

# **ASSIGNMENT #4**

- (1) Find two different composition series for the ring  $R = \mathbb{Z}[x]/(6, x^2)$  as a module over itself.
- (2) The purpose of this problem is to show that the ring

$$R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a \in \mathbb{Z}, b, c \in \mathbb{Q} \right\}$$

is left Noetherian but not right Noetherian.

- (a) Let  $I$  be a left ideal. Show that if there is some  $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in I$  with  $a \neq 0$ , then either

$$I = \begin{bmatrix} a\mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{bmatrix} \quad \text{or} \quad I = \begin{bmatrix} a\mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}.$$

- (b) Show that the left ideals contained in  $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$  are exactly the  $\mathbb{Q}$ -vector subspaces of  $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$ .

Conclude that  $R$  is left Noetherian.

- (c) Show that for every  $n \in \mathbb{N}$ , the subset of  $R$  given by  $\begin{bmatrix} 0 & 0 \\ \frac{1}{n}\mathbb{Z} & 0 \end{bmatrix}$  is a right ideal. Conclude that  $R$  is not right Noetherian.

- (3) Computations with Schur's Lemma.

- (a) Let  $D$  be a division ring and  $n \geq 1$  be an integer. Show<sup>1</sup> that  $\text{End}_{\text{Mat}_n(D)}(D^n) \cong D^{\text{op}}$ .
- (b) Let  $n \geq 3$ , and let  $D_{2n}$  be the dihedral group of order  $2n$ . Show that the standard representation of  $D_{2n}$  on  $V = \mathbb{R}^2$  acting by rotations and reflections is simple.
- (c) With  $V$  as in the previous part, show that  $\text{End}_{\mathbb{R}[D_{2n}]}(V) \cong \mathbb{R}$  as rings.

- (4) The goal of this problem is to find every irreducible  $\mathbb{R}$ -linear representation (up to isomorphism) of the dihedral group  $D_8$  of order 8.

- (a) Prove<sup>2</sup> there are 4 irreducible representations  $M_1, M_2, M_3, M_4$  whose underlying  $\mathbb{R}$ -vector space is just  $\mathbb{R}$  and that are pairwise nonisomorphic.
- (b) Let  $M_5$  be  $\mathbb{R}^2$  with the standard action of  $D_8$  (as in problem 3). Prove that  $M_1, \dots, M_5$  are the only irreducible  $\mathbb{R}$ -linear representations of  $D_8$  up to isomorphism.

- (5) Let  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$  denote the group of quaternions.

- (a) Find the Artin-Wedderburn decomposition of each of  $\mathbb{C}[Q]$  and  $\mathbb{C}[D_8]$  and use these to prove there is a ring isomorphism  $\mathbb{C}[Q] \cong \mathbb{C}[D_8]$ .

<sup>1</sup>Hint: let  $x \in D^{\text{op}}$  correspond to the map given by multiplication on the right by  $x$ .

<sup>2</sup>Recall that a 1-dimensional representation of  $G$  is a homomorphism from  $G$  to  $\mathbb{R}^\times$ , and that any such map factors through the abelianization of  $G$ . It might be helpful to note that the derived subgroup of  $D_8$  is  $\langle r^2 \rangle$  where  $r$  is the "rotation" and the abelianization of  $D_8$  is isomorphic to  $C_2 \times C_2$ .

- (b) Find<sup>3</sup> the Artin-Wedderburn decomposition of each of  $\mathbb{R}[Q]$  and  $\mathbb{R}[D_8]$  and use these to prove these rings are not isomorphic.

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<sup>3</sup>Hint: Use the previous problem to find the Artin-Wedderburn decomposition of  $\mathbb{R}[D_8]$ . To find the decomposition of  $\mathbb{R}[Q]$ , note that the group  $Q$  acts  $\mathbb{R}$ -linearly on the left on the division ring of real quaternions  $\mathbb{H}$ , via the identification of  $Q$  as the subgroup  $\{\pm 1, \pm i, \pm j, \pm k\}$  of  $\mathbb{H}^\times$ , and thus  $\mathbb{H}$  is a left  $\mathbb{R}[Q]$ -module. Thus, there is a surjective ring map  $\mathbb{R}[Q] \rightarrow \mathbb{H}$ .