then there exists a natural number n such that $n > x$.*

This looks really stupid at first. How could it be false? But consider: there are examples of ordered fields, i.e. situations in which Axioms 1–10 hold, in which this Theorem is not true! So, its proof must rely on the Completeness Axiom.

Proof. Let x be any real number. By way of contradiction, suppose there is no natural number n such that $n > x$. That is, suppose that for all $n \in \mathbb{N}$, $n \leq x$. Then $\mathbb{N}$ is a bounded above (by x). Since it is also clearly nonempty, by the Completeness Axiom, $\mathbb{N}$ has a supremum, call it ℓ. Consider the number $y := \ell - 1$. Since $y < \ell$ and ℓ is the supremum of $\mathbb{N}$, y cannot be an upper bound of $\mathbb{N}$. So, there must be some $m \in \mathbb{N}$ such that such that $\ell - 1 < m$. But then by adding 1 to both sides of this inequality we get $\ell < m + 1$ and, since $m + 1 \in \mathbb{N}$, this contradicts that assumption that ℓ is the supremum of $\mathbb{N}$.

We conclude that, given any real number x, there must exist a natural number n such that $n > x$. □

Corollary 6.3 (Archimedean Principle)**.** *If $a \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$, then for some natural number n we have $na > b$.*

"No matter how small a is and how large b is, if we add a to itself enough times, we can overtake b."

Proof. We apply Theorem 6.2 to the real number $x = \frac{b}{a}$. It gives that there is a natural number n such that $n > x = \frac{b}{a}$. Since $a > 0$, upon multiplying both sides by a we get $n \cdot a > b$. □

Theorem 6.4 (Density of the Rational Numbers)**.** *Between any two distinct real numbers there is a rational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. We will prove this by consider two cases: $x \geq 0$ and $x < 0$.

Let us first assume $x \geq 0$. We apply the Archimedean Principle using $a = y - x$ and $b = 1$. (The Principle applies as $a > 0$ since $y > x$.) This gives us that there is a natural number $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1$$

and thus

$$0 < \frac{1}{n} < y - x.$$

Consider the set $S = \{p \in \mathbb{N} \mid p\frac{1}{n} > x\}$. Since $\frac{1}{n} > 0$, using the Archimedean principle again, there is at least one natural number $p \in S$. By the Well Ordering Axiom, there is a smallest natural number $m \in S$.

We claim that $\frac{m-1}{n} \leq x$. Indeed, if $m > 1$, then $m - 1 \in \mathbb{N} \setminus S$ (because $m - 1$ is less than the minimum), so $\frac{m-1}{n} \leq x$; if $m = 1$, then $m - 1 = 0$, so $\frac{m-1}{n} = 0 \leq x$.

So, we have

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

By adding $\frac{1}{n}$ to both sides of $\frac{m-1}{n} \leq x$ and using that $\frac{1}{n} < y - x$, we get

$$\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$$

and hence

$$x < \frac{m}{n} < y.$$

Since $\frac{m}{n}$ is clearly a rational number, this proves the result in this case (when $x > 0$).

We now consider the case $x < 0$. The idea here is to simply "shift" up to the case we've already proven. By Theorem 6.2, we can find a natural number j such that $j > -x$ and thus $0 < x + j < y + j$. Using the first case, which we have already proven, applied to the number $x + j$ (which is positive), there is a rational number q such that $x + j < q < y + j$. We deduce that $x < q - j < y$, and, since $q - j$ is also rational, this proves the theorem in this case. $\square$

7. Wednesday, September 11, 2024 §1.9

Definition: Let S be a set of real numbers. A number ℓ is the **supremum** of S provided

- ℓ is an upper bound of S and
- if b is any upper bound of S, then $\ell \leq b$.

Theorem 6.3: For every real number r, there is a natural number n such that $n > r$.

Corollary 6.4 (Archimedean Principle): For every positive real number a and every real number b, there is some natural number n such that $na > b$.

Theorem 6.5 (Density of rational numbers): For any real numbers x, y with $x < y$, there is some rational number q such that $x < q < y$.

(1) Use the Archimedean Principle to show that for any positive number $\varepsilon > 0$, there is a natural number n such that $0 < \dfrac{1}{n} < \varepsilon$.

Let $\varepsilon > 0$. We apply the Archimedean Principle with $\varepsilon > 0$ and 1 to obtain that there is some natural number n such that $n\varepsilon > 1$. We claim that this is the n we seek. Indeed, after rearranging, we find that $\varepsilon > \dfrac{1}{n}$, and $n > 0$ implies $\dfrac{1}{n} > 0$ as well.

(2) Prove that the supremum of the set $S = \left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$ is 1.

First, we show that 1 is an upper bound for S. Indeed, and $s \in S$ can be written as $1 - \frac{1}{n}$ for some $n \in \mathbb{N}$, and since $\frac{1}{n} > 0$, we have $1 - \frac{1}{n} < 1$. Thus 1 is an upper bound.

Now we show that 1 is the smallest upper bound. Suppose that $b < 1$ is an upper bound. Let $\varepsilon = 1 - b$, which by hypothesis is positive. Then by the previous problem, there is some $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, so $1 - \frac{1}{n} > 1 - \varepsilon = b$. Thus, b is not an upper bound. It follows that 1 is the smallest upper bound, i.e., the supremum of S.

(3) Prove the following:

Corollary 7.1 (Density of irrational numbers): For any real numbers x, y with $x < y$, there is some irrational number z such that $x < z < y$.

Let $x < y$ be real numbers. Then we have $x - \sqrt{2} < y - \sqrt{2}$. By density of rationals, there is some rational number q such that $x - \sqrt{2} < q < y - \sqrt{2}$. Then $x < q + \sqrt{2} < y$. Since q is rational and $\sqrt{2}$ is irrational, $z = q + \sqrt{2}$ is irrational, and hence the number we seek.

(4) Let S be a set of real numbers, and suppose that $\sup(S) = \ell$. Let $T = \{s + 7 \mid s \in S\}$. Prove that $\sup(T) = \ell + 7$.

First, we show that $\ell + 7$ is an upper bound of T. Let $t \in T$. Then there is some $s \in S$ such that $t = s + 7$. Since $s \leq \ell$, we have $t = s + 7 < \ell + 7$, so $\ell + 7$ is indeed an upper bound. Next, let b be an upper bound for T. We claim that $b - 7$ is an upper bound for S. Indeed, if $s \in S$, then $s + 7 \in T$ so $s + 7 \leq b$, so $s \leq b - 7$. Then, by definition of supremum, we have $b - 7 \geq \ell$, so $b \geq \ell + 7$.

8. Friday, September 13, 2024 §1.7 & 1.9

Definition: For a real number x, the **absolute value** of x is

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Theorem 8.1 (Triangle Inequality): Let x, y, z be real numbers. Then

$$|x - z| \leq |x - y| + |y - z|.$$

Theorem 8.2 (Reverse triangle Inequality): Let x, y, z be real numbers. Then

$$|x - z| \geq ||x - y| - |y - z||.$$

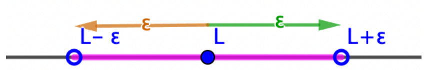
We use often use the Triangle Inequality to show precise versions of "if x is close to y and y is close to z, then x is close to z." We use often use the Reverse triangle Inequality to show precise versions of "if x is far from y and y is close to z, then x is far from z."

(1) If x and y are real numbers, what is the geometric meaning of $|x - y|$?

The distance between x and y on the real number line.

(2) We will often look at conditions like $|x - L| < \varepsilon$, where L and ε are real numbers and x is a variable. Describe $\{x \in \mathbb{R} : |x - L| < \varepsilon\}$ in interval notation. Now draw a picture of this on the real number line, showing the role of L and ε.

$(L - \varepsilon, L + \varepsilon)$.



(3) Describe $\{x \in \mathbb{R} : |3x + 7| < 4\}$ explicitly in interval notation.

Since $|3x + 7| = |3||x + \frac{7}{3}| = 3|x + \frac{7}{3}|$, we have $|3x + 7| < 4$ if and only if $|x + \frac{7}{3}| < \frac{4}{3}$, so our interval goes from $\frac{-7}{3} - \frac{4}{3} = \frac{-11}{3}$ to $\frac{-7}{3} + \frac{4}{3} = -1$. That is $(\frac{-11}{3}, -1)$.

(4) Suppose that $|x - 2| < \frac{1}{5}$, $|y - 2| < \frac{2}{5}$.
   (a) Show that $x > \frac{8}{5}$.
   (b) Show that $|x - y| < \frac{3}{5}$.
   (c) Use the reverse triangle inequality to show that $|y - 3| > \frac{3}{5}$.

(a) Since $|x-2| < \frac{1}{5}$, we have $-(x-2) \leq |x-2| < \frac{1}{5}$, so $x-2 > -\frac{1}{5}$, and $x > 2 - \frac{1}{5} = \frac{9}{5} > \frac{8}{5}$.
(b) By the triangle inequality, $|x-2| < \frac{1}{5}$ and $|y-2| < \frac{2}{5}$ implies $|x-y| < \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$.
(c) Applying the reverse triangle inequality with y, 2, and 3, we have $|y-3| \geq ||3-2|-|y-2|| = |1-|y-2||$. Since $|y-2| < \frac{2}{5}$, $1-|y-2| > 0$, so $|1-|y-2|| = 1-|y-2| > 1 - \frac{2}{5} = \frac{3}{5}$.

(5) True or false & justify[1]: There is a rational number x such that $|x^2 - 2| = 0$.

False! This would imply that $x^2 = 2$, and there is no such rational number.

(6) True or false & justify[8]: There is a rational number x such that $|x^2 - 2| < \frac{1}{1000000}$.

True! There is a rational number x in the interval $(\sqrt{2} - \frac{1}{4000000}, \sqrt{2} + \frac{1}{4000000})$ by density of rational numbers. In particular, $x < \sqrt{2} + \frac{1}{4000000} < 2$ so $|x + \sqrt{2}| < 2 + 2 = 4$. (This is a crude bound, but good enough.) Then $|x^2 - 2| = |x - \sqrt{2}| \cdot |x + \sqrt{2}| < \frac{1}{4000000} \cdot 4 = \frac{1}{1000000}$.

[8]You can use anything we've proven in class, but don't use things we haven't, like decimal expansions.

9. Monday, September 16, 2024 §2.1

We now turn our attention to the next major topic of this class: sequences of real numbers. We will spend the next few weeks developing their properties carefully and rigorously. Sequences form the foundation for much of what we will cover for the rest of the semester.

Definition 9.1. A **sequence** is an infinite list of real numbers indexed by $\mathbb{N}$:

$$a_1, a_2, a_3, \ldots .$$

(Equivalently, a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$: the value of the function at $n \in \mathbb{N}$ is written as a_n.)

We will usually write $\{a_n\}_{n=1}^{\infty}$ for a sequence.

Example 9.2. To describe sequences, we will typically give a formula for the n-th term, a_n, either an explicit one or a recursive one. On rare occasion we'll just list enough terms to make the pattern clear. Here are some examples:

(1) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is the sequence that starts

$$4, \frac{11}{2}, \frac{14}{3}, \frac{21}{4}, \frac{24}{5}, \ldots .$$

(2) Let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. This gives the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

This is an example of a recursively defined sequence. It is the famed **Fibonacci sequence**.

(3) Let $\{c_n\}_{n=1}^{\infty}$ be the sequence whose n-th term is the n-th smallest positive prime integer:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots .$$

Note that here I have not really given an explicit formula for the terms of the sequence, but it is possible to describe an algorithm that lists every term of the sequence in order.

You have all probably seen an "intuitive" definition of the limit of a sequence before. For example, you probably believe that

$$5 + (-1)^n \frac{1}{n}$$

converges to 5. Let's give the rigorous definition.

Definition 9.3. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence and L a real number. We say $\{a_n\}_{n=1}^{\infty}$ **converges** to L provided the following condition is met:

> For every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

This is an extremely important definition for this class. Learn it by heart!

In symbols, the definition is

A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided
$\forall \varepsilon > 0, \exists N \in \mathbb{R} : \forall n \in \mathbb{N}$ s.t. $n > N,\ |a_n - L| < \varepsilon$.

It's a complicated definition — three quantifiers!

Here is what the definition is saying somewhat loosely: No matter how small a number ε you pick, so long as it is positive, if you go far enough out in the sequence, all of the terms from that point on will be within a distance of ε of the limiting value L.

Example 9.4. To say that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5 gives us a different statement for every $\varepsilon > 0$. For example:

- Setting $\varepsilon = 3$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 3$. Namely, we can take $N = 0$, since for *every* term a_n of the sequence, $|a_n - 5| < 3$ holds true.
- Setting $\varepsilon = \frac{1}{3}$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < \frac{1}{3}$. We cannot take $N = 0$ anymore, since $1 > 0$ and $|a_1 - 5| = 1 > \frac{1}{3}$. However, we can take $N = 3$, since for $n > 3$, $|a_n - 5| = \frac{1}{n} < \frac{1}{3}$.
- Setting $\varepsilon = 1/1000000$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 1/1000000$. We need a bigger N; now $N = 1000000$ works.

