SUBSEQUENCES

(1) **True or false; justify.**

(a) The sequence $\left\{\dfrac{1}{2n}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.

(b) The sequence $\left\{\dfrac{1}{3n+7}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.

(c) The constant sequence $\left\{\dfrac{1}{2}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.

(d) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are both subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(e) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are the only two subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(2) Explain how the following Corollary follows from Theorem 15.5.

> **Corollary 15.7:** Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.
> (a) If there is a subsequence of this sequence that diverges, then the sequence itself diverges.
> (b) If there are two subsequences of this sequence that converge to different values, then the sequence itself diverges.

(3) Use Corollary 15.7 to give a quick proof that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

(4) **Prove or disprove:**

(a) Every subsequence of an increasing sequence is increasing.

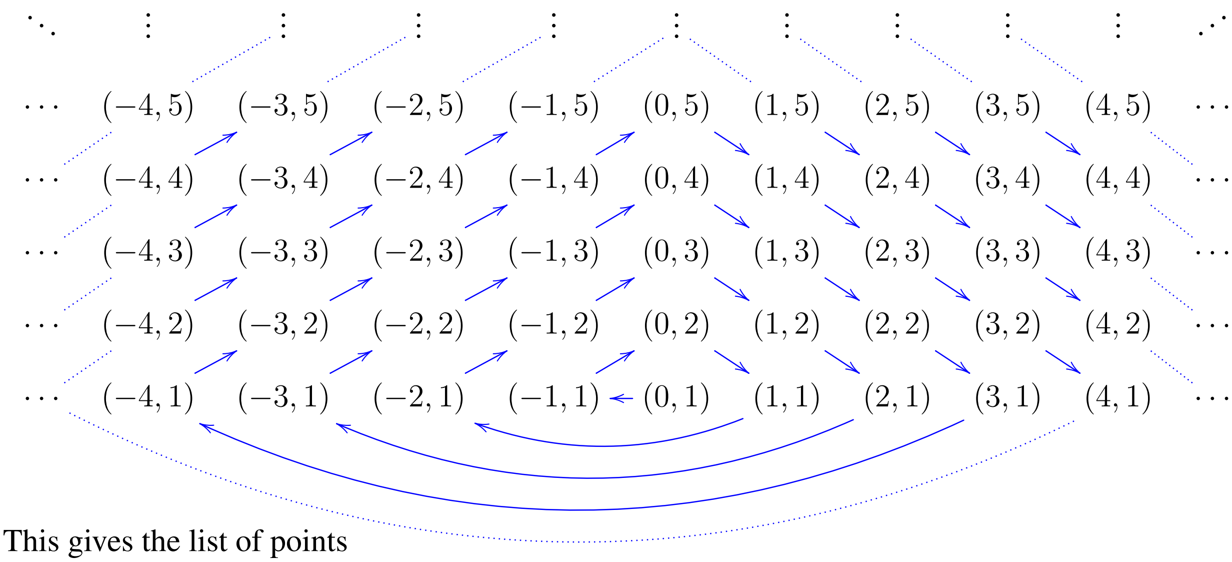
(b) Every subsequence of a bounded sequence is bounded.

(c) Every subsequence of a divergent sequence is divergent.

(d) Every subsequence of a sequence that diverges to $-\infty$ also diverges to $-\infty$.

A wild sequence

Consider the points in the plane whose x-coordinates are integers and y-coordinates are natural numbers. Starting at $(0,1)$, zigzag like so:



This gives the list of points

$$(0,1),(-1,1),(0,2),(1,1),(-2,1),(-1,2),(0,3),(1,2),(2,1),(-3,1),\dots$$

Now convert these to a list of rational numbers by changing (m,n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1},\frac{-1}{1},\frac{0}{2},\frac{1}{1},\frac{-2}{1},\frac{-1}{2},\frac{0}{3},\frac{1}{2},\frac{2}{1},\frac{-3}{1},\dots$$

of rational numbers. Call this sequence $\{w_n\}_{n=1}^{\infty}$.

(6) Explain why every rational number $q \in \mathbb{Q}$ occurs in $\{w_n\}_{n=1}^{\infty}$ infinitely many times.[1]

(7) Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers. Explain why $\{q_n\}_{n=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$. [This is saying that *every* sequence of rational numbers is a subsequence of this single sequence!]

(8) Let r be any real number. Show that there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ converges to r. [This is saying that *every* real number occurs as a limit of a subsequence of this single sequence!]

[1] Hint: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$.