Contents

1. Monday, August 26, 2024 §1.1

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \ldots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not havethought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\ldots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\ldots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number. Right? Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is* 2*.*

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \Longrightarrow \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction.

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. $\square$

Thus, if we would like to have a number corresponding to the length of the diagonal of a square with side length one, it must be a number that is real but not rational. Let's record the common name for such a number.

Definition 1.2. A real number is *irrational* if it is not rational.

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.3 (Arithmetic and order properties of $\mathbb{Q}$)**.** *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

(1) There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p+q$ and $p \cdot q$.
(2) Each of $+$ and $\cdot$ is a commutative operation (i.e., $p+q=q+p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).
(3) Each of $+$ and $\cdot$ is an associative operation (i.e., $(p+q)+r = p+(q+r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q, and r).
(4) The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0+q=q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.
(5) The distributive law holds: $p \cdot (q+r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.
(6) Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.
(7) Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.
(8) There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that
 (a) For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.
 (b) If $p \leq q$ and $q \leq p$, then $p = q$.
 (c) For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.
(9) The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.
(10) The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.

Which of the properties from Proposition 1.3 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$.

We expect everything from Proposition 1.3 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written $\mathbb{R}$, satisfies the following eleven properties:*

(Axiom 1) There are operations $+$ and $\cdot$ defined on $\mathbb{R}$, so that if $x, y \in \mathbb{R}$, then so are $x+y$ and $x \cdot y$.
(Axiom 2) Each of $+$ and $\cdot$ is a commutative operation.
(Axiom 3) Each of $+$ and $\cdot$ is an associative operation.
(Axiom 4) The real number 0 is an identity element for addition and the real number 1 is an identity element for multiplication. This means that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.
(Axiom 5) The distributive law holds: $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.

(Axiom 6) *Every real number has an additive inverse: For any $x \in \mathbb{R}$, there is a number $-x$ satisfying $x + (-x) = 0$.*

(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any $x \in \mathbb{R}$ such that $x \neq 0$, there is a real number x^{-1} satisfying $x^{-1} \cdot x = 1$.*

(Axiom 8) *There is a "total ordering" $\leq$ on $\mathbb{R}$. This means that*

- *(a) For all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.*
- *(b) If $x \leq y$ and $y \leq z$, then $x \leq z$.*
- *(c) For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.*

(Axiom 9) *The total ordering $\leq$ is compatible with addition: If $x \leq y$ then $x + z \leq y + z$ for all z.*

(Axiom 10) *The total ordering $\leq$ is compatible with multiplication by nonnegative real numbers: If $x \leq y$ and $z \geq 0$ then $zx \leq zy$.*

(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

> "Cancellation of Addition": For real numbers, $x, y, z \in \mathbb{R}$, if $x+y = z+y$ then $x = z$.

Let's prove this carefully, using just the list of axioms: Assume that $x + y = z + y$. Then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x + y) + (-y) = (z + y) + (-y)$. This can be rewritten as $x + (y + (-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

For another example, we can deduce the following fact from the axioms:

> For any real number r, we have $r \cdot 0 = 0$.

Let's prove this carefully: Let r be any real number. We have $0+0 = 0$ (Axiom 4) and hence $r \cdot (0+0) = r \cdot 0$. But $r \cdot (0+0) = r \cdot 0 + r \cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

Here is the incredible fact: everything that we know about the real numbers follows from these axioms! In particular, all of the basic facts about arithmetic of real numbers follow from this, like "the product of two negative numbers is positive," as well as everything you encountered in Calculus, like the Mean Value Theorem.

2. Wednesday, August 27, 2024 §1.4 & 1.5

Making sense of if-then statements.

- The statement "If P then Q" is true whenever Q is true or P is false. Equivalently, the statement "If P then Q" is false whenever Q is false and P is true.
- The **converse** of the statement "If P then Q" is the statement "If Q then P".
- The **contrapositive** of the statement "If P then Q" is the statement "If not Q then not P".

- Any if-then statement is equivalent to its *contrapositive*, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Decide if original/contrapositive/converse true or false for real numbers a, b, but don't prove them yet.
  (a) If a is irrational, then $1/a$ is irrational.
  (b) If a and b are irrational, then ab is irrational.
  (c) If $a \geq 3$, then $a^2 \geq 9$.