In general, our choice of N may depend on ε, which is OK since our definition is of the form $\forall \varepsilon > 0, \exists N \ldots$ rather than $\exists N : \forall \varepsilon > 0 \ldots$.

Example 9.5. I claim the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5. I'll give a rigorous proof, along with some commentary and "scratch work" within the parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: Given this ε, our goal is to find N so that if $n > N$, then $|5 + (-1)^n \frac{1}{n} - 5| < \varepsilon$. The latter simplifies to $\frac{1}{n} < \varepsilon$, which in turn is equivalent to $\frac{1}{\varepsilon} < n$ since ε and n are both positive. So, it seems we've found the N that "works". Back to the formal proof....)

Let $N = \frac{1}{\varepsilon}$. Then $\frac{1}{N} = \varepsilon$, since ε is positive.

(Comment: We next show that this is the N that "works" in the definition. Since this involves proving something about every natural number that is bigger than N, we start by picking one.)

Pick any $n \in \mathbb{N}$ such that $n > N$. Then $\frac{1}{n} < \frac{1}{N}$ and hence

$$|a_n - 5| = |5 + (-1)^n \frac{1}{n} - 5| = |(-1)^n \frac{1}{n}| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

This proves that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. □

Remark 9.6. A direct proof that a certain sequence converges to a certain number follows the general outline:

- Let $\varepsilon > 0$ be given. (or, if your prefer, "Pick $\varepsilon > 0$.")
- Let N = [expression in terms of ε from scratch work].
- Let $n \in \mathbb{N}$ be such that $n > N$.
- [Argument that $|a_n - L| < \varepsilon$.]
- Thus $\{a_n\}_{n=1}^{\infty}$ converges to L.

In particular, you usually can't just sit down and write a proof in one fell swoop: you will have to preapre for your proof by figuring out what value of N will beat ε. The work that you use to find N in terms of ε does *not* belong in the final proof.

Example 9.7. I claim that the sequence

$$\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$$

converges to $\frac{2}{5}$. Again I'll give a proof with commentary and scratch work in parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: We need n to be large enough so that

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This simplifies to $\left|\frac{-7}{25n+5}\right| < \varepsilon$ and thus to $\frac{7}{25n+5} < \varepsilon$, which we can rewritten as $\frac{7}{25\varepsilon} - \frac{1}{5} < n$.)

Let $N = \frac{7}{25\varepsilon} - \frac{1}{5}$. We solve this equation for ε: We get $\frac{7}{25\varepsilon} = \frac{5N+1}{5}$ and hence $\frac{25\varepsilon}{7} = \frac{5}{5N+1}$, which gives finally

$$\varepsilon = \frac{7}{25N+5}.$$

(Next we show this value of N works....)

Now pick any $n \in \mathbb{N}$ is such that $n > N$. Then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| = \left|\frac{10n-5-10n-2}{25n+5}\right| = \frac{7}{25n+5}.$$

Since $n > N$, $25n+5 > 25N+5$ and hence

$$\frac{7}{25n+5} < \frac{7}{25N+5} = \varepsilon.$$

We have proven that if $n \in \mathbb{N}$ and $n > N$, then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This proves $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$. $\square$

10. Wednesday, September 18, 2024 §2.1

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to a real number L provided for every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

(1) Consider the sequence $\{a_n\}_{n=1}^{\infty} = \{2 - \frac{50}{n}\}_{n=1}^{\infty}$.
  (a) Write out the first seven terms of this sequence: use a calculator to write the decimal expansions for the numbers.
  (b) Can you find a point N such that every term in the sequence after N (that is, every a_n for $n > N$) is within 20 of 2 (that is, $|a_n - 2| < 20$)?
  (c) Can you find a point N such that every term in the sequence after N (that is, every a_n for $n > N$) is within 10 of 2 (that is, $|a_n - 2| < 10$)?
  (d) Can you find a point N such that every term in the sequence after N (that is, every a_n for $n > N$) is within 1 of 2 (that is, $|a_n - 2| < 1$)?
  (e) If ε is a positive number, can you find a point N such that every term in the sequence after N (that is, every a_n for $n > N$) is within ε of 2 (that is, $|a_n - 2| < \varepsilon$)?
  (f) Write a proof that the sequence $\{a_n\}_{n=1}^{\infty} = \{2 - \frac{50}{n}\}_{n=1}^{\infty}$ converges to 2.

(a) $-48, -23, -14.666, -10.5, -8, -6.333, -5.142$
(b) after the second term, this is true, so $N = 2$ works
(c) after the fourth term, this is true, so $N = 4$ works
(d) Now we have to do a computation: $|a_n - 2| = \frac{50}{n} < 1$ is true when $n > 50$. So $N = 50$ works.
(e) $|a_n - 2| = \frac{50}{n} < \varepsilon$ is true when $n > \frac{50}{\varepsilon}$ so $N = \frac{50}{\varepsilon}$ works.
(f) Let $\varepsilon > 0$. Take $N = \frac{50}{\varepsilon}$. Let $n > N$ be a natural number. Then $n > N$ implies

$$|a_n - 2| = \frac{50}{n} < \frac{50}{N} = \varepsilon.$$

This proves that $\{a_n\}_{n=1}^{\infty}$ converges to 1.

(2) Prove that the sequence $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 0.

(3) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose we know that $\{a_n\}_{n=1}^{\infty}$ converges to 1. Prove[9] that there is a natural number $n \in \mathbb{N}$ such that $a_n > 0$.

[9]Hint: If you know that a "for all" statement is true, you can choose any specific value for that variable and get a more specific true statement.

Take $\varepsilon = 1$. By definition of converges to 1, there is some N such that for all $n > N$, $|a_n - 1| < 1$, and in particular $a_n > 0$. So, take any natural number greater than n, and the conclusion follows.

(4) Prove or disprove: The sequence $\left\{ \dfrac{n+1}{2n} \right\}_{n=1}^{\infty}$ converges to 0.

Take $\varepsilon = 1/2$. We claim that there is no N such that for all $n > N$ we have $|a_n - 0| < 1/2$. Indeed, given N, take any n to be any natural number greater than N. Then $a_n = 1/2 + 1/2n > 1/2$, so $|a_n| > 1/2$. Thus, there is no N satisfying the desired property. This means that the sequence does not converge to $1/2$.

11. Monday, September 23, 2024 §2.1

Definition 11.1. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **convergent** if there is a number L such that $\{a_n\}_{n=1}^{\infty}$ converges to L, and **divergent** otherwise.

Example 11.2. Let's prove the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 6.2. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1 - L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. $\square$

Proposition 11.3. *If a sequence converges, then there is a unique number to which it converges.*

Proof. Recall that to show something satisfying certain properties is unique, one assumes there are two such things and argues that they must be equal. So, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to L and that also converges to M. We will prove $L = M$.

By way of contradiction, suppose $L \neq M$. Then set $\varepsilon = \frac{|L-M|}{3}$. Since we are assuming $L \neq M$, we have $\varepsilon > 0$. According to the definition of convergence, since the sequence converges to L, there is a real number N_1 such that for $n \in \mathbb{N}$ such that $n > N_1$ we have

$$|a_n - L| < \varepsilon.$$

Also according to the definition, since the sequence converges to M, there is a real number N_2 such that for $n \in \mathbb{N}$ and $n > N_2$ we have

$$|a_n - M| < \varepsilon.$$

Pick n to be any natural number larger than $\max\{N_1, N_2\}$ (which exists by Theorem 6.2). For such an n, both $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$ hold. Using the triangle inequality and these two inequalities, we get

$$|L - M| \leq |L - a_n| + |M - a_n| < \varepsilon + \varepsilon.$$

But by the choice of ε, we have $\varepsilon + \varepsilon = \frac{2}{3}|L - M|$. That is, we have deduced that $|L - M| < \frac{2}{3}|L - M|$ which is impossible. We conclude that $L = M$. □

From now on, given a sequence $\{a_n\}_{n=1}^{\infty}$ and a real number L, will we use the shorthand notation

$$\lim_{n\to\infty} a_n = L$$

to mean that the given sequence converges to the given number. For example, we showed above that

$$\lim_{n\to\infty} \frac{2n-1}{5n+1} = \frac{2}{5}.$$

But, to be clear, the statement "$\lim_{n\to\infty} a_n = L$" signifies nothing more and nothing less than the statement "$\{a_n\}_{n=1}^{\infty}$ converges to L".

Here is some terminology we will need:

Definition 11.4. Suppose $\{a_n\}_{n=1}^{\infty}$ is any sequence.

(1) We say $\{a_n\}_{n=1}^{\infty}$ is **bounded above** if there exists at least one real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$.
(2) We say $\{a_n\}_{n=1}^{\infty}$ is **bounded below** if there exists at least one real number m such that $a_n \geq m$ for all $n \in \mathbb{N}$.
(3) We say $\{a_n\}_{n=1}^{\infty}$ is **bounded** if it is both bounded above and bounded below.

Proposition 11.5. *If a sequence $\{a_n\}_{n=1}^{\infty}$ converges then it is bounded.*

Proof. Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number L. Applying the definition of "converges to L" using the particular value $\varepsilon = 1$ gives the following fact: There is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < 1$. The latter inequality is equivalent to $L - 1 < a_n < L + 1$ for all $n > N$.

Let m be any natural number such that $m > N$, and consider the finite list of numbers

$$a_1, a_2, \ldots, a_{m-1}, L + 1.$$

Let b be the largest element of this list. I claim the sequence is bounded above by b. For any $n \in \mathbb{N}$, if $1 \leq n \leq m - 1$, then $a_n \leq b$ since in this case a_n is a member of the above list and b is the largest element of this list. If $n \geq m$ then since $m > N$, we have $n > N$ and hence $a_n < L + 1$ from above. We also have $L + 1 \leq b$ (since $L + 1$ is in the list) and thus $a_n < b$. This proves $a_n \leq b$ for all n as claimed.

Now take p to be the smallest number in the list

$$a_1, a_2, \ldots, a_{m-1}, L - 1.$$

A similar argument shows that $a_n \geq p$ for all $n \in \mathbb{N}$. □

12. Wednesday, September 25, 2024 §2.1

Definition 12.1: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

(1) We say $\{a_n\}_{n=1}^{\infty}$ is **increasing** if for all $n \in \mathbb{N}$ we have $a_n \leq a_{n+1}$.
(2) We say $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if for all $n \in \mathbb{N}$, we have $a_n \geq a_{n+1}$.
(3) We say $\{a_n\}_{n=1}^{\infty}$ is **monotone** if it is either decreasing or increasing.
(4) We say $\{a_n\}_{n=1}^{\infty}$ is **strictly increasing** if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$.

(1) For each of the following sequences which of the following adjectives apply: bounded above, bounded below, bounded, (strictly) increasing, (strictly) decreasing, (strictly) monotone?
  (a) $\{\frac{1}{n}\}_{n=1}^{\infty}$
  (b) The Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ where $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
  (c) $\{(-1)^n\}_{n=1}^{\infty}$
  (d) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$.

(a) bounded, strictly decreasing, strictly monotone
(b) bounded below, increasing, monotone
(c) bounded
(d) bounded

(2) Prove or disprove: Every increasing sequence is bounded above.

False: the Fibonacci sequence is a counterexample, since we saw above that it is increasing but not bounded above.

(3) Prove or disprove: Every increasing sequence is bounded below.

True: it follows from the definition that a_1 is a lower bound.

(4) Prove or disprove: Every bounded sequence is convergent.

The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but divergent, so this is false.

(5) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence that is bounded above by 1000 and below by -1000. Show that the sequence $\left\{\frac{a_n}{n}\right\}_{n=1}^{\infty}$ converges to 0.

Suggestion: First start the way we always do when showing a sequence converges. Then see if you can use the hypothesis that $\{a_n\}_{n=1}^{\infty}$ is bounded in a useful way.

Let $\varepsilon > 0$ be arbitrary. Take $N = \frac{1000}{\varepsilon}$. Then, since $-1000 < a_n < 1000$, we have $|a_n| < 1000$ and $\left|\frac{a_n}{n}\right| < \frac{1000}{n}$ for any $n \in \mathbb{N}$, so for any natural number $n > N$, we have

$$\left|\frac{a_n}{n} - 0\right| < \frac{1000}{n} < \frac{1000}{N} = \varepsilon.$$

This shows that $\left\{\frac{a_n}{n}\right\}_{n=1}^{\infty}$ converges to 0.

13. Friday, September 27, 2024 §2.2

Example 13.1:

(1) A constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.
(2) The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

Theorem 13.2 (Limits and Algebra):
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M.

(1) If c is any real number, then $\{ca_n\}_{n=1}^{\infty}$ converges to cL.
(2) The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.
(3) The sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.
(4) If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{1}{L}$.
(5) If $M \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$.

(1) Use Theorem 13.2 and Example 13.1 to show that the sequence $\{2 + 5/n - 7/n^2\}_{n=1}^{\infty}$ converges to 2. Show every step in your argument.

