ASSIGNMENT #2

(1) Opposites: Let R be a ring.
  (a) Prove that there is an isomorphism[1] $M_n(R^{\text{op}}) \cong M_n(R)^{\text{op}}$.
  (b) Prove that there is an isomorphism $\operatorname{End}_R(R) \cong R^{\text{op}}$.

(2) A module is *finitely generated* if it has a finite generating set, and *finitely presented* if it as a finite generating set for which the module of relations is finitely generated. Let

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

be a short exact sequence of R-modules.
  (a) Show that if M' and M'' are finitely generated, then M is finitely generated.
  (b*) Show that if M' and M'' are finitely presented, then M is finitely presented.

(3) Fix a field K. The collection of pairs (V, W) where $W \subseteq V$ are vector spaces forms a category $\mathscr{C}$, where the morphisms from $(V, W) \to (V', W')$ are linear transformations $\phi : V \to V'$ such that $\phi(W) \subseteq W'$. There are covariant functors $F, G : \mathscr{C} \to K - \mathbf{Vect}$ given by

$$\begin{aligned} F(V, W) &= V & F(\phi) &= \phi \\ G(V, W) &= W \oplus V/W & G(\phi) &= \phi|_W \oplus \overline{\phi} \end{aligned}$$

where $\overline{\phi} : V/W \to V'/W'$ is the induced map $\overline{\phi}(v + W) = \phi(v) + W'$ on the quotient spaces.
  (a) Show that for every $(V, W) \in \mathrm{Ob}(\mathscr{C})$, there is an isomorphism of vector spaces $F(V) \cong G(V)$.
  (b) Let $W = K \oplus \{0\} \subseteq V = K^2$, and take $\phi : K^2 \to K^2$ to be the map given by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Check that ϕ is a morphism in $\mathscr{C}$, and compute $F(\phi)$ and $G(\phi)$.
  (c) Show that there is no natural isomorphism[2] $\eta : F \Rightarrow G$.

(4) A covariant functor $F : R - \mathbf{Mod} \to S - \mathbf{Mod}$ is *additive* if for every $M, N \in R - \mathbf{Mod}$, the map

$$\begin{aligned} \operatorname{Hom}_R(M, N) &\longrightarrow \operatorname{Hom}_S(F(M), F(N)) \\ f &\longmapsto F(f) \end{aligned}$$

is a homomorphism of abelian groups. Show that if F is an additive covariant functor, and

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

is a split exact sequence, then

$$0 \to F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'') \to 0$$

is exact[3].

[1]Hint: Your map should involve transposes.
[2]Moral: Every short exact sequence of vector spaces splits, but *not* naturally!
[3]Moral: Functors (additive or not) between module categories don't always preserve short exact sequences, but (at least additive functors) always preserve *split* exact sequences.

(5) The localization functor:

Let R be a commutative ring. A subset S of R is *multiplicatively closed* if $1 \in S$ and $s, t \in S \Rightarrow st \in S$. Define a new ring $S^{-1}R$ as follows:

$$S^{-1}R = \left\{ \frac{r}{s} \mid r \in R, s \in S \right\} / \sim$$

where $\sim$ is the equivalence relation $\frac{r}{s} \sim \frac{r'}{s'}$ if and only if $t(rs' - r's) = 0$ for some $t \in S$. This[4] set is a ring (a fact you need not check) with respect to the operations

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \qquad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

For an R-module M define

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\} / \sim$$

where $\sim$ is the equivalence relation $\frac{m}{s} \sim \frac{m'}{s'}$ if and only if $t(ms' - m's) = 0$ for some $t \in S$. Then $S^{-1}M$ is an $S^{-1}R$-module (a fact you need not check) via the operations

$$\frac{m}{s} + \frac{m'}{s'} = \frac{ms' + m's}{ss'} \qquad \frac{r}{s} \cdot \frac{m}{s'} = \frac{rm}{ss'}.$$

(a) Show that there is a functor $S^{-1} : R-\mathbf{Mod} \to S^{-1}R-\mathbf{Mod}$ that on objects maps $M \mapsto S^{-1}M$ and on morphisms maps $f \mapsto S^{-1}f$ where $(S^{-1}f)(\frac{m}{s}) = \frac{f(m)}{s}$.

(b) Prove that this functor is exact.

(6*) (a) We only defined a notion of natural transformation/isomorphism for F, G both covariant or F, G both contravariant. Come up with a definition of natural transformation/isomorphism for F covariant and G contravariant.

(b) Show that with this definition, for a field K, the functors $1_{K-\mathbf{vect}}, (-)^* : K-\mathbf{vect} \to K-\mathbf{vect}$ are still not naturally isomorphic.

(c) Let $K-\mathbf{inn}$ where

- objects are finite dimensional K-vector spaces equipped with a nondegenerate[5] symmetric bilinear form $\langle -, - \rangle_V : V \times V \to K$, and the
- morphisms are linear maps $\phi : V \to W$ such that $\langle v, v' \rangle_V = \langle \phi(v), \phi(v') \rangle_W$.

Show that the functors $F, G : K - \mathbf{inn} \to K - \mathbf{vect}$ given by

$$F(V) = V \qquad F(\phi) = \phi$$
$$G(V) = V* \qquad G(\phi) = \phi^*$$

are naturally isomorphic.

[4]This generalizes the construction of the fraction field of a domain R, where $S = R \setminus \{0\}$ gives $S^{-1}R = \mathrm{Frac}(R)$.

[5]That is, for every $v \in V \setminus \{0\}$, there is some $v' \in V$ such that $\langle v, v' \rangle_V \neq 0$.