

ASSIGNMENT #5: DUE THURSDAY, OCTOBER 17 AT MIDNIGHT

This problem set is to be turned in on Canvas. You may reference any result or problem from our worksheets or lectures, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets.

- (1) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences. Either prove¹ or give a counterexample to each of the following:
 - (a) If $\{a_n^2\}_{n=1}^{\infty}$ is divergent, then $\{a_n\}_{n=1}^{\infty}$ is divergent.
 - (b) If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both diverge, then $\{a_n + b_n\}_{n=1}^{\infty}$ also diverges.
 - (c) If $\{a_n\}_{n=1}^{\infty}$ converges and $\{b_n\}_{n=1}^{\infty}$ diverges, then $\{a_n + b_n\}_{n=1}^{\infty}$ diverges.
 - (d) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$. If $\{b_n\}_{n=1}^{\infty}$ converges to 0, then $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ diverges.
- (2) Use the definition to prove that the sequence $\{-\sqrt{n}\}_{n=1}^{\infty}$ diverges to $-\infty$.
- (3) Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$.
 - (a) Use induction to prove that $a_n > 0$ for all $n \in \mathbb{N}$.
 - (b) Prove² that $a_n^2 \geq 2$ for all $n \in \mathbb{N}$.
 - (c) Prove³ that the sequence is decreasing.
 - (d) Show that the sequence is convergent.
 - (e) Determine⁴ to what value the sequence converges.

¹You may use our Theorem on Limits and Algebra whenever convenient, but make sure you are using something the Theorem says, and nothing it doesn't!

²Write $a_n - 2$ in terms of a_{n-1} and factor the expression.

³Consider $a_{n+1} - a_n$ and use (b).

⁴If the sequence converges to L , explain why $L = \frac{L}{2} + \frac{1}{L}$, and solve for L .