The constant sequence $\{2\}_{n=1}^{\infty}$ converges to 2 by Ex 10.1 part 1. The sequence $\{5/n\}_{n=1}^{\infty} = \{5 \cdot 1/n\}_{n=1}^{\infty}$ converges to $5 \cdot 0 = 0$ by Ex 10.1 part 2 and Thm 10.2 part 1. The sequence $\{-7/n^2\}_{n=1}^{\infty} = \{-7 \cdot 1/n \cdot 1/n\}_{n=1}^{\infty}$ converges to $-7 \cdot 0 \cdot 0 = 0$ by Ex 10.1 part 2, Thm 10.2 part 1, and Thm 10.2 part 3. Thus, by Thm 10.2 part 2, the sequence $\{2 + 5/n - 7/n^2\}_{n=1}^{\infty}$ converges to $2 + 0 + 0 = 2$ by Thm 10.2 part 2.

(2) Use Theorem 13.2 and Example 13.1 to show that the sequence $\left\{\dfrac{2n+3}{3n-4}\right\}_{n=1}^{\infty}$ converges to $\dfrac{2}{3}$.

(3) Use Theorem 13.2 to show that if $\{a_n\}_{n=1}^{\infty}$ converges to L, and $\{b_n\}_{n=1}^{\infty}$ converges to M, then $\{a_n - b_n\}_{n=1}^{\infty}$ converges to $L - M$.

By Thm 10.2 part 1, $\{-b_n\}_{n=1}^{\infty}$ converges to $-M$. Then by Thm 10.2 part 2, $\{a_n - b_n\}_{n=1}^{\infty} = \{a_n + (-b_n)\}_{n=1}^{\infty}$ converges $L + (-M) = L - M$.

(4) Prove or disprove the following converse to part (2): If $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$ then $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{b_n\}_{n=1}^{\infty}$ converges to M.

> Take $\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{(-1)^{n+1}\}_{n=1}^{\infty}$. Then $\{a_n + b_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$ converges to 0, but neither $\{a_n\}_{n=1}^{\infty}$ nor $\{b_n\}_{n=1}^{\infty}$ converges.

(5) Prove or disprove: If $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence and $\{b_n\}_{n=1}^{\infty}$ is a divergent sequence, then $\{a_n + b_n\}_{n=1}^{\infty}$ is divergent.

> Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence and $\{b_n\}_{n=1}^{\infty}$ be a divergent sequence. By way of contradiction, assume that $\{a_n + b_n\}_{n=1}^{\infty}$ converges. Say that $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{a_n + b_n\}_{n=1}^{\infty}$ converges to M. Then since $b_n = (a_n + b_n) - a_n$, from the Theorem 13.2 we conclude that $\{b_n\}_{n=1}^{\infty}$ converges to $M - L$; in particular it is convergent. This is a contradiction. We conclude that $\{a_n + b_n\}_{n=1}^{\infty}$ must diverge.

(6) Prove part (1) of Theorem 10.2 in the special case $c = 2$ by following the following steps:
- Assume that $\{a_n\}_{n=1}^{\infty}$ converges to L.
- We now want to show that $\{2a_n\}_{n=1}^{\infty}$ converges to something. You know what goes next!
- Now we do some scratchwork: we want an N such that for $n > N$ we have $|2a_n - 2L| < \varepsilon$. Factor this to get some inequality with a_n. How can we use our assumption to get an N that "works"?
- Complete the proof.

> Assume that $\{a_n\}_{n=1}^{\infty}$ converges to L. Let $\varepsilon > 0$. Since $\varepsilon/2$ is a positive number, by definition of converges, there is some N such that for all $n > N$ we have $|a_n - L| < \varepsilon/2$. Then, for this N and any $n > N$, we have $|(2a_n) - 2L| = 2|a_n - L| < 2\varepsilon/2 = \varepsilon$. This shows that $\{2a_n\}_{n=1}^{\infty}$ converges to $2L$.

(7) Prove part (1) of Theorem 10.2.

(8) Prove part (2) of Theorem 10.2.

(9) Prove part (3) of Theorem 10.2.

14. Monday, September 30, 2024 §2.2

Last time we looked at:

Theorem 13.2 (Limits and algebra)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M.

(1) If c is any real number, then $\{ca_n\}_{n=1}^{\infty}$ converges to cL.
(2) The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.
(3) The sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.
(4) If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{1}{L}$.
(5) If $M \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$.

You will prove part (1) in the homework. Let's prove part (2), and you can read the rest in the notes.

Proof of (2): Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M. Let $\varepsilon > 0$. Since $\varepsilon/2 > 0$, by definition of converges, there is some N_1 such that for all $n > N_1$ we have $|a_n - L| < \varepsilon/2$. Likewise, there is some N_2 such that for all $n > N_2$ we have $|b_n - M| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$. □

First, we prove (3).

Pick $\varepsilon > 0$.

("Scratch work": The goal is to make $|a_n b_n - LM|$ small and the trick is to use that

$$\begin{aligned} |a_n b_n - LM| &= |a_n(b_n - M) + (a_n - L)M| \\ &\leq |a_n(b_n - M)| + |(a_n - L)M| \\ &= |a_n||b_n - M| + |a_n - L||M|. \end{aligned}$$

Our goal will be to take n to be large enough so that each of $|a_n||b_n - M|$ and $|a_n - L||M|$ is smaller than $\varepsilon/2$. We can make $|a_n - L|$ as small as we like and $|M|$ is just a fixed number. So, we can "take care" of the second term by choosing n big enough so that $|a_n - L| < \frac{\varepsilon}{2|M|}$. A irritating technicality here is that $|M|$ could be 0, and so we will use $\frac{\varepsilon}{2|M|+1}$ instead. The other term $|a_n||b_n - M|$ is harder to deal with since each factor varies with n. Here we use that convergent sequences are bounded so that we can find a real number X so that $|a_n| \leq X$ for all n. Then we choose n large enough so that $|b_n - M| < \frac{\varepsilon}{2X}$. Back to the proof.)

Since $\{a_n\}$ converges, it is bounded by Proposition 11.5, which gives that there is a strictly positive real number X so that $|a_n| \leq X$ for all $n \in \mathbb{N}$. Since $\{b_n\}$ converges to M and $\frac{\varepsilon}{2X} > 0$, there is a number N_1 so that if $n > N_1$ then $|b_n - M| <$

$\frac{\varepsilon}{2X}$. Since $\{a_n\}$ converges to L and $\frac{\varepsilon}{2|M|+1} > 0$, there is a number N_2 so that if $n \in \mathbb{N}$ and $n > N_2$, then $|a_n - L| < \frac{\varepsilon}{2|M|+1}$. Let $N = \max\{N_1, N_2\}$. For any $n \in \mathbb{N}$ such that $n > N$, we have

$$\begin{aligned} |a_n b_n - LM| &= |a_n(b_n - M) + (a_n - L)M| \\ &\leq |a_n(b_n - M)| + |(a_n - L)M| \\ &= |a_n||b_n - M| + |a_n - L||M| \\ &< X\frac{\varepsilon}{2X} + \frac{\varepsilon}{2|M|+1}|M| \\ &< \varepsilon. \end{aligned}$$

This proves $\{a_n \cdot b_n\}_{n=1}^{\infty}$ converges to $L \cdot M$.

Now we prove (4).

To prove this claim, pick $\varepsilon > 0$.

(Scratch work: We want to show $\left|\frac{1}{a_n} - \frac{1}{L}\right| < \varepsilon$ holds for n sufficiently large. We have

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|L - a_n|}{|a_n||L|}.$$

We can make the top of this fraction as small as we like, but the problem is that the bottom might be very small too since a_n might get very close to 0. But since a_n converges to L and $L \neq 0$ if we go far enough out, it will be close to L. In particular, if a_n is within a distance of $\frac{|L|}{2}$ of M then $|a_n|$ will be at least $\frac{|L|}{2}$. So for n sufficiently large we have $\frac{|a_n - L|}{|a_n||L|} < 2\frac{|a_n - L|}{|L|^2}$. And then for n sufficiently large we also get $|a_n - L| < \frac{|L|^2}{2\varepsilon}$. Back to the formal proof...)

Since $\{a_n\}$ converges to L and $\frac{|L|}{2} > 0$, there is an N_1 such that for $n > N_1$ we have $|a_n - L| < \frac{|L|}{2}$ and hence $|a_n| > \frac{|L|}{2}$. Again using that $\{a_n\}$ converges to M and that $\frac{\varepsilon|L|^2}{2} > 0$, there is an N_2 so that for $n > N_2$ we have $|a_n - L| < \frac{\varepsilon|L|^2}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then we have

$$\begin{aligned} \left|\frac{1}{a_n} - \frac{1}{L}\right| &= \frac{|a_n - L|}{|a_n||L|} \\ &< \frac{2}{|L|}\frac{|a_n - L|}{|L|} \\ &= 2\frac{|a_n - L|}{|L|^2} \end{aligned}$$

since $|a_n| > |L|/2$ and hence $\frac{1}{|a_n|} < \frac{2}{|L|}$. But then

$$2\frac{|a_n - L|}{|L|^2} < 2\frac{\frac{\varepsilon|L|^2}{2}}{|L|^2} = \varepsilon$$

since $|a_n - L| < \frac{\varepsilon |L|^2}{2}$. Putting these together gives
$$\left|\frac{1}{a_n} - \frac{1}{L}\right| < \varepsilon$$
for all $n > N$. This proves $\{\frac{1}{a_n}\}_{n=1}^\infty$ converges to $\frac{1}{L}$.

Finally, part (5) follows from parts (3) and (4).

The following is another useful technique:

Theorem 14.1 (The "squeeze" principle). *Suppose $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, and $\{c_n\}_{n=1}^\infty$ are three sequences such that $\{a_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ both converge to L, and $a_n \leq b_n \leq c_n$ for all n. Then $\{b_n\}_{n=1}^\infty$ also converges to L.*

Proof. Assume $\{a_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ both converge to L and that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. We need to prove $\{b_n\}_{n=1}^\infty$ converges to L.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^\infty$ converges to L there is a number N_1 such that if $n \in \mathbb{N}$ and $n > N_1$ then $|a_n - L| < \varepsilon$ and hence $L - \varepsilon < a_n < L + \varepsilon$. Likewise, since $\{c_n\}_{n=1}^\infty$ converges to L there is a number N_2 such that if $n \in \mathbb{N}$ and $n > N_2$ then $L - \varepsilon < c_n < L + \varepsilon$. Let
$$N = \max\{N_1, N_2\}.$$
If $n \in \mathbb{N}$ and $n > N$, then $n > N_1$ and hence $L - \varepsilon < a_n$, and $n > N_2$ and hence $c_n < L + \varepsilon$, and also $a_n \leq b_n \leq c_n$. Combining these facts gives that for $n \in \mathbb{N}$ such that $n > N$, we have
$$L - \varepsilon < b_n < L + \varepsilon$$
and hence $|b_n - L| < \varepsilon$. This proves $\{b_n\}_{n=1}^\infty$ converges to L. □

Example 14.2. We can use the Squeeze Theorem to give a short proof that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^\infty$ converges to 5. Note that Theorem 13.2 alone cannot be used in this example. However, from Theorem 13.2, it follows that $\{5 - \frac{1}{n}\}_{n=1}^\infty$ and $\{5 + \frac{1}{n}\}_{n=1}^\infty$ both converge to 5. Then, since
$$5 - \frac{1}{n} \leq 5 + (-1)^n \frac{1}{n} \leq 5 + \frac{1}{n}$$
for all n, our sequence also converges to 5.

When I introduced the Completeness Axiom, I mentioned that, heuristically, it is what tells us that the real number line doesn't have any holes. The next result makes this a bit more precise:

Theorem 14.3. *Every increasing, bounded above sequence converges.*

Proof. Let $\{a_n\}_{n=1}^\infty$ be any sequence that is both bounded above and increasing.

(Commentary: In order to prove it converges, we need to find a candidate number L that it converges to. Since the set of numbers occurring in this sequence is nonempty and bounded above, this number is provided to us by the Completeness Axiom.)

Let S be the set of those real numbers that occur in this sequence. (This is technically different that the sequence itself, since sequences are allowed to have repetitions but sets are not. Also, sequences have an ordering to them, but sets do not.) The set S is clearly nonempty, and it is bounded above since we assume the sequence is bounded above. Therefore, by the Completeness Axiom, S has a supremum L. We will prove the sequence converges to L.

Pick $\varepsilon > 0$. Then $L - \varepsilon < L$ and, since L is the supremum, $L - \varepsilon$ is not an upper bound of S. This means that there is an element of S that is strictly bigger than $L - \varepsilon$. Every element of S is a member of the sequence, and so we get that there is an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$.

(We will next show that this is the N that "works". Note that, in the general definition of convergence of a sequence, N can be any real number, but in this proof it turns out to be a natural number.)

