MATH 901 LECTURE NOTES, FALL 2021

Contents

1. Category Theory

1.1. **Categories.**

Lecture of August 23, 2021

1.1.1. *Definition of category.*

Definition 1.1. A *category* $\mathscr{C}$ consists of the following data:

(1) a collection of *objects*, denoted $\mathrm{Ob}(\mathscr{C})$,
(2) for each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* (also known as *arrows*) from A to B,
(3) for each triple of objects $A, B, C \in \mathrm{Ob}(\mathscr{C})$, a function

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)$$

written as $(\alpha, \beta) \mapsto \beta \circ \alpha$ that we call the *composition rule*.

These data are required to satisfy the following axioms:

(1) (Disjointness) the Hom sets are disjoint: if $A \neq A'$ or $B \neq B'$, then

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \cap \mathrm{Hom}_{\mathscr{C}}(A', B') = \varnothing.$$

(2) (Identities) for every object A, there is an *identity morphism* $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.
(3) (Associativity) composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

Remark 1.2. (1) The word "collection" as opposed to "set" is important here. The point is that there is no set of all sets, but by utilizing bigger collecting objects in set theory, we can sensibly talk about the collection of all sets. We'll sweep all of the set theory under the rug there, but it's worth keeping in mind that the objects of a category don't necessarily form a set. We did assume that the collections of morphisms between a pair of objects form a set, though not everyone does.
(2) The first axiom above guarantees that every morphism α in a category $\mathscr{C}$ has a well-defined *source* and *target* in $\mathrm{Ob}(\mathscr{C})$, namely, the unique A and B (respectively) such that $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

The name arrow dovetails with the common practice of depicting a morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ as

$$A \xrightarrow{\alpha} B.$$

The composition of $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ is $A \xrightarrow{\beta\circ\alpha} C$.

Optional Exercise 1.3. Prove that every element in a category has a unique identity morphism (i.e., a unique morphism that satisfies the hypothesis of axiom (2)).

1.1.2. *Examples of categories.* Many of our favorite objects from algebra naturally congregate in categories!

Example 1.4.
(1) There is a category **Set** where
- Ob(**Set**) is the collection of all sets
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$ is the the set of functions from X to Y
- the composition rule is composition of functions

We observe that every set has an identity function, which behaves as an identity for composition, and that composition of functions is associative.

(2) There is a category **Grp** where
- Ob(**Grp**) is the collection of all groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Grp}}(X,Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

Note that the identity function on a group is a group homomorphism, and that a composition of two group homomorphisms is a group homomorphism.

(3) There is a category **Ab** where
- Ob(**Ab**) is the collection of all abelian groups
- for two sets X, Y, $\mathrm{Hom}_{\mathbf{Ab}}(X,Y)$ is the the set of group homomorphisms from X to Y
- the composition rule is composition of functions

(4) In this class,
- A *semigroup* is a set S with an associative operation $\cdot$ that has an identity element; some may prefer the term *monoid*, but I don't.
- A *semigroup homomorphism* from semigroups $S \to T$ is a function that preserves the operation and maps the identity element to the identity element.

There is a category **Sgrp** where the objects are all semigroups and the morphisms are semigroup homomorphisms. (The composition rule is composition again.)

(5) In this class,
- A *ring* is a set R with two operations $+$ and $\cdot$ such that $(R,+)$ is abelian group, with identity 0, and $(R,\cdot)$ is a semigroup with identity 1, and such that the left and right distributive laws hold: $(r+s)t = rt + st$ and $t(r+s) = tr + ts$.
- A *ring homomorphism* is a function that preserves $+$ and $\cdot$ and sends 1 to 1.

There is a category **Ring** where the objects are all rings and the morphisms are ring homomorphisms.

(6) Let R be a ring. In this class,
- A *left R-module* is an abelian group $(M,+)$ equipped with a pairing $R \times M \to M$, written $(r,m) \mapsto rm$ or $(r,m) \mapsto r \cdot m$ such that
  (a) $r_1(r_2 m) = (r_1 r_2)m$,
  (b) $(r_1 + r_2)m = r_1 m + r_2 m$,
  (c) $r(m_1 + m_2) = rm_1 + rm_2$, and
  (d) $1m = m$.
- A *left module homomorphism* or *R-linear map* between left R-modules $\phi : M \to N$ is a homomorphism of abelian groups from $(M,+) \to (N,+)$ such that $\phi(rm) = r\phi(m)$.

There is a category $R-$**Mod** where the objects are all left R-modules and the morphisms are R-linear maps.

(7) There is a category **Fld** where the objects are all fields and the morphisms are all field homomorphisms.

(8) There is a category **Top** where the objects are all topological spaces and the morphisms are all continuous functions.

Remark 1.5. There are two special cases of the category of R-modules that are worth singling out:

- Every abelian group M is a $\mathbb{Z}$-module in a unique way, by setting

$$n \cdot m = \underbrace{m + \cdots + m}_{n-\text{times}} \quad \text{and} \quad -n \cdot m = -(\underbrace{m + \cdots + m}_{n-\text{times}}) \qquad \text{for } n \geqslant 0.$$

  Thus, **Ab** is basically just $\mathbb{Z} - \mathbf{Mod}$.
- When $R = K$ happens to be a field, we are accustomed to calling K-modules *vector spaces*. Thus, we might write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

Lecture of August 25, 2021

Example 1.6. Here are some variations on the category $K - \mathbf{Vect}$.

(1) The collection of finite dimensional K-vector spaces with all linear transformations is a category; call it $K - \mathbf{vect}$.
(2) The collection of all n-dimensional K-vector spaces with all linear transformations is a category.
(3) The collection of all K-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.
(4) The collection of all K-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it's not closed under composition.
(5) The collection of all n-dimensional vector spaces with singular linear transformations is not a category, since it doesn't have identity maps.

