Geometric Transformations and Image Warping

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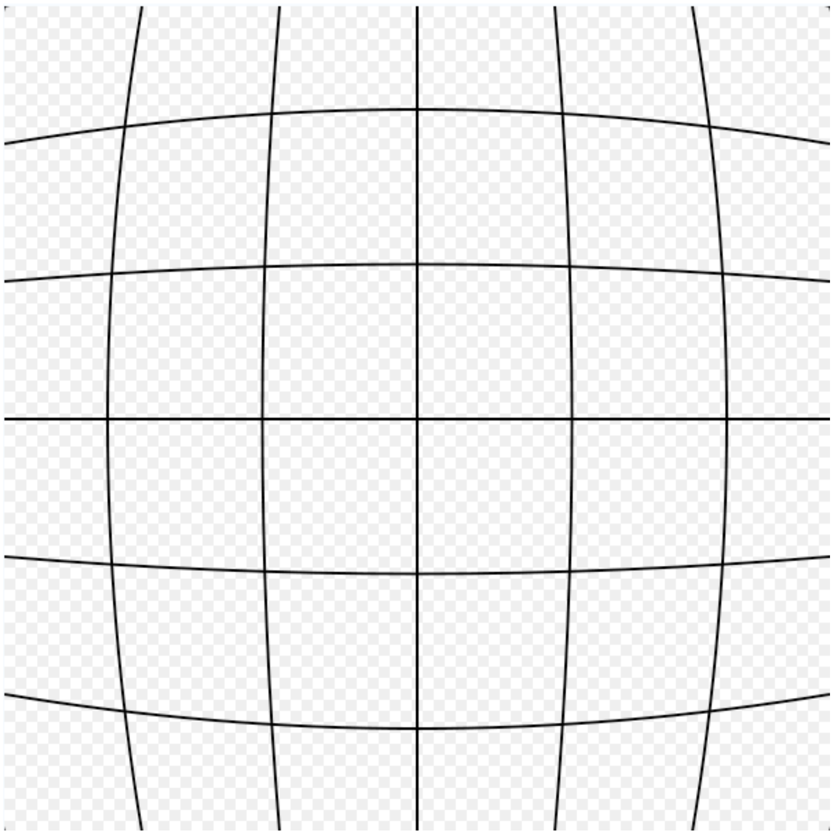
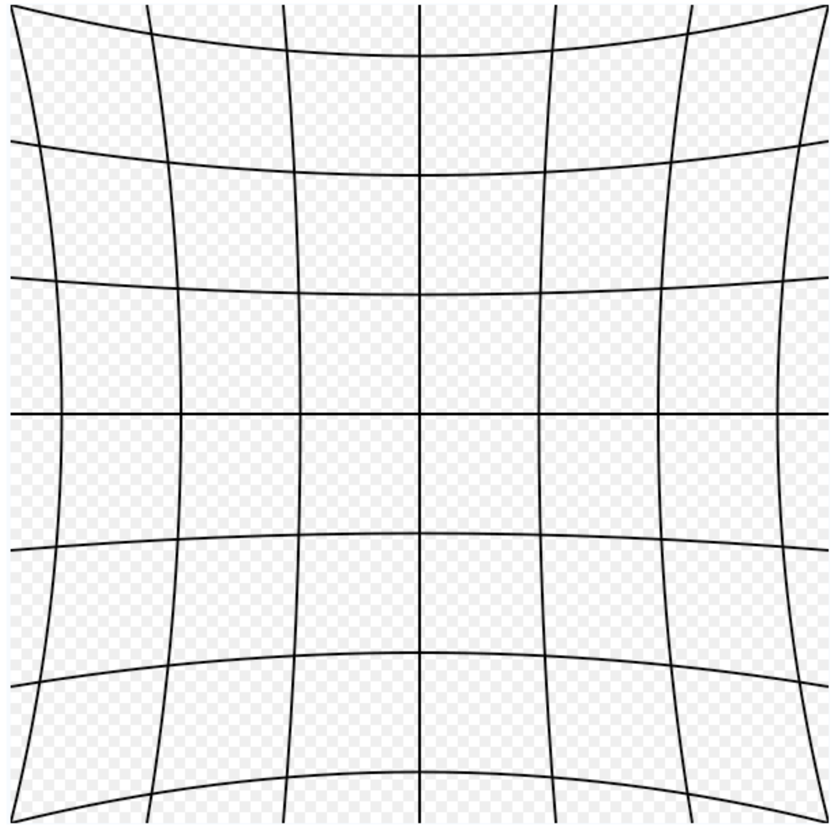
Geometric Transformations

- Greyscale transformations -> operate on range/output
- Geometric transformations -> operate on image domain
  - Coordinate transformations
  - Moving image content from one place to another
- Two parts:
  - Define transformation
  - Resample greyscale image in new coordinates

