Contents

1. August 23, 2022

This class is, as its name makes clear, is all about differential equations. Let's start with an example that is probably similar to something you've seen in Calculus.

Example 1.1. The equation

$$\frac{dy}{dx} = 7y$$

is a differential equation. The unknown in this equation, y, stands for a function. What makes this equation a differential equation is that the equation relates the mystery function and its derivative.

Let's see if we can guess a solution. This equation might remind us of a curious calculus coincidence. If the 7 wasn't there, we would be looking for a function whose derivative is equal to itself; e^x would work.

Let's try $y = 7e^x$ for our original equation. To test it, we plug it in:

$$y = 7e^x \rightsquigarrow y' = (7e^x)' = 7e^x \neq 7y = 49e^x.$$

How about putting the 7 somewhere else:

$$y = e^{7x} \rightsquigarrow y' = (e^{7x})' = e^{7x}(7x)' = 7e^{7x} = 7y.$$

So e^{7x} is a solution!

Could there be any others?

$$y = 5e^{7x} \rightsquigarrow y' = (5e^{7x})' = 5e^{7x}(7x)' = 7(5e^{7x}) = 7y.$$

In general, $y(x) = Ce^{7x}$ is a solution for any constant C.

Of course, at the end of the day, nothing was special about 7. If we replaced 7 by any real number a, for the same reason, we would find that for the differential equation

$$y' = ay$$

the *general solution* is

$$y(x) = Ce^{ax}.$$

Guessing, while successful here, is not going to be our preferred method in the class. Let's savor this victory, and be prepared to collect many methods for solving differential equations as we progress through the course.

Types of differential equations (§1.1). There are many different ways of throwing together functions and derivatives in an equation, so we'll need some terminology to orient ourselves.

Definition 1.2. An *ordinary differential equation (ODE)* is a differential equation involving only one independent variable; i.e., derivatives with respect to just one variable.

For example,

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} = -y + \cos(ty)$$

is an ordinary differential equation.

In general an ODE is an equation of the form

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

for some function F where $y = y(t)$: an equation relating the function y with its derivative(s).

Definition 1.3. A *partial differential equation (PDE)* is a differential equation involving multiple independent variable; i.e., derivatives with respect to different variables.

For example,

$$\frac{\partial u}{\partial t} - 5\frac{\partial u}{\partial x} = 0$$

and

$$\frac{\partial^2 z}{\partial x \partial y} - z^2 = xy$$

are PDEs. A solution of the first PDE would be a function $u(x, t)$ that depends two independent variables x and t.

The "ordinary" vs "partial" refers to what type of derivatives see.

This is a class about ODEs. Almost all of the rest of the differential equations we see this semester will be ordinary!

Definition 1.4. The *order* of a differential equation is the highest order derivative that occurs in the equation.

For example,

$$yy'' + y''' + \frac{1}{y} = 5x$$

is a third order ODE, due to the y''' term and

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} = -y + \cos(ty)$$

is a second order ODE.

Definition 1.5. A *linear* ODE is any ODE of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t).$$

For example,

$$5ty'' + \ln(t)y' + y = \cos(t)$$

is a second order linear ODE, but

$$yy' + 5y = 7$$

and

$$(y')^3 - ty^2 = 3e^t$$

are first order nonlinear ODEs.

We will be especially interested in linear ODEs in this course!

Discussion Questions.

(1) Is the differential equation $y' = y^{2/3}$ ordinary? linear? What is its order?

> Ordinary yes, linear no, order 1.

(2) Which of the following is a solution to the differential equation $y' = y^{2/3}$:
   (a) $y = 8t^2$
   (b) $y = e^{2t/3}$
   (c) $y = \frac{1}{27}t^3$
   (d) $y = 0$ (constant function 0)

> (a) No: $y' = 16t \neq y^{2/3} = 4t^{4/3}$.
> (b) No: $y' = 2/3e^{2t/3} \neq y^{2/3} = (e^{2t/3})^{2/3} = e^{4t/9}$.
> (c) Yes: $y' = \frac{1}{9}t^2 = y^{2/3}$.
> (d) Yes: $y' = 0 = y^{2/3}$.

(3) There is a solution to $xy'' = (4x-4)y$ of the form $y = xe^{ax}$ for some real number a. Find a.

By the product rule,
$$y' = (ax+1)e^{ax} \quad \text{and} \quad y'' = (a^2x+2a)e^{ax},$$
so
$$xy'' - (4x-4)y = (a^2x^2+2ax)e^{ax} - (4x-4)xe^{ax}.$$
If this is zero, we must have
$$a^2x^2 + 2ax = 4x^2 - 4x$$
as functions of x, so $a = -2$.

(4*) If f, g are solutions to $y^{(3)} + 2e^x y^{(2)} - y = \cos(x)$, show that $\frac{f+g}{2}$ is too.

(5*) Using only calculus, justify the claim we made earlier that $y = Ce^{ax}$ is the general solution to $y' = ay$ for any $a \in \mathbb{R}$. That is, explain why there aren't any other solutions (exponential or otherwise).

Initial value problems. In our first example, we saw that there are many solutions to the differential equation $y' = 7y$. To pin one down, we might specify a value for our function at a point. The system
$$\begin{cases} y' = 7y \\ y(2) = 4 \end{cases}$$
is a example of an *initial value problem (IVP)*. Geometrically, $y(2) = 4$ corresponds to the condition that the graph of our solution passes through $(2,4)$.

2. August 25, 2022

Example 2.1.
$$\begin{cases} y' = 7y \\ y(2) = 4 \end{cases}$$
is a example of an *initial value problem*. Geometrically, $y(2) = 4$ corresponds to the condition that the graph of our solution passes through $(2,4)$.

We can solve this using our solution of $y' = 7y$ from earlier. We have
$$y = Ce^{7x} \qquad y(2) = 4$$

so

$$4 = Ce^{7\cdot 2}$$

and

$$C = 4e^{-14}.$$

That is,

$$y = 4e^{-14}e^{7x} = 4e^{7x-14}.$$

Modeling with differential equations (§1.3). Differential equations is one of the most useful areas of math for applications, since so many real life things are described effectively by differential equations. A *mathematical model* is a description of some system or phenomenon by an equation or a formula. A model is rarely perfect, since we can't even know all of the factors that might affect something, but we can often use them to understand things better.

Let's start with a basic example.

Example 2.2. A classical model of human population growth is based on the assumption that the rate at which the population of a country grows is proportional to the population of that country. To express this as a differential equation, let P the population of a country. We are interested in how it changes, so let t be a variable for time and view P as a function of t. To say that two things are *proportional* means that the exists a constant k (called the *constant of proportionality*) such that k times the first quantity is the second quantity. The rate of change of the population is $\frac{dP}{dt}$. Thus, our equation is

$$\frac{dP}{dt} = kP$$

for some constant k. We do not know what k is without further information.

This is the only differential equation we've solved! We must have

$$P(t) = Ce^{kt}.$$

Given two data points for any specific population, we can determine C and k.

Let us try to set up a more complicated model.

Example 2.3. Say that we have a tank of water. At first, it holds 500 liters of pure water (no salt). After switch is flipped, salt water that has 7 grams of salt per liter starts flowing in at a rate of 2 liters per minute, and water from the bottom of the tank starts flowing out at a rate of 2 liters per minute. Let's model the amount of salt in the tank at a given time after the switch is flipped.

Let A be the amount of salt in the tank, in grams, and t be the amount of time since the switch is flipped in minutes. We need to understand the rate of change of A. Salt enters at a rate of $7 \cdot 2 = 14$ grams per minute. To find the rate at which salt exits, the amount of salt in an average liter of water is $A/500$ grams per liter, so the amount of salt exiting is $2 \cdot A/500 = A/250$.

We obtain the differential equation

$$\frac{dA}{dt} = 14 - \frac{A}{250}.$$

We also have the initial condition $A(0) = 0$.

We will learn how to solve systems like this soon.

Discussion Questions. The government of a country wants to remove counterfeit money from circulation. Say there are 20 million total bills in circulation. Every day, 4 million of its bills pass though federal banks, and every counterfeit bill collected is replaced by a legal one. Say that half of the total bills in circulation today are counterfeit. Let's assume that the total number of bills in circulation stays constant and that no more counterfeit bills are being introduced. Our goal is to find an initial value problem modeling the percentage of counterfeit bills in circulation as time passes.

(1) Introduce variables to keep track of the quantities we are interested in. What is the independent variable and what is the dependent variable? What are the units for each?
(2) To set up a differential equation, we want to relate the dependent variable with its rate of change. On average, what is the change in the number of counterfeit bills each day[1]?
(3) Express the previous part as a differential equation.
(4) We also need an initial condition. Write it down.
(5) This is a type of differential equation we've solved already. Find an explicit solution.
(6) Based on your model, when will the total number of counterfeit bills pass below 3 million?

> (1) Take t for time (number of days after today), and C to be the number of millions of counterfeit bills in circulation. C is dependent on t.

[1]Hint: First figure out how many counterfeit bills pass through federal banks each day, on average.

