Let S be a set of real numbers.

- A number b is an *upper bound* for S provided for all $x \in S$ we have $b \geq x$.
- The set S is *bounded above* provided there exists at least one upper bound for S.
- A number m is the *maximum* of S provided
  (1) $m \in S$, and
  (2) m is an upper bound of S.
- A number ℓ is a *supremum* of S provided
  (1) ℓ is an upper bound of S, and
  (2) for any upper bound b for S, we have $\ell \leq b$.

(1) Write, in simplified form, the negation of the statement "b is an upper bound for S".

(2) Write, in simplified form, the negation of the statement "S is bounded above".

(3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
  (a) If $x > \ell$, what is the most concrete thing you can say about x and S?
  (b) If $x < \ell$, what is the most concrete thing[1] you can say about x and S?

(4) Let S be a set of real numbers, and let[2] $T = \{2s \mid s \in S\}$. Prove[3] that if S is bounded above, then T is bounded above.

(5) Let S be a set of real numbers. Show that if S has a supremum, then it is unique.

(6) Let S be a set of real numbers, and let $T = \left\{\frac{s}{2} \mid s \in S\right\}$. Directly[4] prove[5] that if S is unbounded above, then T is unbounded above.

[1] Hint: Use one of the previous problems.

[2] For example, if $S = \{-1, 1, 2\}$, then $T = \{-2, 2, 4\}$.

[3] First, before all else, this is an if then statement: start by assuming the "if" part. We now need to show the "then" part, which is about the existence of an upper bound. Use your assumption about S to find an upper bound for T (and prove that it is indeed an upper bound for T).

[4] Don't try to apply (4), just prove it directly, perhaps using your simplified description of "unbounded above" from (2). This also means don't take the contrapositive and don't proceed by contradiction.

[5] First, before all else, this is an if then statement: start by assuming the "if" part. We now need to show that T is unbounded above, which before everything else, is a for all statement. Use the hypothesis to get the number that you want. . .

WELL-ORDERING AXIOM: Every nonempty subset of $\mathbb{N}$ has a minimum.

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

(1) Prove the following:
THEOREM: For every real number r, there is some natural number n such that $n > r$.

(2) Prove the following:
THEOREM: For every[6] real number r, there exists a unique integer n such that $n - 1 \leq r < n$.

[6]Hint: First deal with the case $r \geq 0$.

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

THEOREM 4.1: For every real number r, there is some natural number n such that $n > r$.

Proof of Theorem 4.1:

(1) Proceed by contradiction: if not, then the set of natural numbers $\mathbb{N}$ is

____.

(2) Since $1 \in \mathbb{N}$, $\mathbb{N}$ is nonempty; thus, by the

____,

the set $\mathbb{N}$ has a

____;

call it ℓ.

(3) The real number $\ell - 1$ is less than $\sup(\mathbb{N})$, so

____.

(4) Adding one,

____.

(5) This contradicts

____.

(6) Thus

____.