Geom Trans: Distortion From Optics



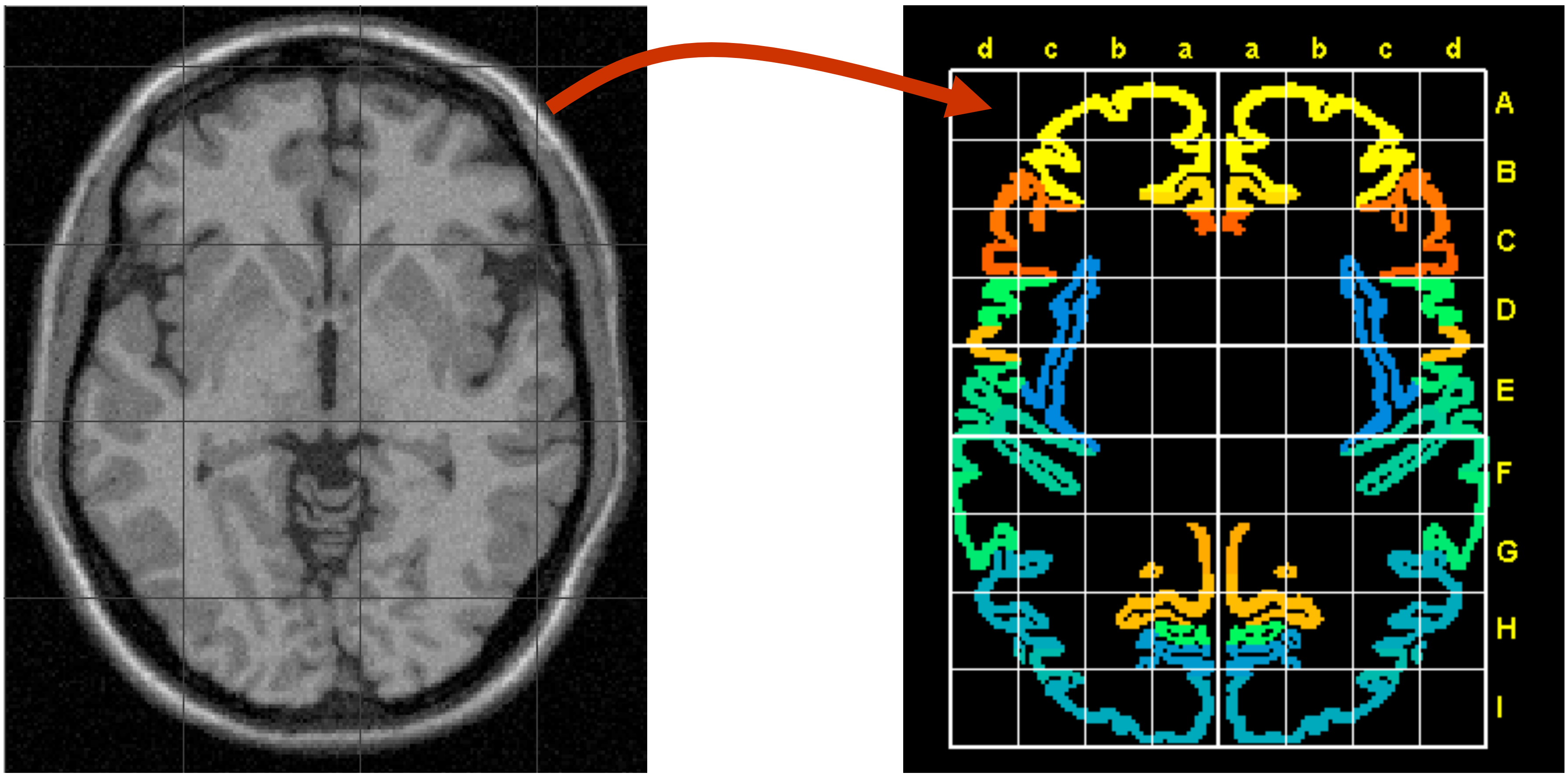
Barrel Distortion



Pincushion Distortion

Geom Trans: Distortion From Optics

Geom. Trans.: Brain Template/Atlas



Geom. Trans.: Mosaicing

Domain Mappings Formulation

$$f \longrightarrow g$$

New image from old one

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T(x,y) = \begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix}$$

Coordinate transformation
Two parts – vector valued

$$g(x,y) = f(x',y')$$
$$g(x,y) = f(x',y') = \tilde{f}(x,y)$$

g is the same image as f, but sampled on these new coordinates

Domain Mappings Formulation

$$\bar{x}' = T(\bar{x})$$

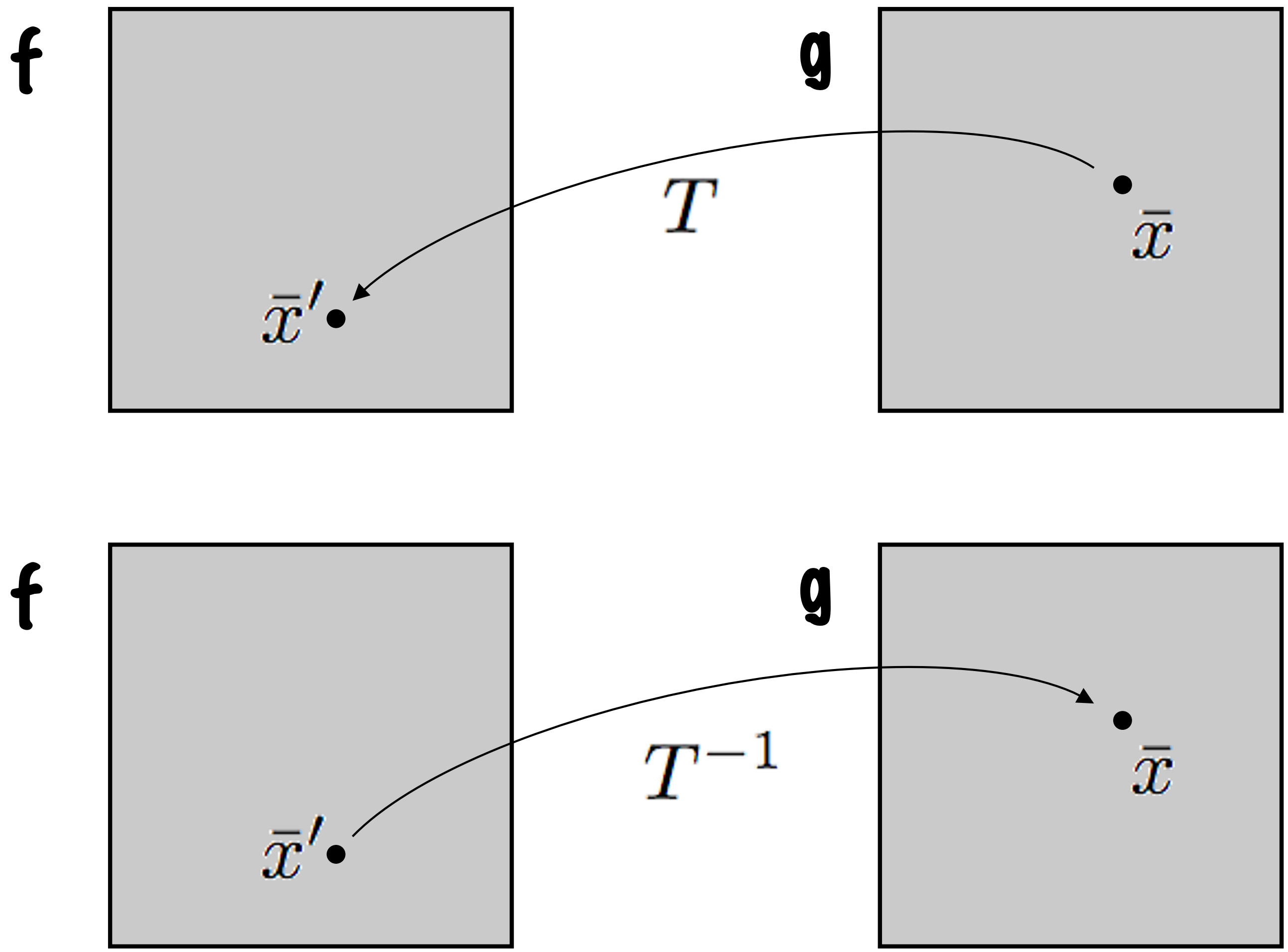
Vector notation is convenient.
Bar used some times, depends on context.

