Theorem 26.1: Let $f(x)$ be a function and let a be a real number. Let $r > 0$ be a positive real number such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$. Let L be any real number.

Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a and satisfies $0 < |x_n - a| < r$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Theorem 26.2. (Algebra and limits of functions): Suppose f and g are two functions and that a is a real number, and assume that

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

for some real numbers L and M. Then

(1) $\lim_{x\to a}(f(x) + g(x)) = L + M$.
(2) For any real number c, $\lim_{x\to a}(c \cdot f(x)) = c \cdot L$.
(3) $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$.
(4) If, in addition, we have that $M \neq 0$, then $\lim_{x\to a}(f(x)/g(x)) = L/M$.

Theorem 26.3. (Squeeze Theorem for limits): Suppose f, g, and h are three functions and a is a real number. Suppose there is a positive real number $r > 0$ such that

- each of f, g, h is defined on $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$,
- $f(x) \leq g(x) \leq h(x)$ for all x such that $0 < |x - a| < r$, and
- $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ for some number L.

Then $\lim_{x\to a} g(x) = L$.

(1) Use Theorem 26.1 to show that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Suggestion: Let $f(x) = \sin(\frac{1}{x})$ and suppose $\lim_{x\to 0} f(x) = L$. Find sequences $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ such that
- $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ both converge to 0,
- $f(x_n) = 1$ for all n, and
- $f(y_n) = -1$ for all n.

You can use any trig facts on the bottom of the page.

(2) Use Theorem 26.2 plus a fact from last time[1] to compute $\lim_{x\to 2} \dfrac{3x^2 - x + 2}{x + 3}$.

(3) Use Theorem 26.3 to show that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$. You can use any trig facts on the bottom of the page.

(4) Use Theorem 26.1 to deduce Theorem 26.3 from our Squeeze Theorem for sequences.

(5) Use Theorem 26.1 to deduce Theorem 26.2 part (1) from our Theorem on algebra and sequences.

(6) Use Theorem 26.1 to deduce Theorem 26.2 part (4) from our Theorem on algebra and sequences.

- $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$
- $\sin(x) = 1 \iff x \in \frac{\pi}{2} + 2\pi\mathbb{Z}$
- $\sin(x) = 0 \iff x \in \pi\mathbb{Z}$
- $\sin(x) = -1 \iff x \in \frac{-\pi}{2} + 2\pi\mathbb{Z}$
- $\pi \notin \mathbb{Q}$
- $\sin(x) = \sin(y) \iff x - y \in 2\pi\mathbb{Z}$ or $x + y \in \pi + 2\pi\mathbb{Z}$

[1] $\lim_{x\to a}\ mx + b = ma + b$. In particular, $\lim_{x\to a}\ x = a$ and $\lim_{x\to b}\ b = b$.