

CLASSIFYING ABELIAN GROUPS, AND OTHERS, UP TO ISOMORPHISM

STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: INVARIANT FACTORS:

Let G be a finitely generated abelian group. There exist integers $r \geq 0$, and $n_i \geq 2$, satisfying $n_1 \mid n_2 \mid \cdots \mid n_t$ such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_t.$$

Moreover, the list r, n_1, \dots, n_t is uniquely determined by G .

STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: ELEMENTARY DIVISORS:

Let G be a finitely generated abelian group. Then there exist integers $r \geq 0$, not necessarily distinct positive prime integers p_1, \dots, p_s , and integers $a_i \geq 1$ for $1 \leq i \leq s$ such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/p_1^{a_1} \times \cdots \times \mathbb{Z}/p_s^{a_s}.$$

Moreover, r and s are uniquely determined by G , and the list of prime powers $p_1^{a_1}, \dots, p_s^{a_s}$ is unique up to the ordering.

(1) Converting between forms:

To convert a cyclic group \mathbb{Z}/a to elementary divisor form, write each $a = p_1^{e_1} \cdots p_s^{e_s}$ as a product of prime powers, and use CRT get

$$\mathbb{Z}/a \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_s^{e_s}.$$

(a) Convert $\mathbb{Z}^2 \times \mathbb{Z}/50 \times \mathbb{Z}/60$ to elementary divisor form.

$$(\mathbb{Z}/2)^2 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/25$$

To convert a group from elementary divisor form to invariant factor form,

- For each distinct prime p_j occurring, take the largest power E_j it has in an elementary divisor, and combine and combine $\prod_j \mathbb{Z}/p_j^{E_j} \cong \mathbb{Z}/(p_1^{E_1} \cdots p_\ell^{E_\ell})$ via CRT. If there's more than one copy of $\mathbb{Z}/p_j^{E_j}$, just take one of the copies and leave the rest.
- Repeat with the remaining factors.

(b) Convert $\mathbb{Z}^3 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/9 \times \mathbb{Z}/27 \times \mathbb{Z}/25$ to invariant factor form.

$$\mathbb{Z}/12 \times \mathbb{Z}/36 \times \mathbb{Z}/2700$$

(2) Which of the following groups are isomorphic or not?

- $\mathbb{Z}/5 \times \mathbb{Z}/12 \times \mathbb{Z}/36$
- $\mathbb{Z}/10 \times \mathbb{Z}/12 \times \mathbb{Z}/18$
- $\mathbb{Z}/30 \times \mathbb{Z}/54$

We use elementary divisor form to decide:

- $\mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/9$
- $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/9$
- $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/5 \times \mathbb{Z}/27$

None are isomorphic. (We could also have used invariant factor form.)

(3) Classify all abelian groups of order 20 up to isomorphism. For each isomorphism class, give its expression in invariant factor form.

Note that $20 = 2^2 \times 5$. Thus, our possible elementary divisors are:

- 2, 2, 5
- 4, 5

This gives the two possibilities in invariant factor form:

- $\mathbb{Z}/2 \times \mathbb{Z}/10$
- $\mathbb{Z}/20$.

(4) Let $p < q$ be primes.

(a) Show that if p does not divide $q - 1$, then any group of order pq is isomorphic to C_{pq} by the following steps:

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for direct products.

By Sylow's Theorem, the number of q -Sylows n_q divides p . Since $p < q$, we have $n_q = 1$, so there is a unique, and hence normal, q -Sylow, Q . Also we have $n_p | q$ and $n_p \equiv 1 \pmod{p}$. By the hypothesis, $q \not\equiv 1 \pmod{p}$. It follows that $n_p = 1$, so there is a normal subgroup P of order p . We must have $P \cap Q = \{e\}$, since any element in the intersection has order dividing p and q , and $|PQ| = pq = |G|$, so $PQ = G$. Recognition Theorem for direct products, we have that $G \cong P \times Q$.

(b) Show from that if p does divide $q - 1$, then there are exactly two groups of order pq up to isomorphism by the following steps:

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for semidirect products.
- Use an Exercise from class about when two semidirect products are isomorphic.

We again have a unique q -Sylow Q for the same reasoning as above. now either $n_p = 1$ or $n_p = q$. If $n_p = 1$, then $G \cong P \times Q$ by the argument above. If $n_p = q$, then (using the same reasoning as above) the Recognition Theorem for semidirect products gives $G \cong Q \rtimes_{\phi} P$ for some ϕ . We claim that every nonabelian semidirect product $Q \rtimes_{\phi} P$ is isomorphic to each other. Indeed, since P is cyclic, by an exercise from class, if $\phi(P)$ and $\phi'(P)$ are conjugate in $\text{Aut}(Q)$, then the corresponding semidirect products are isomorphic. However, $\text{Aut}(Q)$ is cyclic, so it has a unique subgroup of order p . Since we assumed nonabelian, $\phi(P)$ is nontrivial, and hence has order p . This gives the uniqueness.

(5) Let p be a prime integer. Let G be a group of order p^2 .

(a) Show¹ that G is abelian.

(b) Classify all groups of order p^2 up to isomorphism.

(6) Let p, q be primes such that $q = p + 2$ and $p \geq 5$. Show that any group of order $p^2 q^2$ is either isomorphic to a cyclic group or a product of two cyclic groups.

¹Hint: If not, what can you say about $Z(G)$ and $G/Z(G)$?