$$g(\bar{x}) = \tilde{f}(\bar{x}) = f(\bar{x}') = f(T(\bar{x}))$$

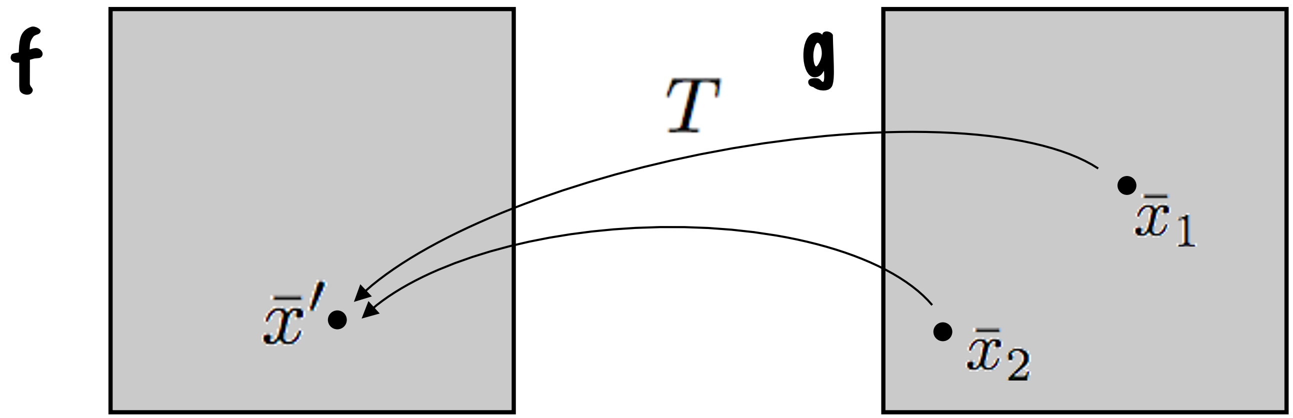
$$\bar{x} = T^{-1}(\bar{x}')$$

T may or may not have an inverse.
If not, it means that information was lost.

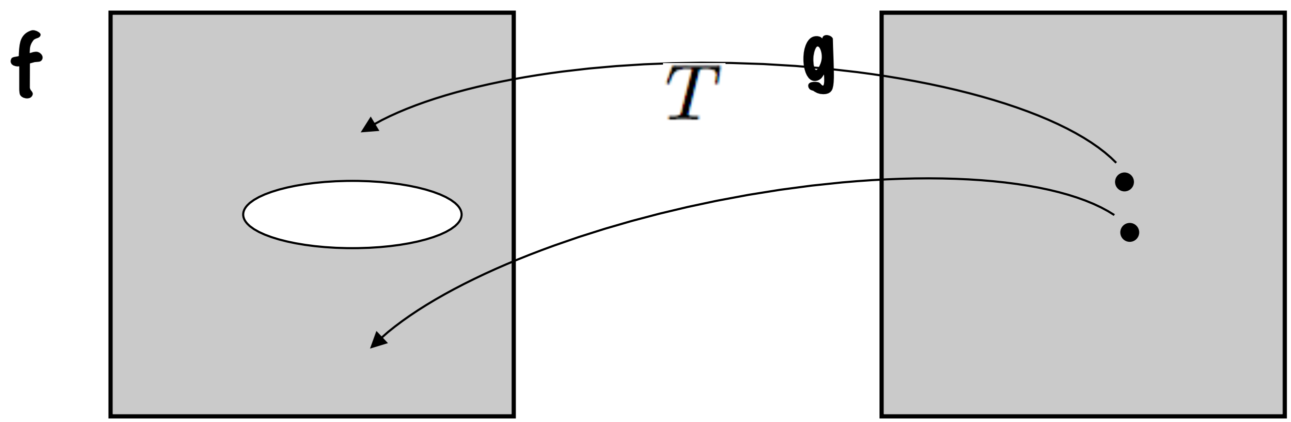
Domain Mappings



No Inverse?