Let n be any natural number such that $n > N$. Since the sequence is increasing, $a_N \leq a_n$ and hence

$$L - \varepsilon < a_N \leq a_n.$$

Also, $a_n \leq L$ since L is an upper bound for the sequence, and thus we have

$$L - \varepsilon < a_n \leq L.$$

It follows that $|a_n - L| < \varepsilon$. We have proven the sequence converges to L. □

Theorem 14.4 (Monotone Convergence Theorem)**.** *Every bounded monotone sequence converges.*

Proof. If $\{a_n\}_{n=1}^{\infty}$ is increasing, then this is the content of Theorem 14.3. If $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, consider the sequence $\{-a_n\}_{n=1}^{\infty}$. If $a_n \leq M$ for all n, then $-a_n \geq -M$ for all n, so $\{-a_n\}_{n=1}^{\infty}$ is bounded below. Also, since $a_n \geq a_{n+1}$ for all n, we have $-a_n \leq -a_{n+1}$ for all n, so $\{-a_n\}_{n=1}^{\infty}$ is increasing. Thus, by Theorem 14.3, $\{-a_n\}_{n=1}^{\infty}$ converges, say to L. Then by Theorem 13.2(1), $\{a_n\}_{n=1}^{\infty} = \{-(-a_n)\}_{n=1}^{\infty}$ converges to $-L$. □

Example 14.5. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by the formula

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

We will use the Monotone Convergence Theorem to prove that this sequence converges.

First, we need to see that the sequence is increasing. Indeed, for every n we have that $a_{n+1} = a_n + \frac{1}{a_{n+1}^2} \geq a_n$.

Next, we need to show that it is bounded above. Observe that

$$\begin{aligned} a_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\ &= 1 + 1 - \frac{1}{n}, \end{aligned}$$

so we have $a_n \leq 2$ for all n. This means that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

Hence, by the Monotone Convergence Theorem, $\{a_n\}_{n=1}^{\infty}$ converges. Leonhard Euler was particularly interested in this sequence, and was able to prove that it converges to $\frac{\pi^2}{6}$. This requires some other ideas, so we won't do that here.

15. Wednesday, October 2, 2024 §2.2

(1) Which of the following implications about sequences hold in general? Either mention a relevant theorem or give a counterexample.

(a) monotone $\implies$ convergent
(b) convergent $\implies$ bounded
(c) bounded + decreasing $\implies$ convergent
(d) increasing + convergent $\implies$ bounded
(e) convergent $\implies$ monotone
(f) bounded $\implies$ convergent

(a) False: $\{n\}_{n=1}^{\infty}$
(b) True: (Every convergent sequence is bounded.)
(c) True: Monotone Convergence Theorem
(d) True: (Every convergent sequence is bounded.)
(e) False: $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$
(f) False: $\{(-1)^n\}_{n=1}^{\infty}$

(2) Show[10] that the sequence $\left\{\dfrac{n^2 - 15\sqrt{n}\sin(n)}{3n^2}\right\}_{n=1}^{\infty}$ converges and determine to what number it converges.

(3) Prove or disprove: If $a_n^2 < 4$ and $a_n < a_{n+1}$ for all n, then $\{a_n\}_{n=1}^{\infty}$ converges.

(4) Prove that for any real number r, there exists a sequence of *rational* numbers that converges to r.
Hint: Show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of rational numbers such that $r - \frac{1}{n} < a_n < r$.

(5) Prove that if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence and $\{b_n\}_{n=1}^{\infty}$ converges to 0, then $\{a_nb_n\}_{n=1}^{\infty}$ converges to 0.

(6) Prove or disprove: The sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 1 + \frac{1}{2^3} + \cdots + \frac{1}{n^3}$ is convergent.

(7) Prove or disprove: The sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ is convergent.

[10] You can use any basic properties about the sine function from trig, like which values of $\sin(x)$ are equal to $0, 1$, or -1, and that $-1 \leq \sin(x) \leq 1$.

16. Friday, October 4, 2024 §2.2

Definition 16.1: A sequence **diverges to** $+\infty$ if for every real number M, there is some $N \in \mathbb{R}$ such that for every natural number $n > N$ we have $a_n > M$.

Definition 16.2: A sequence **diverges to** $-\infty$ if for every real number m, there is some $N \in \mathbb{R}$ such that for every natural number $n > N$ we have $a_n < m$.

(1) Use the definition to prove that the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to $+\infty$.

Let $M \in \mathbb{R}$. [Scratchwork: We need some N such that if $n > N$ then $\sqrt{n} > M$. This inequality is equivalent to $n > M^2$, so take $N = M^2$.] Take $N = M^2$. Then for any natural number $n > N$, we have $\sqrt{n} > \sqrt{M^2} \geq M$. This shows that $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to $+\infty$.

(2) Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$ then it is not bounded above.

Suppose that $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$. By way of contradiction, suppose that $\{a_n\}_{n=1}^{\infty}$ is bounded above. This means that there is some $b \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $a_n \leq b$. Apply the definition of diverges to $+\infty$ with the number $M = b$. Then there is some N such that for all $n > N$ we have $a_n > b$. But for such an n we have $a_n \leq b$ and $a_n > b$, a contradiction. We conclude that $\{a_n\}_{n=1}^{\infty}$ is not bounded above.

(3) Use (2) to show that if a sequence diverges to $+\infty$ then it diverges.

Since any convergent sequence is bounded, an unbounded sequence diverges. In particular, if $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$, then it is not bounded above, and hence divergent.

(4) Prove or disprove: If a sequence diverges, then it diverges to $+\infty$ or it diverges to $-\infty$.

This is false: the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent, but does not diverge to $+\infty$ (take $M = 1$) nor does it diverge to $-\infty$ (take $m = -1$).

(5) Prove or disprove: If a sequence is not bounded above, then it diverges to $+\infty$.

This is false: the sequence $\{(-1)^n \cdot n\}_{n=1}^{\infty}$ is not bounded above or below, but does not diverge to $-\infty$: take $M = 0$; then for any N there is some *odd* natrual number $n > N$, and for this n, $a_n = -n < 0$.

(6) Prove or disprove: If a sequence diverges to $+\infty$ then it is increasing.

(7) Prove or disprove: If a sequence is increasing and not bounded above, it diverges to ∞.

17. Monday, October 7, 2024 §1.5

We will now embark on a bit of detour. I've postponed talking about proofs by induction, but we will need to use that technique on occasion. So let's talk about that idea now.

The technique of proof by induction is used to prove that an infinite sequence of statements indexed by $\mathbb{N}$

$$P_1, P_2, P_3, \ldots$$

are all true. For example the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

holds for all $n \in \mathbb{N}$. We get one statement for each natural number:

$$\begin{aligned} P_1: \quad && 1 &= \frac{1 \cdot 2}{2} \\ P_2: \quad && 1+2 &= \frac{2 \cdot 3}{2} \\ P_3: \quad && 1+2+3 &= \frac{3 \cdot 4}{2} \\ \vdots \quad && \vdots & \end{aligned}$$

Such a fact (for all n) is well-suited to be proven by induction.

Here is the general principle:

Theorem 17.1 (Principle of Mathematical Induction)**.** *Suppose we are given, for each $n \in \mathbb{N}$, a statment P_n. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Then P_n is true for all $n \in \mathbb{N}$.*

"The domino analogy": Think of the statements $P_1, P_2, \ldots$ as dominoes lined up in a row. The fact that $P_k \implies P_{k+1}$ is interpreted as meaning that the dominoes are arranged well enough so that if one falls, then so does the next one in the line. The fact that P_1 is true is interpreted as meaning the first one has been knocked over. Given these assumptions, for every n, the n-th domino will (eventually) fall down.

The Principle of Mathematical Induction (PMI) is indeed a theorem, which we will now prove:

Proof. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Consider the subset

$$S = \{n \in \mathbb{N} \mid \ P_n \text{ is false}\}$$

of $\mathbb{N}$. Our goal is to show S is the empty set.

By way of contradiction, suppose S is not empty. Then by the Well-Ordering Principle, S has a smallest element, call it ℓ. (In other words, P_ℓ is the first statement in the list $P_1, P_2, \ldots,$ that is false.) Since P_1 is true, we must have $\ell > 1$. But then $\ell - 1 < \ell$ and so $\ell - 1$ is not in S. Since $\ell > 1$, we have $\ell - 1 \in \mathbb{N}$ and thus we can say that $P_{\ell-1}$ must be true. Since $P_k \Rightarrow P_{k+1}$ for any k, letting $k = \ell - 1$, we see that, since $P_{\ell-1}$ is true, P_ℓ must also by true. This contradicts the fact that $\ell \in S$. We conclude that S must be the empty set. $\square$

The above proof shows that the Principle of Mathematical Induction is a consequence of the Well-Ordering Principle. The converse is also true.

Example 17.2. Let's prove that the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for every natural number n. Here, P_k is

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

For P_1 we have $1 = \frac{1 \cdot 2}{2}$ is true. Now we show P_k implies P_{k+1}. Let k be a natural number and assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

Then

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

which is P_{k+1}. Thus we have proven the equality for all natural numbers n by induction. $\square$

Proposition 17.3. *Let x be a real number. For any $n \in \mathbb{N}$,*

$$(1-x)(1 + x + x^2 + x^3 + \cdots + x^n) = 1 - x^{n+1}.$$

Proof. We proceed by induction on n. First we check for $n = 1$:

$$(1-x)(1+x) = 1 + x - x - x^2 = 1 - x^{1+1},$$

so the statement is true for $n = 1$. Suppose the equality holds for k:

$$(1-x)(1 + x + x^2 + x^3 + \cdots + x^k) = 1 - x^{k+1}.$$

Then

$$\begin{aligned} &(1-x)(1 + x + x^2 + x^3 + \cdots + x^{k+1}) \\ &\quad = (1-x)((1 + x + x^2 + x^3 + \cdots + x^k) + (x^{k+1})) \\ &\quad = (1-x)(1 + x + x^2 + x^3 + \cdots + x^k) + (1-x)(x^{k+1}) \\ &\quad = 1 - x^{k+1} + x^{k+1} - x^{k+2} = 1 - x^{k+2}, \end{aligned}$$

and it holds for $k+1$. Thus, the statement is true for all n by induction. $\square$

Now, I want to apply what we've done so far to decimal expansions. Let us say that $d_1, d_2, d_3, \dots$ is a "digit sequence" if $d_i \in \{0, 1, \dots, 9\}$ for each $i \in \mathbb{N}$. We will say that an integer k and a digit sequence $d_1, d_2, d_3, \dots$ is a "decimal expansion" and write $k.d_1d_2d_3\dots$ to denote a decimal expansion. We say that the decimal expansion $k.d_1d_2d_3\dots$ "corresponds to a real number r" if the sequence

$$\begin{aligned}
a_1 &= k + \frac{d_1}{10^1} \\
a_2 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} \\
a_3 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \\
&\vdots
\end{aligned}$$

converges to r.

Theorem 17.4. *Every decimal expansion corresponds to a real number.*

Proof. Let k be an integer, and $d_1, d_2, d_3, \dots$ be a sequence such that $d_i \in \{0, 1, \dots, 9\}$ for each $i \in \mathbb{N}$. We need to show that the sequence

$$\begin{aligned}
a_1 &= k + \frac{d_1}{10^1} \\
a_2 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} \\
a_3 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \\
&\vdots
\end{aligned}$$

converges. Observe that $\{a_n\}_{n=1}^{\infty}$ is increasing. Note now that for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
a_n &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \\
&= k + \frac{9}{10^1} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} \\
&= k + \frac{9}{10}\left(1 + \frac{1}{10^1} + \cdots + \frac{1}{10^{n-1}}\right) \\
&= k + \frac{9}{10}\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \\
&= k + \frac{9}{10}\frac{1 - \frac{1}{10^n}}{\frac{9}{10}} \\
&= k + 1 - \frac{1}{10} \leq k + 1.
\end{aligned}$$

Since this holds for all n, $\{a_n\}_{n=1}^{\infty}$ is bounded above by $k + 1$. By the Monotone Convergence Theorem, $\{a_n\}_{n=1}^{\infty}$ converges. $\square$

19. Friday, October 11, 2024 §2.2, 1.5

(1) Let $a \geq 2$ be a real number.

(a) Prove that for any natural number n, the inequality $a^n \geq na$ holds true.

We proceed by induction on n. For the base case, we consider $n = 1$, where we have
$$a^1 = a = 1 \cdot a$$
and the inequality holds true. By way of induction, assume that for some natural number k, the inequality
$$a^k \geq ka$$
holds. Then
$$a^{k+1} = a^k \cdot a \geq (ka) \cdot a \geq 2ka \geq ka + ka \geq ka + a = (k+1)a$$
so the claimed inequality holds for $n = k + 1$. Thus, by induction, the inequality holds for all $n \in \mathbb{N}$.

(b) Prove that the sequence $\left\{\frac{1}{a^n}\right\}_{n=1}^{\infty}$ converges to 0.

