

## Problem Set 4

Due Thursday, February 11

**Instructions:** You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly*. As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

**Problem 1.** Let  $R = \mathbb{Z}/3[x]$  and  $t_A : R^2 \rightarrow R^2$  be the linear transformation given by

$$t_A \left( \begin{bmatrix} p \\ q \end{bmatrix} \right) = A \begin{bmatrix} p \\ q \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} [2] & x + [1] \\ x & [1] \end{bmatrix}.$$

- (a) Give a sequence of elementary row and column operations that transforms  $A$  to the matrix  $A' = \begin{bmatrix} [1] & 0 \\ 0 & x^2 + x + [1] \end{bmatrix}$ .

*Proof.* There are multiple possibilities. One is

- (i) Multiply row 1 by [2].
  - (ii) Add  $-x$  times row 1 to row 2.
  - (iii) Add  $x + [1]$  times column 1 to column 2.  $\square$
- (b) Using your transformations from (a), find invertible matrices  $P, Q$  with entries in  $R$  such that  $PAQ = A'$ .

*Proof.* We recall that applying an ERO to  $M$  is the same as replacing  $M$  by  $EM$ , where  $E$  is the matrix obtained by applying the same operation to the identity matrix; the same holds for a column operation except with  $ME$ . Thus, let

$$E_i = \begin{bmatrix} [2] & 0 \\ 0 & [1] \end{bmatrix}, \quad E_{ii} = \begin{bmatrix} [1] & -x \\ 0 & [1] \end{bmatrix}, \quad \text{and } E_{iii} = \begin{bmatrix} [1] & x + [1] \\ 0 & [1] \end{bmatrix}.$$

We then have  $A' = E_{ii}E_iAE_{iii}$ , so we can take

$$P = E_{ii}E_i = \begin{bmatrix} [2] & 0 \\ x & [1] \end{bmatrix}, \quad \text{and } Q = E_{iii}. \quad \square$$

- (c) Using your transformations from (a), find bases  $B, C$  for  $R^2$  such that  $[t_A]_B^C = A'$ .

*Proof.* Recall that any elementary matrix is a change of basis matrix for an elementary basis change operation. Let  $B_0$  be the standard basis for  $R^2$ . Then we can write

$$A' = E_{ii}E_iAE_{iii} = [\text{id}_{R^2}]_{C_1}^{C_2}[\text{id}_{R^2}]_{B_0}^{C_1}[t_A]_{B_0}^{B_0}[\text{id}_{R^2}]_{B_1}^{B_0} = [t_A]_{B_1}^{C_2}$$

for some bases  $B_1, C_1, C_2$ .

We reinterpret each of our elementary matrices as a change of basis matrix.

- (i)  $E_i$  is the change of basis matrix  $[\text{id}_{R^2}]_D^D$ , where  $D' = \{[2]d_1, d_2\}$ . Setting  $D' = B_0 = \{e_1, e_2\}$ , we have that  $D = C_1$  is given by  $\{2e_1, e_2\}$ .
- (ii)  $E_{ii}$  is the change of basis matrix  $[\text{id}_{R^2}]_D^D$ , where  $D' = \{d_1, d_2 - xd_1\}$ . Setting  $D' = C_1 = \{[2]e_1, e_2\}$ , we have that  $D = C_2 = \{[2]e_1, e_2 + [2]xe_1\}$ .
- (iii)  $E_{iii}$  is the change of basis matrix  $[\text{id}_{R^2}]_D^D$ , where  $D' = \{d_1 + (x + [1])d_2, d_2\}$ . Setting  $D = B_0 = \{e_1, e_2\}$ , we have that  $D' = B_1 = \{e_1 + (x + [1])e_2, e_2\}$ .

Thus,

$$B = \left\{ \begin{bmatrix} 1 \\ x+1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2x \\ 1 \end{bmatrix} \right\}. \quad \square$$

**Problem 2.** Let  $R$  be a commutative ring. We say that two matrices  $A, B \in \text{Mat}_{m \times n}(R)$  are **row equivalent** if there exists a sequence of elementary row operations that transforms  $A$  into  $B$ .

- (a) Prove that row equivalence is an equivalence relation.