(2) First the number of bills that pass through banks on an average day is $4C/20 = C/5$: the proportion of counterfeit bills times the total number of millions of bills passing through the banks. Thus, C decreases by $C/5$ on average each day.
(3) $C' = -C/5$.
(4) $C(0) = 10$.
(5) $C(t) = ke^{-t/5}$ is the general solution. We plug in $C(0) = 10$ to get $k = 10$, so $C(t) = 10e^{-t/5}$.
(6) $C(t) = 3$ gives us $10e^{-t/5} = 3$. Then $t = -5\ln(3/10) \approx 6$. It should take about 6 days.

Let's do an experiment to test our model. Each coin you've been given represents a million bills. Some represent valid coins and some represent counterfeits. Every day, take four random coins; replace the counterfeits with legal ones, and leave the legal ones alone.

(1) Discuss whether your model for the previous situation is relevant to this experiment. What aspects fit the story well, and what ones don't?
(2) Run the experiment, keeping track of the number of counterfeit bills each day, and how long it takes to get down to 3 counterfeits. Even better, run the experiment a few times.

Now let's change our original story. As before, every day, 4 million of its bills pass though federal banks, and every counterfeit bill collected is replaced by a legal one. Say that half of the 20 million total bills in circulation today are counterfeit. But now, let's assume that 1 million new legal bills and 1 million new counterfeit bills are put into circulation each day.

(1) Create a new differential equation and initial value problem to model this situation[2].
(2) Run an experiment similar to the one above adapted to this situation.
(3) Based on the experiment, what do we expect to happen to the currency as time passes?

[2]Hint: You might find it helpful to write a closed formula for the number of total bills in circulation at a given time first.

> (1) Let's use t and C as names again. The total number of bills at time t is now $20+2t$. Now $C'(t) = 1 - \frac{4C}{20+2t} = 1 - \frac{2C}{10+t}$, and $C(0) = 10$ again.

3. August 30, 2022

We will now spend a while focusing on first-order ODEs and corresponding initial value problems: all of Section 1.2 and Chapter 2 will be about this setting.

Existence and uniqueness for initial value problems (§1.2). If our goal in solving an initial value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

is to find *the* function that satisfies the two conditions, for this goal to make sense, there should be a function, and only one function that satisfies the two conditions. The *existence* question for an IVP is whether there is any function that satisfies both conditions; the *uniqueness* question for an IVP is whether there is at most function that satisfies both conditions. (Actually *finding* this function is a different question.)

Let's consider these things in some examples we have seen before.

Example 3.1. In Example 2.1, we considered the IVP

$$\begin{cases} y' = 7y \\ y(2) = 4 \end{cases}$$

and saw that the function $y = 4e^{7x-14}$ was the one and only solution. Thus, there exists a unique solution in this case.

On the other hand, we have the following.

Example 3.2. Consider the IVP

$$\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases} .$$

We saw in the Discussion Questions from Aug 25 that $y = 0$ and $y = \frac{1}{27}t^3$ both satisfy the first differential equation. Both of these functions also satisfy the initial condition, so they are also both solutions to the IVP. Here we have an IVP for which the solution is not unique.

We can even have no solution sometimes.

Example 3.3. Consider the IVP

$$\begin{cases} y' = \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0 \end{cases} \\ y(0) = 0 \end{cases}.$$

This IVP has no solution, no matter how small of an interval! (Challenge: why not?)

Luckily, there is a theorem that guarantees existence and uniqueness of solutions IVP's under certain hypotheses.

Theorem 3.4 (Picard-Lindelöf)**.** *For the IVP*

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

and some rectangle R in the (x,y)-plane containing (x_0, y_0) in its interior, there exists a unique solution on some possibly smaller interval $(x_0 - h, x_0 + h)$, so long as f and $\frac{\partial f}{\partial y}$ are continuous on R.

There's a lot of fine print, but here is the upshot: There exists a unique solution to the IVP

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

near x_0, so long as f and $\frac{\partial f}{\partial y}$ are continuous around (x_0, y_0).

Example 3.5. Consider the differential equation

$$\frac{dy}{dx} = 5xy.$$

We want to use the Picard-Lindelöf Theorem to show that for any initial condition $y(x_0) = y_0$, there is a unique solution near x_0. The $f(x,y)$ of the Theorem is $5xy$; this is continuous on all of $\mathbb{R}^2$. We also need to look at $\frac{\partial f}{\partial y} = 5x$. This is also continuous on all of $\mathbb{R}^2$. We conclude that

$$\begin{cases} \frac{dy}{dx} = 5xy \\ y(x_0) = y_0 \end{cases}$$

has a unique solution, no matter what x_0 and y_0 are.

Example 3.6. Let's consider

$$\begin{cases} y' = y^{2/3} \\ y(x_0) = y_0 \end{cases}.$$

Here $f(x,y) = y^{2/3}$ and $\frac{\partial f}{\partial y}(x,y) = \frac{2}{3}y^{-1/3}$. f is continuous everywhere, but f' is only continuous where $y_0 \neq 0$. Thus, if $y_0 \neq 0$, then there is a unique solution.

However, if $y_0 = 0$, the theorem does not apply. We looked at this example earlier and saw that the solutions were not unique for $(x_0, y_0) = (0,0)$.

Example 3.7. Let's consider

$$\begin{cases} y' = \frac{y^2}{t^2-4} \\ y(-1) = 3 \end{cases} .$$

What is the largest interval on which the Picard-Lindelöf Theorem guarantees the existence of a unique solution? We have, in the notation of the Theorem,

$$f = \frac{y^2}{t^2-4} \qquad \frac{\partial f}{\partial y} = \frac{2y}{t^2-4}.$$

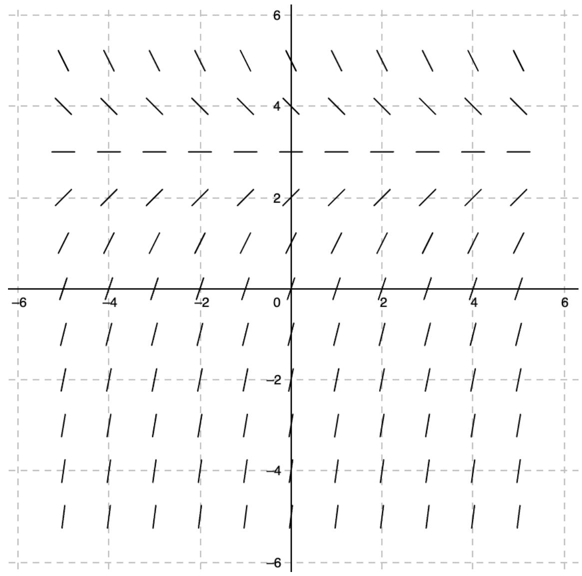
These are continuous except when $t = \pm 2$. Thus, any rectangle whose base is contained in $(-2,2)$ will satisfy the hypotheses of the theorem, so $(-2,2)$ is the interval we seek.

Solution curves from slope fields (§2.1). There is a great way to visualize solutions to differential equations without solving them. The idea is to think of $\frac{dy}{dt}$ geometrically as the slope of the graph of y.

Example 3.8. Consider the differential equation

$$y' = 3 - y.$$

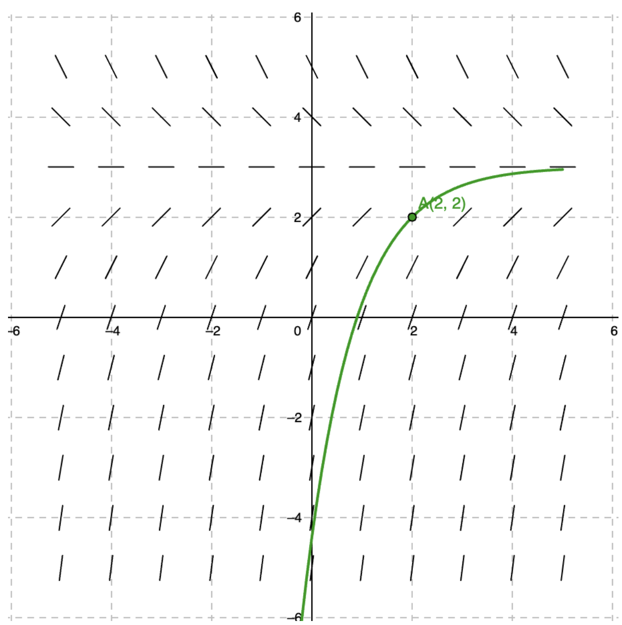
We won't try to write down a solution yet. Instead, on the plane, we draw little lines with slope $3 - y$ at various points.



We can use this to sketch the solution to the IVP

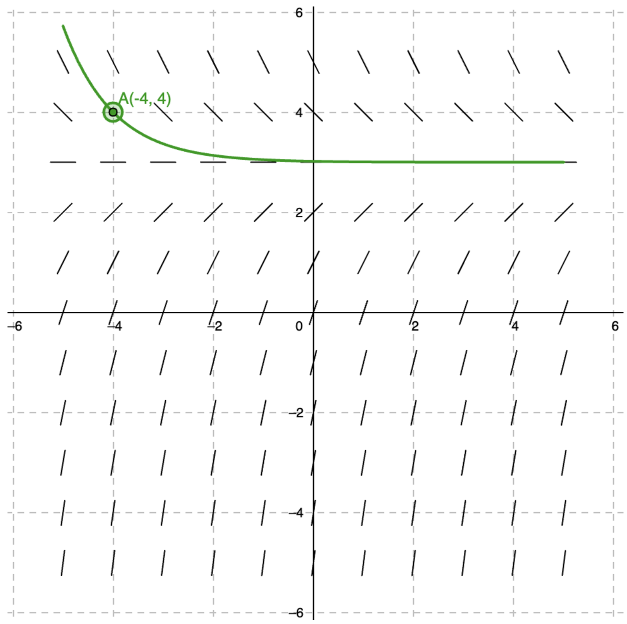
$$\begin{cases} y' = 3 - y \\ y(2) = 2 \end{cases}$$

by starting at $(2, 2)$ and going along with the flow.



