# CONTINUOUS FUNCTIONS §3.3

DEFINITION: A function $f$ is **continuous at** $a$ provided: For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $f(x)$ is defined and $|f(x) - f(a)| < \varepsilon$.

THEOREM: If $f$ is defined at $a$ then $f$ is continuous at $a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

THEOREM: If $f$ and $g$ are both continuous at $a$, and $c$ is any constant, then

(1) $f + g$ is continuous at $a$.
(2) $cf$ is continuous at $a$.
(3) $fg$ is continuous at $a$.
(4) $f/g$ is continuous at $a$, provided $g(a) \neq 0$.

THEOREM: If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$.

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It is tiresome to say "continuous at $a$ for every $a \in \mathbb{R}$". The following definition is then convenient.

DEFINITION 29.1: Let $I$ be an open interval of $\mathbb{R}$ of the form $I = (a, b)$, $I = (a, \infty)$, $I = (-\infty, a)$, or $I = (-\infty, \infty) = \mathbb{R}$. We say $f$ is **continuous on** $I$ if $f$ is continuous at $a$ for all $a \in I$.

(1) Let
$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x + 1 & \text{if } x < 1. \end{cases}$$
Use the $\varepsilon - \delta$ definition to show that $f(x)$ is continuous at 1.

(2) Which of the following functions are continuous on $\mathbb{R}$?

- $f(x) = \sqrt{x^2 + 5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \dfrac{1}{x}$.

(3) Which of the following functions are continuous on $(0, \infty)$?

- $f(x) = \sqrt{x^2 + 5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \dfrac{1}{x}$.

(4) Prove that $j(x) = x \sin(1/x)$ is continuous[1] on $\mathbb{R}$.

(5) Prove or disprove: If $f$ and $g$ are continuous at $a$, then $f/g$ is continuous at $a$.

(6) Prove or disprove: If $f$ and $g$ are two functions, $a \in \mathbb{R}$, and $f(a) = g(a)$, then $f$ is continuous at $a$ if and only if $g$ is continuous at $a$.

(7) Prove or disprove: If $f$ and $g$ are two functions, $a < b$, and $f(x) = g(x)$ for all $x \in (a, b)$, then $f$ is continuous on $(a, b)$ if and only if $g$ is continuous on $(a, b)$.

[1] You can use without proof that $\sin(x)$ is continuous on $\mathbb{R}$.