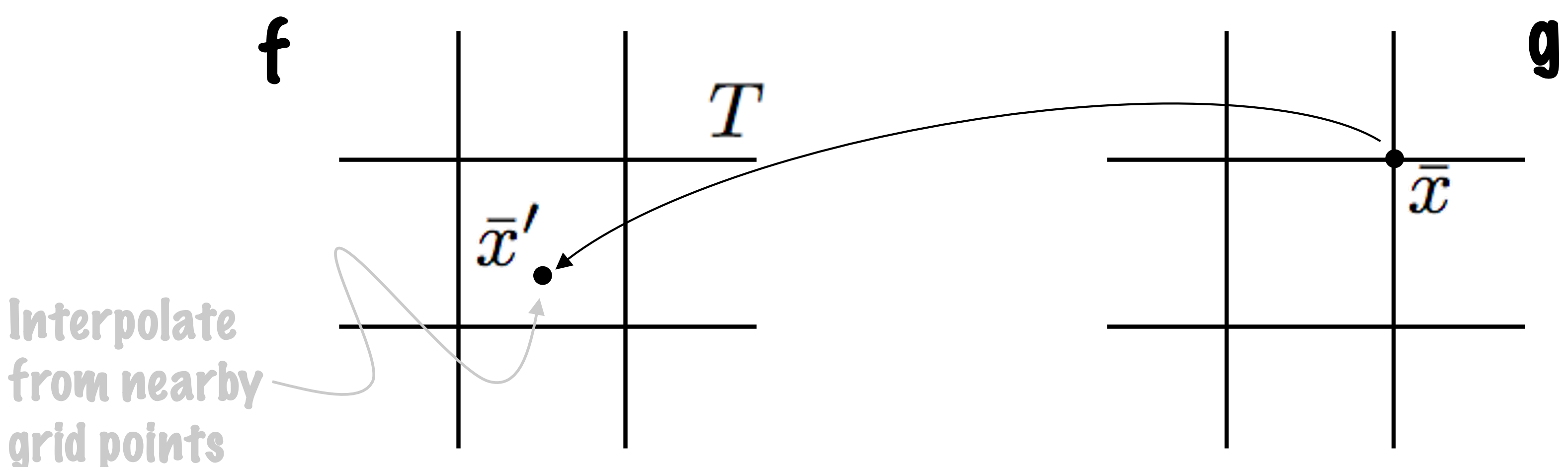
Not "one to one"



Not "onto" - doesn't cover f

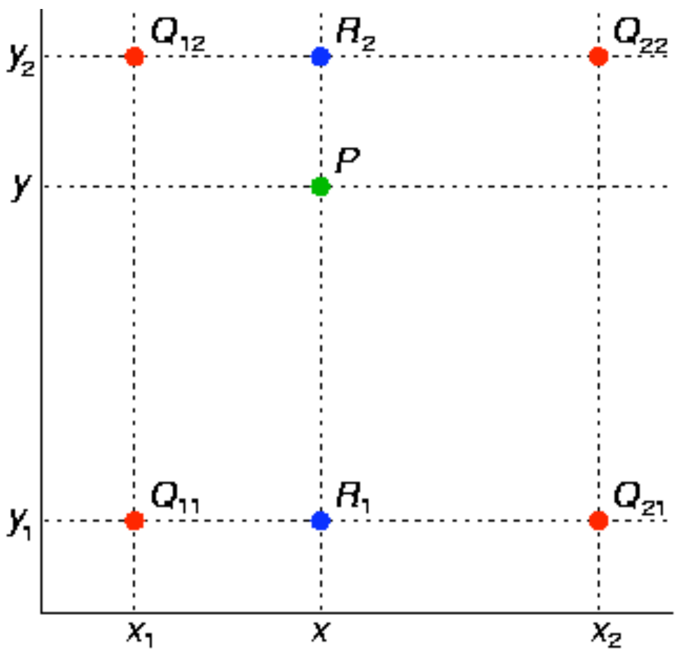
Implementation – Two Approaches

- Pixel filling – backward mapping
  - T() takes you from coords in g() to coords in f()
  - Need random access to pixels in f()
  - Sample grid for g(), interpolate f() as needed



Interpolation: Bilinear

- Successive application of linear interpolation along each axis



$$f(R_1) \approx \frac{x_2 - x}{x_2 - x_1} f(Q_{11}) + \frac{x - x_1}{x_2 - x_1} f(Q_{21})$$

$$f(R_2) \approx \frac{x_2 - x}{x_2 - x_1} f(Q_{12}) + \frac{x - x_1}{x_2 - x_1} f(Q_{22})$$

$$f(P) \approx \frac{y_2 - y}{y_2 - y_1} f(R_1) + \frac{y - y_1}{y_2 - y_1} f(R_2).$$

Source: Wikipedia

Bilinear Interpolation

- Not linear in x, y

$$
\begin{aligned}
f(x,y) \approx\ & \frac{f(Q_{11})}{(x_2-x_1)(y_2-y_1)}(x_2-x)(y_2-y) \\
&+ \frac{f(Q_{21})}{(x_2-x_1)(y_2-y_1)}(x-x_1)(y_2-y) \\
&+ \frac{f(Q_{12})}{(x_2-x_1)(y_2-y_1)}(x_2-x)(y-y_1) \\
&+ \frac{f(Q_{22})}{(x_2-x_1)(y_2-y_1)}(x-x_1)(y-y_1).
\end{aligned}
$$

$$b_1 + b_2 x + b_3 y + b_4 xy$$

$$
\begin{aligned}
b_1 &= f(0,0) \\
b_2 &= f(1,0) - f(0,0) \\
b_3 &= f(0,1) - f(0,0) \\
b_4 &= f(0,0) - f(1,0) \\
&\quad - f(0,1) + f(1,1).
\end{aligned}
$$

Binlinear Interpolation

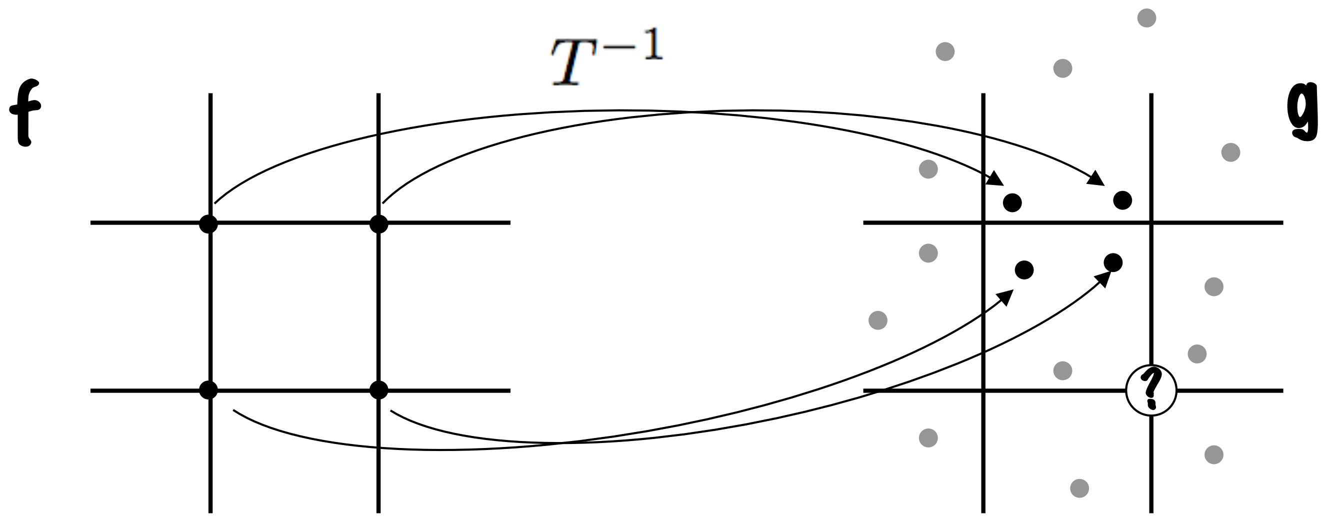
- Convenient form
  - Normalize to unit grid [0,1]x[0,1]