Or

$$\begin{cases} y' = 3 - y \\ y(-4) = 4 \end{cases} \quad :$$



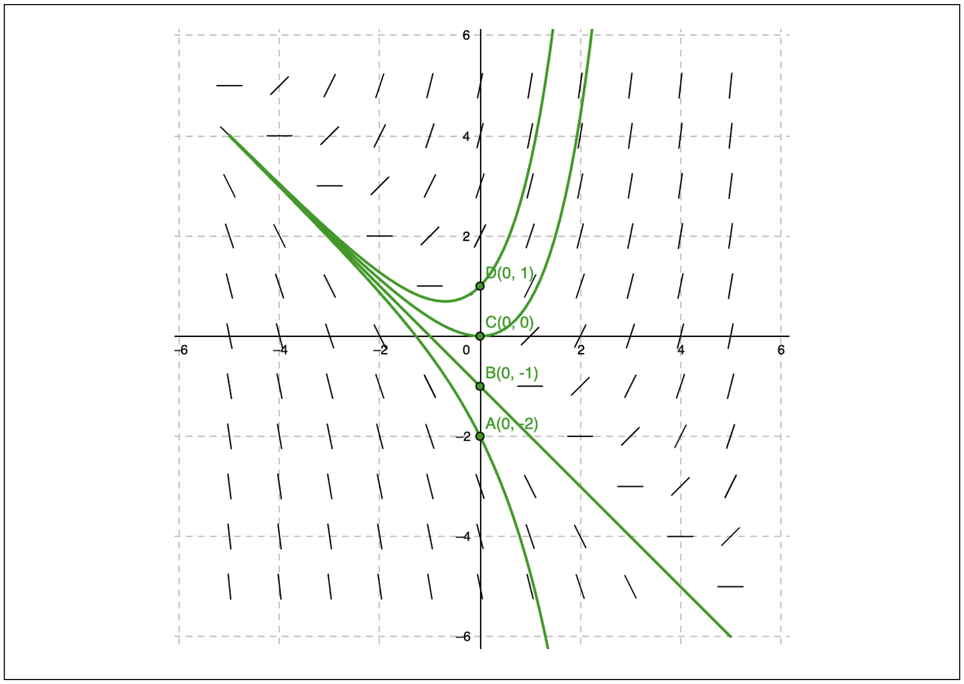
The picture we drew above is called a *slope field.*

Discussion Questions. Draw a slope field for the differential equation

$$y' = x + y$$

and use it to sketch the solutions with initial conditions

(1) $(0, 1)$
(2) $(0, 0)$
(3) $(0, -1)$
(4) $(0, -2)$

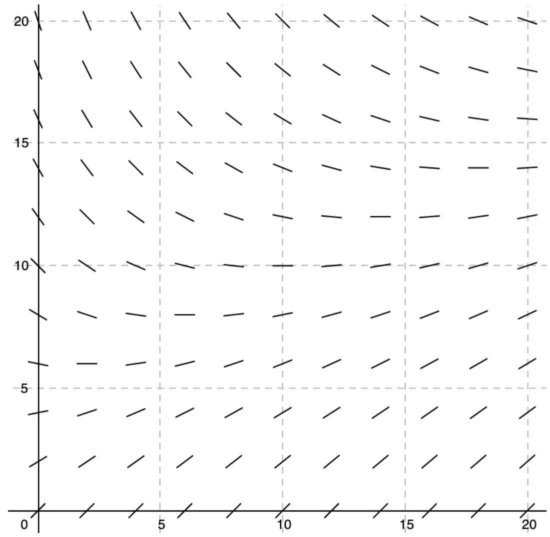


4. September 1, 2022

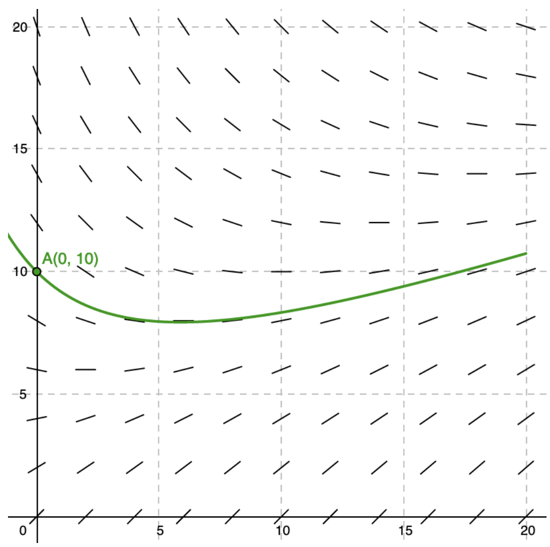
Example 4.1. As we discussed, we can use slope fields to try to understand solutions of an IVP without finding a formula for an explicit solution. Let's return to our counterfeiting example (where new good and bad bills were being introduced, and the banks replaced bad ones with good ones). We found the following IVP for our situation:

$$\begin{cases} C'(t) = 1 - \frac{2C}{10+t} \\ C(0) = 10 \end{cases} .$$

We wanted to know whether the number of counterfeits would continue to grow (and if so, how fast). Let's sketch a slope field for our differential equation:



and sketch a curve for the initial condition:



It looks like the number of counterfeits decreases at first and maybe starts to rebound. We will solve this analytically very soon to confirm our guess.

Autonomous differential equations. Example 3.8 belongs to a class of equations that is worth singling out.

Definition 4.2. A differential equation of the form

$$y' = f(y)$$

is said to be *autonomous*.

The behavior of a solution of an autonomous differential equation is determined entirely by the current value of the function (and not the independent variable).

For an autonomous differential equation, whether a solution is increasing, decreasing, or constant at a point only depends on the y-value at that point. We can determine this either algebraically or using the slope field.

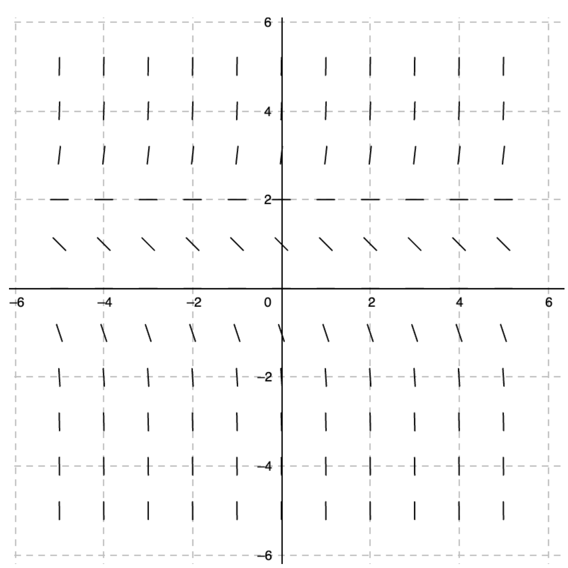
Example 4.3. Consider the autonomous differential equation

$$y' = y^3 - 2y^2.$$

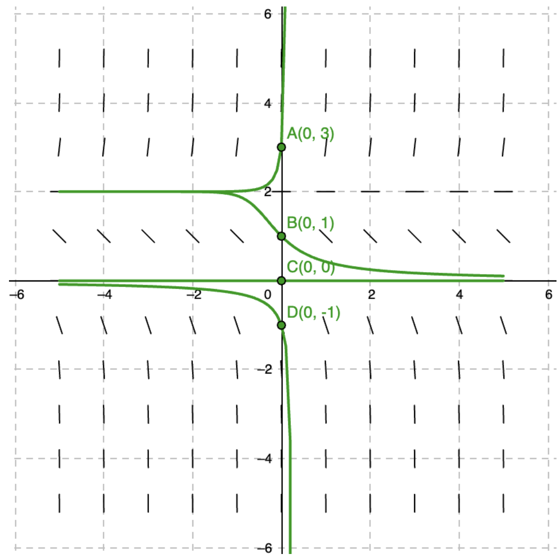
To figure out when y is increasing, we figure out when $0 < y' = y^3 - 2y^2$. We do a little algebra to figure out. Factoring, we see that the zeroes of the right-hand side are 0 and 2, so the sign of y can only change at 0 and 2. On $(-\infty, 0)$, $y^3 - 2y^2 < 0$; at 0, it is zero; on $(0, 2)$ it is negative; at 2, it is 0; on $(2, \infty)$, it is positive.

Thus, a solution y is increasing when $2 < y < \infty$, is decreasing on $(-\infty, 0) \cup (0, \infty)$, and is constant if $y = 0$ or 2.

We can also use the slope field to determine when it is increasing or decreasing.



