MAXIMAL IDEALS AND PRIME IDEALS

DEFINITION: Let R be a ring.

(i) An ideal I of R is a **maximal ideal** if I is proper and for any proper ideal J, $I \subseteq J$ implies $I = J$. That is, I is maximal under containment among all proper ideals of R.

(ii) Let R be commutative. An ideal I of R is a **prime ideal** if I is proper and $ab \in I$ implies $a \in I$ or $b \in I$.

THEOREM 1: Let R be a commutative ring and I an ideal.

(i) The ideal I is maximal if and only if R/I is a field.

(ii) The ideal I is prime if and only if R/I is an integral domain.

(1) Prime ideals vs maximal ideals:

(a) Use Theorem 1 to quickly explain why every maximal ideal in a commutative ring is prime.

(b) Show that the ideal (2) in $\mathbb{Z}[x]$ is prime but not maximal.

(c) Identify a maximal ideal in $\mathbb{Z}[x]$.

(2) Prove[1] Theorem 1.

THEOREM 2: Let R be a ring. Then R has a maximal ideal.

DEFINITION: Let $(P, \leq)$ be a partially ordered set.

(i) A **maximal element** of P is an element $x \in P$ such that for all $y \in P$, one has $y \leq x$.

(ii) A **upper bound** for a subset X is an element $x \in P$ such that for all $y \in X$, one has $y \leq x$.

(iii) A subset X of P is a **chain** if for all $x, y \in X$ either $x \leq y$ or $y \leq x$.

ZORN'S LEMMA: Let $(P, \leq)$ be a nonempty partially ordered set. If every chain $C \subseteq P$ has an upper bound $c \in P$, then P has a maximal element.

(3) Zorn's Lemma warmup.

(a) The most common use of Zorn's Lemma occurs in the following situation: $\mathcal{P}(Y)$ is the collection of all subsets of some set Y ordered by inclusion ($A \leq B$ if and only if $A \subseteq B$), and P is some special family of subsets of $\mathcal{P}(Y)$. Rewrite[2] the statement of Zorn's Lemma in this context.

(b) In the context above, explain how to use Zorn's lemma to try to show the existence of a *minimal element* of P.

(4) Prove Theorem 2.

(5) Prove or disprove: Any group G has a maximal proper subgroup (meaning a proper subgroup that is maximal among all proper subgroups).

(6) Prove that every prime ideal contains a minimal prime ideal.

[1]Hint: For part (i), you might want use a HW problem characterizing fields in terms of ideals.

[2]Meaning replace all $\leq$ with $\subseteq$ and unpackage the definitions of maximal element and upper bound.