# Suprema and consequences

**Definition:** Let $S$ be a set of real numbers. A number $\ell$ is the *supremum* of $S$ provided

- $\ell$ is an upper bound of $S$ and
- if $b$ is any upper bound of $S$, then $\ell \leq b$.

**Theorem 5.3:** For every real number $r$, there is a natural number $n$ such that $n > r$.

**Corollary 5.4: (Archimedean Principle).** For every positive real number $a$ and every real number $b$, there is some natural number $n$ such that $na > b$.

**Theorem 5.5: (Density of rational numbers).** For any real numbers $x, y$ with $x < y$, there is some rational number $q$ such that $x < q < y$.

**Definition:** For a real number $x$, the *absolute value* of $x$ is $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

(1) Let $W$ be the set of real numbers $x$ that satisfy the inequality $x^3 + x < 10$.
   (a) Write $W$ mathematically in set notation.
   (b) Does $W$ have a supremum? Why or why not?
   (c) Is $\sup(W) = 1$? Why or why not?
   (d) Is $\sup(W) = 4$? Why or why not?

(2) Use the Archimedean principle to show that for any positive number $\varepsilon > 0$, there is a natural number $n$ such that $0 < \varepsilon < \dfrac{1}{n}$.

(3) Prove that the supremum of the set $S = \left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$ is 1.

(4) Let $S$ be a set of real numbers, and suppose that $\sup(S) = \ell$. Let $T = \{s + 7 \mid s \in S\}$. Prove that $\sup(T) = \ell + 7$.

(5) Prove the following:

**Corollary 6.1: (Density of irrational numbers).** For any real numbers $x, y$ with $x < y$, there is some irrational number $z$ such that $x < z < y$.

(6) True or false & justify: There is a rational number $x$ such that $|x^2 - 2| = 0$.

(7) True or false & justify: There is a rational number $x$ such that $|x^2 - 2| < \frac{1}{1000000}$.