Let's sketch some solutions:



A constant solution to an autonomous differential equation is also called an *equilibrium solution*.

Review of integration. Now we will start to collect some techniques for solving differential equations. Of course integration will play a big role. Let's review some basic techniques of integration. We will usually be using *indefinite integration* or *antidifferentiation* in this class. First, let's recall a list of building block functions whose integrals we need to know.

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$.
- $\int \frac{1}{x}\,dx = \ln|x| + C$.
- $\int e^x\,dx = e^x + C$.
- $\int \sin(x)\,dx = -\cos(x) + C$.
- $\int \cos(x)\,dx = \sin(x) + C$.

Here are a couple more special ones.

- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$.
- $\int \frac{dx}{1+x^2} = \arctan(x) + C$.

We also have many rule for how to integrate complicated functions in terms of integrating smaller parts. There are two easy rules to get us started.

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$.
- $\int cf(x)\,dx = c\int f(x)\,dx$ for any constant c.

Unlike with derivatives, there is no direct rule for computing the integral of a composition or a product. Instead, we have "u-substitutions" and "integration by parts".

u-substitution is a technique that is likely to work when we can see some function and something like its derivative inside the function we are trying to integrate. In this case, we set the function to be u, we set $du = u'(x)\,dx$, and we try to rewrite our integrand as some function of u times du (and get rid of our starting variable entirely). u-substitutions also work well when instead of some function of x, we have a function of $x - a$ or a function of ax for some constant a. This is the main technique that we will want to use, other than the basic rules.

Example 4.4. To compute $\int x^2 \cos(x^3)\,dx$, take $u = x^3$, so $du = 3x^2\,dx$. Then we have $x^2\,dx = \frac{1}{3}\,du$, so our integral is

$$\int \frac{1}{3}\cos(u)\,du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3) + C.$$

Integration by parts is a technique that might work when our integrand is a product of two things, one of which we know how to integrate (i.e., looks like the derivative of something). The general rule is if we can find functions $u(x), v(x)$ such that our integral is $\int u\,dv$ (where $dv = v'(x)\,dx$), the we have

$$\int u\,dv = uv - \int v\,du;$$

and if we can integrate $\int v\,du$, we are done. A rule of thumb for what to choose for u vs what to choose for dv in integration by parts is LIATE (log, inverse trig, algebraic, trig, exponential): stuff on the left is usually better for u's and stuff on the right is usually better for dv's.

Example 4.5. To compute $\int xe^x\,dx$, take $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. Then we have

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = (x-1)e^x + C.$$

Last but not least, we have a big table of integrals in the front cover of our text.

5. September 6, 2022

Separable first-order equations (§2.2). Now we are ready to solve some differential equations! First we will address first-order equations of a special form: separable eqautions.

Definition 5.1. A first-order ODE is *separable* if it can be written in the form

$$y' = f(x)g(y)$$

for some function f that only involves the independent variable and some function g that only involves the dependent variable.

For example,

$$y' = x^2/y, \quad y' = \frac{3\sqrt{y^2+1}}{\cos(x)}, \quad \text{and } y' = 3 - y$$

are all separable (with $f(x) = x^2$ and $g(y) = 1/y$ in the first one), but

$$y' = x + y \quad \text{and} \quad y' = \sin(xy)$$

are not. Note that a separable equation may or may not be linear, but is always a first order ODE.

Here's how to solve a separable ODE: Write y' as $\frac{dy}{dx}$ and pretend that the dy and dx are separate things. Take

$$\frac{dy}{dx} = f(x)g(y)$$

and multiply by dx and divide by dy on both sides to get

$$\frac{dy}{g(y)} = f(x)\,dx.$$

(This is why we call the equation separable: the independent and dependent variables now occur on separate sides of the equation.) Now integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx,$$

to get an equation for x and y that is no longer differential!

Slogan 5.2. *To solve a separable equation, separate and integrate!*

Example 5.3. Let's start with an equation we saw in Example 3.8:

$$y' = 3 - y.$$

Write this as

$$\frac{dy}{dx} = 3 - y$$

and rearrange to get

$$\frac{dy}{3-y} = dx.$$

Integrate both sides. For the LHS with use the u-sub $u = 3 - y$, so $du = -dy$. The integral becomes

$$\int \frac{dy}{3-y} = \int \frac{-du}{u} = -\ln|u| + C = -\ln|3-y| + C = \ln\frac{1}{|3-y|} + C.$$

We then get

$$\ln \frac{1}{|3-y|} = \int \frac{dy}{3-y} = \int dx = x + C.$$

Note that we only need the constant of integration on one side. Then to solve for y, take exponentials:

$$e^{\ln(\frac{1}{|3-y|})} = e^{x+C},$$

$$\frac{1}{|3-y|} = e^{x+C}.$$

Then

$$|3-y| = e^{-x-C}$$

$$3-y = \pm e^{-x-C}$$

$$y = 3 \pm e^{-x-C}$$

Since $e^{-x-C} = e^{-x}e^{-C}$ is just some other constant C' times e^{-x}, we could also write

$$y = 3 + C'e^{-x}.$$

Example 5.4. Take

$$\frac{dy}{dt} = t + ty^2.$$

It doesn't look separable yet, but once we factor

$$\frac{dy}{dt} = t(1+y^2),$$

it becomes clear that it's separable. Separate variables:

$$\frac{dy}{1+y^2} = t\,dt,$$

integrate:

$$\int \frac{dy}{1+y^2} = \int t\,dt,$$

$$\arctan(y) = \frac{t^2}{2} + C,$$

and solve for y by taking tangent of both sides:

$$y = \tan(\arctan(y)) = \tan\left(\frac{t^2}{2} + C\right).$$

This is our general solution!

Discussion Questions.

(1) Which of the following equations is separable?
   (a) $y' + 2y = x^3$.
   (b) $\cos(x)y' = \sin(y)$.
   (c) $y' - xy = 1 + x + y$.

(b) and (c), but not (a)

(2) Find the particular solution to the IVP

$$\begin{cases} e^x\, y' = \frac{1}{y} \\ y(0) = 1 \end{cases}.$$

This is separable since we rewrite as $y' = e^{-x}\frac{1}{y}$. Then

$$yy' = e^{-x} \rightsquigarrow y\ dy = e^{-x}\ dx \rightsquigarrow \int y\ dy = \int e^{-x}\ dx,$$

$$\rightsquigarrow \frac{y^2}{2} = -e^{-x} + C \rightsquigarrow y = \sqrt{C - 2e^{-x}},$$

and using the initial condition,

$$1 = y(0) = \sqrt{C - 2e^0} = \sqrt{C - 2} \rightsquigarrow C = 3,$$

so we have

$$y = \sqrt{3 - 2e^{-x}}.$$

Sometimes we encounter integrals that are just impossible:

Linear equations (§2.3). Now we solve another large class of differential equations: first order linear ODE's. Let's recall that a differential equation is a first order linear ODE if we can write it in the form

$$a_1(t)y' + a_0(t)y = f(t)$$

for some functions $a_1(t), a_0(t), f(t)$ that only involve the independent variable t.

We are going to learn a magic trick to solve these: this really is a rabbit-in-the-hat idea. To motivate this wacky idea, I want to consider a couple of small examples to get the idea.

Example 5.5. Consider the differential equation

$$ty' + y = t^2.$$

After trying a few things, we realize this isn't separable. However, the left-hand side, as written, is interesting. Notice that, by the product rule, we have

$$(ty)' = ty' + t'y = ty' + y.$$

Thus, we have

$$\begin{aligned} (ty)' &= t^2, \\ \rightsquigarrow \quad ty &= \int (ty)'\,dt = \int t^2\,dt = \frac{t^3}{3} + C, \end{aligned}$$

so

$$y = \frac{t^2}{3} + \frac{C}{t}.$$

That was lucky! What if we had

$$y' + \frac{3}{t}y = t$$

instead? This is still not separable. I will magically multiply by t^3 to get

$$t^3y' + 3t^2y = t^4.$$

By reverse product rule on the left-hand side, we have

$$(t^3y)' = t^4.$$

Now we integrate to get

$$t^3y = \int t^4\,dt = \frac{t^5}{5} + C,$$

so

$$y = \frac{t^2}{5} + \frac{C}{t^3}.$$

This is the idea we will use: multiply our equation through by something to turn it into an equation where the left-hand side comes from the product rule! But how do we come up with the magic multiplier?

6. September 8, 2022

Idea 6.1. Given a linear first order ODE of the form

$$y' + p(t)y = q(t),$$

multiply by an *integrating factor* $\mu(t)$ so the left-hand side collapses via the product rule:

$$\begin{array}{rcccl} & \mu(t)y' & + & \mu(t)p(t)y & = \mu(t)q(t) \\ (?(t)y)' = & ?(t)y' & + & ?'(t)y & \end{array}.$$

The mystery function $?(t)$ has to be the same as our integrating factor μ, and we also have to have

$$\mu' = \mu p(t).$$

This is separable now: we solve

$$\frac{\mu'}{\mu} = p(t)$$

by integrating

$$\ln|\mu| = \int p(t)\,dt,$$

and isolating μ

$$\mu = \pm e^{\int p(t)\,dt}.$$

Let's take the positive one: Set $\mu = e^{\int p(t)\,dt}$.

