§2.9: NOETHERIAN RINGS

DEFINITION: A ring R is **Noetherian** if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some N such that $I_n = I_N$ for all $n \geq N$.

HILBERT BASIS THEOREM: If R is a Noetherian ring, then the polynomial ring $R[X]$ and power series ring $R[\![X]\!]$ are also Noetherian.

We will return to the proof of Hilbert Basis Theorem after discussing Noetherian modules next time.

COROLLARY: Every finitely generated algebra over a field is Noetherian.

(1) Equivalences for Noetherianity.
- **(a)** Show[1] that R is Noetherian if and only if every ideal is finitely generated.
- **(b)** Show[2] that R is Noetherian if and only if every nonempty collection of ideals has a maximal[3] element.

(2) Some Noetherian rings:
- **(a)** Show that fields and PIDs are Noetherian.
- **(b)** Show that if R is Noetherian and $I \subseteq R$, then R/I is Noetherian.
- **(c)** Is[4] every subring of a Noetherian ring Noetherian?

(3) Use the Hilbert Basis Theorem to deduce the Corollary.

(4) Some nonNoetherian rings:
- **(a)** Let K be a field. Show that $K[X_1, X_2, \dots]$ is not Noetherian.
- (b) Let K be a field. Show that $K[X, XY, XY^2, \dots]$ is not Noetherian.
- (c) Show that $\mathcal{C}([0,1], \mathbb{R})$ is not Noetherian.

(5) Let R be a Noetherian ring. Show that for every ideal I, there is some n such that $\sqrt{I}^n \subseteq I$. In particular, there is some n such that for every nilpotent element z, $z^n = 0$.

(6) Let R be Noetherian. Show that every element of R admits a decomposition into irreducibles.

(7) Prove the principle of **Noetherian induction**: Let $\mathcal{P}$ be a property of a ring. Suppose that "For every nonzero ideal I, $\mathcal{P}$ is true for R/I implies that $\mathcal{P}$ is true for R" and $\mathcal{P}$ holds for all fields. Then $\mathcal{P}$ is true for every Noetherian ring.

(8)
- (a) Suppose that every maximal ideal of R is finitely generated. Must R be Noetherian?
- (b) Suppose that every ascending chain of prime ideals stabilizes. Must R be Noetherian?
- (c) Suppose that every prime ideal of R is finitely generated. Must R be Noetherian?

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[1] For the backward direction, consider $\bigcup_{n\in\mathbb{N}} I_n$

[2] Hint: For the forward direction, show the contrapositive.

[3] This means that if $\mathcal{S}$ is our collection of ideals, there is some $I \in \mathcal{S}$ such that no $J \in \mathcal{S}$ properly contains I. It does not mean that there is a maximal ideal in $\mathcal{S}$.

[4] Hint: Every domain has a fraction field, even the domain from (4a).