DEFINITION: A function f is **continuous at** a provided: For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $f(x)$ is defined and $|f(x) - f(a)| < \varepsilon$.

THEOREM: If f is defined at a then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.

THEOREM: If f and g are both continuous at a, and c is any constant, then
1. $f + g$ is continuous at a.
2. cf is continuous at a.
3. fg is continuous at a.
4. f/g is continuous at a, provided $g(a) \neq 0$.

THEOREM: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

It is tiresome to say "continuous at a for every $a \in \mathbb{R}$". The following definition is then convenient.

DEFINITION 29.1: Let I be an open interval of $\mathbb{R}$ of the form $I = (a, b)$, $I = (a, \infty)$, $I = (-\infty, a)$, or $I = (-\infty, \infty) = \mathbb{R}$. We say f is **continuous on** I if f is continuous at a for all $a \in I$.

(1) Let

$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x + 1 & \text{if } x < 1. \end{cases}$$

Use the $\varepsilon - \delta$ definition to show that $f(x)$ is continuous at 1.

(2) Which of the following functions are continuous on $\mathbb{R}$?

- $f(x) = \sqrt{x^2 + 5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \dfrac{1}{x}$.

(3) Which of the following functions are continuous on $(0, \infty)$?

- $f(x) = \sqrt{x^2 + 5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \dfrac{1}{x}$.

(4) Prove that $j(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous[1] on $\mathbb{R}$.

(5) Prove or disprove: If f and g are continuous at a, then f/g is continuous at a.

(6) Prove or disprove: If f and g are two functions, $a \in \mathbb{R}$, and $f(a) = g(a)$, then f is continuous at a if and only if g is continuous at a.

(7) Prove or disprove: If f and g are two functions, $a < b$, and $f(x) = g(x)$ for all $x \in (a, b)$, then f is continuous on (a, b) if and only if g is continuous on (a, b).

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[1]You can use without proof that $\sin(x)$ is continuous on $\mathbb{R}$.