USING THEOREMS ON CONVERGENT SEQUENCES §2.2

(1) Which of the following implications about sequences hold in general? Either mention a relevant theorem or give a counterexample.

(a) monotone $\Longrightarrow$ convergent
(b) convergent $\Longrightarrow$ bounded
(c) bounded + decreasing $\Longrightarrow$ convergent
(d) increasing + convergent $\Longrightarrow$ bounded
(e) convergent $\Longrightarrow$ monotone
(f) bounded $\Longrightarrow$ convergent

(2) Show[1] that the sequence $\left\{ \dfrac{n^2 - 15\sqrt{n}\sin(n)}{3n^2} \right\}_{n=1}^{\infty}$ converges and determine to what number it converges.

(3) Prove or disprove: If $a_n^2 < 4$ and $a_n < a_{n+1}$ for all n, then $\{a_n\}_{n=1}^{\infty}$ converges.

(4) Prove that for any real number r, there exists a sequence of *rational* numbers that converges to r.
Hint: Show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of rational numbers such that $r - \frac{1}{n} < a_n < r$.

(5) Prove that if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence and $\{b_n\}_{n=1}^{\infty}$ converges to 0, then $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0.

(6) Prove or disprove: The sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 1 + \frac{1}{2^3} + \cdots + \frac{1}{n^3}$ is convergent.

(7) Prove or disprove: The sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is convergent.

[1] You can use any basic properties about the sine function from trig, like which values of $\sin(x)$ are equal to $0, 1,$ or -1, and that $-1 \leq \sin(x) \leq 1$.