ASSIGNMENT #3: DUE FRIDAY, OCTOBER 11 AT 7PM

This problem set is to be turned in by Gradescope. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

DEFINITION: Let R be an Noetherian $\mathbb{N}$-graded ring, with $R_0 = K$ a field. The **Hilbert function** of R is the function $H_R(t) = \dim_K(R_i)$, where $\dim_K$ denotes vector space dimension.

(1) Let K be a field, and $R = K[X^2, X^3] \subseteq S = K[X]$. Show that for the ideal I of R generated by X^2, we have $IS \cap R \supsetneqq I$.

(2) Let K be an infinite field, $m \geqslant 1$, and let $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let $G = (K^\times)^m$ act on R by (degree-preserving) K-algebra automorphisms as follows:
$$(\lambda_1, \dots, \lambda_m) \cdot X_i = \lambda_1^{a_{1,i}} \cdots \lambda_m^{a_{m,i}} X_i \qquad i = 1, \dots, n$$
for some $m \times n$ matrix of integers $A = [a_{i,j}]$.
  (a) Show[1] R^G has a K-vector space basis given by the set of monomials $X_1^{b_1} \cdots X_n^{b_n}$ such that, for $b = (b_1, \dots, b_n)$, $Ab = 0$.
  (b) Consider the polynomial ring R with a (nonstandard) $\mathbb{Z}^m$-grading given by setting
$$\deg(X_i) = (a_{1,i}, \dots, a_{m,i})$$
  for each i. Show that R^G is the degree zero piece of R under this grading.
  (c) Show that R^G is a direct summand of R, and deduce that R^G is a finitely generated K-algebra.
  (d) Let $R = K[X, Y, Z, W]$ and consider the K-algebra action of $G = K^\times$ on R given by
$$\lambda \cdot X = \lambda X \qquad \lambda \cdot Y = \lambda Y \qquad \lambda \cdot Z = \lambda^{-1} Z \qquad \lambda \cdot W = \lambda^{-1} W.$$
  Find a finite set of generators for R^G as a K-algebra.

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(3) Let $U, \dots, Z$ be indeterminates over $\mathbb{C}$, and $R = \mathbb{C}\begin{bmatrix} U & V & W \\ X & Y & Z \end{bmatrix} \Big/ I$ where $I = I_2\left(\begin{bmatrix} U & V & W \\ X & Y & Z \end{bmatrix}\right)$.
In this problem, we will show that $A = \mathbb{C}[u, v - x, w - y, z] \subseteq R$ is a Noether normalization, where $u, \dots, z$ denote the images of $U, \dots, Z$ in R.
  (a) Apply Graded NAK[2] to R as a graded A-module to show that $A \subseteq R$ is module-finite.
  (b) Show that for any $\alpha, \beta, \gamma, \zeta \in \mathbb{C}$, there is a rank one 2×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ with $a, \dots, f \in \mathbb{C}$ such that $a = \alpha, b - d = \beta, c - e = \gamma, f = \zeta$.
  (c) Suppose that $F(T_1, T_2, T_3, T_4)$ is a $\mathbb{C}$-algebraic relation on $u, v - x, w - y, z$. Use the previous part to show that $F(\alpha, \beta, \gamma, \zeta) = 0$ for all $\alpha, \beta, \gamma, \zeta \in \mathbb{C}$, and deduce[1] that $u, v - x, w - y, z$ are algebraically independent.

(4) Suppose that R is a finitely-generated $\mathbb{Z}$-algebra, and that $\mathfrak{m}$ is a maximal ideal of R. Show that $R/\mathfrak{m}$ is a finite field.

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[1] You can use without proof that any nonzero (multivariate) polynomial over an infinite field is a nonzero function.

[2] One could instead show that the generators of R are integral over A, but we will try out the power of Graded NAK.