Making sense of if then statements and quantifier statements.

- The *converse* of the statement “If P then Q” is the statement “If Q then P”.
- The *contrapositive* of the statement “If P then Q” is the statement “If not Q then not P”.
- Any if then statement is equivalent to its contrapositive, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Is the original/contrapositive/converse true or false for real numbers a, b? Explain why (but don’t prove).
  (a) If a is irrational, then $1/a$ is irrational.
  (b) If a and b are irrational, then ab is irrational.
  (c) If $a > 3$, then $a^2 > 9$.

- The symbol for “for all” is $\forall$ and the symbol for there exists is $\exists$.
- The negation of “For all $x \in S$, P” is “There exists $x \in S$ such that $\mathrm{not}\ P$”.
- The negation of “There exists $x \in S$ such that P” is “For all $x \in S$, $\mathrm{not}\ P$”.

(2) Rewrite each statement with symbols in place of quantifiers, and write its negation. Is the original statement true or false? Explain why (but don’t prove them).
  (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
  (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
  (c) For all $x \in \mathbb{R}$ such that[1] $x \neq 0$, $x^2 > 0$.
  (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
  (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

[1]In a statement of the form “For all $x \in S$ such that Q, P”, the “such that Q” part is part of the hypothesis: it is restricting the set S that we are “alling”’ over.

Proving if then statements and quantifier statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if then statement than the original, especially when the negation of the hypothesis or conclusion is something negative.
- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

(3) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[2] that $x \geq y$ if and only if $-y \geq -x$.

(4) Let x be a real number. Show that if x^2 is irrational, then x is irrational.

(5) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have proven in class to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

(6) Prove that[3] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

(7) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.

(8) Prove[4] that (2d) is true and (2e) is false.

(9) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.

(10) Prove that $1 > 0$.

[2]Hint: You may want to add something to both sides.
[3]Hint: Use (5).
[4]You can "work out of order here" and *use* (10) now.