Making sense of quantifier statements.

- The symbol for "for all" is \forall and the symbol for "there exists" is \exists .
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that not P".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, not P".

A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both)!

(1) Match each informal story (i)–(iv) below with a precise quantifier statement (A)–(D).

Informal stories:

- (i) Every number past some point is red.
- (ii) There are arbitrarily big red numbers.
- (iii) All positive numbers are red.
- (iv) There are positive red number(s).

Precise statements:

- (A) For every y > 0, y is red.
- (B) There exists y > 0 such that y is red.
- (C) For every $x \in \mathbb{R}$, there is some y > x such that y is red.
- (D) There exists $x \in \mathbb{R}$ such that for every y > x, y is red.
- (2) Draw a picture where (A) is false and (B) is true.
- (3) Draw a picture where (C) is true and (D) is false.
- (4) Suppose that (C) is true. Which of the following statements must also be true? Why?
 - (a) There is some y > 1000000000 such that y is red.
 - (b) For every $\mu \in \mathbb{R}$, there is some $\theta > \mu$ such that θ is red.
 - (c) For every $x \in \mathbb{R}$, there is some y > 2x such that y is red.

The next problem is no longer about a spraypainting of the real number line.

- (5) Rewrite each statement with symbols in place of quantifiers, and write its negation. Do you think the original statement is true or false (but don't prove them yet)?.
 - (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
 - (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
 - (c) For all $x \in \mathbb{R}$ such that $x \neq 0, x^2 > 0$.
 - (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that x < y.
 - (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, x < y.

¹In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling" over.

Proving quantifier statements and using the axioms of \mathbb{R} .

| • | The general | outline | of a | proof | of "For | all x | $\in S$, | P" goes |
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- (1) Let $x \in S$ be arbitrary.
- (2) Do some stuff.
- (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
 - (1) Consider² x = [some specific element of S].
 - (2) Do some stuff.
 - (3) Conclude that P holds for x.

Note: explaining *how* you found your example "x" is *not* a logically necessary part of the proof.

- (6) Complete each of the sentences below with either "GENERAL ARGUMENT" or "SPECIFIC EXAMPLE".
 - To prove a "for all" statement, you need to give a _____.
 - To disprove a "for all" statement, you need to give a ______.
 - To prove a "there exists" statement, you need to give a _____.
 - To disprove a "there exists" statement, you need to give a _____
- (7) Complete each of the sentences below with either "CAN CHOOSE A SPECIFIC VALUE" or "MUST USE A MYSTERY VALUE".
 - If you want to *use* a "for all" statement that you know is true, you .
 - If you want to *use* a "there exists" statement that you know is true, you

(8) Prove or disprove each of the statements in (5) (skip (c) for now) using the axioms of \mathbb{R} and facts we have already proven.

More practice with quantifier statements. Using the axioms of \mathbb{R} and statements that we've already proven (like cancellation of addition, or any problem on the list above the given one), prove the following:

- (9) Prove that there exists some $x \in \mathbb{R}$ such that 2x + 5 = 3.
- (10) Prove² that for any real number r, we have $r \cdot 0 = 0$.
- (11) Let x be a real number. Use the axioms of \mathbb{R} and facts we have already proven to show that if there exists a real number y such that xy = 1, then $x \neq 0$.
- (12) Prove that³ for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

²Hint: You might find it useful to write 0 = 0 + 0 and, in a later step, use cancellation of addition.

³Hint: Use (11).