Contents

1. August 23, 2021

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \ldots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not have thought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\ldots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\ldots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number.

Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is 2.*

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \Longrightarrow \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction. (Some mathematicians would actually not consider this to be a proof by contradiction. For some, a proof by contradiction refers to when the truth of a statement P is established by assuming the statement "not P" and deducing from that a falsity.)

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. $\square$

A version of the previous proof was known even to the ancient Greeks.

2. August 25, 2021

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 2.1. *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

(1) *There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p+q$ and $p \cdot q$.*
(2) *Each of $+$ and $\cdot$ is a commutative operation (i.e., $p+q = q+p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*
(3) *Each of $+$ and $\cdot$ is an associative operation (i.e., $(p+q)+r = p+(q+r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q, and r).*
(4) *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0 + q = q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.*
(5) *The distributive law holds: $p \cdot (q + r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.*
(6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.*
(7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.*
(8) *There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that*
    (a) *For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
    (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
    (c) *For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
(9) *The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.*
(10) *The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

Which of the properties from Proposition 2.1 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$. There is one other important property of $\mathbb{N}$, which we accept to be true without proof. Such a property is called an axiom.

Axiom 2.2 (Well-ordering axiom)**.** *Every nonempty subset of* $\mathbb{N}$ *has a smallest element (which we call its minimum).*

As we will discuss later, the well-ordering axiom is closely related to the principle of induction.

Example 2.3. For the set of all even multiples of 7, $S = \{7 \cdot (2n) \mid n \in \mathbb{N}\}$, we have $\min(S) = 14$.

We expect everything from Proposition 2.1 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written* $\mathbb{R}$*, satisfies the following eleven properties:*

(Axiom 1) *There are operations* $+$ *and* $\cdot$ *defined on* $\mathbb{R}$*, so that if* $x, y \in \mathbb{R}$*, then so are* $x + y$ *and* $x \cdot y$*.*

(Axiom 2) *Each of* $+$ *and* $\cdot$ *is a commutative operation.*

(Axiom 3) *Each of* $+$ *and* $\cdot$ *is an associative operation.*

(Axiom 4) *The real number* 0 *is an identity element for addition and the real number* 1 *is an identity element for multiplication. This means that* $0 + x = x$ *and* $1 \cdot x = x$ *for all* $x \in \mathbb{R}$*.*

(Axiom 5) *The distributive law holds:* $x \cdot (y + z) = x \cdot y + x \cdot z$ *for all* $x, y, z \in \mathbb{R}$*.*

(Axiom 6) *Every real number has an additive inverse: For any* $x \in \mathbb{R}$*, there is a number* $-x$ *satisfying* $x + (-x) = 0$*.*

(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any* $x \in \mathbb{R}$ *such that* $x \neq 0$*, there is a real number* x^{-1} *satisfying* $x^{-1} \cdot x = 1$*.*

(Axiom 8) *There is a "total ordering"* $\leq$ *on* $\mathbb{R}$*. This means that*

- (a) *For all* $x, y \in \mathbb{R}$*, either* $x \leq y$ *or* $y \leq x$*.*
- (b) *If* $x \leq y$ *and* $y \leq z$*, then* $x \leq z$*.*
- (c) *For all* $x, y, z \in \mathbb{R}$*, if* $x \leq y$ *and* $y \leq z$*, then* $x \leq z$*.*

(Axiom 9) *The total ordering* $\leq$ *is compatible with addition: If* $x \leq y$ *then* $x + z \leq y + z$ *for all* z*.*

(Axiom 10) *The total ordering* $\leq$ *is compatible with multiplication by non-negative real numbers: If* $x \leq y$ *and* $z \geq 0$ *then* $zx \leq zy$*.*

(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

$$\text{“Cancellation of Addition”: If } x+y=z+y \text{ then } x=z.$$

Let's prove this carefully, using just the list of axioms: If $x+y=z+y$ then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x+y)+(-y)=(z+y)+(-y)$. This can be rewritten as $x+(y+(-y))=z+(y+(-y))$ (Axiom 3) and hence as $x+0=z+0$ (Axiom 6), which gives $x=z$ (Axiom 4 and Axiom 2).

For another example, we can deduce the following fact from the axioms:

$$r\cdot 0=0 \text{ for any real number } r.$$

Let's prove this carefully: Let r be any real number. We have $0+0=0$ (Axiom 4) and hence $r\cdot(0+0)=r\cdot 0$. But $r\cdot(0+0)=r\cdot 0+r\cdot 0$ (Axiom 5) and so $r\cdot 0=r\cdot 0+r\cdot 0$. We can rewrite this as $0+r\cdot 0=r\cdot 0+r\cdot 0$ (Axiom 4). Now apply the Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0=r\cdot 0$.

As I said, there are many other familiar properties of the real numbers that follow from these axioms, but I will not list them all. The great news is that all of these familiar properties follow from this short list of axioms. We will prove a couple, but for the most part, I'll rely on your innate knowledge that facts such as $r\cdot 0=0$ hold.

I owe you a description of the very important Completeness Axiom, and it will take a bit of time to do so. Before we get to this, it will be helpful to review set notation, and some basics of proof-writing.

Often, sets are described as subsets of other larger sets, by specifying properties. For example, when I write

$$S=\{m\in\mathbb{Z}\mid m=a^2 \text{ for some } a\in\mathbb{Z}\}$$

I am specifying a subset of the set of all integers $\mathbb{Z}$. In words, S is: “the set of those integers that are equal to the square of some integer”. We could also write this set out by listing its elements:

$$S=\{0,1,4,9,16,25,36,\dots\}.$$

It's safer in general to use the former description, since you don't have to worry about the reader getting the pattern.

The previous is an example of a subset of $\mathbb{Z}$, but we will mostly be concerned with subsets of $\mathbb{R}$. For example, we might consider the set

$$\{x\in\mathbb{R}\mid x^2<2\}.$$

We will also deal with "intervals" a lot. When I write $(0, 1)$ I mean the set $\{x \in \mathbb{R} \mid 0 < x < 1\}$. That is, it is all real numbers strictly between 0 and 1.

More generally, if a, b are real numbers and $a < b$, then

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

(What if $b \leq a$?) The set (a, b) is called an *open interval*.

We also have $[a, b]$, known as a *closed interval* and defined to be

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

We also have $[a, b)$, $(a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, and $(-\infty, b]$, all of which you probably have seen before.

We will also have need to consider sets defined in more complicated ways such as

$$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}.$$

The latter is a bit different than the previous examples. The previous ones had form $\{$ element of a set $\mid$ property holds $\}$, but this one has the form $\{$ expression involving symbols $\mid$ allowable values of these symbols $\}$. Explicitly, this example is the set $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.

Recall also a few ways of making sets from others:

- union : $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$
- intersection : $S \cup T = \{x \mid x \in S \text{ and } x \in T\}$
- set difference : $S \setminus T = \{x \mid x \in S \text{ and } x \notin T\}$.

Let's now talk a bit more about rules of logic, methods of proof, quantification, etc. Our book has a very nice treatment of these topics in Sections 1.4 and 1.5. Part of the next problem set will involve your reading these sections on your own and doing some of the exercises. Here, I'll just give some highlights.

Let me start with some rules of logic, and how that affects proofs. First, a *statement* is a sentence (or sometimes sequence of sentences) that is either true or false. Things like "Jack's shirt is ugly" is not a statement, nor is "Go Huskers!". But "All odd numbers are prime" is a statement — it happens to be false. The sentence

> "The digit 9 occurs infinitely often in the decimal expansion of π."

is a statement, as it is surely either true or false. But, no one knows which!

An odder example is "This sentence is false". Is it a statement? (Is it true? Is it false?) No!

If P and Q are any two statements, then we can form compound statements from them such as

- P and Q.
- P or Q.
- Not P.
- If P then Q.

The "truth values" for the first three are pretty clear, but be careful about the last.

- "P and Q" is a true statement when both P and Q are true statements.
- "P or Q" is a true statement when either P or Q is a true statement.
- "Not P" is true when P is a false statement.
- "If P then Q" is true when P is false or Q is true. In other words "If P then Q" is logically equivalent to "not P or Q".

Which of the following are true?

(1) If $1 + 1 = 1$, then I am the pope.
(2) If 8 is prime then every real number is an integer.
(3) If my name is Jack then I am the pope.
(4) If it had been raining this morning then I would have brought an umbrella with me to class.

All but the third are true.

Most of the statements that we consider are, or can be framed as if-then statements: anything with hypotheses and a conclusion is an if-then statement. How do we prove such a statement? To give a "direct proof" of "if P then Q" we:

(1) Assume P,
(2) Do some stuff, then
(3) Conclude Q.

For example, the Goldbach Conjecture posits that if n is an even integer greater than 2, then n is a sum of two primes. (A conjecture is a statement that people believe to be true based on some evidence, but is not proven.) I can't prove this conjecture, but I can tell you the first and last sentence of a proof: "Assume that n is an even integer. ... Thus, n is a sum of two primes."

3. August 27, 2021

As I said earlier, "If P then Q" is the same as "not P or Q". It follows that "If not Q then not P" is the same as "not not Q or not P" and hence is the same as "not P or Q". That is:

> "If P then Q" is logically equivalent to "If not Q then not P".

"If not Q then not P" is known as the *contrapositive* of "If P then Q". So, an if-then statement and its contrapositive are logically equivalent.

Often when proving an if-then statement, it works a bit better to give a "direct" proof of the contrapositive. That is, in a proof of "If P then Q" *by contraposition* we:

(1) Assume not Q,
(2) Do some stuff, then
(3) Conclude not P.

Example 3.1. An *irrational number* is a real number that is not rational. Consider the following assertion:

> Let r be any rational number and let x be any real number. If x is irrational then $x + r$ is irrational.

This is logically equivalent to:

> Let r be any rational number and let x be any real number. If $x + r$ is rational then x is rational.

Let us prove the latter statement "directly": Let r be any rational number and let x be any real number. Suppose $x + r$ is rational. Then since r is rational, $-r$ is also rational (by Proposition 2.1, part (6)). It follows that $(x + r) + (-r)$ is also rational (by Proposition 2.1, part (1)) and hence $(x + r) + (-r) = x + (r + (-r)) = x + 0 = x$ is rational.

Never, ever, ever, ever confuse the contrapositive of an if-then statement with its *converse*. The converse of "If P then Q" is "If Q then P".

Example 3.2. Give examples of statements that are true whose converses are false.

Recall that when we say "P if and only if Q" we mean "If P then Q, and if Q then P". In other words, an "if and only if" statement includes both an if-then statement and its converse. The statement "P if and only if Q" is true when either P and Q are both true or P and Q are both false, and it is false in the other two cases, when one is true and the other is false. A proof of such a statement generally has two parts, one where we prove P implies Q (either directly or by contraposition) and one where we prove Q implies P (again either directly or by contraposition).

Let me also say a bit about quantification: This refers to usage of "for every" or "there exists". For example, "For every real number x, x^2 is strictly positive" and "There exists an even integer that is prime".

"For every" statements are sometimes better cast as if-then statements. For example, the first one above is equivalent to "If x is a real number, then x^2 is strictly positive". So, be aware that sometimes, as in this example, there is an implicit "For every" clause lurking about even if you don't see those words written.

The negation of a "for every" clause usually involves "there exists". For example the negation of "For every real number x, x^2 is strictly positive." is "There is a real number x such that x^2 is not strictly positive".

The negation of a "there exists" statement usually involves "for every". For example, the negation of "There is an even integer that is prime" is "For every even integer n, n is not prime" or better "If n is an even integer, then n is not prime".

In general,

- the negation of "For every $x \in S$, P" is "There exists $x \in S$ such that not P";
- the negation of "There exists $x \in S$ such that P" for some statement P is "For every $x \in S$, not P".

How do we prove statements with quantifiers? To prove "For every $x \in S$, P" , we

(1) Take an arbitrary $x \in S$,
(2) Do some stuff, then
(3) Conclude that P holds for x.

In the first step we specify one element of S, but we *don't get to decide which one*. In particular, its name should be a variable, rather than the name of any specific element in S.

To *disprove* "For every $x \in S$, P" we can give a counterexample. That means that we get to choose an element of S, and show that P fails for our choice.

To prove "There exists $x \in S$ such that P" , we just need to give an example: we can choose any element of S and show that P holds for that element.

Things get harder when we combine "for every" and "there exist" clauses in one statement. One very important point here is that order matters a lot. For example,

"For every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $n < m$"

and

"There is an $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n < m$"

have very different meanings. In fact, the first is clearly true (since given an n one could, for example, take $m = n + 1$) and the second is false.

Never interchange the positions of "for every" and "there exist" unless you intend to change the meaning!

When we combine "for every" and "there exist" clauses with a negation things can also get confusing. For example: the negation of "For every integer m there is an integer n such that $n > m$" is "There exists an integer m such that for every integer n, $n \leq m$."

Using symbols sometimes helps focus attention on the underlying logic. We write $\forall$ and $\exists$ in place of "for every" and "there exists", sometimes. For example the negation of "$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$" is "$\exists x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, n \leq x$".

Here are a couple more statements with multiple quantifiers related to where we're going. Consider the following two properties for a set of natural numbers

P : for every $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $n > N$ such that $n \in S$

Q : there exists $n \in \mathbb{N}$ such that for every $N \in \mathbb{N}$ with $n > N$, $n \in S$

For the set of all numbers bigger than 3, both P and Q are true, where for the set of even numbers, P is true and Q is false.

In more natural terms, P holds for S means that "there are arbitrarily large numbers in S" while Q holds for S means "all sufficiently large numbers are in S".

4. August 30, 2021

- Write the contrapositive, and the converse of each statement. Is the statement true or false? Is the converse true or false? Explain why (but don't write a full proof). For each statement below, a, b are real numbers.

  ♢ If a is irrational, then $1/a$ is irrational.

> Contrapositive: "If $1/a$ is rational, then a is rational. [True]
> Converse: "If $1/a$ is irrational, then a is irrational. [True]

  ♢ If $x > 3$ then $x^2 > 9$.

> Contrapositive: "If $x^2 \leq 9$, then $x \leq 3$. [True]
> Converse: "If $x^2 > 9$, then $x > 3$. [False]

◇ If a and b are both irrational, then ab is irrational.

> Contrapositive: "If ab is rational, then either a or b is rational. [False]
> Converse: "If ab is irrational, then a and b are both irrational. [False]

- Write the negation of each statement. Is the statement true or false? Explain why (but don't write a full proof).
  ◇ $\exists x \in \mathbb{Q}$: $x^2 = 2$.

> Negation: $\forall x \in \mathbb{Q}$, $x^2 \neq 2$. [The original is false.]

◇ $\forall x \in \mathbb{Q}$, $x^2 > 0$.

> Negation: $\exists x \in \mathbb{Q}$, $x^2 \leq 0$. [The original is false.]

◇ $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$: $xy = 1$.

> Negation: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, xy \neq 1$. [The original is false.]

◇ $\exists x \in \mathbb{R}$: $\forall y \in \mathbb{R}$, $e^y < x$.

> Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : e^y \geq x$. [The original is false.]

◇ $\exists x \in \mathbb{R}$: $\forall y \in \mathbb{R}$, $\sin(y) < x$.

> Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : \sin(y) \geq x$. [The original is true.]

- Prove the following statements using the axioms of $\mathbb{R}$ and facts we have proven in class.
  ◇ Let x be a real number. If there is a real number y such that $xy = 1$, then x is nonzero.[1]

> We argue the contrapositive. Let x be zero. Then, for any $y \in \mathbb{R}$, we have $xy = 0$, by a fact we proved in class. In particular, we have $xy \neq 0$, as required.

[1]Hint: Consider the contrapositive of this statement.

♢ If x and y is a nonzero real numbers, then xy is also nonzero.[2]

> Let x and y be nonzero real numbers. By Axiom 7, there are element $x^{-1}, y^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$ and $yy^{-1} = 1$. Then $xy \cdot (x^{-1}y^{-1}) = (xx^{-1})(yy^{-1}) = 1$, using Axioms 2 and 3 in the first equality and Axiom 5 in the second. By the previous fact (applies to xy) we conclude that $xy \neq 0$.

♢ For any real numbers x and y, $x \geq y$ if and only if $-x \leq -y$.[3]

> Let $x \geq y$. Adding $(-x) + (-y)$ to both sides (which exists by Axiom 6), we obtain $-y = x+((-x)+(-y)) \geq y + ((-x) + (-y)) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x+y)+(-x) \leq (x+y)+(-y) = x$ (by Axiom 9 and Axiom 5).

♢ For any real numbers x and y, $(-x) \cdot y = -xy$.

> Observe that
>
> $$(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0.$$
>
> Thus,
>
> $$-x \cdot y = -x \cdot y + xy + -xy = 0 + -xy = -xy.$$

♢ The product of two negative real numbers is nonnegative.

> Assume that $x < 0$ and $y < 0$. Then, we also have $0 < -x$. It follows that $-xy = (-x) \cdot y \leq 0 \cdot y = 0$, Thus, $xy \geq 0$. Moreover, since $x \neq 0$ and $y \neq 0$, we have that $xy \neq 0$, so we must have $xy > 0$.

5. September 1, 2021

Definition 5.1. Let S be any subset of $\mathbb{R}$. A real number b is called an *upper bound* of S provided that for every $s \in S$, we have $s \leq b$.

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[2]Hint: Use x^{-1} and the previous statement.

[3]Hint: Add something to both sides.

For example, the number 1 is an upper bound for the set $(0,1)$. The number 182 is also an upper bound of this set and so is π. It is pretty clear that 1 is the "best" (i.e., smallest) upper bound for this set, in the sense that every other upper bound of $(0,1)$ must be at least as big as 1. Let's make this official:

Proposition 5.2. *If b is an upper bound of the set $(0,1)$, then $b \geq 1$.*

I will prove this claim using just the axioms of the real numbers (in fact, I will only use the first 10 axioms):

Proof. Suppose b is an upper bound of the set $(0,1)$. By way of contradiction, suppose $b < 1$. (Our goal is to derive a contradiction from this.)

Consider the number $y = \frac{b+1}{2}$ (the average of b and 1). I will argue that $b < y$ and $b \geq y$, which is not possible.

Since we are assuming $b < 1$, we have $\frac{b}{2} < \frac{1}{2}$ and hence

$$b = \frac{2b}{2} = \frac{b}{2} + \frac{b}{2} < \frac{b}{2} + \frac{1}{2} = \frac{b+1}{2} = y.$$

So, $b < y$.

Similarly,

$$1 = \frac{1+1}{2} > \frac{b+1}{2} = y$$

so that

$$y < 1.$$

Since $\frac{1}{2} \in S$ and b is an upper bound of S, we have $\frac{1}{2} \leq b$. Since we already know that $b < y$, it follows that $\frac{1}{2} < y$ and hence $0 < y$. We have proven that $y \in (0,1)$. But, remember that b is an upper bound of $(0,1)$, and so we get $y \leq b$ by definition.

