Definition: Let S be a set of real numbers. A number ℓ is the supremum of S provided

- ℓ is an upper bound of S and
- if b is any upper bound of S, then $\ell \leq b$.

Theorem 5.3: For every real number r, there is a natural number n such that $n > r$.

Corollary 5.4: (Archimedean Principle). For every positive real number a and every real number b, there is some natural number n such that $na > b$.

Theorem 5.5: (Density of rational numbers). For any real numbers x, y with $x < y$, there is some rational number q such that $x < q < y$.

(1) Use the Archimedean principle to show that for any positive number $\varepsilon > 0$, there is a natural number n such that $0 < \varepsilon < \dfrac{1}{n}$.

(2) Prove that the supremum of the set $S = \left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$ is 1.

(3) Let S be a set of real numbers, and suppose that $\sup(S) = \ell$. Let $T = \{s + 7 \mid s \in S\}$. Prove that $\sup(T) = \ell + 7$.

(4) Prove the following:
Corollary 6.1: (Density of irrational numbers). For any real numbers x, y with $x < y$, there is some irrational number z such that $x < z < y$.