- The symbol for "for all" is $\forall$ and the symbol for "there exists" is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that $\text{not } P$".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, $\text{not } P$".

Making sense of quantifier statements.

(1) Rewrite the following symbolic statements in sentences:

(2) A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both).

(a) Match each[1] informal description (i)–(vi) below with the precise quantifier statement (A)–(D).

Informal stories:

(i) Every number past some point is red.
(ii) Not every positive number is blue.
(iii) There are arbitrarily large red numbers.
(iv) Every positive number is red.
(v) You never get to a point where past that point every number is blue.
(vi) There are positive red number(s).

Precise statements:

(A) For every $y > 0$, y is red.
(B) There exists $y > 0$ such that y is red.
(C) For every $x \in \mathbb{R}$, there is some $y > x$ such that y is red.
(D) There exists $x \in \mathbb{R}$ such that for every $y > x$, y is red.

(b) Does (A) imply (B)? Does (B) imply (A)?
(c) Draw a picture where (A) is false and (B) is true.
(d) Does (C) imply (D)? Does (D) imply (C)?
(e) Draw a picture where (C) is true and (D) is false.

(3) Rewrite each statement with symbols in place of quantifiers, and write its negation. Is the original statement true or false? Discuss why (but don't prove them).

(a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
(b) For all $x \in \mathbb{R}$, $x^2 > 0$.
(c) For all $x \in \mathbb{R}$ such that[2] $x \neq 0$, $x^2 > 0$.
(d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
(e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

[1]Note: some precise statements correspond to multiple informal stories.

[2]In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

Proving quantifier statements and using the axioms of $\mathbb{R}$.

- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

Note: explaining *how* you found your example"x" is *not* a logically necessary part of the proof.

(4) Circle the correct answer in each of the blanks below:
- To prove a "for all" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *dis*prove a "for all" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To prove a "there exists" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *dis*prove a "there exists" statement, you need to give a GENERAL ARGUMENT / SPECIFIC EXAMPLE.

(5) Prove or disprove each of the statements in 3 using the axioms of $\mathbb{R}$ and facts we have already proven.

(6) Prove that there exists some $x \in \mathbb{R}$ such that $2x + 5 = 3$.

(7) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.

(8) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have already proven to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

(9) Prove that[3] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

(10) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.

[3]Hint: Use (7).