To summarize: given an upper bound b of $(0,1)$, starting with the assumption that $b < 1$, we have deduced the existence of a number y such that both $b < y$ and $y \leq b$ hold. As this is not possible, it must be that $b < 1$ is false, and hence $b \geq 1$. $\square$

This claim proves the (intuitively obvious) fact that 1 is "least upper bound" of the set $(0,1)$. The notion of "least upper bound" will be an extremely important one in this class.

Definition 5.3. A subset S of $\mathbb{R}$ is called *bounded above* if there exists at least one upper bound for S. That is, S is bounded above provided there is a real number b such that $s \leq b$ for all $s \in S$.

For example, $(0,1)$ is bounded above, by for example 50.

The subset $\mathbb{N}$ of $\mathbb{R}$ is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of $\mathbb{R}$ that is bounded above.

Example 5.4. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 5.5. Suppose S is subset of $\mathbb{R}$ that is bounded above. A *supremum* (also known as a *least upper bound*) of S is a number ℓ such that

(1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
(2) if b is any upper bound of S, then $\ell \leq b$.

Example 5.6. 1 is a supremum of $(0, 1)$. Indeed, it is clearly an upper bound, and in the "Claim" above, we proved that if b is any upper bound of $(0, 1)$ then $b \geq 1$. Note that this example shows that a supremum of S does not necessarily belong to S.

Example 5.7. I claim 1 is a supremum of $(0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$. It is by definition an upper bound. If b is any upper bound of $(0, 1]$ then, since $1 \in (0, 1]$, by definition we have $1 \leq b$. So 1 is the supremum of $(0, 1]$.

The subset $\mathbb{N}$ does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly:

the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term "supremum", I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom)**.** *Every nonempty, bounded-above subset of* $\mathbb{R}$ *has a supremum.*

Proposition 5.8. *If a subset of* $\mathbb{R}$ *has a supremum, then it is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P, then x and y must be the same thing.

Proof. Suppose both x and y are both suprema of the same subset S of $\mathbb{R}$. Then, since y is an upper bound of S and x is a supremum of S, by part (2) of the definition of "supremum" we have $y \geq x$. Likewise, since x is an upper bound of S and y is a supremum of S, we have $x \geq y$ by definiton. Since $x \leq y$ and $y \leq x$, we conclude $x = y$. $\square$

From now on we will speak of *the* supremum of a set (when it exists).

6. September 3, 2021

Let us now explore consequences of the completeness axiom. First up, we show that it implies that $\sqrt{2}$ really exists:

Proposition 6.1. *There is a positive real number whose square is* 2.

Proof. Define S to be the subset

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

S is nonempty since, for example, $1 \in S$, and it is bounded above, since, for example, 2 is an upper bound for S, as we showed earlier. So, by the Completeness Axiom, S has a least upper bound, and we know it is unique from the proposition above. Let us call it ℓ. I will prove $\ell^2 = 2$.

We know one of $\ell^2 > 2$, $\ell^2 < 2$ or $\ell^2 = 2$ must hold. We prove $\ell^2 = 2$ by showing that both $\ell^2 > 2$ and $\ell^2 < 2$ are impossible.

We start by observing that $1 \leq \ell \leq 2$. The inequality $1 \leq \ell$ holds since $1 \in S$ and ℓ is an upper bound of S, and the inequality $\ell \leq 2$ holds since 2 is an upper bound of S and ℓ is the least upper bound of S.

Suppose $\ell^2 < 2$. We show this leads to a contradiction by showing that ℓ is not an upper bound of S in this case. We will do this by constructing a number that is ever so slightly bigger than ℓ and belongs to S. Let $\varepsilon = 2 - \ell^2$. Then $0 < \varepsilon \leq 1$ (since $\ell^2 < 2$ and $\ell^2 \geq 1$). We will now show that $\ell + \varepsilon/5$ is in S: We have

$$(\ell + \varepsilon/5)^2 = \ell^2 + \frac{2}{5}\ell\varepsilon + \frac{\varepsilon^2}{25} = \ell^2 + \varepsilon(\frac{2\ell}{5} + \frac{\varepsilon}{25}).$$

Now, using $\ell \leq 2$ and $0 < \varepsilon \leq 1$, we deduce

$$0 < \frac{2\ell}{5} + \frac{\varepsilon}{25} \leq \frac{4}{5} + \frac{\varepsilon}{25} < 1.$$

Putting these equations and inequalities together yields

$$(\ell + \frac{\varepsilon}{5})^2 < \ell^2 + \varepsilon = 2.$$

So, $\ell + \frac{\varepsilon}{5} \in S$ and yet $\ell + \frac{\varepsilon}{5} > \ell$, contradicting the fact that l is an upper bound of S. We conclude $\ell^2 < 2$ is not possible.

Assume now that $\ell^2 > 2$. Our strategy will be to construct a number ever so slightly smaller than ℓ, which therefore cannot be an upper bound of S, and use this to arrive at a contradiction. Let $\delta = \ell^2 - 2$. Then $0 < \delta \leq 2$ (since $\ell \leq 2$ and hence $\ell^2 - 2 \leq 2$). Since $\delta > 0$, we have $\ell - \frac{\delta}{5} < \ell$. Since ℓ is the least upper bound of S, $\ell - \frac{\delta}{5}$ must not be an upper bound of S. By definition, this means that there is $r \in S$ such that $\ell - \frac{\delta}{5} < r$. Since $\delta \leq 2$ and $\ell \geq 1$, it follows that $\ell - \frac{\delta}{5}$ is positive and hence so is r. We may thus square both sides of $\ell - \frac{\delta}{5} < r$ to obtain

$$(\ell - \frac{\delta}{5})^2 < r^2.$$

Now

$$(\ell - \frac{\delta}{5})^2 = \ell^2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = \delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25}$$

since $\ell^2 = \delta + 2$. Moreover,

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = 2 + \delta(1 - \frac{2\ell}{5} + \frac{\delta}{25}) \geq 2 + \delta(1 - \frac{4}{5} + \frac{\delta}{25})$$

since $\ell \leq 2$. We deduce that

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} \geq 2 + \delta(\frac{1}{5}) \geq 2.$$

Putting these inequalities together gives $r^2 > 2$, contrary to the fact that $r \in S$. We conclude that $\ell^2 > 2$ is also not possible.

Since $\ell^2 < 2$ and $\ell^2 > 2$ are impossible, we must have $\ell^2 = 2$. $\square$

The collection of rational numbers does not satisfy the completeness axiom and indeed it is precisely the completeness axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Example 6.2. Within the set $\mathbb{Q}$ the subset $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum. That is, no matter which rational number you pick that is an upper bound for S, you may always find an even smaller one that is also an upper bound of S.

It is precisely the completeness axiom that assures us that everything that ought to be a number (like the length of the hypotenuse in an isosceles right triangles of side length 1) really is a number. It gives us that there are "no holes" in the real number line — the real numbers are *complete*.

For example, we can use it to prove that $\sqrt[8]{147}$ exists: Let $S = \{x \in \mathbb{R} \mid x^8 < 147\}$. Then S is nonempty (e.g., $0 \in S$) and bounded above (e.g., 50 is an upper bound) and so it must have a supremum ℓ. A proof similar to (but even messier than) the proof of Proposition 6.1 above shows that ℓ satisfies $\ell^8 = 147$.

The completeness axiom is also at the core of the Intermediate Value Theorem and many of the other major theorems we will cover in this class.

7. September 8, 2021

We now discuss a few consequence of the completeness axiom.

Theorem 7.1. *If x is any real number, then there exists a natural number n such that $n > x$.*

This looks really stupid at first. How could it be false? But consider: there are examples of ordered fields, i.e. situations in which Axioms 1–10 hold, in which this Theorem is not true! So, its proof must rely on the Completeness Axiom.

Proof. Let x be any real number. By way of contradiction, suppose there is no natural number n such that $n > x$. That is, suppose that for all $n \in \mathbb{N}$, $n \leq x$. Then $\mathbb{N}$ is a bounded above (by x). Since it is also clearly nonempty, by the Completeness Axiom, $\mathbb{N}$ has a supremum, call it ℓ. Consider the number $y := \ell - 1$. Since $y < \ell$ and ℓ is the supremum of $\mathbb{N}$, y cannot be an upper bound of $\mathbb{N}$. So, there must be some $m \in \mathbb{N}$ such that such that $\ell - 1 < m$. But then by adding 1 to both sides of this inequality we get $\ell < m + 1$ and, since $m + 1 \in \mathbb{N}$, this contradicts that assumption that ℓ is the supremum of $\mathbb{N}$.

We conclude that, given any real number x, there must exist a natural number n such that $n > x$. $\square$

Corollary 7.2 (Archimedean Principle)**.** *If $a \in R$, $a > 0$, and $b \in \mathbb{R}$, then for some natural number n we have $na > b$.*

"No matter how small a is and how large b is, if we add a to itself enough times, we can overtake b."

Proof. We apply Theorem 7.1 to the real number $x = \frac{b}{a}$. It gives that there is a natural number n such that $n > x = \frac{b}{a}$. Since $a > 0$, upon multiplying both sides by a we get $n \cdot a > b$. □

Theorem 7.3 (Density of the Rational Numbers)**.** *Between any two distinct real numbers there is a rational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. We will prove this by consider two cases: $x \geq 0$ and $x < 0$.

Let us first assume $x \geq 0$. We apply the Archimedean Principle using $a = y - x$ and $b = 1$. (The Principle applies as $a > 0$ since $y > x$.) This gives us that there is a natural number $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1$$

and thus

$$0 < \frac{1}{n} < y - x.$$

Consider the set $S = \{p \in \mathbb{N} \mid p\frac{1}{n} > x\}$. Since $\frac{1}{n} > 0$, using the Archimedean principle again, there is at least one natural number $p \in S$. By the Well Ordering Axiom, there is a smallest natural number $m \in S$.

We claim that $\frac{m-1}{n} \leq x$. Indeed, if $m > 1$, then $m - 1 \in \mathbb{N} \setminus S$ (because $m - 1$ is less than the minimum), so $\frac{m-1}{n} \leq x$; if $m = 1$, then $m - 1 = 0$, so $\frac{m-1}{n} = 0 \leq x$.

So, we have

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

By adding $\frac{1}{n}$ to both sides of $\frac{m-1}{n} \leq x$ and using that $\frac{1}{n} < y - x$, we get

$$\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$$

and hence

$$x < \frac{m}{n} < y.$$

Since $\frac{m}{n}$ is clearly a rational number, this proves the result in this case (when $x > 0$).

We now consider the case $x < 0$. The idea here is to simply "shift" up to the case we've already proven. By Theorem 7.1, we can find a natural number j such that $j > -x$ and thus $0 < x + j < y + j$. Using the first case, which we have already proven, applied to the number $x+j$ (which is positive), there is a rational number q such that

$x + j < q < y + j$. We deduce that $x < q - j < y$, and, since $q - j$ is also rational, this proves the theorem in this case. ☐

Let me say a bit more about the density of the rationals: it is a consequence of this result that between any two distinct real numbers there are *infinitely many* rational numbers: For if $x, y \in \mathbb{R}$ and $x < y$, them by the Corollary there is a rational number q_1 with $x < q_1 < y$. But then we can apply the Corollary again using x and q_1, to obtain the existence of a rational number q_2 with $x < q_2 < q_1$, and yet again using x and q_2 to obtain $q_3 \in \mathbb{Q}$ with $x < q_3 < q_2$, and so on forever.

8. September 10, 2021

(1) Let $S \subseteq \mathbb{R}$ be nonempty and bounded above, and set $\ell = \sup(S)$. Let $T = \{3x \mid x \in S\}$. Show that $\sup(T) = 3\ell$.

> First, we show that 3ℓ is an upper bound for T. Let $t \in T$. We can write $t = 3s$ for some $s \in S$. Since $s \leq \ell$, we have $t = 3s \leq 3\ell$, so 3ℓ is indeed an upper bound.
>
> Next, we show that if b is any upper bound for T, then $b \geq \ell$. Let b be an upper bound for T. This means that $b \geq t$ for any $t \in T$. Note that $b/3$ is an upper bound for S: if $s \in S$, then $t = 3s \in T$, so $3s = t \leq b$, so $s \leq b/3$. By definition of supremum, $b/3 \geq \ell$, but then $b \geq 3\ell$, as required.

Corollary 8.1 (Density of the Irrational Numbers)**.** *Between any two distinct real numbers there is an irrational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists an irrational number z such that $x < z < y$.*

(2) Prove[4] Corollary 8.1.

> Let $x < y$. We have $x - \sqrt{2} < y - \sqrt{2}$, so by Density of Rational Numbers, there is some rational number q such that $x - \sqrt{2} < q < y - \sqrt{2}$. Then $x < q + \sqrt{2} < y$. Since q is rational and $\sqrt{2}$ is irrational, $q + \sqrt{2}$ is irrational, as we proved in class. Thus, $q + \sqrt{2}$ is the number we seek.

(3) Explain why there are *infinitely many* irrational numbers between x and y.

---

[4]Hint: What can you say about the numbers $x - \sqrt{2} < y - \sqrt{2}$.

Given $x < y$, by Density of Irrational Numbers, there is some $z_1 \in \mathbb{R} \smallsetminus \mathbb{Q}$ such that $x < z_1 < y$. By Density of Irrational Numbers applied to $z_1 < y$, there is some $z_1 \in \mathbb{R} \smallsetminus \mathbb{Q}$ such that $x < z_1 < z_2 < y$. We can continue like this forever.

Theorem 8.2. *For any real number r, there is a unique integer n such that $n \leq r < n + 1$.*

(4) Proof of Theorem 8.2:

(a) First, assume that $r \geq 0$. Complete the following sentence: "The number $n+1$ should be the smallest natural number that ________________."

is larger than r.

(b) Take your sentence and turn it into a recipe for n to prove that such an integer n exists in this case.

Assume that $r \geq 0$. Consider the set $S = \{m \in \mathbb{N} \mid m > r\}$. By Theorem 7.1, S is nonempty, so by the Well-Ordering Axiom for $\mathbb{N}$, there is a minimum element in S. Set $n = \min(S) - 1$; note that $r < n + 1$ because $n+1 \in S$. If $\min(S) > 1$, $n \in \mathbb{N} \smallsetminus S$, since n is less than the minimum for S, so $n \leq r$. If $\min(S) = 1$, then $n = 0$, so by our assumption, $n = 0 \leq r$. Either way, $n \in \mathbb{N} \cup \{0\} \subseteq \mathbb{Z}$ and $n \leq r \leq n + 1$, as required.

(c) Now, assume that $r < 0$. Explain why there is some $j \in \mathbb{N}$ such that $j + r > 0$. Deduce that an integer n as in the statement exists in this case too.

Now assume that $r < 0$. By Theorem 7.1, there is some $j \in \mathbb{N}$ such that $j > -r$, so $j + r > 0$. By the case we already established, there is some integer $n \in \mathbb{Z}$ such that $n \leq j + r < n + 1$. We then have $n - j \leq r < (n - j) + 1$, so $n - j \in \mathbb{Z}$ is the integer we seek.

(d) Finally, prove that n is unique. You can use without proof that there are no integers in between 0 and 1.

To see that n is unique, suppose that $n, m \in \mathbb{Z}$ with $n \leq r < n+1$ and $m \leq r < m+1$. We then have $n < m+1$, so $n \leq m$, and, switching roles, $m \leq n$. Thus, $m = n$.

9. September 13, 2021

We will move on to next main topic of this class soon: sequences. But first, it is useful to talk a bit about absolute values.

Definition 9.1. If x is any real number we define the *absolute value* of x, written $|x|$, to be the real number

$$|x| = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0. \end{cases}$$

It will be important for us to interpret absolute values in terms of distance. For any two real numbers x and y, the number $|x-y|$ is the distance between them. By the Proposition $|x-y| = |y-x|$, which in geometric language says that the distance from x to y is the same as the distance from y to x.

Example 9.2. The set of all real numbers x such that $|x-7| \leq 2$ is the closed interval $[5, 9]$. To see this using the Proposition, note that $|x-7| \leq 2$ if and only if $-2 \leq x-7 \leq 2$ by Part (3). Now add through by 7 to get $5 \leq x \leq 9$. So $\{x \in \mathbb{R} \mid |x-7| \leq 2\} = \{x \in \mathbb{R} \mid 5 \leq x \leq 9\} = [5, 9]$.

Similarly, the set of all real numbers x such that $|x-7| < 2$ is the open interval $(5, 9)$.

Theorem 9.3 (The Triangle Inequality)**.** *For any real numbers a and b we have*

$$|a+b| \leq |a| + |b|.$$

Remark 9.4. Setting $a = x-y$ and $b = y-z$, we get that for all $x, y, z \in \mathbb{R}$,

$$|x-z| \leq |x-y| + |y-z|.$$

We also have

Corollary 9.5 (The Reverse Triangle Inequality)**.** *For any real numbers x and y we have*

$$|a-b| \geq ||a| - |b||\,.$$

Remark 9.6. Setting $a = x - y$ and $b = z - y$, we get that for all $x, y, z \in \mathbb{R}$,
$$|x - z| \geq ||x - y| - |y - z||.$$

Between the two triangle inequalities, we get both a way to bound $|x - z|$ from above and from below.

We now turn our attention to the next major topic of this class: sequences of real numbers. We will spend the next few weeks developing their properties carefully and rigorously. Sequences form the foundation for much of what we will cover for the rest of the semester.

Definition 9.7. A *sequence* is an infinite list of real numbers indexed by $\mathbb{N}$:
$$a_1, a_2, a_3, \ldots.$$
(Equivalently, a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$: the value of the function at $n \in \mathbb{N}$ is written as a_n.)

We will usually write $\{a_n\}_{n=1}^{\infty}$ for a sequence.

Example 9.8. To describe sequences, we will typically give a formula for the n-th term, a_n, either an explicit one or a recursive one. On rare occasion we'll just list enough terms to make the pattern clear. Here are some examples:

(1) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is the sequence that starts
$$4, \frac{11}{2}, \frac{14}{3}, \frac{21}{4}, \frac{24}{5}, \ldots.$$
(2) Let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. This gives the sequence
$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$
This is an example of a recursively defined sequence. It is the famed *Fibonacci sequence*.

(3) Let $\{c_n\}_{n=1}^{\infty}$ be the sequence whose n-th term is the n-th smallest positive prime integer:
$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots.$$
Note that here I have not really given an explicit formula for the terms of the sequence, but it is possible to describe an algorithm that lists every term of the sequence in order.