Definition 6.2. Given a linear first order ODE of the form

$$y' + p(t)y = q(t),$$

we call the function

$$\mu = e^{\int p(t)\,dt}$$

the *integrating factor* of the equation. For the integrating factor, we can take just one antiderivative (i.e., ignore the constant of integration), since this is just something we're using to get a solution and not a solution.

Slogan 6.3. *For the linear first order ODE*

$$y' + p(t)y = q(t),$$

multiply by the integrating factor

$$\mu = e^{\int p(t)\,dt},$$

realize the left-hand side as $(\mu y)'$*, and integrate.*

Then we multiply by the integrating factor, realize the left-hand side as a result of the product rule, and integrate to solve.

Example 6.4. Let's solve the ODE that arose in our model of counterfeit currency from Aug 25. We found the equation

$$C'(t) = 1 - \frac{2C}{10+t}$$

which we can write as

$$C' + \frac{2}{10+t}C = 1.$$

The integrating factor is $\mu = e^{\int \frac{2}{10+t}\,dt}$. We should simplify this. Set $u = 10+t$, so $du = dt$, then

$$\int \frac{2}{10+t}\,dt = \int \frac{2}{u}\,du = 2\ln|u| = \ln((10+t)^2)$$

so

$$\mu = e^{\ln((10+t)^2)} = (10+t)^2.$$

Now we multiply by this integrating factor:

$$(10+t)^2 C' + (10+t)^2 \frac{2}{10+t} C = (10+t)^2$$

$$\rightsquigarrow (10+t)^2 C' + 2(10+t)C = (10+t)^2.$$

Now we recognize the left-hand side as coming from the product rule:

$$\big((10+t)^2 C\big)' = (10+t)^2 C' + (10+t)^2 \frac{2}{10+t} C,$$

so we have

$$\big((10+t)^2 C\big)' = (10+t)^2.$$

Now we integrate:

$$(10+t)^2 C = \int (10+t)^2 \; dt.$$

Use the sub $u = 10+t$, $du = dt$ to get

$$\int u^2 \; du = \frac{u^3}{3} + k = \frac{(10+t)^3}{3} + k.$$

We used k for the constant of integration since the name C is taken! Thus,

$$(10+t)^2 C = \frac{(10+t)^3}{3} + k$$

$$\rightsquigarrow C = \frac{10+t}{3} + \frac{k}{(10+t)^2}.$$

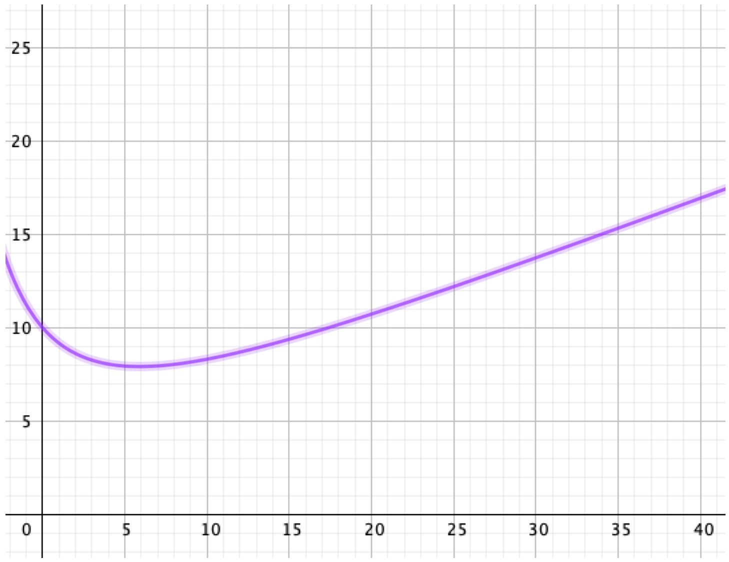
We also recall our initial condition $C(0) = 10$ and plug it in to get

$$10 = C(0) = \frac{10+0}{3} + \frac{k}{(10+0)^2} = \frac{10}{3} + \frac{k}{100}$$

and solve to get $k = \frac{2000}{3}$. Thus,

$$C(t) = \frac{10+t}{3} + \frac{2000}{3(10+t)^2}.$$

We can now graph this in a calculator:



and based on the formula, we know that for $t \gg 0$, the second term goes to zero, and the first term gets larger and larger. We conclude that the number of counterfeits will continue to grow in the long run!

Example 6.5. Let's solve

$$xy' = (x+3)y = x^2 e^{-x}.$$

We need to put it in the correct form first, so divide by x:

$$y' + (1 + \frac{3}{x})y = xe^{-x}.$$

In the earlier notation, $p(x) = (1 + \frac{3}{x})$. The integrating factor is

$$\mu(x) = e^{\int (1+\frac{3}{x})\,dx} = e^{x+3\ln x} = e^x e^{\ln x^3} = e^x x^3 = x^3 e^x.$$

Multiply by $\mu(x)$:

$$x^3 e^x y' + (1 + \frac{3}{x})x^3 e^x = (x^3 e^x)(xe^{-x})$$

and realize LHS as coming from product rule

$$(x^3 e^x y)' = x^4.$$

Now we integrate:

$$x^3 e^x y = \frac{x^5}{5} + C$$

$$y = \frac{x^2}{5} e^{-x} + \frac{C}{x^3} e^{-x}.$$

Discussion Questions. Find the general solution of the differential equation
$$y' = x + y.$$

Rearrange as $y' - y = x$. Then the integrating factor is $e^{\int -1\,dx} = e^{-x}$. Multiply through and do the product rule backwards to get
$$(e^{-x}y)' = e^{-x}y - e^{-x}y' = xe^{-x}.$$
We now integrate both sides. On the right, we can use integration by parts with $u = x$, $dv = e^{-x}\,dx$, $du = dx$, $v = -e^{-x}$. We get (leaving aside the constant of integration)
$$\begin{aligned}\int xe^{-x}\,dx &= -xe^{-x} - \int (-e^{-x})\,dx \\ &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} \\ &= -(x+1)e^{-x}\end{aligned}$$
so
$$e^{-x}y = -(x+1)e^{-x} + C.$$
Then multiply through to get
$$y = -(x+1) + Ce^{x}.$$

7. September 13, 2022

Let's do one more example of solving a first-order linear equation with integrating factors.

Example 7.1. Let's find the general solution to
$$x^2y' - y = 5.$$
First, we put it in the form from which we can read off the integrating factor:
$$y' - \frac{1}{x^2}y = \frac{5}{x^2}.$$
Now, the integrating factor is
$$\mu(t) = e^{\int -1/x^2\,dx} = e^{1/x}.$$
Multiply through to get
$$e^{1/x}y' - e^{1/x}x^2y = \frac{5e^{1/x}}{x^2}.$$

We realize the left-hand side as $(e^{1/x}y)'$ (and we double-check it using the product rule and the chain rule):

$$(e^{1/x}y)' = \frac{5e^{1/x}}{x^2}.$$

Thus

$$e^{1/x}y = \int \frac{5e^{1/x}}{x^2}\,dx = -5\int e^u\,du = -5e^{1/x} + C.$$

using the u-substitution $u = e^{1/x}$, $du = -e^{1/x}/x^2\,dx$. Finally, we get

$$y = -5 + Ce^{-1/x}$$

as the general solution.

Euler's Method (§2.6). We now have great methods to solve certain first order ODEs, namely separable and linear ones. But this doesn't encompass all of them. There is a way to solve any IVP for first order ODE of the form

$$\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases},$$

at least approximately. The idea is that

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = \frac{\text{small change in } y}{\text{small change in } t},$$

which we can rewrite as

$$\Delta y \cong \frac{dy}{dt}\,\Delta t.$$

So, to approximate our solution, we start at our initial value $y(t_0) = y_0$, keep adding small amounts Δt to t, and each time we add

$$\frac{dy}{dt}\,\Delta t = f(t,y)\,\Delta t$$

to our y-value. This is the idea behind *Euler's method.*

More concretely, we fix a *step size* h, which plays the role of our small change in t that we called Δt. This should be a small positive number. (Though how small is a good choice can be difficult to pin down.)

We start with the values t_0 and y_0 from the initial condition, and make a list a t-values $t_0, t_1, t_2, t_3, \ldots$ given by the rule

$$t_n = t_0 + nh.$$

For the corresponding y-values, we go from one to the next by the rule

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Example 7.2. Take the IVP

$$\begin{cases} y' = x + y^2 \\ y(0) = 0 \end{cases}.$$

Let's use Euler's method with step size 1 to approximate a solution. We start with $x_0 = 0$ and $y_0 = 0$. Then

$$x_1 = 1, \quad y_1 = 0 + (0 + 0^2)1 = 0$$

$$x_2 = 2, \quad y_2 = 0 + (1 + 0^2)1 = 1 \dots$$

$$x_3 = 3, \quad y_3 = 1 + (2 + 1^2)1 = 4$$

$$x_4 = 4, \quad y_4 = 4 + (3 + 4^2)1 = 23 \dots$$

Let's also try step size $h = .5$.

$$x_1 = .5, \quad y_1 = 0 + (0 + 0^2).5 = 0$$

$$x_2 = 1, \quad y_2 = 0 + \big(.5 + 0^2\big).5 = .25$$

$$x_3 = 1.5, \quad y_3 = .25 + \big(1 + (.25)^2\big).5 = .78125$$

$$x_4 = 2, \quad y_4 = .78125 + \big(1.5 + (.78125)^2\big).5 \approx 1.836425$$

Note that the different step sizes gave different answers for $y(2)$. In general, smaller step sizes give better approximations, but take longer to compute.

