

$$S = [R[x]]_{n \text{ of these}}$$

$$\text{Hom}_{D_{SIR}}(H_{SIR}^n(S), M) = 0.$$

for

$$M = \left\{ \begin{array}{l} R[x] \\ R[[x]] \\ R\{x\} \\ C^\infty(R^n) \end{array} \right\} \quad \text{because the only function (in } M \text{) with } x_i f = 0 \text{ all } x_i \text{ is zero function.}$$

Equally well can look for solutions to differential equations in any D-module:

$$\textcircled{X} \quad \left\{ \begin{bmatrix} S_{11} & \dots & S_{1m} \\ \vdots & & \vdots \\ S_{n1} & \dots & S_{nm} \end{bmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right. \quad \left. \begin{array}{l} \text{linear system of PDE's} \\ A \end{array} \right\}$$

$$N_A = \text{oker} \left(D_{SIR}^n \xrightarrow{A} D_{SIR}^m \right) \quad \begin{array}{l} \text{fin pres} \\ \text{D-module} \end{array}$$

$$\text{Sol}_{N_A}(M) := \text{Hom}_{D_{S(R)}}(N_A, M)$$

$\text{Sol}_{N_A}(-) : D\text{-mod} \rightarrow \mathbb{R}\text{-vector space}$
 can look for solutions to ~~sys~~ in any
 D -module M .

D -ideals & D -simplicity

Recall: A D -ideal is a D -submodule of R .

Def: A comm. ring R is D -module simple (over $A \rightarrow R$) if it is simple as a D -module.
 I.e., the only D -ideals of R are (0) and R .

Remark: Many sources say " D -simple"
 for what we call " D -module simple."

Lem: R is D -mod-simple $\Leftrightarrow \forall r \in R \setminus \{0\}$
 $\exists S \in D_{RIA} : S(r) = 1$.

pf: (\Leftarrow) If $I \neq (0)$ is a D -ideal, $r \in I \setminus \{0\}$,
 then $\exists S : S(r) = 1$. Then $1 \in I$, so $I = R$.

(\Rightarrow) For $r \in R$, $D_{RIA}(r) \subseteq R$ is a D -ideal. If $r \neq 0$,
 then $D_{RIA}(r) = R$, so $\exists S$ st. $S(r) = 1$. \square

Prop: If R is Noeth., reduced, and
 D -mod. simple, (for any A), then R
 is a domain.

pf: In this case, (0) is radical,
 so it is the intersection of all the
 minimal primes of R . Each of these
 is then a D -ideal, so there can only
 be just (0) ; i.e., (0) is prime. \square

Recall: In the Noeth. case, minimal
 primary of D -ideals are D -ideals:

$$(I :_{D_{RIA}} I) \subseteq (Q :_{D_{RIA}} Q)$$

if Q is a min primary cpt. of I .

Ex: Let R be a poly ring over a field k of char 0. Then R is D-mod simple. In fact, we proved something much stronger:

$$D_{Rk}^i \xrightarrow{\text{res.}} \text{Hom}_k([R]_{\leq i}, R)$$

is bijective. Thus, given $r \neq 0$, we have $r \in [R]_{\leq i}$ for some i , and there is some $s \in D_{Rk}^i$ with $s(r) = 1$.

In fact, the same argument works for any field of any characteristic.

(Exercise: double-check that that argument works char. free).

Ex: A Stanley-Reisner ring (other than a poly ring) is not D-mod simple.

E.g., for $R = k[x, y]/(xy)$, $(0), (x), (y), (x, y)$ are all D-ideals.

Ex: The cubic $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$
 is not D-mod simple. Indeed, there
 are no diff'l ops of negative degree,
 so each ideal $[R]_{\geq i}$ is a D-ideal.

Lem: Let $A \rightarrow R \rightarrow S$ comm. rings
 M, N S -modules. Then

$$D_{SIR}^i(M, N) \subseteq D_{SIA}^i(M, N) \subseteq D_{RA}^i(M, N)$$

trif Exercise.

Lem: Let $A \rightarrow R \xrightarrow{i} S$ comm. rings,
 and $\beta: S \rightarrow R$ R -linear. Then

there is a map $\rho: D_{SIA}^i \rightarrow D_{RA}^i$

$$\rho: D_{SIA}^i \xrightarrow{\quad M=N=S \quad} D_{RA}^i \xleftarrow{\quad M=N=R \quad}$$

$$S \longmapsto \beta \circ \text{id}_S,$$

(Warning: not a ring homomorphism).

Pf: We have $D_{SIA}^i \subseteq D_{RA}^i(S, S)$,

$\beta \in D_{RA}^i(S, R)$, $i \in D_{RA}^i(R, S)$,

so follows from general composition rule. \square

Thm (Smith): Let $A \rightarrow R \rightarrow S$ comm. rings,

S D -module simple (over A), and

R a direct summand of S . Then
 R is D -module simple.

Pf: $R \hookrightarrow S$, $S \twoheadrightarrow R$ splitting

so $\beta(I) = \beta(c(I)) = I$. Then $\forall r \in R \setminus \{0\}$,
 $c(r) \neq 0$ and $\exists S \in D_{SIA}$ with $S(c(r)) = I$.

