

WORKSHEET #1

Definition 1. A triple (a, b, c) of natural numbers is a **Pythagorean triple** if they form the side lengths of a right triangle, where c is the length of the hypotenuse.

Theorem 2 (Fundamental Theorem of Arithmetic). Every natural number $n \geq 1$ can be written as a product of prime numbers:

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}.$$

This expression is unique up to reordering. □

Definition 3. We call the number e_i the **multiplicity** of the prime p_i in the prime factorization of

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}.$$

Definition 4. Let m, n be integers and $K \geq 1$ be a natural number. We say that m **is congruent to n modulo K** , written as $m \equiv n \pmod{K}$, if $m - n$ is a multiple of K .

Theorem 5. Let n be an integer and $K \geq 1$ a natural number. Then n is congruent to exactly one nonnegative integer between 0 and $K - 1$: this number is the “remainder” when you divide n by K . □

Proposition 6. Let m, m', n, n' and K be natural numbers. Suppose that

$$m \equiv m' \pmod{K} \quad \text{and} \quad n \equiv n' \pmod{K}.$$

Then

$$m + n \equiv m' + n' \pmod{K} \quad \text{and} \quad mn \equiv m'n' \pmod{K}. \quad \square$$

Definition 7. A triple (a, b, c) of natural numbers is a **primitive Pythagorean triple (PPT)** if $a^2 + b^2 = c^2$, and there is no common factor of a, b, c greater than 1; equivalently, a, b, c have no common prime factor.

Theorem 8. The set of primitive Pythagorean triples (a, b, c) with a odd is given by the formula

$$a = st, \quad b = \frac{s^2 - t^2}{2}, \quad c = \frac{s^2 + t^2}{2},$$

where $s > t \geq 1$ are odd integers with no common factors.

Theorem 9. The set of points on the unit circle $x^2 + y^2 = 1$ with positive rational coordinates is given by the formula

$$(x, y) = \left(\frac{2v}{v^2 + 1}, \frac{v^2 - 1}{v^2 + 1} \right)$$

where v ranges through rational numbers greater than one.

WORKSHEET #2

Definition 10. The **greatest common divisor** of two integers a and b , denoted $\gcd(a, b)$, is the largest integer that divides a and b .

Definition 11. Two integers a and b are **coprime** if $\gcd(a, b) = 1$.

Theorem 12. The Euclidean algorithm terminates and outputs the correct value of $\gcd(a, b)$.

Definition 13. An expression of the form $ra + sb$ with $r, s \in \mathbb{Z}$ is a **linear combination** of a and b .

Corollary 14. If a, b are integers, then $\gcd(a, b)$ can be realized as a linear combination of a and b . Concretely, we can use the Euclidean algorithm to do this.

Theorem 15. Let a, b, c be integers. The equation

$$ax + by = c$$

has an integer solution if and only if c is divisible by $d := \gcd(a, b)$. If this is the case, there are infinitely many solutions. If (x_0, y_0) is a one particular solution, then the general solution is of the form

$$x = x_0 - (b/d)n, \quad y = y_0 + (a/d)n$$

as n ranges through all integers.

PROBLEM SET #1

Lemma 16. Let a, b, c be integers. If a and b are coprime, and a divides bc , then a divides c .

WORKSHEET #3

Definition 17. A **congruence class modulo K** is a set of the form

$$[a] := \{n \in \mathbb{Z} \mid n \equiv a \pmod{K}\}$$

for some $a \in \mathbb{Z}$.

Definition 18. A **representative** for a congruence class is an element of the congruence class.

Proposition 19. Given $K > 0$, the set of integers \mathbb{Z} is the disjoint union of K congruence classes:

$$\mathbb{Z} = [0] \sqcup [1] \sqcup \cdots \sqcup [K - 1].$$

Definition 20. The ring \mathbb{Z}_K is the set of congruence classes modulo K :

$$\{[0], [1], \dots, [K - 1]\}$$

equipped with the operations

$$[a] + [b] = [a + b] \quad \text{and} \quad [a][b] = [ab].$$

Definition 21. We say that a number a is a **unit modulo K** if there is an integer solution x to $ax \equiv 1 \pmod{K}$, and we say that such a number x is an **inverse modulo K** to a .

