ISOMORPHISM WRAPUP

PROPOSITION: Let $f : G \to H$ be a group homomorphism. Then f is an isomorphism if and only if f is bijective[1].

LEMMA: Let $f : G \to H$ be an isomorphism. Then for any $g \in G$, we have $|g| = |f(g)|$.

DEFINITION: A property $\mathcal{P}$ of a group is an **isomorphism invariant** if whenever $G \cong H$ and $\mathcal{P}$ holds for G, then $\mathcal{P}$ also holds for H.

THEOREM: The following are isomorphism invariants:

(1) The order of the group.
(2) The set of orders of elements of the group.
(3) Being abelian.
(4) The order of the center of the group.
(5) Being finitely generated.

(1) Use the Theorem to show that none of the following groups are pairwise isomorphic:

$$S_3 \qquad S_4 \qquad \mathbb{Z}/6$$

Since $|S_3| = |\mathbb{Z}/6| = 6$ and $|S_4| = 24$, we have $S_3 \ncong S_4$ and $\mathbb{Z}/6 \ncong S_4$. Since $\mathbb{Z}/6$ is abelian and S_3 is not, we also have $\mathbb{Z}/6 \ncong S_3$.

(2) Prove the Proposition.

Suppose that f is an isomorphism, so there is an inverse homomorphism $g : H \to G$. In particular, g is an inverse function to f, so f is bijective.

Conversely, suppose that f is a bijective homomorphism, and let f^{-1} be the inverse function. We claim that $f^{-1} : H \to G$ is a group homomorphism. Indeed, let $x, y \in H$. Since f is surjective, we can write $x = f(a)$ and $y = f(b)$. Then $f^{-1}(xy) = f^{-1}(f(a)f(b)) = f^{-1}(f(ab))$ since f is a homomorphism. Then $f^{-1}(xy) = ab = f^{-1}(a)f^{-1}(b)$ by definition of inverse function. Thus, f^{-1} is an inverse homomorphism to f, so f is an isomorphism.

(3) Prove the Lemma.

(4) Prove the Thoerem.

[1] Reminder: by definition a function is **bijective** if it is injective and surjective (i.e. a one-to-one correspondence). It is a theorem from set theory that a function $f : X \to Y$ is bijective if and only if there exists an inverse function $g : Y \to X$.