Proposition 1.2 (Arithmetic and order properties of $\mathbb{Q}$)**.** The set of rational numbers form an "ordered field". This means that the following ten properties hold:

(1) There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$:

$$\text{for all } p, q \in \mathbb{Q}, \quad p + q \in \mathbb{Q} \text{ and } p \cdot q \in \mathbb{Q}.$$

(2) Each of $+$ and $\cdot$ is a commutative operation:

$$\text{for all } p, q \in \mathbb{Q}, \quad p + q = q + p \text{ and } p \cdot q = q \cdot p.$$

(3) Each of $+$ and $\cdot$ is an associative operation:

$$\text{for all } p, q, r \in \mathbb{Q}, \quad (p + q) + r = p + (q + r) \text{ and } (p \cdot q) \cdot r = p \cdot (q \cdot r).$$

(4) The number 0 is an identity element for addition and the number 1 is an identity element for multiplication:

$$\text{for all } p \in \mathbb{Q}, \quad 0 + p = p \text{ and } 1 \cdot p = p.$$

(5) The distributive law holds:

$$\text{for all } p, q, r \in \mathbb{Q}, \quad p \cdot (q + r) = p \cdot q + p \cdot r.$$

(6) Every number has an additive inverse:

$$\text{for each } p \in \mathbb{Q}, \text{ there is some ``}-p\text{''} \in \mathbb{Q} \text{ such that} \quad p + (-p) = 0.$$

(7) Every nonzero number has a multiplicative inverse:

$$\text{for each } p \in \mathbb{Q}, p \neq 0, \text{ there is some ``}p^{-1}\text{''} \in \mathbb{Q} \text{ such that} \quad p \cdot p^{-1} = 1.$$

(8) There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that
 (a) for all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.
 (b) for all $p, q \in \mathbb{Q}$, if $p \leq q$ and $q \leq p$, then $p = q$.
 (c) for all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.

(9) The total ordering $\leq$ is compatible with addition:

$$\text{for all } p, q, r \in \mathbb{Q}, \quad \text{if } p \leq q \text{ then } p + r \leq q + r.$$

(10) The total ordering $\leq$ is compatible with multiplication by nonnegative numbers:

$$\text{for all } p, q, r \in \mathbb{Q}, \quad \text{if } p \leq q \text{ and } r \geq 0 \text{ then } pr \leq qr.$$

Axioms of $\mathbb{R}$. By axiom, the collection of real numbers, $\mathbb{R}$, is a *complete ordered field*. This means the following ten properties hold:

(Axiom 1) There are operations $+$ and $\cdot$ defined on $\mathbb{R}$:

$$\text{for all } p, q \in \mathbb{R}, \quad p + q \in \mathbb{R} \text{ and } p \cdot q \in \mathbb{R}.$$

(Axiom 2) Each of $+$ and $\cdot$ is a commutative operation:

$$\text{for all } p, q \in \mathbb{R}, \quad p + q = q + p \text{ and } p \cdot q = q \cdot p.$$

(Axiom 3) Each of $+$ and $\cdot$ is an associative operation:

$$\text{for all } p, q, r \in \mathbb{R}, \quad (p + q) + r = p + (q + r) \text{ and } (p \cdot q) \cdot r = p \cdot (q \cdot r).$$

(Axiom 4) The number 0 is an identity element for addition and the number 1 is an identity element for multiplication:

$$\text{for all } p \in \mathbb{R}, \quad 0 + p = p \text{ and } 1 \cdot p = p.$$

(Axiom 5) The distributive law holds:

$$\text{for all } p, q, r \in \mathbb{R}, \quad p \cdot (q + r) = p \cdot q + p \cdot r.$$

(Axiom 6) Every number has an additive inverse:

$$\text{for each } p \in \mathbb{R}, \text{ there is some "}{-p}\text{"} \in \mathbb{R} \text{ such that } \quad p + (-p) = 0.$$

(Axiom 7) Every nonzero number has a multiplicative inverse:

$$\text{for each } p \in \mathbb{R}, p \neq 0, \text{ there is some "}p^{-1}\text{"} \in \mathbb{R} \text{ such that } \quad p \cdot p^{-1} = 1.$$

(Axiom 8) There is a "total ordering" $\leq$ on $\mathbb{R}$. This means that
(a) for all $p, q \in \mathbb{R}$, either $p \leq q$ or $q \leq p$.
(b) for all $p, q \in \mathbb{R}$, if $p \leq q$ and $q \leq p$, then $p = q$.
(c) for all $p, q, r \in \mathbb{R}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.

(Axiom 9) The total ordering $\leq$ is compatible with addition:

$$\text{for all } p, q, r \in \mathbb{R}, \quad \text{if } p \leq q \text{ then } p + r \leq q + r.$$

(Axiom 10) The total ordering $\leq$ is compatible with multiplication by nonnegative numbers:

$$\text{for all } p, q, r \in \mathbb{R}, \quad \text{if } p \leq q \text{ and } r \geq 0 \text{ then } pr \leq qr.$$

(Axiom 11) The COMPLETENESS AXIOM