Problem Set 4

Due Thursday, September 25

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly*. As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let G be a group, and let H and K be finite subgroups of G of relatively prime order; i.e., $\gcd(|H|, |K|) = 1$. Show that $H \cap K = \{e\}$.

Problem 2. For $k \in \mathbb{Z}_{\geq 2}$, let C_k denote the cyclic group of order k. Show that for any relatively prime $m, n \geq 2$, there is an isomorphism $C_m \times C_n \cong C_{mn}$.

Problem 3. Let S_n denote the symmetric group on n symbols.

(3.1) Show that[1] the sign map $S_n \to \{\pm 1\}$ is a group homomorphism, where $\{\pm 1\}$ is considered as a subgroup of $\mathbb{R}^\times$. The kernel of this map is called the **alternating group** on n symbols and denoted A_n.

(3.2) Let $n \geq 3$. Show that A_n is generated by the set of 3-cycles $(i\ j\ k)$ and disjoint pairs[2] of transpositions $(i\ j)(k\ \ell)$ in S_n.

Problem 4. Consider the set $\mu = \{z \in \mathbb{C}^\times \mid z^n = 1 \text{ for some integer } n \geq 1\}$, which is a subgroup of $\mathbb{C}^\times$; you do not need to prove this. Concretely, $\mu = \{e^{2\pi i m/n} \mid m, n \in \mathbb{Z}\}$.

(4.1) Prove that for each integer $m \geq 1$, there is a unique subgroup $H_m \leq \mu$ with $|H_m| = m$ and that H_m is cyclic.

(4.2) Prove that every finitely generated subgroup of μ has finitely many elements.

(4.3) Prove that μ is not finitely generated.

Defintion: Let G be a group and N be a subgroup. We say that N is a **normal** subgroup of G if for all g in G, we have $gNg^{-1} \subseteq N$; that is, for any $g \in G$ and any $n \in N$, we have that $gng^{-1} \in N$. We write $N \trianglelefteq G$ to say N is a normal subgroup of G.

[1] Your proof should be no more than a few lines.
[2] For $n = 3$, there are no disjoint pairs of transpositions.

Problem 5. Let $f : G \to H$ be a group homomorphism.

(5.1) Show[3] that $\ker(f) \trianglelefteq G$.

(5.2) Show that if $K \trianglelefteq H$, then $f^{-1}(H) \trianglelefteq G$.

Problem 6. Let G be a group, S a subset of G, and $H = \langle S \rangle$.

(6.1) Prove that $H \trianglelefteq G$ if and only if $gsg^{-1} \in H$ for every $s \in S$ and $g \in G$.

(6.2) Consider the commutator subgroup of G

$$[G, G] := \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$$

generated by all the commutators of elements in G. Prove that $[G, G] \trianglelefteq G$.

[3] In particular, $A_n \trianglelefteq S_n$!