Problem Set #2

(1) Use the Shephard-Todd Theorem to determine for which of the following group actions on $S = \mathbb{C}[x,y]$ the invariant ring is a polynomial ring.

(a) $G = \langle \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix} \rangle$, where $\omega = e^{2\pi i/3}$.

(b) $G = \langle \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \rangle$.

(c) $G = \langle \begin{bmatrix} \omega & 0 \\ 0 & -1 \end{bmatrix} \rangle$.

(2) Find primary and secondary invariants for the following group actions.

(a) $S = \mathbb{C}[x_1, x_2]$, $G = \mathbb{Z}/2 = \langle g \rangle$ with $g(x_i) = -x_i$.

(b) $S = \mathbb{C}[x_1, x_2]$, $G = \mathbb{Z}/3 = \langle g \rangle$ with $g(x_i) = \omega x_i$, where $\omega = e^{2\pi i/3}$.

(3) Let $K = \mathbb{F}_2$, and let $G = \mathbb{Z}/2$ act on $S = K[x_1, x_2, x_3, y_1, y_2, y_3]$ by swapping x_i with y_i for each i. In this problem, we will verify directly (without the use of Kemper's Theorem) that $R = S^G$ is not Cohen-Macaulay.

(a) Show that $\{x_i + y_i, x_i y_i \mid i = 1, 2, 3\}$ is a homogeneous system of parameters for R.

(b) Show that $H_R(t) = \frac{1+3t^2}{(1-t)^3(1-t^2)^3}$.

(c) Show that R if Cohen-Macaulay then R is a free $K[x_i + y_i, x_i y_i]$-module with basis 1 and $\{x_i y_j - x_j y_i \mid 1 \le i < j \le 3\}$.

(d) Find a relation on the elements above and deduce that R is not Cohen-Macaulay.

(4) Let K be a field and G a finite group such that $\operatorname{Hom}(G, K^\times) = \{1\}$. In this problem we will show that $R = S^G$ is a unique factorization domain.

- First, show that if G is a p-group and $\operatorname{char}(K) = p$, then the hypothesis applies.

Let $r \in R$. Take an S-irreducible decomposition $r = s_1 \cdots s_t$. The group G partitions the principal ideals $(s_i)S$ into orbits, and let $t_1, \dots, t_\ell$ be the orbit products, so $r = t_1 \cdots t_\ell$.

- Show that $t_i \in R$. Hint: For each $g \in G$, there is $\theta(g) \in S^\times$ such that $g(t_i) = \theta(g) t_i$. Show that θ is a group homomorphism.
- Show that $t_i \in R$ is irreducible.
- Show that $r = t_1 \cdots t_\ell$ is the unique irreducible decomposition of r.

(5) Let $G = \mathbb{Z}/4$ act on $\mathbb{F}_2[x_1, x_2, x_3, x_4]$ by cyclically permuting the variables. Use the results above to deduce that S^G is a unique factorization domain that is not Cohen-Macaulay.

(6) Let K be a finite field and

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Is $G = \langle A, B, C \rangle$ generated by pseudoreflections? Does Shephard-Todd apply?

(b) Show that $S^{\langle A,B \rangle} = K[x_1, x_2, N(x_3), N(x_4)]$.

(c) Show that if S^G is a polynomial ring, the generators live in degrees $1, 1, p, p^2$.

(d) Show that S^G has no generator of degree p and deduce that S^G is not a polynomial ring.
(e) Show that, moreover, every point stabilizer of G is generated by pseudoreflections.

(7) Modify the proof of Shephard-Todd to show that if $R = S^G$ is a polynomial ring, then for each $v \in V$, the group $\mathrm{Stab}(v) \le G$ is generated by pseudoreflections. (You can keep the hypothesis $K = \mathbb{C}$ as we did in the proof.) Now show that the previous example is such that for each $v \in V$, the group $\mathrm{Stab}(v) \le G$ is generated by pseudoreflections.

(8) Let $f_1, \dots, f_n$ be a homogeneous system of parameters for $R = S^G$. Show that $\deg(f_1) \cdots \deg(f_t) = m|G|$ for some integer $m \ge 1$, and when $m = 1$, this system of parameters generates R as an algebra.

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