Proving if-then statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if-then statement than the original, especially when the conclusion is something negative. We sometimes call this an *indirect proof* or a *proof by contraposition.*

(2) Consider the following proof of the claim "For real numbers x, y, z, if $x + y = z + y$, then $x = z$" from the axioms of $\mathbb{R}$. Match the parts of this proof with the general outline above. Which sentences are *assumptions* and which are *assertions*? Is it clear *just from reading each sentence on its own* whether it is an assumption or an assertion? Is it clear *why* each assertion is true?

Proof. Suppose that $x + y = z + y$. Then adding $-y$ (which exists by Axiom 6) we get

$$(x + y) + (-y) = (z + y) + (-y).$$

This can be rewritten (by Axiom 3) as

$$x + (y + (-y)) = z + (y + (-y)),$$

and hence (by Axiom 6) as

$$x + 0 = z + 0,$$

which gives $x = z$ (by Axioms 4 and 2). $\square$

(3) Consider the following purported proof of the true fact "If $2x + 5 \geq 7$ then $x \geq 1$." Is this a good proof? Is it a correct proof?

Proof.

$$x \geq 1.$$

Multiply both sides by two.

$$2x \geq 2.$$

Add five to both sides.

$$2x + 5 \geq 7.$$

□

Proving if-then statements.

(4) Prove or disprove each of the statements in (1). You might consider a proof by contraposition for some of these!

(5) Prove or disprove the *converse* of each of the statements in (1).

Using the axioms of $\mathbb{R}$ to prove basic arithmetic facts.

(6) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[1] that if $x \geq y$ then $-x \leq -y$.

(7) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove that $x \geq y$ if and only if $-x \leq -y$.

(8) Let x, y be real numbers. Use the axioms of $\mathbb{R}$ and facts we have already proven[2] to prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

(9) Use[3] the axioms of $\mathbb{R}$ and facts we have already proven to prove that $1 > 0$.

3. Friday, August 29, 2024 §1.4 & 1.5

Making sense of quantifier statements.

- The symbol for "**for all**" is $\forall$ and the symbol for "**there exists**" is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that not P".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, not P".

A prankster has spraypainted the real number line red and blue, so every real number is red or blue (but not both)!

(1) Match each informal story (i)–(iv) below with a precise quantifier statement (A)–(D).

[1]Hint: You may want to add something to both sides.
[2]Be careful: are you using any facts that we have not already proven?
[3]Hint: Try a proof by contradiction.

Informal stories:

(i) Every number past some point is red.
(ii) There are arbitrarily big red numbers.
(iii) All positive numbers are red.
(iv) There are positive red number(s).

Precise statements:

(A) For every $y > 0$, y is red.
(B) There exists $y > 0$ such that y is red.
(C) For every $x \in \mathbb{R}$, there is some $y > x$ such that y is red.
(D) There exists $x \in \mathbb{R}$ such that for every $y > x$, y is red.

(2) Draw a picture where (A) is false and (B) is true.

(3) Draw a picture where (C) is true and (D) is false.

(4) Suppose that (C) is true. Which of the following statements must also be true? Why?
  (a) There is some $y > 1000000000$ such that y is red.
  (b) For every $\mu \in \mathbb{R}$, there is some $\theta > \mu$ such that θ is red.
  (c) For every $x \in \mathbb{R}$, there is some $y > 2x$ such that y is red.

The next problem is no longer about a spraypainting of the real number line.

(5) Rewrite each statement with symbols in place of quantifiers, and write its negation. Do you think the original statement is true or false (but don't prove them yet)?.
  (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
  (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
  (c) For all $x \in \mathbb{R}$ such that[4] $x \neq 0$, $x^2 > 0$.
  (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
  (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

Proving quantifier statements and using the axioms of $\mathbb{R}$.

- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

[4]In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

Note: explaining *how* you found your example "x" is *not* a logically necessary part of the proof.

