LIMITS

FROM LAST TIME:

Theorem 18.4: Let $f(x)$ be a function and let a be a real number. Let $r > 0$ be a positive real number such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$. Let L be any real number.

Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a and satisfies $0 < |x_n - a| < r$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Corollary 18.5: Let f be a function and a and L be real numbers. Suppose that the domain of f is all of $\mathbb{R}$ or $\mathbb{R} \setminus \{a\}$. Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a such that $x_n \neq a$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

NEW STUFF:

Theorem 19.1. (Algebra and limits of functions): Suppose f and g are two functions and that a is a real number, and assume that

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

for some real numbers L and M. Then

(1) $\lim_{x\to a}(f(x) + g(x)) = L + M$.
(2) For any real number c, $\lim_{x\to a}(c \cdot f(x)) = c \cdot L$.
(3) $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$.
(4) If, in addition, we have that $M \neq 0$, then $\lim_{x\to a}(f(x)/g(x)) = L/M$.

Theorem 19.2. (Squeeze Theorem for limits): Suppose f, g, and h are three functions and a is a real number. Suppose there is a positive real number $r > 0$ such that

- each of f, g, h is defined on $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$,
- $f(x) \leq g(x) \leq h(x)$ for all x such that $0 < |x - a| < r$, and
- $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ for some number L.

Then $\lim_{x\to a} g(x) = L$.

(1) Use the $\varepsilon - \delta$ definition to show that $\lim_{x\to 0} |x| = 0$.

(2) Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

Use the $\varepsilon - \delta$ definition to show that $\lim_{x\to a} f(x)$ does not exist for any real number $a \in \mathbb{R}$.

Suggestion: Fix $a \in \mathbb{R}$ and suppose that $\lim_{x\to a} f(x) = L$. Take $\varepsilon = \frac{1}{2}$. You may find Density of Rational/Irrational numbers helpful here.

(3) Use Corollary 18.5 to show that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Suggestion: Let $f(x) = \sin(\frac{1}{x})$ and suppose $\lim_{x\to 0} f(x) = L$. Find sequences $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ such that

- $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ both converge to 0,
- $f(x_n) = 1$ for all n, and
- $f(y_n) = -1$ for all n.

You can use any trig facts on the bottom of the page.

(4) Use Theorem 19.1 plus a homework problem[1] to compute $\lim_{x\to 2} \frac{3x^2 - x + 2}{x+3}$.

(5) Use Theorem 19.2 to show that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. You can use any trig facts on the bottom of the page.

(6) Use Theorem 18.4 to deduce Theorem 19.2 from our Squeeze Theorem for sequences.

(7) Use Theorem 18.4 to deduce Theorem 19.1 part (1) from our Theorem 10.2 on algebra and sequences.

(8) Use Theorem 18.4 to deduce Theorem 19.1 part (4) from our Theorem 10.2 on algebra and sequences.

- $-1 \le \sin(x) \le 1$ for all $x \in \mathbb{R}$
- $\sin(x) = 1 \iff x \in \frac{\pi}{2} + 2\pi\mathbb{Z}$
- $\sin(x) = 0 \iff x \in \pi\mathbb{Z}$
- $\sin(x) = -1 \iff x \in \frac{-\pi}{2} + 2\pi\mathbb{Z}$
- $\pi \notin \mathbb{Q}$
- $\sin(x) = \sin(y) \iff x - y \in 2\pi\mathbb{Z}$ or $x + y \in \pi + 2\pi\mathbb{Z}$

[1] $\lim_{x\to a} mx + b = ma + b$. In particular, $\lim_{x\to a} x = a$ and $\lim_{x\to b} b = b$.