0. UNIVERSAL MAPPING PROPERTY FOR QUOTIENT GROUPS: Let $f : G \to H$ be a group homomorphism. If $N \trianglelefteq G$ and $N \subseteq \ker(f)$, then there is a unique group homomorphism $\overline{f} : G/N \to H$ such that $f = \overline{f} \circ \pi$; i.e., the diagram

$$\begin{tikzcd}
 & G \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\
G/N \arrow[rr, dashed, "\overline{f}"'] & & H
\end{tikzcd}$$

commutes. Moreover, $\mathrm{im}(\overline{f}) = \mathrm{im}(f)$, and $\ker(\overline{f}) = \{gN \mid g \in \ker(f)\}$.

1. FIRST ISOMORPHISM THEOREM: Let $f : G \to H$ be a group homomorphism. Then

$$\begin{array}{ccc} G/N & \xrightarrow{\overline{f}} & \mathrm{im}(f) \\ gN & \mapsto & f(g) \end{array}$$

is an isomorphism.

2. DIAMOND ISOMORPHISM THEOREM: Let G be a group, $H \leq G$, and $N \trianglelefteq G$. Then

$$HN \leq G, \quad N \cap H \trianglelefteq H, \quad N \trianglelefteq HN$$

and

$$\begin{array}{ccc} \dfrac{H}{N \cap H} & \longrightarrow & \dfrac{HN}{N} \\ h(N \cap H) & \mapsto & hN \end{array}$$

is an isomorphism.

$$\begin{tikzcd}
 & G \arrow[ddl, dash, "\leq"'] \arrow[d, dash, "\leq"] \arrow[ddr, dash, "\trianglelefteq"] & \\
 & HN \arrow[dl, dash, "\leq"] \arrow[dr, dash, "\trianglelefteq"'] & \\
H \arrow[dr, dash, "\trianglelefteq"'] & & N \arrow[dl, dash, "\leq"] \\
 & N \cap H &
\end{tikzcd}$$

3. CANCELLING ISOMORPHISM THEOREM Let G be a group, $M \leq N \leq G$, $M \trianglelefteq G$, and $N \trianglelefteq G$. Then

$$M \trianglelefteq N, \qquad N/M \trianglelefteq G/M,$$

and the map

$$\begin{array}{ccc} \dfrac{(G/M)}{(N/M)} & \longrightarrow & G/N \\ gM & \mapsto & gN \end{array}$$

is an isomorphism.

4. LATTICE ISOMORPHISM THEOREM
Let G be a group and N a normal subgroup of G, and let $\pi : G \twoheadrightarrow G/N$ be the quotient map. There is an order-preserving bijection of posets (a lattice isomorphism)

$$\{\text{subgroups of } G \text{ that contain } N\} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \{\text{subgroups of } G/N\}$$

$$H \longmapsto \pi(H) = H/N$$

$$\pi^{-1}(A) = \{x \in G \mid \pi(x) \in A\} \longleftarrow A$$

Then this bijection enjoys the following properties:

(1) Subgroups correspond to subgroups:

$$H \leq G \iff H/N \leq G/N.$$

(2) Normal subgroups correspond to normal subgroups:

$$H \trianglelefteq G \iff H/N \trianglelefteq G/N.$$

(3) Indices are preserved:

$$[G : H] = [G/N : H/N].$$

(4) Intersections and generated subgroups are preserved:

$$H/N \cap K/N = (H \cap K)/N \quad \text{and} \quad \langle H/N \cup K/N \rangle = \langle H \cup K \rangle / N.$$