DEFINITION: Let S be a subset of a ring R. The **ideal generated by** S, denoted (S), is the smallest ideal containing S. Equivalently,

$$(S) = \left\{ \sum r_i s_i \mid r_i \in R, s_i \in S \right\} \quad \text{is the set of } R\text{-linear combinations[1] of elements of } S.$$

We say that S **generates** an ideal I if $(S) = I$.

DEFINITION: Let I, J be ideals of a ring R. The following are ideals:

- $IJ := (ab \mid a \in I, b \in J)$.
- $I^n := \underbrace{I \cdot I \cdots I}_{n \text{ times}} = (a_1 \cdots a_n \mid a_i \in I)$ for $n \geq 1$.
- $I + J := \{a + b \mid a \in I, b \in J\} = (I \cup J)$.
- $rI := (r)I = \{ra \mid a \in I\}$ for $r \in R$.
- $I : J := \{r \in R \mid rJ \subseteq I\}$.

DEFINITION: Let I be an ideal in a ring R. The **radical** of I is $\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \geq 1\}$. An ideal I is **radical** if $I = \sqrt{I}$.

DIVISION ALGORITHM: Let A be a ring, and $R = A[X]$ be a polynomial ring. Let $g \in R$ be a **monic** polynomial; i.e., the leading coefficient of f is a unit. Then for any $f \in R$, there exist unique polynomials $q, r \in R$ such that $f = gq + r$ and the top degree of r is less than the top degree of g.

(1) Briefly discuss why the two characterizations of (S) are equal.

(2) Finding generating sets for ideals: Let S be a subset of a ring R, and I an ideal.

- **(a)** To show that $(S) = I$, which containment do you think is easier to verify? How would you check?
- **(b)** To show that $(S) = I$ given $(S) \subseteq I$, explain why it suffices to show that $I/(S) = 0$ in $R/(S)$; i.e., that every element of I is equivalent to 0 modulo S.
- **(c)** Let K be a field, $R = K[U, V, W]$ and $S = K[X, Y]$ be polynomial rings. Let $\phi : R \to S$ be the ring homomorphism that is constant on K, and maps $U \mapsto X^2, V \mapsto XY, W \mapsto Y^2$. Show that the kernel ϕ is generated by $V^2 - UW$ as follows:
  - Show that $(V^2 - UW) \subseteq \ker(\phi)$.
  - Think of R as $K[U, W][V]$. Given $F \in \ker(\phi)$, use the Division Algorithm to show that $F \equiv F_1 V + F_0$ modulo $(V^2 - UW)$ for some $F_1, F_0 \in K[U, W]$ with $F_1 V + F_0 \in \ker(\phi)$.
  - Use $\phi(F_1 V + F_0) = 0$ to show that $F_1 = F_0 = 0$, and conclude that $F \in \ker(\phi)$.

(3) Radical ideals:

- **(a)** Fill in the blanks and convince yourself:
  - R/I is a field $\iff$ I is ________________
  - R/I is a domain $\iff$ I is ________________
  - R/I is reduced $\iff$ I is ________________
- **(b)** Show that the radical of an ideal is an ideal.
- **(c)** Show that a prime ideal is radical.
- **(d)** Let K be a field and $R = K[X, Y, Z]$. Find a generating set[2] for $\sqrt{(X^2, XYZ, Y^2)}$.

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[1]Linear combinations always means *finite* linear combinations: the axioms of a ring can only make sense of finite sums.

[2]Hint: To show your set generates, you might consider the bottom degree of F considered as a polynomial in X and Y.

(4) Evaluation ideals in polynomial rings: Let K be a field and $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in K^n$.

(a) Let $\mathrm{ev}_\alpha : R \to K$ be the map of evaluation at α: $\mathrm{ev}_\alpha(f) = f(\alpha_1, \dots, \alpha_n)$, or $f(\alpha)$ for short. Show that $\mathfrak{m}_\alpha := \ker \mathrm{ev}_\alpha$ is a maximal ideal and $R/\mathfrak{m}_\alpha \cong K$.

(b) Apply division repeatedly to show that $\mathfrak{m}_\alpha = (X_1 - \alpha_1, \dots, X_n - \alpha_n)$.

(c) For $K = \mathbb{R}$ and $n = 1$, find a maximal ideal that is not of this form. Same question with $n = 2$.

(d) With K arbitrary again, show that every maximal ideal $\mathfrak{m}$ of R for which $R/\mathfrak{m} \cong K$ is of the form $\mathfrak{m}_\alpha$ for some $\alpha \in K^n$. Note: this is *not* a theorem with a fancy German name.