Definition 22. We say that a congruence class $[a]$ is a **unit in \mathbb{Z}_K** if there is a congruence class $x \in \mathbb{Z}_K$ such that $[a]x = [1]$, and we say that such a class x is an **inverse** to $[a]$ in \mathbb{Z}_K .

Theorem 23. Let a and n be integers, with n positive. Then a is a unit modulo n if and only if a and n are coprime.

Theorem 24 (Chinese Remainder Theorem). Given $m_1, \dots, m_k > 0$ integers such that m_i and m_j are coprime for each $i \neq j$, and $a_1, \dots, a_k \in \mathbb{Z}$, the system of congruences

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \vdots \\ x \equiv a_k \pmod{m_k} \end{cases}$$

has a solution $x \in \mathbb{Z}$. Moreover, the set of solutions forms a unique congruence class modulo $m_1 m_2 \cdots m_k$.

PROBLEM SET #2

Lemma 25. Let a, b, c be integers. If a and b are coprime, a divides c , and b divides c , then ab divides c .

Definition 26. Given integers a_1, \dots, a_m , the **greatest common divisor** of a_1, \dots, a_m is the largest integer that divides all of them.

Theorem 27. Let a, b, n be integers, with $n > 0$. Then $[a]x = [b]$ has a solution x in \mathbb{Z}_n if and only if $\gcd(a, n)$ divides b . In this case, the number of distinct solutions is exactly $\gcd(a, n)$.

WORKSHEET #4

Definition 28. A **group** is a set G equipped with a product operation

$$G \times G \rightarrow G \quad (g, h) \mapsto gh$$

and an **identity** element $1 \in G$ such that

- the product is associative: $(gh)k = g(hk)$ for all $g, h, k \in G$,
- $g1 = 1g = g$ for all $g \in G$, and
- for every $g \in G$, there is an inverse element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.

Definition 29. A group is **abelian** if the product is commutative: $gh = hg$ for all $g, h \in G$.

Definition 30. A **finite group** is a group G that is a finite set.

Definition 31. Let G be a group and $g \in G$. The **order** of g is the smallest positive integer n such that $g^n = e$, if some such n exists, and ∞ if no such integer exists.

Theorem 32 (Lagrange's Theorem). Let G be a finite group and $g \in G$. Then the order of g is finite and divides the cardinality of the group G .

Theorem 33 (Fermat's Little Theorem). Let p be a prime number and a an integer. If p does not divide a , then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Definition 34. Let n be a positive integer. We define $\varphi(n)$ to be the number of elements of \mathbb{Z}_n^\times . We call this **Euler's phi function**.

Proposition 35. Euler's phi function satisfies the following properties.

- (1) If p is a prime and n is a positive integer, then $\varphi(p^n) = p^{n-1}(p-1)$.
- (2) If m, n are coprime positive integers, then $\varphi(mn) = \varphi(m)\varphi(n)$.

Theorem 36 (Euler's Theorem). Let a, n be coprime integers, with n positive. Then

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

WORKSHEET #5

Proposition 37. Let p be a prime. Let $p(x)$ be a polynomial of degree d with coefficients in \mathbb{Z}_p . Then $p(x)$ has at most d roots in \mathbb{Z}_p . □

Lemma 38. If G is a group, $g \in G$, and n a positive integer such that $g^n = 1$, then the order of g divides n .

Definition 39. Let n be a positive integer. An element $x \in \mathbb{Z}_n^\times$ is a **primitive root** if the order of x in \mathbb{Z}_n^\times equals $\phi(n)$ (the cardinality of \mathbb{Z}_n^\times).

Theorem 40. Let p be a prime number. Then there exists a primitive root in \mathbb{Z}_p^\times .

Definition 41. If $[a]$ is a primitive root in \mathbb{Z}_p , the function

$$\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_{p-1} \quad [b] \mapsto [m] \text{ such that } [b] = [a]^m$$

is called the **discrete logarithm** or **index** of \mathbb{Z}_p^\times with base $[a]$.

Lemma 42. Let p be a prime and $[a]$ a primitive root in \mathbb{Z}_p . The corresponding discrete logarithm function $I : \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_{p-1}$ satisfies the property

$$I(xy) = I(x) + I(y) \quad \text{and} \quad I(x^n) = [n]I(x)$$

for $x, y \in \mathbb{Z}_p^\times$ and $n \in \mathbb{N}$.

