

CONTINUED FRACTIONS

DEFINITION: A **finite continued fraction** is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

for some integers $a_0 \in \mathbb{Z}$, $a_1, \dots, a_n \in \mathbb{Z}_{>0}$.

We write $[a_0; a_1, \dots, a_n]$ as shorthand for this.

By a **continued fraction** we mean either an infinite or finite continued fraction. We call the numbers a_i the **partial quotients** in the continued fraction.

An **infinite continued fraction** is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots}}}}$$

for some integers $a_0 \in \mathbb{Z}$, $a_1, a_2, a_3, \dots \in \mathbb{Z}_{>0}$.

We write $[a_0; a_1, a_2, \dots]$ as shorthand for this.

(1) Evaluating finite continued fractions:

(a) Evaluate $2 + \frac{1}{13 + \frac{1}{2}}$.

(b) Evaluate $[3; 2, 1, 4]$

(c) Explain why every finite continued fraction evaluates to a rational number.

(a) $\frac{56}{27}$.

(b) $\frac{47}{14}$.

(c) A finite continued fraction is made out of integers from addition and division.

(2) Using the Euclidean algorithm to compute finite continued fractions:

(a) What type of computation is the computation below?

$$250 = 2 \cdot 117 + 16$$

$$117 = 7 \cdot 16 + 5$$

$$16 = 3 \cdot 5 + 1$$

$$5 = 5 \cdot 1$$

(b) How does one obtain $\frac{250}{117} = 2 + \frac{1}{\frac{117}{16}}$ from the computation above?

(c) Repeat (b) to obtain a finite continued fraction expansion for $\frac{250}{117}$.

(d) Use the steps above to obtain a finite continued fraction expansion for $\frac{7}{5}$.

(e) Use the steps above to obtain a finite continued fraction expansion for $\frac{39}{314}$.

(f) What is the general formula for the continued fraction $[a_0; a_1, \dots, a_n]$ for m/n in terms of the Euclidean algorithm?

(a) Euclidean algorithm.

(b) Divide the first line by 117 and flip the last fraction.

(c) $\frac{250}{117} = 2 + \frac{1}{7 + \frac{1}{3 + \frac{1}{5}}}$.

(d) $\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}}$.

(e) $\frac{39}{314} = \frac{1}{8 + \frac{1}{19 + \frac{1}{2}}}$.

(f) The a_i 's are just the quotients in the Euclidean algorithm.

(3) Euclidean algorithm and continued fraction algorithm:

(a) In the computation from (2a) above, check that

$$2 = \left\lfloor \frac{250}{117} \right\rfloor \text{ and that } \frac{117}{16} = \left(\frac{250}{117} - \left\lfloor \frac{250}{117} \right\rfloor \right)^{-1}.$$

(b) More generally, in the Euclidean algorithm

$$\begin{array}{ccccccc} \vdots & \vdots & \vdots & \vdots & & & \\ \textcolor{red}{u}_i & = & q_i & \cdot & \textcolor{blue}{v}_i & + & \textcolor{red}{r}_i & (u_{i+1} = v_i) \\ \textcolor{blue}{u}_{i+1} & = & q_{i+1} & \cdot & \textcolor{blue}{v}_{i+1} & + & \textcolor{red}{r}_{i+1} & (v_{i+1} = r_i) \\ \vdots & \vdots & \vdots & \vdots & & & & \end{array}$$

show that

$$q_i = \left\lfloor \frac{\textcolor{red}{u}_i}{\textcolor{blue}{v}_i} \right\rfloor \text{ and } \frac{\textcolor{blue}{u}_{i+1}}{\textcolor{blue}{v}_{i+1}} = \left(\frac{\textcolor{red}{u}_i}{\textcolor{blue}{v}_i} - \left\lfloor \frac{\textcolor{red}{u}_i}{\textcolor{blue}{v}_i} \right\rfloor \right)^{-1}.$$

