

SUBSEQUENCES

(1) **True or false; justify.**

- (a) The sequence $\left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.
- (b) The sequence $\left\{\frac{1}{3n+7}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.
- (c) The constant sequence $\left\{\frac{1}{2}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.
- (d) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are both subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.
- (e) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are the only two subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(2) Explain how the following Corollary follows from Theorem 15.5.

Corollary 15.7: Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.

- (a) If there is a subsequence of this sequence that diverges, then the sequence itself diverges.
- (b) If there are two subsequences of this sequence that converge to different values, then the sequence itself diverges.

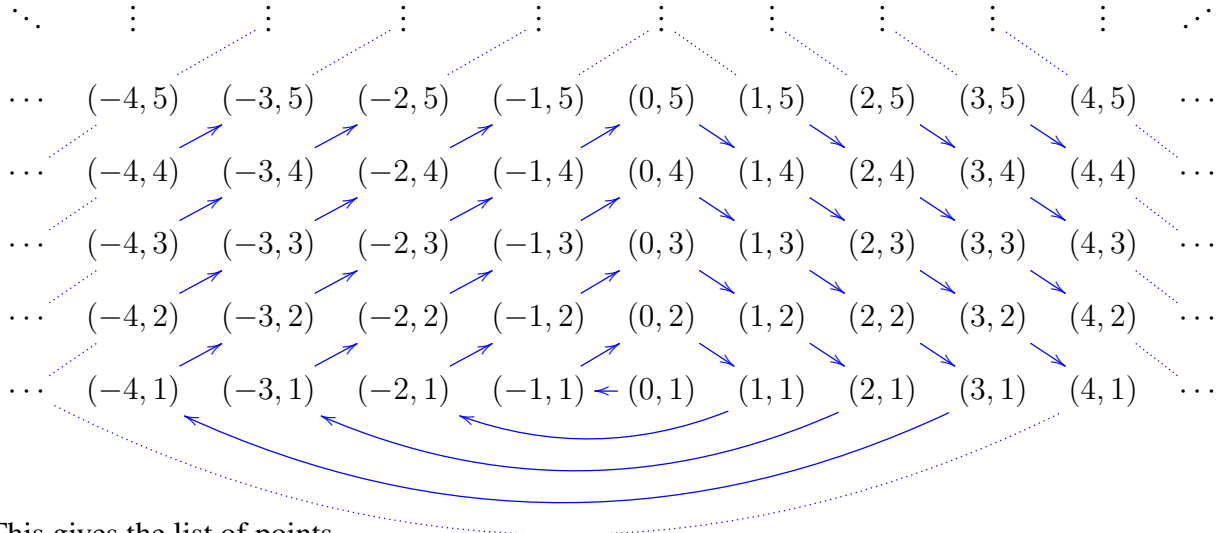
(3) Use Corollary 15.7 to give a quick proof that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

(4) **Prove or disprove:**

- (a) Every subsequence of a bounded sequence is bounded.
- (b) Every subsequence of a divergent sequence is divergent.
- (c) Every subsequence of a sequence that diverges to $-\infty$ also diverges to $-\infty$.

A WILD SEQUENCE

Consider the points in the plane whose x -coordinates are integers and y -coordinates are natural numbers. Starting at $(0, 1)$, zigzag like so:



This gives the list of points

$$(0, 1), (-1, 1), (0, 2), (1, 1), (-2, 1), (-1, 2), (0, 3), (1, 2), (2, 1), (-3, 1), \dots$$

Now convert these to a list of rational numbers by changing (m, n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1}, \frac{-1}{1}, \frac{0}{2}, \frac{1}{1}, \frac{-2}{1}, \frac{-1}{2}, \frac{0}{3}, \frac{1}{2}, \frac{2}{1}, \frac{-3}{1}, \dots$$

of rational numbers. Call this sequence $\{w_n\}_{n=1}^{\infty}$.

- (5) Explain why every rational number $q \in \mathbb{Q}$ occurs in $\{w_n\}_{n=1}^{\infty}$ infinitely many times.¹
- (6) Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers. Explain why $\{q_n\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$. [This is saying that *every* sequence of rational numbers is a subsequence of this single sequence!]
- (7) Let r be any real number. Show that there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ converges to r . [This is saying that *every* real number occurs as a limit of a subsequence of this single sequence!]

¹Hint: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$.