You have all probably seen an "intuitive" definition of the limit of a sequence before. For example, you probably believe that
$$\lim_{n\to\infty} 5 + (-1)^n \frac{1}{n} = 5.$$
Let's give the rigorous definition.

Definition 9.9. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence and L a real number. We say $\{a_n\}_{n=1}^{\infty}$ *converges* to L provided the following condition is met:

> For every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

This is an extremely important definition for this class. Learn it by heart!

The definition of convergence can be rewritten in a number of ways to make it read better. Here is one such way:

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided for every real number $\varepsilon > 0$, there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < \varepsilon$.

In symbols, the definition is

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$ satisfying $n > N$, we have $|a_n - L| < \varepsilon$.

It's a complicated definition — three quantifiers!

Here is what the definition is saying somewhat loosely: No matter how small a number ε you pick, so long as it is positive, if you go far enough out in the sequence, all of the terms from that point on will be within a distance of ε of the limiting value L.

Example 9.10. To say that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5 gives us a different statement for every $\varepsilon > 0$. For example:

- Setting $\varepsilon = 3$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 3$. Namely, we can take $N = 0$, since for *every* term a_n of the sequence, $|a_n - 5| < 3$ holds true.
- Setting $\varepsilon = \frac{1}{3}$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < \frac{1}{3}$. We cannot take $N = 0$ anymore, since $1 > 0$ and $|a_1 - 5| = 1 > \frac{1}{3}$. However, we can take $N = 3$, since for $n > 3$, $|a_n - 5| = \frac{1}{n} < \frac{1}{3}$.

In general, our choice of N may depend on ε, which is OK since our definition is of the form $\forall \varepsilon > 0, \exists N \ldots$ rather than $\exists N : \forall \varepsilon > 0 \ldots$.

10. September 15, 2021

Example 10.1. I claim the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5. I'll give a rigorous proof, along with some commentary and "scratch work" within the parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: Given this ε, our goal is to find N so that if $n > N$, then $|5+(-1)^n\frac{1}{n}-5| < \varepsilon$. The latter simplifies to $\frac{1}{n} < \varepsilon$, which in turn is equivalent to $\frac{1}{\varepsilon} < n$ since ε and n are both positive. So, it seems we've found the N that "works". Back to the formal proof....)

Let $N = \frac{1}{\varepsilon}$. Then $\frac{1}{N} = \varepsilon$, since ε is positive.

(Comment: We next show that this is the N that "works" in the definition. Since this involves proving something about every natural number that is bigger than N, we start by picking one.)

Pick any $n \in \mathbb{N}$ such that $n > N$. Then $\frac{1}{n} < \frac{1}{N}$ and hence

$$|a_n - 5| = |5 + (-1)^n\frac{1}{n} - 5| = |(-1)^n\frac{1}{n}| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

This proves that $\{5 + (-1)^n\frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. □

Remark 10.2. A direct proof that a certain sequence converges to a certain number follows the general outline:

- Let $\varepsilon > 0$ be given. (or, if your prefer, "Pick $\varepsilon > 0$.")
- Let $N =$ [insert appropriate expression in terms of from scratch work here].
- Let $n \in \mathbb{N}$ be such that $n > N$.

- [Argument that $|a_n - L| < \varepsilon$]
- Thus $\{a_n\}_{n=1}^{\infty}$ converges to L.

Example 10.3. I claim that the sequence

$$\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$$

congerges to $\frac{2}{5}$. Again I'll give a proof with commentary and scratch work in parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: We need n to be large enough so that

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This simplifies to $\left|\frac{-7}{25n+5}\right| < \varepsilon$ and thus to $\frac{7}{25n+5} < \varepsilon$, which we can rewrite as $\frac{7}{25\varepsilon} - \frac{1}{5} < n$.)

Let $N = \frac{7}{25\varepsilon} - \frac{1}{5}$. We solve this equation for ε: We get $\frac{7}{25\varepsilon} = \frac{5N+1}{5}$ and hence $\frac{25\varepsilon}{7} = \frac{5}{5N+1}$, which gives finally

$$\varepsilon = \frac{7}{25N+5}.$$

(Next we show this value of N works....)

Now pick any $n \in \mathbb{N}$ is such that $n > N$. Then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| = \left|\frac{10n-5-10n-2}{25n+5}\right| = \frac{7}{25n+5}.$$

Since $n > N$, $25n + 5 > 25N + 5$ and hence

$$\frac{7}{25n+5} < \frac{7}{25N+5} = \varepsilon.$$

We have proven that if $n \in \mathbb{N}$ and $n > N$, then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This proves $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$. □

Definition 10.4. We say a sequence $\{a_n\}_{n=1}^{\infty}$ *converges* or *is convergent* if there is (at least one) number L such that it converges to L. Otherwise, of no such L exists, we say the sequence *diverges* or *is divergent*.

(We'll show soon that if a sequence converges to a number L, then L is the *only* number to which in converges.)

Example 10.5. Let's prove the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent. This means that there is no L to which it converges.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 7.1. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1-L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. □

11. September 17, 2021

Proposition 11.1. *If a sequence converges, then there is a unique number to which it converges.*

Proof. Recall that to show something satisfying certain properties is unique, one assumes there are two such things and argues that they must be equal. So, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to L and that also converges to M. We will prove $L = M$.

By way of contradiction, suppose $L \neq M$. Then set $\varepsilon = \frac{|L-M|}{3}$. Since we are assuming $L \neq M$, we have $\varepsilon > 0$. According to the definition of convergence, since the sequence converges to L, there is a real number N_1 such that for $n \in \mathbb{N}$ such that $n > N_1$ we have

$$|a_n - L| < \varepsilon.$$

Also according to the definition, since the sequence converges to M, there is a real number N_2 such that for $n \in \mathbb{N}$ and $n > N_2$ we have

$$|a_n - M| < \varepsilon.$$

Pick n to be any natural number larger than $\max\{N_1, N_2\}$ (which exists by Theorem 7.1. For such an n, both $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$ hold. Using the triangle inequality and these two inequalities, we get

$$|L - M| \leq |L - a_n| + |M - a_n| < \varepsilon + \varepsilon.$$

But by the choice of ε, we have $\varepsilon + \varepsilon = \frac{2}{3}|L - M|$. That is, we have deduced that $|L - M| < \frac{2}{3}|L - M|$ which is impossible. We conclude that $L = M$. □

From now on, given a sequence $\{a_n\}_{n=1}^{\infty}$ and a real number L, will we use the short-hand notation

$$\lim_{n\to\infty} a_n = L$$

to mean that the given sequence converges to the given number. For example, we showed above that

$$\lim_{n\to\infty} \frac{2n-1}{5n+1} = \frac{2}{5}.$$

But, to be clear, the statement "$\lim_{n\to\infty} a_n = L$" signifies nothing more and nothing less than the statement "$\{a_n\}_{n=1}^{\infty}$ converges to L".

Here is some terminology we will need:

Definition 11.2. Suppose $\{a_n\}_{n=1}^{\infty}$ is any sequence.

(1) We say $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if there exists at least one real number M such that $a_n \le M$ for all $n \in \mathbb{N}$; we say $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if there exists at least one real number m such that $a_n \ge m$ for all $n \in \mathbb{N}$; and we say $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below.
(2) We say $\{a_n\}_{n=1}^{\infty}$ is *increasing* if for all $n \in \mathbb{N}$, $a_n \le a_{n+1}$; we say $\{a_n\}_{n=1}^{\infty}$ is *decreasing* if for all $n \in \mathbb{N}$, $a_n \ge a_{n+1}$; and we say $\{a_n\}_{n=1}^{\infty}$ is *monotone* if it is either decreasing or increasing.
(3) We say $\{a_n\}_{n=1}^{\infty}$ is *strictly increasing* if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$. I leave the definition of *strictly decreasing* and *strictly monotone* to your imaginations.

Remark 11.3. Be sure to interpret "monotone" correctly. It means

$$(\forall n \in \mathbb{N}, a_n \le a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_n \ge a_{n+1});$$

it does *not* mean

$$\forall n \in \mathbb{N}, (a_n \le a_{n+1}) \text{ or } (a_n \ge a_{n+1}).$$

Do you see the difference?

Example 11.4. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is strictly increasing and bounded (above by, e.g., 1 and below by, e.g., 0).

The Fibonacci sequence $\{f_n\}_{n=1}^{\infty} = 1, 1, 2, 3, 5, 8, \ldots$ is strictly increasing and bounded below, but not bounded above.

The sequence $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is not monotone, but it is bounded (above by, e.g., 6 and below by, e.g., 4).

Is the sequence of quotients of Fibonacci numbers $\{\frac{f_{n+1}}{f_n}\}_{n=1}^{\infty} = \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$ monotone? Is it bounded? Convergent?

Proposition 11.5. *If a sequence $\{a_n\}_{n=1}^{\infty}$ converges then it is bounded.*

Proof. Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number L. Applying the definition of "converges to L" using the particular value $\varepsilon = 1$ gives the following fact: There is a real number N such that if

$n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < 1$. The latter inequality is equivalent to $L - 1 < a_n < L + 1$ for all $n > N$.

Let m be any natural number such that $m > N$, and consider the finite list of numbers

$$a_1, a_2, \ldots, a_{m-1}, L + 1.$$

Let b be the largest element of this list. I claim the sequence is bounded above by b. For any $n \in \mathbb{N}$, if $1 \leq n \leq m - 1$, then $a_n \leq b$ since in this case a_n is a member of the above list and b is the largest element of this list. If $n \geq m$ then since $m > N$, we have $n > N$ and hence $a_n < L + 1$ from above. We also have $L + 1 \leq b$ (since $L + 1$ is in the list) and thus $a_n < b$. This proves $a_n \leq b$ for all n as claimed.

Now take p to be the smallest number in the list

$$a_1, a_2, \ldots, a_{m-1}, L - 1.$$

A similar argument shows that $a_n \geq p$ for all $n \in \mathbb{N}$. $\square$

Remark 11.6. The converse of the previous proposition is false; the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is a counterexample.

12. September 20, 2021

Proposition 12.1.
(1) *If c is any real number, then the constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.*
(2) *The sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0.*

Theorem 12.2 (Limits and algebra)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M.*

(1) *If c is any real number, then $\{ca_n\}_{n=1}^{\infty}$ converges to cL.*
(2) *The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.*
(3) *The sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.*
(4) *If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{1}{L}$.*
(5) *If $M \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$.*

(1) Use the two results above to give a short proof[5] that $\left\{\dfrac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\dfrac{2}{5}$.

We can rewrite $\dfrac{2n-1}{5n+1} = \dfrac{(2n-1)/n}{(5n+1)/n} = \dfrac{2-1/n}{5+1/n}$. Since $\{1/n\}_{n=1}^{\infty}$ converges to 0, $\{-1/n\}_{n=1}^{\infty}$ converges to $-1 \cdot 0 = 0$ by Theorem 12.2(1). Since $\{2\}_{n=1}^{\infty}$ converges to 2 and $\{-1/n\}_{n=1}^{\infty}$ converges to 0, $\{2-1/n\}_{n=1}^{\infty}$ converges to 2 by Theorem 12.2(2). Since $\{5\}_{n=1}^{\infty}$ converges to 5 and $\{1/n\}_{n=1}^{\infty}$ converges to 0, $\{5+1/n\}_{n=1}^{\infty}$ converges to 5 by Theorem 12.2(2). Then, by Theorem 12.2(5), which applies since $5 \neq 0$ and every term of $\{5+1/n\}_{n=1}^{\infty}$ is nonzero, the sequence converges to $2/5$.

(2) Prove part (1) of Proposition 12.1.

Let $\varepsilon > 0$. Take $N = 0$. For any natural number $n > N$, we have $|c - c| = 0 < \varepsilon$. This shows that the sequence converges to c.

(3) Prove part (1) of Theorem 12.2:

- First, assume that $c = 0$. Explain why the result is true in this case.
- Now, assume that $c \neq 0$. We need to prove that $\{ca_n\}_{n=1}^{\infty}$ converges to cL. Write the first sentence of the proof of this.
- We have assumed that $\{a_n\}_{n=1}^{\infty}$ converges to L. Explain what this means when applied to the positive number $\dfrac{\varepsilon}{|c|}$ (in the place of what we usually call ε).
- Complete the proof.

First, if $c = 0$, then $\{ca_n\}_{n=1}^{\infty}$ is the constant sequence $\{0\}_{n=1}^{\infty}$, which converges to $0 = 0 \cdot L$, so the result holds in this case.

Now, let $\varepsilon > 0$. Applying the definition of $\{a_n\}_{n=1}^{\infty}$ converges to L with the positive number $\varepsilon/|c|$, there is some

[5] Hint: Rewrite $\dfrac{2n-1}{5n+1} = \dfrac{(2n-1)/n}{(5n+1)/n} = \dfrac{2-1/n}{5+1/n}$.

$N \in \mathbb{R}$ such that for all natural numbers $n > N$, we have $|a_n - L| < \varepsilon/|c|$. Then $|ca_n - cL| = |c||a_n - L| < |c|\varepsilon/|c| = \varepsilon$. This shows that $\{ca_n\}_{n=1}^{\infty}$ converges to cL.

(4) Prove[6] part (2) of Theorem 12.2.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L and $\frac{\varepsilon}{2}$ is positive, there is a number N_1 such that for all $n \in \mathbb{N}$ with $n > N_1$ we have

$$|a_n - L| < \frac{\varepsilon}{2}.$$

Likewise, since $\{b_n\}_{n=1}^{\infty}$ converges to M, there is a number N_2 such that for all $n \in \mathbb{N}$ with $n > N_2$ we have

$$|b_n - M| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. If $n \in \mathbb{N}$ and $n > N$, then $n > N_1$ and $n > N_2$ and hence we have

$$|a_n - L| < \frac{\varepsilon}{2} \text{ and } |b_n - M| < \frac{\varepsilon}{2}.$$

Using these inequalities and the triangle inequality we get

$$\begin{aligned}|(a_n + b_n) - (M + L)| &= |(a_n - M) + (b_n - L)| \\ &\leq |(a_n - M)| + |(b_n - L)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This proves that $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.

(5) Prove[7] part (3) of Theorem 12.2.

Pick $\varepsilon > 0$.

("Scratch work": The goal is to make $|a_n b_n - LM|$ small and the trick is to use that

$$\begin{aligned}|a_n b_n - LM| &= |a_n(b_n - M) + (a_n - L)M| \\ &\leq |a_n(b_n - M)| + |(a_n - L)M| \\ &= |a_n||b_n - M| + |a_n - L||M|.\end{aligned}$$

[6]Hint: Given $\varepsilon > 0$, can you find N such that $|a_n - L| < \varepsilon/2$ and $|b_n - M| < \varepsilon/2$ for all natural numbers $n > N$?

[7]Hint: Write $|a_n b_n - LM| = |a_n(b_n - M) + M(a_n - L)|$ and apply the triangle inequality. Since $\{a_n\}_{n=1}^{\infty}$ converges, it is bounded.

Our goal will be to take n to be large enough so that each of $|a_n||b_n - M|$ and $|a_n - L||M|$ is smaller than $\varepsilon/2$. We can make $|a_n - L|$ as small as we like and $|M|$ is just a fixed number. So, we can "take care" of the second term by choosing n big enough so that $|a_n - L| < \frac{\varepsilon}{2|M|}$. A irritating technicality here is that $|M|$ could be 0, and so we will use $\frac{\varepsilon}{2|M|+1}$ instead. The other term $|a_n||b_n - M|$ is harder to deal with since each factor varies with n. Here we use that convergent sequences are bounded so that we can find a real number X so that $|a_n| \leq X$ for all n. Then we choose n large enough so that $|b_n - M| < \frac{\varepsilon}{2X}$. Back to the proof.)

Since $\{a_n\}$ converges, it is bounded by Proposition 11.5, which gives that there is a strictly positive real number X so that $|a_n| \leq X$ for all $n \in \mathbb{N}$. Since $\{b_n\}$ converges to M and $\frac{\varepsilon}{2X} > 0$, there is a number N_1 so that if $n > N_1$ then $|b_n - M| < \frac{\varepsilon}{2X}$. Since $\{a_n\}$ converges to L and $\frac{\varepsilon}{2|M|+1} > 0$, there is a number N_2 so that if $n \in \mathbb{N}$ and $n > N_2$, then $|a_n - L| < \frac{\varepsilon}{2|M|+1}$. Let $N = \max\{N_1, N_2\}$. For any $n \in \mathbb{N}$ such that $n > N$, we have

$$\begin{aligned}
|a_n b_n - LM| &= |a_n(b_n - M) + (a_n - L)M| \\
&\leq |a_n(b_n - M)| + |(a_n - L)M| \\
&= |a_n||b_n - M| + |a_n - L||M| \\
&< X\frac{\varepsilon}{2X} + \frac{\varepsilon}{2|M|+1}|M| \\
&< \varepsilon.
\end{aligned}$$

This proves $\{a_n \cdot b_n\}_{n=1}^{\infty}$ converges to $L \cdot M$.

Here is the proof of parts (4) and (5) of the theorem:

Proof. We start with (4).

To prove this claim, pick $\varepsilon > 0$.

(Scratch work: We want to show $\left|\frac{1}{a_n} - \frac{1}{L}\right| < \varepsilon$ holds for n sufficiently large. We have

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|L - a_n|}{|a_n||L|}.$$

We can make the top of this fraction as small as we like, but the problem is that the bottom might be very small too since a_n might get very close to 0. But since a_n converges to L and $L \neq 0$ if we go far enough out, it will be close to L. In particular, if a_n is within a distance of $\frac{|L|}{2}$ of M then $|a_n|$ will be at least $\frac{|L|}{2}$. So for n sufficiently large we have $\frac{|a_n-L|}{|a_n||L|} < 2\frac{|a_n-L|}{|L|^2}$. And then for n sufficiently large we also get $|a_n - L| < \frac{|L|^2}{2\varepsilon}$. Back to the formal proof...)

Since $\{a_n\}$ converges to L and $\frac{|L|}{2} > 0$, there is an N_1 such that for $n > N_1$ we have $|a_n - L| < \frac{|L|}{2}$ and hence $|a_n| > \frac{|L|}{2}$. Again using that $\{a_n\}$ converges to M and that $\frac{\varepsilon|L|^2}{2} > 0$, there is an N_2 so that for $n > N_2$ we have $|a_n - L| < \frac{\varepsilon|L|^2}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then we have

$$\begin{aligned}\left|\frac{1}{a_n} - \frac{1}{L}\right| &= \frac{|a_n - L|}{|a_n||L|} \\ &< \frac{2}{|L|}\frac{|a_n - L|}{|L|} \\ &= 2\frac{|a_n - L|}{|L|^2}\end{aligned}$$

since $|a_n| > |L|/2$ and hence $\frac{1}{|a_n|} < \frac{2}{|L|}$. But then

$$2\frac{|a_n - L|}{|L|^2} < 2\frac{\frac{\varepsilon|L|^2}{2}}{|L|^2} = \varepsilon$$

since $|a_n - L| < \frac{\varepsilon|L|^2}{2}$. Putting these together gives

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| < \varepsilon$$

for all $n > N$. This proves $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ converges to $\frac{1}{L}$.

