MATH 902 LECTURE NOTES, SPRING 2022

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In this class, all rings are assumed to be commutative, with associative multiplication and containing 1.

1. Finiteness conditions

1.1. **Finitely generated algebras.** We start by recalling a definition from last semester, specialized to the setting of commutative rings.

Definition 1.1 (Algebra)**.** Given a ring A, an A-*algebra* is a ring R equipped with a ring homomorphism $\phi : A \to R$. This defines an A-module structure on R given by restriction of scalars, that is, for $a \in A$ and $r \in R$, $ar := \phi(a)r$ that is compatible with the internal multiplication of R i.e.,

$$a(rs) = (ar)s = r(as) \text{ for all } a \in A, rs \in R.$$

We will call ϕ the *structure homomorphism* of the A-algebra R.

Example 1.2.

- If A is a ring and $x_1, \ldots, x_n$ are indeterminates, the inclusion map $A \hookrightarrow A[x_1, \ldots, x_n]$ makes the polynomial ring into an A-algebra.
- When $A \subseteq R$ the inclusion map makes R an A-algebra. In this case the A-module multiplication ar coincides with the internal (ring) multiplication on R.
- Any ring comes with a unique structure as a $\mathbb{Z}$-algebra.

The collection of A-algebras forms a category where the morphisms are ring homomorphisms $f : R \to S$ such that the following diagram commutes

$$\begin{tikzcd} & A \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ R \arrow[rr, "f"'] & & S \end{tikzcd}$$

for structural homomorphisms $\varphi : A \to R$ and $\psi : A \to S$.

Definition 1.3 (Algebra generation)**.** Let R be an A-algebra and let $\Lambda \subseteq R$ be a set. The A-*algebra generated by* a subset Λ of R, denoted $A[\Lambda]$, is the smallest (w.r.t containment) subring of R containing Λ and $\varphi(A)$.

A set of elements $\Lambda \subseteq R$ *generates* R as an A-algebra if $R = A[\Lambda]$.

Note that there are two different meanings for the notation $A[S]$ for a ring A and set S: one calls for a polynomial ring, and the other calls for a subring of something.

This can be unpackaged more concretely in a number of equivalent ways:

Lemma 1.4. *The following are equivalent*

(1) Λ *generates* R *as an* A*-algebra.*
(2) *Every element in* R *admits a polynomial expression in* Λ *with coefficients in* $\phi(A)$*, i.e.*
$$R = \left\{ \sum_{\text{finite}} \phi(a)\lambda_1^{i_1} \cdots \lambda_n^{i_n} \mid a \in A, \lambda_j \in \Lambda, i_j \in \mathbb{N} \right\}.$$
(3) *The* A*-algebra homomorphism* $\psi : A[X] \to R$*, where* $A[X]$ *is a polynomial ring on a set of indeterminates* X *in bijection with* Λ *and* $\psi(x_i) = \lambda_i$*, is surjective.*

Proof. Let $S = \left\{\sum_{\text{finite}} \phi(a)\lambda_1^{i_1} \cdots \lambda_n^{i_n} \mid a \in A, \lambda_j \in \Lambda, i_j \in \mathbb{N}\right\}$. For the equivalence between (2) and (3) we note that S is the image of ψ. In particular, S is a subring of R. It then follows from the definition that (1) implies (2). Conversely, any subring of R containing $\phi(A)$ and Λ certainly must contain S, so (2) implies (1). $\square$

Example 1.5. We may have also seen these brackets used in $\mathbb{Z}[\sqrt{d}]$ for some $d \in \mathbb{Z}$ to describe the ring
$$\{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}.$$
In fact, this is a special instance of generating: the $\mathbb{Z}$-algebra generated by $\sqrt{d}$ in the most natural place, the algebraic closure of $\mathbb{Q}$, is exactly the set above. The point is that for any power $(\sqrt{2})^n$, write $n = 2q + r$ with $r \in \{0, 1\}$, so $(\sqrt{2})^n = 2^d(\sqrt{2})^r$. Similarly, the ring $\mathbb{Z}[\sqrt[3]{d}]$ can be written as
$$\{a + b\sqrt[3]{d} + c\sqrt[3]{d^2} \mid a, b, c \in \mathbb{Z}\}.$$

Note that the homomorphism ψ in part (3) need not be injective.

- If the homomorphism ψ is injective (so an isomorphism) we say that A is a *free* algebra.
- the set $\ker(\psi)$ measures how far R is from being a free A-algebra and is called the set of *relations* on Λ.

Definition 1.6 (Algebra-finite)**.** We say that $\varphi : A \to R$ is *algebra-finite*, or R is a *finitely generated* A*-algebra*, if there exists a finite set of elements $f_1, \ldots, f_d$ that generates R as an A-algebra. We write $R = A[f_1, \ldots, f_d]$ to denote this.

The term *finite-type* is also used to mean this.

Remark 1.7. Note that, by the lemma on generating sets, an A-algebra is finitely generated if and only if it is isomorphic to a quotient of a polynomial ring over A in finitely many variables. The choice of an isomorphism with a quotient of a polynomial ring is equivalent to a choice of generating set.

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