

## UPPER BOUNDS AND THE COMPLETENESS AXIOM

Let  $S$  be a set of real numbers.

- A number  $b$  is an *upper bound* for  $S$  provided for all  $x \in S$  we have  $b \geq x$ .
- The set  $S$  is *bounded above* provided there exists at least one upper bound for  $S$ .
- A number  $m$  is the *maximum* of  $S$  provided
  - (1)  $m \in S$ , and
  - (2)  $m$  is an upper bound of  $S$ .
- A number  $\ell$  is a *supremum* of  $S$  provided
  - (1)  $\ell$  is an upper bound of  $S$ , and
  - (2) for any upper bound  $b$  for  $S$ , we have  $\ell \leq b$ .

- (1) Write, in simplified form, the negation of the statement “ $b$  is an upper bound for  $S$ ”.
- (2) Write, in simplified form, the negation of the statement “ $S$  is bounded above”.
- (3) Let  $S$  be a set of real numbers and suppose that  $\ell = \sup(S)$ .
  - (a) If  $x > \ell$ , what is the most concrete thing you can say about  $x$  and  $S$ ?
  - (b) If  $x < \ell$ , what is the most concrete thing<sup>1</sup> you can say about  $x$  and  $S$ ?
- (4) Let  $S$  be a set of real numbers, and let<sup>2</sup>  $T = \{2s \mid s \in S\}$ . Prove<sup>3</sup> that if  $S$  is bounded above, then  $T$  is bounded above.
- (5) Let  $S$  be a set of real numbers. Show that if  $S$  has a supremum, then it is unique.
- (6) Let  $S$  be a set of real numbers, and let  $T = \left\{\frac{s}{2} \mid s \in S\right\}$ . Directly<sup>4</sup> prove<sup>5</sup> that if  $S$  is unbounded above, then  $T$  is unbounded above.

<sup>1</sup>Hint: Use one of the previous problems.

<sup>2</sup>For example, if  $S = \{-1, 1, 2\}$ , then  $T = \{-2, 2, 4\}$ .

<sup>3</sup>First, before all else, this is an if then statement: start by assuming the “if” part. We now need to show the “then” part, which is about the existence of an upper bound. Use your assumption about  $S$  to find an upper bound for  $T$  (and prove that it is indeed an upper bound for  $T$ ).

<sup>4</sup>Don’t try to apply (4), just prove it directly, perhaps using your simplified description of “unbounded above” from (2). This also means don’t take the contrapositive and don’t proceed by contradiction.

<sup>5</sup>First, before all else, this is an if then statement: start by assuming the “if” part. We now need to show that  $T$  is unbounded above, which before everything else, is a for all statement. Use the hypothesis to get the number that you want. . .

WELL-ORDERING AXIOM: Every nonempty subset of  $\mathbb{N}$  has a minimum.

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

(1) Prove the following:

THEOREM: For every real number  $r$ , there is some natural number  $n$  such that  $n > r$ .

(2) Prove the following:

THEOREM: For every<sup>6</sup> real number  $r$ , there exists a unique integer  $n$  such that  $n - 1 \leq r < n$ .

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<sup>6</sup>Hint: First deal with the case  $r \geq 0$ .

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

THEOREM 4.1: For every real number  $r$ , there is some natural number  $n$  such that  $n > r$ .

*Proof of Theorem 4.1:*

(1) Proceed by contradiction: if not, then the set of natural numbers  $\mathbb{N}$  is

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(2) Since  $1 \in \mathbb{N}$ ,  $\mathbb{N}$  is nonempty; thus, by the

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the set  $\mathbb{N}$  has a

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call it  $\ell$ .

(3) The real number  $\ell - 1$  is less than  $\sup(\mathbb{N})$ , so

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(4) Adding one,

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(5) This contradicts

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(6) Thus

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