Finally, part (5) follows from parts (3) and (4). □

13. September 22, 2021

The following is another useful technique:

Theorem 13.1 (The "squeeze" principle)**.** *Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ are three sequences such that*

- *$\{a_n\}_{n=1}^{\infty}$ converges to L,*
- *$\{c_n\}_{n=1}^{\infty}$ also converges to L (same value), and*
- *there is a real number M such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ such that $n > M$.*

Then $\{b_n\}_{n=1}^{\infty}$ also converges to L.

The heuristic version of this theorem is:

> If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ and b_n is "eventually" between a_n and c_n, then $\lim_{n\to\infty} b_n = L$ too.

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ both converge to L and that there is a real number M such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ such that $n > M$. We need to prove $\{b_n\}_{n=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L there is a number N_1 such that if $n \in \mathbb{N}$ and $n > N_1$ then $|a_n - L| < \varepsilon$ and hence $L - \varepsilon < a_n < L + \varepsilon$. Likewise, since $\{c_n\}_{n=1}^{\infty}$ converges to L there is a number N_2 such that if $n \in \mathbb{N}$ and $n > N_2$ then $L - \varepsilon < c_n < L + \varepsilon$. Let

$$N = \max\{N_1, N_2, M\}$$

where M is defined as in the statement of the Theorem. If $n \in \mathbb{N}$ and $n > N$, then $n > N_1$ and hence $L - \varepsilon < a_n$, and $n > N_2$ and hence $c_n < L + \varepsilon$, and $n > M$ and hence $a_n < b_n < c_n$. Combining these facts gives that for $n \in \mathbb{N}$ such that $n > N$, we have

$$L - \varepsilon < b_n < L + \varepsilon$$

and hence $|b_n - L| < \varepsilon$. This proves $\{b_n\}_{n=1}^{\infty}$ converges to L. $\square$

14. September 24, 2021

Example 14.1. We can use the Squeeze Theorem to give a short proof that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. Note that Theorem 12.2 alone cannot be used in this example. However, from Theorem 12.2, it follows that $\{5 - \frac{1}{n}\}_{n=1}^{\infty}$ and $\{5 + \frac{1}{n}\}_{n=1}^{\infty}$ both converge to 5. Then, since

$$5 - \frac{1}{n} \leq 5 + (-1)^n \frac{1}{n} \leq 5 + \frac{1}{n}$$

for all n, our sequence also converges to 5.

When I introduced the Completeness Axiom, I mentioned that, heuristically, it is what tells us that the real number line doesn't have any holes. The next result makes this a bit more precise:

Theorem 14.2. *Every increasing, bounded above sequence converges.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence that is both bounded above and increasing.

(Commentary: In order to prove it converges, we need to find a candidate number L that it converges to. Since the set of numbers occurring in this sequence is nonempty and bounded above, this number is provided to us by the Completeness Axiom.)

Let S be the set of those real numbers that occur in this sequence. (This is technically different that the sequence itself, since sequences are allowed to have repetitions but sets are not. Also, sequences have an ordering to them, but sets do not.) The set S is clearly nonempty, and it is bounded above since we assume the sequence is bounded above.

Therefore, by the Completeness Axiom, S has a supremum L. We will prove the sequence converges to L.

Pick $\varepsilon > 0$. Then $L - \varepsilon < L$ and, since L is the supremum, $L - \varepsilon$ is not an upper bound of S. This means that there is an element of S that is strictly bigger than $L - \varepsilon$. Every element of S is a member of the sequence, and so we get that there is an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$.

(We will next show that this is the N that "works". Note that, in the general definition of convergence of a sequence, N can be any real number, but in this proof it turns out to be a natural number.)

Let n be any natural number such that $n > N$. Since the sequence is increasing, $a_N \leq a_n$ and hence

$$L - \varepsilon < a_N \leq a_n.$$

Also, $a_n \leq L$ since L is an upper bound for the sequence, and thus we have

$$L - \varepsilon < a_n \leq L.$$

It follows that $|a_n - L| < \varepsilon$. We have proven the sequence converges to L. $\square$

Remark 14.3. Note that any increasing sequence is bounded below, for example, by its first term. Thus, an increasing sequence is bounded if and only if it is bounded below. Likewise, and decreasing sequence is bounded if and only if it is bounded above.

Theorem 14.4 (Monotone Converge Theorem)**.** *Every bounded monotone sequence converges.*

Proof. If $\{a_n\}_{n=1}^{\infty}$ is increasing, then this is the content of Theorem **??**. If $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, consider the sequence $\{-a_n\}_{n=1}^{\infty}$. If $a_n \leq M$ for all n, then $-a_n \geq -M$ for all n, so $\{-a_n\}_{n=1}^{\infty}$ is bounded below. Also, since $a_n \geq a_{n+1}$ for all n, we have $-a_n \leq -a_{n+1}$ for all n, so $\{-a_n\}_{n=1}^{\infty}$ is increasing. Thus, by Theorem **??**, $\{-a_n\}_{n=1}^{\infty}$ converges, say to L. Then by Theorem 12.2(1), $\{a_n\}_{n=1}^{\infty} = \{-(-a_n)\}_{n=1}^{\infty}$ converges to $-L$. $\square$

Example 14.5. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by the formula

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

We will use the Monotone Convergence Theorem to prove that this sequence converges.

First, we need to see that the sequence is increasing. Indeed, for every n we have that $a_{n+1} = a_n + \frac{1}{a_{n+1}^2} \geq a_n$.

Next, we need to show that it is bounded above. Observe that

$$\begin{aligned} a_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\le 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\ &= 1 + 1 - \frac{1}{n}, \end{aligned}$$

so we have $a_n \le 2$ for all n. This means that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

Hence, by the Monotone Convergence Theorem, $\{a_n\}_{n=1}^{\infty}$ converges. Leonhard Euler was particularly interested in this sequence, and was able to prove that it converges to $\frac{\pi^2}{6}$. This requires some other ideas, so we won't do that here.

We will discuss a bit the notion of "diverging to infinity", a concept that you might have seen before in Calculus.

It is sometimes useful to distinguish between sequences like

$$\{(-1)^n\}_{n=1}^{\infty}$$

that diverge because they "oscillate", and sequences like

$$\{n\}_{n=1}^{\infty}$$

that diverge because they "head toward infinity".

Definition 14.6. A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to* ∞ if for every real number M, there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then we have $a_n > M$.

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$ if for every real number L, there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $a_n < L$.

Intuitively, a sequence diverges to ∞ provided that, no matter how big M is, if you go far enough along the sequence, eventually all of the terms are bigger than M. Similarly for diverges to $-\infty$.

Example 14.7. (1) $\{n^2+3\}_{n=1}^{\infty}$ diverges to ∞.
(2) $\{-\sqrt{n}\}_{n=1}^{\infty}$ diverges to $-\infty$.
(3) $\{(-1)^n\}_{n=1}^{\infty}$ diverges, but not to ∞ nor to $-\infty$.
(4) $\{(-1)^n n\}_{n=1}^{\infty}$ diverges, but not to ∞ nor to $-\infty$.

15. September 27, 2021

Example 15.1. The sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to ∞. Let us prove this using the definition: Pick $M \in \mathbb{R}$. (Scratch work: I need $\sqrt{n} > M$ which will occur if $n > M^2$.) Let $N = M^2$. If $n \in \mathbb{N}$ and $n > N$, then $\sqrt{n} > \sqrt{N} = \sqrt{M^2} = |M| \geq M$. (Note that M could conceivably be negative.) This proves $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to ∞.

Proposition 15.2. *If a sequence $\{a_n\}_{n=1}^{\infty}$ diverges to ∞ or diverges to $-\infty$, then it diverges.*

Proof. We prove the contrapositive. (What is the contrapositive? If a sequence converges, then it does not diverge to ∞ and it des not diverge to $-\infty$.) Suppose $\{a_n\}_{n=1}^{\infty}$ converges to some number L. Then since it converges, it is bounded, so that there are real numbers b and c such that $b \leq a_n \leq c$ for all n.

In particular, this means that there is no $N \in \mathbb{R}$ such that $a_n > c$ for all natural numbers $n > N$. Thus, taking "$M = c$" in the definition of diverges to ∞, we see that $\{a_n\}_{n=1}^{\infty}$ does not diverge to ∞.

Similarly, that there is no $N \in \mathbb{R}$ such that $a_n < b$ for all natural numbers $n > N$. Thus, taking "$M = b$" in the definition of diverges to $-\infty$, we see that $\{a_n\}_{n=1}^{\infty}$ does not diverge to $-\infty$. $\square$

As a matter of shorthand, we write $\lim_{n\to\infty} a_n = \infty$ to indicate that $\{a_n\}_{n=1}^{\infty}$ diverges to ∞. But note unlike when we wrote things such as $\lim_{n\to\infty} a_n = 17$, when we write $\lim_{n\to\infty} a_n = \infty$ we are asserting that $\{a_n\}_{n=1}^{\infty}$ *diverges* (in a specific way). Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ to indicate that $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

Example 15.3. Take the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

This is known as the "harmonic series". We will show that this sequence diverges to ∞.

Observe that

$$\begin{aligned} a_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots . \end{aligned}$$

For most natural numbers n, it may be a little messy to deal with the last terms in the sum. But, if $k \in \mathbb{N}$, and $n = 2^k$, we can do this nicely:

$$\begin{aligned}
a_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^k} \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \underbrace{\left(\frac{1}{2^3} + \cdots + \frac{1}{2^3}\right)}_{\text{from } 2^2+1 \text{ to } 2^3} + \cdots + \underbrace{\left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right)}_{\text{from } 2^{k-1}+1 \text{ to } 2^k} \\
&= 1 + \frac{1}{2} + 2^1 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + \cdots + 2^{k-1} \cdot \frac{1}{2^k} = 1 + \frac{k}{2}.
\end{aligned}$$

Let $M \in \mathbb{R}$ be given. Let M' be the smallest natural number greater than M (why does such a number exist?) and take $N = 2^{2M'}$. By the computation above, taking $k = 2M'$, we see that $a_N \geq 1 + \frac{2M'}{2}$. Then, for $n > N$, since $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence, we have

$$a_n \geq a_N \geq 1 + \frac{2M'}{2} = M' + 1 > M' > M,$$

which shows that $\{a_n\}_{n=1}^{\infty}$ diverges to ∞.

We will now embark on a bit of detour. I've postponed talking about proofs by induction, but we will need to use that technique on occasion. So let's talk about that idea now.

The technique of proof by induction is used to prove that an infinite sequence of statements indexed by $\mathbb{N}$

$$P_1, P_2, P_3, \ldots$$

are all true. For example, for any real number x, the equation

$$(1 - x)(1 + x + \cdots + x^n) = 1 - x^{n+1}$$

holds for all $n \in \mathbb{N}$. Fixing x, we get one statement for each natural number:

$$\begin{aligned}
P_1 &: & (1 - x)(1 + x) &= 1 - x^2 \\
P_2 &: & (1 - x)(1 + x + x^2) &= 1 - x^3 \\
P_3 &: & (1 - x)(1 + x + x^2 + x^3) &= 1 - x^4 \\
&\vdots & &\vdots
\end{aligned}$$

Such a fact (for all n) is well-suited to be proven by induction.

Here is the general principle:

Theorem 15.4 (Principle of Mathematical Induction)**.** *Suppose we are given, for each $n \in \mathbb{N}$, a statment P_n. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Then P_n is true for all $n \in \mathbb{N}$.*

"The domino analogy": Think of the statements $P_1, P_2, \ldots$ as dominoes lined up in a row. The fact that $P_k \implies P_{k+1}$ is interpreted as meaning that the dominoes are arranged well enough so that if one falls, then so does the next one in the line. The fact that P_1 is true is interpreted as meaning the first one has been knocked over. Given these assumptions, for every n, the n-th domino will (eventually) fall down.

The Principle of Mathematical Induction (PMI) is indeed a theorem, which we will now prove:

Proof. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Consider the subset

$$S = \{n \in \mathbb{N} \mid \ P_n \text{ is false}\}$$

of $\mathbb{N}$. Our goal is to show S is the empty set.

By way of contradiction, suppose S is not empty. Then by the Well-Ordering Principle, S has a smallest element, call it ℓ. (In other words, P_ℓ is the first statement in the list $P_1, P_2, \ldots,$ that is false.) Since P_1 is true, we must have $\ell > 1$. But then $\ell - 1 < \ell$ and so $\ell - 1$ is not in S. Since $\ell > 1$, we have $\ell - 1 \in \mathbb{N}$ and thus we can say that $P_{\ell-1}$ must be true. Since $P_k \Rightarrow P_{k+1}$ for any k, letting $k = \ell - 1$, we see that, since $P_{\ell-1}$ is true, P_ℓ must also by true. This contradicts the fact that $\ell \in S$. We conclude that S must be the empty set. $\square$

The above proof shows that the Principle of Mathematical Induction is a consequence of the Well-Ordering Principle. The converse is also true.

16. September 29, 2021

(1) Show that if $\{a_n\}_{n=1}^{\infty}$ is increasing and not bounded above, then $\{a_n\}_{n=1}^{\infty}$ diverges to ∞.

Suppose that $\{a_n\}_{n=1}^{\infty}$ is increasing and not bounded above. Let $M \in \mathbb{R}$ be arbitrary. Since $\{a_n\}_{n=1}^{\infty}$ is not bounded above, M is not an upper bound, so there exists some $N \in \mathbb{N}$ such that $a_N > M$. We claim that for all natural numbers $n > N$, we have $a_n > M$ Indeed, if $n > N$, since the sequence is increasing, we have $a_n \geq a_N > M$, as desired. Thus, for every $M \in \mathbb{R}$, there is some $N \in \mathbb{R}$ such that for all natural numbers $n > N$, we have $a_n > M$. This means that $\{a_n\}_{n=1}^{\infty}$ diverges to ∞.

Alternative argument:

Suppose that $\{a_n\}_{n=1}^{\infty}$ is increasing and not bounded above. By way of contradiction, suppose that $\{a_n\}_{n=1}^{\infty}$ does not diverge to ∞. This means that there exists some $M \in \mathbb{R}$ such that for all $N \in \mathbb{R}$, there exists a natural number $n > N$ with $a_n \leq M$. We claim that M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, which will be the desired contradiction. Let $k \in \mathbb{N}$; we need to show that $a_k \leq M$. Taking $k = N$ in the statement above, there is some natural number $n > k$ with $a_n \leq M$. But since $\{a_n\}_{n=1}^{\infty}$ is increasing, we have $a_k \leq a_n \leq M$, as desired, so M is an upper bound. We have obtained a contradiction, so we must have that $\{a_n\}_{n=1}^{\infty}$ does indeed diverge to ∞.

(2) Let S be a nonempty bounded above set and $\ell = \sup(S)$.

• Show that, for every $n \in \mathbb{N}$, there is some $x \in S$ such that $\ell - \frac{1}{n} < x \leq \ell$.

Since $\ell - \frac{1}{n} < \ell = \sup(S)$, $\ell - \frac{1}{n}$ is not an upper bound for S, so there exists some $x \in S$ with $\ell - \frac{1}{n} < x$. Since ℓ is an upper bound of S, we also have $x \leq \ell$.

• Show that there is a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in S$ for all $n \in \mathbb{N}$ that converges to ℓ.

For each n, pick some $a_n \in S$ such that $\ell - \frac{1}{n} < a_n \leq \ell$. We claim that this sequence converges to ℓ. Note that the sequence $\{\ell\}_{n=1}^{\infty}$ converges to ℓ since it is constant, and $\{\ell - \frac{1}{n}\}_{n=1}^{\infty}$ converges to ℓ also by Theorem 12.2. Thus, by the Squeeze Theorem, $\{\ell\}_{n=1}^{\infty}$ converges to ℓ too.

Alternative argument:

For each n, pick some $a_n \in S$ such that $\ell - \frac{1}{n} < a_n \leq \ell$. We claim that this sequence converges to ℓ. Let $\varepsilon > 0$. Take $N = 1/\varepsilon$. Then for a natural number $n > N$, using the inequalities above, we have that $|a_n - \ell| < \frac{1}{n} < \frac{1}{N} = \varepsilon$.

17. October 1, 2021

Example 17.1. Let's show that for every $x \in \mathbb{R}$, the equality

$$(1-x)(1+x+\cdots+x^n) = 1-x^{n+1}$$

holds for all $n \in \mathbb{N}$. Fix a real number $x \in \mathbb{R}$. We will show by induction on n that the equality

$$(1-x)(1+x+\cdots+x^n) = 1-x^{n+1}$$

holds this real number x and for all $n \in \mathbb{N}$. For $n=1$, we have

$$(1-x)(1+x) = 1-x^2,$$

so the equality holds in this case. Assume that the equality holds for some natural number $n=k$. Then

$$\begin{aligned}
(1-x)(1+x+\cdots+x^{k+1}) &= (1-x)((1+x+\cdots+x^k)+x^{k+1}) \\
&= (1-x)(1+x+\cdots+x^k)+x^{k+1}-x^{k+2} \\
&= 1-x^{k+1}+x^{k+1}-x^{k+2} \\
&= 1-x^{k+2}
\end{aligned}$$

so the equality holds for $n=k+1$. Thus, by induction, the equality holds for all $n \in \mathbb{N}$. □

Example 17.2. Let's show that every finite set has a minimum element. It's not obvious that induction makes any sense here, but we can rephrase the statement as saying that for every $n \in \mathbb{N}$, a set of real numbers with n elements has a minimum. So, let us prove by induction on n that for every set of n real numbers, the set has a minimum. When $n=1$, any set with one element clearly has its only element as a minimum, so the statement is true in this case. Assume that for some natural number $n=k$, every set of k elements has a minimum element; we need to show that every set of $k+1$ elements has a minimum. Let S be a set of $k+1$ elements, and fix some $x \in S$. Then $T = S \setminus \{x\}$ has k elements, so it has a minimum, which we will call y. Either $x<y$ or $x \geq y$. If $x<y$, and $z \in S$, either $z \in T$ so $z \geq y > x$, or else $z = x$ so $z \geq x$; thus, $x = \min(S)$ in this case. If $x \geq y$, and $z \in S$, if $z \in T$ then $z \geq y$, and otherwise $z = x \geq y$; thus $y = \min(S)$ in this case. We have shown that S has a minimum element. This completes the inductive step, and hence completes the proof. □

Now, I want to apply what we've done so far to decimal expansions. Let us say that $d_1, d_2, d_3, \ldots$ is a "digit sequence" if $d_i \in \{0, 1, \ldots, 9\}$ for each $i \in \mathbb{N}$. We will say that an integer k and a digit sequence $d_1, d_2, d_3, \ldots$ is a "decimal expansion" and write $k.d_1d_2d_3\ldots$ to denote a decimal expansion. We say that the decimal expansion $k.d_1d_2d_3\ldots$ "corresponds to a real number r" if the sequence

$$\begin{aligned} a_1 &= k + \frac{d_1}{10^1} \\ a_2 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} \\ a_3 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \\ &\vdots \end{aligned}$$

converges to r.

