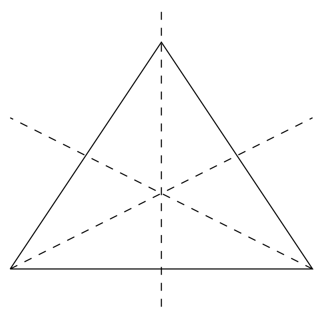
Dihedral Groups

Definition:

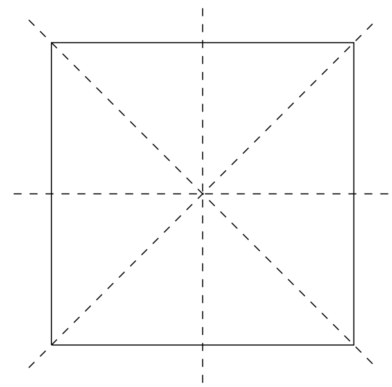
- A **isometry** of $\mathbb{R}^2$ is a bijective function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distances between pairs of points; examples include rotations around a point, translations, and reflections over a line.
- Let $X \subseteq \mathbb{R}^2$. A **symmetry** of X is an isometry of $\mathbb{R}^2$ such that $f(X) = X$ as a set.
- The **dihedral group** D_n is the group of symmetries of a regular n-gon P_n in the plane, with composition of functions as the group operation.

Theorem 1: The dihedral group D_n is indeed a group. It has exactly $2n$ elements consisting of:

- The identity map e,
- $n-1$ rotations $r, r^2, \ldots, r^{n-1}$, where r is counterclockwise rotation by $2\pi/n$ (so r^j is counterclockwise rotation by $2\pi j/n$),
- n reflections. More precisely,
  - for n is odd, there are n distinct reflections over lines between a vertex and opposite edge;
  - for n is even, there are $n/2$ distinct reflections over lines between opposite pairs of vertices, and another $n/2$ distinct reflections over lines between opposite pairs of edges.



The reflection lines in D_3



The reflection lines in D_4

(1) Why is the dihedral group a group? I.e., why are the group axioms true for this set and operation?

(2) What is the order of the rotation r? What is the order of a reflection in D_n?

(3) Proof of Theorem 1:

(a) Show that[1] if f is a symmetry of P_n and c is the center of P_n, then $f(c) = c$.

(b) Show that[2] if f is a symmetry of P_n and v is a vertex of P_n, then $f(v)$ is a vertex of P_n.

(c) Show that if f is a symmetry of P_n and v, v' are adjacent vertices, then $f(v)$ and $f(v')$ are adjacent vertices of P_n.

(d) Prove[3] the Theorem.

---

[1] Hint: After rescaling, we can assume that $d(c, v) = 1$ for any vertex v. Then observe that
(i) $d(c, x) \leq 1$ for any $x \in P_n$, and
(ii) if $q \in P_n$ is *not* the center, then $d(q, x) > 1$ for some $x \in P_n$.

[2] Hint: Again assume $d(c, v) = 1$ for any vertex v. Observe that v is a vertex of P_n if and only if $d(c, v) = 1$.

[3] You can use the following fact from geometry: if f, f' are two isometries of the plane, $p_1, p_2, p_3 \in \mathbb{R}^2$ are three points not on a line, and $f(p_i) = f'(p_i)$ for $i = 1, 2, 3$, then $f = f'$.

LEMMA: Let $v \in P_n$ be a vertex, and $s \in D_n$ the reflection through the axis containing v. Let $r \in D_n$ be counterclockwise rotation by $2\pi/n$. Then $srs = r^{-1}$.

THEOREM 2: Let $v \in P_n$ be a vertex, and $s \in D_n$ the reflection through the axis containing s. Let $r \in D_n$ be counterclockwise rotation by $2\pi/n$.

(1) Every element of D_n can be written uniquely in the form

$$r^j \quad \text{for } j = 0, \ldots, n-1, \qquad \text{or} \qquad r^j s \quad \text{for } j = 0, \ldots, n-1.$$

(2) D_n is generated by r, s.

(3) D_n has the group presentation $\langle r, s \mid r^n = e, s^2 = e, srs^{-1} = r^{-1} \rangle$.

(4) Show that the elements $r^j s$ for $j = 0, \ldots, n-1$ are n distinct reflections. Deduce Theorem 2(1) from this and Theorem 1.

(5) Use the Theorem 2(1) to prove Theorem 2(2).

(6) Every element can be written as r^j or $r^j s$; in particular, every element is a multiple of powers of r and s, and thus r, s generate.

(7) Prove the Lemma.

(8) Discuss Theorem 2(3).

---

(9) Consider a circle in the plane.
   (a) Compute the symmetry group G of the circle; give an answer in a similar form to Theorem 1.
   (b) What are all of the possible orders of elements in this group?
   (c) Find two elements of order 2 in G whose product has infinite order.
   (d) Does G has a finite generating set?