Discussion Questions. Use Euler's method with step size $h = 0.5$ to approximate a solution to

$$\begin{cases} y' = -ty^2 \\ y(0) = 1 \end{cases}$$

up to $t = 2$.

$$t_0 = 0, \quad y_0 = 1.$$
$$t_1 = .5, \quad y_1 = 1 + (-0 \cdot 1^2).5 = 1.$$
$$t_2 = 1, \quad y_2 = 1 + (-.5 \cdot 1^2).5 = .75$$
$$t_3 = 1.5, \quad y_3 = .75 + (-1 \cdot .75^2).5 = .46875$$
$$t_4 = 2, \quad y_4 = .46875 + (-1.5 \cdot .46875^2).5 \approx .30395$$

Linear models (§3.1). We now want to apply our ability to solve many first-order ODEs arising from real world situations, like the ones we saw in the first couple weeks.

Example 7.3. Newton's law of cooling says that the rate at which an object cools/heats up is proportional to the difference between its temperature and the temperature of its surroundings. First, let's state Newton's law of cooling as a general model with a differential equation.

Take T to be the temperature of the heating or cooling object. We are thinking of this as changing in time, so let t denote time, and think of T as a function of t. Also, the temperature of the surroundings plays a role, so take T_0 to the be ambient temperature. Then the rule is

$$\frac{dT}{dt} = k(T - T_0)$$

where k is some constant of proportionality.

8. September 15, 2022

Let's now apply it to a specific situation. Say we place a cake in a 400°F oven. At first, the cake is 70°F and after 10 minutes it is 100°F. How long will it take for the cake to reach 150°F?

Based on our story, $T_0 = 400$. We find the general solution of our differential equation

$$\frac{dT}{dt} = k(T - 400).$$

This is separable and linear, so we can solve it two different ways. Let's use an integrating factor.

$$\begin{gathered}
T' - kT = -400k \\
\mu = e^{\int -k\,dt} = e^{-kt} \\
\rightsquigarrow e^{-kt}T' - ke^{-kt}T = -400ke^{-kt} \\
\rightsquigarrow (e^{-kt}T)' = -400ke^{-kt} \\
\rightsquigarrow e^{-kt}T = \int -400ke^{-kt}\,dt = -400k\int e^{-kt}\,dt = \frac{-400k}{-k}e^{-kt} + C = 400e^{-kt} + C \\
\rightsquigarrow T = 400 + Ce^{kt}.
\end{gathered}$$

We now have to find C and k for a particular function of T. Plug in $T(0) = 70$ to get

$$\begin{gathered}
70 = T(0) = 400 + Ce^0 = 400 + C \\
\rightsquigarrow C = 70 - 400 = -330,
\end{gathered}$$

so

$$T = 400 - 330e^{kt}.$$

Then, plug in $T(10) = 75$ to get

$$100 = 400 - 330e^{10k}$$
$$\rightsquigarrow -300 = -330e^{10k}$$
$$\rightsquigarrow \frac{10}{11} = e^{10k}$$
$$\rightsquigarrow 10k = \ln(\frac{10}{11})$$
$$\rightsquigarrow k = \ln(\frac{10}{11})/10$$

so

$$T = 400 - 330e^{t(\ln\frac{10}{11})/10}.$$

Now we can set $T(t) = 150$ and solve

$$150 = 400 - 330e^{t(\ln\frac{10}{11})/10}$$
$$\rightsquigarrow -250 = -330e^{t(\ln\frac{10}{11})/10}$$
$$\rightsquigarrow e^{t(\ln\frac{10}{11})/10} = 25/33$$
$$\rightsquigarrow t(\ln\frac{10}{11})/10 = \ln(25/33)$$
$$\rightsquigarrow t = 10\ln(\frac{25}{33})/\ln(\frac{10}{11}) \approx 29.$$

Example 8.1. We can also come up with explicit solutions for the mixing models of the type we considered earlier. Say that we have a tank of water with 300 gallons of fresh water right now. Water starts flowing out at a rate of 3 gallons per minute, and water with 5 grams of salt per gallon flows in at a rate of 6 gallons per minute. Let's find the amount of salt in the tank as a function of time.

Let S be the amount of salt in the tank in grams and t be time in minutes. First we note that the tank has $300+3t$ gallons of water after t minutes. The rate of salt coming in is $5 \times 6 = 30$ grams per minute. The rate of salt going out is $3 \times \frac{S}{300+3t} = \frac{3}{300+3t}S$. The differential equation modeling the situation is

$$S' = 30 - \frac{3}{300+3t}S.$$

We also have the initial condition $S(0) = 0$.

Our ODE is linear first-order.

$$S' + \frac{3}{300+3t}S = 30.$$

The integrating factor is

$$\mu = e^{\int \frac{3}{300+3t}\,dt} = e^{\int du/u} = e^{\ln u} = u = 300 + 3t$$

with the u-sub $u = 300 + 3t$, $du = 3\,dt$.

$$
\begin{aligned}
(300+3t)S' + 3S &= 30(300+3t) \\
\rightsquigarrow \big((300+3t)S\big)' &= 30(300+3t) \\
\rightsquigarrow (300+3t)S &= \int 9000 + 90t\,dt = 9000t + 45t^2 + C \\
\rightsquigarrow S &= \frac{9000t + 45t^2 + C}{300+3t}.
\end{aligned}
$$

Now we plug in the initial condition $S(0) = 0$:

$$0 = S(0) = \frac{C}{300},$$

so $C = 0$. Thus, we have

$$S = \frac{9000t + 45t^2}{300+3t}.$$

Logistic models (§3.2). We explore an important type nonlinear first-order ODEs that commonly show up in mathematical models. One place it arises is in population growth.

Example 8.2. We discussed earlier a model of population growth wherein the rate of growth of a population of proportional to the population itself. This led to the model

$$\frac{dP}{dt} = kP$$

where P is the population considered as a function of time t, and k is some constant of proportionality. The general solution to this is of the form

$$P(t) = Ce^{kt}.$$

This grows larger and larger and larger! A model such as this will not describe the long-term growth effectively if there is some constraint on the size of the population.

Suppose we consider a population, let's say of squirrels in the city of Lincoln. There is only so much food to go around so there should be a maximum capacity for the population; let's say that there's food enough for at most 100000 squirrels; let's call this the *total capacity*. Then $100000 - P$ is the amount of room for growth in the squirrel population; let's call this the *remaining capacity*. A *logistic growth model* for the squirrel population supposes that the rate of growth of the population is *jointly proportional* to the population P and the remaining capacity $100000-P$, meaning it is proportional to the product $P(100000 - P)$. That is, the logistic model says

$$\frac{dP}{dt} = kP(100000 - P).$$

The constant k we take here should be positive.

9. September 20, 2022

Example 9.1. Let's use our techniques from §2.1 to analyze this equation without solving it. This is an autonomous equation, so the behavior (increasing vs decreasing vs constant) depends only on the value of P and not on t. The right-hand side is positive when $0 < P < 100000$, negative when $P < 0$ or $P > 100000$, and constant for $P = 0$ or $P = 100000$. Thus, the population grows when it is positive and below capacity, decreases when above capacity, and stays constant if zero or at capacity.

Definition 9.2. A *logistic model* is a model that supposes that the rate of growth of some quantity P is jointly proportional to P and $P_c - P$ for some constant P_c.

A *logistic equation* is a differential equation of the form

$$\frac{dP}{dt} = kP(P_c - P)$$

for some constants k and P_c. The constant P_c is called the *total capacity* or *carrying capacity*.

Every logistic model can be expressed as a logistic equation.

Let's see another example.

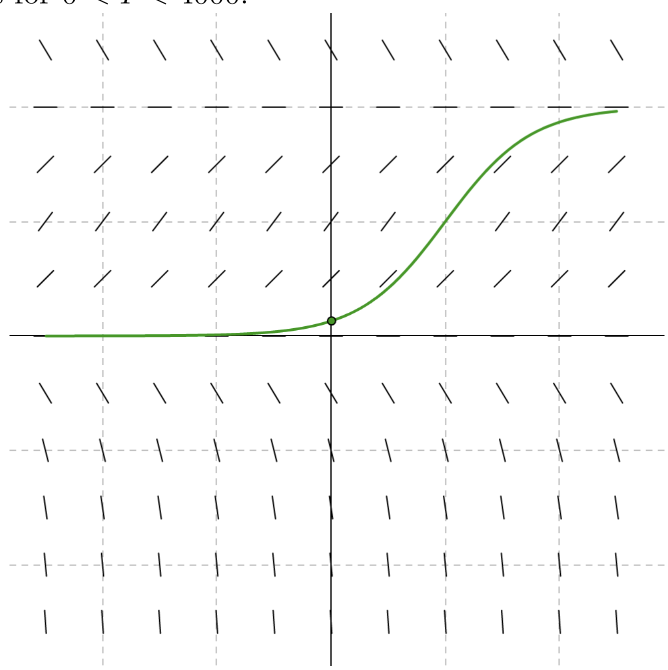
Example 9.3. A simple model for the spread of a rumor throughout a population assumes that the rate at which the rumor spreads is jointly proportional to the number of people informed and the number of people not informed. Suppose the rumor spreads among a population of 4000 people.

