

### CONTINUOUS FUNCTIONS §3.3

**DEFINITION:** A function  $f$  is **continuous at**  $a$  provided: For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $|x - a| < \delta$  then  $f(x)$  is defined and  $|f(x) - f(a)| < \varepsilon$ .

**THEOREM:** If  $f$  is defined at  $a$  then  $f$  is continuous at  $a$  if and only if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

**THEOREM:** If  $f$  and  $g$  are both continuous at  $a$ , and  $c$  is any constant, then

- (1)  $f + g$  is continuous at  $a$ .
- (2)  $cf$  is continuous at  $a$ .
- (3)  $fg$  is continuous at  $a$ .
- (4)  $f/g$  is continuous at  $a$ , provided  $g(a) \neq 0$ .

**THEOREM:** If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$ , then  $f \circ g$  is continuous at  $a$ .

It is tiresome to say “continuous at  $a$  for every  $a \in \mathbb{R}$ ”. The following definition is then convenient.

**DEFINITION 29.1:** Let  $I$  be an open interval of  $\mathbb{R}$  of the form  $I = (a, b)$ ,  $I = (a, \infty)$ ,  $I = (-\infty, a)$ , or  $I = (-\infty, \infty) = \mathbb{R}$ . We say  $f$  is **continuous on**  $I$  if  $f$  is continuous at  $a$  for all  $a \in I$ .

(1) Let

$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x + 1 & \text{if } x < 1. \end{cases}$$

Use the  $\varepsilon - \delta$  definition to show that  $f(x)$  is continuous at 1.

(2) Which of the following functions are continuous on  $\mathbb{R}$ ?

- $f(x) = \sqrt{x^2 + 5}$ .
- Every polynomial function.
- $f(x) = \sqrt{x}$ .
- $f(x) = \frac{1}{x}$ .

(3) Which of the following functions are continuous on  $(0, \infty)$ ?

- $f(x) = \sqrt{x^2 + 5}$ .
- Every polynomial function.
- $f(x) = \sqrt{x}$ .
- $f(x) = \frac{1}{x}$ .

(4) Prove that  $j(x) = x \sin(1/x)$  is continuous<sup>1</sup> on  $\mathbb{R}$ .

(5) Prove or disprove: If  $f$  and  $g$  are continuous at  $a$ , then  $f/g$  is continuous at  $a$ .

(6) Prove or disprove: If  $f$  and  $g$  are two functions,  $a \in \mathbb{R}$ , and  $f(a) = g(a)$ , then  $f$  is continuous at  $a$  if and only if  $g$  is continuous at  $a$ .

(7) Prove or disprove: If  $f$  and  $g$  are two functions,  $a < b$ , and  $f(x) = g(x)$  for all  $x \in (a, b)$ , then  $f$  is continuous on  $(a, b)$  if and only if  $g$  is continuous on  $(a, b)$ .

<sup>1</sup>You can use without proof that  $\sin(x)$  is continuous on  $\mathbb{R}$ .