BASICS OF DERIVATIVES

Definition: Let f be a function and r be a real number. We say that f is *differentiable at* r if f is defined at r and the limit

$$\lim_{x \to r} \frac{f(x) - f(r)}{x - r}$$

exists. In this case, we call the limit *the derivative of* f *at* r and write $f'(r)$ for this limit.

(1) Use the definition to show that the function $f(x) = x$ is differentiable at any $x = r$ and compute its derivative.

(2) Use the definition to show that the function $f(x) = |x|$ is *not* differentiable at $x = 0$.

(3) Prove[1] that if f is differentiable at $x = r$, then f is continuous at $x = r$.

(4) Prove or disprove the converse of the previous statement.

Theorem (Derivatives and algebra: Let f, g be functions that are differentiable at $x = r$, and c be a real number. Then,

(1) $f + g$ is differentiable at $x = r$ and $(f + g)'(r) = f'(r) + g'(r)$;
(2) cf is differentiable at $x = r$ and $(cf)'(r) = cf'(r)$;
(3) fg is differentiable at $x = r$ and $(fg)'(r) = f'(r)g(r) + f(r)g'(r)$.

(6) Prove[2] that if $f(x) = x^n$, then f is differentiable at any value of x and $f'(x) = nx^{n-1}$ for every $n \in \mathbb{N}$.

(7) Use the Theorem plus the previous problem to compute the derivative of $f(x) = 5x^7 - \sqrt{19}\, x^4$.

(8) Prove the Theorem.

[1]Hint: Write down the function in the definition of derivative, and multiply both sides by $(x - r)$ and consider the limit.
[2]You many want to use part (3) of the Theorem above.

Theorem: Let f be a function that is differentiable at $x = r$.

(1) If $f'(r) > 0$, then there is some $\delta > 0$ such that
- if $x \in (r, r+\delta)$ then $f(r) < f(x)$;
- if $x \in (r-\delta, r)$ then $f(x) < f(r)$.

(2) If $f'(r) < 0$, then there is some $\delta > 0$ such that
- if $x \in (r, r+\delta)$ then $f(r) > f(x)$;
- if $x \in (r-\delta, r)$ then $f(x) > f(r)$.

Corollary (Derivatives and optimization): Let f be a function that is continuous on a closed interval $[a, b]$. If f attains a maximum or minimum value on $[a, b]$ at $r \in (a, b)$, and f is differentiable at r, then $f'(r) = 0$.

(1) Find the values of x on $[0, 2]$ at which f achieves its minimum and maximum values.

(2) Explain why the Corollary follows from the Theorem.

(3) Prove part (1) of the Theorem:
- Consider the function $h(x) = \frac{f(x)-f(r)}{x-r}$. Apply the definition of limit to this function with $\varepsilon = f'(r)$. What does the definition give you?
- If $h(x) > 0$ and $x > r$, what can you say about $f(x) - f(r)$?
- If $h(x) > 0$ and $x < r$, what can you say about $f(x) - f(r)$?

(4) Prove part (2) of the Theorem.

A function f is *increasing* on an interval (a, b) if for any $r, s \in (a, b)$ with $r < s$, we have $f(r) < f(s)$.

(5) Prove or disprove: If $f'(r) > 0$, then there is some $\delta > 0$ such that f is increasing on $(r - \delta, r + \delta)$.