Example 1.7. (1) There is a category $\mathbf{Set}_*$ of *pointed sets* where
- the objects are pairs (X, x) where X is a set and $x \in X$,
- for two pointed sets, the morphisms from (X, x) to (Y, y) are functions $f : X \to Y$ such that $f(x) = y$,
- usual composition.

(2) For a commutative ring A,
- A *commutative A-algebra* is a commutative ring R plus a homomorphism $\phi : A \to R$.
- Slightly more generally, an *A-algebra* is a ring R plus a homomorphism $\phi : A \to R$ such that $\phi(A)$ lies in the center of R: $r \cdot \phi(a) = \phi(a) \cdot r$ for any $a \in A$ and $r \in R$. (In the more general situation, A is still commutative but R may not be.)
- An *A-algebra homomorphism* between two A-algebras (R, ϕ) and (S, ψ) is a ring homomorphism $\alpha : R \to S$ such that $\alpha \circ \phi = \psi$.

The category of A-algebras is denoted $A - \mathbf{Alg}$, and the category of commutative A-algebras is $A - \mathbf{cAlg}$.

(3) Fix a field K, and define a category $\mathbf{Mat}_K$ as follows: the objects are the positive natural numbers $n \in \mathbb{N}_{>0}$, and $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is the set of $b \times a$ matrices with entries in K. To see this as a category, we need a composition rule. Given $B \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ and $A \in \mathrm{Hom}_{\mathscr{C}}(a, b)$, take the composition $A \circ B \in \mathrm{Hom}_{\mathscr{C}}(a, c)$ to be the product AB. Since matrix multiplication is associative, axiom (3) holds, and the $n \times n$ identity matrix serves as an identity morphism in the sense of axiom (2). Finally, if $A \in \mathrm{Hom}_{\mathscr{C}}(a, b) \cap \mathrm{Hom}_{\mathscr{C}}(a', b')$, then A is a $b \times a$ matrix and a $b' \times a'$ matrix, so $a = a'$ and $b = b'$. Notably, the morphisms in this category are not functions.

We can also make a bunch of categories in a hands-on way as follows:

Example 1.8. Let $(P, \leqslant)$ be a poset. We define a category $\mathbf{PO}(P)$ from P as follows. The objects of $\mathbf{PO}(P)$ are just the elements of P. For each pair $a, b \in P$ with $a \leqslant b$, form a symbol f_a^b. Then we set

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a,b) = \begin{cases} \{f_a^b\} & \text{if } a \leqslant b \\ \varnothing & \text{otherwise.} \end{cases}$$

There is only one possible composition rule:

$$\mathrm{Hom}_{\mathbf{PO}(P)}(a,b) \times \mathrm{Hom}_{\mathbf{PO}(P)}(b,c) \longrightarrow \mathrm{Hom}_{\mathbf{PO}(P)}(a,c)$$

when $a \leqslant b$ and $b \leqslant c$ we also have $a \leqslant c$, and the unique pair on the left must map to the unique element on the right, so $f_b^c \circ f_a^b = f_a^c$; when either $a \nleqslant b$ or $b \nleqslant c$, there is nothing to compose!

Each morphism f_a^b is in only one Hom set (with source a and target b). Composition is associative since there is at most one function between one element sets. For any a, $f_a^a \in \mathrm{Hom}_{\mathbf{PO}(P)}(a,a)$ is the identity morphism.

For a specific example, we can think of $\mathbb{N}_{>0}$ as a category this way. Drawing all of the morphisms would be a mess, but any morphism is a composition of the ones depicted:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots .$$

Note that the objects of this category are exactly the same as in Example **??**(3), but with much fewer morphisms!

Example 1.9. A category with one object is nothing but a semigroup.

1.1.3. *Constructions of categories.* Here are a few more basic constructions of categories:

Definition 1.10. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{\mathrm{op}}$ is the category with $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$, and $\mathrm{Hom}_{\mathscr{C}}(A,B) = \mathrm{Hom}_{\mathscr{C}}(B,A)$ for all $A, B \in \mathrm{Ob}(\mathscr{C})$.

That is, the opposite category is the "same category with the arrows reversed." To avoid confusion, we might write α^{op} for the morphism $B \xrightarrow{\alpha^{\mathrm{op}}} A$ in $\mathscr{C}^{\mathrm{op}}$ corresponding to $A \xrightarrow{\alpha} B$ in $\mathscr{C}$.

Definition 1.11. Given two categories $\mathscr{C}$ and $\mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ is the category with $\mathrm{Ob}(\mathscr{C} \times \mathscr{D})$ given by the collection of pairs (C, D) with $C \in \mathrm{Ob}(\mathscr{C})$ and $D \in \mathrm{Ob}(\mathscr{D})$, and $\mathrm{Hom}_{\mathscr{C}\times\mathscr{D}}((A,B),(C,D)) = \mathrm{Hom}_{\mathscr{C}}(A,C) \times \mathrm{Hom}_{\mathscr{D}}(B,D)$. We leave it to you to pin down the composition rule.

Definition 1.12. A category $\mathscr{D}$ is a *subcategory* of another category $\mathscr{C}$ provided

(1) every object of $\mathscr{D}$ is an object of $\mathscr{C}$
(2) for every $A, B \in \mathrm{Ob}(\mathscr{D})$, $\mathrm{Hom}_{\mathscr{D}}(A,B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A,B)$, and
(3) for every $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ in $\mathscr{D}$, the composition of α and β in $\mathscr{D}$ equals the composition of α and β in $\mathscr{C}$.

If equality hold in (2) (for all A, B), we say that $\mathscr{D}$ is a *full subcategory* of $\mathscr{C}$.

Example 1.13. Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, since not every function between groups is a homomorphism, **Grp** is not a full subcategory of **Set**. Similarly, **Ab**, **Ring**, $R-$**Mod**, and **Top** are all subcategories of **Set**.

