## UPPER BOUNDS AND THE COMPLETENESS AXIOM

Let $S$ be a set of real numbers.

- A number $b$ is an *upper bound* for $S$ provided for all $x \in S$ we have $b \geq x$.
- The set $S$ is *bounded above* provided there exists at least one upper bound for $S$.
- A number $m$ is the *maximum* of $S$ provided
  (1) $m \in S$, and
  (2) $m$ is an upper bound of $S$.
- A number $\ell$ is a *supremum* of $S$ provided
  (1) $\ell$ is an upper bound of $S$, and
  (2) for any upper bound $b$ for $S$, we have $\ell \leq b$.

(1) Write, in simplified form, the negation of the statement "$b$ is an upper bound for $S$".

(2) Write, in simplified form, the negation of the statement "$S$ is bounded above".

(3) Let $S$ be a set of real numbers and suppose that $\ell = \sup(S)$.
  (a) If $x > \ell$, what is the most concrete thing you can say about $x$ and $S$?
  (b) If $x < \ell$, what is the most concrete thing[1] you can say about $x$ and $S$?

(4) Let $S$ be a set of real numbers, and let[2] $T = \{2s \mid s \in S\}$. Prove[3] that if $S$ is bounded above, then $T$ is bounded above.

(5) Let $S$ be a set of real numbers. Show that if $S$ has a supremum, then it is unique.

(6) Let $S$ be a set of real numbers, and let $T = \{s/2 \mid s \in S\}$. Directly[4] prove[5] that if $S$ is unbounded above, then $T$ is unbounded above.

[1] Hint: Use one of the previous problems.

[2] For example, if $S = \{-1, 1, 2\}$, then $T = \{-2, 2, 4\}$.

[3] First, before all else, this is an if then statement: start by assuming the "if" part. We now need to show the "then" part, which is about the existence of an upper bound. Use your assumption about $S$ to find an upper bound for $T$ (and prove that it is indeed an upper bound for $T$).

[4] Don't try to apply (4), just prove it directly.

[5] First, before all else, this is an if then statement: start by assuming the "if" part. We now need to show that $T$ is unbounded above, which before everything else, is a for all statement. Use the for all statement in the hypothesis to get the number that you want…

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

THEOREM 4.1: For every real number $r$, there is some natural number $n$ such that $n > r$.

*Proof of Theorem 4.1:*

(1) Proceed by contradiction: if not, then the set of natural numbers $\mathbb{N}$ is

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(2) Since $1 \in \mathbb{N}$, $\mathbb{N}$ is nonempty; thus, by the

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the set $\mathbb{N}$ has a

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call it $\ell$.

(3) The real number $\ell - 1$ is less than $\sup(\mathbb{N})$, so

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(4) Adding one,

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(5) This contradicts

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(6) Thus

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