$$f(x,y) \approx f(0,0)\,(1-x)(1-y) + f(1,0)\,x(1-y) + f(0,1)\,(1-x)y + f(1,1)xy.$$

$$f(x,y) \approx \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} \begin{bmatrix} 1-y \\ y \end{bmatrix}.$$

Implementation – Two Approaches

- Splatting – backward mapping
  - T-1() takes you from coords in f() to coords in g()
  - You have f() on grid, but you need g() on grid
  - Push grid samples onto g() grid and do interpolation from unorganized data (kernel)



Nearby points are not organized – "scattered"

Scattered Data Interpolation With Kernels
Shepard's method

- Define kernel
  - Falls off with distance, radially symmetric

$$K(\bar{x}_1, \bar{x}_2) = K(|\bar{x}_1 - \bar{x}_2|)$$

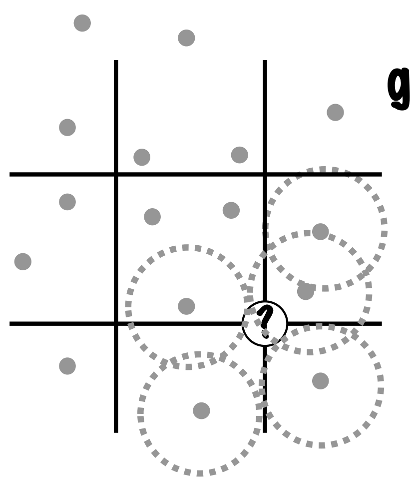
$$g(x) = \frac{1}{\sum_{j=1}^{N} w_j} \sum_{i=1}^{N} w_i f\left(x'_i\right)$$

$$w_j = K\left(|\bar{x} - T^{-1}(\bar{x}'_j)\right)$$

Kernel examples

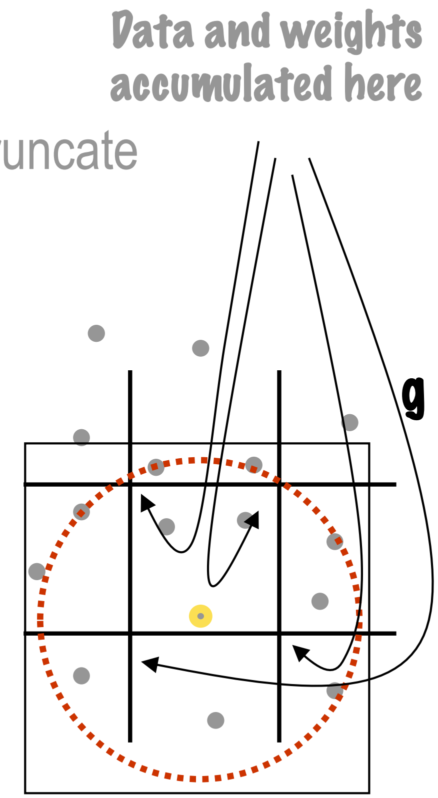
$$K(\bar{x}_1, \bar{x}_2) = \frac{1}{2\pi\sigma^2} e^{\frac{|\bar{x}_1 - \bar{x}_2|^2}{2\sigma^2}}$$

$$K(\bar{x}_1, \bar{x}_2) = \frac{1}{|\bar{x}_1 - \bar{x}_2|^p}$$



Shepard's Method Implementation

- If points are dense enough
  - Truncate kernel
  - For each point in f()
    - Form a small box around it in g() – beyond which truncate
    - Put weights and data onto grid in g()
  - Divide total data by total weights: B/A



$$A = \sum_{j=1}^{N} w_j \qquad B = \sum_{i=1}^{N} w_i f\left(T^{-1}(x_i')\right)$$

Transformation Examples

- Linear $\bar{x}' = A\bar{x} + \bar{x}_0$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$x' = ax + by + x_0$$
$$y' = cx + dy + y_0$$

- Homogeneous coordinates

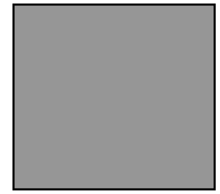
$$\bar{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad A = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{x}' = A\bar{x}$$

Special Cases of Linear

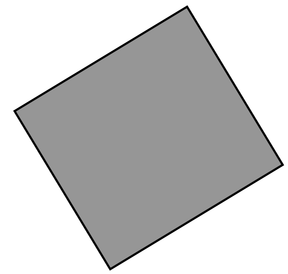
- Translation)

$$A = \begin{pmatrix} 0 & 0 & x_0 \\ 0 & 0 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