On the other hand, **Ab** is a full subcategory of **Grp**, and **Grp** is a full subcategory of **Sgrp**: a morphism of abelian groups is a morphism of groups that happens to be between abelian groups (and likewise for groups and semigroups)!

Lecture of August 27, 2021

1.2. Basic notions with morphisms.

Definition 1.14. A *diagram* in a category $\mathscr{C}$ is a directed multigraph whose vertices are objects in $\mathscr{C}$ and whose arrows/edges are morphisms in $\mathscr{C}$. A *commutative diagram* in $\mathscr{C}$ is a diagram in which for each pair of vertices A, B, any two paths from A to B compose to the same morphism.

Example 1.15. To say that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

commutes is to say that $\beta \circ \alpha = \delta \circ \gamma$ in $\mathrm{Hom}_{\mathscr{C}}(A, D)$.

Definition 1.16. Let $\mathscr{C}$ be any category and $A \xrightarrow{\alpha} B$ a morphism.

- α is an *isomorphism* if there exists $B \xrightarrow{\beta} A$ such that $\beta \circ \alpha = 1_A$ and $\alpha \circ \beta = 1_B$. Such an β is called the *inverse* of f.
- α has β as a *left inverse* if $\beta \circ \alpha = 1_A$. Similarly define *right inverse*.
- α is a *monomorphism* or is *monic* if for all arrows

$$\begin{tikzcd} C \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & A \arrow[r, "\alpha"] & B \end{tikzcd}$$

  if $\alpha\beta_1 = \alpha\beta_2$ then $\beta_1 = \beta_2$. That is, α can be cancelled from the left.
- α is an *epimorphism* or is *epic* if for all arrows

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & C \end{tikzcd}$$

  if $\beta_1\alpha = \beta_2\alpha$ then $\beta_1 = \beta_2$. That is, α can be cancelled from the right.

Remark 1.17. Note that α has a left inverse in $\mathscr{C}$ if and only if α^{op} has a right inverse in $\mathscr{C}^{\mathrm{op}}$, and that α is monic in $\mathscr{C}$ if and only if α^{op} is epic in $\mathscr{C}^{\mathrm{op}}$. We say that these are *dual* notions in category theory.

Lemma 1.18. *If α has a left inverse, then α is monic. Similarly for "right inverse" and "epic".*

Proof. If $\beta \circ \alpha = 1_A$ and γ_1, γ_2 are two morphisms from $C \to A$ such that $\alpha \circ \gamma_1 = \alpha \circ \gamma_2$, then

$$\gamma_1 = (\beta \circ \alpha) \circ \gamma_1 = \beta \circ (\alpha \circ \gamma_1) = \beta \circ (\alpha \circ \gamma_2) = (\beta \circ \alpha) \circ \gamma_2 = \gamma_2.$$

Similarly for "right inverse" and "epic". $\square$

Example 1.19. In **Set**, the monomorphisms and left-invertible morphisms agree, and these are the injective functions. The epimorphisms and right-invertible morphisms agree, and there are the surjective functions.

Optional Exercise 1.20. For any poset P, in $\mathbf{PO}(P)$, every morphism is both monic and epic, but no nonidentity morphism has a left or right-inverse.

1.3. **Category-theoretic constructions of objects.** A property or construction is *category theoretic* if can be described just in terms of the data of the category rather than aspects of a particular category.

Example 1.21. Can we identify $\varnothing$ in **Set** without looking at the objects' and morphisms' names? We can: for every set S, there is a unique function $f : \varnothing \to S$; $\varnothing$ is the only set with this property.

Definition 1.22.
(1) An object X in a category $\mathscr{C}$ is *initial* if there for every $Y \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $X \to Y$.
(2) An object X in a category $\mathscr{C}$ is *terminal* if there for every $Y \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $Y \to X$.

Example 1.23.
(1) We just saw that $\varnothing$ is initial in **Set**. Any singleton is terminal.
(2) A group with only one element $\{e\}$ is both initial and terminal in **Grp**.
(3) $\mathbb{Z}$ is initial in **Ring**.

1.3.1. *Definitions of product and coproduct.*

Definition 1.24. Let $\mathscr{C}$ be a category, and $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects. A *product* of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by an object P and a family of morphisms $\{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda}$ that is universal in the following sense:

Given an object Y and a family of morphisms $\{f_\lambda : Y \to X_\lambda\}_{\lambda\in\Lambda}$, there is a unique morphism $\phi : Y \to P$ such that $p_\lambda \circ \phi = f_\lambda$ for all λ.

Here is a diagram for the (first few) maps involved when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
& P \arrow[dr, "p_1"] \arrow[ddr, "p_2"] \arrow[dddr, "p_3"] \arrow[ddddr] & \\
& & X_1 \\
Y \arrow[uur, dashed, "\phi"] \arrow[urr, "f_1"] \arrow[rr, "f_2"] \arrow[drr, "f_3"] \arrow[ddrr] & & X_2 \\
& & X_3 \\
& & \vdots
\end{tikzcd}$$

We can also take a "big picture" view of this universal property:

$$\begin{tikzcd}
& P \arrow[dr, rightsquigarrow, "\{p_\lambda\}"] & \\
Y \arrow[ur, dashed, "\phi"] \arrow[rr, rightsquigarrow, "\{f_\lambda\}"'] & & \{X_\lambda\},
\end{tikzcd}$$

where the squiggly arrows are again collections of maps instead of maps. The data of Y with a family of maps to each X_λ is the sort of thing a product might be, so we may think of it as a "product candidate." In this way, we can think of a product as a "terminal product candidate."