(a) ✓

(b) The formula for q_i is the general formula in the division algorithm (since $u_i/v_i - 1 < \lfloor u_i/v_i \rfloor \leq u_i/v_i$ implies $v_i > u_i - \lfloor u_i/v_i \rfloor v_i \geq 0$.) We then have

$$\frac{u_{i+1}}{v_{i+1}} = \frac{v_i}{r_i} = \frac{v_i}{u_i - \lfloor \frac{u_i}{v_i} \rfloor v_i} = \frac{1}{\frac{u_i}{v_i} - \lfloor \frac{u_i}{v_i} \rfloor}.$$

DEFINITION: Given an infinite continued fraction $[a_0; a_1, a_2, \dots]$, the k -th **convergent** of the continued fraction is the value C_k of the finite continued fraction $[a_0; a_1, \dots, a_k]$.

THEOREM (CONVERGENCE OF CONTINUED FRACTIONS): Every infinite continued fraction converges to a real number; i.e., for any $[a_0; a_1, a_2, a_3, \dots]$ with $a_0 \in \mathbb{Z}$ and $a_1, a_2, \dots \in \mathbb{Z}_{>0}$, the sequence of convergents C_1, C_2, C_3, \dots converges. We call this limit the value of the infinite continued fraction.

CONTINUED FRACTION ALGORITHM: Given a real number r ,

(I) Start with $\beta_0 := r$ and $n := 0$.

(II) Set $a_n := \lfloor \beta_n \rfloor$.

(III) If $a_n = \beta_n$, **STOP**; the continued fraction is $[a_0; a_1, \dots, a_n]$.

Else, set $\beta_{n+1} := (\beta_n - a_n)^{-1}$, and return to Step (II).

If the algorithm does not terminate, the continued fraction is $[a_0; a_1, a_2, \dots]$.

THEOREM (CORRECTNESS OF CONTINUED FRACTION ALGORITHM): For any real number r , the continued fraction obtained from the Continued Fraction Algorithm with input r converges to r .

PROPOSITION: Let r be a real number. The Continued Fraction Algorithm with input r terminates in finitely many steps if and only if r is rational.

DIRICHLET APPROXIMATION THEOREM: Let $r = [a_0; a_1, a_2, a_3, \dots]$ be a real number. Then for every convergent $C_k = \frac{p_k}{q_k}$ (in lowest terms), we have $\left| r - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}$.

In particular, if r is irrational, there are infinitely many rational numbers $\frac{p}{q}$ such that $\left| r - \frac{p}{q} \right| < \frac{1}{q^2}$.

- (4) Use the continued fraction algorithm to find the first four ($n \leq 3$) partial quotients and convergents for $\sqrt{2}$, and π . Can you find the whole continued fraction for either of these?

$\sqrt{2} = [1; 2, 2, 2, \dots]$ and 2's forever, since $\beta_i = \sqrt{2} + 1$ for all $i > 0$, with $C_0, C_1, C_2, C_3 = 1, 3/2, 7/5, 12/5$. $\pi = [3; 7, 15, 1, \dots]$ and a mysterious pattern, with $C_0, C_1, C_2, C_3 = 3, 22/7, 333/106, 355/113$.

- (5) Find¹ the value of the continued fraction $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$.

We have $L = 1 + 1/L$, so $L^2 = L + 1$. This has two roots $\frac{1 \pm \sqrt{5}}{2}$. Since $L > 0$, we must have $L = \frac{1 + \sqrt{5}}{2}$, the golden ratio.

- (6) Continued fraction algorithm and rational numbers.
 (a) Explain why the continued fraction algorithm just creates a continued fraction in the same way the Euclidean algorithm does as we did in problem (2).
 (b) Explain why the Proposition above is true.

- (a) This was the point of problem (3).
 (b) If the algorithm terminates, then r has a finite continued fraction, and hence is rational. Conversely, if r is rational, the continued fraction algorithm follows the Euclidean algorithm and after finitely many steps returns a finite continued fraction.

