# ORBIT-STABILIZER THEOREM

DEFINITION: Let $G$ be a group acting on a set $X$, and $x \in X$.

- The **orbit** of $x$ is $\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\} \subseteq X$.
- The **stabilizer** of $x$ is $\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\} \leq G$.

ORBIT-STABILIZER THEOREM: Let $G$ be a group acting on a set $X$, and $x \in X$. Then

$$|\mathrm{Orb}_G(x)| = [G : \mathrm{Stab}_G(x)].$$

COROLLARY OF ORBIT-STABILIZER THEOREM: Let $G$ be a finite group acting on a set $X$, and $x \in X$. Then

$$|\mathrm{Orb}_G(x)| \cdot |\mathrm{Stab}_G(x)| = |G|.$$

In particular, the size of any orbit divides the order of $G$.

**(1)** Use the Orbit-Stabilizer Theorem and/or its corollary above to quickly explain why the following are *impossible*:

- $S_4 \curvearrowright X$ transitively for a set $X$ with 5 elements.
- $G \curvearrowright X$ with $|G| = 16$, $|X|$ odd, and the action has no fixed point.

**(2)** Proof of Theorem/Corollary.

**(a)** Prove the Orbit-Stabilizer Theorem by showing that the map

$$\{\text{left cosets of } \mathrm{Stab}_G(x) \text{ in } G\} \longrightarrow \mathrm{Orb}_G(x)$$
$$g \cdot \mathrm{Stab}_G(x) \mapsto g \cdot x$$

is a well-defined bijective function.

**(b)** Deduce the Corollary from the Theorem.

**(3)** Let $G$ be the group of rotational symmetries of a cube.

**(a)** Explain very briefly why $G$ acts on the set $F$ of faces of the cube.

**(b)** Explain why $G \curvearrowright F$ is transitive.

**(c)** Compute $\mathrm{Stab}_G(f)$ for $f \in F$.

**(d)** Compute $|G|$.