CONTINUOUS FUNCTIONS

FROM LAST TIME:

Definition: A function f is *continuous at a* provided: For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $f(x)$ is defined and $|f(x) - f(a)| < \varepsilon$.

Theorem: If f is defined at a then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.

Theorem: If f and g are both continuous at a, and c is any constant, then

(1) $f + g$ is continuous at a.
(2) cf is continuous at a.
(3) fg is continuous at a.
(4) f/g is continuous at a, provided $g(a) \neq 0$.

Theorem: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

(1) Let
$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x + 1 & \text{if } x < 1 \end{cases}.$$
Use the $\varepsilon - \delta$ definition to show that $f(x)$ is continuous at 1.

(2) Let
$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
Show that $g(x)$ is continuous at 0 and is *not* continuous at any other real number. You can use any theorems you like and anything relevant from the homework.

(3) Let $h(x) = \sqrt{x^2 + 5}$. Show that h is continuous at a for every $a \in \mathbb{R}$.

It is tiresome to say "continuous at a for every $a \in \mathbb{R}$". The following definition is then convenient.

Definition 21.1: Let S be an open interval of $\mathbb{R}$ of the form $S = (a, b)$, $S = (a, \infty)$, $S = (-\infty, a)$, or $S = (-\infty, \infty) = \mathbb{R}$. We say f is *continuous on S* if f is continuous at a for all $a \in S$.

(4) Which of the following functions are continuous on $\mathbb{R}$?

- $f(x) = \sqrt{x^2 + 5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \frac{1}{x}$.

(5) Which of the following functions are continuous on $(0, \infty)$?

- $f(x) = \sqrt{x^2 + 5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \frac{1}{x}$.

(6) Prove that $j(x) = x \sin(1/x)$ is continuous on $\mathbb{R}$. (You can use wiithout proof that $\sin(x)$ is continuous on $\mathbb{R}$).

(7) Prove or disprove: If f and g are two functions, $a \in \mathbb{R}$, and $f(a) = g(a)$, then f is continuous at a if and only if g is continuous at a.

(8) Prove or disprove: If f and g are two functions, $a < b$, and $f(x) = g(x)$ for all $x \in (a, b)$, then f is continuous on (a, b) if and only if g is continuous on (a, b).

The definition of continuous on a closed interval $[a, b]$ is actually a bit different: we shouldn't necessarily ask that f be continuous at a, since to know that would have to use something about f on input values outside of our interval!

Definition 21.2: Given a function $f(x)$ and real numbers $a < b$, we say f is *continuous on the closed interval* $[a, b]$ provided

(1) for every $r \in (a, b)$, f is continuous at r in the sense defined already,
(2) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $a \leq x < a + \delta$, then $|f(x) - f(a)| < \varepsilon$.
(3) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $b - \delta < x \leq b$, then $|f(x) - f(b)| < \varepsilon$.

(9) Explain why if f is continuous at x for every $x \in [a, b]$, then f is continuous on the closed interval $[a, b]$. Conclude that every polynomial is continuous on every closed interval.

(10) Show that the function $f(x) = \sqrt{1 - x^2}$ is continuous on the closed interval $[-1, 1]$:

- For showing condition (1), I recommend using a Theorem from last class.
- For condition (2), it may help to write $\sqrt{1 - x^2} = \sqrt{1 - x}\sqrt{1 + x}$.
- Condition (3) is similar to condition (2) so you can just say "Similar to (2)" for that.