PROBLEM SET #1

(1) * Basic rules with derivations:
  (a) Prove the generalized product rule for derivations: if $\partial : R \to M$ is a derivation, then $\partial(a_1 \cdots a_n) = \sum_{j=1}^{n} (\prod_{j \neq i} a_i) \partial(a_j)$.
  (b) Prove the power rule for derivations: if $\partial : R \to M$ is a derivation, then $\partial(r^n) = nr^{n-1}\partial(r)$.
  (c) Show that if R is a ring of characteristic p, then the subring $R^p := \{r^p \mid r \in R\}$ is in the kernel of every derivation.

(2) * Let A be a ring and $S = A[x_1, \ldots, x_n]$ be a polynomial ring.
  (a) Let R be an $\mathbb{N}$-graded A-algebra such that A lives in degree zero. Show that there is a derivation on R such that for every homogeneous element f of degree d, $\partial(f) = df$. This derivation is called the *Euler operator* associated to the grading.
  (b) Let $S = A[x_\lambda \mid \lambda \in \Lambda]$ be a polynomial ring over A endowed with the $\mathbb{N}$-grading by the rule $\deg(x_\lambda) = n_\lambda$. Express the Euler operator of the grading as an S-linear combination of the partial derivatives.

(3) Let A be a ring and $R = A[x_1, \ldots, x_n]$ be a polynomial ring.
  (a) Give an explicit formula for the Lie algebra bracket on $\mathrm{Der}_{R|A}(R)$.
  (b) Does $\mathrm{Der}_{R|A}(R)$ have any nontrivial proper Lie ideals (i.e., A-submodules B such that $[d, b] \in B$ for all $b \in B$ and $d \in \mathrm{Der}_{R|A}(R)$)?

(4) Let R be a ring of characteristic $p > 0$ and $\partial : R \to R$ be a derivation. Show that ∂^p, i.e., the p-fold self composition of ∂, is a derivation on R.

(5) Let $R = \mathcal{C}^\infty(\mathbb{R}^n)$ be the ring of smooth functions on $\mathbb{R}^n$, and $\mathfrak{m}$ be the maximal ideal consisting of functions that vanish at some point $x_0 \in \mathbb{R}^n$.
  (a) * Show that $\mathfrak{m}^t$ consists of the functions $f \in R$ such that $\frac{d^{a_1}}{dx_1^{a_1}} \cdots \frac{d^{a_n}}{dx_n^{a_n}}(f)|_{x=x_0} = 0$ for all $a_1, \ldots, a_n$ with $0 \leqslant a_1 + \cdots + a_n < t$.
  (b) Show that $\mathrm{Der}_{R|\mathbb{R}}(R/\mathfrak{m}) \cong (\mathfrak{m}/\mathfrak{m}^2)^* \cong \mathbb{R}^n$ as vector spaces.

As a moral, we conclude that $\mathrm{Der}_{R|\mathbb{R}}(R/\mathfrak{m})$ serves as a model for the tangent space of $\mathbb{R}^n$ at x_0 constructed from the ring of smooth functions.

(6) * Let R be an A-algebra and I an ideal. Show that if the identity map on I/I^2 is in the image of $\mathrm{Der}_{R|A}(I/I^2) \xrightarrow{\text{res}} \mathrm{Hom}_A(I/I^2, I/I^2)$, then there is an A-algebra right inverse to the quotient map $\pi : R/I^2 \to R/I$. Conclude that the following are equivalent:
  - $\mathrm{Der}_{R|A}(M) \xrightarrow{\text{res}} \mathrm{Hom}_A(I/I^2, M)$ is surjective for all R/I-modules M;
  - $\mathrm{Der}_{R|A}(I/I^2) \xrightarrow{\text{res}} \mathrm{Hom}_A(I/I^2, I/I^2)$ is surjective;
  - The quotient map $R/I^2 \to R/I$ has an A-algebra right inverse.

(7) Let R be a ring and M an R-module. Recall that $R \ltimes M$ denotes the Nagata idealization of M: the ring with additive structure $R \oplus M$ and multiplication $(r, m)(s, n) = (rs, rn + sm)$. Show that $\alpha : R \to M$ is a derivation if and only if $(1, \alpha) : R \to R \otimes M$ $(r \mapsto (r, \alpha(r)))$ is a ring homomorphism.