

## PROBLEM SET #1

- (1) \* Basic rules with derivations:
  - (a) Prove the generalized product rule for derivations: if  $\partial : R \rightarrow M$  is a derivation, then  $\partial(a_1 \cdots a_n) = \sum_{i=1}^n (\prod_{j \neq i} a_j) \partial(a_i)$ .
  - (b) Prove the power rule for derivations: if  $\partial : R \rightarrow M$  is a derivation, then  $\partial(r^n) = nr^{n-1}\partial(r)$ .
  - (c) Show that if  $R$  is a ring of characteristic  $p$ , then the subring  $R^p := \{r^p \mid r \in R\}$  is in the kernel of every derivation.
  
- (2) \* Let  $A$  be a ring and  $S = A[x_1, \dots, x_n]$  be a polynomial ring.
  - (a) Let  $R$  be an  $\mathbb{N}$ -graded  $A$ -algebra such that  $A$  lives in degree zero. Show that there is a derivation on  $R$  such that for every homogeneous element  $f$  of degree  $d$ ,  $\partial(f) = df$ . This derivation is called the *Euler operator* associated to the grading.
  - (b) Let  $S = A[x_\lambda \mid \lambda \in \Lambda]$  be a polynomial ring over  $A$  endowed with the  $\mathbb{N}$ -grading by the rule  $\deg(x_\lambda) = n_\lambda$ . Express the Euler operator of the grading as an  $S$ -linear combination of the partial derivatives.
  
- (3) Let  $A$  be a ring and  $R = A[x_1, \dots, x_n]$  be a polynomial ring.
  - (a) Give an explicit formula for the Lie algebra bracket on  $\text{Der}_{R|A}(R)$ .
  - (b) Does  $\text{Der}_{R|A}(R)$  have any nontrivial proper Lie ideals (i.e.,  $A$ -submodules  $B$  such that  $[d, b] \in B$  for all  $b \in B$  and  $d \in \text{Der}_{R|A}(R)$ )?
  
- (4) Let  $R$  be a ring of characteristic  $p > 0$  and  $\partial : R \rightarrow R$  be a derivation. Show that  $\partial^p$ , i.e., the  $p$ -fold self composition of  $\partial$ , is a derivation on  $R$ .
  
- (5) Let  $R = C^\infty(\mathbb{R}^n)$  be the ring of smooth functions on  $\mathbb{R}^n$ , and  $\mathfrak{m}$  be the maximal ideal consisting of functions that vanish at some point  $x_0 \in \mathbb{R}^n$ .
  - (a) \* Show that  $\mathfrak{m}^t$  consists of the functions  $f \in \mathbb{R}$  such that  $\frac{d^{a_1}}{dx_1^{a_1}} \cdots \frac{d^{a_n}}{dx_n^{a_n}}(f)|_{x=x_0} = 0$  for all  $a_1, \dots, a_n$  with  $a_1 + \cdots + a_n < t$ .
  - (b) Show that  $\text{Der}_{R|\mathbb{R}}(R/\mathfrak{m}) \cong (\mathfrak{m}/\mathfrak{m}^2)^* \cong \mathbb{R}^n$  as vector spaces.

As a moral, we conclude that  $\text{Der}_{R|\mathbb{R}}(R/\mathfrak{m})$  serves as a model for the tangent space of  $\mathbb{R}^n$  at  $x_0$  constructed from the ring of smooth functions.
  
- (6) \* Let  $R$  be an  $A$ -algebra and  $I$  an ideal. Show that if the identity map on  $I/I^2$  is in the image of  $\text{Der}_{R|A}(I/I^2) \xrightarrow{\text{res}} \text{Hom}_A(I/I^2, I/I^2)$ , then there is an  $A$ -algebra right inverse to the quotient map  $\pi : R/I^2 \rightarrow R/I$ . Conclude that the following are equivalent:
  - $\text{Der}_{R|A}(M) \xrightarrow{\text{res}} \text{Hom}_A(I/I^2, M)$  is surjective for all  $R/I$ -modules  $M$ ;
  - $\text{Der}_{R|A}(I/I^2) \xrightarrow{\text{res}} \text{Hom}_A(I/I^2, I/I^2)$  is surjective;
  - The quotient map  $R/I^2 \rightarrow R/I$  has an  $A$ -algebra right inverse.