(5) Lots of generators:

(a) Let K be a field and $R = K[X_1, X_2, \dots]$ be a polynomial ring in countably many variables. Explain[3] why the ideal $\mathfrak{m} = (X_1, X_2, \dots)$ cannot be generated by a finite set.

(b) Show that the ideal $(X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n) \subseteq K[X, Y]$ cannot be generated by fewer than $n + 1$ generators.

(c) Let $R = \mathcal{C}([0, 1], \mathbb{R})$ and $\alpha \in (0, 1)$. Show that for any element $g \in (f_1, \dots, f_n) \subseteq \mathfrak{m}_\alpha$, there is some $\varepsilon > 0$ and some $C > 0$ such that $|g| < C \max_i\{|f_i|\}$ on $(\alpha - \varepsilon, \alpha + \varepsilon)$. Use this to show that $\mathfrak{m}_\alpha$ cannot be generated by a finite set.

(6) Evaluation ideals in function rings: Let $R = \mathcal{C}([0, 1], \mathbb{R})$. Let $\alpha \in [0, 1]$.

(a) Let $\mathrm{ev}_\alpha : \mathcal{C}([0, 1]) \to \mathbb{R}$ be the map of evaluation at α: $\mathrm{ev}_\alpha(f) = f(\alpha)$. Show that $\mathfrak{m}_\alpha := \mathrm{ev}_\alpha$ is a maximal ideal and $R/\mathfrak{m}_\alpha \cong \mathbb{R}$.

(b) Show that $(x - \alpha) \subseteq \mathfrak{m}_\alpha$.

(c) Show that every maximal ideal R is of the form $\mathfrak{m}_\alpha$ for some $\alpha \in [0, 1]$. You may want to argue by contradiction: if not, there is an ideal I such that the sets $U_f := \{x \in [0, 1] \mid f(x) \neq 0\}$ for $f \in I$ form an open cover of $[0, 1]$. Take a finite subcover $U_{f_1}, \dots, U_{f_t}$ and consider $f_1^2 + \cdots + f_t^2$.

(7) Division Algorithm.

(a) What fails in the Division Algorithm when g is not monic? Uniqueness? Existence? Both?

(b) Review the proof of the Division Algorithm.

(8) Let K be a field and $R = K[\![X_1, \dots, X_n]\!]$ be a power series ring in n indeterminates. Let $R' = K[\![X_1, \dots, X_{n-1}]\!]$, so we can also think of $R = R'[\![X_n]\!]$. In this problem we will prove the useful analogue of division in power series rings:

WEIERSTRASS DIVISION THEOREM: Let $r \in R$, and write $g = \sum_{i \geq 0} a_i X_n^i$ with $a_i \in R'$. For some $d \geq 0$, suppose that $a_d \in R'$ is a unit, and that $a_i \in R'$ is *not* a unit for all $i < d$. Then, for any $f \in R$, there exist unique $q \in R$ and $r \in R'[X_n]$ such that $f = gq + r$ and the top degree of r as a polynomial in X_n is less than d.

(a) Show the theorem in the very special case $g = X_n^d$.

(b) Show the theorem in the special case $a_i = 0$ for all $i < d$.

(c) Show the uniqueness part of the theorem.[4]

(d) Show the existence part of the theorem.[5]

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[3]Hint: You might find it convenient to show that $(f_1, \dots, f_m) \subseteq (X_1, \dots, X_n)$ for some n, and then show that $(X_1, \dots, X_n) \subsetneq \mathfrak{m}$

[4]Hint: For an element of R' or of R, write ord' for the order in the $X_1, \dots, X_{n-1}$ variables; that is, the lowest total $X_1, \dots, X_{n-1}$-degree of a nonzero term (not counting X_n in the degree). If $qg + r = 0$, write $q = \sum_i b_i X_n^i$. You might find it convenient to pick i such that $\mathrm{ord}'(b_i)$ is minimal, and in case of a tie, choose the smallest such i among these.

[5]Hint: Write $g_- = \sum_{i=0}^{t-1} a_i X_n^i$ and $g_+ = \sum_{i=t}^{\infty} a_i X_n^i$. Apply (b) with g_+ instead of g, to get some q_0, r_0; write $f_1 = f - (q_0 g + r_0)$, and keep repeating to get a sequence of q_i's and r_i's. Show that $\mathrm{ord}'(q_i), \mathrm{ord}'(r_i) \geq i$, and use this to make sense of $q = \sum_i q_i$ and $r = \sum_i r_i$.