Proposition 43. Let n be a positive integer. Then $\sum_{d|n} \varphi(d) = n$.

Theorem 44. Let p be a prime. Suppose that there are n distinct solutions to $x^n = 1$ in \mathbb{Z}_p . Then \mathbb{Z}_p^\times has exactly $\varphi(n)$ elements of order n .

WORKSHEET #6

Definition 45. We say that an element $x \in \mathbb{Z}_n$ is a **square** or a **quadratic residue** if there is some $y \in \mathbb{Z}_n$ such that $y^2 = x$, and in this case, we call y a **square root** of x .

Definition 46. Let p be an odd prime. For $r \in \mathbb{Z}$ not a multiple of p we define the **Legendre symbol** of r with respect to p as

$$\left(\frac{r}{p}\right) = \begin{cases} 1 & \text{if } [r] \text{ is a square in } \mathbb{Z}_p, \\ -1 & \text{if } [r] \text{ is a not square in } \mathbb{Z}_p. \end{cases}$$

Theorem 47 (Euler's Criterion). For p an odd prime and $r \in \mathbb{Z}$ not a multiple of p , we have

$$\left(\frac{r}{p}\right) \equiv r^{(p-1)/2} \pmod{p}.$$

Theorem 48 (Quadratic Reciprocity part -1). If p is odd, then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases}.$$

Proposition 49. Let p be an odd prime and a, b integers not divisible by p . Then

$$(1) \ a \equiv b \pmod{p} \text{ implies that } \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right).$$

$$(2) \ \left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right).$$

$$(3) \ \left(\frac{a^2}{p}\right) = 1.$$

PROBLEM SET #3

Theorem 50. If p is an odd prime and $n > 0$, then \mathbb{Z}_{p^n} has a primitive root.

WORKSHEET #7

Theorem 51 (Quadratic Reciprocity). Let p and q be distinct odd primes. Then

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) \quad \text{if either } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4},$$

$$\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right) \quad \text{if both } p \equiv 3 \pmod{4} \text{ and } q \equiv 3 \pmod{4}.$$

Theorem 52 (Quadratic Reciprocity part 2). Let p be an odd prime. Then

$$\left(\frac{2}{p}\right) = 1 \quad \text{if } p \equiv \pm 1 \pmod{8},$$

$$\left(\frac{2}{p}\right) = -1 \quad \text{if } p \equiv \pm 3 \pmod{8}.$$

Lemma 53 (Gauss' Lemma). *Let p be an odd prime and set $p' = \frac{p-1}{2}$. Note that every integer coprime to p is congruent modulo p to a unique integer in the set $S = \{\pm 1, \pm 2, \dots, \pm p'\}$.*

Let a be an integer coprime to p . Consider the sequence

$$a, 2a, 3a, \dots, p'a$$

and replace each element in the sequence with element of S that is congruent with modulo p to get a list L of p' -many elements of S .

Then $\left(\frac{a}{p}\right) = (-1)^\nu$, where ν is the number of negative integers in L .

Lemma 54. *Let p and q be two coprime odd positive integers. Then*

$$\sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor + \sum_{\ell=1}^{\frac{q-1}{2}} \left\lfloor \frac{\ell p}{q} \right\rfloor = \frac{p-1}{2} \cdot \frac{q-1}{2}.$$

WORKSHEET #8

Theorem 55 (Euclid). *There are infinitely many primes.*

Proposition 56. *For each of the following conditions, there are infinitely many primes p :*

- $p \equiv 1 \pmod{3}$
- $p \equiv 2 \pmod{3}$
- $p \equiv 1 \pmod{4}$
- $p \equiv 3 \pmod{4}$

WORKSHEET #9

Theorem 57. *An odd prime is a sum of two squares if and only if it is congruent to 1 modulo 4.*

Theorem 58 (Sums of Two Squares Theorem). *A positive integer n is a sum of two squares if and only if: for every prime p such that $p \equiv 3 \pmod{4}$ and p divides n , the multiplicity of p in the prime factorization of n is even.*