Lecture of August 30, 2021

Remark 1.25. Note that $(P, \{p_\lambda\}_{\lambda\in\Lambda})$ is a product of $\{X_\lambda\}_{\lambda\in\Lambda}$ if and only if the function

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(Y,P) \arrow[r] & \bigtimes_{\lambda\in\Lambda} \operatorname{Hom}_{\mathscr{C}}(Y,X_\lambda) \\
\phi \arrow[r, mapsto] & (p_\lambda \circ \phi)_{\lambda\in\Lambda}
\end{tikzcd}$$

is a bijection for all objects Y: the universal property says that everything in the target comes from a unique thing in the source.

Definition 1.26. Let $\mathscr{C}$ be a category, and $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects. A *coproduct* of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by an object C and a family of morphisms $\{i_\lambda : X_\lambda \to C\}_{\lambda\in\Lambda}$ that is universal in the following sense:

Given an object Y and a family of morphisms $\{f_\lambda : X_\lambda \to Y\}_{\lambda\in\Lambda}$, there is a unique morphism $\phi : C \to Y$ such that $\phi \circ i_\lambda = f_\lambda$ for all λ.

Here is a diagram for the (first few) maps involved when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & C \arrow[ddr, dashed, "\phi"] & \\
\vdots \arrow[ur] \arrow[drr] & & \\
X_3 \arrow[uur, "i_3"] \arrow[rr, "f_3"] & & Y \\
X_2 \arrow[uuur, "i_2"] \arrow[urr, "f_2"] & & \\
X_1 \arrow[uuuur, "i_1"] \arrow[uurr, "f_1"] & &
\end{tikzcd}$$

We can also take a "big picture" view of the universal property:

$$\begin{tikzcd}
 & C \arrow[dr, dashed, "\phi"] & \\
\{X_\lambda\} \arrow[ur, squiggly, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_\lambda\}"'] & & Y,
\end{tikzcd}$$

where the squiggly arrows are now collections of maps instead of maps. We can again think of the coproduct as the "initial coproduct candidate."

Remark 1.27. Note that $(C, \{i_\lambda\}_{\lambda\in\Lambda})$ is a coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ if and only if the function

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(C,Y) \arrow[r] & \bigtimes_{\lambda\in\Lambda} \operatorname{Hom}_{\mathscr{C}}(X_\lambda,Y) \\
\phi \arrow[r, mapsto] & (\phi \circ i_\lambda)_{\lambda\in\Lambda}
\end{tikzcd}$$

is a bijection for all objects Y: the universal property says that everything in the target comes from a unique thing in the source.

Proposition 1.28. *If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda\in\Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a category $\mathscr{C}$, then there is a unique isomorphism $\alpha : P \xrightarrow{\sim} P'$ such that $p'_\lambda \circ \alpha = p_\lambda$ for all λ. The analogous statement holds for coproducts.*

Proof. We will just deal with products. The following picture is a rough guide:

$$\begin{tikzcd}
P \arrow[r, dashed, "\alpha"] \arrow[drr, squiggly, "\{p_\lambda\}"'] & P' \arrow[r, dashed, "\beta"] \arrow[dr, squiggly, "\{p'_\lambda\}"] & P \arrow[d, squiggly, "\{p_\lambda\}"] \\
& & \{X_\lambda\}
\end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta : P' \to P$ such that $p_\lambda \circ \beta = p'_\lambda$. Switching roles, we obtain a unique map $\alpha : P \to P'$ such that $p'_\lambda \circ \alpha = p_\lambda$.

Consider the composition $\beta \circ \alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ for all λ. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ for all λ, so by the uniqueness property of products, $\beta \circ \alpha = 1_P$. We can again switch roles to see that $\alpha \circ \beta = 1_{P'}$. Thus α is an isomorphism. The uniqueness of α in the statement is part of the universal property. $\square$

Optional Exercise 1.29. Prove the analogous statement for coproducts.

We use the notation $\prod_{\lambda\in\Lambda} X_\lambda$ to denote the (object part of the) product of $\{X_\lambda\}$ and $\coprod_{\lambda\in\Lambda} X_\lambda$ to denote the (object part of the) coproduct of $\{X_\lambda\}$.

Observe that products and coproducts are dual notions in the same way as monic versus epic morphisms. The product of a family in $\mathscr{C}$ is the coproduct of the same family in $\mathscr{C}^{\mathrm{op}}$.

1.3.2. *Products in familiar categories.* The familiar notion of Cartesian product or direct product serves as a product in many of our favorite categories. Let's note first that given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in any of the categories **Set**, **Sgrp**, **Grp**, **Ring**, $R -$ **Mod**, **Top**, the direct product $\times_{\lambda\in\Lambda} X_\lambda$ is an object of the same category:

- for sets, this is clear;
- for semigroups, groups, and rings, take the operation coordinate by coordinate: $(x_\lambda)_{\lambda\in\Lambda} \cdot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda}$;
- for modules, addition is coordinate by coordinate, and the action is the same on each coordinate: $r \cdot (x_\lambda)_{\lambda\in\Lambda} = (r \cdot x_\lambda)_{\lambda\in\Lambda}$;
- for topological spaces, use the product topology.

Note that this is not true for fields!

Proposition 1.30. *In each of the categories* **Set**, **Sgrp**, **Grp**, **Ring**, $R-$**Mod**, **Top**, *given a family $\{X_\lambda\}_{\lambda\in\Lambda}$, the direct product $\times_{\lambda\in\Lambda} X_\lambda$ along with the projection maps $\pi_\lambda : \times_{\gamma\in\Lambda} X_\gamma \to X_\lambda$ forms a product in the category.*

Proof. We observe that in each category, the direct product is an object, and the projection maps π_λ are morphisms in the category.

