UPPER BOUNDS AND THE COMPLETENESS AXIOM

Let S be a set of real numbers.

- A number b is an **upper bound** for S provided for all $x \in S$ we have $b \geq x$.
- The set S is **bounded above** provided there exists at least one upper bound for S.
- A number m is the **maximum** of S provided
  (1) $m \in S$, and
  (2) m is an upper bound of S.
- A number ℓ is a **supremum** of S provided
  (1) ℓ is an upper bound of S, and
  (2) for any upper bound b for S, we have $\ell \leq b$.

(1) Write, in simplified form, the negation of the statement "b is an upper bound for S".

(2) Write, in simplified form, the negation of the statement "S is bounded above".

(3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
   (a) If $x > \ell$, what is the most concrete thing you can say about x and S?
   (b) If $x < \ell$, what is the most concrete thing[1] you can say about x and S?

(4) Let $S = \{x \in \mathbb{R} \mid x^3 + x < 5\}$. Use the definition of supremum to answer the followng:
   (a) Is 1 the supremum of S? Why or why not?
   (b) Is 2 the supremum of S? Why or why not?

(5) Consider the open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.
   (a) Prove[2] that $(0, 1)$ has no maximum element.
   (b) Prove that $\sup(\,(0, 1)\,) = 1$.

(6) Let S be a set of real numbers, and let[3] $T = \{2s \mid s \in S\}$. Prove[4] that if S is bounded above, then T is bounded above.

(7) Let S be a set of real numbers. Show that if S has a supremum, then it is unique.

[1]Hint: Use one of the previous problems.
[2]Hint: Try a proof by contradiction!
[3]For example, if $S = \{-1, 1, 2\}$, then $T = \{-2, 2, 4\}$.
[4]First, before all else, this is an if then statement: start by assuming the "if" part. We now need to show the "then" part, which is about the existence of an upper bound. Use your assumption about S to find an upper bound for T (and prove that it is indeed an upper bound for T).

WELL-ORDERING AXIOM: Every nonempty subset of $\mathbb{N}$ has a minimum.

COMPLETENESS AXIOM: Every nonempty bounded above set of real numbers has a supremum.

(1) Prove the following:
THEOREM: For every[5] real number r, there exists a unique integer n such that $n-1 \leq r < n$.

(2) Prove the following:
THEOREM: For every real number r, there is some natural number n such that $n > r$.

[5] Hint: First deal with the case $r \geq 0$.