DEFINITION: For a real number x, the **absolute value** of x is $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

THEOREM 8.1 (TRIANGLE INEQUALITY): Let x, y, z be real numbers. Then

$$|x - z| \leq |x - y| + |y - z|.$$

We use often use the Triangle Inequality to show precise versions of "if x is close to y and y is close to z, then x is close to z."

THEOREM 8.2 (REVERSE TRIANGLE INEQUALITY): Let x, y, z be real numbers. Then

$$|x - z| \geq ||x - y| - |y - z||.$$

We use often use the Reverse triangle Inequality to show precise versions of "if x is far from y and y is close to z, then x is far from z."

(1) If x and y are real numbers, what is the geometric meaning of $|x - y|$?

(2) We will often look at conditions like $|x - L| < \varepsilon$, where L and ε are real numbers and x is a variable. Describe $\{x \in \mathbb{R} \ : \ |x - L| < \varepsilon\}$ in interval notation. Now draw a picture of this on the real number line, showing the role of L and ε.

(3) Describe $\{x \in \mathbb{R} \ : \ |3x + 7| < 4\}$ explicitly in interval notation.

(4) Suppose that $|x - 2| < \frac{1}{5}$, $|y - 2| < \frac{2}{5}$.
   (a) Show that $x > \frac{8}{5}$.
   (b) Show that $|x - y| < \frac{3}{5}$.
   (c) Use the reverse triangle inequality to show that $|y - 3| > \frac{3}{5}$.

(5) True or false & justify[1]: There is a rational number x such that $|x^2 - 2| = 0$.

(6) True or false & justify[1]: There is a rational number x such that $|x^2 - 2| < \frac{1}{1000000}$.

Here is another important fact in the relationship between $\mathbb{R}$ and $\mathbb{Z}$:

THEOREM 8.3: For every real number r, there is a unique integer $n \in \mathbb{Z}$ such that $n \leq r < n + 1$.

(7) Proof of Theorem 8.3:
   (a) First, assume that $r \geq 0$. Complete the following sentence: "The number $n + 1$ should be the smallest natural number that ________________."
   (b) Take your sentence and turn it into a recipe for n to prove that such an integer n exists in this case.
   (c) Now, assume that $r < 0$. Explain why there is some $j \in \mathbb{N}$ such that $j + r > 0$. Deduce that an integer n as in the statement exists in this case too.
   (d) Finally, prove that n is unique. You can use without proof that there are no integers in between 0 and 1.

[1] You can use anything we've proven in class, but don't use things we haven't, like decimal expansions.