Let $\mathscr{C}$ be one of these categories, and suppose that we have morphisms $g_\lambda : Y \to X_\lambda$ for all λ in $\mathscr{C}$. We need to show there is a unique morphism $\phi : Y \to \times_{\lambda\in\Lambda} X_\lambda$ such that $\pi_\lambda \circ \phi = g_\lambda$ for all λ. The last condition is equivalent to $(\phi(y))_\lambda = (\pi_\lambda \circ \phi)(y) = g_\lambda(y)$ for all λ, which is equivalent to $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$, so if this is a valid morphism, it is unique. Thus, it suffices to show that the map $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$ is a morphism in $\mathscr{C}$, which is easy to see in each case. $\square$

1.3.3. *Coproducts in familiar categories.*

Example 1.31. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The product of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the cartesian product along with the projection maps. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the "disjoint union" with the various inclusion maps. By disjoint union, we simply mean union if the sets are disjoint; in general do something like replace X_λ with $X_\lambda \times \{\lambda\}$ to make them disjoint.

Proposition 1.32. *Let R be a ring, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left R-modules. A coproduct for the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is $(\bigoplus_{\lambda\in\Lambda} M_\lambda, \{\iota_\lambda\}_{\lambda\in\Lambda})$, where*

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \text{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda$$

is the direct sum of the modules M_λ, and ι_λ is the inclusion map to the λ coordinate.

Lecture of September 1, 2021

Remark 1.33. If the index set Λ is finite, then the objects $\prod_{\lambda\in\Lambda} M_\lambda$ and $\bigoplus_{\lambda\in\Lambda} M_\lambda$ are identical, but the product and coproduct are not the same since one involves projection maps and the other involves inclusion maps.

Proof. Given R-module homomorphisms $g_\lambda : M_\lambda \to N$ for each λ, we need to show that there is a unique R-module homomorphism $\alpha : \bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ such that $\alpha \circ \iota_\lambda = g_\lambda$. We define

$$\alpha((m_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} g_\lambda(m_\lambda).$$

Note that since $(m_\lambda)_{\lambda\in\Lambda}$ is in the direct sum, at most finitely many m_λ are nonzero, so the sum on the right hand side is finite, and hence makes sense in N. We need to check that α is R-linear; indeed,

$$\alpha((m_\lambda) + (n_\lambda)) = \alpha((m_\lambda + n_\lambda)) = \sum g_\lambda(m_\lambda + n_\lambda) = \sum g_\lambda(m_\lambda) + \sum g_\lambda(n_\lambda) = \alpha((m_\lambda)) + \alpha((n_\lambda)),$$

and the check for scalar multiplication is similar. For uniqueness of α, note that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is generated by the elements $\iota_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha' \circ \iota_\lambda = g_\lambda$ for all λ, then $\alpha(\iota_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(\iota_\lambda(m_\lambda))$ so the maps must be equal. $\square$

Remark 1.34. For any indexing set Λ, $\coprod_{\lambda\in\Lambda} R$ is a free R-module. If $R = K$ happens to be a field, then $\prod_{\lambda\in\Lambda} K$ is free, since all vector spaces are free modules, but in general, $\prod_{\lambda\in\Lambda} R$ is not free for an infinite set Λ.

Remark 1.35.
- In **Top**, disjoint unions serve as coproducts.
- In **Sgrp** and **Grp**, coproducts exist, and are given as free products. You may see or have seen them in topology in the context of Van Kampen's theorem.
- In **Ring**, the story is more complicated. Let's note first that disjoint unions won't work, since they aren't rings. Direct sums of infinitely many rings don't have 1, so aren't rings, but even finite direct sums/products won't work, since the inclusion maps don't send 1 to 1. We will later on construct coproducts in **cRing** , the full subcategory of **Ring** consisting of commutative rings.

1.4. Functors.

Definition 1.36. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ is a mapping that assigns to each object $A \in \mathrm{Ob}(\mathscr{C})$ an object $F(A) \in \mathrm{Ob}(\mathscr{D})$ and to each morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ such that

(1) F preserves compositions, meaning $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ for all morphisms α, β in $\mathscr{C}$, and
(2) F preserves identity morphisms, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

A *contravariant functor* $F : \mathscr{C} \to \mathscr{D}$ is a mapping that assigns to each object $A \in \mathrm{Ob}(\mathscr{C})$ an object $F(A) \in \mathrm{Ob}(\mathscr{D})$ and to each morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ a morphism $F(\alpha) \in \mathrm{Hom}_{\mathscr{D}}(F(B), F(A))$ such that

(1) F preserves compositions, meaning $F(\alpha \circ \beta) = F(\beta) \circ F(\alpha)$ for all morphisms α, β in $\mathscr{C}$, and
(2) F preserves identity morphisms, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

Remark 1.37. One can also interpret a contravariant functor as a covariant functor from $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$, or as a covariant functor from $\mathscr{C} \to \mathscr{D}^{\mathrm{op}}$.

Example 1.38. Here are some examples of functors.

