

SMITH NORMAL FORM

THEOREM (SMITH NORMAL FORM): Let R be a PID. Let $A \in \text{Mat}_{m \times n}(R)$.

- (i) There exist invertible matrices P, Q such that
 - $PAQ = D$ is diagonal, meaning $d_{ij} = 0$ whenever $i \neq j$, and
 - $d_{11} \mid d_{22} \mid \cdots \mid d_{tt}$, where d_{tt} is the last nonzero diagonal entry.
- (ii) The elements d_{ii} are unique up to associate, meaning that if $D' = [d'_{ij}]$ is another diagonal matrix as in (1), then for each d'_{ii} is a unit times d_{ii} .
- (iii) If R is a Euclidean domain, then P, Q can be taken to be products of elementary matrices.

STRUCTURE THEOREM FOR FINITELY GENERATED MODULES OVER PIDS (INVARIANT FACTOR FORM): Let R be a PID. Let M be a finitely generated R -module. Then there exist $r, t \geq 0$ and $a_1, \dots, a_t \in R$ such that

- $M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_t)$, and
- $a_1 \mid a_2 \mid \cdots \mid a_t$.

Moreover, each a_i is uniquely determined up to associates.

- (1) Let R be a commutative ring, and $D \in \text{Mat}_{m \times n}(R)$ be a diagonal matrix with nonzero diagonal entries d_{11}, \dots, d_{tt} . Prove that the module presented by D is

$$R/(d_{11}) \oplus R/(d_{22}) \oplus \cdots \oplus R/(d_{tt}) \oplus R^{m-t}.$$
- (2) Use the Smith Normal Form Theorem to deduce the Structure Theorem for Finitely Generated Modules over PIDs (Invariant Factor Form).
- (3) State the Structure Theorem for Finitely Generated Abelian Groups (Invariant Factor Form), and deduce it from the PID Theorem.
- (4) Let R be a Euclidean domain. Use the Smith Normal Form Theorem to deduce that any invertible matrix over R is a product of elementary matrices.
- (5) Prove¹ the uniqueness part of Smith Normal Form Theorem.

DEFINITION: Let R be a domain and M be an R -module. We say that M is **torsionfree** if for $r \in R$ and $m \in M$, $rm = 0$ implies $r = 0$ or $m = 0$.

- (6) Let R be a PID.
 - (a) Show that any finitely generated torsionfree R -module is free.
 - (b) Prove or disprove: any torsionfree R -module is free.

¹Hint: In HW#4, you proved that $I_t(PAQ) = I_t(A)$ for invertible matrices P, Q .

STRUCTURE THEOREM FOR FINITELY GENERATED MODULES OVER PIDS (ELEMENTARY DIVISOR FORM): Let R be a PID. Let M be a finitely generated R -module. Then there exist $r, s \geq 0$ and prime elements $p_1, \dots, p_s \in R$ such that $M \cong R^r \oplus R/(p_1^{e_1}) \oplus \dots \oplus R/(p_s^{e_s})$, and Moreover, the list $p_1^{e_1}, \dots, p_s^{e_s}$ is unique up to reordering and associates.