INDUCTION AND SEQUENCES §1.5 & §2.2

(1) Let $a \geq 2$ be a real number.

  (a) Prove that for any natural number n, the inequality $a^n \geq na$ holds true.

  (b) Prove that the sequence $\left\{\frac{1}{a^n}\right\}_{n=1}^{\infty}$ converges to 0.

(2) Define a sequence $\{b_n\}_{n=1}^{\infty}$ recursively by the rule $b_1 = 0$, and $b_n = \dfrac{1+b_{n-1}}{2}$ for $n > 1$. Prove that $b_n = 1 - \dfrac{1}{2^n}$ for all $n \in \mathbb{N}$ and compute $\lim\limits_{n\to\infty} b_n$.

(3) Define a sequence $\{c_n\}_{n=1}^{\infty}$ recursively by the rule $c_1 = 1$, and $c_n = \sqrt{2c_{n-1}}$ for $n > 1$.

  (a) Use a calculator to write down the first 5 terms of this sequence.

  (b) Prove that the sequence $\{c_n\}_{n=1}^{\infty}$ is bounded above by 2.

  (c) Use the previous part to show that $\{c_n\}_{n=1}^{\infty}$ is an increasing sequence.

  (d) Prove that the sequence $\{c_n\}_{n=1}^{\infty}$ is convergent.

  (e) What value does $\{c_n\}_{n=1}^{\infty}$ converge to? Can you prove it?