

THE ISOMORPHISM THEOREMS

0. UNIVERSAL MAPPING PROPERTY FOR QUOTIENT GROUPS: Let $f : G \rightarrow H$ be a group homomorphism. If $N \trianglelefteq G$ and $N \subseteq \ker(f)$, then there is a unique group homomorphism $\bar{f} : G/N \rightarrow H$ such that $f = \bar{f} \circ \pi$; i.e., the diagram

$$\begin{array}{ccc} & G & \\ \pi \swarrow & & \searrow f \\ G/N & \xrightarrow{\quad \bar{f} \quad} & H \end{array}$$

commutes. Moreover, $\text{im}(\bar{f}) = \text{im}(f)$, and $\ker(\bar{f}) = \{gN \mid g \in \ker(f)\}$.

1. FIRST ISOMORPHISM THEOREM: Let $f : G \rightarrow H$ be a group homomorphism. Then

$$\begin{aligned} G/\ker(f) &\xrightarrow{\bar{f}} \text{im}(f) \\ g \cdot \ker(f) &\mapsto f(g) \end{aligned}$$

is an isomorphism.

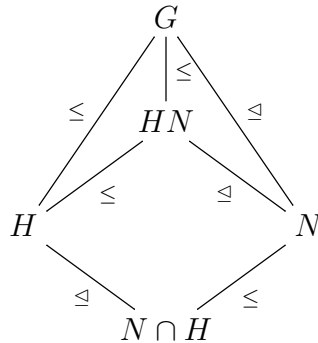
2. DIAMOND ISOMORPHISM THEOREM: Let G be a group, $H \leq G$, and $N \trianglelefteq G$. Then

$$HN \leq G, \quad N \cap H \trianglelefteq H, \quad N \trianglelefteq HN$$

and

$$\begin{aligned} \frac{H}{N \cap H} &\longrightarrow \frac{HN}{N} \\ h(N \cap H) &\mapsto hN \end{aligned}$$

is an isomorphism.



3. CANCELLING ISOMORPHISM THEOREM: Let G be a group, $M \leq N \leq G$, $M \trianglelefteq G$, and $N \trianglelefteq G$. Then

$$M \trianglelefteq N, \quad N/M \trianglelefteq G/M,$$

and the map

$$\begin{aligned} \frac{(G/M)}{(N/M)} &\longrightarrow G/N \\ gM(N/M) &\mapsto gN \end{aligned}$$

is an isomorphism.

4. LATTICE ISOMORPHISM THEOREM: Let G be a group and N a normal subgroup of G , and let $\pi: G \twoheadrightarrow G/N$ be the quotient map. There is an order-preserving bijection of posets (a lattice isomorphism)

$$\begin{aligned} \{\text{subgroups of } G \text{ that contain } N\} &\xrightleftharpoons[\Phi]{\Psi} \{\text{subgroups of } G/N\} \\ H &\longmapsto \pi(H) = H/N \\ \pi^{-1}(A) = \{x \in G \mid \pi(x) \in A\} &\longleftarrow A \end{aligned}$$

Then this bijection enjoys the following properties:

(1) Subgroups correspond to subgroups:

$$H \leq G \iff H/N \leq G/N.$$

(2) Normal subgroups correspond to normal subgroups:

$$H \trianglelefteq G \iff H/N \trianglelefteq G/N.$$

(3) Indices are preserved:

$$[G : H] = [G/N : H/N].$$

(4) Intersections and generated subgroups are preserved:

$$H/N \cap K/N = (H \cap K)/N \quad \text{and} \quad \langle H/N \cup K/N \rangle = \langle H \cup K \rangle / N.$$