

§2.9: NOETHERIAN RINGS

DEFINITION: A ring R is **Noetherian** if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some N such that $I_n = I_N$ for all $n \geq N$.

HILBERT BASIS THEOREM: If R is a Noetherian ring, then the polynomial ring $R[X]$ and power series ring $R[[X]]$ are also Noetherian.

We will return to the proof of Hilbert Basis Theorem after discussing Noetherian modules next time.

COROLLARY: Every finitely generated algebra over a field is Noetherian.

(1) Equivalences for Noetherianity.

- (a) Show¹ that R is Noetherian if and only if every ideal is finitely generated.
- (b) Show² that R is Noetherian if and only if every nonempty collection of ideals has a maximal³ element.

- (a) (\Leftarrow) Suppose that every nonempty collection of ideals has a maximal element. Then a chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is, in particular, a nonempty collection of ideals, hence has a maximal element, say I_n . Then for $n \geq n$, $I_n \subseteq I_n$ and maximality of I_n imply $I_n = I_n$. (\Rightarrow) Suppose that there is a nonempty collection of ideals without a maximal element, say \mathcal{S} . Let I_1 be any element of \mathcal{S} . Then, by definition, there is some I_2 that properly contains I_1 , and so on, yielding a chain that does not stabilize.
- (b) (\Leftarrow) Suppose that every ideal is finitely generated, and take a chain $I_1 \subseteq I_2 \subseteq \cdots$. Consider $I = \bigcup_n I_n$. This is an ideal (it was important that we had a chain, not an arbitrary collection of ideals for this step), and by hypothesis we have $I = (f_1, \dots, f_m)$. For each i , there is some n_i such that $f_i \in I_{n_i}$. Let $N = \max\{n_i\}$. Then $I = (f_1, \dots, f_m) \subseteq I_N \subseteq I$, so equality holds, and the chain stabilizes at N . (\Rightarrow) Suppose that there is an ideal I that is not finitely generated. Then we construct an infinite chain as follows: let $f_1 \in I \setminus 0$ (0 is finitely generated so $I \neq 0$), and set $I_1 = (f_1)$, and for each n take $f_{n+1} \in I \setminus I_n = (f_1, \dots, f_n)$, (I_n is finitely generated so $I \neq I_n$).

(2) Some Noetherian rings:

- (a) Show that fields and PIDs are Noetherian.
- (b) Show that if R is Noetherian and $I \subseteq R$, then R/I is Noetherian.
- (c) Is⁴ every subring of a Noetherian ring Noetherian?

- (a) Every element of a field is generated by no elements; every element of a PID is generated by one element.
- (b) The ideals of R/I are in containment-preserving bijection with ideals of R containing I . A chain of ideals in R containing I must stabilize, so the corresponding chain in R/I must stabilize as well.

¹For the backward direction, consider $\bigcup_{n \in \mathbb{N}} I_n$

²Hint: For the forward direction, show the contrapositive.

³This means that if \mathcal{S} is our collection of ideals, there is some $I \in \mathcal{S}$ such that no $J \in \mathcal{S}$ properly contains I . It does not mean that there is a maximal ideal in \mathcal{S} .

⁴Hint: Every domain has a fraction field, even the domain from (4a).

(c) No: $K[X_1, X_2, \dots]$ is not Noetherian, but it is a subring of its fraction field $K(X_1, X_2, \dots)$, which is a field, hence Noetherian.

(3) Use the Hilbert Basis Theorem to deduce the Corollary.

From the Hilbert Basis Theorem and induction, if R is Noetherian, then $R[X_1, \dots, X_n]$ is as well. In particular, if K is a field, then $K[X_1, \dots, X_n]$ is too. Since a finitely generated K -algebra is a quotient of some $K[X_1, \dots, X_n]$, then any such ring is Noetherian as well.

(4) Some nonNoetherian rings:

(a) Let K be a field. Show that $K[X_1, X_2, \dots]$ is not Noetherian.

(b) Let K be a field. Show that $K[X, XY, XY^2, \dots]$ is not Noetherian.

(c) Show that $\mathcal{C}([0, 1], \mathbb{R})$ is not Noetherian.

(a) The ideal (X_1, X_2, \dots) is not finitely generated.

(b) The ideal (X, XY, \dots) is not finitely generated.

(c) The ideal $\sqrt{(x)} = \mathfrak{m}_0$ is not finitely generated.

(5) Let R be a Noetherian ring. Show that for every ideal I , there is some n such that $\sqrt{I}^n \subseteq I$. In particular, there is some n such that for every nilpotent element z , $z^n = 0$.

Let $\sqrt{I} = (f_1, \dots, f_m)$. For each i , there is some n_i such that $f_i^{n_i} \in I$. Then for $n \geq n_1 + \dots + n_m - m + 1$, any generator $f_1^{a_1} \dots f_m^{a_m}$ with $\sum a_i = n$ must have $a_j \geq n_j$ for some j , and hence $f_1^{a_1} \dots f_m^{a_m} \in I$.

For the particular case, we consider $\sqrt{0}$.

(6) Let R be Noetherian. Show that every element of R admits a decomposition into irreducibles.

We argue the contrapositive. Suppose that $r \in R$ does not admit a decomposition into irreducibles. Then in particular, r is reducible, so $r = r_1 r'_1$, with r'_1 not a unit, so $(r) \subsetneq (r_1)$. Likewise, r_1 is reducible, so $r_1 = r_2 r'_2$, with r'_2 not a unit, so $(r_1) \subsetneq (r_2)$. We can continue like this forever to obtain an infinite ascending chain of *principal* ideals even.

(7) Prove the principle of **Noetherian induction**: Let \mathcal{P} be a property of a ring. Suppose that “For every nonzero ideal I , \mathcal{P} is true for R/I implies that \mathcal{P} is true for R ” and \mathcal{P} holds for all fields. Then \mathcal{P} is true for every Noetherian ring.

- (8) (a) Suppose that every maximal ideal of R is finitely generated. Must R be Noetherian?
 (b) Suppose that every ascending chain of prime ideals stabilizes. Must R be Noetherian?
 (c) Suppose that every prime ideal of R is finitely generated. Must R be Noetherian?

(a) No. Here is one possibility: let K be a field, and R be the subring of $K(X, Y)$ consisting of elements that can be written as f/g with $f = aX^n + bY$ and $g = uX^n + cY$ for some $n \geq 0$, $a, b, c \in K[X, Y]$, and $u \in K[X, Y]$ with nonzero constant term. I leave it to you to show that

- R is indeed a subring of $K(X, Y)$,

- the ideal (X) is a maximal ideal,
- any $r \in R \setminus (X)$ is a unit, so (X) is the unique maximal ideal, and
- the ideal $(Y, Y/X, Y/X^2, \dots)$ is not finitely generated.

(b) No.

(c) Yes.