Definition: Let S be a set of real numbers. A number ℓ is the *supremum* of S provided
- ℓ is an upper bound of S and
- if b is any upper bound of S, then $\ell \leq b$.

Theorem 5.3: For every real number r, there is a natural number n such that $n > r$.

Corollary 5.4: (Archimedean Principle). For every positive real number a and every real number b, there is some natural number n such that $na > b$.

Theorem 5.5: (Density of rational numbers). For any real numbers x, y with $x < y$, there is some rational number q such that $x < q < y$.

Definition: For a real number x, the *absolute value* of x is $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

(1) Let W be the set of real numbers x that satisfy the inequality $x^3 + x < 10$.
  (a) Write W mathematically in set notation.
  (b) Does W have a supremum? Why or why not?
  (c) Is $\sup(W) = 1$? Why or why not?
  (d) Is $\sup(W) = 4$? Why or why not?

(2) Use the Archimedean Principle to show that for any positive number $\varepsilon > 0$, there is a natural number n such that $0 < \varepsilon < \dfrac{1}{n}$.

(3) Prove that the supremum of the set $S = \left\{ 1 - \dfrac{1}{n} \mid n \in \mathbb{N} \right\}$ is 1.

(4) Let S be a set of real numbers, and suppose that $\sup(S) = \ell$. Let $T = \{s + 7 \mid s \in S\}$. Prove that $\sup(T) = \ell + 7$.

(5) Prove the following:

Corollary 6.1: (Density of irrational numbers). For any real numbers x, y with $x < y$, there is some irrational number z such that $x < z < y$.

(6) True or false & justify[1]: There is a rational number x such that $|x^2 - 2| = 0$.

(7) True or false & justify[1]: There is a rational number x such that $|x^2 - 2| < \frac{1}{1000000}$.

[1]You can use anything we've proven in class, but don't use things we haven't, like decimal expansions.