

ASSIGNMENT #3

- (1) Let R be a Noetherian ring and W^{-1} be a multiplicatively closed subset. Show that $W^{-1}R$ is also Noetherian.
- (2) Let $K \subseteq L$ be fields, and $R = K[x_1, \dots, x_n] \subseteq S = L[x_1, \dots, x_n]$ be polynomial rings. Let $I \subseteq R$ be an ideal.
- (a) Show¹ that if $I \subseteq R$ is a proper ideal, then the expansion $IS \subseteq S$ is also a proper ideal.
- (b) Show² that $IS \cap R = I$.

- (3) Let K be a field, $R = K[x_1, \dots, x_n]$ be a polynomial ring, and $I \subseteq R$ be a proper ideal.
- (a) Show that $\mathcal{Z}_{\overline{K}}(I) \neq \emptyset$.
- (b) Show that $f \in \sqrt{I}$ if and only if $\mathcal{Z}_{\overline{K}}(f) \supseteq \mathcal{Z}_{\overline{K}}(I)$.
- (c) Show³ that

$$\sqrt{I} = \bigcap_{\mathfrak{m} \in \text{Max}(I) \cap V(I)} \mathfrak{m}.$$

- (4) Describe all of the elements of $\text{Spec}(\mathbb{C}[x])$ and $\text{Spec}(\mathbb{R}[x])$, and describe the map $\text{Spec}(\mathbb{C}[x]) \rightarrow \text{Spec}(\mathbb{R}[x])$ induced by the inclusion map $\mathbb{R}[x] \subseteq \mathbb{C}[x]$.
- (5) Let M be an R -module, not necessarily finitely generated. Show that the support of M is Zariski closed if and only if $\text{Ass}_R(M)$ has finitely many minimal elements.
- (B) Show⁴ that $\text{Spec}(R)$ is connected if and only if R cannot be written (up to isomorphism) as a direct product of two rings.

¹Hint: It might be useful to think of $L[x_1, \dots, x_n]$ as $L \otimes_K K[x_1, \dots, x_n]$.

²Hint: Consider the map $R/I \rightarrow S/IS$.

³Hint: For the containment “ \supseteq ”, if $f \notin \sqrt{I}$, take $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{Z}_{\overline{K}}(I) \setminus \mathcal{Z}_{\overline{K}}(f)$. Show that the image of the ring homomorphism $\phi_{\mathbf{a}} : R \rightarrow \overline{K}$ sending $f \mapsto f(a_1, \dots, a_n)$ is a field, and hence the kernel is a maximal ideal containing I but not f .

⁴Hint: You might end up wanting to show along the way that if the class of e in $R/\sqrt{(0)}$ is idempotent, then some element in $e + \sqrt{(0)}$ is idempotent in R . It may be helpful to apply the Chinese Remainder Theorem to $e^n(1 - e)^n$ for this.