SIMILARITY, INVARIANT FACTORS, AND MINIMAL/CHARACTERISTIC POLYNOMIALS FOR MATRICES

Throughout, F is a field, and V is an n-dimensional F-vectorspace.

- For two bases B, B' for V and a linear transformation $\phi : V \to V$, the matrices $[\phi]_B^B$ and $[\phi]_{B'}^{B'}$ are similar. Conversely, if A and A' are similar matrices, $A' = [t_A]_B^B$ for some basis B of F^n.
- Given a linear transformation $\phi : V \to V$, there is an $F[x]$-module V_ϕ that is just V as an F-vector space and with $F[x]$-action determined by $x \cdot v = \phi(v)$.
- The $F[x]$-modules $(F^n)_{t_A}$ and $(F^n)_{t_B}$ are isomorphic if and only if A and B are similar.
- The $F[x]$-module V_ϕ is presented by the matrix $xI_n - [\phi]_B^B$ for any basis B of V. In particular, the $F[x]$-module V_{t_A} is presented by the matrix $xI_n - A$.
- (INVARIANT FACTORS) The invariant factors of ϕ are the invariant factors of the $F[x]$-module V_ϕ. This consists of monic polynomials $g_1 | \cdots | g_k$.
- The companion matrix of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is

$$C(f) = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ & & & -a_1 \\ & I_{n-1} & & \vdots \\ & & & -a_{n-1} \end{bmatrix}.$$

- (RATIONAL CANONICAL FORM) There exists a basis B for V such that

$$[\phi]_B^B = \begin{bmatrix} C(g_1) & 0 & \cdots & 0 \\ 0 & C(g_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(g_s) \end{bmatrix},$$

  where $g_1 | \cdots | g_s$ are the invariant factors. If there is some basis B where this formula holds with the division condition, then $g_1, \ldots, g_s$ must be the invariant factors.
- The following are equivalent:
  1. A and B are similar matrices.
  2. A and B have the same invariant factors.
  3. A and B have the same rational canonical form.
- (CHARACTERISTIC POLYNOMIAL) The characterisic polynomial of ϕ is $\det(xI_n - [\phi]_B^B)$ for some basis B, denoted c_ϕ.
- (MINIMAL POLYNOMIAL) The minimal polynomial of ϕ is $\mathrm{ann}_{F[x]}(V_\phi)$, denoted m_ϕ.
- (CAYLEY-HAMILTON) $m_\phi | c_\phi$.
- Let $g_1, \ldots, g_s$ be the invariant factors of ϕ. Then
  1. $\deg(g_1) + \cdots + \deg(g_s) = n$.
  2. $m_\phi = g_s$.
  3. $c_\phi = g_1 \cdots g_s$.
  4. The irreducible factors of m_ϕ are the same as the irreducible factors of c_ϕ.
- The following are equivalent:
  1. λ is a root of m_ϕ
  2. λ is a root of c_ϕ
  3. λ is an eigenvalue of ϕ.