

MATH 902 LECTURE NOTES, SPRING 2022

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Lecture of January 19, 2022

In this class, all rings are assumed to be commutative, with associative multiplication and containing 1.

1. FINITENESS CONDITIONS

1.1. Finitely generated algebras. We start by recalling a definition from last semester, specialized to the setting of commutative rings.

Definition 1.1 (Algebra). Given a ring A , an A -algebra is a ring R equipped with a ring homomorphism $\phi : A \rightarrow R$. This defines an A -module structure on R given by restriction of scalars, that is, for $a \in A$ and $r \in R$, $ar := \phi(a)r$ that is compatible with the internal multiplication of R i.e.,

$$a(rs) = (ar)s = r(as) \text{ for all } a \in A, rs \in R.$$

We will call ϕ the *structure homomorphism* of the A -algebra R .

Example 1.2.

- If A is a ring and x_1, \dots, x_n are indeterminates, the inclusion map $A \hookrightarrow A[x_1, \dots, x_n]$ makes the polynomial ring into an A -algebra.
- When $A \subseteq R$ the inclusion map makes R an A -algebra. In this case the A -module multiplication ar coincides with the internal (ring) multiplication on R .

- Any ring comes with a unique structure as a \mathbb{Z} -algebra.

The collection of A -algebras forms a category where the morphisms are ring homomorphisms $f : R \rightarrow S$ such that the following diagram commutes

$$\begin{array}{ccc} & A & \\ \phi \swarrow & & \searrow \psi \\ R & \xrightarrow{f} & S \end{array}$$

for structural homomorphisms $\varphi : A \rightarrow R$ and $\psi : A \rightarrow S$.

Definition 1.3 (Algebra generation). Let R be an A -algebra and let $\Lambda \subseteq R$ be a set. The A -algebra generated by a subset Λ of R , denoted $A[\Lambda]$, is the smallest (w.r.t containment) subring of R containing Λ and $\varphi(A)$.

A set of elements $\Lambda \subseteq R$ generates R as an A -algebra if $R = A[\Lambda]$.

Note that there are two different meanings for the notation $A[S]$ for a ring A and set S : one calls for a polynomial ring, and the other calls for a subring of something.

This can be unpackaged more concretely in a number of equivalent ways:

Lemma 1.4. *The following are equivalent*

- (1) Λ generates R as an A -algebra.
- (2) Every element in R admits a polynomial expression in Λ with coefficients in $\phi(A)$, i.e.

$$R = \left\{ \sum_{\text{finite}} \phi(a) \lambda_1^{i_1} \cdots \lambda_n^{i_n} \mid a \in A, \lambda_j \in \Lambda, i_j \in \mathbb{N} \right\}.$$

- (3) The A -algebra homomorphism $\psi : A[X] \rightarrow R$, where $A[X]$ is a polynomial ring on a set of indeterminates X in bijection with Λ and $\psi(x_i) = \lambda_i$, is surjective.

Proof. Let $S = \{ \sum_{\text{finite}} \phi(a) \lambda_1^{i_1} \cdots \lambda_n^{i_n} \mid a \in A, \lambda_j \in \Lambda, i_j \in \mathbb{N} \}$. For the equivalence between (2) and (3) we note that S is the image of ψ . In particular, S is a subring of R . It then follows from the definition that (1) implies (2). Conversely, any subring of R containing $\phi(A)$ and Λ certainly must contain S , so (2) implies (1). \square

Example 1.5. We may have also seen these brackets used in $\mathbb{Z}[\sqrt{d}]$ for some $d \in \mathbb{Z}$ to describe the ring

$$\{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}.$$

In fact, this is a special instance of generating: the \mathbb{Z} -algebra generated by \sqrt{d} in the most natural place, the algebraic closure of \mathbb{Q} , is exactly the set above. The point is that for any power $(\sqrt{2})^n$, write $n = 2q + r$ with $r \in \{0, 1\}$, so $(\sqrt{2})^n = 2^q(\sqrt{2})^r$. Similarly, the ring $\mathbb{Z}[\sqrt[3]{d}]$ can be written as

$$\{a + b\sqrt[3]{d} + c\sqrt[3]{d^2} \mid a, b, c \in \mathbb{Z}\}.$$

Note that the homomorphism ψ in part (3) need not be injective.

- If the homomorphism ψ is injective (so an isomorphism) we say that A is a *free* algebra.
- the set $\ker(\psi)$ measures how far R is from being a free A -algebra and is called the set of *relations* on Λ .

Definition 1.6 (Algebra-finite). We say that $\varphi : A \rightarrow R$ is *algebra-finite*, or R is a *finitely generated A -algebra*, if there exists a finite set of elements f_1, \dots, f_d that generates R as an A -algebra. We write $R = A[f_1, \dots, f_d]$ to denote this.

The term *finite-type* is also used to mean this.

Remark 1.7. Note that, by the lemma on generating sets, an A -algebra is finitely generated if and only if it is isomorphic to a quotient of a polynomial ring over A in finitely many variables. The choice of an isomorphism with a quotient of a polynomial ring is equivalent to a choice of generating set.

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