Discussion Questions:

(1) Find an ODE modeling the number of people informed I based the information above. (You may have an unknown constant of proportionality k.)
(2) Based on the story, should your constant k be positive or negative?

(3) Based on your model (and not just the story), determine whether I is increasing, decreasing, or constant, based on the number of people who know at the start.
(4) Suppose that 10 people know at the start. Without solving the equation, try to sketch a graph of the solution to the IVP (even without knowing k!).

We should have the equation
$$\frac{dI}{dt} = kI(4000 - I).$$
Since this is presumably increasing for values of I between 0 and 4000, then k should be positive. Now, the function $kI(4000 - I)$ is zero when $I = 0$ or $I = 4000$, and is positive when $0 < I < 4000$, so we have equilibrium solutions for $I = 0, 4000$ and increasing solutions for $0 < I < 4000$.



The general logistic equation is separable, and we can find a general solution to it. Take
$$\frac{dP}{dt} = kP(P_c - P)$$
and separate to get
$$\frac{dP}{P(P_c - P)} = k\,dt.$$

To integrate the left-hand side, we recall the trick of *partial fractions*: we write

$$\frac{1}{P(P_c - P)} = \frac{a}{P} + \frac{b}{P_c - P}$$

for constants a, b and solve:

$$\begin{aligned} 1 &= (P_c - P)a + Pb = P_c a + P(b - a) \\ &\rightsquigarrow a = 1/P_c,\ b - a = 0,\ b = 1/P_c,\ \text{so} \end{aligned}$$

$$\begin{aligned} \int \frac{dP}{P(P_c - P)} &= \frac{1}{P_c}\int \frac{dP}{P} + \frac{1}{P_c}\int \frac{dP}{P_c - P} \\ &= \frac{1}{P_c}\ln|P| - \frac{1}{P_c}\ln|P_c - P| + C \\ &= \frac{1}{P_c}(\ln|P| - \ln|P_c - P|) + C = \frac{1}{P_c}\ln\left|\frac{P}{P_c - P}\right| + C. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{P_c}\ln\left|\frac{P}{P_c - P}\right| &= kt + C \\ \ln\left|\frac{P}{P_c - P}\right| &= kP_c t + C \\ \left|\frac{P}{P_c - P}\right| &= e^{kP_c t + C} = e^C e^{kP_c t} \\ \frac{P}{P_c - P} &= \pm e^C e^{kP_c t} \end{aligned}$$

Since $\pm e^C$ is just a constant in any case, let's reuse the name C for that other constant. Then

$$\frac{P}{P_c - P} = Ce^{kP_c t}$$

for some new constant C'. Rearranging and solving gives

$$\begin{aligned} P &= (P_c - P)Ce^{kP_c t} = P_c Ce^{kP_c t} - PCe^{kP_c t} \\ &\rightsquigarrow P + PCe^{kP_c t} = P_c Ce^{kP_c t} \\ &\rightsquigarrow P(1 + Ce^{kP_c t}) = P_c Ce^{kP_c t} \\ &\rightsquigarrow P = P_c \frac{Ce^{kP_c t}}{1 + Ce^{kP_c t}}. \end{aligned}$$

In particular, a *logistic function* can be written in the explicit form

$$f(t) = a\frac{be^{ct}}{1 + be^{ct}}$$

for some constants a, b, c.

In this class, you don't need to know how to solve a logistic equation or this general formula for a logistic function, just how to set one up and analyze it at we did in Sections 2.1 and 2.6.

Higher-order differential equations (§4.1). For higher order differential equations, we will focus especially on linear equations.

Definition 9.4. An ODE that can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where $y^{(r)} = \frac{d^n y}{dx^r}$ is an *n-th order linear equation*. If $g(x) = 0$, we say that the equation is *homogeneous*; otherwise, it is *nonhomogeneous*.

With a first-order equation, to get a particular solution, we specified an initial condition of one value. For a higher-order ODE, we typically need more initial conditions (or boundary conditions).

Example 9.5. Consider the differential equation

$$y''' = 5.$$

To find the general solution, we can just integrate three times:

$$y'' = \int 5\,dt = 5t + c_1$$

$$y' = \int 5t + c_1\,dt = \frac{5}{2}t^2 + c_1 t + c_2$$

$$y = \int \frac{5}{2}t^2 + c_1 t + c_2\,dt = \frac{5}{6}t^3 + \frac{c_1}{2}t^2 + c_2 t + c_3.$$

To get to a particular solution, we need to know more than one value of the function: we need three pieces of information of some sort to determine the three c's.

One natural possibility is to specify the values at three different points. (This is an example of what we will call a *boundary condition* soon.) Another possibility is to specify the values of y, of y', and of y'' at the same point. (This is what we will call an *initial condition* in this setting.)

For example, given the IVP

$$\begin{cases} y^{(3)} = 5 \\ y(0) = 1 \\ y'(0) = -2 \\ y''(0) = 5 \end{cases}$$

we solve

$$1 = y(0) = \frac{5}{6} \cdot 0^3 + \frac{c_1}{2} \cdot 0^2 + c_2 \cdot 0 + c_3 = c_3$$

$$-2 = y'(0) = \frac{5}{2} \cdot 0^2 + c_1 \cdot 0 + c_2 = c_2$$
$$5 = y''(0) = 5 \cdot 0 + c_1 = c_1$$

so we get a unique solution

$$y(t) = \frac{5}{6}t^3 + \frac{5}{2}t^2 - 2t + 1.$$

Example 9.6. Suppose that we know that the solution IVP

$$\begin{cases} y^{(3)} + y' = 0 \\ y(\pi) = 0 \\ y'(\pi) = 2 \\ y''(\pi) = -1 \end{cases}$$

is of the form

$$y(x) = c_1 + c_2 \sin(x) + c_3 \cos(x).$$

Let's find a particular solution using the initial conditions.

$$0 = y(\pi) = c_1 + c_2 \sin(\pi) + c_3 \cos(\pi) = c_1 - c_3$$

so $c_1 = c_3$.

$$2 = y'(\pi) = c_2 \cos(\pi) - c_3 \sin(\pi) = -c_2$$

so $c_2 = -2$.

$$-1 = y''(\pi) = -c_2 \sin(\pi) - c_3 \cos(\pi) = c_3$$

so $c_3 = -1$. So, our solution must be

$$y(x) = -1 - 2\cos(x) - \sin(x).$$

10. September 22, 2022

A *boundary value problem* poses conditions at different points to get a particular solution. For example, a system of equations like

$$\begin{cases} y'' + y = 0 \\ y(a) = c \\ y(b) = d \end{cases}$$

is a boundary value problem. We will be focusing on initial value conditions, like

$$\begin{cases} y'' + y = 0 \\ y(a) = c \\ y'(a) = d \end{cases}$$

rather than boundary value conditions in this course.

The great news is that for a linear ODE, there exist unique solutions to IVPs in the same sense we discussed for first-order equations with the Picard-Lindelöf Theorem.

Theorem 10.1 (Existence and uniqueness theorem for linear IVPs)**.** *Given a linear ODE*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where $g(x), a_0(x), \ldots, a_n(x)$ are continuous and $a_n(x) \neq 0$ for all x, then there exists a unique solution

$$\begin{cases} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ \quad \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

on some interval containing t_0.

Example 10.2. Consider the ODE

$$(x-7)y'' + 3y = x^2\cos(x).$$

Since $x - 7 = 0$ when $x = 7$, the theorem doesn't apply yet, so we should divide through by $x - 7$:

$$y'' + \frac{3}{x-7}y = \frac{x^2\cos(x)}{x-7}.$$

The Theorem then says that when $\frac{3}{x-7}$ and $\frac{x^2\cos(x)}{x-7}$ are continuous, there is a unique solution near that x-value: this is OK as long as $x \neq 7$. For example, with the initial condition

$$\begin{cases} y(-1) = 3 \\ y'(-1) = -.7 \end{cases}$$

starting from $x = -1$, we know that there exists a unique solution near $x = -1$. Moreover, we can go to the left forever and to the right up until $x = 7$ without getting into trouble, so this IVP has a unique solution on $(-\infty, 7)$.

Principle of superposition.

Example 10.3. Consider the linear homogeneous ODE

$$3y'' + 2ty' + t^7 y = 0.$$

Suppose we have two solutions $y_1(t), y_2(t)$. Then for any constants c_1, c_2, the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is also a solution. (For example, things like $-y_1, 47y_2, 3y_1 - 5y_2$ are solutions.) Let's check it:

$$\begin{aligned} 3y'' + 2ty' + t^7 y &= 3(c_1y_1 + c_2y_2)'' + 2t(c_1y_1 + c_2y_2)' + t^7(c_1y_1 + c_2y_2) \\ &= 3(c_1y_1'' + c_2y_2'') + 2t(c_1y_1' + c_2y_2') + t^7(c_1y_1 + c_2y_2) \\ &= c_1(3y_1'' + 2ty_1' + t^7y_1) + c_2(3y_2'' + 2ty_2' + t^7y_2) \\ &= c_1(\qquad 0 \qquad) + c_2(\qquad 0 \qquad) = 0, \end{aligned}$$

where the last equality is because y_1 and y_2 are solutions; this means that y is a solution!

