MORE CONVERGENCE OF SEQUENCES

(1) For each of the following sequences which of the following adjectives apply: bounded above, bounded below, bounded, (strictly) increasing, (strictly) decreasing, (strictly) monotone?
   (a) $\{\frac{1}{n}\}_{n=1}^{\infty}$
   (b) The Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ where $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
   (c) $\{(-1)^n\}_{n=1}^{\infty}$
   (d) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$.

(2) Prove or disprove the converse[1] to Proposition 9.5.

Example 10.1:
(1) A constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.
(2) The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

Theorem 10.2 (Limits and algebra): Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M.
(1) If c is any real number, then $\{ca_n\}_{n=1}^{\infty}$ converges to cL.
(2) The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.
(3) The sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.
(4) If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{1}{L}$.
(5) If $M \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$.

(3) Use Theorem 10.2 and Example 10.1 to show that the sequence $\{2 + 5/n - 7/n^2\}_{n=1}^{\infty}$ converges to 2.

(4) Use Theorem 10.2 and Example 10.1 to show[2] that the sequence $\dfrac{2n+3}{3n-4}$ converges to $\dfrac{2}{3}$.

(5) Use Theorem 10.1 to show that if $\{a_n\}_{n=1}^{\infty}$ converges to L, and $\{b_n\}_{n=1}^{\infty}$ converges to M, then $\{a_n - b_n\}_{n=1}^{\infty}$ converges to $L - M$.

(6) Prove or disprove the following converse to part (2): If $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$ then $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{b_n\}_{n=1}^{\infty}$ converges to M.

(7) Prove part (1) of Theorem 10.2 in the special case $c = 2$ by following the following steps:
   - Assume that $\{a_n\}_{n=1}^{\infty}$ converges to L.
   - We now want to show that $\{2a_n\}_{n=1}^{\infty}$ converges to something. We know what we have to write next!
   - Now we do some scratchwork: we want an N such that for $n > N$ we have $|2a_n - 2L| < \varepsilon$. Factor this to get some inequality with a_n. How can we use our assumption to get an N that "works"?
   - Complete the proof.

(8) Prove[3] part (1) of Theorem 10.1.

(9) Prove[4] part (2) of Theorem 10.1.

(10) Prove[5] part (3) of Theorem 10.1.

[1] **Proposition 9.5:** Any convergent sequence is bounded.
[2] Hint: Divide the top and bottom by the same thing to get a sequence where part (5) above applies.
[3] Hint: You might deal with the cases $c = 0$ and $c \neq 0$ separately.
[4] Hint: The Triangle Inequality might be helpful in the form $|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M|$.
[5] Hint: You may need to use Proposition 9.5.