NOETHER NORMALIZATION: Let K be a field, and R be a finitely-generated K-algebra. Then there exists a finite[1] set of elements $f_1, \dots, f_m \in R$ that are algebraically independent over K such that $K[f_1, \dots, f_m] \subseteq R$ is module-finite; equivalently, there is a module-finite injective K-algebra map from a polynomial ring $K[X_1, \dots, X_m] \hookrightarrow R$. Such a ring S is called a **Noether normalization** for R.

LEMMA: Let A be a ring, and $F \in R := A[X_1, \dots, X_n]$ be a nonzero polynomial. Then there exists an A-algebra automorphism ϕ of R such that $\phi(F)$, viewed as a polynomial in X_n with coefficients in $A[X_1, \dots, X_{n-1}]$, has top degree term aX_n^t for some $a \in A \smallsetminus 0$ and $t \geq 0$.

- If $A = K$ is an infinite field, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + \lambda_i X_n$ for some $\lambda_1, \dots, \lambda_{n-1} \in K$.
- In general, if the top degree of F (with respect to the standard grading) is D, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + X_n^{D^{n-i}}$ for $i < n$.

ZARISKI'S LEMMA: An algebra-finite extension of fields is module-finite.

USEFUL VARIATIONS ON NOETHER NORMALIZATION:

- NN FOR DOMAINS: Let $A \subseteq R$ be a module-finite inclusion of domains[2]. Then there exists $a \in A \smallsetminus 0$ and $f_1, \dots, f_m \in R[1/a]$ that are algebraically independent over $A[1/a]$ such that $A[1/a][f_1, \dots, f_m] \subseteq R[1/a]$ is module-finite.
- GRADED NN: Let K be an infinite field, and R be a standard graded K-algebra. Then there exist algebraically independent elements $L_1, \dots, L_m \in R_1$ such that $K[L_1, \dots, L_m] \subseteq R$ is module-finite.
- NN FOR POWER SERIES: Let K be an infinite field, and $R = K[\![X_1, \dots, X_n]\!]/I$. Then there exists a module-finite injection $K[\![Y_1, \dots, Y_m]\!] \hookrightarrow R$ for some power series ring in m variables.

(1) Examples of Noether normalizations: Let K be a field.

(a) Show that $K[x, y]$ is a Noether normalization of $R = \dfrac{K[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$, where x, y are the classes of X and Y in R, respectively.

(b) Show that $K[x]$ is *not* a Noether normalization of $R = \dfrac{K[X, Y]}{(XY)}$. Then show that $K[x + y] \subseteq R$ *is* a Noether normalization.

(c) Show that $K[X^4, Y^4]$ is a Noether normalization for $R = K[X^4, X^3Y, XY^3, Y^4]$.

(a) From the equation $z^3 + x^3 + y^3 = 0$, we have $K[x, y] \subseteq R$ is integral, and since z generates as an algebra, hence module-finite. We need to check that x, y are algebraically independent in R. Suppose that $p(x, y) = 0$ in R, so $p(X, Y) \in (X^3 + Y^3 + Z^3)$ in $K[X, Y, Z]$. By considering $K[X, Y, Z] = K[X, Y][Z]$ as polynomials in Z, the Z-degree of such a p, which forces $p = 0$. Thus x, y are algebraically independent.

[1] Possibly empty!

[2] The assumption that R is a domain is actually not necessary, but can't quite state the general statement yet. We assume that R is a domain so that there is fraction field of R in which to take $R[1/a]$.

> **(b)** y is not integral over $K[x]$: this would imply $Y^n + a_1(X)Y^{n-1} + \cdots a_n(X) = XYb(X,Y)$ in $K[X,Y]$, but no monomial from any term can cancel Y^n. Alternatively, if the inclusion is module-finite, go mod x to get $K \subseteq K[X,Y]/(XY,X) = K[Y]$ module-finite, which it isn't.
> **(c)** It is easy to check that X^4, Y^4 are algebraically independent, and $(X^3Y)^4 = (X^4)^3Y^4$, $(XY^3)^4 = X^4(Y^4)^3$ give integral dependence relations for the algebra generators.