Since $a^n \geq na$ for all n, we have
$$\frac{1}{an} \geq \frac{1}{a^n} \geq 0$$
for all n. The sequence $\frac{1}{an} = \frac{1}{a}\frac{1}{n}$ converges to 0, as does the constant sequence 0, so the given sequence converges to 0 by the Squeeze Theorem.

(2) Define a sequence $\{b_n\}_{n=1}^{\infty}$ recursively by the rule $b_1 = 0$, and $b_n = \dfrac{1 + b_{n-1}}{2}$ for $n > 1$. Prove that $b_n = 1 - \dfrac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$ and compute $\lim\limits_{n\to\infty} b_n$.

We prove the first equality by induction. For the base case $n = 1$, we have $b_1 = 0 = 1 - 1 = 0$. Suppose that the equality holds for $n = k$, so
$$b_k = 1 - 1 - \frac{1}{2^{k-1}}.$$
Then, since $k + 1 > 1$, we have
$$b_{k+1} = \frac{1 + b_k}{2} = \frac{1 + 1 - \frac{1}{2^{k-1}}}{2} = \frac{2}{2} - \frac{1}{2^{k-1} \cdot 2} = 1 - \frac{1}{2^k} = 1 - \frac{1}{2^{(k+1)-1}}.$$
Thus the claimed equality holds for $k + 1$. By induction, the equality holds for all $n \in \mathbb{N}$.

Now, we can use the previous problem plus Algebra and Limits to deduce that the sequence converges to 1.

(3) Define a sequence $\{c_n\}_{n=1}^{\infty}$ recursively by the rule $c_1 = 1$, and $c_n = \sqrt{2c_{n-1}}$ for $n > 1$.

(a) Use a calculator to write down the first 5 terms of this sequence.
(b) Prove that the sequence $\{c_n\}_{n=1}^{\infty}$ is bounded above by 2.

> Again, we use induction. $c_1 = 1 < 2$. Suppose that $c_k < 2$. Then $c_{k+1} = \sqrt{2c_k} < \sqrt{2 \cdot 2} = 2$. Thus, by induction, $c_n < 2$ for all n.

(c) Use the previous part to show that $\{c_n\}_{n=1}^{\infty}$ is an increasing sequence.

> Since $c_n < 2$, we have $c_{n+1}^2 = 2c_n > c_n^2$, and since c_n and c_{n+1} are both positive, $c_n < c_{n+1}$.

(d) Prove that the sequence $\{c_n\}_{n=1}^{\infty}$ is convergent.

> Because the sequence is monotone and bounded above, it is convergent by Monotone Convergence Theorem.

(e) What value does $\{c_n\}_{n=1}^{\infty}$ converge to? Can you prove it?

> Because the sequence is convergent, we can write the limit as L. Since c_n is a positive increasing bounded above sequence, so is $\{\sqrt{c_n}\}_{n=1}^{\infty}$. Setting $\lim \sqrt{c_n} = M$, we have $M^2 = \lim \sqrt{c_n}\sqrt{c_n} = \lim c_n = L$, and since $M > 0$, we have $M = \sqrt{L}$. Then
>
> $$L = \lim c_n = \lim c_{n+1} = \lim \sqrt{2c_n} = \lim \sqrt{2}\sqrt{c_n} = \sqrt{2}\sqrt{L}$$
>
> so $L = \sqrt{2}\sqrt{L}$, and $L^2 = 2L$; since $L > 0$, we must have $L = 2$.

20. Monday, October 14, 2024 §2.3

We next discuss the important concept of a "subsequence".

Informally speaking, a subsequence of a given sequence is a sequence one forms by skipping some of the terms of the original sequence. In other words, it is a sequence formed by taking just some of the terms of the original sequence, but still infinitely many of them, without repetition.

We'll cover the formal definition soon, but let's give a few examples first, based on this informal definition.

Example 20.1. Consider the sequence

$$a_n = \begin{cases} 7 & \text{if } n \text{ is divisible by 3 and} \\ \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

If we pick off every third term starting with the term a_3 we get the subsequence

$$a_3, a_6, a_9, \dots$$

which is the constant sequence

$$7, 7, 7, \dots.$$

If we pick off the other terms we form the subsequence

$$a_1, a_2, a_4, a_5, a_7, a_8, a_{10}, \dots$$

which gives the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{10}, \dots.$$

Note that it is a little tricky to find an explicit formula for this sequence.

On the other hand,

$$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots$$

is *not* a subsequence, because $\frac{1}{2}$ only appears once in the original sequence.

Here is the formal definition:

Definition 20.2. A **subsequence** of a given sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence of the form

$$\{a_{n_k}\}_{k=1}^{\infty}$$

where

$$n_1, n_2, n_3, \dots$$

is any strictly increasing sequence of natural numbers — that is $n_k \in \mathbb{N}$ and $n_{k+1} > n_k$ for all $k \in \mathbb{N}$, so that

$$n_1 < n_2 < n_3 < \cdots.$$

Note that k is the index of the subsequence; i.e., the first term in the subsequence is when $k = 1$, the second is when $k = 2$ and so on. The integer sequence $\{n_k\}_{k=1}^{\infty}$ is the sequence of indices of the original sequence we choose to make the subsequence.

Example 20.3. Let $\{a_n\}_{n=1}^\infty$ be any sequence.

Setting $n_k = 2k - 1$ for all $k \in \mathbb{N}$ gives the subsequence of just the odd-indexed terms of the original sequence.

Setting $n_k = 2k$ for all $k \in \mathbb{N}$ gives the subsequence of just the even-indexed terms of the original sequence.

Setting $n_k = 3k - 2$ for all $k \in \mathbb{N}$ gives the subsequence of consising of every third term of the original sequence, starting with the first.

Setting $n_k = 100 + k$ gives the subsequence that is that "tail end" of the original, obtained by skipping the first 100 terms:

$$a_{101}, a_{102}, a_{103}, a_{104}, \dots .$$

Of course, there is nothing special about 100 in this example.

The following result is important:

Theorem 20.4. *If a sequence $\{a_n\}_{n=1}^\infty$ converges to L, then every subsequence of this sequence also converges to L.*

We prepare with a lemma.

Lemma 20.5. *Let $b_1, b_2, \dots$ be any strictly increasing sequence of natural numbers; that is, assume $b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ and that $b_k < b_{k+1}$ for all $k \in \mathbb{N}$. Then $b_k \geq k$ for all k.*

Proof. Suppose $b_1, b_2, \dots$ is a strictly increasing sequence of natural numbers. We prove $b_n \geq n$ for all n by induction on n. That is, for each $n \in \mathbb{N}$, let P_n be the statement that $b_n \geq n$.

P_1 is true since $b_1 \in \mathbb{N}$ and so $b_1 \geq 1$. Given $k \in \mathbb{N}$, assume P_k is true; that is, assume $b_k \geq k$. Since $b_{k+1} > b_k$ and both are natural numbers, we have $b_{k+1} \geq b_k + 1 \geq k + 1$; that is, P_{k+1} is true too. By induction, P_n is true for all $n \in \mathbb{N}$. □

Proof of Theorem 20.4. Let the sequence $\{a_n\}_{n=1}^\infty$ converge to L, and take a subsequence $\{a_{n_k}\}_{k=1}^\infty$ for some strictly increasing sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers.

Let $\varepsilon > 0$. Since $\{a_n\}_{n=1}^\infty$ converges to L, there is some $N \in \mathbb{R}$ such that for all natural numbers $n > N$ we have $|a_n - L| < \varepsilon$. We claim that the same N works to verify the definition of $\{a_{n_k}\}_{k=1}^\infty$ converges to L for this ε. Indeed, if $k > N$, then $n_k > N$, so $|a_{n_k} - L| < \varepsilon$. Thus, $\{a_{n_k}\}_{k=1}^\infty$ converges to L. □

(1) **True or false; justify.**
   (a) The sequence $\left\{\dfrac{1}{3n+7}\right\}_{n=1}^\infty$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^\infty$.
   (b) The constant sequence $\left\{\dfrac{1}{2}\right\}_{n=1}^\infty$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^\infty$.
   (c) The constant sequences $\{-1\}_{n=1}^\infty$ and $\{1\}_{n=1}^\infty$ are both subsequences of the sequence $\{(-1)^n\}_{n=1}^\infty$.
   (d) The constant sequences $\{-1\}_{n=1}^\infty$ and $\{1\}_{n=1}^\infty$ are the only two subsequences of the sequence $\{(-1)^n\}_{n=1}^\infty$.
   (e) The sequence $\{\sin(\pi n)\}_{n=1}^\infty$ is a subsequence of $\{\sin(n)\}_{n=1}^\infty$.

(a) True: Take $n_k = 3k + 7$.
(b) False: The term $1/2$ occurs only for $n = 2$, so we can't choose an increasing sequence of indices that yield this value.
(c) True: take $n_k = 2k + 1$ and take $n_k = 2k$, respectively.
(d) False: The sequence itself is a subsequence ($n_k = k$) for example.
(e) False: $\sin(\pi n)$ is $1, 0, -1, 0, 1, 0, -1, 0, \ldots$, but the values of $\sin(n)$ are never 0 or ± 1.

(2) Explain how the following Corollary follows from Theorem 20.4.
Corollary 20.6: Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.
(a) If there is a subsequence of this sequence that diverges, then the sequence itself diverges.
(b) If there are two subsequences of this sequence that converge to different values, then the sequence itself diverges.

These are special cases of the contrapositive.

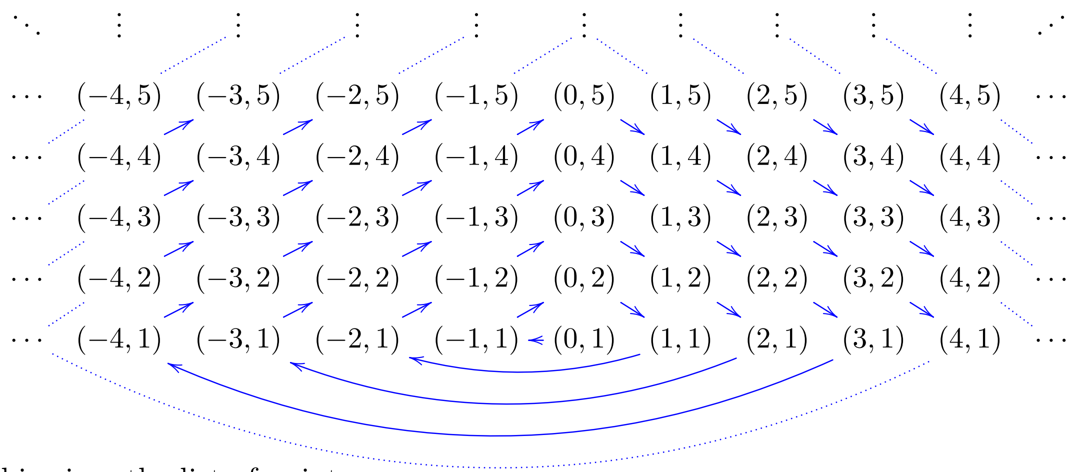
(3) Use Corollary 20.6 to give a quick proof that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

It has subsequences that converge to different values, as we saw above.

(4) **Prove or disprove:**
(a) Every subsequence of a bounded sequence is bounded.
(b) Every subsequence of a divergent sequence is divergent.
(c) Every subsequence of a sequence that diverges to $-\infty$ also diverges to $-\infty$.

(a) True: if $m < a_n < M$ for all n and $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, then $m < a_{n_k} < M$ for all k.
(b) False: The divergent sequence $\{(-1)^n\}_{n=1}^{\infty}$ has a convergent subsequence $\{1\}_{n=1}^{\infty}$.
(c) True: Let $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of natural numbers, and $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence. Let $m \in \mathbb{R}$. There is some N such that $a_n < m$ for all $n > N$. We claim that this N works (for this m) to show that $\{a_{n_k}\}_{k=1}^{\infty}$ diverges to $-\infty$. Indeed, if $k > N$, then $n_k \geq k > N$, so $a_{n_k} < m$. Thus $\{a_{n_k}\}_{k=1}^{\infty}$ diverges to $-\infty$.

21. Wednesday, October 16, 2024 §2.3



This gives the list of points

$$(0,1),(-1,1),(0,2),(1,1),(-2,1),(-1,2),(0,3),(1,2),(2,1),(-3,1),\dots$$

Now convert these to a list of rational numbers by changing (m,n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1},\frac{-1}{1},\frac{0}{2},\frac{1}{1},\frac{-2}{1},\frac{-1}{2},\frac{0}{3},\frac{1}{2},\frac{2}{1},\frac{-3}{1},\dots$$

of rational numbers. Call this sequence $\{w_n\}_{n=1}^{\infty}$.

(1) True or false: Every rational number occurs in this sequence. That is, for every $q \in \mathbb{Q}$, there is some $n \in \mathbb{N}$ such that $w_n = q$.

True: The idea is that writing $q = \frac{m}{n}$, the point (m,n) with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ gets passed through by the zigzag at least once.

(2) True or false: Every rational number occurs in this sequence infinitely many times. That is, for every $q \in \mathbb{Q}$, there are *infinitely many* natural numbers $n \in \mathbb{N}$ such that $w_n = q$.

