Math 817 Review Sheet #1

A (not necessarily complete) list of important things to know for Exam I

Examples of Groups: Cyclic groups (i.e., $\mathbf{Z}$ and $\mathbf{Z}_n$), matrix groups (e.g., $\mathrm{GL}_n(\mathbf{R})$), the dihedral groups, the quaternions, permuation groups (i.e., S_n and A_n), free groups.

Orders of elements: If $|x| = n$ then $x^s = 1$ if and only if n divides s. The order of an element equals the order of the cyclic subgroup it generates.

Subgroups of Cyclic Groups: Let $G = \langle x \rangle$ be a cyclic group of finite order n. Then every subgroup of G is cyclic. Moreover, for every positive divisor d of n there is a unique subgroup of G order d, namely $\langle x^{\frac{n}{d}} \rangle$.

Cosets: Let G be a group, H a subgroup, and $x \in G$. The left coset xH is defined to be $\{xh \in H\}$. The set of left cosets of H partitions the group G. Also, $|xH| = |H|$ for every $x \in H$. (This gives us Lagrange's Theorem below.) We let $[G : H]$ (the index of H in G) be the number of left cosets. Another useful fact: $xH = yH$ if and only if $y^{-1}x \in H$.

Cayley's Theorem: Let G be a group and H a subgroup such that $[G : H] = n$. Then, by letting G act on the set of left cosets of H we obtain a homomorphism $\phi : G \to S_n$. The $\ker \phi$ is the largest normal subgroup contained in H. Also, $\ker \phi = \bigcap_{x \in G} xHx^{-1}$.

Lagrange's Theorem: If G is a finite group and H is a subgroup then $|G| = |H|[G : H]$. In particular, $|H|$ divides $|G|$.

Products of Subgroups: Let H and K be subgroups of G. Then $HK := \{hk \mid h \in H, k \in K\}$.

1. $|HK| = \frac{|H||K|}{|H \cap K|}$.
2. HK is a subgroup if and only if $HK = KH$.
3. If either H or K is normal then HK is a subgroup.

Quotient Groups: Let K be a normal subgroup of G.

1. The set of left cosets of K form a group under coset multiplication. We denote this group by G/K.
2. $|G/K| = |G|/|K|$.
3. If H is a subgroup of G containing K then H/K is a subgroup of G/K. Moreover, any subgroup of G/K can be written uniquely in the form H/K where H is subgroup of G containing K.
4. If H is a subgroup of G containing K then $H/K \triangleleft G/K$ if and only if $H \triangleleft G$.

Isomorphism Theorems:

1. Let G be a group and $K \subseteq H$ normal subgroups of G. Then $H/K \triangleleft G/K$ if and only if $H \triangleleft G$.
2. Let $\phi : G \to G'$ be a group homomorphism. Then $G/\ker \phi \cong \operatorname{im} \phi$.
3. Let H and K be subgroups of G with $K \triangleleft G$. Then $H \cap K \triangleleft H$ and $HK/K \cong H/H \cap K$.

Generators and presentations: What it means for a subset to generate a group, the group generated by a subset, the group with a given presentation.

Group Actions: Suppose G acts on a set X.

1. For $a \in X$, the orbit of a is $\mathcal{O}(a) := \{ga \mid g \in G\}$.
2. For $a \in X$, the stabilizer of a is $G_a := \{g \in G \mid ga = a\}$.
3. (Orbit-Stabilizer Theorem) For any $a \in X$, $|\mathcal{O}(a)| = [G : G_a]$. In particular, the size of any orbit divides the order of G.
4. The action is called *faithful* if $\bigcap_{a \in X} G_a = \{1\}$; equivalently, for any $g \in G$ there exists an $a \in X$ such that $ga \neq a$.
5. The action is called *transitive* if there is only one orbit; equivalently, for any $a, b \in X$ there exists a $g \in G$ such that $ga = b$.