WARMING UP WITH IF-THEN STATEMENTS AND REAL NUMBERS

Making sense of if-then statements.

- The *converse* of the statement "If P then Q" is the statement "If Q then P".
- The *contrapositive* of the statement "If P then Q" is the statement "If not Q then not P".
- Any if then statement is equivalent to its contrapositive, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Is the original/contrapositive/converse true or false for real numbers a, b? Explain why (but don't prove).
  (a) If a is irrational, then $1/a$ is irrational.
  (b) If a and b are irrational, then ab is irrational.
  (c) If $a > 3$, then $a^2 > 9$.

Proving if-then statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if-then statement than the original, especially when the conclusion is something negative. We sometimes call this an *indirect proof* or a *proof by contraposition*.

(2) Consider the following proof of the claim "For real numbers x, y, z, if $x+y = z+y$, then $x = z$" from the list of axioms of $\mathbb{R}$. Match the parts of this proof with the general outline above. Which sentences are *assumptions* and which sentences are *assertions* (i.e., say something is true)? Is it clear *just from reading each sentence on its own* whether it is an assumption or an assertion?

Proof. Suppose that $x + y = z + y$. Then we can add $-y$ (which exists by Axiom 6) to get

$$(x + y) + (-y) = (z + y) + (-y).$$

This can be rewritten (by Axiom 3) as

$$x + (y + (-y)) = z + (y + (-y)),$$

and hence (by Axiom 6) as

$$x + 0 = z + 0,$$

which gives $x = z$ (by Axioms 4 and 2). $\square$

(3) Consider the following purported proof of the true fact "If $2x + 5 \geq 7$ then $x \geq 1$." Is this a good proof? Is it a correct proof?

Proof.

$$x \geq 1.$$

Multiply both sides by two.

$$2x \geq 2.$$

Add five to both sides.

$$2x + 5 \geq 7. \quad \square$$

Proving if-then statements about real numbers.

(4) Let x be a real number. Show that if x^2 is irrational, then x is irrational.

(5) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[1] that if $x \geq y$ then $-x \leq -y$.

(6) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[2] that $x \geq y$ if and only if $-x \leq -y$.

(7) Let x, y be real numbers. Use the axioms of $\mathbb{R}$ and facts we have already proven[3] to prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

(8) Use[4] the axioms of $\mathbb{R}$ and facts we have already proven to prove that $1 > 0$.

[1] Hint: You may want to add something to both sides.
[2] Hint: You may want to add something to both sides.
[3] Be careful: are you using any facts that we have not already proven?
[4] Hint: Try a proof by contradiction.