SUBSEQUENCES

(1) **True or false; justify.**

(a) The sequence $\left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

(b) The sequence $\left\{\frac{1}{3n+7}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

(c) The constant sequence $\left\{\frac{1}{2}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

(d) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are both subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(e) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are the only two subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(2) Explain how the following Corollary follows from Theorem 15.5.

Corollary 15.7: Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.

(a) If there is a subsequence of this sequence that diverges, then the sequence itself diverges.

(b) If there are two subsequences of this sequence that converge to different values, then the sequence itself diverges.

(3) Use Corollary 15.7 to give a quick proof that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

(4) **Prove or disprove:**

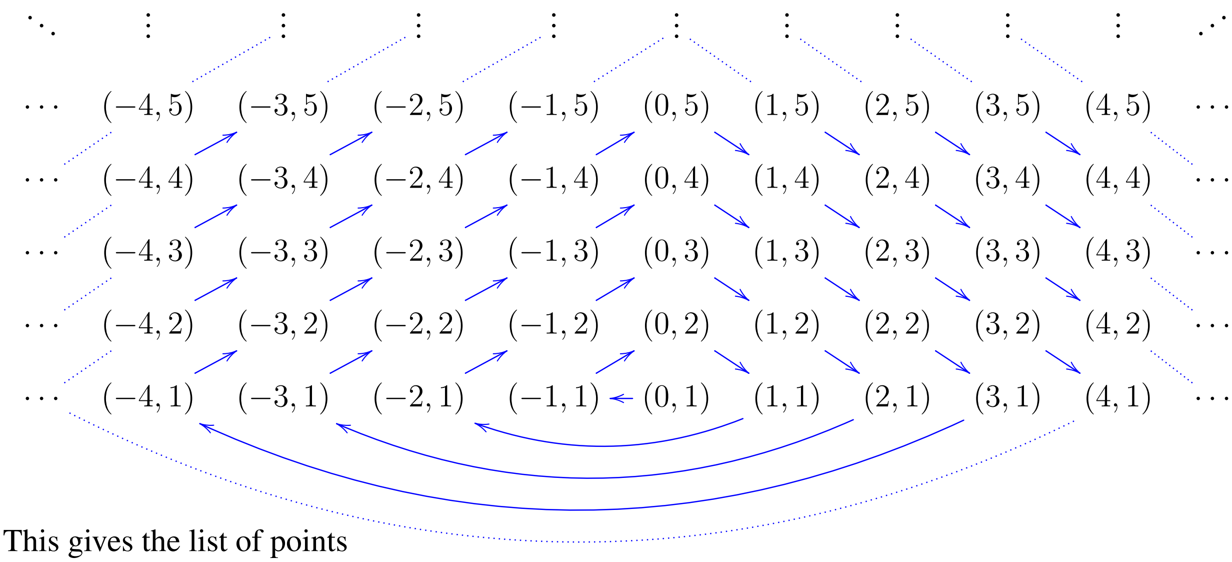
(a) Every subsequence of a bounded sequence is bounded.

(b) Every subsequence of a divergent sequence is divergent.

(c) Every subsequence of a sequence that diverges to $-\infty$ also diverges to $-\infty$.

A WILD SEQUENCE

Consider the points in the plane whose x-coordinates are integers and y-coordinates are natural numbers. Starting at $(0,1)$, zigzag like so:



This gives the list of points

$$(0,1),(-1,1),(0,2),(1,1),(-2,1),(-1,2),(0,3),(1,2),(2,1),(-3,1),\dots$$

Now convert these to a list of rational numbers by changing (m,n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1},\frac{-1}{1},\frac{0}{2},\frac{1}{1},\frac{-2}{1},\frac{-1}{2},\frac{0}{3},\frac{1}{2},\frac{2}{1},\frac{-3}{1},\dots$$

of rational numbers. Call this[1] sequence $\{w_n\}_{n=1}^{\infty}$.

(5) Explain why every rational number $q \in \mathbb{Q}$ occurs in $\{w_n\}_{n=1}^{\infty}$ infinitely many times.[2]

(6) Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers. Explain why $\{q_n\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$. [This is saying that *every* sequence of rational numbers is a subsequence of this single sequence!]

(7) Let r be any real number. Show that there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ converges to r. [This is saying that *every* real number occurs as a limit of a subsequence of this single sequence!]

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[1]Even though you didn't want to know, we can give w_n by a formula as

$$w_n = \begin{cases} \dfrac{n-t^2+2t-1}{n-t^2+t-1} & \text{if } n \le t^2-t \\[2ex] \dfrac{-n+t^2+1}{-n+t^2-t+1} & \text{if } n > t^2-t \end{cases}, \quad \text{where } t = \min\{m \in \mathbb{N} \mid m^2 \ge n\}.$$

[2]Hint: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$.