True: We can write $q = \frac{m}{n} = \frac{2m}{2n} = \frac{3m}{3n} = \cdots$, and each point (km, kn) gets passed through by the zigzag.

(3) True or false: For every rational number $q \in \mathbb{Q}$, the constant sequence $\{q\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.

True: Since q occurs infinitely many times, there is a strictly increasing sequence of indices n_k such that $w_{n_k} = q$.

(4) True or false: For every real number $r \in \mathbb{R}$, the constant sequence $\{r\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.

> False: Every term in the sequence is rational.

(5) True or false: Every sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$.

> True: there is some n_1 such that $w_{n_1} = q_1$. Then since q_2 occurs infinitely many times, it must occur with some index larger than n_1, so there is some $n_2 > n_1$ with $w_{n_2} = q_2$. Continuing like this, we obtain some $\{n_k\}$ strictly increasing such that $q_k = w_{n_k}$.

(6) True or false: For every real number $r \in \mathbb{R}$, there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ that converges to r.

> True: For every real number r, there is a sequence of rational numbers that converges to r, and any such sequence is a subsequence of $\{w_n\}$.

THEOREM 21.1: There is a sequence of rational numbers $\{w_n\}_{n=1}^{\infty}$ such that

(1) every rational number occurs in $\{w_n\}_{n=1}^{\infty}$ infinitely many times;
(2) every sequence of rational numbers is a subsequence of $\{w_n\}_{n=1}^{\infty}$; and
(3) every real number occurs as the limit of some subsequence of $\{w_n\}_{n=1}^{\infty}$.

Theorem 21.2 (Cantor's Theorem): There is no sequence of real numbers such that every real number r occurs in the sequence.

By contradiction, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that every real number occurs as some a_n and write out the decimal expansions (where each $d_{i,j} \in \{0, 1, \ldots, 9\}$ is a digit).

$$\begin{aligned}
a_1 &= (\text{integer part}).d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}d_{1,6}\cdots \\
a_2 &= (\text{integer part}).d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}d_{2,6}\cdots \\
a_3 &= (\text{integer part}).d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}d_{3,6}\cdots \\
a_4 &= (\text{integer part}).d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}d_{4,6}\cdots \\
a_5 &= (\text{integer part}).d_{5,1}d_{5,2}d_{5,3}d_{5,4}d_{5,5}d_{5,6}\cdots \\
a_6 &= (\text{integer part}).d_{6,1}d_{6,2}d_{6,3}d_{6,4}d_{6,5}d_{6,6}\cdots \\
a_7 &= (\text{integer part}).d_{7,1}d_{7,2}d_{7,3}d_{7,4}d_{7,5}d_{7,6}\cdots \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Proof. Form a real number x as $0.e_1e_2e_3\cdots$ where the e_i's are digits chosen as follows: Let

$$e_i = \begin{cases} 7 & \text{if } d_{i,i} \le 5 \\ 3 & \text{if } d_{i,i} > 5. \end{cases}$$

In particular, $e_i \neq d_{i,i}$ for every i. This means that the digit sequence $e_1, e_2, e_3, \ldots$ is not equal to any of the other digit sequences $d_{i,1}, d_{i,2}, d_{i,3}, \ldots$ for any i, because the i-th values are different. Moreover, the number x has a unique decimal expansion (since the only time two decimal expansions give the same number is one is eventually all 0's and the other is eventually all 9's), so $a_i \neq x$ for all $i \in \mathbb{N}$.

Thus x is not a member of this sequence, contrary to what we assumed. □

22. Friday, October 18, 2024 §2.3

Our next big theorem has a very short statement, but is surprisingly tricky to prove.

Theorem 22.1 (Bolzano-Weierstrass Theorem)**.** *Every sequence has a monotone subsequence.*

The proof of this theorem requires a preliminary lemma.

Lemma 22.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.*

(1) If the set of values of the sequence $\{a_n \mid n \in \mathbb{N}\}$ does not have a maximum value, then $\{a_n\}_{n=1}^{\infty}$ has a subsequence that is strictly increasing.
(2) If the set of values of the sequence $\{a_n \mid n \in \mathbb{N}\}$ does not have a minimum value, then $\{a_n\}_{n=1}^{\infty}$ has a subsequence that is strictly decreasing.

Proof. (1) Assume that the set of values $\{a_n \mid n \in \mathbb{N}\}$ does not have a maximum value.

We define a subsequence recursively. We will recursively choose natural numbers $n_1, n_2, n_3, \ldots$ so that $n_k < n_{k+1}$ for all k and $a_{n_k} \leq a_{n_{k+1}}$.

We start by setting $n_1 = 1$. Once we have chosen n_k, we need to find some natural number $n_{k+1} > n_k$ such that $a_{n_{k+1}} \leq a_{n_k}$. Let $b = \max\{a_1, a_2, \ldots, a_{n_k}\}$. This is some value of the sequence; since the sequence attains no maximum value, there is some m such that $a_m > b$. We must have $m > n_k$, because if $m \leq n_k$, then a_m is on the list $a_1, a_2, \ldots, a_{n_k}$, so cannot be larger than b. Thus, we can take $m = n_{k+1}$.

This gives a recursive definition for a strictly increasing subsequence.

(2) Similar to (1), or apply (1) to $\{-a_n\}_{n=1}^{\infty}$. □

Proof of Bolzano-Weierstrass Theorem 22.1. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence. Recall that our goal is to prove it either has an increasing subsequence or it has a decreasing subsequence. This is equivalent to showing that if it has no increasing subsequences, then it does have at least one decreasing subsequence. So, let us assume it has no increasing subsequences.

We will prove it has at least one decreasing subsequence by constructing the indices $n_1 < n_2 < \cdots$ of such a subsequence recursively. By the contrapositive of part (1) Lemma 22.2 , since $\{a_n\}_{n=1}^{\infty}$ does not contain any increasing subsequences, we know that $\{a_n \mid n \in \mathbb{N}\}$ has a maximum value. That is, there exists a natural number n_1 such that $a_{n_1} \geq a_m$ for all $m \geq 1$.

Once we have chosen up through n_k, note that the subsequence $a_{n_k+1}, a_{n_k+2}, a_{n_k+3}, \ldots$ also has no increasing subsequence, since a subsequence of such a sequence is a subsequence of the original sequence too. Thus, it must have a maximum value again by part (1) Lemma 22.2; choose n_{k+1} such that $a_{n_{k+1}} = \max\{a_{n_k+1}, a_{n_k+2}, a_{n_k+3}, \ldots\}$. By construction, we have $n_{k+1} > n_k$. Furthermore, a_{n_k} is the maximum of a set that contains $a_{n_{k+1}}$ (since it is later in the sequence). It follows that $a_{n_k} \geq a_{n_{k+1}}$. Thus, we have constructed recursively a decreasing subsequence of the original sequence. □

Corollary 22.3 (Main Corollary of Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem 22.1 it admits a monotone subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, and it too is bounded (since any subsequence of a bounded sequence is also bounded.) The result follows since every monotone bounded sequence converges by the Monotone Convergence Theorem 14.4. $\square$

You can use any basic trig facts below to answer the following questions.

- $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$
- $\cos(x) = 1 \iff x \in 2\pi\mathbb{Z}$
- $\cos(x) = 0 \iff x \in \frac{\pi}{2} + \pi\mathbb{Z}$
- $\cos(x) = -1 \iff x \in \pi + 2\pi\mathbb{Z}$
- $\pi \notin \mathbb{Q}$
- $\cos(x) = \cos(y) \iff x - y \in 2\pi\mathbb{Z}$ or $x + y \in 2\pi\mathbb{Z}$

(1) **Prove or disprove:** Is $\{\cos(\pi n)\}_{n=1}^{\infty}$ a subsequence of $\{\cos(n)\}_{n=1}^{\infty}$?

(2) **Prove or disprove:** The sequence $\{\cos(n)\}_{n=1}^{\infty}$ has a convergent subsequence.

(3) **Prove or disprove:** The sequence $\{\cos(n)\}_{n=1}^{\infty}$ has a constant subsequence.

(4) **Prove or disprove:** The sequence $\{\cos(n)\}_{n=1}^{\infty}$ has a subsequence that converges to some $x > 1$.

(a) No; to get a subsequence we would need have natural numbers inside the cosine, not multiples of π.
(b) True: $\cos(n)$ is bounded, so there is a convergent subsequence by Main Corollary of Bolzano-Weierstrass.
(c) False: in fact, $\cos(n)$ never takes the same value twice. If it did, we would have $\cos(n) = \cos(m)$ for natural numbers $m \neq n$, so $m - n = 2\pi k$ or $m + n = 2\pi k$, for some integer k, which would make $\pi = \frac{m-n}{2k}$ or $\pi = \frac{m+n}{2k}$, contradicting that π is irrational.
(d) False: if there is a subsequence converging to $x > 1$, let $\varepsilon = x - 1 > 0$. Then for some K, for all $k > K$, $|\cos(n_k) - 1| < \varepsilon$, which implies $\cos(n_k) > 1$, which is a contradiction.

23. Wednesday, October 23, 2024 §3.1

Reminder on functions: Given any two sets S and T, a **function** from S to T, written $f : S \to T$, is an assignment to each element $s \in S$ a unique element $t \in T$. The set S is called the **domain** of f. We will usually consider functions from some set of real numbers to $\mathbb{R}$. We often[11] specify functions by formulas; when we do this we take the domain to be the set of all real numbers for which the formula evaluates to a unique real number. In particular,

$$f(x) = 2x + 2 \quad \text{and} \quad g(x) = \frac{2x^2 - 2}{x - 1}$$

are *not* the same function, even though their values agree for all $x \neq 1$, since their domains differ.

Definition 23.1: Let S be a subset of $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be a function, and a and L be real numbers. We say that **the limit of** f **as** x **approaches** a **is** L provided:

> for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then x is in the domain of f and $|f(x) - L| < \varepsilon$.

If this happens, we write $\lim_{x \to a} f(x) = L$ to denote this.

(1) Unpackaging parts of the definition.
- (a) Describe $\{x \in \mathbb{R} \mid 0 < |x - 2| < 1\}$ as a union of two open intervals.
- (b) For a general $a \in \mathbb{R}$ and $\delta > 0$, describe $\{x \in \mathbb{R} \mid 0 < |x - a| < \delta\}$ as a union of two open intervals.
- (c) Focusing on the "domain" part of the definition, if the limit of f as x approaches a is L, then f must at least be defined ____________ (where?).

- (a) $(1, 2) \cup (2, 3)$
- (b) $(a - \delta, a) \cup (a, a + \delta)$.
- (c) on some open intervals to the left and to the right of a.

(2) The $\varepsilon - \delta$ game.
- (a) Player 0 starts by graphing a function f (like a familiar one from calculus) and specifies an x-value a and a y-value L that (based on previous calculus knowledge) they think makes $\lim_{x \to a} f(x) = L$ **true**. [The graph should be large.]
- (b) Player 1 choses an ε. This is how close we would like our function to be to L. Thus, ε goes up and down from L (corresponding to $|f(x) - L| < \varepsilon$). Draw horizontal dotted lines with y-values $L - \varepsilon$ and $L + \varepsilon$. [The ε should be large enough for people to see and have room to work in the picture.]
- (c) Player 2 must find a δ such that every $x \in (a - \delta, a) \cup (a, a + \delta)$ is
  - in the domain of f, and

[11] Beware: not every function has a formula!

- has an output $f(x)$ within $(L-\varepsilon, L+\varepsilon)$.

Draw vertical dotted lines for the x-values $a-\delta$ and $a+\delta$. [Everyone in the team can assist player 2!]

(d) Repeat with the same graph, players 1& 2 switching roles (and a new ε).

(3) Draw the graph of $g(x) = \dfrac{2x^2-2}{x-1}$. Play the $\varepsilon-\delta$ game with this function, $a=1$ and $L=-3$. What happens?

So long as $\varepsilon < 7$, it is impossible for Player 2.

(4) Consider the function $g(x) = \dfrac{2x^2-2}{x-1}$. It is true that $\lim_{x\to 1} g(x) = 4$.
(a) I claim that for $\varepsilon = 3$, the choice $\delta = 1.5$ "works" to make the rest of the definition true. Verify this.
(b) Find a δ that "works" for $\varepsilon = 1$.
(c) Find a δ that "works" for $\varepsilon = 1/2$.
(d) Find a δ that "works" for $\varepsilon > 0$.

(a) Let $0 < |x-1| < 1.5$, so $-.5 < x < 2.5$ and $x \neq 1$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2 \cdot 1.5 = 3 = \varepsilon$ since $|x-1| < 1.5$.
(b) Take $\delta = .5$. Let $0 < |x-1| < .5$, so $.5 < x < 1.5$ and $x \neq 1$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2 \cdot .5 = 1 = \varepsilon$ since $|x-1| < .5$.
(c) Take $\delta = .25$. Let $0 < |x-1| < .25$, so $.75 < x < 1.25$ and $x \neq 1$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2 \cdot .25 = .5 = \varepsilon$ since $|x-1| < .25$.
(d) Take $\delta = \varepsilon/2$. Let $0 < |x-1| < \varepsilon/2$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2 \cdot \varepsilon/2 = \varepsilon$ since $|x-1| < \varepsilon/2$.

