ASSIGNMENT #2

(1) Let K be a field, and x be an indeterminate. Let $R = K[x^2, x^3] \subseteq S = K[x]$. Find an ideal $I \subseteq R$ for which $IS \cap R \supsetneq I$.

(2) Let K be an infinite field, and $R = K[x_1, \dots, x_n]$ be a polynomial ring. Let $G = (K^\times)^m$ act on R as follows:

$$(\lambda_1, \dots, \lambda_m) \cdot k = k \qquad\qquad k \in K,$$
$$(\lambda_1, \dots, \lambda_m) \cdot x_i = \lambda_1^{a_{1i}} \cdots \lambda_m^{a_{mi}} x_i \qquad\qquad i = 1, \dots, n$$

for some $m \times n$ matrix of integers $A = [a_{ij}]$.

(a) Show that R^G has a K-vector space basis given by the set of monomials $x_1^{b_1} \cdots x_n^{b_n}$ such that, for $b = (b_1, \dots, b_n)$, $Ab = 0$.

(b) Consider the polynomial ring R with a (nonstandard) $\mathbb{Z}^m$-grading given by setting

$$|x_i| = (a_{1i}, \dots, a_{mi})$$

for each i. Show that R^G is the degree zero piece of R under this grading.

(c) Show that R^G is a direct summand of R, and conclude that R^G is a finitely generated K-algebra.

(A combinatorial consequence of this: for any integer matrix A, there is a finite set of solution vectors $v_1, \dots, v_t$ such that every solution with nonnegative entries can be written as a nonnegative linear combination of $v_1, \dots, v_t$.)

(3) Let $X \subseteq \mathbb{A}_K^m$ be an affine varieties over an infinite field K.

(a) If $\phi : X \to \mathbb{A}_K^n$ is an algebraic map, show that $\mathcal{I}(\operatorname{im} \phi) = \ker(\phi^*)$ as ideals in $K[y_1, \dots, y_n]$, where $y_1, \dots, y_n$ are the coordinates of $\mathbb{A}_K^n$.

(b) Use (a) to compute $\mathcal{I}(\{(t, t^2, t^3) \in \mathbb{A}_K^3 \mid t \in K\})$.

(c) Use (a) to show[1] $\mathcal{I}(\{(t^3, t^4, t^5) \in \mathbb{A}_K^3 \mid t \in K\}) = (x^3 - yz, y^2 - xz, z^2 - x^2y)$.

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(4) Compute the irreducible decompositions of the following varieties over $\mathbb{C}$:

(a) $\mathcal{Z}(y^3 - x^2y^2)$.

(b) $\mathcal{Z}(x_1x_2, x_1x_3, x_2x_3x_4)$.

(c) $\mathcal{Z}(x_1x_3 + x_2x_4, x_1x_5 + x_2x_6)$.

(5) Let R be a finitely generated $\mathbb{Z}$-algebra and $\mathfrak{m}$ be a maximal ideal of R. Show that $R/\mathfrak{m}$ is finite.

[1]Suggestion: The homomorphism $K[x, y, z] \to K[t]$ sending $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$ is a graded homomorphism if we set $|x| = 3$, $|y| = 4$, $|z| = 5$. Show that, if J is the ideal on the right hand side, the nth graded piece of $K[x, y, z]/J$ is a K-vector space of dimension at most 1 for $n \geqslant 3$ and $n = 0$, and is zero for $n = 1, 2$.

(B) In this problem we will prove the **Ax-Grothendieck Theorem:** Any injective algebraic morphism $\phi : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$ is surjective.

(a) First, left K be an arbitrary field, and $\phi(a_1, \dots, a_n) = (f_1(a_1, \dots, a_n), \dots, f_n(a_1, \dots, a_n))$ be an algebraic morphism, for polynomials $f_1, \dots, f_n$. Show that ϕ is *not* surjective if and only if there is some $(a_1, \dots, a_n) \in \mathbb{A}^n_K$ such that

$$\mathcal{Z}_K(f_1(\underline{x}) - a_1, \dots, f_n(\underline{x}) - a_n) = \varnothing.$$

(b) Consider $\mathbb{A}^{2n}_K$ with variables $x_1, \dots, x_n, y_1, \dots, y_n$. Show that ϕ is injective if and only if

$$\mathcal{Z}_K(f_1(\underline{x}) - f_1(\underline{y}), \dots, f_n(\underline{x}) - f_n(\underline{y})) \subseteq \mathcal{Z}_K(x_1 - y_1, \dots, x_n - y_n) \quad \text{in } \mathbb{A}^{2n}_K.$$

(c) Now, let $K = \mathbb{C}$ and suppose that ϕ is injective but not surjective. Show that there exist $g_i(\underline{x}), h_{i,j}(\underline{x}, \underline{y}) \in \mathbb{C}[\underline{x}, \underline{y}]$, and integers t_j such that

$$\sum_i g_i(\underline{x})(f_i(\underline{x}) - a_i) = 1, \quad (x_j - y_j)^{t_j} = \sum_i h_{i,j}(\underline{x}, \underline{y})(f_i(\underline{x}) - f_i(\underline{y})) \text{ in } \mathbb{C}[\underline{x}, \underline{y}].$$

Setting $R = \mathbb{Z}[\{\text{coefficients of } f_i's, g_i's, h'_{i,j}s\}, a_1, \dots, a_n]$, conclude that the same equations hold in a polynomial ring $R[\underline{x}, \underline{y}]$ over a finitely generated $\mathbb{Z}$-subalgebra $R \subseteq \mathbb{C}$.

(d) Go modulo a maximal ideal of R, and complete the proof of the theorem.