- (7) Dirichlet Approximation Theorem.
 (a) Let r be any real number. Explain why for *any* positive integer q , there is some integer p such that $|r - \frac{p}{q}| < \frac{1}{q}$. Conclude that $|r - \frac{p}{q}| < \frac{1}{q}$ is “not very impressive”.
 (b) For $r = \sqrt{2}$, find all rational numbers p/q with $|r - \frac{p}{q}| < \frac{1}{q^2}$ with $q \leq 6$ and compare to the list of convergents C_0, C_1, C_2 . What about $|r - \frac{p}{q}| < \frac{1}{2q^2}$? Conclude that $|r - \frac{p}{q}| < \frac{1}{q^2}$ is “pretty impressive”.
 (c) Discuss $\pi \approx \frac{22}{7}$ in the context of the results above. Give a better approximation.

- (a) Set $p = \lfloor r/q \rfloor$.
 (b) For the first, we just have C_0, C_1, C_2 along with $\frac{2}{1}$ and $\frac{4}{3}$. For the second, just C_0, C_1, C_2 . We are impressed.
 (c) This is a good approximation in the sense of Dirichlet Approximation Theorem, since it comes from the continued fraction. $\pi \approx 355/113$ is a very good approximation.

PROPOSITION: Let $[a_0; a_1, a_2, \dots]$ be a continued fraction. Set

$$\begin{aligned} p_0 &:= a_0, & p_1 &:= a_0 a_1 + 1, & p_k &:= a_k p_{k-1} + p_{k-2} \\ q_0 &:= 1, & q_1 &:= a_1, & q_k &:= a_k q_{k-1} + q_{k-2}. \end{aligned}$$

Then,

- (1) $C_k = \frac{p_k}{q_k}$ for all $k \geq 0$, and
 (2) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ for all $k \geq 1$.

¹Hint: This limit has a value L . Find an equation that L satisfies by recognizing L as a smaller piece of this continued fraction.

(8) Proof of convergence Theorem and Dirichlet Approximation Theorem.

- (a) Use the Proposition above to show that $C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$ for all $k \geq 1$.
- (b) Use the Proposition above to show that $C_k - C_{k-2} = \frac{(-1)^k a_k}{q_k q_{k-2}}$ for all $k \geq 2$.
- (c) Use (8b) to show that the sequence C_0, C_2, C_4, \dots is increasing, that the sequence C_1, C_3, C_5, \dots is decreasing; use (8a) to show that $C_{2k} < C_{2\ell+1}$ for all k, ℓ . Deduce that $\lim_{k \rightarrow \infty} C_{2k} = \sup\{C_{2k} \mid k \in \mathbb{N}\}$ and $\lim_{\ell \rightarrow \infty} C_{2\ell+1} = \inf\{C_{2\ell+1} \mid \ell \in \mathbb{N}\}$ both exist.
- (d) Use (8a) to show that $\sup\{C_{2k} \mid k \in \mathbb{N}\} = \inf\{C_{2\ell+1} \mid \ell \in \mathbb{N}\}$, and hence that $\lim_{n \rightarrow \infty} C_n$ exists and is equal to both of these values. Thus, every continued fraction converges.
- (e) Suppose that β is the value of our continued fraction. Use (8d) to show that $|\beta - C_n| \leq |C_{n+1} - C_n|$, and use (8a) to deduce Dirichlet's Approximation.