(5) Consider the function $g(x) = \dfrac{2x^2-2}{x-1}$. It is not true that $\lim_{x\to 1} g(x) = -3$. I claim that for $\varepsilon = 1$, there is no choice of $\delta > 0$ that "works" to make the rest of the definition true. Verify this.

Let $\delta > 0$. Take $x = 1 + \delta/2$. Then $|x-1| = \delta/2$ is between 0 and δ, and $f(x) = 2x+2 = 4+\delta > 4$, so $|f(x)-(-3)| = |f(x)+3| > 7 > 1 = \varepsilon$.

24. Friday, October 25, 2024 §3.1

Example 24.1. Let f be the function given by the formula

$$f(x) = \frac{5x^2 - 5}{x - 1}.$$

Recall our convention that we interpret the domain of f to be all real numbers where this rule is defined. So, $f : S \to \mathbb{R}$ where $S = \mathbb{R} \setminus \{1\}$. I claim that the limit of $f(x)$ as x approaches 1 is 10. To prove it:

Pick $\varepsilon > 0$.

(Scratch work: Since f is defined at all points other that 1, the condition about f being defined for all x such that $0 < |x - a| < \delta$ will be met for any choice of δ. We need $|f(x) - 10| < \varepsilon$ to hold. Manipulating this a bit, we see that it is equivalent to $|x - 1| < \frac{\varepsilon}{5}$. Thus setting $\delta = \frac{\varepsilon}{5}$ is the way to go. Back to the proof....)

Let $\delta = \frac{\varepsilon}{5}$. Pick x such that $0 < |x - 1| < \delta$. Then $x \neq 1$ and hence f is defined at x. We have

$$\begin{aligned} |f(x) - 10| &= \left|\frac{5x^2 - 5}{x - 1} - 10\right| = \left|\frac{5x^2 - 5 - 10x + 10}{x - 1}\right| = \left|\frac{5x^2 - 10x + 5}{x - 1}\right| \\ &= \left|\frac{5(x^2 - 2x + 1)}{x - 1}\right| = \left|\frac{5(x - 1)^2}{x - 1}\right| = |5x - 5| = 5|x - 1| < 5\delta = \varepsilon. \end{aligned}$$

We have shown that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then f is defined at x and $|f(x) - 10| < \varepsilon$. This proves $\lim_{x\to 1} f(x) = 10$.

Example 24.2. Let's do a more complicated example: Let $f(x) = x^2$ with domain all of $\mathbb{R}$. I claim that $\lim_{x\to 2} x^2 = 4$. This is intuitively obvious but we need to prove it using just the definition.

Proof. Pick $\varepsilon > 0$.

(Scratch work: The domain of f is all of $\mathbb{R}$ and so we don't need to worry at all about whether f is defined at all. We need to figure out how small to make δ so that if $0 < |x - 2| < \delta$ then $|x^2 - 4| < \varepsilon$. The latter is equivalent to $|x - 2||x + 2| < \varepsilon$. We can make $|x - 2|$ arbitrarily small by making δ aribitrarily small, but how can we handle $|x+2|$? The trick is to bound it appropriately. This can be done in many ways. Certainly we can choose δ to be at most 1, so that if $|x - 2| < \delta$ then $|x - 2| < 1$ and hence $1 < x < 3$, so that $|x + 2| < 5$. So, we will be allowed to assume $|x + 2| < 5$. Then $|x-2||x+2| < 5|x-2|$ and $5|x-2| < \varepsilon$ provided $|x-2| < \frac{\varepsilon}{5}$. Back to the formal proof...)

Let $\delta = \min\{\frac{\varepsilon}{5}, 1\}$. Let x be any real number such that $0 < |x - 2| < \delta$. Then certainly f is defined at x. Since $\delta \leq 1$ we get $|x - 2| < 1$ and hence $|x + 2| \leq 5$. Since $\delta \leq \frac{\varepsilon}{5}$ we have $|x - 2| < \frac{\varepsilon}{5}$. Putting these together gives

$$|f(x) - 4| = |x^2 - 4| = |x - 2||x + 2| < |x - 2|5 < \frac{\varepsilon}{5}5 = \varepsilon.$$

This proves $\lim_{x\to 2} x^2 = 4$. $\square$

25. Monday, October 28, 2024 §3.1

(1) Let $b \in \mathbb{R}$ be a real number. Use the $\varepsilon - \delta$ definition of limit to prove that for any $a \in \mathbb{R}$,
$$\lim_{x \to a} b = b.$$

Let $\varepsilon > 0$. Take $\delta = 1$ (or $\delta = \varepsilon$, or δ is literally anything positive). Then if x is a real number such that $0 < |x - a| < \delta$, then f is defined at x because the domain is all real numbers, and $|f(x) - b| = |b - b| = 0 < \varepsilon$. This shows that $\lim\limits_{x \to a} b = b$.

(2) Let $m, b \in \mathbb{R}$ be real numbers. Use[12] the $\varepsilon - \delta$ definition of limit to prove that for any $a \in \mathbb{R}$,
$$\lim_{x \to a} mx + b = ma + b.$$

The case where $m = 0$ follows from the previous part, so assume $m \neq 0$. Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{|m|}$. Since $\varepsilon > 0$ and $|m| > 0$, this is greater than zero. Then if x is a real number such that $0 < |x - a| < \delta$, then f is defined at x because the domain is all real numbers, and
$$|f(x) - (ma + b)| = |(mx + b) - (ma + b)| = |m(x - a)| = |m| \cdot |x - a| < |m|\delta = \varepsilon.$$
This shows that $\lim\limits_{x \to a} f(x) = ma + b$.

(3) In this problem we will prove that the function $f(x) = \dfrac{1}{x-3}$ does not have a limit as x approaches 3.
   (a) What proof technique should we use? Write down the start of the proof.
   (b) If $\lim_{x \to a} f(x) = L$ then for any positive number ε that we choose, we get a more specific true statement as a consequence of the definition. Write down what statement we get when $\varepsilon = 1$.
   (c) Explain why there exists some real number x such that $3 < x < \min\{4, 3 + \delta\}$.
   (d) Use the number x from the previous part to show that $L > 0$.
   (e) Do something else to show that $L < 0$ and conclude the proof.

By way of contradiction, suppose that there is some $L \in \mathbb{R}$ such that $\lim_{x \to 3} f(x) = L$. In particular, taking $\varepsilon = 1$, by definition of limit there is some $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|\frac{1}{x-3} - L| < 1$.

Since $\delta > 0$, both $3 + \delta$ and 4 are larger than 3, so there is some real number x such that $3 < x < \min\{3+\delta, 4\}$. For this x, we have $0 < x-3 < 1$, so $\frac{1}{x-3} > 1$. For this same x, we also have $|\frac{1}{x-3} - L| < 1$. Rearranging the latter, we have $\frac{1}{x-3} - 1 < L < \frac{1}{x-3} + 1$, so in particular $L > 0$.

[12]Suggestion: You may want to consider the case where $m = 0$ separately..

On the other hand, both $3-\delta$ and 2 are less than 3, so there is some real number x such that $3 > x > \max\{3-\delta, 2\}$. For this x, we have $-1 < x-3 < 0$, so $\frac{1}{x-3} < -1$. For this same x, we also have $|\frac{1}{x-3} - L| < 1$. Rearranging the latter, we have $\frac{1}{x-3} - 1 < L < \frac{1}{x-3} + 1$, so in particular $L < 0$.

We have obtained mutually exclusive conditions on L, which is a contradiction. It follows that no such L can exist.

(4) Prove that the limit of f as x approaches a, if it exists, is unique.

By way of contradiction, suppose that there is some function f, and real numbers a, L, M, such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, with $L \neq M$. Set $\varepsilon = \frac{|L-M|}{3}$, which is positive. Then there is some $\delta_1 > 0$ such that if $0 < |x-a| < \varepsilon$, then f is defined and $|f(x) - L| < \varepsilon$. There is also some $\delta_2 > 0$ such that if $0 < |x-a| < \varepsilon$, then f is defined and $|f(x) - M| < \varepsilon$. Take x such that $0 < |x-a| < \min\{\delta_1, \delta_2\}$. Then for this x, both $|f(x) - L| < \frac{|L-M|}{3}$ and $|f(x) - M| < \frac{|L-M|}{3}$ are true. But them by the triangle inequality, $|L-M| \leq |f(x) - L| + |f(x) - M| < 2\frac{|L-M|}{3}$, a contradiction. Thus the limit of f as x approaches a, if it exists, must be unique.

(5) Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Use the definition to show that $\lim_{x\to a} f(x)$ does not exist for any real number a.

26. Wednesday, October 30, 2024 §3.2

Theorem 26.1: Let $f(x)$ be a function and let a be a real number. Let $r > 0$ be a positive real number such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$. Let L be any real number.

Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a and satisfies $0 < |x_n - a| < r$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Theorem 26.2. (Algebra and limits of functions): Suppose f and g are two functions and that a is a real number, and assume that

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

for some real numbers L and M. Then

(1) $\lim_{x\to a}(f(x) + g(x)) = L + M$.
(2) For any real number c, $\lim_{x\to a}(c \cdot f(x)) = c \cdot L$.
(3) $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$.
(4) If, in addition, we have that $M \neq 0$, then $\lim_{x\to a}(f(x)/g(x)) = L/M$.

Theorem 26.3. (Squeeze Theorem for limits): Suppose f, g, and h are three functions and a is a real number. Suppose there is a positive real number $r > 0$ such that

- each of f, g, h is defined on $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$,
- $f(x) \leq g(x) \leq h(x)$ for all x such that $0 < |x-a| < r$, and
- $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ for some number L.

Then $\lim_{x\to a} g(x) = L$.

(1) Use Theorem 26.1 to show that $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist.

Suggestion: Let $f(x) = \sin(\frac{1}{x})$ and suppose $\lim_{x\to 0} f(x) = L$. Find sequences $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ such that
- $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ both converge to 0,
- $f(x_n) = 1$ for all n, and
- $f(y_n) = -1$ for all n.

You can use any trig facts on the bottom of the page.

Suppose $\lim_{x\to 0} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty} = \{\frac{1}{\frac{\pi}{2}+2\pi n}\}_{n=1}^{\infty}$. This sequence converges to 0 and $f(x_n) = 1$ for all n, so $\{f(x_n)\}_{n=1}^{\infty}$ converges to 1. Thus, $L = 1$. Now let $\{y_n\}_{n=1}^{\infty} = \{\frac{1}{\frac{-\pi}{2}+2\pi n}\}_{n=1}^{\infty}$. This sequence converges to 0 and $f(y_n) = -1$ for all n, so $\{f(y_n)\}_{n=1}^{\infty}$ converges to -1. Thus, $L = -1$. This is a contradiction, so no such L exists.

(2) Use Theorem 26.2 plus a fact from last time[13] to compute $\lim_{x\to 2} \dfrac{3x^2 - x + 2}{x+3}$.

We have $\lim_{x\to 2} x = 2$ and the limit of a constant is the value of that constant. Thus $\lim_{x\to 2} x + 3 = 2 + 3 = 5$, and $\lim_{x\to 2} x^2 = (\lim_{x\to 2} x)^2 = 4$, so $\lim_{x\to 2} 3x^2 - x + 2 = 3 \cdot 4 - 2 + 2 = 12$, and hence $\lim_{x\to 2} \dfrac{3x^2 - x + 2}{x+3} = 5$.

(3) Use Theorem 26.3 to show that $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$. You can use any trig facts on the bottom of the page.

We have $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so $-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$. We know that $\lim_{x\to 0} |x| = 0$ and hence $\lim_{x\to 0} -|x| = 0$ by the Theorem on algebra of limits of functions. Then by the Squeeze theorem for functions, $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$.

(4) Use Theorem 26.1 to deduce Theorem 26.3 from our Squeeze Theorem for sequences.

Proof. Let f, g, h, a, r, L be as in the statement. Let $\{x_n\}_{n=1}^\infty$ be a sequence that converges to a and such that $0 < |x_n - a| < r$ for all n. By Theorem 26.1, it suffices to show that $\lim_{n\to\infty} g(x_n) = L$. By Theorem 26.1, we know that $\lim_{n\to\infty} f(x_n) = L = \lim_{n\to\infty} h(x_n)$. Since $f(x_n) \leq g(x_n) \leq h(x_n)$ for all n, we have $\lim_{n\to\infty} g(x_n) = L$ by the Squeeze Theorem (for sequences). $\square$

(5) Use Theorem 26.1 to deduce Theorem 26.2 part (1) from our Theorem on algebra and sequences.