- Rotation

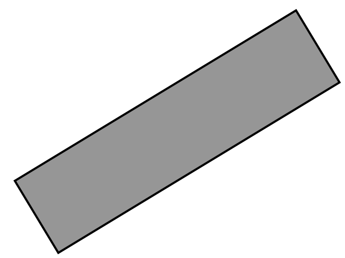
$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- Rigid = rotation translation
- Scaling
  - Include forward and backward rotation for arbitary axis

$$A = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$p, q < 1$: expand



- Skew

- Reflection

Linear Transformations

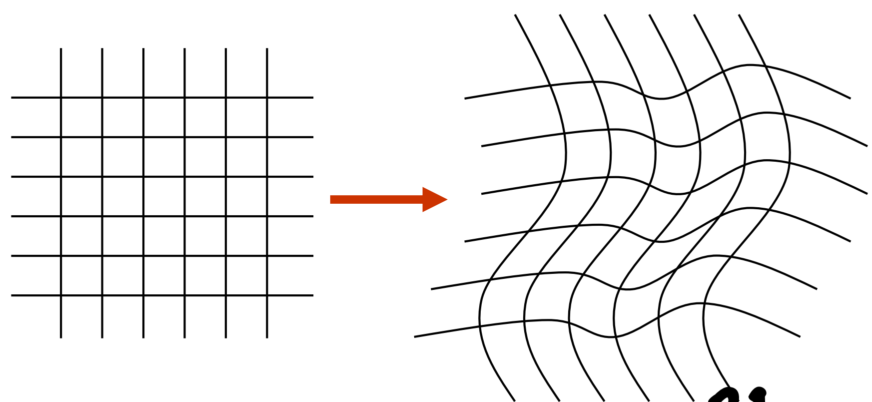
- Also called “affine”
  - 6 parameters
- Rigid -> 3 parameters
- Invertability
  - Invert matrix

$$T^{-1}(\bar{x}) = A^{-1}\bar{x}$$

- What does it mean if A is not invertible?

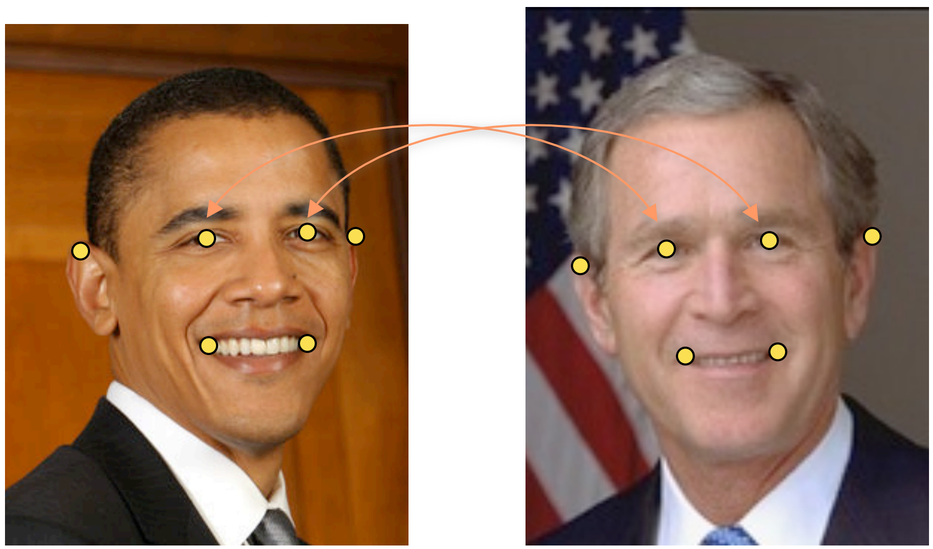
Other Transformations

- All polynomials of (x,y)
- Any vector valued function with 2 inputs
- How to construct transformations
  - Define form or class of a transformation
  - Choose parameters within that class
    - Rigid - 3 parameters
    - Affine - 6 parameters



Correspondences

- Also called "landmarks" or "fiducials"



$$\bar{c}_1, \bar{c}'_1$$
$$\bar{c}_2, \bar{c}'_2$$
$$\bar{c}_3, \bar{c}'_3$$
$$\bar{c}_4, \bar{c}'_4$$
$$\bar{c}_5, \bar{c}'_5$$
$$\bar{c}_6, \bar{c}'_6$$

Transformations/Control Points — Strategy

- Define a functional representation for T with k parameters (B) $T(\beta, \bar{x})$ $\beta = (\beta_1, \beta_2, \dots, \beta_K)$
- Define (pick) N correspondences
- Find B so that

$$\bar{c}'_i = T(\beta, \bar{c}_i) \quad i = 1, \dots, N$$

- If overconstrained (K < 2N) then solve

$$\arg\min_{\beta} \left[\sum_{i=1}^{N} (\bar{c}'_i - T(\beta, \bar{c}_i))^2 \right]$$

Example: Quadratic

Transformation

$$T_x = \beta_x^{00} + \beta_x^{10}x + \beta_x^{01}y + \beta_x^{11}xy + \beta_x^{20}x^2 + \beta_x^{02}y^2$$
$$T_y = \beta_y^{00} + \beta_y^{10}x + \beta_y^{01}y + \beta_y^{11}xy + \beta_y^{20}x^2 + \beta_y^{02}y^2$$

Denote $\bar{c}_i = (c_{x,i}, c_{y,i})$

Correspondences must match

$$c'_{y,i} = \beta_y^{00} + \beta_y^{10}c_{x,i} + \beta_y^{01}c_{y,i} + \beta_y^{11}c_{x,i}c_{y,i} + \beta_y^{20}c_{x,i}^2 + \beta_y^{02}c_{y,i}^2$$
$$c'_{x,i} = \beta_x^{00} + \beta_x^{10}c_{x,i} + \beta_x^{01}c_{y,i} + \beta_x^{11}c_{x,i}c_{y,i} + \beta_x^{20}c_{x,i}^2 + \beta_x^{02}c_{y,i}^2$$

Note: these equations are linear in the unkowns

Write As Linear System

$$\begin{pmatrix} 1 & c_{x,1} & c_{y,1} & c_{x,1}c_{y,1} & c_{x,1}^2 & c_{y,1}^2 & & & & & & \\ 1 & c_{x,2} & c_{y,2} & c_{x,2}c_{y,2} & c_{x,2}^2 & c_{y,2}^2 & & & & & & \\ & & & \vdots & & & & & & 0 & & \\ 1 & c_{x,N} & c_{y,N} & c_{x,N}c_{y,N} & c_{x,N}^2 & c_{y,N}^2 & & & & & & \\ & & & & & & 1 & c_{x,1} & c_{y,1} & c_{x,1}c_{y,1} & c_{x,1}^2 & c_{y,1}^2 \\ & & & & & & 1 & c_{x,2} & c_{y,2} & c_{x,2}c_{y,2} & c_{x,2}^2 & c_{y,2}^2 \\ & & & 0 & & & & & & \vdots & & \\ & & & & & & 1 & c_{x,N} & c_{y,N} & c_{x,N}c_{y,N} & c_{x,N}^2 & c_{y,N}^2 \end{pmatrix} \begin{pmatrix} \beta_x^{00} \\ \beta_x^{10} \\ \beta_x^{01} \\ \beta_x^{11} \\ \beta_x^{20} \\ \beta_x^{02} \\ \beta_y^{00} \\ \beta_y^{10} \\ \beta_y^{01} \\ \beta_y^{11} \\ \beta_y^{20} \\ \beta_y^{02} \end{pmatrix} = \begin{pmatrix} c'_{x,1} \\ c'_{x,2} \\ \vdots \\ c'_{x,N} \\ c'_{y,1} \\ c'_{y,2} \\ \vdots \\ c'_{y,N} \end{pmatrix}$$

