MATH 901 LECTURE NOTES, FALL 2021

Contents

1. Category Theory

1.1. **Categories.**

Lecture of August 23, 2021

1.1.1. *Definition of category.*

Definition 1.1. A *category* $\mathscr{C}$ consists of the following data:

(1) a collection of *objects*, denoted $\mathrm{Ob}(\mathscr{C})$,
(2) for each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* (also known as *arrows*) from A to B,
(3) for each triple of objects $A, B, C \in \mathrm{Ob}(\mathscr{C})$, a function

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)$$

written as $(\alpha, \beta) \mapsto \beta \circ \alpha$ that we call the *composition rule*.

These data are required to satisfy the following axioms:

(1) (Disjointness) the Hom sets are disjoint: if $A \neq A'$ or $B \neq B'$, then

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \cap \mathrm{Hom}_{\mathscr{C}}(A', B') = \varnothing.$$

(2) (Identities) for every object A, there is an *identity morphism* $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.
(3) (Associativity) composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

Remark 1.2. (1) The word “collection” as opposed to “set” is important here. The point is that there is no set of all sets, but by utilizing bigger collecting objects in set theory, we can sensibly talk about the collection of all sets. We’ll sweep all of the set theory under the rug there, but it’s worth keeping in mind that the objects of a category don’t necessarily form a set. We did assume that the collections of morphisms between a pair of objects form a set, though not everyone does.
(2) The first axiom above guarantees that every morphism α in a category $\mathscr{C}$ has a well-defined *source* and *target* in $\mathrm{Ob}(\mathscr{C})$, namely, the unique A and B (respectively) such that $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

The name arrow dovetails with the common practice of depicting a morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ as

$$A \xrightarrow{\alpha} B.$$

The composition of $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ is $A \xrightarrow{\beta\circ\alpha} C$.

Optional Exercise 1.3. Prove that every element in a category has a unique identity morphism (i.e., a unique morphism that satisfies the hypothesis of axiom (2)).

1.1.2. *Examples of categories.* Many of our favorite objects from algebra naturally congregate in categories!

Example 1.4.

(1) There is a category **Set** where
- Ob(**Set**) is the collection of all sets
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the the set of functions from X to Y
- the composition rule is composition of functions

We observe that every set has an identity function, which behaves as an identity for composition, and that composition of functions is associative.

(2) There is a category **Grp** where
- Ob(**Grp**) is the collection of all groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Grp}}(X, Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

Note that the identity function on a group is a group homomorphism, and that a composition of two group homomorphisms is a group homomorphism.

(3) There is a category **Ab** where
- Ob(**Ab**) is the collection of all abelian groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Ab}}(X, Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

(4) In this class,
- A *semigroup* is a set S with an associative operation $\cdot$ that has an identity element; some may prefer the term *monoid*, but I don't.
- A *semigroup homomorphism* from semigroups $S \to T$ is a function that preserves the operation and maps the identity element to the identity element.

There is a category **Sgrp** where the objects are all semigroups and the morphisms are semigroup homomorphisms. (The composition rule is composition again.)

(5) In this class,
- A *ring* is a set R with two operations $+$ and $\cdot$ such that $(R, +)$ is abelian group, with identity 0, and $(R, \cdot)$ is a semigroup with identity 1, and such that the left and right distributive laws hold: $(r + s)t = rt + st$ and $t(r + s) = tr + ts$.
- A *ring homomorphism* is a function that preserves $+$ and $\cdot$ and sends 1 to 1.

There is a category **Ring** where the objects are all rings and the morphisms are ring homomorphisms.

