Upper bounds and the completeness axiom

Let S be a set of real numbers.

- A number b is an *upper bound* for S provided for all $x \in S$ we have $b \geq x$.
- The set S is *bounded above* provided there exists at least one upper bound for S.
- A number m is the *maximum* of S provided
  (1) $m \in S$, and
  (2) m is an upper bound of S.
- A number ℓ is a *supremum* of S provided
  (1) ℓ is an upper bound of S, and
  (2) for any upper bound b for S, we have $\ell \leq b$.

(1) Write, in simplified form, the negation of the statement "b is an upper bound for S".

(2) Write, in simplified form, the negation of the statement "S is bounded above".

(3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
  (a) If $x > \ell$, what is the most concrete thing you can say about x and S?
  (b) If $x < \ell$, what is the most concrete thing[1] you can say about x and S?

(4) Let S be a set of real numbers, and set[2] $T = \{2s \mid s \in S\}$. Prove[3] that if S is bounded above, then T is bounded above.

[1] Hint: Use one of the previous problems.

[2] For example, if $S = \{-1, 1, 2\}$, then $T = \{-2, 2, 4\}$.

[3] First, before all else, this is an if then statement: start by assuming the "if" part. We now need to show the "then" part, which is about the existence of an upper bound. Use your assumption about S to find an upper bound for T (and prove that it is indeed an upper bound for T).