SMITH NORMAL FORM

THEOREM (SMITH NORMAL FORM): Let R be a PID. Let $A \in \mathrm{Mat}_{m\times n}(R)$.

(i) There exist invertible matrices P, Q such that
- $PAQ = D$ is diagonal, meaning $d_{ij} = 0$ whenever $i \neq j$, and
- $d_{11} \mid d_{22} \mid \cdots \mid d_{tt}$, where d_{tt} is the last nonzero diagonal entry.

(ii) The elements d_{ii} are unique up to associate, meaning that if $D' = [d'_{ij}]$ is another diagonal matrix as in (1), then for each d'_{ii} is a unit times d_{ii}.

(iii) If R is a Euclidean domain, then P, Q can be taken to be products of elementary matrices.

STRUCTURE THEOREM FOR FINITELY GENERATED MODULES OVER PIDS (INVARIANT FACTOR FORM): Let R be a PID. Let M be a finitely generated R-module. Then there exist $r, t \geq 0$ and $a_1, \dots, a_t \in R$ such that
- $M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_t)$, and
- $a_1 \mid a_2 \mid \cdots \mid a_t$.

Moreover, each a_i is uniquely determined up to associates.

(1) Let R be a commutative ring, and $D \in \mathrm{Mat}_{m\times n}(R)$ be a diagonal matrix with nonzero diagonal entries $d_{11}, \dots, d_{tt}$. Prove that the module presented by D is

$$R^{m-t} \oplus R/(d_{11}) \oplus R/(d_{22}) \oplus \cdots \oplus R/(d_{tt}).$$

(2) Use the Smith Normal Form Theorem to deduce the Structure Theorem for Finitely Generated Modules over PIDs (Invariant Factor Form).

(3) State the Structure Theorem for Finitely Generated Abelian Groups (Invariant Factor Form), and deduce it from the PID Theorem.

(4) Let R be a Euclidean domain. Use the Smith Normal Form Theorem to deduce that any invertible matrix over R is a product of elementary matrices.

(5) Prove[1] the uniqueness part (ii) of Smith Normal Form Theorem.

DEFINITION: Let R be a domain and M be an R-module. We say that M is **torsionfree** if for $r \in R$ and $m \in M$, we have $rm = 0$ implies $r = 0$ or $m = 0$.

(6) Let R be a PID.
- **(a)** Show that any finitely generated torsionfree R-module is free.
- **(b)** Show that any submodule of a finitely generated free R-module is free.
- **(c)** Prove or disprove: any torsionfree R-module is free.
- (d) Prove or disprove: any submodule of a free R-module is free.

[1]Hint: In HW#4, you proved that $I_t(PAQ) = I_t(A)$ for invertible matrices P, Q.

STRUCTURE THEOREM FOR FINITELY GENERATED MODULES OVER PIDS (ELEMENTARY DIVISOR FORM): Let R be a PID. Let M be a finitely generated R-module. Then there exist $r, s \geq 0$ and prime elements $p_1, \ldots, p_s \in R$ such that $M \cong R^r \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_s^{e_s})$, and Moreover, the list $p_1^{e_1}, \ldots, p_s^{e_s}$ is unique up to reordering and associates.