Midterm Exam

Instructions: Solve *two* problems from Part 1 and *two* problems from Part 2. You may use any results proved in class or in the problem sets, except for the specific question being asked. You should clearly state any facts you are using. You are also allowed to use anything stated in one problem to solve a different problem, even if you have not yet proved it. Remember to show all your work, and to write clearly and using complete sentences. No calculators, notes, cellphones, smartwatches, or other outside assistance allowed.

Part 1: Old problems

Choose *two* of the following problems.

(1) Prove that there is no nontrivial group homomorphism $\mathbb{Z}/n \to \mathbb{Z}$.

(2) Let G be a group and H a subgroup of G. The centralizer of H in G is the set of elements of G that commute with each element of H:

$$C_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

Prove that if H is normal in G, then $C_G(H)$ is a normal subgroup of G and that $G/C_G(H)$ is isomorphic to a subgroup of the automorphism group of H.

(3) Let G be a group of order p^n, for some prime p and some $n \geq 1$, acting on a finite set X.

(a) Suppose p does not divide $\#X$. Prove that there exists some element of X that is fixed by all elements of G.

(b) Suppose G acts faithfully[1] on X. Prove that $\#X \geq n \cdot p$.

alt Let G be a group and $Z(G)$ denote the center. Show that if $G/Z(G)$ is cyclic, then G is abelian.

alt A subgroup H of a group G is a **characteristic subgroup** if for every $\sigma \in \mathrm{Aut}(G)$, we have $\sigma(H) = H$.

(a) Show that a characteristic subgroup is normal.

(b) Show that if G is a group, H is a normal subgroup of G, and K is a characteristic subgroup of H, then K is a normal subgroup of G.

alt Let G be a group (not necessarily finite) and H a nonempty subset of G that is closed under multiplication. Suppose that for all $g \in G$ we have $g^2 \in H$.

(a) Show that H must be a subgroup of G.

(b) Show that H must be a normal subgroup of G.

(c) Show that G/H must be abelian.

[1] Recall this means that if $g \cdot x = x$ for all $x \in X$, then $g = e_G$.

Part 2: New problems

Choose *two* of the following problems.

(4) Let G be a nontrivial finite group.

   (a) Show that if $|G|$ is prime, then G is cyclic.

   (b) Show that[2] G has no nontrivial proper subgroups if and only if $|G|$ is prime.

(5) Let G be a group, and H, H' be two subgroups of G. Prove that $H \cup H'$ is a subgroup of G if and only if $H \subseteq H'$ or $H' \subseteq H$.

(6) Consider $\mathbb{Z} \leq \mathbb{Q}$ as groups with addition as the operation. Prove that $\mathbb{Q}/\mathbb{Z}$ is not a finitely generated group.

alt Let G be a group and $N \leq H$ two subgroups of G.

   (a) Give an example such that $N \trianglelefteq H$ and $H \trianglelefteq G$ but N is not normal in G.

   (b) Suppose that $N \trianglelefteq G$ and G/N is abelian. Prove that $H \trianglelefteq G$ and H/H is abelian.

[2] Note that you cannot use Cauchy's Theorem, since we have not shown it yet.