

THE MAIN THEOREM OF SYLOW THEORY

RECALL: Let G be a finite group and p be a prime number. Write $|G| = p^e m$ with $e \geq 0$ and $p \nmid m$.

- A p -subgroup of G is a subgroup of order p^k for some $k \geq 0$.
- A Sylow p -subgroup of G is a subgroup of order p^e .
- We write $\text{Syl}_p(G)$ for the set of Sylow p -subgroups of G . We often write n_p for $\#\text{Syl}_p(G)$.

MAIN THEOREM OF SYLOW THEORY: Let G be a finite group and p be a prime number. Write $|G| = p^e m$ with $e \geq 0$ and $p \nmid m$.

- (1) There exists a Sylow p -subgroup of G .
- (2) Every Sylow subgroup is conjugate. Moreover, for any p -subgroup Q and any Sylow p -subgroup P , there is some $g \in G$ such that $Q \leq gPg^{-1}$.
- (3) The number of Sylow p -subgroups of G is congruent to 1 modulo p .
- (4) The number of Sylow p -subgroups of G divides m .

LEMMA: Let G be a finite group and p be a prime number. Let P be a Sylow p -subgroup of G and Q be any p -subgroup of G . Then $Q \cap N_G(P) = Q \cap P$.

- (1) Let $p < q$ be distinct primes and G be a group of order pq . Use the Sylow Theorem to show that G is not simple.
- (2) Consider $G = S_4$.
 - (a) Show¹ that G has a subgroup isomorphic to D_4 , the symmetry group of the square.
 - (b) Show that S_4 has exactly three subgroups isomorphic to D_4 , that these three are conjugate, and that any subgroup of S_4 of order 8 is isomorphic to D_4 .
 - (c) Describe the subgroups of order 3 of S_4 .
- (3) Proof of part (1) of Sylow's Theorem: Fix p . We will argue by induction on n that every group of n has a Sylow p -subgroup.
 - (a) Write $n = p^e m$. Address the case $e = 0$. Henceforth assume $e > 0$, so $p \mid n$.
 - (b) Case 1: Assume that p divides $|Z(G)|$. Explain why there is some $N \trianglelefteq G$ with $|N| = p$.
 - (c) Apply the induction hypothesis to G/N . How can you use this to find a Sylow p -subgroup in G ?
 - (d) Case 2: Assume that p does not divide $|Z(G)|$. Show that there is some $g \in G$ such that $[G : C_G(g)]$ is not a multiple of p and not one. What does this say about $|C_G(g)|$? What do you get from the induction hypothesis?
- (4) Proof of parts (2) and (3) of Sylow's Theorem: Fix a Sylow p -subgroup P . Let \mathcal{S}_P be the set of conjugates of P , namely $\{gPg^{-1} \mid g \in G\} \subseteq \text{Syl}_p(G)$. We need to show that (2) $\text{Syl}_p(G) = \mathcal{S}_P$ and that (3) $\#\text{Syl}_p(G) \equiv 1 \pmod{p}$.
 - (a) Let Q be any p -subgroup of G , and let Q act on \mathcal{S}_P by conjugation. Use the Lemma to show that for any $P_i \in \mathcal{S}_P$, $\text{Stab}_Q(P_i) = Q \cap P_i$.
 - (b) Show that $|\mathcal{S}_P| = \sum_{i=1}^s [Q : Q \cap P_i]$ where P_i ranges through a set of representatives of distinct orbits for the action of Q on \mathcal{S}_P .
 - (c) Take $Q = P$ and WLOG $P_1 = P$. Deduce that $|\mathcal{S}_P| \equiv 1 \pmod{p}$.
 - (d) To show (2) by contradiction, suppose that Q is not contained in any conjugate of P . Observe that $Q \cap P_i \subsetneq Q$ for all i . Revisit the equation in part (b) and the conclusion of part (c) to obtain a contradiction.
 - (e) Deduce part (3) from part (c) and part (2).

¹Hint: D_4 acts on the vertices of a square.