

DEFINITION: Let  $R$  be a commutative ring. Let  $V$  be a free  $R$ -module with ordered basis  $\mathcal{B} = \{b_1, \dots, b_n\}$  and let  $W$  be a free  $R$ -module with ordered basis  $\mathcal{C} = \{c_1, \dots, c_m\}$ . Given an  $R$ -module homomorphism  $T: V \rightarrow W$ , for each  $j = 1, \dots, n$ , write

$$(\clubsuit) \quad T(b_j) = r_{1,j}c_1 + \dots + r_{m,j}c_m$$

for some elements  $r_{i,j} \in R$ . The matrix

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{bmatrix}$$

is the **matrix representing  $T$  in the bases  $\mathcal{B}$  and  $\mathcal{C}$** .

LEMMA: Let  $R$  be a commutative ring. Consider the standard bases  $\mathcal{E}$  on  $R^n$  and  $\mathcal{E}'$  on  $R^m$ . For any linear transformation  $T: R^n \rightarrow R^m$ , we have

$$T(v) = [T]_{\mathcal{E}}^{\mathcal{E}'} \cdot v, \quad \text{where the RHS is usual matrix-times-vector multiplication.}$$

(1) Warming up with the definition:

- (a) If  $R$  is a field  $F$ , translate everything<sup>1</sup> in the definition into linear algebra terms.
- (b) Use the equation  $(\clubsuit)$  to explain as concretely as possible what the  $j$ -th column of  $[T]_{\mathcal{B}}^{\mathcal{C}}$  means in terms of  $T$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ .
- (c) Explain why the entries  $r_{i,j}$  are well-defined.
- (d) Just using your answer for part (b) and not looking at the formula, describe the dimensions of the matrix  $[T]_{\mathcal{B}}^{\mathcal{C}}$  in terms of the rank of  $V$  and the rank of  $W$ .
- (e) Let  $V$  be the  $\mathbb{R}$ -vector space of polynomials in  $\mathbb{R}[x]$  of degree at most 3 along with the zero polynomial. The derivative map  $\frac{d}{dx}$  is a linear transformation from  $V$  to  $V$ . Choose a basis  $\mathcal{B}$  for  $V$  and compute the matrix  $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{B}}$ .
- (f) Find another basis  $\mathcal{C}$  for  $V$  such that  $[\frac{d}{dx}]_{\mathcal{C}}^{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .

(2) Prove the Lemma above.

PROPOSITION: Let  $R$  be a commutative ring. Let  $V$  be a free  $R$ -module with ordered basis  $\mathcal{B} = \{b_1, \dots, b_n\}$  and let  $W$  be a free  $R$ -module with ordered basis  $\mathcal{C} = \{c_1, \dots, c_m\}$ . Then the map

$$\begin{array}{ccc} \text{Hom}_R(V, W) & \longrightarrow & \text{Mat}_{m \times n}(R) \\ T & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{C}} \end{array}$$

is bijective. Moreover, this is an isomorphism of  $R$ -modules.

When  $V = W$  and  $\mathcal{B} = \mathcal{C}$ , the same map

$$\begin{array}{ccc} \text{End}_R(V) & \longrightarrow & \text{Mat}_{n \times n}(R) \\ T & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{B}} \end{array}$$

is an isomorphism of rings.

(3) Prove that the map  $T \mapsto [T]_{\mathcal{B}}^{\mathcal{C}}$  in the Proposition is bijective.

(4) Suppose that  $V$  is a free module with a countably infinite basis  $\mathcal{B} = \{b_1, b_2, b_3, \dots\}$ , and  $W$  is free with a countably infinite basis  $\mathcal{C} = \{c_1, c_2, c_3, \dots\}$ . What is the correct analogue of the Proposition in this case?

(5) Prove the Proposition.

<sup>1</sup>You can do this aloud instead of rewriting everything.