

DEFINITION: Let  $R$  be a ring and  $M$  be a (left)  $R$ -module. A linear combination of finitely many elements  $m_1, \dots, m_n$  of  $M$  is an element of the form  $r_1 m_1 + \dots + r_n m_n \in M$  for some  $r_1, \dots, r_n \in R$ .

DEFINITION: Let  $R$  be a ring and  $M$  be a (left)  $R$ -module. Let  $A$  be a subset of  $M$ . The submodule of  $M$  **generated by**  $A$  is the submodule  $RA$  of  $M$  given by the three equivalent following descriptions:

- $RA$  is the unique smallest  $R$ -submodule of  $M$  containing  $A$ .
- $RA = \bigcap N_\lambda$ , where  $N_\lambda$  ranges over all submodules of  $M$  containing  $A$ .
- $RA = \{r_1 m_1 + \dots + r_t m_t \mid r_i \in R, m_i \in A\}$ , the set of linear combinations of elements of  $A$ .

DEFINITION: Let  $R$  be a ring and  $M$  be a (left)  $R$ -module. Let  $A$  be a subset of  $M$ .

- We say that  $A$  **generates**  $M$  if  $RA = M$ .
- We say that  $A$  is **linearly independent** if for  $m_1, \dots, m_t \in A$  distinct and any  $r_1, \dots, r_t \in R$ ,  

$$r_1 m_1 + \dots + r_t m_t = 0 \quad \text{implies} \quad r_1 = \dots = r_t = 0.$$
- We say that  $A$  is a **basis** of  $M$  if  $A$  is linearly independent and generates  $M$ .
- We say that  $M$  is **free** if there exists a basis  $A$  for  $M$ .

- (1) Let  $R = \mathbb{Z}$  and consider the  $R$ -module  $M = \mathbb{Z}/n$  for some  $n > 1$ .
  - (a) Explain why any nonempty subset of  $M$  is *not* linearly independent.
  - (b) Explain why  $M$  is *not* a free module.
  - (c) An  $R$ -module is **cyclic** if it is generated by a single element. Show that  $M$  is cyclic.
  - (d) Does every generating set of  $M$  consist of a single element?
- (2) Let  $R$  be a commutative ring. Let  $R[x]$  be a polynomial ring over  $R$ .
  - (a) Explain why  $\{1, x, x^2, x^3, \dots\}$  is a basis for  $R[x]$  as an  $R$ -module.
  - (b) Give an example of a set that is  $R$ -linearly independent in  $R[x]$  that is not a basis.
  - (c) Give an example of a set that generates  $R[x]$  that is not a basis.
  - (d) Give a different example of a basis for  $R[x]$ .
- (3) Show that an  $R$ -module  $M$  is cyclic if and only if  $M \cong R/I$  for some left ideal  $I$ .
- (4) Let  $R = \mathbb{Z}[x]$  and  $I$  be the ideal  $(2, x)$ , considered as an  $R$ -module.
  - (a) Explain<sup>2</sup> why  $I$  is not cyclic.
  - (b) Show that  $I$  is not free.
  - (c) Give an example of a pair of modules  $N \subseteq M$  where  $N$  requires more generators than  $M$ .
  - (d) Give an example of a pair of modules  $N \subseteq M$  where  $M$  is free and  $N$  is not.
- (5) We say that an  $R$ -module  $M$  is **simple** if the only submodules of  $M$  are  $0$  and  $M$ . Let  $R$  be a commutative ring. Show that  $M$  is simple if and only if  $M \cong R/\mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$  of  $R$ .
- (6) Let  $R$  be a commutative ring, and  $x, y$  be two indeterminates.
  - (a) Show that  $R[x, y]$  is a free  $R[x]$ -module, and find a basis.
  - (b) Show that  $R[x, y]$  is a free  $R$  module, and find a basis.

<sup>1</sup>This includes  $0$  as the “empty sum”.

<sup>2</sup>Reuse something from 817!