Proof. First, as a technical matter, we note that since we assume $\lim_{x\to a} f(x) = L$ there is a positive real number r_1 such that $f(x)$ is defined for all x satisfying $0 < |x - a| < r_1$, and likewise since $\lim_{x\to a} g(x) = M$ there is a positive real number r_2 such that $g(x)$ is defined for all x satisfying $0 < |x - a| < r_2$. Letting $r = \min\{r_1, r_2\}$, we have that $r > 0$ and $f(x)$ and $g(x)$ and hence $f(x) + g(x)$ are defined for all x satisfying $0 < |x - a| < r$. (We needed to prove this in order to apply Theorem 26.1.)

Let $\{x_n\}_{n=1}^\infty$ be any sequence converging to a such that $0 < |x_n - a| < r$ for all n. By Theorem 26.1 in the "forward direction", we have that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. By limits and algebra theorem for sequences,

[13] $\lim_{x\to a}\ mx + b = ma + b$. In particular, $\lim_{x\to a}\ x = a$ and $\lim_{x\to b}\ b = b$.

$\lim_{n\to\infty} f(x_n) + g(x_n) = L + M$. So, by Theorem 26.1 again (this time applying it to $f(x) + g(x)$ and using the "backward implication"), it follows that $\lim_{x\to a}(f(x) + g(x)) = L + M$. $\square$

(6) Use Theorem 26.1 to deduce Theorem 26.2 part (4) from our Theorem on algebra and sequences.

Proof. Since we assume $\lim_{x\to a} f(x) = L$ there is a positive real number r_1 such that $f(x)$ is defined for all x satisfying $0 < |x - a| < r_1$. Since $\lim_{x\to a} g(x) = M$ there is a positive real number r_2 such that $g(x)$ is defined for all x satisfying $0 < |x - a| < r_2$. Since $M \neq 0$, $|M| > 0$, and applying definition of limit, there is some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < |M|$, and hence by the reverse triangle inequality, $|g(x)| \geq ||M| - |g(x) - M|| > 0$, so $g(x) \neq 0$.

Letting $r = \min\{r_1, r_2, \delta\}$, we have that $r > 0$ and $f(x)$, $g(x)$, $1/g(x)$, and hence $f(x)/g(x)$ are defined for all x satisfying $0 < |x - a| < r$.

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence converging to a such that $0 < |x_n - a| < r$ for all n. By Theorem 26.1 in the "forward direction", we have that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. Since $M \neq 0$ and $g(x_n) \neq 0$ for all $n \in \mathbb{N}$, by the limits and algebra theorem for sequences,
$\lim_{n\to\infty} f(x_n)/g(x_n) = L/M$. So, by Theorem 26.1 again, it follows that $\lim_{x\to a}(f(x)/g(x)) = L/M$. $\square$

- $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$
- $\sin(x) = 1 \iff x \in \frac{\pi}{2} + 2\pi\mathbb{Z}$
- $\sin(x) = 0 \iff x \in \pi\mathbb{Z}$
- $\sin(x) = -1 \iff x \in \frac{-\pi}{2} + 2\pi\mathbb{Z}$
- $\pi \notin \mathbb{Q}$
- $\sin(x) = \sin(y) \iff x - y \in 2\pi\mathbb{Z}$ or $x + y \in \pi + 2\pi\mathbb{Z}$

Proof of Theorem 26.1. Let f be a function, $a \in \mathbb{R}$, and $r > 0$ a positive real number such that f is defined on $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$.

($\Rightarrow$) Assume $\lim_{x\to a} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence that converges to a and is such that $0 < |x_n - a| < r$ for all n. We need to prove that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. By definition of the limit of a function, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - L| < \varepsilon$. Since $\delta > 0$ and $\{x_n\}_{n=1}^{\infty}$ converges to a, by the definition of convergence, there is an N such that if $n \in \mathbb{N}$ and $n > N$ then $|x_n - a| < \delta$. I claim that this N "works" to prove $\{f(x_n)\}_{n=1}^{\infty}$ converges to L too: If $n \in \mathbb{N}$ and $n > N$, then $|x_n - a| < \delta$ and, since $x_n \neq a$ for all n, we have $0 < |x_n - a| < \delta$. It follows that $|f(x_n) - L| < \varepsilon$. This shows that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

($\Leftarrow$) We prove the contrapositive. That is, assume $\lim_{x\to a} f(x)$ is not L (including the case where the limit does not exist). We need to prove that there is at least one sequence $\{x_n\}_{n=1}^{\infty}$ such that (a) it converges to a, (b) $0 < |x_n - a| < r$ for all n and yet (c) the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L.

The fact that $\lim_{x\to a} f(x)$ is not L means:

> There is an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$, but either f is not defined at x or $|f(x) - L| \geq \varepsilon$.

For this ε, for any natural number n, set $\delta_n = \min\{\frac{1}{n}, r\}$. We get that there is a $x_n \in \mathbb{R}$ such that $0 < |x_n - a| < \delta_n$ and $|f(x_n) - L| \geq \varepsilon$. (Note that f is necessarily defined at x_n since $\delta_n \leq r$.) I claim that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the needed three conditions. (a) Since $\delta_n \leq \frac{1}{n}$, we have $a - \frac{1}{n} < x_n < a + \frac{1}{n}$ for all n, and hence by the Squeeze Lemma, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a. (b) This holds by construction, since $\delta_n \leq r$. (c) Since, for the positive number ε above, we have $|f(x_n) - L| \geq \varepsilon$ for all n, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. $\square$

27. Friday, November 1, 2024 §3.3

Prove or disprove:

(1) $\lim_{x\to -1}\sqrt{1-x^2}=0$.

This is false. To disprove it, suppose by way of contradiction that $\lim_{x\to -1}\sqrt{1-x^2}=0$. Take $\varepsilon=1$. Then by assumption there is some $\delta>0$ such that for all real numbers x such that $0<|x+1|<\delta$, then $\sqrt{1-x^2}$ is defined and $|\sqrt{1-x^2}|<\varepsilon$. But, given $\delta>0$, the real number $x=-1-\delta/2$ satisfies $0<|x+1|<\delta$, but $1-x^2<0$, so $\sqrt{1-x^2}$ is not defined. This is a contradiction. We conclude that $\lim_{x\to -1}\sqrt{1-x^2}=0$ is false.

(2) If $f:\mathbb{R}\smallsetminus\{0\}\to\mathbb{R}$ is a function with $\lim_{x\to 0}f(x)=5$, then the sequence $\left\{f\left(\frac{2}{n}\right)^2\right\}_{n=1}^{\infty}$ converges to $\dfrac{4}{25}$.

This is false. In fact, $\left\{f\left(\frac{2}{n}\right)^2\right\}_{n=1}^{\infty}$ converges to 25. Indeed, note that $\{\frac{2}{n}\}_{n=1}^{\infty}$ converges to 0, and is not equal to zero for any n. Thus, by Theorem 26.1 (applied with $r=2$, for example) we deduce that $\{f(\frac{2}{n})\}_{n=1}^{\infty}$ converges to 5. Then by limits and algebra, $\{f(\frac{2}{n})^2\}_{n=1}^{\infty}$ converges to $5^2=25$.

(3) If $\lim_{x\to 1}f(x)$ does not exist, then $\lim_{x\to 1}(x^2+f(x))$ also does not exist.

This is true. We prove the contrapositive. Suppose that $\lim_{x\to 1}(x^2+f(x))$ exists, and call it L. Note that $\lim_{x\to 1}x^2=1$. Then $\lim_{x\to 1}f(x)=L-1$ by limits of functions and algebra. In particular it exists.

(4) Let f,g be two functions with domain all real numbers x such that $x\neq 1$. If $\lim_{x\to 1}f(x)$ and $\lim_{x\to 1}g(x)$ both exist, then $\lim_{x\to 1}\dfrac{f(x)}{g(x)}$ exists.

28. Monday, November 4, 2024 §3.3

We come to the formal definition of continuity. We first define what it means for a function to be continuous *at a single point*, but ultimately we will be interested in functions that are continuous on entire intervals.

Definition 28.1. Suppose f is a function and a is a real number. We say f *is continuous at* a provided the following condition is met:

> For every $\varepsilon > 0$ there is a $\delta > 0$ such that if x is a real number such that $|x - a| < \delta$ then f is defined at x and $|f(x) - f(a)| < \varepsilon$.

Remark 28.2. If f is continuous at a, then by applying the definition using any positive number $\varepsilon > 0$ you like (e.g., $\varepsilon = 1$) we get that there exists a $\delta > 0$ such that f is defined for all x such that $a - \delta < x < a + \delta$. That is, in order for f to be continuous at a it is necessary (but not sufficient) that f is defined at all points near a *including at* a *itself*. In particular, unlike in the definition of "limit", f must be defined at a in order for it to possibly be continuous at a.

Example 28.3. I claim $f(x) = 3x$ is continuous at a for every value of a. Pick $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{3}$. If $|x - a| < \delta$ then f is defined at x (since the domain of f is all of $\mathbb{R}$) and

$$|f(x) - f(a)| = |3x - 3a| = 3|x - a| < 3\delta = \varepsilon.$$

Example 28.4. The function $f(x)$ with domain $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x - 7 & \text{if } x \geq 3 \text{ and} \\ -x & \text{if } x < 3 \end{cases}$$

is not continuous at 3. Since the domain of f is all of $\mathbb{R}$, the negation of the definition of "continuous at 3" is:

> there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is a real number x such that $|x - 3| < \delta$ and $|f(x) - f(3)| \geq \varepsilon$.

Set $\varepsilon = 1$. For any $\delta > 0$, we may choose a real number x so that $3 - \delta < x < 3$ and $2.9 < x < 3$. For such an x, we have

$$|f(x) - f(3)| = |-x + 1| = x - 1 > 1.9 > \varepsilon.$$

The proves f is not continuous at 3.

The definition of continuous looks a lot like the definition of limit, with L replaced by $f(a)$. This is not just superficial:

Theorem 28.5. *Suppose that f is a function and a is a real number and assume that f is defined at a. Then f is continuous at a if and only if* $\lim_{x \to a} f(x) = f(a)$.

Proof. ($\Rightarrow$) This is immediate from the definitions.

($\Leftarrow$) This is almost immediate from the definitions: Suppose $\lim_{x \to a} f(x) = f(a)$. Pick $\varepsilon > 0$. Then there is a δ such that if $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| < \varepsilon$. This nearly gives that f is continuous at a by definition, except that we need to know that if $|x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| < \varepsilon$. The only "extra" case is the case $x = a$. But if $x = a$, then f is defined at a by assumption and we have $|f(x) - f(a)| = 0 < \varepsilon$. □

Remark 28.6. Remember, when we write $\lim_{x\to a} f(x) = f(a)$ we mean that the limit exists and is equal to the number $f(a)$. So, by this Lemma, if $\lim_{x\to a} f(x)$ does not exist, then f is not continuous at a.

Example 28.7. The function $f(x) = \sqrt{x}$ is continuous at a for every $a > 0$. This holds since for any $a > 0$, as you proved on the homework we have

$$\lim_{x\to a} \sqrt{x} = \sqrt{a}.$$

Theorem 28.8. *Let $a \in \mathbb{R}$ and suppose f and g are two functions that are both continuous at a. Then so are*

(1) $f(x) + g(x)$*,*
(2) $c \cdot f(x)$*, for any constant* c*,*
(3) $f(x) \cdot g(x)$*, and*
(4) $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$*.*

Proof. Follows from Theorems 28.5 and 26.2. □

Example 28.9. Polynomials are continuous everywhere. The function x is continuous everywhere (since $\lim_{x\to a} x = a$). By part (3) above and a simple induction, x^n is continuous everywhere for every n. Then by parts (1) and (2), it follows that every polynomial is continuous everywhere.

Recall that for functions f and g, $f \circ g$ is the *composition*: it is the function that sends x to $f(g(x))$. The domain of $f \circ g$ is

$$\{x \in \mathbb{R} \mid \ x \text{ is the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}.$$

Theorem 28.10. *Suppose g is continuous at a point a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Proof. Let $a \in \mathbb{R}$ be such that that g is continuous at a and f is continuous at $g(a)$. We will that prove $f \circ g$ is continuous at a using the definition.

Pick $\varepsilon > 0$. Since f is continuous at $g(a)$, there is a $\gamma > 0$ such that if $|y - g(a)| < \gamma$ then f is defined at y and $|f(y) - f(g(a))| < \varepsilon$. (We are using y in place of the usual x for clarity below, and we are calling this number γ, and not δ, since it is not the δ we are seeking.) Since $\gamma > 0$ and g is continuous at a, there is a $\delta > 0$ such that if $|x - a| < \delta$ then g is defined at x and $|g(x) - a| < \gamma$.

This δ "works" to prove $f \circ g$ is continuous at a: Let x be any real number such that $|x - a| < \delta$. Then g is defined at x and $|g(x) - g(a)| < \gamma$. Taking $y = g(x)$ above, this gives that f is defined at $g(x)$ and $|f(g(x)) - f(g(a))| < \varepsilon$. This proves $f \circ g$ is continuous at a. □

INDEX