(6) Let R be a ring. In this class,
- A *left R-module* is an abelian group $(M, +)$ equipped with a pairing $R \times M \to M$, written $(r, m) \mapsto rm$ or $(r, m) \mapsto r \cdot m$ such that
  (a) $r_1(r_2m) = (r_1r_2)m$,
  (b) $(r_1 + r_2)m = r_1m + r_2m$,
  (c) $r(m_1 + m_2) = rm_1 + rm_2$, and
  (d) $1m = m$.
- A *left module homomorphism* or *R-linear map* between left R-modules $\phi : M \to N$ is a homomorphism of abelian groups from $(M, +) \to (N, +)$ such that $\phi(rm) = r\phi(m)$.

There is a category $R-$**Mod** where the objects are all left R-modules and the morphisms are R-linear maps.

(7) There is a category **Fld** where the objects are all fields and the morphisms are all field homomorphisms.

(8) There is a category **Top** where the objects are all topological spaces and the morphisms are all continuous functions.

Remark 1.5. There are two special cases of the category of R-modules that are worth singling out:

- Every abelian group M is a $\mathbb{Z}$-module in a unique way, by setting
$$n \cdot m = \underbrace{m + \cdots + m}_{n\text{-times}} \quad \text{and} \quad -n \cdot m = -(\underbrace{m + \cdots + m}_{n\text{-times}}) \qquad \text{for } n \geqslant 0.$$
Thus, **Ab** is basically just $\mathbb{Z} - \mathbf{Mod}$.
- When $R = K$ happens to be a field, we are accustomed to calling K-modules *vector spaces*. Thus, we might write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

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Example 1.6. Here are some variations on the category $K - \mathbf{Vect}$.

(1) The collection of finite dimensional K-vector spaces with all linear transformations is a category; call it $K - \mathbf{vect}$.
(2) The collection of all n-dimensional K-vector spaces with all linear transformations is a category.
(3) The collection of all K-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.
(4) The collection of all K-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it's not closed under composition.
(5) The collection of all n-dimensional vector spaces with singular linear transformations is not a category, since it doesn't have identity maps.

Example 1.7. (1) There is a category $\mathbf{Set}_*$ of *pointed sets* where
- the objects are pairs (X, x) where X is a set and $x \in X$,
- for two pointed sets, the morphisms from (X, x) to (Y, y) are functions $f : X \to Y$ such that $f(x) = y$,
- usual composition.

(2) For a commutative ring A,
- A *commutative A-algebra* is a commutative ring R plus a homomorphism $\phi : A \to R$.
- Slightly more generally, an *A-algebra* is a ring R plus a homomorphism $\phi : A \to R$ such that $\phi(A)$ lies in the center of R: $r \cdot \phi(a) = \phi(a) \cdot r$ for any $a \in A$ and $r \in R$. (In the more general situation, A is still commutative but R may not be.)
- An *A-algebra homomorphism* between two A-algebras (R, ϕ) and (S, ψ) is a ring homomorphism $\alpha : R \to S$ such that $\alpha \circ \phi = \psi$.

The category of A-algebras is denoted $A - \mathbf{Alg}$, and the category of commutative A-algebras is $A - \mathbf{cAlg}$.

(3) Fix a field K, and define a category $\mathbf{Mat}_K$ as follows: the objects are the positive natural numbers $n \in \mathbb{N}_{>0}$, and $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is the set of $b \times a$ matrices with entries in K. To see this as a category, we need a composition rule. Given $B \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ and $A \in \mathrm{Hom}_{\mathscr{C}}(a, b)$, take the composition $A \circ B \in \mathrm{Hom}_{\mathscr{C}}(a, c)$ to be the product AB. Since matrix multiplication is associative, axiom (3) holds, and the $n \times n$ identity matrix serves as an identity morphism in the sense of axiom (2). Finally, if $A \in \mathrm{Hom}_{\mathscr{C}}(a, b) \cap \mathrm{Hom}_{\mathscr{C}}(a', b')$, then A is a $b \times a$ matrix and a $b' \times a'$ matrix, so $a = a'$ and $b = b'$. Notably, the morphisms in this category are not functions.

