THEOREM 38.1: Let f be a function that is differentiable at $x = r$.

(1) If $f'(r) > 0$, then there is some $\delta > 0$ such that
- if $x \in (r, r+\delta)$ then $f(r) < f(x)$;
- if $x \in (r-\delta, r)$ then $f(x) < f(r)$.

(2) If $f'(r) < 0$, then there is some $\delta > 0$ such that
- if $x \in (r, r+\delta)$ then $f(r) > f(x)$;
- if $x \in (r-\delta, r)$ then $f(x) > f(r)$.

COROLLARY 38.2 (MIN-MAX THEOREM): Let f be a function that is continuous on a closed interval $[a, b]$. If f attains a maximum or minimum value on $[a, b]$ at $r \in (a, b)$, and f is differentiable at r, then $f'(r) = 0$.

(1) Find the values of x on $[0, 2]$ at which the function $f(x) = x^3 - x^2 - 2x$ achieves its minimum and maximum values. Justify your answer carefully using the results above.

(2) Explain why the Corollary follows from the Theorem.

(3) Use the Corollary to deduce: If f is continuous on the closed interval $[a, b]$, then f attains its minimum and maximum values at some value on the list
- $x = a$
- $x = b$
- some $x = r \in (a, b)$ with $f'(r) = 0$
- some $x = r \in (a, b)$ with $f'(r)$ undefined.

(4) Give examples of continuous functions on $[0, 2]$ such that
   (a) $f(x)$ attains its maximum at $x = 0$;
   (b) $g(x)$ attains its maximum at $x = 2$;
   (c) $h(x)$ attains its maximum at $x = 1$ and h is differentiable at $x = 1$;
   (d) $j(x)$ attains its maximum at $x = 1$ and j is not differentiable at $x = 1$.

(5) Prove part (1) of the Theorem:
- Consider the function $h(x) = \dfrac{f(x) - f(r)}{x - r}$. Apply the definition of limit to this function with $\varepsilon = f'(r)$. What does the definition give you?
- If $h(x) > 0$ and $x > r$, what can you say about $f(x) - f(r)$?
- If $h(x) > 0$ and $x < r$, what can you say about $f(x) - f(r)$?

(6) Prove part (2) of the Theorem.

(7) True or false: If $f'(7) > 0$, then $f(7.0000001) > f(7)$.

(8) True or false: If $f'(7) > 0$, then there exists some $N \in \mathbb{N}$ such that for all natural numbers $n > N$, $f\left(7 + \frac{1}{10^n}\right) > f(7)$.

DEFINITION 38.3: Let f be a function. We say that f attains a **local minimum** at r if there exists some $\delta > 0$ such that f is defined on $(r-\delta, r+\delta)$ and f achieves its minimum value on the interval $(r-\delta, r+\delta)$ at the input value $x = r$. We define **local maximum** analogously.

COROLLARY 38.4 (LOCAL MIN-MAX THEOREM): If f attains a local maximum or local minimum at $x = r$ and f is differentiable at $x = r$, then $f'(r) = 0$.

(9) Prove Corollary 38.4.

(10) Draw graphs of functions are continuous on the closed interval $[-3, -1]$ satisfying the following:
- (a) $f(x)$, that has a local maximum at -2 but does not attain its maximum at -2.
- (b) $g(x)$, such that $g'(-2) = 0$ but g does not attain a local minimum or local maximum at -2.
- (c) $h(x)$, such that h is not differentiable at -2 but h does not attain a local minimum or local maximum at -2.
- (d) $j(x)$ that has no local maximum at all on $[-3, -1]$.

(11) "THE ZEROTH DERIVATIVE TEST" Suppose that f is continuous on the closed interval $[a, b]$ and that there finitely many points $r_1 < r_2 < \cdots < r_{t-1}$ in (a, b) where either f' is zero or undefined. Set $r_0 = a$ and $r_t = b$.
- (a) Show that the maximum value of f on $[a, b]$ is $\max\{f(r_0), \cdots, f(r_t)\}$, and likewise the minimum value of f on $[a, b]$ is $\min\{f(r_0), \cdots, f(r_t)\}$.
- (b) Show that, for $i = 1, \ldots, t-1$, f attains a local maximum at r_i if and only if $f(r_{i-1}) < f(r_i) > f(r_{i+1})$; likewise, f attains a local minimum at r_i if and only if $f(r_{i-1}) > f(r_i) < f(r_{i+1})$.