*Proof.* Let  $\sim$  denote row equivalence. Note that  $A \sim B$  if and only if  $B = E_r \cdots E_1 A$  for some elementary matrices  $E_1, \dots, E_r$ .  $A \sim A$  since the identity map is an elementary matrix (e.g., it's  $E_1(1)$ .) If  $A \sim B$ , then  $B = E_r \cdots E_1 A$  for some elementary matrices  $E_1, \dots, E_r$  and hence  $A = E_1^{-1} \cdots E_r^{-1} B$  and, since the inverse of an elementary matrix is an elementary matrix (shown in class), this proves  $B \sim A$ . If  $A \sim B$  and  $B \sim C$  then  $B = E_r \cdots E_1 A$  and  $C = E_s \cdots E_{r+1} B$  for some elementary matrices  $E_1, \dots, E_r, E_{r+1}, \dots, E_s$  and hence  $C = E_s \cdots E_1 A$  so that  $A \sim C$ .  $\square$

- (b) Suppose that every invertible matrix over  $R$  is a product of elementary matrices over  $R$ . (We stated in class that this is true over any field.) Prove that every invertible  $n \times n$  matrix over  $R$  is row equivalent to the  $n \times n$  identity matrix.

*Proof.* We show by induction on  $m$  that any product  $E_m E_{m-1} \cdots E_1$  of elementary matrices is row equivalent to the identity matrix. For  $m = 1$ , we noted this in class (alternatively, we check the three cases). If the claim holds for some  $m$ , consider a product of the form  $E_{m+1} E_m E_{m-1} \cdots E_1$ . Then  $E_m E_{m-1} \cdots E_1$  is row equivalent to  $I$ , and  $E_{m+1}(E_m E_{m-1} \cdots E_1)$  is row equivalent to  $E_m E_{m-1} \cdots E_1$ , so the claim follows from part (a).  $\square$

- (c) Give an example of two  $n \times n$  matrices over  $\mathbb{R}$  that have the same rank<sup>1</sup> but are not row equivalent.

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<sup>1</sup>DEFINITION: For a matrix  $A \in \text{Mat}_{m \times n}(F)$  over a field  $F$  we define the **rank** of  $A$  to be the rank of the linear transformation  $t_A: F^n \rightarrow F^m$  given by  $t_A(v) = Av$ .

*Proof.* One can take  $A = [1 \ 0]$  and  $B = [0 \ 1]$  in  $\text{Mat}_{1 \times 2}(\mathbb{R})$ . No row operation can turn  $A$  into a matrix with a nonzero second entry.  $\square$

**Problem 3.** Let  $R$  be a commutative ring. For a matrix  $M$  with entries in  $R$ , a  $t \times t$  **minor** of  $M$  is a determinant  $\det(M')$  of a matrix  $M'$  obtained from selecting  $t$  rows and  $t$  columns of  $M$ . We<sup>2</sup> write  $I_t(M)$  for the ideal of  $R$  generated by all  $t \times t$  minors of  $M$ .

- (a) Let  $A$  be an  $m \times k$  matrix and  $B$  be a  $k \times n$  matrix. Let  $t \leq \min\{m, k, n\}$ . Show that  $I_t(AB) \subseteq I_t(A)$ .

*Proof.* By definition  $I_t(AB)$  is generated by all the determinants of matrices obtained from  $AB$  by choosing a subset  $I \subseteq \{1, \dots, m\}$  of size  $t$ , a subset  $J \subseteq \{1, \dots, n\}$  of size  $t$ , and taking  $(AB)^{I,J}$  as the submatrix of  $AB$  with rows  $I$  and columns  $J$ . It suffices to show that  $\det((AB)^{I,J}) \in I_t(A)$  for any such  $I, J$ , since  $I_t(AB)$  is the unique smallest ideal containing all of these elements.

Let  $A'$  be the  $t \times k$  submatrix of  $A$  with rows  $I$ , and let  $B'$  be the  $k \times t$  submatrix of  $B$  with columns  $J$ . Note that  $(AB)^{I,J} = A'^T B'$ . Then by a Proposition from class, we have  $\det(A'^T B') \in I_t(A')$ , and by definition, we have  $I_t(A') \subseteq I_t(A)$ . This shows the claim.  $\square$

- (b) In the same setting as part (a), show that  $I_t(AB) \subseteq I_t(B)$ .

