

GENERATORS, LINEAR DEPENDENCE, AND BASES

DEFINITION: Let R be a ring and M be a (left) R -module. A linear combination of finitely many elements m_1, \dots, m_n of M is an element of the form $r_1m_1 + \dots + r_nm_n \in M$ for some $r_1, \dots, r_n \in R$.

DEFINITION: Let R be a ring and M be a (left) R -module. Let A be a subset of M . The submodule of M generated by A is the submodule RA of M given by the three equivalent following descriptions:

- RA is the unique smallest R -submodule of M containing A .
- $RA = \bigcap N_\lambda$, where N_λ ranges over all submodules of M containing A .
- $RA = \{r_1m_1 + \dots + r_tm_t \mid r_i \in R, m_i \in A\}$, the set of linear combinations of elements of A .

DEFINITION: Let R be a ring and M be a (left) R -module. Let A be a subset of M .

- We say that A generates M if $RA = M$.
- We say that A is linearly independent if for $m_1, \dots, m_t \in A$ distinct and any $r_1, \dots, r_t \in R$,

$$r_1m_1 + \dots + r_tm_t = 0 \quad \text{implies} \quad r_1 = \dots = r_t = 0.$$

- We say that A is a basis of M if A is linearly independent and generates M .
- We say that M is free if there exists a basis A for M .

(1) Let $R = \mathbb{Z}$ and consider the R -module $M = \mathbb{Z}/n$ for some $n > 1$.

- (a) Explain why any nonempty subset of M is not linearly independent.
- (b) Explain why M is not a free module.
- (c) An R -module is cyclic if it is generated by a single element. Show that M is cyclic.
- (d) Does every generating set of M consist of a single element?

(2) Let R be a commutative ring. Let $R[x]$ be a polynomial ring over R .

- (a) Explain why $\{1, x, x^2, x^3, \dots\}$ is a basis for $R[x]$ as an R -module.
- (b) Give an example of a set that is R -linearly independent in $R[x]$ that is not a basis.
- (c) Give an example of a set that generates $R[x]$ that is not a basis.
- (d) Give a different example of a basis for $R[x]$.

(3) Show that an R -module M is cyclic if and only if $M \cong R/I$ for some left ideal I .

(4) Let $R = \mathbb{Z}[x]$ and I be the ideal $(2, x)$, considered as an R -module.

- (a) Explain² why I is not cyclic.
- (b) Show that I is not free.
- (c) Give an example of a pair of modules $N \subseteq M$ where N requires more generators than M .
- (d) Give an example of a pair of modules $N \subseteq M$ where M is free and N is not.

(5) We say that an R -module M is simple if the only submodules of M are 0 and M . Let R be a commutative ring. Show that M is simple if and only if $M \cong R/\mathfrak{m}$ for some maximal ideal \mathfrak{m} of R .

(6) Let R be a commutative ring, and x, y be two indeterminates.

- (a) Show that $R[x, y]$ is a free $R[x]$ -module, and find a basis.
- (b) Show that $R[x, y]$ is a free R module, and find a basis.

¹This includes 0 as the “empty sum”.

²Reuse something from 817!