(1) Many of the categories we considered before are sets with extra structure, and the morphisms are functions that preserve the extra structure. The *forgetful functor* from such a category $\mathscr{C}$ to **Set**, is the covariant functor that forgets that extra structure are returns the underlying set of the object. For example the forgetful functor **Grp** $\to$ **Set** sends each group to its set of elements, and each homomorphism to its corresponding function of sets. Along similar lines, a ring is a group under addition with the bonus structure of multiplication, and we can talk about the forgetful functor from **Ring** to **Grp**, etc.
(2) The *identity functor* $1_{\mathscr{C}}$ on any category $\mathscr{C}$ sends each object to itself and each morphism to itself. It is covariant.
(3) There is a covariant functor $(-)^{\times 2}$: **Set** $\to$ **Set** that sends every set S to its cartesian square $S \times S$, and every function $f : S \to T$ to the function $(f, f) : S \times S \to T \times T$ that sends $(s_1, s_2) \mapsto (f(s_1), f(s_2))$. Let's check the axioms: given $g : S \to T$ and $f : T \to U$, we need to see that $(f, f) \circ (g, g) = (f \circ g, f \circ g)$, which is clear, and that $(1_S, 1_S)$ is the identity map on $S \times S$, which is also clear.
(4) Given a group G, the subgroup $G' \leqslant G$ generated by the set of commutators $\{ghg^{-1}h^{-1} \mid g, h \in G\}$ is a normal subgroup, and the quotient $G^{\mathrm{ab}} := G/G'$ is called the *abelianization* of G. The group G^{ab} is abelian. Given a group homomorphism $\phi : G \to H$, ϕ automatically takes commutators to commutators, so it induces a homomorphism $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. Put together, abelianization gives a covariant functor from **Grp** to **Ab**.
(5) Given any topological space X, the set of continuous functions from X to $\mathbb{R}$, $\mathrm{Cont}(X, \mathbb{R})$ is a ring with pointwise addition and multiplication. Given a continuous map $X \xrightarrow{\alpha} Y$, and a continuous map $Y \xrightarrow{f} \mathbb{R}$, the composition $X \xrightarrow{\alpha \circ f} \mathbb{R}$ is a continuous function. In this way, we get a map from $\mathrm{Cont}(Y, \mathbb{R})$ to $\mathrm{Cont}(X, \mathbb{R})$. In fact, this map is a ring homomorphism. Put together, we obtain a contravariant functor from **Top** to **Ring**.
(6) Fix a field K. Given a vector space V, the collection V^* of linear transformations from V to K is again a K-vector space, the *dual vector space* of V. If $\phi : W \to V$ is a linear transformation and $\ell : V \to K$ is in V^* then $\ell \circ \phi : W \to K$ is in W^*, so there is a map $V^* \to W^*$. You can check that this together forms a functor $(-)^*$ that is contravariant.

(7) You may be familiar with the fundamental group of a pointed topological space; this is a group $\pi_1(X, x)$ assigned to a topological space and a point in it. The rule π_1 gives a functor from pointed topological spaces to groups.

(8) The unit group functor **Ring** $\to$ **Grp** sends each ring to its group of units. A homomorphism of rings restricts to a group homomorphism on the units: if $x \in R$ is a unit, so $xy = 1$, and $\phi : R \to S$ is a group homomorphism, then $1 = \phi(xy) = \phi(x)\phi(y)$, so $\phi(x)$ is a unit; ϕ preserves multiplication as well. This is covariant.

Lecture of September 3, 2021

It follows from the definition of covariant functor that if we apply a covariant functor F to a commutative diagram, we get another commutative diagram of the same shape, e.g.:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] \arrow[d, "F(\gamma)"'] & F(B) \arrow[d, "F(\beta)"] \\ F(C) \arrow[r, "F(\delta)"] & F(D). \end{tikzcd}$$

If we apply a contravariant functor G to a commutative diagram, we get a commutative diagram of the same shape with the arrows reversed, e.g.:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd} \quad \overset{G}{\rightsquigarrow} \quad \begin{tikzcd} G(A) & G(B) \arrow[l, "G(\alpha)"'] \\ G(C) \arrow[u, "G(\gamma)"] & G(D). \arrow[l, "G(\delta)"'] \arrow[u, "G(\beta)"'] \end{tikzcd}$$

Remark 1.39. A composition of two covariant functors, or of two contravariant functors, is a covariant functor. The composition of a covariant functor and a contravariant functor, or vice versa, is a contravariant functor.

1.5. **Natural transformations.**

Definition 1.40. Let F and G be covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A *natural transformation* η between F and G is a mapping that to each object A in $\mathscr{C}$ assigns a morphism $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes. We sometimes write $\eta : F \implies G$.

A *natural isomorphism* is a natural transformation η where each η_A is an isomorphism.

In short, a natural transformation is a rule to turn F of whatever into G of whatever in a reasonable way.

Optional Exercise 1.41. Let $F, G : \mathscr{C} \to \mathscr{D}$ be covariant functors. Show that a natural transformation $\eta : F \implies G$ is a natural isomorphism if and only if there is another natural transformation $\mu : G \implies F$ such that $\mu \circ \eta$ is the identity natural isomorphism on F and $\eta \circ \mu$ is the identity natural transformation on G.

We can make make a similar definition for contravariant functors.

Definition 1.42. Let F and G be contravariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A *natural transformation* between F and G is a mapping that to each object A in $\mathscr{C}$ assigns a morphism $\eta_A \in \operatorname{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] & G(A) \\
F(B) \arrow[u, "F(f)"] \arrow[r, "\eta_B"'] & G(B) \arrow[u, "G(f)"']
\end{tikzcd}$$

commutes.

Example 1.43. Let's describe a natural transformation of functors $\eta : (-)^{\times 2} \implies 1_{\mathbf{Set}}$, namely we take

$$\eta_S : S \times S \to S \qquad (s_1, s_2) \mapsto s_1.$$

We need to check for every map $f : S \to T$ the commutativity of a diagram:

$$\begin{tikzcd}
S \times S \arrow[r, "\eta_S"] \arrow[d, "(f,f)"'] & S \arrow[d, "f"] \\
T \times T \arrow[r, "\eta_T"'] & T.
\end{tikzcd}$$

Going either down then left or right then down, (s_1, s_2) maps to $f(s_1)$, so this does commute, and we indeed have a natural transformation. This is not a natural isomorphism, since the map η_S is not always (almost never) an isomorphism of sets.