For about the same reason, this works for any linear homogeneous ODE: the key points were that we could "pull out the sum and constants" form the differential equation, which is a consequence of having a linear equation, and that a sum of constants times zero is zero, which is why we needed a homogeneous equation. Namely, we have:

Theorem 10.4 (Principle of superposition for homogeneous ODEs)**.** *For any linear homogeneous ODE*

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

given any solutions $y_1, y_2, \ldots, y_t$, and any constants $c_1, c_2, \ldots, c_t$, the function $y = c_1y_1 + \cdots + c_ty_t$ is also a solution to the same equation.

This is only true for linear homogeneous equations, so be sure to only apply it in that setting. We give the recipe that appears above a name.

Definition 10.5. If $y_1, y_2, \dots, y_n$ are functions, we say that a function of the form $y = c_1 y_1 + \cdots + c_t y_t$ for some constants $c_1, c_2, \dots, c_t$ is a *linear combination* or *superposition* of the functions $y_1, y_2, \dots, y_t$.

Let's return to our example and see what happens if the equation is nonhomogeneous instead.

Example 10.6. Consider the linear nonhomogeneous ODE

$$3y'' + 2ty' + t^7 y = \sin(t).$$

Suppose we have a solution $y_1(t)$ to this equation and we want to get another one. Based on the homogeneous case, we might the try something like $y = c_1 y_1 + c_2 y_2$ for some constants c_1, c_2 and some other function y_2. What changes? In the last line of computation of Example 10.3, the first and last zeroes should now be $\sin(x)$. The c_1 is going to mess things up, so we better hold off of it; the rest is OK as long as we still the second 0. This means we are OK to take $y = y_1 + c_2 y_2$ for a solution y_2 of the homogeneous equation with the same "left-hand side"!

Again, for about the same reason, this works for any linear ODE. Namely, we have:

Theorem 10.7 (Principle of superposition for nonhomogeneous ODEs)**.** *For any linear nonhomogeneous ODE*

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x),$$

given one particular solution y_p, and some solutions $y_1, \dots, y_t$ of the corresponding homogeneous equation

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

and any constants $c_1, c_2, \dots, c_t$, the function $y = y_p + c_1 y_1 + \cdots + c_t y_t$ is also a solution to the first equation.

Discussion Questions. The functions e^x and e^{2x} are solutions to

$$y'' - 3y' + 2y = 0,$$

and $-xe^x$ is a solution to

$$y'' - 3y' + 2y = e^x.$$

Consider the following functions:

(a) $y = 5e^x$
(b) $y = e^x - e^{2x}$
(c) $y = 7xe^x$
(d) $y = 12e^x - 3e^{2x}$
(e) $y = -2e^x - xe^x$
(f) $y = -2xe^x + 3e^{2x}$
(g) $y = -xe^x + 9e^{2x}$
(h) $y = 4e^{2x} - xe^x + 15e^{2x}$

According to the superposition principle determine:

(1) Which of the functions are solutions to
$$y'' - 3y' + 2y = 0 \text{ ?}$$
(2) Which of the functions are solutions to
$$y'' - 3y' + 2y = e^x \text{ ?}$$

(1) Since this is a linear homogeneous equation, we know that any superpositions of our given solutions e^x and e^{2x} are also solutions of the equation. This means (a), (b), and (d) are solutions too.
(2) Since this is a linear nonhomogeneous equation, we know that our particular solution plus any superposition of our homogeneous solutions is also a solution. This means (e), (g), and (h) are solutions too.

The other functions (c) and (f) are not solutions to either!

11. October 4, 2022

Discussion Questions. Consider the differential equations

$$(\clubsuit) \qquad y'' + \sin(t)y' + e^{t^2}y = 0$$

$$(\diamondsuit) \qquad y'' + \sin(t)y' + e^{x^2}y = \tan(t)$$

(1) What is the order of these equations? Are they linear? Are the homogeneous?

Second order, linear, first is homogeneous, second is not.

(2) Say that we have solutions $f(t)$ and $g(t)$ to equation ($\clubsuit$), and a solution $h(y)$ to equation ($\diamondsuit$). Which of the following definitely are solutions to ($\clubsuit$)? Which definitely are solutions to ($\diamondsuit$)?

(a) $y = 2f$
(b) $y = 2h$
(c) $y = 3f - g$
(d) $y = f^2$
(e) $y = 0$
(f) $y = g + h$
(g) $y = tg$
(h) $y = h - 4f$

2a, 2c, 2e are solutions to ($\clubsuit$) and 2f, 2h are solutions to ($\diamondsuit$).

(3) What can you say about existence and uniqueness of the following initial value problems? Are they true on some interval? If so, what's the biggest such interval?

(a)

$$\begin{cases} y'' + \sin(t)y' + e^{t^2}y = 0 \\ y(0.2) = 4 \\ y'(0.2) = \pi \end{cases}$$

Exists and unique on $(-\infty, \infty)$.

(b)

$$\begin{cases} y'' + \sin(t)y' + e^{t^2}y = \tan(t) \\ y(0.2) = 4 \\ y'(-0.1) = \pi \end{cases}$$

Exists and unique on $(-\pi/2, \pi/2)$.

(c)

$$\begin{cases} y'' + \sin(t)y' + e^{t^2}y = 0 \\ y(0.3) = 7 \end{cases}$$

Exists but not unique on $(-\infty, \infty)$.

We are going to put the principle of superposition to use to write general solutions of linear ODEs in terms of a few solutions. Namely, for a homogeneous linear ODE, we will express the general solution as

$$y = C_1 y_1 + \cdots + C_t y_t$$

for functions $y_1, \ldots, y_t$ (that we have to go find in each case). For a nonhomogeneous linear ODE, we will express the general solution as

$$y = y_p + C_1 y_1 + \cdots + C_t y_t$$

where y_p is any *particular solution* and $C_1 y_1 + \cdots + C_t y_t$ is the general solution to the associated homogeneous ODE.

We would like a way to figure out when we have found enough solutions y_i to make all of them by superposition. Before we can do it, we need a way to say that an extra solution is new.

Linear dependence and Wronskians.

Example 11.1. Consider the ODE

$$y'' - y = 0$$

(equivalently, $y'' = y$). Certainly $y_1 = e^x$ is a solution, since its derivative is itself. Can we guess another? $y_2 = e^{-x}$ has its negative for its derivative, and then the sign flips again with the second derivative, so this is also a solution.

By the principle of superposition, $7e^x$ and $-15e^x + \pi e^{-x}$ are also solutions. Note that something like xe^x is not a solution: the principle only applies to constant multipliers.

Let's recall a couple more functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

are functions that satisfy $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$, so $y_3 = \sinh(x)$ and $y_4 = \cosh(x)$ are also solutions.

However, $y_3 = \frac{1}{2}y_1 - \frac{1}{2}y_2$ and $y_4 = \frac{1}{2}y_1 + \frac{1}{2}y_2$ are already explained by y_1 and y_2 using the principle of superposition, so we don't really need y_3 or y_4 to create the other solutions. That is, using linear combinations, $\{y_1, y_2\}$ can be used to build everything that $\{y_1, y_2, y_3, y_4\}$ can. Note that we can also build everything using $\{y_3, y_4\}$.

The upshot is that $\{y_1, y_2, y_3, y_4\}$ has redundancy for forming linear combinations.

We want a tool for detecting redundancy like this.

In the last example, the equation

$$\frac{1}{2}y_1 + \frac{1}{2}y_2 = y_4$$

can be rewritten as

$$\frac{1}{2}y_1 + \frac{1}{2}y_2 + 0y_3 + (-1)y_4 = 0.$$

Definition 11.2. We say that a set of functions $\{y_1, \ldots, y_n\}$ is *linearly independent* (i.e., no redundancy) if whenever some linear combination yields the zero function

$$c_1y_1 + \cdots + c_ny_n = 0$$

the coefficients $c_1, \ldots, c_n$ must be 0. Otherwise $\{y_1, \ldots, y_n\}$ is *linearly dependent*.

For example,

$$\frac{1}{2}e^x + \frac{1}{2}e^{-x} + (-1)\cosh(x) = 0$$

means that $\{e^x, e^{-x}, \cosh(x)\}$ is linearly dependent.

We now want a tool to detect linear (in)dependence of functions. Here is Wronski's clever idea: if $c_1 f + c_2 g = 0$ (as functions) then, taking derivatives,

$$c_1 f' + c_2 g' = 0$$

also. Then

$$fg' - f'g = f(\frac{-c_1}{c_2}g') - f'(\frac{-c_1}{c_2}g) = 0.$$

Definition 11.3. The *determinant* of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The *determinant* of a 3×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi.$$

Larger square matrices have determinants too, but we won't discuss them in this class. Do not attempt to generalize the formula for 3×3 to $n \times n$ matrices; consult a linear algebra text instead.

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