We can also make a bunch of categories in a hands-on way as follows:

Example 1.8. Let $(P, \leqslant)$ be a poset. We define a category $\mathbf{PO}(P)$ from P as follows. The objects of $\mathbf{PO}(P)$ are just the elements of P. For each pair $a, b \in P$ with $a \leqslant b$, form a symbol f_a^b. Then we set

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a,b) = \begin{cases} \{f_a^b\} & \text{if } a \leqslant b \\ \varnothing & \text{otherwise.} \end{cases}$$

There is only one possible composition rule:

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a,b) \times \mathrm{Hom}_{\mathbf{PO}(P)}(b,c) \longrightarrow \mathrm{Hom}_{\mathbf{PO}(P)}(a,c)$$

when $a \leqslant b$ and $b \leqslant c$ we also have $a \leqslant c$, and the unique pair on the left must map to the unique element on the right, so $f_b^c \circ f_a^b = f_a^c$; when either $a \nleqslant b$ or $b \nleqslant c$, there is nothing to compose!

Each morphism f_a^b is in only one Hom set (with source a and target b). Composition is associative since there is at most one function between one element sets. For any a, $f_a^a \in \mathrm{Hom}_{\mathbf{PO}(P)}(a,a)$ is the identity morphism.

For a specific example, we can think of $\mathbb{N}_{>0}$ as a category this way. Drawing all of the morphisms would be a mess, but any morphism is a composition of the ones depicted:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots .$$

Note that the objects of this category are exactly the same as in Example **??**(3), but with much fewer morphisms!

Example 1.9. A category with one object is nothing but a semigroup.

1.1.3. *Constructions of categories.* Here are a few more basic constructions of categories:

Definition 1.10. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{\mathrm{op}}$ is the category with $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$, and $\mathrm{Hom}_{\mathscr{C}}(A,B) = \mathrm{Hom}_{\mathscr{C}}(B,A)$ for all $A, B \in \mathrm{Ob}(\mathscr{C})$.

That is, the opposite category is the "same category with the arrows reversed." To avoid confusion, we might write α^{op} for the morphism $B \xrightarrow{\alpha^{\mathrm{op}}} A$ in $\mathscr{C}^{\mathrm{op}}$ corresponding to $A \xrightarrow{\alpha} B$ in $\mathscr{C}$.

Definition 1.11. Given two categories $\mathscr{C}$ and $\mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ is the category with $\mathrm{Ob}(\mathscr{C} \times \mathscr{D})$ given by the collection of pairs (C, D) with $C \in \mathrm{Ob}(\mathscr{C})$ and $D \in \mathrm{Ob}(\mathscr{D})$, and $\mathrm{Hom}_{\mathscr{C}\times\mathscr{D}}((A,B),(C,D)) = \mathrm{Hom}_{\mathscr{C}}(A,C) \times \mathrm{Hom}_{\mathscr{D}}(B,D)$. We leave it to you to pin down the composition rule.

Definition 1.12. A category $\mathscr{D}$ is a *subcategory* of another category $\mathscr{C}$ provided

(1) every object of $\mathscr{D}$ is an object of $\mathscr{C}$
(2) for every $A, B \in \mathrm{Ob}(\mathscr{D})$, $\mathrm{Hom}_{\mathscr{D}}(A,B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A,B)$, and
(3) for every $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ in $\mathscr{D}$, the composition of α and β in $\mathscr{D}$ equals the composition of α and β in $\mathscr{C}$.

If equality hold in (2) (for all A, B), we say that $\mathscr{D}$ is a *full subcategory* of $\mathscr{C}$.

Example 1.13. Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, since not every function between groups is a homomorphism, **Grp** is not a full subcategory of **Set**. Similarly, **Ab**, **Ring**, $R-$**Mod**, and **Top** are all subcategories of **Set**.

On the other hand, **Ab** is a full subcategory of **Grp**, and **Grp** is a full subcategory of **Sgrp**: a morphism of abelian groups is a morphism of groups that happens to be between abelian groups (and likewise for groups and semigroups)!