Example 1.44. Let $\mathscr{C}$ be the full subcategory of **Set** consisting of countable sets. For every $S \in \operatorname{Ob}(\mathscr{C})$, there is a $\mathscr{C}$-isomorphism, i.e., a bijection, $\eta_S : S \to S \times S$. Namely, we can take η_S as follows: enumerate S as $S = \{s_1, s_2, s_3, \dots\}$, and do the usual zigzag trick

$$\begin{array}{l}
s_1 \mapsto (s_1, s_1) \\
s_2 \mapsto (s_2, s_1) \\
s_3 \mapsto (s_1, s_2) \\
s_4 \mapsto (s_1, s_3) \\
s_5 \mapsto (s_2, s_2) \\
s_6 \mapsto (s_3, s_1) \\
\quad \vdots
\end{array}
\qquad
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
(s_1, s_4) & (s_2, s_4) \arrow[dr] & (s_3, s_4) & (s_4, s_4) & \cdots \\
(s_1, s_3) \arrow[dr] & (s_2, s_3) \arrow[ul] & (s_3, s_3) \arrow[dr] & (s_4, s_3) \arrow[ul] & \cdots \\
(s_1, s_2) \arrow[u] & (s_2, s_2) \arrow[dr] & (s_3, s_2) \arrow[ul] & (s_4, s_2) & \cdots \\
(s_1, s_1) \arrow[r] & (s_2, s_1) \arrow[ul] & (s_3, s_1) \arrow[r] & (s_4, s_1) \arrow[ul] & \cdots
\end{tikzcd}$$

However, the bijections η_S do not form a natural bijection (in fact, if we just choose η_S like so for one set S, no matter what the other choices are, we can't get a natural transformation). Let $f : S \to S$ satisfy $f(s_1) = s_2$ and $f(s_2) = s_1$. Then in the diagram

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[d, "f"'] & S \times S \arrow[d, "(f,f)"] \\
S \arrow[r, "\eta_S"'] & S \times S,
\end{tikzcd}$$

we have $\eta_S(f(s_1)) = (s_2, s_1)$ while $(f, f)(\eta_S(s_1)) = (s_2, s_2)$, so the diagram does not commute.

Intuitively, we can blame the fact that our map η decided on a *choice* of enumeration of the set.

Example 1.45. Recall the contravariant functor $(-)^* : K-\mathbf{vect} \to K-\mathbf{vect}$; here we are restricting to finite dimensional vector spaces.

For every $V \in K-\mathbf{vect}$, there is an isomorphism $V \cong V^*$: if we fix a basis $\mathcal{B}$ for V, there is a dual basis for V^* (the $\mathcal{B}$-coordinate functions) of the same size, so they are isomorphic. However, there is no natural isomorphism $\eta : 1_{K-\mathbf{vect}} \implies (-)^*$, since $1_{K-\mathbf{vect}}$ is covariant and $(-)^*$ is contravariant. We will actually see a more compelling version of this nonnatruality statement in the homework.

Composing the dual functor with itself twice we get the covariant double-dual functor $(-)^{**} : K-\mathbf{vect} \to K-\mathbf{vect}$. We will show that there is a natural isomorphism $1_{K-\mathbf{vect}} \implies (-)^{**}$.

For every $v \in V$, there is a map $\mathrm{ev}_v : V^* \to V$ given by evaluation at v: $\mathrm{ev}_v(\ell) = \ell(v)$. So, $\mathrm{ev}_v \in V^{**}$. Since we have one for each v, there is a function $\mathrm{ev} : V \to V^{**}$ given by $\mathrm{ev}(v) = \mathrm{ev}_v$.

The map ev is a linear transformation:

$$\mathrm{ev}_{cv+w}(\ell) = \ell(cv + w) = c\ell(v) + \ell(w) = c\,\mathrm{ev}_v(\ell) + \mathrm{ev}_w(\ell).$$

It is injective, since any nonzero vector takes on a nonzero value for some linear functional. It is then a bijection since $\dim(V) = \dim(V^*) = \dim(V^{**})$.

We just need to check commutativity of the square:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "\phi"] & V^{**} \arrow[d, "\phi^{**}"] \\ W \arrow[r, "\mathrm{ev}"] & W^{**} \end{tikzcd}$$

This translates to

$$\mathrm{ev}_{\phi(v)} = (\mathrm{ev} \circ \phi)(v) \stackrel{?}{=} (\phi^{**} \circ \mathrm{ev})(v) = \phi^{**}(\mathrm{ev}_v)$$

in W^{**} for all $v \in V$. But, for all $\ell \in W^*$,

$$\mathrm{ev}_{\phi(v)}(\ell) = \ell(\phi(v)) = \phi^*(\ell)(v) = (\mathrm{ev}_v \circ \phi^*)(\ell) = \phi^{**}(\mathrm{ev}_v)(\ell),$$

so the equality holds.

In the homework, we will discuss some more examples from linear algebra. For example, for a pair of vector spaces $W \leqslant V$, there are isomorphisms $V \cong W \oplus V/W$, but no natural isomorphism of the sort. On the bright side, we will see that if V has an inner product, then V and V^* are naturally isomorphic in a suitable sense.

2. R-Modules

Lecture of September 8, 2021

2.0.1. *Left vs right vs both.* Recall that a left R-module is an abelian group M with an action map $R \times M \to M$ written $(r, m) \mapsto rm$ such that $r(sm) = (rs)m$, along with two distributive properties and the condition that 1 acts as the identity. A *right module* over R is defined similarly; we usually write the action as $(r, m) \mapsto mr$, and we have $(mr)s = m(rs)$, along with distributive and identity properties. The point is that when we act by a product $rs \in R$, we can think of it as an iterated action; in a left module, the left factor acts last while in a right module the right factor acts last.

Definition 2.1. If R is a ring, the *opposite ring* R^{op} is the ring with the same underlying set and same addition, but with multiplication given by $r \cdot_{R^{\mathrm{op}}} s = s \cdot_R r$.

A right R-module is exactly the same thing as a left R^{op}-module (except our convention for writing the action). In particular, if R is commutative, then a left R-module is exactly the same thing as a right R-module, and we will just say "module" in this case. By default, in general, when we say module, we will mean left R-module.

Example 2.2. Let R be a ring. The collection $M_n(R)$ of $n \times n$ matrices with entries in R forms a ring that in general is not commutative. The collection of column vectors of length n with entries in R is naturally a left $M_n(R)$-module. The collection of row vectors of length n with entries in R is naturally a right $M_n(R)$-module.