Theorem 17.3. *Every decimal expansion corresponds to a real number.*

Proof. Let k be an integer, and $d_1, d_2, d_3, \ldots$ be a sequence such that $d_i \in \{0, 1, \ldots, 9\}$ for each $i \in \mathbb{N}$. We need to show that the sequence

$$\begin{aligned} a_1 &= k + \frac{d_1}{10^1} \\ a_2 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} \\ a_3 &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \\ &\vdots \end{aligned}$$

converges. Observe that $\{a_n\}_{n=1}^{\infty}$ is increasing. Note now that for any $n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &= k + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \\ &= k + \frac{9}{10^1} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} \\ &= k + \frac{9}{10}\left(1 + \frac{1}{10^1} + \cdots + \frac{1}{10^{n-1}}\right) \\ &= k + \frac{9}{10}\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \\ &= k + \frac{9}{10}\frac{1 - \frac{1}{10^n}}{\frac{9}{10}} \\ &= k + 1 - \frac{1}{10} \leq k + 1. \end{aligned}$$

Since this holds for all n, $\{a_n\}_{n=1}^{\infty}$ is bounded above by $k+1$. By the Monotone Convergence Theorem, $\{a_n\}_{n=1}^{\infty}$ converges. $\square$

Example 17.4. The decimal expansion 0.25 corresponds to $\frac{1}{4}$. The decimal expansion $0.99999\ldots$ corresponds to 1.

18. October 8, 2021

The next example of a proof by induction will establish a fact that is perhaps intuitively obvious. Since it will play an important role in later proofs, we state it as a Lemma here:

Lemma 18.1. *Let $b_1, b_2, \ldots$ be any strictly increasing sequence of natural numbers; that is, assume $b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ and that $b_k < b_{k+1}$ for all $k \in \mathbb{N}$. Then $b_k \geq k$ for all k.*

Proof. Suppose $b_1, b_2, \ldots$ is a strictly increasing sequence of natural numbers. We prove $b_n \geq n$ for all n by induction on n. That is, for each $n \in \mathbb{N}$, let P_n be the statement that $b_n \geq n$.

P_1 is true since $b_1 \in \mathbb{N}$ and so $b_1 \geq 1$. Given $k \in \mathbb{N}$, assume P_k is true; that is, assume $b_k \geq k$. Since $b_{k+1} > b_k$ and both are natural numbers, we have $b_{k+1} \geq b_k + 1 \geq k+1$; that is, P_{k+1} is true too. By induction, P_n is true for all $n \in \mathbb{N}$. $\square$

We next discuss the important concept of a "subsequence".

Informally speaking, a subsequence of a given sequence is a sequence one forms by skipping some of the terms of the original sequence. In other words, it is a sequence formed by taking just some of the terms of the original sequence, but still infinitely many of them, without repetition.

We'll cover the formal definition soon, but let's give a few examples first, based on this informal definiton.

Example 18.2. Consider the sequence

$$a_n = \begin{cases} 7 & \text{if } n \text{ is divisible by 3 and} \\ \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

If we pick off every third term starting with the term a_3 we get the subsequence

$$a_3, a_6, a_9, \ldots$$

which is the constant sequence

$$7, 7, 7, \ldots.$$

If we pick off the other terms we form the subsequence

$$a_1, a_2, a_4, a_5, a_7, a_8, a_{10}, \ldots$$

which gives the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{10}, \ldots.$$

Note that it is a little tricky to find an explicit formula for this sequence.

Example 18.3. For another, simpler, example, consider the sequence $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$. Taking just the odd-indexed terms gives the sequence

$$-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, \ldots$$

and taking the even-indexed terms gives the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$$

This time we can easily give a formula for each of these sequences: the first is

$$\{-\frac{1}{2n-1}\}_{n=1}^{\infty}$$

and the second is

$$\{\frac{1}{2n}\}_{n=1}^{\infty}.$$

Here is the formal definiton:

Definition 18.4. A *subsequence* of a given sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence of the form

$$\{a_{n_k}\}_{k=1}^{\infty}$$

where

$$n_1, n_2, n_3, \ldots$$

is any strictly increasing sequence of natural numbers — that is $n_k \in \mathbb{N}$ and $n_{k+1} > n_k$ for all $k \in \mathbb{N}$, so that

$$n_1 < n_2 < n_3 < \cdots.$$

Example 18.5. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.

Setting $n_k = 2k - 1$ for all $k \in \mathbb{N}$ gives the subsequence of just the odd-indexed terms of the original sequence.

Setting $n_k = 2k$ for all $k \in \mathbb{N}$ gives the subsequence of just the even-indexed terms of the original sequence.

Setting $n_k = 3k - 2$ for all $k \in \mathbb{N}$ gives the subsequence of consising of every third term of the original sequence, starting with the first.

Setting $n_k = 100 + k$ gives the subsequence that is that "tail end" of the original, obtained by skipping the first 100 terms:

$$a_{101}, a_{102}, a_{103}, a_{104}, \ldots.$$

Of course, there is nothing special about 100 in this example.

The following result is important:

Theorem 18.6. *If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, then every subsequence of this sequence also converges to L.*

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ converges to L and let $n_1 < n_2 < \cdots$ be any strictly increasing sequence of natural numbers. We need to prove $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L, there is an N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < \varepsilon$. (We will show that the same N also "works" for the subsequence.)

If $k \in \mathbb{N}$ and $k > N$, then $n_k \geq k$ by Lemma 18.1, and hence $n_k > N$. It follows that $|a_{n_k} - L| < \varepsilon$. This proves $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L. □

Corollary 18.7. *Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.*

(1) *If there is a subsequence of this sequence that diverges, then the sequence itself diverges.*
(2) *If there are two subsequences of this sequence that converge to different values, then the sequence itself diverges.*

Proof. These are both immediate consequences of the theorem. □

Example 18.8. Consider the sequence

$$a_n = \begin{cases} 7 & \text{if } n \text{ is divisible by 3 and} \\ \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

Let $n_k = 3k$. Then the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is

$$a_3, a_6, a_9, \ldots$$

which is the constant sequence

$$7, 7, 7, \ldots.$$

It converges to 7.

Now let $n_k = 3k - 2$. Then the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is

$$a_1, a_4, a_7, \ldots$$

which is the sequence $\{\frac{1}{3k-2}\}_{k=1}^{\infty}$. It converges to 0.

Since the original sequence admits two subsequences that converge to different values, by the Corollary, the original sequence diverges.

I want to give a *crazy* example of a sequence:

Lemma 18.9. *There exists a sequence of strictly positive rational numbers $\{a_n\}_{n=1}^{\infty}$ such that every strictly positive rational number occurs in it infinitely many times.*

Proof. Consider the points in the first quadrant whose Cartesian coordinates are positive integers: (m, n) for some $m, n \in \mathbb{N}$. Starting at $(1, 1)$ travel back and forth along diagonal lines of slope -1 as shown:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
(1,4) \arrow[u, dotted, no head] & (2,4) \arrow[ul, dotted, no head] \arrow[dr] & (3,4) \arrow[ul, dotted, no head] & (4,4) \arrow[ul, dotted, no head] \arrow[dr, dotted, no head] & \cdots \\
(1,3) \arrow[dr] & (2,3) \arrow[ul] & (3,3) \arrow[dr] & (4,3) \arrow[ul] \arrow[dr, dotted, no head] & \cdots \\
(1,2) \arrow[u] & (2,2) \arrow[dr] & (3,2) \arrow[ul] & (4,2) \arrow[dr, dotted, no head] & \cdots \\
(1,1) \arrow[r] & (2,1) \arrow[ul] & (3,1) \arrow[r] & (4,1) \arrow[ul] & \cdots
\end{tikzcd}
$$

This gives the list of points

$$(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1), (4, 1), (3, 2), (2, 3), (1, 4), \ldots$$

Now convert these to a list of rational numbers by changing (m, n) to $\frac{m}{n}$ to get the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \ldots$$

of positive rational numbers.

I claim every strictly positive rational number occurs infinitely many times in this sequence: Let q be any strictly positive rational number. Then $q = \frac{m}{n}$ for some $m, n \in \mathbb{N}$. Moreover, $q = \frac{jm}{jn}$ for all $j \in \mathbb{N}$, and since $\frac{jm}{jn}$ occurs in the sequence for all $j \in \mathbb{N}$, the number q appears infinitely many times. $\square$

We can improve this a bit:

Corollary 18.10. *There exists a sequence $\{q_n\}_{n=1}^{\infty}$ of rational numbers such that every rational number occurs infinitely many times.*

Proof. Starting with a sequence $\{a_n\}_{n=1}^{\infty}$ as in Lemma 18.9, such that every strictly positive rational number occurs infinitely many times, define a new sequence by

$$a_1, -a_1, 0, a_2, -a_2, 0, a_3, -a_3, 0, a_3, -a_3, 0, \ldots$$

More formally, let

$$q_n = \begin{cases} a_{(n-1)/3}, & \text{if } n \text{ is congruent to 1 modulo 3,} \\ -a_{(n-2)/3}, & \text{if } n \text{ is congruent to 2 modulo 3, and} \\ 0, & \text{if } n \text{ is congruent to 0 modulo 3.} \end{cases}$$

It is clear that every rational number occurs infinitely many times in this new sequence. □

In particular, the sequence $\{q_n\}_{n=1}^{\infty}$ in this Corollary has the following property: For each rational number q, there is a subsequence of it that converges to q. Namely, for any $q \in \mathbb{Q}$, form the constant subsequence $q, q, q, \ldots$ of the sequence, which is possible since q occurs an infinite number of times.

19. October 11, 2021

In fact, we can do even better: I claim that every *real* number occurs as a limit of the sequence of the Corollary!

First a Lemma.

Lemma 19.1. *For any $x \in \mathbb{R}$ there is a sequence of rational numbers that converges to x.*

Proof. For each $n \in \mathbb{N}$ we have $x < x + \frac{1}{n}$, and hence by the Density of the Rationals, there is a rational number $a_n \in \mathbb{Q}$ so that $x < \alpha_n < x + \frac{1}{n}$. Since both the constant sequence $\{x\}_{n=1}^{\infty}$ and the sequence $\{x + \frac{1}{n}\}_{n=1}^{\infty}$ converge to x. By the Squeeze Theorem, $\{a_n\}_{n=1}^{\infty}$ also converges to x. □

Theorem 19.2. *There exists a sequence of rational numbers having the property that every real number is the limit of some subsequence of it.*

Proof. Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational nubmers as in Corollary 18.10, so that that every rational number occurs infinitely many times. Let x be any real number. I will construct a subsequence that converges to x.

By Lemma 19.1 there is a sequence of rational numbers $\{a_n\}_{n=1}^{\infty}$ that converges to x. Since $a_1 \in \mathbb{Q}$, a_1 occurs (infinitely many times) in $\{q_n\}_{n=1}^{\infty}$ and hence there is $n_1 \in \mathbb{N}$ such that $a_1 = q_{n_1}$.

Given $n_1 < \cdots < n_k$ such that $q_{n_k} = a_k$, we claim that we can find some $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $q_{n_{k+1}} = a_{k+1}$. Indeed, there are infinitely many natural numbers m such that $q_m = a_{k+1}$, and only finitely many of them can be less than or equal to n_k, so there must be such a number that is greater than n_k.

Thus, we can recursively define an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ such that $q_{n_k} = a_k$ for all k, and hence $\{a_k\}_{k=1}^{\infty}$ is a subsequence of $\{q_n\}_{n=1}^{\infty}$. $\square$

On the other hand, there is no sequence that actually contains every real number. To prove this, we will use decimal expansions, as discussed on the homework.

Recall that if $d_1, d_2, d_3, \ldots$ is a sequence of "digits", where $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for every i, then the sequence $\{q_n\}_{n=1}^{\infty}$, where

$$q_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

converges, and we say that $.d_1d_2d_3\cdots$ is a *decimal expansion* for the real number $r = \lim_{n\to\infty} q_n$.

Theorem 19.3 (Cantor's Theorem)**.** *There is no sequence that contains every real number.*

Proof. By way of contradiction, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence in which every real number appears at least once. Write each member of this sequence in its decimal form, so that

$$\begin{aligned} a_1 &= (\text{whole part}).d_{1,1}d_{1,2}d_{1,3}\cdots \\ a_2 &= (\text{whole part}).d_{2,1}d_{2,2}d_{2,3}\cdots \\ a_3 &= (\text{whole part}).d_{3,1}d_{3,2}d_{3,3}\cdots \\ &\vdots \end{aligned}$$

where each $d_{i,j} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a digit. Now form a real number x as $0.e_1e_2e_3\cdots$ where the e_i's are digits chosen as follows: Let

$$e_i = \begin{cases} 9 & \text{if } d_{i,i} \le 5 \\ 0 & \text{if } d_{i,i} > 5. \end{cases}$$

In particular, $e_i \ne d_{i,i}$ for every i. Then $x \ne a_1$ since these two numbers have different first digits, $x \ne a_2$ since these two numbers have different second digits, etc.; in general, for any n, $x \ne a_n$ since these two numbers have different digits in the n-th position.

Thus x is not a member of this sequence, contrary to what we assumed. $\square$

It is worth noting that the proof given above needs a little more justification because two different decimal expansions can converge to the same number. I'll leave it as a challenge to you to prove that

Our next big theorem has a very short statement, but is surprisingly hard to prove.

Theorem 19.4 (Bolzano-Weierstrass Theorem)**.** *Every sequence has a monotone subsequence.*

The proof of this theorem requires two preliminary lemmas.

Lemma 19.5. *If a sequence is not bounded above, then it has a subsequence that is increasing. If a sequence is not bounded below, then it has a subsequence that is decreasing.*

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ is not bounded above. This means that for each real number M, there is a natural number such that $a_n > M$. Using this assumption, we will build a strictly increasing sequence of natural numbers $n_1 < n_2 < \cdots$ so that the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is increasing. We will define this sequence recursively.

We start by just letting $n_1 = 1$.

If we have chosen n_k, then let $b = \max\{a_1, \ldots, a_{n_k}\}$. Since b is not an upper bound of the sequence $\{a_n\}_{n=1}^{\infty}$, there exists some $m \in \mathbb{N}$ such that $a_m > b$. We must have $m > n_k$, since otherwise a_m is on the list $a_1, \ldots, a_{n_k}$ so that $a_m \leq b$. Thus, we can take $n_{k+1} = m$, and we have $n_{k+1} > n_k$ and $a_{n_{k+1}} > a_{n_k}$. Thus, we have defined the desired subsequence recursively.

The proof for the case of sequences that are not bounded below is similar. □

Lemma 19.6. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Assume $\{a_n\}_{n=1}^{\infty}$ is bounded above and let β be the supremum of the numbers appearing in the sequence. If β does not occur in the sequence, then the sequence contains an increasing subsequence.*

20. October 13, 2021

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ is bounded above, β is the supremum of the numbers appearing in the sequence, and β does not occur in the sequence. Notice that these conditions mean: (1) $a_n < \beta$ for all n and (2) if y is any real number such that $y < \beta$, then there exists a natural number n such that $a_n > y$. Moreover, from (1) it follows that for all n, we have $\max\{a_1, a_2, \ldots, a_n\} < \beta$.

We again define our subsequence recursively. We start by setting $n_1 = 1$.

If we have chosen n_k, then let $b = \max\{a_1, \ldots, a_{n_k}\}$. By (1) above, $b < \beta$, and by (2) above, there is some $m \in \mathbb{N}$ for which $b < a_m$. We must have $m > n_k$, since otherwise a_m is on the list $a_1, \ldots, a_{n_k}$ so that $a_m \leq b$. Thus, we can take $n_{k+1} = m$, and we have $n_{k+1} > n_k$ and $a_{n_{k+1}} > a_{n_k}$. Thus, we have defined the desired subsequence recursively. □

Remark 20.1. We will actually use the contrapositive of the Lemma in the proof of the Theorem:

> If $\{a_n\}_{n=1}^{\infty}$ is a bounded above sequence that does not contain any increasing subsequences, then the supremum of the terms of the sequence must occur somewhere in the sequence.

Proof of Bolzano-Weierstrass Theorem 19.4. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence. Recall that our goal is to prove it either has an increasing subsequence or it has a decreasing subsequence. We consider two cases.

Case I: The sequence is not bounded.

Then it is not bounded above or it is not bounded below, and in either situation, Lemma 19.5 gives the result.

Case II: The sequence is bounded.

We need to prove that it either has an increasing subsequence or a decreasing subsequence. This is equivalent to showing that if it has no increasing subsequences, then it does have at least one decreasing subsequence. So, let us assume it has no increasing subsequences.

We will prove it has at least one decreasing subsequence by constructing the indices $n_1 < n_2 < \cdots$ of such a subsequence one at a time.

Let β_1 be the supremum of all the terms of the sequence. By Lemma 19.6 (see the remark following it), since $\{a_n\}_{n=1}^{\infty}$ is bounded above and does not contain any increasing subsequences, we know that β_1 must be *in the sequence*. That is, there exists a natural number n_1 such that $\beta_1 = a_{n_1}$. Note that it follows that $a_{n_1} \geq a_m$ for all $m \geq 1$.

For any k, given n_k, the subsequence $a_{n_k+1}, a_{n_k+2}, a_{n_k+3}, \ldots$ is also bounded above and has no increasing subsequence. Thus, it must contain its supremum β_{k+1} by Lemma 19.6. So, $\beta_{k+1} = a_m$ for some $m > n_k$. Choose $n_{k+1} = m$ for such a value m. This gives a recursive definition for n_k.

By construction, we have $n_{k+1} > n_k$ for all k. Note that $a_{n_k} = \beta_k$ is the supremum of a set that contains $a_{n_{k+1}} = \beta_{k+1}$. It follows that $a_{n_k} \geq a_{n_{k+1}}$. That is, we have constructed a decreasing subsequence of the original sequence. □

21. October 15, 2021

Corollary 21.1 (Main Corollary of Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem 19.4 it admits a monotone subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, and it too is bounded (since any subsequence of a bounded sequence is also bounded.) The result follows since every monotone bounded sequence converges by the Monotone Convergence Theorem 14.4. □

Definition 21.2. A sequence $\{a_n\}_{n=1}^{\infty}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there is some $N \in \mathbb{R}$ such that for all $m, n \in \mathbb{N}$ such that $m > n > N$, we have $|a_m - a_n| < \varepsilon$.