$$Ax = b$$

A - matrix that depends on the (unprimed) correspondences and the transformation

x - unknown parameters of the transformation

b - the primed correspondences

Linear Algebra Background

$$Ax = b$$

$$\begin{array}{rcl} a_{11}x_1 + \ldots + a_{1N}x_N & = & b_1 \\ a_{21}x_1 + \ldots + a_{2N}x_N & = & b_2 \\ \ldots & & \ldots \\ a_{M1}x_1 + \ldots + a_{MN}x_N & = & b_M \end{array}$$

Simple case: A is sqaure (M=N) and invertable (det[A] not zero)

$$A^{-1}Ax = Ix = x = A^{-1}b$$

Numerics: Don' t find A inverse. Use Gaussian elimination or some kind of decomposition of A

Linear Systems – Other Cases

- $M<N$ or $M = N$ and the equations are degenerate or singular
  - System is underconstrained – lots of solutions
- Approach
  - Impose some extra criterion on the solution
  - Find the one solution that optimizes that criterion
  - Regularizing the problem

Linear Systems – Other Cases

- M > N
  - System is overconstrained
  - No solution
- Approach
  - Find solution that is best compromise
  - Minimize squared error (least squares)

$$x = \arg\min_{\mathrm{x}} |\mathrm{Ax} - \mathrm{b}|^2$$

Solving Least Squares Systems

- Psuedoinverse (normal equations)

$$A^T A x = A^T b$$

$$x = (A^T A)^{-1} A^T b$$

  - Issue: often not well conditioned (nearly singular)
- Alternative: singular value decomposition

Singular Value Decomposition

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} = UWV^T = \begin{pmatrix} & & \\ & U & \\ & & \end{pmatrix} \begin{pmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \dots & \\ & & \dots & \\ 0 & & & w_N \end{pmatrix} \begin{pmatrix} & V^T & \end{pmatrix}$$

$$I = U^T U = U U^T = V^T V = V V^T$$

Invert matrix A with SVD

$$A^{-1} = V W^{-1} U^T \qquad W^{-1} = \begin{pmatrix} \frac{1}{w_1} & & & 0 \\ & \frac{1}{w_2} & & \\ & & \dots & \\ & & \dots & \\ 0 & & & \frac{1}{w_N} \end{pmatrix}$$

SVD for Singular Systems

- If a system is singular, some of the w's will be zero

$$x = VW^*U^Tb$$

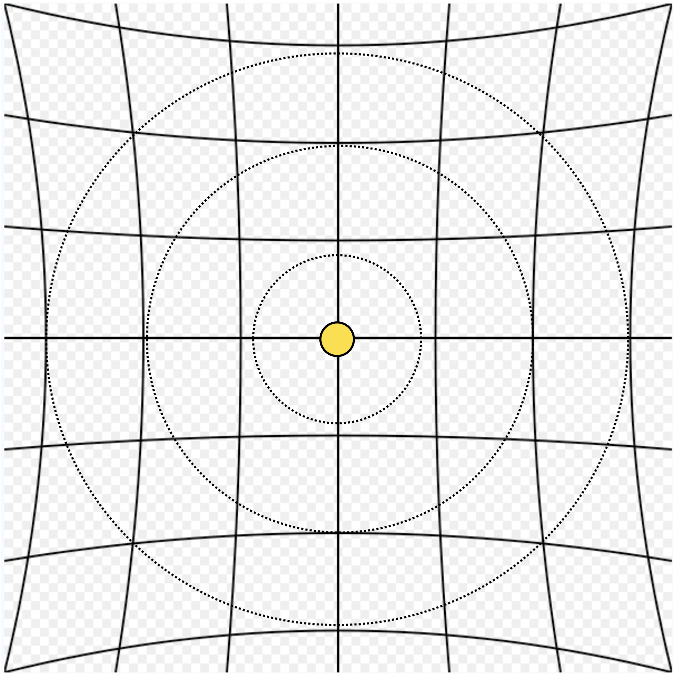
$$w_j^* = \begin{cases} 1/w_j & |w_j| > \epsilon \\ 0 & \text{otherwise} \end{cases}$$

- Properties:
  - Underconstrained: solution with shortest overall length
  - Overconstrained: least squares solution

Warping Application: Lens Distortion

- Radial transformation – lenses are generally circularly symmetric
  - Optical center is known

