Math 905 Fall 2024: Commutative Algebra I

MWF 11:30am–12:20pm, Burnett 203

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Office Hours. Monday 9:30am–11pm, Thursdays 1pm–2pm, and by appointment.

What is Commutative Algebra? In short, Commutative Algebra is the study of commutative rings. Wherever there are numbers or functions, rings are generally lurking in the background. Commutative Algebra largely developed to explain the strange similarities between integers and polynomials, and has deep connections to Algebraic Geometry, Number Theory, Complex Analysis, and Representation Theory.

Three major themes in this course will be:

(1) Every ring is a geometric object! We will come to think of rings in this way, in addition to their more obvious nature as bags of knickknacks that can be added, subtracted, and multiplied.
(2) The hypothesis studied by and named in honor of Emmy Noether has incredible finiteness properties. Investigating these will be a recurring feature.
(3) While there will be many definitions and theorems, we will stay grounded and become quite familiar with many actual rings.

Prerequisites. We will lean upon some basics of rings, ideals, and modules, as covered in the Math 817–818 sequence. In particular, you should be familiar with the definitions of ring, ideal, and module; quotient rings and quotient modules; the First, Second, and Third Isomorphism Theorems for rings and modules; polynomial rings; PIDs and UFDs; free modules; the structure theory of finitely generated modules over PIDs; and Gauss' and Eisenstein's criteria. If you are unfamiliar with these things, I recommend reaching out to me to discuss.

We will also use some of the basic notions from point set topology (abstract topological space, continuous function, and connectedness) and elementary real analysis (ε/δ definition of continuity). If you are unfamiliar with any of these things, I recommend meeting me so I can point you to some background reading before the course.

Relationship to other courses. This course is a prerequisite for Math 953: Algebraic Geometry, which will be offered in Spring 2025, Math 918: Topics in Commutative Algebra, offered Spring 2025 & 2026, and Math 906: Commutative Algebra II, offered in Spring 2026. Math 905 runs on a two-year cycle, and will not run again until Fall 2026.

Style of class. *This will be an active learning course.* Class time will be dedicated to working in groups instead of lecture. Each class day, we will have a worksheet exploring new definitions, examples, and theorems. Math is learned by working through proofs and examples. I could tell you about all of the interesting commutative algebra I know, and I could mix it in with funny anecdotes and obscure puns, but my algebra will never be your own until you do it. This style of class may stretch our comfort zone more than a conventional lecture, but it is a better approximation of doing research and writing a thesis than the latter.

Requirements. Attendance is required. We will cover a lot of material in this course. To facilitate effective group work, you will be expected to read a 1–2 page summary of the day's new definitions and theorems before class, and to review the new material after class. We will also have fortnightly homework assignments, which may include some problems from the worksheets, and will involve some basic computations with the computer algebra system Macaulay2.

Textbooks and other resources. There is no required text for the course. The worksheets we go through will be largely self-contained, but the worksheet previews will give pointers to related sections of Eloísa Grifo's Math 905 lecture notes from Fall 2022 and the seminal *Introduction to commutative algebra* by Atiyah and MacDonald. Other sources that cover similar material include *A term of commutative algebra* by Altman and Kleiman, *Math 614 Lecture notes* by Hochster, and *Commutative Algebra with a view towards Algebraic Geometry* by Eisenbud.

Grading. Grades will be assigned based on assignments (80%) and participation (20%).

UNL Course Policies and Resources. Students are responsible for knowing the university policies and resources found on this page (`https://go.unl.edu/coursepolicies`):

- University-wide Attendance Policy
- Academic Honesty Policy
- Services for Students with Disabilities
- Mental Health and Well-Being Resources
- Final Exam Schedule
- Fifteenth Week Policy
- Emergency Procedures
- Diversity & Inclusiveness
- Title IX Policy
- Other Relevant University-Wide Policies

Tentative daily list of topics.

(1) Rings, Ideals, and Modules
- 1.1. Rings
- 1.2. Ideals
- 1.3. Modules
- 1.4. Algebras
- 1.5. Determinants

(2) Finiteness conditions
- 2.6. Algebra-finite and module-finite maps
- 2.7. Integral extensions
- 2.8. Noetherian rings
- 2.9. Noetherian modules
- 2.10. UFDs and integral closure

(3) Graded rings
- 3.11. Graded rings
- 3.12. Graded modules
- 3.13. Finiteness theorem for invariant rings

(4) Nullstellensatz and spectrum
- 4.14. Noether normalization
- 4.15. Nullstellensatz
- 4.16. Varieties and radical ideals
- 4.17. Spectrum of a ring
- 4.18. Spectrum and radical ideals

(5) Localization
- 5.19. Local rings and NAK
- 5.20. Localization of rings
- 5.21. Localization and spectrum
- 5.22. Localization of modules
- 5.23. Local properties

(6) Decompositions of ideals and modules
- 6.24. Minimal primes
- 6.25. Associated primes
- 6.26. More associated primes
- 6.27. Primary decomposition: existence
- 6.28. Primary decomposition: uniqueness

(7) Dimension and affine algebras
- 7.29. Dimension
- 7.30. Krull-Seidenberg theorems
- 7.31. More Krull-Seidenberg theorems
- 7.32. Dimension of affine algebras
- 7.33. Transcendence degree and dimension

(8) Local dimension theory
- 8.34. Length and simple modules
- 8.35. Dimension zero
- 8.36. Krull height theorem
- 8.37. Systems of parameters
- 8.38. Regular local rings

(9) Graded dimension theory
- 9.39. Hilbert functions for graded rings
- 9.40. Artin-Rees
- 9.41. Hilbert function theorem for local rings

(10) Normal rings
- 10.42. Dedekind domains
- 10.43. Finiteness Theorem for integral closures