

ASSIGNMENT #3: DUE FRIDAY, OCTOBER 11 AT 7PM

This problem set is to be turned in by Gradescope. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

- (1) Let K be a field, and $R = K[X^2, X^3] \subseteq S = K[X]$. Show that for the ideal I of R generated by X^2 , we have $IS \cap R \supsetneq I$. Conclude that R is not a direct summand of S .
- (2) Something Nullstellensatz
- (3) Something graded
- (4) Suppose that R is a finitely-generated \mathbb{Z} -algebra, and that \mathfrak{m} is a maximal ideal of R . Show that R/\mathfrak{m} is a finite field.
- (5) Let U, \dots, Z be indeterminates over \mathbb{C} , and $R = \mathbb{C} \begin{bmatrix} U & V & W \\ X & Y & Z \end{bmatrix} / I$ where $I = I_2 \left(\begin{bmatrix} U & V & W \\ X & Y & Z \end{bmatrix} \right)$.
In this problem, we will show that $A = \mathbb{C}[u, v - x, w - y, z] \subseteq R$ is a Noether normalization, where u, \dots, z denote the images of U, \dots, Z in R .
 - (a) Use Graded NAK¹ to R as a graded A -module to show that $A \subseteq R$ is module-finite.
 - (b) Show that for any $\alpha, \beta, \gamma, \zeta \in \mathbb{C}$, there is a rank one 2×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ with $a, \dots, f \in \mathbb{C}$ such that $a = \alpha, b - d = \beta, c - e = \gamma, f = \zeta$.
 - (c) Suppose that $F(u, v - x, w - y, z)$ is a \mathbb{C} -algebraic relation on $u, v - x, w - y, z$, so $F(U, V - X, W - Y, Z) \in I$. Use the previous part to show that $F(\alpha, \beta, \gamma, \zeta) = 0$ for all $\alpha, \beta, \gamma, \zeta \in \mathbb{C}$, and deduce² that $u, v - x, w - y, z$ are algebraically independent.

The remaining problems are to be solved with Macaulay2. Copy or screenshot the inputs and outputs (e.g., from the right-hand pane on the web interface) you used to solved these problems.

- (6) Read the discussion of modules in Macaulay2 from §1.1 of Grifo's Lecture Notes. Then let

$$R = \frac{\mathbb{Q}[U, V, W, X, Y, Z]}{(UY - VX, UZ - WX, VZ - WY)}$$

- the ideal (U, V, W) ,
- the submodule of R^2 generated by $\begin{bmatrix} U \\ X \end{bmatrix}$, $\begin{bmatrix} V \\ Y \end{bmatrix}$, and $\begin{bmatrix} W \\ Z \end{bmatrix}$.
- the submodule of R^2 generated by $\begin{bmatrix} U + V \\ W + X \end{bmatrix}$, $\begin{bmatrix} V + W \\ X + Y \end{bmatrix}$, and $\begin{bmatrix} W + X \\ Y + Z \end{bmatrix}$.

¹One could instead show that the generators of R are integral over A , but we will try out the power of Graded NAK.

²You can use without proof that any nonzero (multivariate) polynomial over an infinite field is a nonzero function.