So, $\beta(S)(r) = \beta(S(c(r))) = \beta(I) = I$. \square

Cor: A direct summand of a poly. ring
over a field is D -mod. simple.

In particular,
rings of invariants
of linearly reductive
groups,

$R \hookrightarrow S$

direct summand

$\Rightarrow IS \cap R = I$ all ideals I .

I D_R -ideal

e.g. finite groups
with $1 \in K^\times$,
 $(K^\times)^t$, and others,
are D -module simple.

Also $R = \mathbb{C}[\{A_{ij} \mid i < j\}]$ is D -mod simple.

Now, want to relate D -mod simplicity to
classes of singularities in positive characteristic.

Recall: $R \xrightarrow{F} R$

$$F(r) = r^p \quad \begin{array}{l} \text{is a ring homom.} \\ \text{if } \text{char } p > 0. \end{array}$$

Likewise

$$R \xrightarrow{F^e} R \quad F^e(r) = r^{p^e}$$

its ~~reates~~.

We will write ${}^e R$ for R with the
 R -module structure via rest. of scalars
through F^e . That is,

$$s \cdot r = s^e r$$

$\uparrow \quad \uparrow \quad \uparrow$

$R \curvearrowright {}^e R \quad {}^e R$

Thus, $R \xrightarrow{F^e} R$ is R -linear.

R is reduced $\Leftrightarrow F$ is injective

$\Leftrightarrow F^e$ is injective
for some \Leftrightarrow all e .

In this case (R reduced),

$\text{Frac}(R) := W^{-1}R$, W set of nonzero divisors
on R

is a product of fields, and

$R \hookrightarrow \text{Frac}(R) \cong \prod_{P \in \text{Min}(R)} \text{Frac}(R_P)$

↓
alg. closure

$\widehat{\text{Frac}(R)} := \prod_{P \in \text{Min}(R)} \widehat{\text{Frac}(R_P)}$

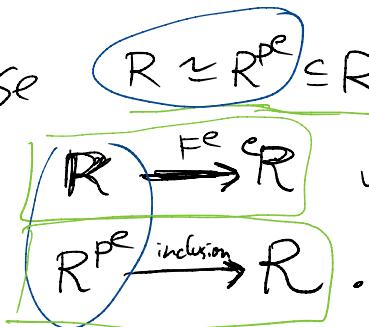
Then, we have $R^{1/p} := \{r \in \widehat{\text{Frac}(R)} / r^p \in R\}$.

is a subring of $\widehat{\text{Frac}(R)}$, and

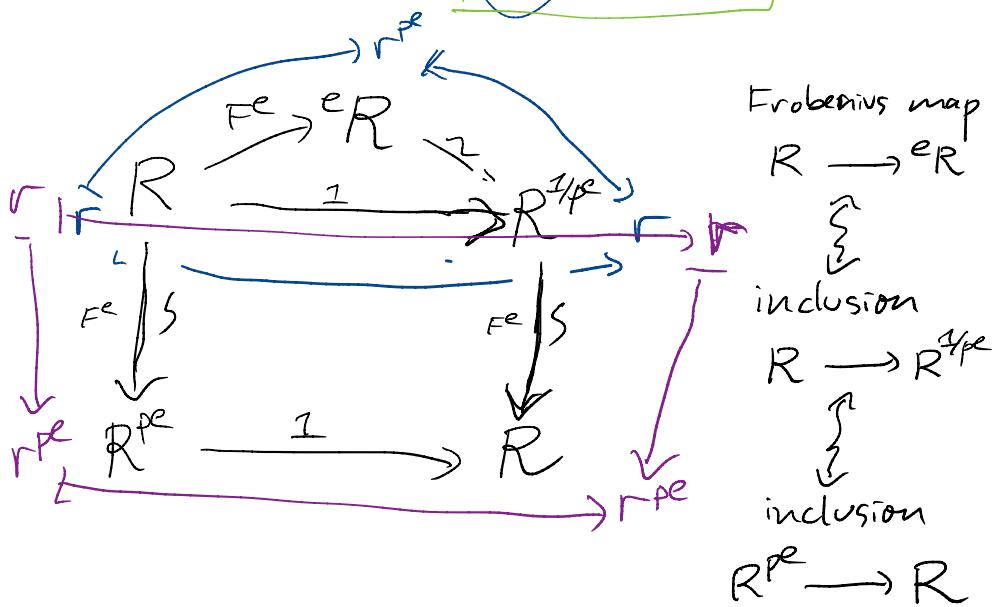
the R -module structure on $R^{1/p}$ via

restriction of scalars $R \subseteq R^{1/p}$ agrees with
 R -module $\widehat{\text{Frac}(R)}$ via $r \mapsto r^{1/p} := \begin{cases} \text{element } s \text{ such that} \\ s^p = r, s \in \text{Frac}(R) \end{cases}$.

Also in this case $R \cong R^{pe} \subseteq R$ and we can identify



with



$$\dots \cong R \xleftarrow{F} R \cong R^P \xleftarrow{F} R \cong R^{1_P} \xleftarrow{F} R \cong R^{1_{P2}} \cong \dots$$

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