Loosely speaking sequence is Cauchy if eventually all the terms are very close together. The most important fact about Cauchy sequences is the following:

Theorem 21.3. *A sequence is a Cauchy sequence if and only if it converges.*

(1) Prove that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence using the definition.

Pick $\varepsilon > 0$. (Scratch work: we need to figure out how big m and n need to be in order that $|\frac{1}{m} - \frac{1}{n}| < \varepsilon$. Note that if both $\frac{1}{n}$ and $\frac{1}{m}$ are between 0 and ε, then the distance between them will be at most ε, and this occurs so long as $n, m > \frac{1}{\varepsilon}$. So we will set $N = \frac{1}{\varepsilon}$. Back to the proof.) Let $N = \frac{1}{\varepsilon}$. Let $m, n \in \mathbb{N}$ be such that $m > n > N$. Then $0 < \frac{1}{n} < \frac{1}{N} = \varepsilon$ and $0 < \frac{1}{m} < \frac{1}{N} = \varepsilon$. It follows that

$$|a_m - a_n| = |\frac{1}{m} - \frac{1}{n}| < \varepsilon$$

and this proves the sequence is Cauchy.

(2) Prove that every convergent sequence is a Cauchy sequence.

Assume $\{a_n\}_{n=1}^{\infty}$ converges to L. Pick $\varepsilon > 0$. We apply the definition of "converges" to the sequence $\{a_n\}_{n=1}^{\infty}$, which converges to L, using the positive number $\frac{\varepsilon}{2}$. We get that there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < \frac{\varepsilon}{2}$. I claim that this same number N "works" to prove the sequence is Cauchy: Assume m and n are natural numbers such that $m > N$ and $n > N$. By the triangle inequality

$$|a_m - a_n| \leq |a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(3) Prove that every Cauchy sequence is bounded.

Using $\varepsilon = 1$ in the definition of "Cauchy", we get that there is an N such that if $m, n \in \mathbb{N}$ are such that $m > N$ and $n > N$, then we have $|a_m - a_n| < 1$. Let k be the smallest natural number that is bigger than N, and let M be the maximum of the numbers

$$a_1, \ldots, a_{k-1}, a_k + 1.$$

I claim M is an upper bound of this sequence. Given $n \in \mathbb{N}$, if $n < k$, then $a_n \leq M$ since in this case a_n occurs in the above list. If $n \geq k$ then we have $|a_n - a_k| < 1$ and thus $a_n < a_k + 1 \leq M$. So, M is indeed an upper bound of the sequence.

A similar argument shows that the sequence is bounded below by the minimum number in the list $a_1, \ldots, a_{k-1}, a_k - 1$.

(4) Prove that every Cauchy sequence has a convergent subsequence.

Since a Cauchy sequence is bounded, it has a convergent subsequence by the Main Corollary to Bolzano-Weierstrass.

(5) Prove that every Cauchy sequence converges.

Given a Cauchy sequence, it has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$; let's say that this subsequence converges to L. We will prove $\{a_n\}_{n=1}^{\infty}$ itself converges to L, using that it is Cauchy.

Pick $\varepsilon > 0$. Since the sequence is Cauchy and $\frac{\varepsilon}{2} > 0$, by definition there is a number N such that if $m > n > N$ then $|a_m - a_n| < \frac{\varepsilon}{2}$. We will prove that this N "works" to show $\{a_n\}_{n=1}^{\infty}$ converges to L; that is, I claim that if $n > N$, then $|a_n - L| < \varepsilon$.

So, let $n > N$. Since the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L and $\frac{\varepsilon}{2}$ is positive, there is a real number K such that if $k > K$ then $|a_{n_k} - L| < \varepsilon/2$. Let k be any natural number such that $k > \max\{K, n\}$. We have $n_k > n > N$ and, since $n_k \geq k$, $n_k > K$.

Thus, by the bounds above and the triangle inequality,

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\{a_n\}_{n=1}^{\infty}$ converges to L.

22. October 20, 2021

We are now going to start talking about functions, limits of functions, continuity of functions, etc.

In general, if S and T are any two sets, a *function* from S to T, written

$$f : S \to T$$

is any "rule" that assigns to each element $s \in S$ and unique element $t \in T$. The element of T that is assigned to s by this rule is written $f(s)$.

This is not really a very good definition since "rule" itself is not defined. A more careful definiton is: Given sets S and T, let $S \times T$ denote the set consisting of all ordered pairs (s, t) where $s \in S$ and $t \in T$. Then a *function* f from S to T is a subset G of $S \times T$ having the following property: For each $s \in S$ there is a unique $t \in T$ such that $(s, t) \in G$. In other words, a function is by definition given by its graph.

In this class, we will almost always consider functions of the form

$$f : S \to \mathbb{R}$$

where S is a subset of $\mathbb{R}$. Indeed, henceforth, let us agree that if I say "function" I mean a function of the form $f : S \to \mathbb{R}$ for some subset S of $\mathbb{R}$. Recall that the *domain* of a function refers to the subset S of $\mathbb{R}$ on which it is defined.

Often, but certainly not always, f will indeed by given by a formula, such as $f(x) = \frac{x^2-1}{x-1}$. In such cases, we will usually be a bit sloppy in specifying its domain S. For example, if I say "consider the function $f(x) = \frac{x^2-1}{x-1}$", it is understood that its domain is every real number on which this formula is well-defined. In this example, that would be $S = \mathbb{R} \setminus \{1\} = \{x \in \mathbb{R} \mid x \neq 1\}$. Following this convention, $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$ are two different functions, since their domains are different.

It is also worth noting that while most of the functions we consider will be given by formulas, there are many functions that cannot be expressed in terms of formulas. Imagine for every real number x flipping a coin and setting $f(x) = 1$ if coin x turns up heads and $f(x) = 0$ if coin x turns up tails. The result will certainly be a function, albeit an unimaginably wild one.

Many times, the domain of the functions we talk about will be intervals:

$$(a,b),(a,b],[a,b),[a,b],(a,\infty),(-\infty,b),[a,\infty),(-\infty,b],(-\infty,\infty)$$

Definition 22.1. Let $f : S \to \mathbb{R}$ be a function (where S is a subset of $\mathbb{R}$) and let $a \in \mathbb{R}$ be any real number. For a real number L, we say *the limit of $f(x)$ as x approaches a is L* provided the following condition is met:

> For every $\varepsilon > 0$, there is $\delta > 0$ such that if x is any real number such that $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - L| < \varepsilon$.

Example 22.2. Let f be the function given by the formula

$$f(x) = \frac{5x^2 - 5}{x - 1}.$$

Recall our convention that we interpret the domain of f to be all real numbers where this rule is defined. So, $f : S \to \mathbb{R}$ where $S = \mathbb{R} \setminus \{1\}$.

I claim that the limit of $f(x)$ as x approaches 1 is 10. Pick $\varepsilon > 0$.

Let $\delta = \frac{\varepsilon}{5}$. Pick x such that $0 < |x - 1| < \delta$. Then $x \neq 1$ and hence f is defined at x. We have

$$\begin{aligned} |f(x) - 10| &= \left|\frac{5x^2 - 5}{x - 1} - 10\right| = \left|\frac{5x^2 - 5 - 10x + 10}{x - 1}\right| = \left|\frac{5x^2 - 10x + 5}{x - 1}\right| \\ &= \left|\frac{5(x^2 - 2x + 1)}{x - 1}\right| = \left|\frac{5(x-1)^2}{x - 1}\right| = |5x - 5| = 5|x - 1| < 5\delta = \varepsilon \end{aligned}$$

We have shown that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then f is defined at x and $|f(x) - 10| < \varepsilon$. This proves $\lim_{x\to 1} f(x) = 10$.

23. October 23, 2021

Here is an equivalent formulation of the definition of limit:

> For every $\varepsilon > 0$, there is $\delta > 0$ such that if x is any real number such that either $a - \delta < x < a$ or $a < x < a + \delta$, then f is defined at x and $|f(x) - L| < \varepsilon$.

As a matter of shorthand, we write $\lim_{x\to a} f(x) = L$ to mean the limit of $f(x)$ as x approaches a is L.

Note that in order for the limit of f at a to exist, we need in particular that there is a $\delta > 0$ such that f is defined at every point on $(a - \delta, a)$ and $(a, a + \delta)$. Loosely, f needs to be defined at all points near, but not necessarily equal to, a. If the domain of f is $\mathbb{R}$ or $\mathbb{R} \setminus \{a\}$, this condition is automatic.

The more important condition is that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Intuitively, this is saying that no matter how small of a positive number ε you pick, if you only look at inputs very close to (but not equal to) a, the function values at these inputs are within a distance of ε of the limiting value L.

Example 23.1. Let's do a more complicated example: Let $f(x) = x^2$ with domain all of $\mathbb{R}$. I claim that $\lim_{x\to 2} x^2 = 4$. This is intuitively obvious but we need to prove it using just the definition.

Proof. Pick $\varepsilon > 0$.

(Scratch work: The domain of f is all of $\mathbb{R}$ and so we don't need to worry at all about whether f is defined at all. We need to figure out how small to make δ so that if $0 < |x - 2| < \delta$ then $|x^2 - 4| < \varepsilon$. The latter is equivalent to $|x-2||x+2| < \varepsilon$. We can make $|x-2|$ arbitrarily small by making δ aribitrarily small, but how can we handle $|x + 2|$? The trick is to bound it appropriately. This can be done in many ways. Certainly we can choose δ to be at most 1, so that if $|x - 2| < \delta$ then $|x - 2| < 1$ and hence $1 < x < 3$, so that $|x + 2| < 5$. So, we will be allowed to assume $|x + 2| < 5$. Then $|x - 2||x + 2| < 5|x - 2|$ and $5|x - 2| < \varepsilon$ provided $|x - 2| < \frac{\varepsilon}{5}$. Back to the formal proof...)

Let $\delta = \min\{\frac{\varepsilon}{5}, 1\}$. Let x be any real number such that $0 < |x - 2| < \delta$. Then certainly f is defined at x. Since $\delta \le 1$ we get $|x - 2| < 1$ and hence $|x + 2| \le 5$. Since $\delta \le \frac{\varepsilon}{5}$ we have $|x - 2| < \frac{\varepsilon}{5}$. Putting these together gives

$$|f(x) - 4| = |x^2 - 4| = |x - 2||x + 2| < |x - 2|5 < \frac{\varepsilon}{5}5 = \varepsilon.$$

This proves $\lim_{x\to 2} x^2 = 4$. ☐

Let's give an example of a function that does *not* have a limiting value as x approaches some number a.

Example 23.2. Let $f(x) = \frac{1}{x-3}$ with domain $\mathbb{R} \setminus \{3\}$. I claim that the limit of $f(x)$ as x approaches 3 does not exist. To prove this, by way of contradiction, suppose the limit of $f(x)$ as x approaches 3 does exist and is equal to L. Taking $\varepsilon = 1$ in the definition, there is a $\delta > 0$ so that if $0 < |x - 3| < \delta$, then $\left|\frac{1}{x-3} - L\right| < 1$. We can find a real number x so that both $3 < x < 3.05$ and $0 < |x - 3| < \delta$ hold. For such an x we have $\left|\frac{1}{x-3} - L\right| < 1$ and so

$$\frac{1}{x-3} - 1 < L < \frac{1}{x-3} + 1,$$

and we also have $0 < x - 3 < .05$ and so $\frac{1}{x-3} > 20$. It follows that

$$L > 19.$$

Now pick x such that $2.95 < x < 3$ and $0 < |x - 3| < \delta$. We get

$$\frac{1}{x-3} - 1 < L < \frac{1}{x-3} + 1,$$

and $\frac{1}{x-3} < -20$ and hence

$$L < -19.$$

This is not possible; so the limit of $f(x)$ as x approaches 3 does not exist.

24. October 25, 2021

(1) Explain why $f(x) = \sqrt{x}$ does not have a limit as x approaches 0.

> The domain of $f(x)$ consists of all nonnegative numbers. For any $\delta > 0$, there is some negative number x such that $0 < |x| < \delta$, so there is no $\delta > 0$ such that $f(x)$ is defined for all numbers satisfying that inequality.

(2) Let $f(x)$ be any linear function: $f(x) = mx + b$ where m and b are fixed real numbers. Let a be any real number.

- Prove that $\lim_{x\to a} f(x) = f(a)$ in the case $m = 0$.

> Suppose $m = 0$. Then set $\delta = 10^{100}$ (or any positive number you like). If $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| = |b - b| = 0 < \varepsilon$. This proves $\lim_{x\to a} f(x) = f(a)$.

- Prove that $\lim_{x\to a} f(x) = f(a)$ in the case $m \neq 0$.

> Suppose $m \neq 0$. Set $\delta = \frac{\varepsilon}{|m|}$. If $0 < |x - a| < \delta$, then f is defined at x and
> $$\begin{aligned}|f(x) - f(a)| &= |mx + b - (ma + b)| \\ &= |m(x - a)| = |m||x - a| < |m|\delta = \varepsilon.\end{aligned}$$
> This proves $\lim_{x\to a} f(x) = f(a)$.

(3) Let f be a function. Prove that the limit as of f as x approaches a, if it exists, is unique.

> Suppose that the limit of f as approaches a is both L and M. By way of contradiction, suppose that $L \neq M$. Take $\varepsilon = \frac{|L-M|}{2}$. By definition, there exist $\delta_1, \delta_2 > 0$ such that if

$0 < |x-a| < \delta_1$ then $|f(x)-L| < \varepsilon$, and if $0 < |x-a| < \delta_2$ then $|f(x)-M| < \varepsilon$. Let x be some real number such that $0 < |x-a| < \min\{\delta_1, \delta_2\}$. Then

$$|L-M| \leq |f(x)-L| + |f(x)-M| < \varepsilon + \varepsilon = |L-M|.$$

This is a contradiction, so we must in fact have $L = M$.

(4) Using any basic facts from trig, compute $\lim_{x\to 0} \sin(\frac{1}{x})$.

We will show that there is no limit of $\sin(\frac{1}{x})$ as x approaches 0. By way of contradiction, suppose that there is such a limit L. Take $\varepsilon = 1$. By definition, there is some $\delta > 0$ such that $|\sin(\frac{1}{x}) - L| < \varepsilon$ for all x such that $0 < |x| < \delta$.

We consider two cases. First, assume that $L \geq 0$. There is some natural number N greater than $1/\delta$. Then

$$2\pi N + \frac{3\pi}{2} > \frac{1}{\delta}$$

as well, so

$$x = \frac{1}{2\pi N + \frac{3\pi}{2}} < \delta,$$

and

$$\sin(1/x) = \sin(2\pi N + \frac{3\pi}{2}) = -1$$

so $|\sin(1/x) - L| \geq 1 = \varepsilon$. Thus $L \geq 0$ is impossible.

Now, assume that $L < 0$. There is some natural number N greater than $1/\delta$. Then

$$2\pi N + \frac{\pi}{2} > \frac{1}{\delta}$$

as well, so

$$x = \frac{1}{2\pi N + \frac{\pi}{2}} < \delta,$$

and

$$\sin(1/x) = \sin(2\pi N + \frac{\pi}{2}) = 1,$$

so $|\sin(1/x) - L| \geq 1 = \varepsilon$. Thus $L < 0$ is impossible too.

We conclude that no such L exists.

(5) Come up with a definition of $\lim_{x\to\infty} f(x) = L$ and prove, using your definition, that

$$\lim_{x\to\infty} \frac{2x-1}{x+3} = 2.$$

25. October 27, 2021

The following result gives an important connection between limits of functions and limits of sequences. This Lemma will allow us to translate statements we have proven about limits of sequences to limits of functions.

Theorem 25.1. *Let $f(x)$ be a function and let a be a real number. Let $r > 0$ be a positive real number such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$. Let L be any real number.*

$\lim_{x\to a} f(x) = L$ *if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a and satisfies $0 < |x_n - a| < r$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.*

Loosely, the condition that there is an $r > 0$ such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$ says that "f is defined near, but not necessarily at, a".

Proof. Let f be a function, $a \in \mathbb{R}$, and $r > 0$ a positive real number such that f is defined on $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$.

($\Rightarrow$) Assume $\lim_{x\to a} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence that converges to a and is such that $0 < |x_n - a| < r$ for all n. We need to prove that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. By definition of the limit of a function, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then f is defined at x and $|f(x) - L| < \varepsilon$. Since $\delta > 0$ and $\{x_n\}_{n=1}^{\infty}$ converges to a, by the definition of convergence, there is an N such that if $n \in \mathbb{N}$ and $n > N$ then $|x_n - a| < \delta$. I claim that this N "works" to prove $\{f(x_n)\}_{n=1}^{\infty}$ converges to L too: If $n \in \mathbb{N}$ and $n > N$, then $|x_n - a| < \delta$ and, since $x_n \neq a$ for all n, we have $0 < |x_n - a| < \delta$. It follows that $|f(x_n) - L| < \varepsilon$. This shows that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

($\Leftarrow$) We prove the contrapositive. That is, assume $\lim_{x\to a} f(x)$ is not L (including the case where the limit does not exist). We need to prove that there is at least one sequence $\{x_n\}_{n=1}^{\infty}$ such that (a) it converges to a, (b) $0 < |x_n - a| < r$ for all n and yet (c) the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L.

The fact that $\lim_{x\to a} f(x)$ is not L means:

> There is an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$, but either f is not defined at x or $|f(x) - L| \geq \varepsilon$.

For this ε, for any natural number n, set $\delta_n = \min\{\frac{1}{n}, r\}$. We get that there is a $x_n \in \mathbb{R}$ such that $0 < |x_n - a| < \delta_n$ and $|f(x_n) - L| \geq \varepsilon$. (Note that f is necessarily defined at x_n since $\delta_n \leq r$.) I claim that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the needed three conditions. (a) Since $\delta_n \leq \frac{1}{n}$, we have $a - \frac{1}{n} < x_n < a + \frac{1}{n}$ for all n, and hence by the Squeeze Lemma, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a. (b) This holds by construction, since $\delta_n \leq r$. (c) Since, for the positive number ε above, we have $|f(x_n) - L| \geq \varepsilon$ for all n, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. $\square$

26. October 29, 2021

Corollary 26.1. *Let f be a function and a and L be real numbers. Suppose that the domain of f is all of $\mathbb{R}$ or $\mathbb{R} \setminus \{a\}$. Then $\lim_{x \to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a such that $x_n \neq a$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.*

Proof. ($\Rightarrow$) Assume $\lim_{x \to a} f(x) = L$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges to a such that $x_n \neq a$ for all n. Since $\{x_n\}_{n=1}^{\infty}$ is convergent, it is bounded, so there is some $M > 0$ such that $|x_n| < M$ for all n. Then $|x_n - a| < M + |a|$ by the Triangle Inequality. Thus, $0 < |x_n - a| < M + |a|$ for all n, so we can apply Theorem 25.1 (with "r"$= M + |a|$), so $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

($\Leftarrow$) The point is that if the "right hand side" condition holds in this statement, then for any $r > 0$, the "right hand side" condition of Theorem 25.1 holds. Thus, by Theorem 25.1, $\lim_{x \to a} f(x) = L$. $\square$

We can use this Corollary to compute limits.

