Math 445 — Problem Set #1
Due: Friday, September 1 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Which of the following are true?
   (a) $10 \equiv 45 \pmod 5$
   (b) $19 \equiv 2 \pmod{12}$
   (c) $150974 \equiv 6 \pmod 8$.

(2) Let m, m', n, n', K be integers with $K > 0$. Prove that if

$$m \equiv m' \pmod K \quad \text{and} \quad n \equiv n' \pmod K$$

then

$$m + n \equiv m' + n' \pmod K \quad \text{and} \quad mn \equiv m'n' \pmod K.$$

(3) Divisibility tests and congruences:
   (a) Show that any natural number is congruent modulo 4 to the two digit number (in base ten) that corresponds to its last two digits. Use this to show that a number is divisible by 4 if and only if its last "two digit part" is divisible by 4.
   (b) Show that any natural number is congruent modulo 8 to the three digit number (in base ten) that corresponds to its last three digits. Use[1] this to show that a number is divisible by 8 if and only if its "last three digit part" is divisible by 8.
   (c) Show[2] that any natural number is congruent modulo 3 to the sum of its digits. Use this to show that a number is divisible by 3 if and only the sum of its digits is divisible by 3.
   (d) Show that any natural number is congruent modulo 9 to the sum of its digits. Use this to show that a number is divisible by 9 if and only the sum of its digits is divisible by 9.
   (e) Show that any natural number is congruent modulo 11 to the alternating sum of its digits, i.e.

$$1\text{s digit} - 10\text{s digit} + 100\text{s digit} \pm \cdots .$$

   Use this to show that a number is divisible by 11 if and only the alternating sum of its digits is divisible by 11.

[1] The step from the first sentence to the second sentence is similar to that in part (a); once you are convinced of this, you can just say this instead of repeating the argument.

[2] Hint: Start by showing that $10^k \equiv 1 \pmod 3$ for any k.

(4) The number 150974 is a sum of three squares:

$$362^2 + 141^2 + 7^2 = 150974.$$

In this problem we will show that 150975 is *not* a sum of three squares; i.e., there are no integers a, b, c such that

$$a^2 + b^2 + c^2 = 150975.$$

(a) Show that if a is odd, then $a^2 \equiv 1 \pmod 8$.
(b) Show[3] that if a is even, then either $a^2 \equiv 0 \pmod 8$ or $a^2 \equiv 4 \pmod 8$.
(c) Show that if $n = a^2 + b^2 + c^2$, then $n \equiv 7 \pmod 8$ is impossible.
(d) Conclude that 150975 is not a sum of three squares.

(5) Let a, b, c be integers. Use prime factorization to show that if a and b have no common prime factor and a divides bc, then a divides c.

The remaining problems are only required for Math 845 students, though all are encouraged to think about them.

(6) Recall that the Fibonacci sequence is given by the formula

$$f_{n+2} = f_{n+1} + f_n, \;\; f_0 = f_1 = 1.$$

For which n is f_n a multiple of 2? A multiple of 4? A multiple of 5?

(7) Find a formula for all of the rational points (x, y) on the hyperbola $x^2 - 2y^2 = 1$.

[3] Hint: Every even number is congruent to 0 mod 4 or to 2 mod 4.