(6) Circle the correct answer in each of the blanks below:
- To prove a "for all" statement, you need to give a
  GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *dis*prove a "for all" statement, you need to give a
  GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To prove a "there exists" statement, you need to give a
  GENERAL ARGUMENT / SPECIFIC EXAMPLE.
- To *dis*prove a "there exists" statement, you need to give a
  GENERAL ARGUMENT / SPECIFIC EXAMPLE.

(7) Prove or disprove each of the statements in (5) using the axioms of $\mathbb{R}$ and facts we have already proven.

More practice with quantifier statements.

(8) Prove that there exists some $x \in \mathbb{R}$ such that $2x + 5 = 3$.

(9) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.

(10) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have already proven to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

(11) Prove that[5] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.

(12) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.

4. Wednesday, September 3, 2024 §1.8

I owe you a statement of the very important Completeness Axiom. Before we get there, I want to recall an axiom of $\mathbb{N}$ that we haven't discussed yet. It pertains to minimum elements in sets. Let's be precise and define minimum element.

Definition 4.1. Let S be a set of real numbers. A **minimum** element of S is a real number x such that

(1) $x \in S$, and
(2) for all $y \in S$, $x \leq y$.

In this case, we write $x = \min(S)$.

The definition of **maximum** is the same except with the opposite inequality.

[5] Hint: Use (10).

Axiom 4.2 (Well-ordering axiom)**.** *Every nonempty subset of* $\mathbb{N}$ *has a minimum element.*

Example 4.3. If S is the set of even multiples of 7, then S has 14 as its minimum.

We generally like to say *the* minimum, rather than *a* minimum. To justify this, let's prove the following.

Proposition 4.4. *Let S be a set of real numbers. If S has a minimum, then the minimum is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P, then x and y must be the same thing.

Proof of Proposition 4.4. Let S be a set of real numbers, and let x and y be two minima of S. Applying part (1) of the definition of minimum to y, we have $y \in S$. Applying part (2) of the definition of minimum to x and the fact that $y \in S$, we get that $x \leq y$. Switching roles, we get that $y \leq x$. Thus $x = y$.

We conclude that if a minimum exists, it is necessarily unique. □

The previous proposition plus the Well-Ordering Axiom together imply that every nonempty subset of $\mathbb{N}$ has exactly one minimum element. A similar proof shows that if a maximum exists, it is necessarily unique. Could a set fail to have a maximum or a minimum? Yes!

Example 4.5.
(1) The empty set $\varnothing$ has no minimum and no maximum element. (There is no $s \in \varnothing$!)
(2) The set of natural numbers $\mathbb{N}$ has 1 as a minimum, but has no maximum. (Suppose there was: if $n = \max(\mathbb{N})$ was the maximum, then $n < n + 1 \in \mathbb{N}$ gives a contradiction.)
(3) The open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has no minimum and no maximum. (Exercise later.)

Definition 4.6. Let S be any subset of $\mathbb{R}$. A real number b is called an **upper bound** of S provided that for every $s \in S$, we have $s \leq b$.

Definition 4.7. A subset S of $\mathbb{R}$ is called **bounded above** if there exists at least one upper bound for S. That is, S is bounded above provided there is a real number b such that for all $s \in S$ we have $s \leq b$.

For example, the open interval $(0, 1)$ is bounded above, by for example 50.

The subset $\mathbb{N}$ of $\mathbb{R}$ is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of $\mathbb{R}$ that is bounded above.

Example 4.8. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 4.9. Suppose S is subset of $\mathbb{R}$ that is bounded above. A **supremum** (also known as a **least upper bound**) of S is a number ℓ such that

(1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
(2) if b is any upper bound of S, then $\ell \leq b$.

In this case we write $\sup(S) = \ell$.

Example 4.10. I claim 1 is a supremum of

$$(0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}.$$

It is by definition an upper bound. If b is any upper bound of $(0, 1]$ then, since $1 \in (0, 1]$, by definition we have $1 \leq b$. So 1 is the supremum of $(0, 1]$.

Observation 4.11. *Let S be a set of real numbers. Suppose that $b \in S$ and that b is an upper bound for S. Then*

(1) b is the maximum of S, and
(2) b is a supremum of S.

The subset $\mathbb{N}$ does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly: the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term "supremum", I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom). *Every nonempty, bounded-above subset of $\mathbb{R}$ has a supremum.*