§6.25: ASSOCIATED PRIMES

DEFINITION: Let R be a ring and M be a module. A prime ideal $\mathfrak{p}$ of R is an **associated prime** of M if $\mathfrak{p} = \operatorname{ann}_R(m)$ for some $m \in M$. The element m is called a **witness** for the associated prime $\mathfrak{p}$. We write $\operatorname{Ass}_R(M)$ for the set of associated primes of a module.

LEMMA: Let R be a Noetherian ring and M be a module. For any nonzero element $m \in M$, the ideal $\operatorname{ann}_R(m)$ is contained in an associated prime of M. In particular, if $M \neq 0$, then M has an associated prime.

DEFINITION: Let R be a ring and M be an R-module. We say that an element $r \in R$ is a **zerodivisor** on M if there is some $m \in M \smallsetminus 0$ such that $rm = 0$.

PROPOSITION: Let R be a Noetherian ring and M an R-module. The set of zerodivisors on M is the union of the associated primes of M.

THEOREM: Let R be a Noetherian ring, W be a multiplicatively closed set, and M be a module. Then

$$\operatorname{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R(M), \mathfrak{p} \cap W = \varnothing\}.$$

COROLLARY: Let R be a Noetherian ring and I be an ideal. Then $\operatorname{Min}(I) \subseteq \operatorname{Ass}_R(R/I)$.

(1) Proof of Lemma and Proposition: Let R be a Noetherian ring and M be a nonzero module.
- **(a)** Let $\mathcal{S} = \{\operatorname{ann}_R(m) \mid m \in M \smallsetminus 0\}$. Explain why $\mathcal{S}$ has a maximal element J.
- **(b)** Let $J = \operatorname{ann}_R(m)$ and suppose that $rs \in J$ but $s \notin J$. Explain why $J = \operatorname{ann}_R(sm)$.
- **(c)** Conclude the proof of the Lemma.
- **(d)** Deduce the Proposition from the Lemma.
- **(e)** What does the Proposition say in the special case when $M = R$?

(2) Working with associated primes.
- **(a)** Let R be a domain and M be a torsionfree module. Show that $\operatorname{Ass}_R(M) = \{(0)\}$.
- **(b)** Let R be a ring and $\mathfrak{p}$ be a prime ideal. Show that for any nonzero element $\overline{r} \in R/\mathfrak{p}$ that $\operatorname{ann}_R(\overline{r}) = \mathfrak{p}$ and use the definition to deduce that $\operatorname{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$.
- **(c)** Let K be a field and $R = K[X,Y]/(X^2Y, XY^2)$. Use[1] the definition to show that (x,y), (x), and (y) are associated primes of R.
- **(d)** Let M be a module. Explain why $\mathfrak{p} \in \operatorname{Ass}_R(M)$ if and only if there is an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow M$.

(3) Using the Theorem. Let R be a Noetherian ring.
- **(a)** Restate the Theorem in the special case $W = R \smallsetminus \mathfrak{p}$ with our standard notation for this setting.
- **(b)** Show (either using the Theorem or 2(d) above) that $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$.
- **(c)** Use the Theorem (and the previous part or otherwise) to prove the Corollary.
- **(d)** Show the more general statement: if M is a nonzero module, then the primes that are minimal within the support of I are associated to M.

[1]Hint: Consider xy and y^2.

(4) The ring of Puiseux series is $R = \bigcup_{n\geq 1} \mathbb{C}[\![X^{1/n}]\!]$: elements consist of power series with fractional exponents that have a common denominator (though different elements can have different common denominators).
   (a) Show that every nonzero element of R can be written in the form $X^{m/n} \cdot u$ for some unit u.
   (b) Show that the R-module $R/(X)$ is nonzero but has no associated primes.

(5) Proof of Theorem: Let R be a Noetherian ring, W be a multiplicatively closed set, and M be a module.
   (a) Suppose that $\mathfrak{p}$ is an associated prime of M with $W \cap \mathfrak{p} = \varnothing$, and let m be a witness for $\mathfrak{p}$ as an associated prime of M. Show that $W^{-1}\mathfrak{p}$ is an associated prime of $W^{-1}M$ with witness $\frac{m}{1}$.
   (b) Suppose that $W^{-1}\mathfrak{p} \in \operatorname{Spec}(W^{-1}R)$ is an associated prime of $W^{-1}M$. Explain why there is a witness of the form $\frac{m}{1}$.
   (c) Let $\mathfrak{p} = (f_1, \ldots, f_t)$. Explain why there exist $w_1, \ldots, w_t \in W$ such that $w_i f_i m = 0$ in M for all i.
   (d) Show that $w_1 \cdots w_t m$ is a witness for $\mathfrak{p}$ as an associated prime of M.

(6) Let R be a Noetherian ring. Is every minimal prime of a zerodivisor a minimal prime of R?