Convergence of Sequences

Definition: Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence and L a real number. We say $\{a_n\}_{n=1}^{\infty}$ *converges* to L provided if for every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

To prove that a particular sequence $\{a_n\}_{n=1}^{\infty}$ converges to a particular real number L directly from the definition:

- Let $\varepsilon > 0$ be arbitrary.
- Take $N =$ [expression from scratchwork outside of the proof, maybe in terms of ε, that makes $|a_n - L| < \varepsilon$ whenever $n > N$].
- Let $n > N$ be a natural number.
- [Argument that $|a_n - L| < \varepsilon$ (that cannot refer to the previous scratchwork outside the proof)]
- Thus $\{a_n\}_{n=1}^{\infty}$ converges to L.

(1) Let c be a real number. Prove that the constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.

(2) Prove that[1] the sequence $\left\{\dfrac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 0.

(3) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose we know that $\{a_n\}_{n=1}^{\infty}$ converges to 1. Prove that there is a natural number $n \in \mathbb{N}$ such that $a_n > 0$.

(4) Prove or disprove: The sequence $\left\{\dfrac{n+1}{2n}\right\}_{n=1}^{\infty}$ converges to 0.

(5) Prove or disprove: The sequence[2] $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = \begin{cases} 1 & \text{if } n = 10^m \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

converges to 0.

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is *convergent* if there is a real number L such that $\{a_n\}_{n=1}^{\infty}$ converges to L. Otherwise, it is said to be *divergent*.

(6) In this problem, we will prove that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.
- Proceed by contradiction and suppose it converges to L.
- Apply the definition of "converges to L" with $\varepsilon = \frac{1}{2}$, so we get some N.
- Take an odd integer n bigger than N: what does this say about L?
- Take an even integer n bigger than N: what does this say about L?
- Conclude the proof.

[1] By $\sqrt{n}$, we mean the positive number whose square is n. Such a number exists by a proof similar to the one that $\sqrt{2}$ exists.

[2] That is, the sequence 1,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,…; the 1's get further and further apart.