Contents

1. August 23, 2022

This class is, as its name makes clear, is all about differential equations. Let's start with an example that is probably similar to something you've seen in Calculus.

Example 1.1. The equation

$$\frac{dy}{dx} = 7y$$

is a differential equation. The unknown in this equation, y, stands for a function. What makes this equation a differential equation is that the equation relates the mystery function and its derivative.

Let's see if we can guess a solution. This equation might remind us of a curious calculus coincidence. If the 7 wasn't there, we would be looking for a function whose derivative is equal to itself; e^x would work.

Let's try $y = 7e^x$ for our original equation. To test it, we plug it in:

$$y = 7e^x \rightsquigarrow y' = (7e^x)' = 7e^x \neq 7y = 49e^x.$$

How about putting the 7 somewhere else:

$$y = e^{7x} \rightsquigarrow y' = (e^{7x})' = e^{7x}(7x)' = 7e^{7x} = 7y.$$

So e^{7x} is a solution!

Could there be any others?

$$y = 5e^{7x} \rightsquigarrow y' = (5e^{7x})' = 5e^{7x}(7x)' = 7(5e^{7x}) = 7y.$$

In general, $y(x) = Ce^{7x}$ is a solution for any constant C.

Of course, at the end of the day, nothing was special about 7. If we replaced 7 by any real number a, for the same reason, we would find that for the differential equation

$$y' = ay$$

the *general solution* is

$$y(x) = Ce^{ax}.$$

Types of differential equations. There are many different ways of throwing together functions and derivatives in an equation, so we'll need some terminology to orient ourselves.

Definition 1.2. An *ordinary differential equation (ODE)* is a differential equation involving only one independent variable; i.e., derivatives with respect to just one variable.

For example,

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} = -y + \cos(ty)$$

is an ordinary differential equation.

In general an ODE is an equation of the form

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$

for some function F where $y = y(t)$: an equation relating the function y with its derivative(s).

Definition 1.3. The *order* of a differential equation is the highest order derivative that occurs in the equation.

For example,

$$yy'' + y''' + \frac{1}{y} = 5x$$

is a third order ODE, due to the y''' term and

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} = -y + \cos(ty)$$

is a second order ODE.

Definition 1.4. A *linear* ODE is any ODE of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t).$$

For example,

$$5ty'' + \ln(t)y' + y = \cos(t)$$

is a second order linear ODE, but

$$yy' + 5y = 7$$

and

$$(y')^3 - ty^2 = 3e^t$$

are first order nonlinear ODEs.

We will be especially interested in linear ODEs in this course!

Definition 1.5. A *partial differential equation (PDE)* is a differential equation involving multiple independent variable; i.e., derivatives with respect to different variables.

For example,

$$\frac{\partial u}{\partial t} - 5\frac{\partial u}{\partial x} = 0$$

is a PDE. A solution of this PDE would be a function $u(x, t)$ that depends two independent variables x and t.

Discussion Questions.

(1) Is the differential equation $y' = y^{2/3}$ ordinary? linear? What is its order?

(2) Which of the following is a solution to the differential equation $y' = y^{2/3}$:
  (a) $y = 4t^2$
  (b) $y = e^{2t/3}$
  (c) $y = \frac{1}{27}t^3$
  (d) $y = 0$ (constant function 0)

(3) There is a solution to $y'' - y' = 2y$ of the form $y = ae^{2x} + be^{-x}$ for some real numbers a, b. Find it!

(4*) If f, g are solutions to ?, show that ? is too.

Initial value problems; existence and uniqueness. In our first example, we saw that there are many solutions to the differential equation $y' = 7y$. To pin one down, we might specify a value for our function at a point. The system

$$\begin{cases} y' = 7y \\ y(2) = 4 \end{cases}$$

is a example of an *initial value problem.* Geometrically, $y(2) = 4$ corresponds to the condition that the graph of our solution passes through $(2, 4)$.

We can solve this using our solution of $y' = 7y$ from earlier. We have

$$y = Ce^{7x} \;\; y(2) = 4$$

so

$$4 = Ce^{7 \cdot 2}$$

and

$$C = 4e^{-14}.$$

That is,

$$y = 4e^{-14}e^{7x} = 4e^{7x-14}.$$

Not only did we find a solution to our IVP, but along the way we saw that our solution was the only one; i.e, is unique (as long as we believe we started off with all of the solutions to $y' = 7y$).

On the other hand, if we consider the two solutions $y = 0$ and $y = \frac{1}{27}t^3$, we see that they are also both soltuions to the IVP

$$\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases},$$

so here we have an IVP for which the solution is not unique.

What if we are so unlucky as to try to solve an IVP and there's no solution at all?

Theorem 1.6 (Picard-Lindelöf)**.** *For the IVP*

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

and some rectangle R in the (x, y)-plane containing (x_0, y_0) in its interior, there exists a unique solution on some possibly smaller interval $(x_0 - h, x_0 + h)$, so long as f and $\frac{\partial f}{\partial y}$ are continuous on R.

Example 1.7. Consider the differential equation

$$\frac{dy}{dx} = 5xy.$$

We want to use the Picard-Lindelöf Theorem to show that for any initial condition $f(x_0) = y_0$, there is a unique solution near x_0. The $f(x, y)$ of the Theorem is $5xy$; this is continuous on all of $\mathbb{R}^2$. We also need to look at $\frac{\partial f}{\partial y} = 5x$. This is also continuous on all of $\mathbb{R}^2$. We conclude that

$$\begin{cases} \frac{dy}{dx} = 5xy \\ y(x_0) = y_0 \end{cases}$$

has a unique solution, no matter what x_0 and y_0 are.

Example 1.8. Let's consider

$$\begin{cases} y' = y^{2/3} \\ y(x_0) = y_0 \end{cases}.$$

Here $f(x, y) = y^{2/3}$ and $\frac{\partial f}{\partial y}(x, y) = 2/3y^{-1/3}$. f is continuous everywhere, but f' is only continuous where $y \neq 0$. Thus, if $y_0 \neq 0$, then there is a unique solution.

However, if $y_0 = 0$, the theorem say nothing. We looked at this example earlier and saw that the solutions were not unique for $(x_0, y_0) = (0, 0)$.