Example 26.2. Let $f(x) = \sin(1/x)$. We claim that $\lim_{x \to 0} f(x)$ does not exist. (We proved this using the definition on a worksheet earlier, but we give a second proof here.) Since the domain of f is $\mathbb{R} \setminus \{0\}$, Corollary 26.1 applies. Consider the sequence given by $x_n = \dfrac{1}{\pi n + \frac{\pi}{2}}$. We have $x_n \neq 0$ for all x, and $\{x_n\}_{n=1}^{\infty}$ converges to 0. Thus, by Corollary 26.1, if $\lim_{x \to 0} f(x) = L$, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to L, and in particular, is convergent. But

$$f(x_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

so $\{f(x_n)\}_{n=1}^{\infty}$ is divergent. It follows that $\lim_{x\to 0} f(x) = L$ does not exist.

Example 26.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}.$$

We will show that, for every real number a, $\lim\limits_{x\to a} f(x)$ does not exist.

Fix a and, by way of contradiction, suppose $\lim_{x\to a} f(x)$ exists and is equal to L. Let $\{q_n\}_{n=1}^{\infty}$ be any sequence of rational numbers that converges to a such that $q_n \neq a$ for all n. (Such a sequence exists by Lemma 19.1 above; technically this Lemma does not include the statement that $q_n \neq a$ for all n, but the proof makes it clear that there is a sequence that also has this property.) Then by Corollary 26.1, $L = \lim_{n\to\infty} f(q_n)$. But $f(q_n) = 1$ for all n by definition, and hence $L = 1$.

On the other hand, as you proved on the homework, there also exists a sequence $\{y_n\}_{n=1}^{\infty}$ that converges to a such that y_n is *irrational* for each n and $y_n \neq a$ for all n. (Likewise, the homework problem did not include the fact that $y_n \neq a$ for all n, but your proof should give that too.) By Corollary 26.1, $L = \lim_{n\to\infty} f(y_n)$. But $f(y_n) = 1$ for all n by definition, and hence $L = 0$. This is impossible.

We can also use Theorem 25.1 to deduce some results about limits that are analogous to earlier results about sequences.

Theorem 26.4. *Suppose f and g are two functions and that a is a real number, and assume that*

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

for some real numbers L and M. Then

1. $\lim_{x\to a}(f(x) + g(x)) = L + M$.
2. *For any real number c,* $\lim_{x\to a}(c \cdot f(x)) = c \cdot L$.
3. $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$.
4. *If, in addition, we have that $M \neq 0$, then* $\lim_{x\to a}(f(x)/g(x)) = L/M$.

Proof. Each part follows from Theorem 25.1 and the corresponding theorem about sums, products, and quotients of sequences (Theorem 12.2). We give the details just for one of them, part (3):

First, as a technical matter, we note that since we assume $\lim_{x\to a} f(x) = L$ there is a positive real number r_1 such that $f(x)$ is defined for all x satisfying $0 < |x - a| < r_1$, and likewise since $\lim_{x\to a} g(x) = L$ there is a positive real number r_2 such that $g(x)$ is defined for all x satisfying $0 < |x - a| < r_1$. Letting $r = \min\{r_1, r_2\}$, we have that $r > 0$ and $f(x)$ and $g(x)$ and hence $f(x) \cdot g(x)$ are defined for all x satisfying $0 < |x - a| < r$. (We needed to prove this in order to apply Theorem 25.1.)

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence converging to a such that $0 < |x_n - a| < r$ for all n. By Theorem 25.1 in the "forward direction", we have that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. By Theorem 12.2, $\lim_{n\to\infty} f(x_n)g(x_n) = L \cdot M$. So, by Theorem 25.1 again (this time applying it to $f(x)g(x)$ and using the "backward implication"), it follows that $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$. $\square$

Example 26.5. We can use this theorem to compute $\lim\limits_{x\to 2} \dfrac{3x^2 - x + 2}{x + 3}$. Indeed, since $\lim_{x\to 2} x = 2$, and $\lim_{x\to 2} 1 = 1$, we have $\lim_{x\to 2} 3x^2 = 3\lim_{x\to 2} x^2 = 3(\lim_{x\to 2} x)(\lim_{x\to 2} x) = 12$, and from earlier computations, $\lim_{x\to 2}(-x+2) = (-2)+2 = 0$ and $\lim_{x\to 2} x+3 = 2+3 = 5$, so $\lim_{x\to 2} 3x^2 - x + 2 = 12$, and $\lim\limits_{x\to 2} \dfrac{3x^2 - x + 2}{x + 3} = \dfrac{12}{5}$.

Here is another example of the same idea.

Theorem 26.6. *Suppose f, g, and h are three functions and a is a real number. Suppose there is a positive real number $r > 0$ such that*

(1) *each of f, g, h is defined on $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$,*
(2) *$f(x) \le g(x) \le h(x)$ for all x such that $0 < |x - a| < r$,*
(3) *$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ for some number L.*

Then $\lim_{x\to a} g(x) = L$.

Proof. Let f, g, h, a, r, L be as in the statement. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges to a and such that $0 < |x_n - a| < r$ for all n. By Theorem 25.1, it suffices to show that $\lim_{n\to\infty} g(x_n) = L$. By Theorem 25.1, we know that $\lim_{n\to\infty} f(x_n) = L = \lim_{n\to\infty} h(x_n)$. Since $f(x_n) \le g(x_n) \le h(x_n)$ for all n, we have $\lim_{n\to\infty} g(x_n) = L$ by the Squeeze Theorem (for sequences). $\square$

27. November 1, 2021

We come to the formal definition of continuity. We first define what it means for a function to be continuous *at a single point*, but ultimately we will be interested in functions that are continuous on entire intervals.

Definition 27.1. Suppose f is a function and a is a real number. We say *f is continuous at a* provided the following condition is met:

> For every $\varepsilon > 0$ there is a $\delta > 0$ such that if x is a real number such that $|x - a| < \delta$ then f is defined at x and $|f(x) - f(a)| < \varepsilon$.

Remark 27.2. If f is continuous at a, then by applying the definition using any postive number $\varepsilon > 0$ you like (e.g., $\varepsilon = 1$) we get that there exists a $\delta > 0$ such that f is defined for all x such that $a - \delta < x < a + \delta$. That is, in order for f to be continuous at a it is necessary (but not sufficient) that f is defined at all points near a *including at a itself*. In particular, unlike in the definition of "limit", f must be defined at a in order for it to possibly be continuous at a.

Example 27.3. I claim $f(x) = 3x$ is continuous at a for every value of a. Pick $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{3}$. If $|x - a| < \delta$ then f is defined at x (since the domain of f is all of $\mathbb{R}$) and

$$|f(x) - f(a)| = |3x - 3a| = 3|x - a| < 3\delta = \varepsilon.$$

Example 27.4. The function $f(x)$ with domain $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x - 7 & \text{if } x \geq 3 \text{ and} \\ -x & \text{if } x < 3 \end{cases}$$

is not continuous at 3. Since the domain of f is all of $\mathbb{R}$, the negation of the defintion of "continuous at 3" is:

> there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is a real number x such that $|x - 3| < \delta$ and $|f(x) - f(3)| \geq \varepsilon$.

Set $\varepsilon = 1$. For any $\delta > 0$, we may choose a real number x so that $3 - \delta < x < 3$ and $2.9 < x < 3$. For such an x, we have

$$|f(x) - f(3)| = |-x + 1| = x - 1 > 1.9 > \varepsilon.$$

The proves f is not continuous at 3.

The definition of continuous looks a lot like the definition of limit, with L replaced by $f(a)$. This is not just superficial:

Theorem 27.5. *Suppose f is a function and a is a real number and assume that f is defined at a. f is continous at a if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof. ($\Rightarrow$) This is immediate from the definitions.

($\Leftarrow$) This is almost immediate from the definitions: Suppose $\lim_{x \to a} f(x) = f(a)$. Pick $\varepsilon > 0$. Then there is a δ such that if $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| < \varepsilon$. This nearly gives that f is continuous at a by definition, except that we need to know that if $|x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| < \varepsilon$. The only "extra" case is the case $x = a$. But if $x = a$, then f is defined at a by assumption and we have $|f(x) - f(a)| = 0 < \varepsilon$. $\square$

Remark 27.6. Remember, when we write $\lim_{x\to a} f(x) = f(a)$ we mean that the limit exists and is equal to the number $f(a)$. So, by this Lemma, if $\lim_{x\to a} f(x)$ does not exist, then f is not continuous at a.

Example 27.7. Define a function f whose domain is all of $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

As I proved, $\lim_{x\to a} f(x)$ does not exist for any a. So, this function is continuous nowhere.

Example 27.8. Recall the function f whose domain is all of $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

As you showed, $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to a} f(x)$ does not exist for all $a \neq 0$. Since $f(0) = 0$ this shows that $f(x)$ is continuous at $x = 0$, but not continuous at all other points. So, somewhat counterintuitively, it is possible for a function defined on all of $\mathbb{R}$ to be continuous at one and only one spot!

Example 27.9. The function $f(x) = \sqrt{x}$ is continuous at a for every $a > 0$. This holds since for any $a > 0$, as you proved on the homework we have

$$\lim_{x\to a} \sqrt{x} = \sqrt{a}.$$

28. November 3, 2021

Theorem 28.1. *Let $a \in \mathbb{R}$ and suppose f and g are two functions that are both continuous at a. Then so are*

(1) *$f(x) + g(x)$,*
(2) *$c \cdot f(x)$, for any constant c,*
(3) *$f(x) \cdot g(x)$, and*
(4) *$\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.*

Proof. Follows from Theorems 27.5 and 26.4. $\square$

Example 28.2. Polynomials are continuous everywhere. The function x is continuous everywhere (since $\lim_{x\to a} x = a$). By part (3) above and a simple induction, x^n is continuous everywhere for every n. Then by parts (1) and (2), it follows that every polynomial is continuous everywhere.

Recall that for functions f and g, $f \circ g$ is the *composition*: it is the function that sends x to $f(g(x))$. The domain of $f \circ g$ is

$$\{x \in \mathbb{R} \mid\ x \text{ is the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}.$$

Theorem 28.3. *Suppose g is continuous at a point a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Proof. Let $a \in \mathbb{R}$ be such that that g is continuous at a and f is continuous at $g(a)$. I prove $f \circ g$ is continuous at a using the definition.

Pick $\varepsilon > 0$. Since f is continuous at $g(a)$, there is a $\gamma > 0$ such that if $|y - g(a)| < \gamma$ then f is defined at y and $|f(y) - f(g(a))| < \varepsilon$. (I am using y in place of the usual x for clarity below, and I am calling this number γ, and not δ, since it is not the δ I am seeking.) Since $\gamma > 0$ and g is continuous at a, there is a $\delta > 0$ such that if $|x - a| < \delta$ then g is defined at x and $|g(x) - a| < \gamma$.

This δ "works" to prove $f \circ g$ is continuous at a: Let x be any real number such that $|x - a| < \delta$. Then g is defined at x and $|g(x) - g(a)| < \gamma$. Taking $y = g(x)$ above, this gives that f is defined at $g(x)$ and $|f(g(x)) - f(g(a))| < \varepsilon$. This proves $f \circ g$ is continuous at a. ☐

Example 28.4. The function $\sqrt{x^2+5}$ is continuous at every real number: let $g(x) = x^2 + 5$ and $f(x) = \sqrt{x}$. Then g is continuous at a for every $a \in \mathbb{R}$ since it is a polynomial. For each $x \in \mathbb{R}$, $g(x) > 0$ and hence f is continuous at $g(x)$. So $\sqrt{x^2+5}$ is continuous at every $a \in \mathbb{R}$ by the Theorem.

Example 28.5. You can apply the Theorem to the compositions of more than two functions too. For example $\sqrt{|x^3|+1}$ is continuous at a for and $a \in \mathbb{R}$.

We are probably getting a bit tired of saying "continuous at a for every $a \in \mathbb{R}$". The following definition will then be convenient.

Definition 28.6. Let S be an open interval of $\mathbb{R}$ of the form $S = (a, b)$, $S = (a, \infty)$, $S = (-\infty, a)$, or $S = (-\infty, \infty) = \mathbb{R}$. We say f is *continuous on* S if f is continuous at a for all $a \in S$.

Example 28.7.
- Every polynomial is continuous on $\mathbb{R}$.
- Every polynomial is continuous on $(-13, 5)$.
- The function $\sqrt{x}$ is continuous on $(0, \infty)$.
- The function $1/x$ is continuous on $(0, \infty)$. It is also continuous on $(-\infty, 0)$.

Since the function $\sqrt{x}$ does not jump at $x = 0$, we would like to say the function is continuous on all of $[0, \infty)$. However, the definition for

f to be continuous at a point a requires that f is defined on $(a-\delta, a+\delta)$ for some $\delta > 0$, so we have to change our definition a bit.

29. November 5, 2021

Definition 29.1. Given a function $f(x)$ and real numbers $a < b$, we say f is *continous on the closed interval* $[a, b]$ provided

(1) for every $r \in (a, b)$, f is continuous at r in the sense defined already,
(2) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in [a, b]$ and $a \leq x < a + \delta$, then $|f(x) - f(a)| < \varepsilon$.
(3) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in [a, b]$ and $b - \delta < x \leq b$, then $|f(x) - f(b)| < \varepsilon$.

In short, condition (1) says that f is continuous on the open interval (a, b), condition (2) says that f is "continuous from the right" at a, and condition (3) says f is "continuous from the left" at b.

Theorem 29.2 (Intermediate Value Theorem)**.** *Suppose f is a function, that $a < b$ are real numbers, and that f is continuous on the closed interval $[a, b]$. If y is any number between $f(a)$ and $f(b)$ (i.e., $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$), then there is a $c \in [a, b]$ such that $f(c) = y$.*

(1) Explain why if f is continuous at x for every $x \in [a, b]$, then f is continuous on the closed interval $[a, b]$. Conclude that every polynomial is continuous on every closed interval.

> Suppose that f is continuous at r for all $r \in [a, b]$. Then, in particular f is continuous at r for all $r \in (a, b)$ so (1) holds. Since f is continuous at a, then condition (2) holds (with the same δ for any given ε); since f is continuous at b, then condition (3) holds (with the same δ for any given ε).

(2) Show that the function $f(x) = \sqrt{1 - x^2}$ is continuous on the closed interval $[-1, 1]$:
- For showing condition (1), I recommend using a Theorem from last class.
- For condition (2), it may help to write $\sqrt{1 - x^2} = \sqrt{1 - x}\sqrt{1 + x}$. [8]

[8] You can ask me for a δ if you get stuck.

- Condition (3) is similar to condition (2) so you can just say "Similar to (2)" for that.

It is continuous at a for all a such that $-1 < a < 1$ since $1-x^2$ is continuous on all of $\mathbb{R}$, $\sqrt{y}$ is continuous on $(0,\infty)$, and $1-x^2 > 0$ for all $-1 < x < 1$.

To prove the condition at the endpoint -1, pick $\varepsilon > 0$. (Scratch work: $\sqrt{1-x^2} < \varepsilon$ if and only if $|1-x||1+x| < \varepsilon^2$. So long as $-1 \leq x \leq 1$ we have $|1-x| \leq 2$.) Let $\delta = \varepsilon^2/2$. Assume x is any real number such that $-1 \leq x < -1+\delta$ and $x \in [-1,1]$. We have that $|1+x| \leq 2$ and hence $\sqrt{|1+x|} \leq \sqrt{2}$. We have $\sqrt{|1-x|} \leq \sqrt{\varepsilon^2/2} = \varepsilon/\sqrt{2}$. Thus we obtain

$$\sqrt{1-x^2} = \sqrt{|1-x|}\sqrt{|1+x|} < \sqrt{2}\varepsilon/\sqrt{2} = \varepsilon.$$

The other endpoint 1 is dealt with in a similar way.

(3) Explain why if f is continuous on the closed interval $[a,b]$, then it does not follow in general that f is continuous at x for every $x \in [a,b]$.

The function $f(x) = \sqrt{1-x^2}$ is continuous on the closed interval $[-1,1]$, but not continuous at -1.

(4) Use the Intermediate Value Theorem to give a quick proof that there is a positive real number x such that $x^2 = 2$.

Let $f(x) = x^2$. This is continuous on $[1,2]$. We have $f(1) = 1 < 2 < 4 = f(2)$, so there is some x (between 1 and 2) such that $f(x) = 2$.

(5) Use the Intermediate Value Theorem to give a quick proof that every real number has a cube root.

Let $f(x) = x^3$.

I claim that there are numbers a and b such that $f(a) \leq r \leq f(b)$: If $r \geq 1$, then $a = 0$ and $b = r$ work; if $r \leq -1$, then $a = r$ and $b = 0$ work; and if $-1 \leq r < 1$, then $a = -1$ and $b = 1$ work.

Since f is a polynomial, it is continuous on all of $\mathbb{R}$ and hence on the closed interval $[a,b]$. Thus, since $f(a) \leq r \leq$

> $f(b)$ by the Intermediate Value Theorem, there is a real number s such that $a \le s \le b$ and $f(s) = r$; that is, $s^3 = r$.

(6) Explain why the statement of the Intermediate Value Theorem would be false if we did not assume that f is continuous on $[a, b]$.

> Suppose $f(x) = x$ for $x < 0$ and $f(x) = x + 1$ for $x \ge 0$. Then f is defined on $[-1, 1]$, $f(-1) = -1$ and $f(1) = 2$. So $f(-1) < 1/2 < f(1)$, there there is clearly no x such that $f(x) = 1/2$.

(7) True or False: The polynomial $f(x) = x^3 - 4x + 1$ has two roots in the interval $[0, 2]$.