*Proof.* Note that  $(AB)^T = B^T A^T$ . Since determinants are preserved by transposes, using part (a) we have

$$I_t(AB) = I_t((AB)^T) \subseteq I_t(B^T A^T) \subseteq I_t(B^T) = I_t(B). \quad \square$$

- (c) Let  $A$  be an  $m \times n$  matrix. Let  $P$  be an invertible  $m \times m$  matrix and  $Q$  be an invertible  $n \times n$  matrix. Let  $t \leq \min\{m, n\}$ . Show that  $I_t(A) = I_t(PAQ)$ .

*Proof.* Using part (a) and part (b) we have

$$I_t(PAQ) = I_t(P(AQ)) \subseteq I_t(AQ) \subseteq I_t(A).$$

Writing  $A = P^{-1}(PAQ)Q^{-1}$  and applying the same argument we also get  $I_t(A) \subseteq I_t(PAQ)$ . Thus, equality holds.  $\square$

**Problem 4.** Let  $F$  be a field and  $A \in \text{Mat}_{m \times n}(F)$  be a nonzero matrix.

- (a) Prove that

$$\text{rank}(A) = \min\{t \geq 1 \mid A = BC \text{ for some } B \in \text{Mat}_{m \times t}(F), C \in \text{Mat}_{t \times n}(F)\}.$$

*Proof.* To show the inequality ( $\leq$ ) it suffices to find such matrices  $BC$  of the appropriate size. Let  $a_1, \dots, a_n$  be the columns of  $A$  and  $t$  be the rank of  $A$ . Let  $v_1, \dots, v_t$  be a basis for  $\text{im}(t_A)$  and set  $C$  to be the matrix with columns  $v_1, \dots, v_t$ . We can write  $v_i = \sum_j b_{ij}$  for some  $b_{ij} \in F$ . Then  $A = BC$ .

For the other inequality, note that  $\text{im}(t_{BC}) \subseteq \text{im}(t_B)$ , and by Rank-Nullity,  $\dim(\text{im}(t_B)) \leq t$ .  $\square$

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<sup>2</sup>You can use Proposition 12.4.9 and Exercise 12.4.10 without proof.

(b) Prove that

$$\text{rank}(A) = \max\{t \geq 1 \mid I_t(A) \neq 0\}.$$

*Proof.* From Problem #5 of HW#3, we can find bases  $\mathcal{B}$  and  $\mathcal{C}$  such that  $[t_A]_{\mathcal{B}}^{\mathcal{C}}$  is of the form  $D = \begin{bmatrix} \mathbb{1}_{k \times k} & 0_{k \times (n-k)} \\ 0_{(m-k) \times k} & 0_{(m-k) \times (n-k)} \end{bmatrix}$  for some  $k$ .

Note that  $\text{rank}(t_A) = k$ . Indeed, if  $\mathcal{C} = \{c_1, \dots, c_m\}$ , then  $\{c_1, \dots, c_k\}$  is a basis for  $\text{im}(t_A)$ .

We claim that  $k = \max\{t \mid I_t(A) \neq 0\}$ . Indeed, we can write  $D = PAQ$  for some invertible change of basis matrices  $P, Q$ ; concretely  $D = [t_A]_{\mathcal{B}}^{\mathcal{C}} = [\text{id}]_{\text{std}}^{\mathcal{C}} [t_A]_{\text{std}}^{\text{std}} [\text{id}]_{\mathcal{B}}^{\text{std}} = PAQ$  for  $P = [\text{id}]_{\text{std}}^{\mathcal{C}}$  and  $Q = [\text{id}]_{\mathcal{B}}^{\text{std}}$ , where std denotes the standard basis. Thus  $I_t(D) = I_t(A)$  for all  $t > 0$  by Problem #3. Then  $1 \in I_t(D)$  for  $t \leq k$  (since  $D$  has an identity  $t \times t$  submatrix), and  $I_t(D) = 0$  for  $t > k$  (since any  $t \times t$  submatrix of  $D$  has a zero row). This shows the claim and completes the proof.  $\square$