EISENSTEIN'S CRITERION

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$$f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a monic polynomial of degree at least one. If there is a prime ideal P of R such that $a_0, \ldots, a_{n-1} \in P$ but $a_0 \notin P^2$, then f is irreducible in $R[x]$.

COROLLARY: Let R be a UFD and

$$f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a monic polynomial of degree at least one. If there is an irreducible element $p \in R$ such that $p \mid a_i$ for $i = 0, \ldots, n-1$ and $p^2 \nmid a_0$, then f is irreducible in $R[x]$.

(1) Examples:
- **(a)** Show that the polynomial $x^5 - 6x + 18$ is irreducible in $\mathbb{Z}[x]$.
- **(b)** Let p be a prime number and $n \geq 1$. Show that $x^n - p$ is irreducible in $\mathbb{Z}[x]$.
- **(c)** Show that the polynomials from the previous parts are irreducible over $\mathbb{Q}[x]$.
- **(d)** Let F be a field. Show that $x^2 + xy + y$ is irreducible in $F[x, y]$.

(2) Proof of Eisenstein's Criterion:
- **(a)** Prove the following Lemma: If T is an integral domain and $g, h \in T[x]$ are polynomials such that $gh = x^n$ for some $n \geq 1$, then $g = x^i$ and $h = x^j$ for some $0 \leq i, j, \leq n$ with $i + j = n$.
- **(b)** In the setting of Eisenstein's criterion, suppose that $f = GH$ for some $G, H \in R[x]$ of positive degree. Apply the Lemma with $T = R/P$. What can you deduce about G, H?
- **(c)** Consider the constant coefficient of GH, and obtain a decisive contradiction.

(3) More Examples:
- (a) Show that the polynomial $x^3 + y^3 + z^3$ is irreducible in $\mathbb{C}[x, y, z]$.
- (b) Show that[1] the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.
- (c) Let p be a prime integer. Show that $x^{p-1} + x^{p-2} + \cdots + 1$ is irreducible in $\mathbb{Q}[x]$.

[1] Hint: Show that $f(x + 1)$ is irreducible.