(2) Use Noether Normalization[3] to prove Zariski's Lemma.

> Let $K \subseteq L$ be an algebra-finite extension of fields. Take a NN of L: say $K \subseteq K[\ell_1,\dots,\ell_t] \subseteq L$, with ℓ_i algebraically independent and $R := K[\ell_1,\dots,\ell_t] \subseteq L$ module-finite and a fortiori integral. From the Integral Extensions worksheet, since L and R are domains, the extension is integral, and L is a field, we know that R is a field. This means that $t = 0$, so $K \subseteq L$ is module-finite.

(3) Proof of Noether Normalization (using the Lemma): Proceed by induction on the number of generators of R as a K-algebra; write $R = K[r_1,\dots,r_n]$.

- **(a)** Deal with the base case $n = 0$.
- **(b)** For the inductive step, first do the case that $r_1,\dots,r_n$ are algebraically independent over K.

(3 ctd.)

- **(c)** Let $\alpha : K[X_1,\dots,X_n] \to R$ be the K-algebra homomorphism such that $\alpha(X_i) = r_i$, and let ϕ be a K-algebra automorphism of $K[X_1,\dots,X_n]$. Let $r'_i = \alpha(\phi(X_i))$ for each i. Explain[4] why $R = K[r'_1,\dots,r'_n]$, and for any K-algebra relation F on $r_1,\dots,r_n$, the polynomial $\phi^{-1}(F)$ is a K-algebra relation on $r'_1,\dots,r'_n$.
- **(d)** Use the Lemma to find a K-subalgebra R' of R with $n-1$ generators such that the inclusion $R' \subseteq R$ is module-finite.
- **(e)** Conclude the proof.

> **(a)** This means that R is a quotient of K, but K is a field, so $R = K$; the identity map is module-finite.
> **(b)** If we have an algebraically independent set of generators for R, then R works: the identity map is module-finite.
> **(c)** First we claim that $R = K[r'_1,\dots,r'_n]$: indeed, the map $\alpha' = \alpha\circ\phi$ is the K-algebra map that sends X_i to r'_i, and since α and ϕ are surjective, α' is surjective, verifying the claim. The relations on the r'_i are of the elements of the kernel of α'; if F is a relation on the originals, then $\alpha(F) = 0$, so $\alpha'(\phi^{-1}(F)) = 0$ as well.
> **(d)** Take a map ϕ as in the Lemma, and n generators $r_1,\dots,r_n$. Set $r'_i = \phi^{-1}(r_i)$. By the previous part, these generate, and there is a relation on these that is monic in X_n, so $R' = K[r'_1,\dots,r'_{n-1}] \subseteq R$ is module-finite.
> **(e)** Apply IH to R' to get $K[f_1,\dots,f_t] \subseteq R'$ with f_i alg indep't and the inclusion module-finite. Then $K[f_1,\dots,f_t]$ is a Noether normalization.

(4) Proof of Lemma:

(a) in the General case: Consider the automorphism ϕ from the general case of the Lemma. Show that for a monomial, we have $\phi(aX_1^{d_1}\cdots X_n^{d_n})$ is a polynomial with unique highest

[3] and a suitable fact about integral extensions…

[4] Say α' is the K-algebra map given by $\alpha'(X_i) = r'_i$. Observe that $\alpha' = \alpha\circ\phi$. Why is this surjective?

degree term $aX_n^{d_1D^{n-1}+d_2D^{n-2}+\cdots+d_n}$. Can two monomials μ, ν in F, have $\phi(\mu)$ and $\phi(\nu)$ with the same highest degree term? Complete the proof.

(b) Prove the Lemma in the infinite field case.

(5) Variations on NN.
  (a) Adapt the proof of NN to show Graded NN.
  (b) Adapt the proof of NN to show NN for domains.
  (c) Adapt the proof of NN to show NN for power series.