> Note that $f(0) = 1$ and $f(1) = -2$. Since f is continuous on $[0, 1]$, there is some $c_1 \in [0, 1]$ such that $f(c_1) = 0$. Now note that $f(2) = 1$. Since f is continuous on $[1, 2]$, there is some $c_2 \in [1, 2]$ such that $f(c_2) = 0$. We must have $c_1 < 1$ since $f(c_1) \ne 0$, so $c_1 \ne c_2$; these are two distinct roots in $[0, 2]$.

30. November 8, 2021

Proof of Intermediate Value Theorem. Assume f is continuous on $[a, b]$ and y is a real number such that $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$. We need to prove there is a $c \in [a, b]$ such that $f(c) = y$.

Let us assume $f(a) \le y \le f(b)$ — the other case may be proved in a very similar manner, or by appealing to this case using the function $-f(x)$ instead.

If $f(a) = y$ then we may take $c = a$ and if $f(b) = y$ then we may take $c = b$. So, we may assume $f(a) < y < f(b)$.

Consider the set

$$S = \{z \in \mathbb{R} \mid a \le z \le b \text{ and } f(x) < y \text{ for all } x \in [a, z]\}$$

This set is nonempty, since $a \in S$, and it is bounded above, by b. It therefore has a supremum, which we will call c. I claim $f(c) = y$.

Let us first show that $c > a$. By way of contradiction, suppose $c \le a$. Since $c \ge a$, we must have $c = a$. Since f is continuous on $[a, b]$, taking $\varepsilon = y - f(a) > 0$ in the definition, we get that there is a $\delta > 0$ such that if $a \le x < a + \delta$, then $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$. In particular, if $a \le x \le a + \delta/2$, then $f(x) < f(a) + \varepsilon = y$. This proves that $a + \delta/2 \in S$. But $a + \delta/2 > a = c$, contrary to the fact that c is the supremum of S. We conclude that $c > a$.

Similarly, one may show that $c < b$ — I leave this to you as an exercise.

We now know that $a < c < b$, and we next prove that $f(c) = y$ by showing that $f(c) > y$ and $f(c) < y$ are each impossible.

Suppose $f(c) > y$. Setting $\varepsilon = f(c) - y$ and applying the definition of continuous at c, there is a $\delta > 0$ such that if x is any number such that $c - \delta < x < c + \delta$ then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. In particular, for any z such that $c - \delta < z \le c$, we have

$$f(z) > f(c) - \varepsilon = y.$$

In particular, z is not in the set S. It follows that $c - \delta$ is an upper bound of S, contrary to the fact that c is the least upper bound of S.

Suppose $f(c) < y$. Setting $\varepsilon = y - f(c)$ and applying the definition of continuous at c, there is a $\delta > 0$ such that if $c - \delta < x < c + \delta$, then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. In particular, if x is any real number such that $c \le x \le c + \delta/2$, then $f(x) < f(c) + \varepsilon = y$. Moreover, if $x < c$, then x is not an upper bound of S, and hence there is a $z \in S$ such that $x < z$. If follows that $f(x) \le y$. So, we have shown that if $x \le c + \delta/2$, then $f(x) < y$. This shows that $c + \delta/2 \in S$, contrary to c being an upper bound of S. $\square$

Our next goal is to prove the Boundedness Theorem and the Extreme Value Theorem. You might recall the latter from Calculus class. Here are the statements:

Theorem 30.1 (Boundedness Theorem)**.** *Suppose f is continuous on the closed interval $[a, b]$ for some real numbers a, b with $a < b$. Then f is bounded on $[a, b]$ — that is, there are real numbers m and M so that $m \le f(x) \le M$ for all $x \in [a, b]$.*

Remark 30.2. The statement of this theorem would become false if either we omit the continuous assumption or if we changed the closed interval $[a, b]$ to an open one.

For example, consider

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \text{ and} \\ 5 & \text{if } x = 0. \end{cases}$$

Simialarly, $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ but not bounded on it.

Theorem 30.3 (Extreme Value Theorem)**.** *Assume f is continuous on the closed interval $[a, b]$ for some real numbers a and b with $a < b$. Then*

f attains a minimum and a maximum value on $[a, b]$ — that is, there exists a number $r \in [a, b]$ such that $f(x) \leq f(r)$ for all $x \in [a, b]$ and there exists a number $s \in [a, b]$ such that $f(x) \geq f(s)$ for all $x \in [a, b]$.

Remark 30.4. The statement of the Extreme Value Theorem would become false if we either omitted the continous assumption or replaced $[a, b]$ with, for example, (a, b).

Remark 30.5. The Boundedness Theorem is an immediate consequence of the Extreme Value Theorem. The reason we state it first as a separate theorem is that we need the Boundedness Theorem in order to prove the Extreme Value Theorem.

For the proofs of both of these theorems, we will need the following Lemma.

Lemma 30.6. *Assume f is continuous on $[a, b]$ and that $\{x_n\}_{n=1}^{\infty}$ is any sequence such that $a \leq x_n \leq b$ for all n. If $\{x_n\}_{n=1}^{\infty}$ converges to some number r, then*

(1) *$r \in [a, b]$ and*
(2) *$\lim_{n\to\infty} f(x_n) = f(r)$.*

We will return to the proof soon.

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Proof of the Boundedness Theorem. Assume $f(x)$ is continuous on the closed interval $[a, b]$. By way of contradiction, suppose $f(x)$ is not bounded above. Then for each natural number n, the function $f(x)$ is bigger than n somewhere on the interval. So, for each $n \in \mathbb{N}$, there is a real number x_n such that $a \leq x_n \leq b$ and $f(x_n) > n$. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ formed by these chosen numbers. It need not converge, but it is bounded (above by b and below by a) and so the Main Corollary to the Bolzano-Weierstrass Thereom ensures that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges, say to the number r. By the Lemma, $r \in [a, b]$ and $\lim_{k\to\infty} f(x_{n_k}) = f(r)$. In particular, $\{f(x_{n_k})\}_{k=1}^{\infty}$ converges. But by construction $f(x_{n_k}) > n_k$ for all k, and so it cannot converge. We have reached a contradiction. So f must be bounded above.

To show $f(x)$ is bounded below, using what we have already proven, we have that $-f(x)$ is also bounded above by some number N, and it follows that $f(x)$ is bounded below by $-N$. □

Proof of the Extreme Value Theorem. We first prove f attains a maximum value on $[a, b]$.

Let R be the range of f; that is,

$$R = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in [a, b]\}.$$

By the Boundedness Theorem, R is bounded above, and it is clearly nonempty. So, by the Completeness Axiom it has a supremum, call it M. We can find a sequence $\{y_n\}$ with $y_n \in R$ for all n that converges to M. (This follows from the definition of supremum: for each $n \in \mathbb{N}$ since $M - \frac{1}{n}$ is not an upper bound of R, there exists a $y_n \in R$ with $M - \frac{1}{n} < y_n \leq M$. By the Squeeze Theorem, $\{y_n\}$ converges to M.)

Since $y_n \in R$, for each n, we may pick $x_n \in [a, b]$ such that $f(x_n) = y_n$. The sequence $\{x_n\}$ might not converge, but it is bounded, and so thanks to the Main Corollary of the Bolzano-Weierstrass Theorem it has a subsequence $\{x_{n_k}\}$ that does converge. Say this subsequence converges to r. By the Lemma, we have $r \in [a, b]$ and

$$f(r) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k}.$$

Since $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$ and the latter converges to M, so does the former. So, $f(r) = M$. By definition of M, we have $f(x) \leq M = f(r)$ for all $x \in [a, b]$.

To prove f attains a minimum value on $[a, b]$, apply what we just proven to the function $-f(x)$. This gives us that $-f(x)$ attains its maximum value N at some point s. It follows that $f(x)$ attains its minimum value (which is $-N$) at s. □

Corollary 31.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous. The range of f is either a single point or a closed interval.*

Proof. By the Extreme Value Theorem, the range of f contains a minimum value m and a maximum value M. If $m = M$, then the range is the single point $m = M$. Otherwise, we have $m < M$. We claim that the range of f is equal to $[m, M]$. Clearly the range of f is contained in $[m, M]$ by definition of minimum and maximum. For the other containment, take $y \in [m, M]$. Take $r, s \in [a, b]$ such that $f(r) = m$ and $f(s) = M$.

Case 1: $r < s$. In this case, f is continuous on $[r, s]$, and by the Intermediate Value Theorem, there is some $c \in [r, s] \subseteq [a, b]$ such that $f(c) = y$, so y is in the range of f.

Case 2: $s < r$. In this case, f is continuous on $[s, r]$, and by the Intermediate Value Theorem, there is some $c \in [s, r] \subseteq [a, b]$ such that $f(c) = y$, so y is in the range of f.

We conclude that the range is the closed interval $[m, M]$. □

End of material for exam 2

32. November 19, 2021

Definition 32.1. Suppose f is a function and r is a real number. We say f is *differentiable at* r if f is defined at r and the limit

$$\lim_{x\to r}\frac{f(x)-f(r)}{x-r}$$

exists. In this case, the value of this limit is the *derivative of* f *at* r and is written as $f'(r)$ for short.

Remark 32.2. Notice that $\frac{f(x)-f(r)}{x-r}$ is undefined when $x=r$. But this is OK since in the definition of a limit, the function is not necessarily defined at the limiting point.

Remark 32.3. For the limit $\lim_{x\to r}\frac{f(x)-f(r)}{x-r}$ there must be a positive real number $\delta>0$ such that f is defined for all x satisfying $0<|x-r|<\delta$. Since we assume f is defined at r too, it follows that if f is differentiable at r, then it is defined on $(r-\delta,r+\delta)$ for some $\delta>0$.

Example 32.4. Let $f(x)=x^3$ and let r be arbitrary. Then using that $x^3-r^3=(x-r)(x^2+xr+r^2)$ we get

$$\lim_{x\to r}\frac{f(x)-f(r)}{x-r}=\lim_{x\to r}\frac{x^3-r^3}{x-r}=\lim_{x\to r}x^2+xr+r^2=3r^2.$$

This proves that $f'(r)=3r^2$ for any real number r.

Example 32.5. Let $f(x)=\sqrt{x}$. For any $r>0$ we have

$$\lim_{x\to r}\frac{f(x)-f(r)}{x-r}=\lim_{x\to r}\frac{\sqrt{x}-\sqrt{(r)}}{x-r}=\lim_{x\to r}\frac{x-r}{(x-r)(\sqrt{x}+\sqrt{r})}=\lim_{x\to r}\frac{1}{\sqrt{x}+\sqrt{r}}=\frac{1}{2\sqrt{r}}$$

where the last step uses that $\lim_{x\to r}\sqrt{x}=\sqrt{r}$ and other properties of limits we have established.

Note that $\sqrt{x}$ is not differentiable at 0.

As you well know, the derivative of a function may again be regarded as another function. In detail, if f is any function then its derivative is the function f' whose value at x is $f'(x)$. The domain of $f'(x)$ is

$$\{x\in\mathbb{R}\mid\ f \text{ is differentiable at } x\}$$

In our fist example above, we have shown that the derivative of $f(x)=x^3$ is $f'(x)=3x^2$, and the domain of $f'(x)$ is all of $\mathbb{R}$. In the second we showed that $g(x)=\sqrt{x}$ is differentiable for all $x<0$ and its derivative is $\frac{1}{2\sqrt{x}}$ for $x>0$.

A somewhat technical but nevertheless useful result is:

Proposition 32.6. *If f is differentiable at r, then f is continuous at r.*

Proof. Using the Theorem about limits of sums, products etc. we get

$$\begin{aligned}\lim_{x\to r} f(x) &= \lim_{x\to r}(f(r) + f(x) - f(r)) \\ &= f(r) + \lim_{x\to r}\frac{f(x)-f(r)}{x-r}\cdot(x-r) \\ &= f(r) + \lim_{x\to r}\frac{f(x)-f(r)}{x-r}\cdot\lim_{x\to r}(x-r) \\ &= f(r) + f'(r)\cdot 0 \\ &= f(r).\end{aligned}$$

This proves f is continuous at r. □

Let us give an example of a function that is not differentiable at a point:

Example 32.7. Let $f(x) = |x|$. It is pretty clear at an intuitive level that f is not differentiable at the point $x = 0$. To prove this carefully, we need to show that $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist. Note

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \text{ and} \\ -1, & \text{if } x < 0. \end{cases}$$

Letting $x_n = \frac{1}{n}$ we have $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) = 1$ and letting $y_n = \frac{-1}{n}$ we have $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} f(y_n) = -1$. This proves the limit does not exist.

Example 32.8. The derivative of a constant function is 0 at all points. Also, if $h(x) = x$ then $h'(x) = 1$. These both follow immediately from the definition.

Theorem 32.9. *Suppose f and g are two functions that are both differentiable at a number r. Then:*

(1) *$f + g$ is differentiable at r and $(f + g)'(r) = f'(r) + g'(r)$.*
(2) *For any constant c, cf is differentiable at r and $(cf)'(r) = cf'(r)$.*
(3) *("The Product Rule") $f\cdot g$ is differentiable at r and $(f\cdot g)'(r) = f'(r)g(r) + f(r)g'(r)$.*
(4) *If $g(r) \neq 0$, then $\frac{1}{g}$ is differentiable at r and*

$$\left(\frac{1}{g}\right)'(r) = -\frac{1}{g^2(r)}g'(r).$$

Proof. For part (1), we note that

$$\frac{(f+g)(x)-(f+g)(r)}{x-r} = \frac{f(x)+g(x)-f(r)-g(r)}{x-r} = \frac{f(x)-f(r)}{x-r} + \frac{g(x)-g(r)}{x-r}.$$

When we take the limit as x approaches r, this is $f'(r) + g'(r)$, using the definition of $f'(r)$ and $g'(r)$ and the fact that the limit of a sum of two functions is the sum of the limits (when they both exist).

For (2), we note that

$$\frac{(cf)(x) - (cf)(r)}{x - r} = c\frac{f(x) - f(r)}{x - r},$$

and it follows from our limit theorems that the limit as x approaches r is $cf'(r)$.

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For (3), using what we know about limits we get

$$\lim_{x\to r} \frac{f(x)g(x) - f(r)g(r)}{x - r} = \lim_{x\to r} \left(\frac{f(x)g(x) - f(r)g(x)}{x - r} + \frac{f(r)g(x) - f(r)g(r)}{x - r} \right)$$
$$= \lim_{x\to r} g(x) \cdot \lim_{x\to r} \left(\frac{f(x) - f(r)}{x - r} \right) + f(r) \cdot \lim_{x\to r} \left(\frac{g(x) - g(r)}{x - r} \right)$$
$$= g(r)f'(r) + f(r)g'(r),$$

where for the last step we use that $\lim_{x\to r} g(x) = g(r)$ since g is continuous at r (since differentiable implies continuous).

For (4), we have

$$\begin{aligned} \lim_{x\to r} \frac{\frac{1}{g(x)} - \frac{1}{g(r)}}{1 - r} &= \lim_{x\to r} \frac{g(r) - g(x)}{g(x)g(r)(1 - r)} \\ &= -\lim_{x\to r} \frac{g(x) - g(r)}{1 - r} \frac{1}{g(x)g(r)} \\ &= -\lim_{x\to r} \frac{g(x) - g(r)}{1 - r} \frac{1}{g(r) \lim_{x\to r} g(x)} \\ &= -g'(r)\frac{1}{g(r)g(r)} \\ &= -\frac{1}{g^2(r)}g'(r). \end{aligned}$$

In this chain of equalities, we have used that $g(x)$ is continuous at r to get $\lim_{x\to r} g(x) = g(r)$. $\square$

Remark 33.1. We can deduce the "Quotient rule" from parts (5) and (6) above. Indeed, if f and g are differentiable at r and $g'(r) \neq 0$, then $(f/g)'(r) = (f \cdot 1/g)'(r) = f(r)(1/g)'(r) + f'(r)(1/g)(r) = \frac{f(r)(-g'(r))}{g^2(r)} + \frac{f'(r)}{g(r)} = \frac{f'(r)g(r) - f(r)g'(r)}{g^2(r)}$.

Remark 33.2. Using this Theorem and the previous example, we can show (by induction) that if $f(x) = x^n$ for any integer $n \geq 0$, then f is differentiable on all of $\mathbb{R}$ and we have $f'(x) = nx^{n-1}$ for all x.

Using that differentiation is linear, if f is a polynomial, so that $f(x) = a_n x^n + \cdots + a_1 x + a_0$ for constants $a_0, \ldots, a_n$, we get that f is differentiable on all of $\mathbb{R}$ and

$$f'(x) = na_n x^{n-1} + \cdots + a_1.$$

For $n \in \mathbb{Z}$ such that $n < 0$, and (6) of the Theorem allows us to conclude that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ in this case too.

Theorem 33.3 (Chain Rule)**.** *Suppose g is differentiable at s and f is differentiable at $g(s)$. Then $f \circ g$ is differentiable at s and*

$$(f \circ g)'(s) = f'(g(s))g'(s).$$

Example 33.4. Say $h(x) = \sqrt{x^4 + 1}$. Then $h = f \circ g$ where $g(x) = x^4 + 1$ and $f(x) = \sqrt{x}$. Since g is a polynomial, it is differentiable on all of $\mathbb{R}$ and $g'(x) = 4x^3$. The range of g is $[1, \infty)$ and as we showed above $f(x)$ is differentiable on all of $(0, \infty)$ and that $f'(x) = \frac{1}{2\sqrt{x}}$ at such points. Using the Chain Rule we get that $h(x)$ is differentiable on all of $\mathbb{R}$ and that

$$h'(x) = \frac{4x^3}{2\sqrt{x^4 + 1}}.$$

Before we prove the chain rule, it's worth discussing the most natural approach: we can write

$$\lim_{x \to s} \frac{(f \circ g)(x) - (f \circ g)(s)}{x - s} = \lim_{x \to s} \frac{f(g(x)) - f(g(s))}{g(x) - g(s)} \frac{g(x) - g(s)}{x - s}.$$

Then $\lim\limits_{x \to s} \dfrac{g(x) - g(s)}{x - s} = g'(s)$, and we would be done if we could also show that $\lim\limits_{x \to s} \dfrac{f(g(x)) - f(g(s))}{g(x) - g(s)} = f'(g(s))$. This certainly looks right at first, especially if we let $r = g(s)$, we think of $g(x)$ as y and observe that the limit as x approaches s of $y = g(x)$ is r, since g is continuous, so we would *want* to say that

$$\lim_{x \to s} \frac{f(g(x)) - f(g(s))}{g(x) - g(s)} = \lim_{y \to r} \frac{f(y) - f(r)}{y - r} = f'(r) = f'(g(s)).$$

But there is a problem with this: when $g(x) - g(r) = 0$, the function $\dfrac{f(g(x)) - f(g(s))}{g(x) - g(s)}$ is not defined, so if $g(x) = g(x)$ for values of x that are arbitrarily close to s, there is no limit!