(a)

$$C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

(b)

$$\begin{aligned} C_k - C_{k-2} &= C_k - C_{k-1} + C_{k-1} - C_{k-2} = \frac{(-1)^{k-1}}{q_k q_{k-1}} + \frac{(-1)^{k-2}}{q_{k-1} q_{k-2}} \\ &= (-1)^k \frac{-q_{k-2} + q_k}{q_k q_{k-1} q_{k-2}} = \frac{(-1)^k a_k q_{k-1}}{q_k q_{k-1} q_{k-2}} = \frac{(-1)^k a_k}{q_k q_{k-2}} \end{aligned}$$

(c) From (8b), we have $C_k - C_{k-2} > 0$ (so $C_k > C_{k-2}$) if k is even and $C_k - C_{k-2} < 0$ (so $C_k < C_{k-2}$) if k is odd. Thus, the sequence C_0, C_2, C_4, \dots is increasing and the sequence C_1, C_3, C_5, \dots is decreasing. By (8a), $C_{2\ell+1} - C_{2\ell} > 0$, so $C_{2\ell+1} > C_{2\ell}$; if $\ell \leq k$, then $C_{2\ell+1} > C_{2\ell} > C_{2k}$; if $\ell \geq k$, then $C_{2\ell+1} > C_{2k+1} > C_{2k}$. Then the sequence $(C_{2k})_{k=1}^{\infty}$ is increasing and bounded above (by, e.g., C_1), and the sequence $(C_{2\ell+1})_{\ell=1}^{\infty}$ is decreasing and bounded below (by, e.g., C_0). By the monotone convergence theorem, these sequences converge to their sup and inf, respectively.

(d) Suppose $\sup\{C_{2k}\} < \inf\{C_{2\ell+1}\}$, and let $\delta = \inf\{C_{2\ell+1}\} - \sup\{C_{2k}\}$. Let $2n$ be an even number larger than $1/\delta$. Then

$$C_{2n} < \sup\{C_{2k}\} < \inf\{C_{2\ell+1}\} < C_{2n+1}$$

implies $C_{2n+1} - C_{2n} > 1/(2n)$, but we also have $C_{2n+1} - C_{2n} = 1/(q_{2n} q_{2n-1})$. Since $q_{2n} > 2n$, this is a contradiction. It follows that the sequence of convergents converges.

(e) If n is even, then we have $C_n < \sup\{C_{2k}\} = \beta = \inf\{C_{2\ell+1}\} < C_{n+1}$, and if n is odd, we have $C_{n+1} < \sup\{C_{2k}\} = \beta = \inf\{C_{2\ell+1}\} < C_n$. This shows that $|\beta - C_n| \leq |C_{n+1} - C_n|$. Then from (8a), $|C_{n+1} - C_n| = 1/(q_n q_{n+1}) < 1/q_n^2$.

(9) Prove the Proposition above.

(10) Proof of Correctness of Continued Fraction Algorithm:

If r is rational, the algorithm terminates and returns r , so we can assume that r is irrational and that the algorithm does not terminate. Given r , let $a_0, a_1, a_2, a_3, \dots$ and $\beta_0, \beta_1, \beta_2, \dots$ be the sequences arising from the continued fraction algorithm.

- (a) Explain why $r = [a_0; a_1, \dots, a_k, \beta_{k+1}]$. (Note, β_{k+1} is not an integer, but we can plug it into a finite continued fraction anyway.)
- (b) Explain why $r = \frac{\beta_{k+1} p_k + p_{k-1}}{\beta_{k+1} q_k + q_{k-1}}$ where p_k, q_k , where p_k, q_k are the numbers coming from the continued fraction (with an irrational number snuck in) $[a_0; a_1, \dots, a_k, \beta_{k+1}]$ as in the Proposition above.

(c) Show that $|r - C_k| < \frac{1}{q_k q_{k+1}}$ for all $k \geq 1$ and deduce the result.

(11) Prove the following theorem, which basically says that the convergents are the *best* approximations of a rational number.

THEOREM: Let r be a real number, $C_k = \frac{p_k}{q_k}$ be the k -th convergent of r , and $\frac{p}{q} \neq r$ be a rational number. If $q \geq q_k$, then $\left| r - \frac{p}{q} \right| \geq \left| r - \frac{p_k}{q_k} \right|$.