

## DERIVATIVES AND OPTIMIZATION §4.3

**THEOREM 38.1:** Let  $f$  be a function that is differentiable at  $x = r$ .

- (1) If  $f'(r) > 0$ , then there is some  $\delta > 0$  such that
  - if  $x \in (r, r + \delta)$  then  $f(r) < f(x)$ ;
  - if  $x \in (r - \delta, r)$  then  $f(x) < f(r)$ .
- (2) If  $f'(r) < 0$ , then there is some  $\delta > 0$  such that
  - if  $x \in (r, r + \delta)$  then  $f(r) > f(x)$ ;
  - if  $x \in (r - \delta, r)$  then  $f(x) > f(r)$ .

**COROLLARY 38.2 (MIN-MAX THEOREM):** Let  $f$  be a function that is continuous on a closed interval  $[a, b]$ . If  $f$  attains a maximum or minimum value on  $[a, b]$  at  $r \in (a, b)$ , and  $f$  is differentiable at  $r$ , then  $f'(r) = 0$ .

- (1) Find the values of  $x$  on  $[0, 2]$  at which the function  $f(x) = x^3 - x^2 - 2x$  achieves its minimum and maximum values. Justify your answer carefully using the results above.
- (2) Explain why the Corollary follows from the Theorem.
- (3) Use the Corollary to deduce: if  $f$  is continuous on the closed interval  $[a, b]$ , then  $f$  attains its minimum and maximum values at some value on the list
  - $x = a$
  - $x = b$
  - some  $x = r \in (a, b)$  with  $f'(r) = 0$
  - some  $x = r \in (a, b)$  with  $f'(r)$  undefined.
- (4) Give examples of continuous functions on  $[0, 2]$  such that
  - (a)  $f(x)$  attains its maximum at  $x = 0$ ;
  - (b)  $g(x)$  attains its maximum at  $x = 2$ ;
  - (c)  $h(x)$  attains its maximum at  $x = 1$  and  $h$  is differentiable at  $x = 1$ ;
  - (d)  $j(x)$  attains its maximum at  $x = 1$  and  $j$  is not differentiable at  $x = 1$ .
- (5) Prove part (1) of the Theorem:
  - Consider the function  $h(x) = \frac{f(x) - f(r)}{x - r}$ . Apply the definition of limit to this function with  $\varepsilon = f'(r)$ . What does the definition give you?
  - If  $h(x) > 0$  and  $x > r$ , what can you say about  $f(x) - f(r)$ ?
  - If  $h(x) > 0$  and  $x < r$ , what can you say about  $f(x) - f(r)$ ?
- (6) Prove part (2) of the Theorem.
- (7) True or false: If  $f'(7) > 0$ , then  $f(7.0000001) > f(7)$ .
- (8) True or false: If  $f'(7) > 0$ , then there exists some  $N \in \mathbb{N}$  such that for all natural numbers  $n > N$ ,  $f\left(7 + \frac{1}{10^n}\right) > f(7)$ .

**DEFINITION 38.3:** Let  $f$  be a function. We say that  $f$  attains a **local minimum** at  $r$  if there exists some  $\delta > 0$  such that  $f$  is defined on  $(r - \delta, r + \delta)$  and  $f$  achieves its minimum value on the interval  $(r - \delta, r + \delta)$  at the input value  $x = r$ . We define **local maximum** analogously.

**COROLLARY 38.4 (LOCAL MIN-MAX THEOREM):** If  $f$  attains a local maximum or local minimum at  $x = r$ , then either  $f$  is not differentiable at  $x = r$  or  $f'(r) = 0$ .

- (9) Prove Corollary 38.4.
- (10) Draw graphs of functions are continuous on the closed interval  $[-3, -1]$  satisfying the following:
- (a)  $f(x)$ , that has a local maximum at  $-2$  but does not attain its maximum at  $-2$ .
  - (b)  $g(x)$ , such that  $g'(-2) = 0$  but  $g$  does not attain a local minimum or local maximum at  $-2$ .
  - (c)  $h(x)$ , such that  $h$  is not differentiable at  $-2$  but  $h$  does not attain a local minimum or local maximum at  $-2$ .
  - (d)  $j(x)$  that has no local maximum at all on  $[-3, -1]$ .
- (11) “THE ZEROth DERIVATIVE TEST” Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and that there finitely many points  $r_1 < r_2 < \cdots < r_{t-1}$  in  $(a, b)$  where either  $f'$  is zero or undefined. Set  $r_0 = a$  and  $r_t = b$ .
- (a) Show that the maximum value of  $f$  on  $[a, b]$  is  $\max\{f(r_0), \dots, f(r_t)\}$ , and likewise the minimum value of  $f$  on  $[a, b]$  is  $\min\{f(r_0), \dots, f(r_t)\}$ .
  - (b) Show that, for  $i = 1, \dots, t-1$ ,  $f$  attains a local maximum at  $r_i$  if and only if  $f(r_{i-1}) < f(r_i) > f(r_{i+1})$ ; likewise,  $f$  attains a local minimum at  $r_i$  if and only if  $f(r_{i-1}) > f(r_i) < f(r_{i+1})$ .