Reminder on functions

Given any two sets S and T, a *function* from S to T, written $f : S \to T$, is a "rule"[1] that assigns to each element $s \in S$ a unique element $t \in T$. The set S is called the *domain* of f. We will generally consider functions from some set of real numbers to $\mathbb{R}$. We often specify functions by formulas; when we do this the take the domain to be the set of all real numbers for which the formula evaluates to a unique real number. In particular,

$$f(x) = 2x + 2 \quad \text{and} \quad g(x) = \frac{2x^2 - 2}{x - 1}$$

are *not* the same function, even though their values agree for all $x \neq 1$, since their domains are different.

Limits of functions

Definition 17.1: Let S be a subset of $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be a function, and a and L be real numbers. We say that *the limit of f as x approaches a is L* provided:

> for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then x is in the domain of f and $|f(x) - L| < \varepsilon$.

If this happens, we write $\lim_{x \to a} f(x) = L$ to denote this.

(1) Unpackaging parts of the definition.
  (a) Describe $\{x \in \mathbb{R} \mid 0 < |x - 2| < 1\}$ as a union of two open intervals.
  (b) For a general $a \in \mathbb{R}$ and $\delta > 0$, describe $\{x \in \mathbb{R} \mid 0 < |x - a| < \delta\}$ as a union of two open intervals.
  (c) Focusing on the "domain" part of the definition, if the limit of f as x approaches a is L, then f must at least be defined ______________________________ (where?).

(2) The $\varepsilon - \delta$ game.
  (a) Player 0 starts by graphing a function f (like a familiar one from calculus) and specifies an x-value a and a y-value L that (based on previous calculus knowledge) they think makes $\lim_{x \to a} f(x) = L$ **true**. [The graph should be large.]
  (b) Player 1 choses an ε. This is how close we would like our function to be to L. Thus, ε goes up and down from L (corresponding to $|f(x) - L| < \varepsilon$). Draw horizontal dotted lines with y-values $L - \varepsilon$ and $L + \varepsilon$. [The ε should be large enough for people to see and have room to work in the picture.]
  (c) Player 2 must find a δ such that every $x \in (a - \delta, a) \cup (a, a + \delta)$ is
    - in the domain of f, and
    - has an output $f(x)$ within $(L - \varepsilon, L + \varepsilon)$.

    Draw vertical dotted lines for the x-values $a - \delta$ and $a + \delta$. [Everyone in the team can assist player 2!]
  (d) Repeat with the same graph, players 1& 2 switching roles (and a new ε).

(3) Draw the graph of $g(x) = \dfrac{2x^2 - 2}{x - 1}$. Play the $\varepsilon - \delta$ game with this function, $a = 1$ and $L = -3$. What happens?

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[1] Here's a real definition: a *function* from S to T is a subset $G \subset S \times T$ of ordered pairs of elements of S and T with the property that for all $s \in S$ there is a unique $t \in T$ such that $(s, t) \in G$; we write $f(s)$ for this element t.

(4) Consider the function $g(x) = \dfrac{2x^2 - 2}{x - 1}$. It is true that $\lim_{x \to 1} g(x) = 4$.

   (a) I claim that for $\varepsilon = 3$, the choice $\delta = 1.5$ "works" to make the rest of the definition true. Verify this.
   (b) Find a δ that "works" for $\varepsilon = 1$.
   (c) Find a δ that "works" for $\varepsilon = 1/2$.
   (d) Find a δ that "works" for $\varepsilon > 0$.

(5) Consider the function $g(x) = \dfrac{2x^2 - 2}{x - 1}$. It is not true that $\lim_{x \to 1} g(x) = -3$. I claim that for $\varepsilon = 1$, there is no choice of $\delta > 0$ that "works" to make the rest of the definition true. Verify this.

(6) Repackage your work from (4) to *prove* that $\lim_{x \to 1} g(x) = 4$.

(7) Repackage your work from (5) to *disprove* that $\lim_{x \to 1} g(x) = -3$.
(Warning: Until we prove something else, the conclusion of (6) is irrelevant to this problem. . . prove what?)