We can think of R-module structures in a different way. To prepare, let's record a lemma.

Lemma 2.3. *If M is an abelian group, then* $\mathrm{End}_{\mathbf{Ab}}(M) := \mathrm{Hom}_{\mathbf{Ab}}(M, M)$ *forms a ring with pointwise addition and composition as multiplication. More generally, if M is a left R-module, then* $\mathrm{End}_R(M) := \mathrm{Hom}_{R-\mathbf{Mod}}(M, M)$ *forms a ring (with the aforementioned operations).*

Proof. The first statement is a special case of the first, since an abelian group is the same thing as a $\mathbb{Z}$-module, so we'll prove the second. Let $f, g \in \mathrm{End}_R(M)$. Since

$$(f + g)(rm + n) = f(rm + n) + g(rm + n) = rf(m) + f(n) + rg(m) + g(n) = r(f + g)(m) + (f + g)(n)$$

we see that $f + g \in \mathrm{End}_R(M)$. It's easy to see that $\mathrm{End}_R(M)$ is an abelian group under $+$. Associativity of multiplication is a special case of associativity of composition of functions. For distributive laws, we have

$$((f + g)h)(m) = (f + g)(h(m)) = f(h(m)) + g(h(m)) = (fh)(m) + (gh)(m)$$
$$(f(g + h))(m) = f(g(m) + h(m)) = f(g(m)) + f(h(m)) = (fg)(m) + (fh)(m);$$

for the latter distributive law, it was crucial that we are dealing with homomorphisms of abelian groups. We also have the identity map on M as a mutliplicative identity. □

Optional Exercise 2.4. Show that there is a ring isomorphism $\mathrm{End}_R(R) \cong R^{\mathrm{op}}$.

Proposition 2.5. *Let R be a ring and $(M, +)$ an abelian group. There is a bijective correspondence*

$$\{R - \textit{module actions } R \times M \to M \textit{ (with given } +)\} \longleftrightarrow \{\textit{ring homomorphisms } \rho : R \to \mathrm{End}_{\mathbb{Z}}(M)\}$$

$$\cdot \longmapsto \rho(r)(m) = r \cdot m$$

$$r \cdot m = \rho(r)(m) \longleftarrow\!\shortmid\ \rho.$$

Proof. We clearly have a bijection as long as the maps are well-defined.

Given an R-module action $\cdot$, one distributive property translates to the condition that $\rho(r)$ is $\mathbb{Z}$-linear; the identity condition means $\rho(1_R)$ is the identity function on M, which is the 1 element in $\mathrm{End}_{\mathbb{Z}}(M)$; the other distributive condition means ρ preserves addition; and the associativity condition means ρ preserves multiplication. Thus, ρ is a ring homomorphism. And conversely. □

It turns out that we often have a left module structure and a right module structure on something in a compatible way.

Definition 2.6. Let R and S be rings. An (R, S)-*bimodule* is an abelian group M equipped with a left R-module structure and a right S-module structure that commute with each other:

$$(r \cdot m) \cdot s = r \cdot (m \cdot s) \qquad \text{for all } m \in M, r \in R, s \in S.$$

Example 2.7. Here are some basic sources of bimodules:

(1) If R is a ring, then $M = R$ is an (R, R)-bimodule in the obvious way. More generally, if $\phi : A \to R$ is a ring homomorphism, then R is an (R, A)-bimodule by

$$s \cdot r \cdot a = sr\phi(a) \qquad \text{for } r, s \in R, a \in A;$$

equally well, R is an (A, R) or (A, A)-bimodule.

(2) If R is a commutative ring and M is any left module, then M is also a right module by the same action, and M is an (R, R)-bimodule with these structures. I.e., starting with an action $r \cdot m$, we set $m \cdot s$ to be $s \cdot m$, and

$$(r \cdot m) \cdot s = s \cdot (r \cdot m) = sr \cdot m = rs \cdot m = r \cdot (s \cdot m) = r \cdot (m \cdot s).$$

(3) Every left R-module is automatically an $(R, \mathbb{Z})$-bimodule in a unique way:

$$(r \cdot m) \cdot n = \underbrace{(r \cdot m) + \cdots + (r \cdot m)}_{n \text{ times}} = r \cdot (\underbrace{m + \cdots + m}_{n \text{ times}}) = r \cdot (m \cdot n) \qquad \text{for } n \in \mathbb{Z}_{\geqslant 0},$$

and similarly for $n \leqslant 0$. Likewise, every right R-module is automatically a $(\mathbb{Z}, R)$-bimodule.

Example 2.8. For a ring R, the set of columns vectors of length n, R^n, is a $(\mathrm{Mat}_n(R), R)$-bimodule.

Sometimes, when we want to keep track of various module and bimodule structures, we may write something like ${}_R M_S$ to indicate that M is an (R, S)-bimodule, or ${}_R M$ to indicate that M is a left R-module.

2.1. **Kernels, images, and exact sequences.** To every homomorphism $\phi : M \to N$ in $R - \mathbf{Mod}$, the kernel $\ker(\phi) \subseteq M$ and image $\mathrm{im}(\phi) \subseteq N$ are in $R - \mathbf{Mod}$, and the inclusion maps are homomorphisms of R-modules. It is surprisingly convenient to keep track of and compare these data in terms of exact sequences.

Definition 2.9. A sequence of R-modules and R-module maps of the form

$$\cdots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

(possible infinite, possibly not) is a *chain complex*, or just *complex* for short, if $d_i \circ d_{i+1} = 0$ for all i or, equivalently, $\mathrm{im}(d_{i+1}) \subseteq \ker(d_i)$ for all i.

A chain complex is *exact* at M_i if $\mathrm{im}(d_{i+1}) = \ker(d_i)$; it is exact if it is exact at every module that has a map in and a map out.