$$\bar{x}' = \bar{x}\left(1 + k_1 r^2 + k_2 r^4 + k_3 r^6 + \ldots\right)$$



Correspondences

- Take picture of known grid – crossings

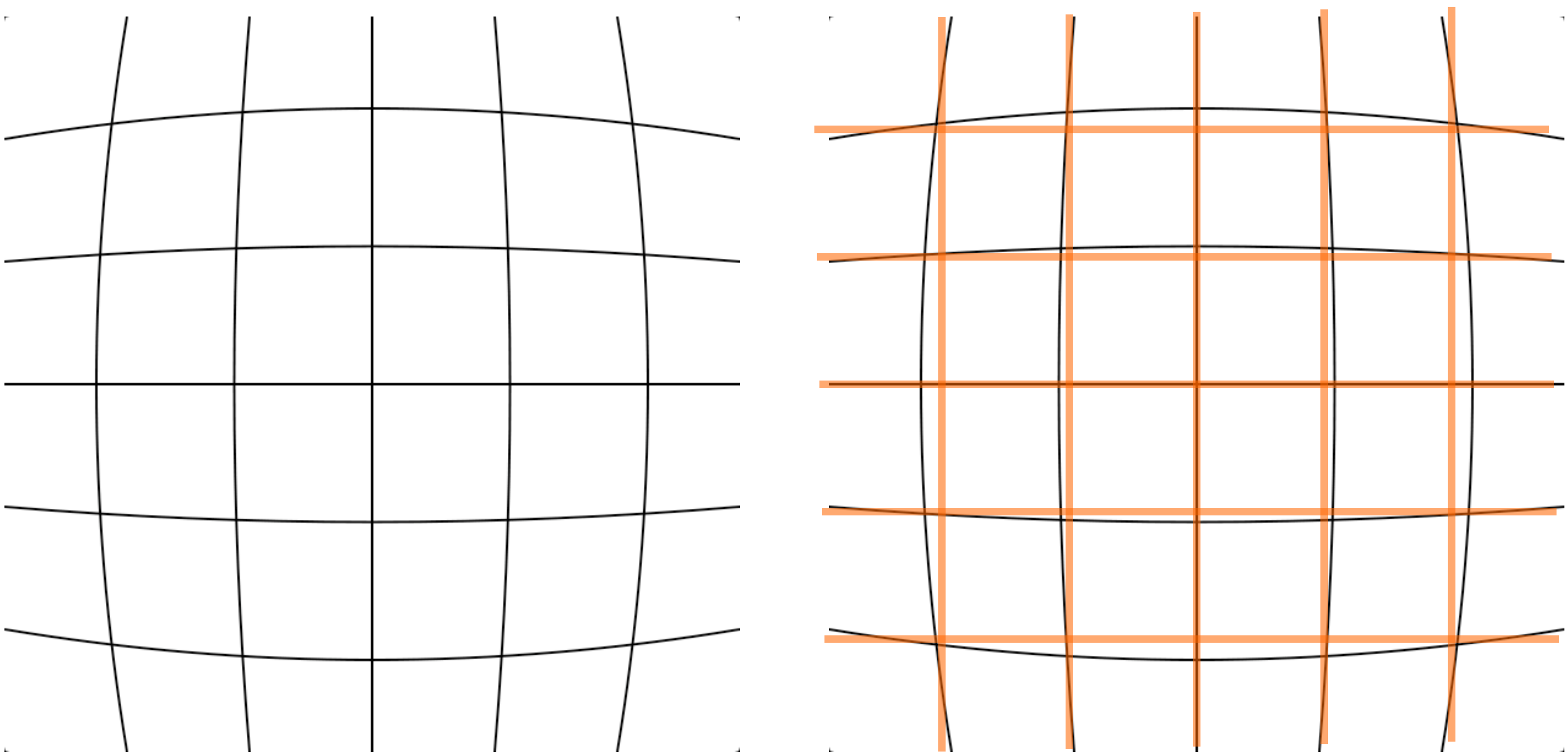


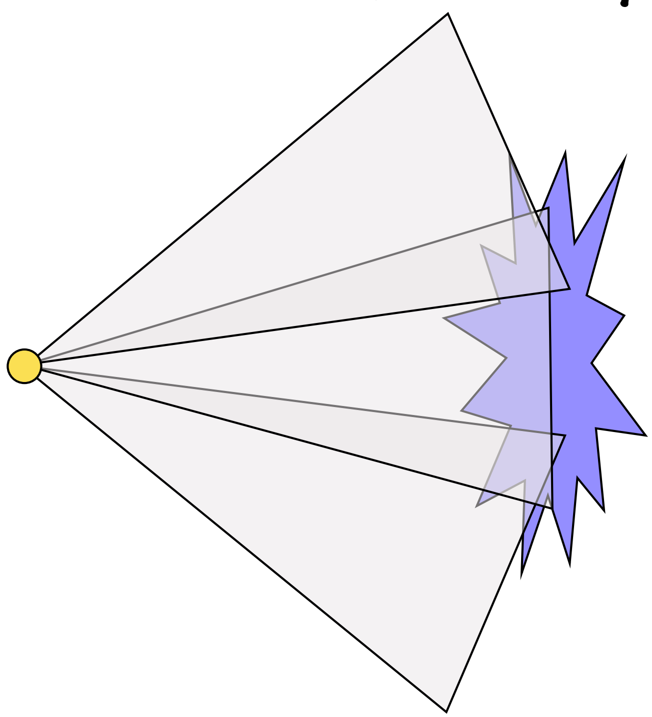
Image Mosaicing

- Piecing together images to create a larger mosaic
- Doing it the old fashioned way
  - Paper pictures and tape
  - Things don't line up
  - Translation is not enough
- Need some kind of warp
- Constraints
  - Warping/matching two regions of two different images only works when…

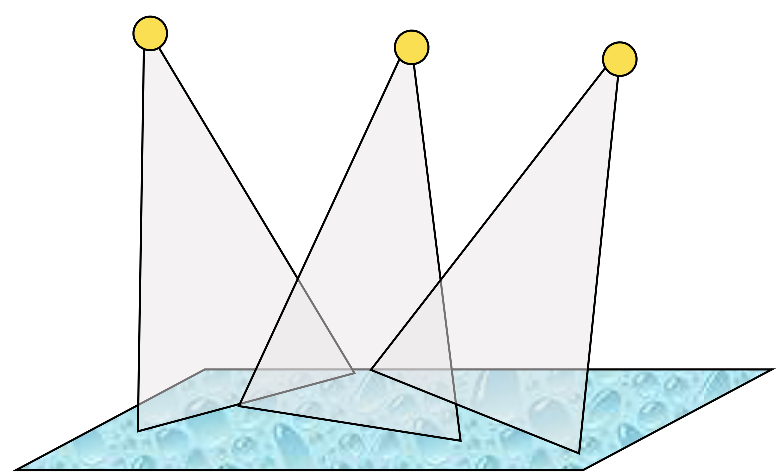
Special Cases

- Nothing new in the scene is uncovered in one view vs another
  - No ray from the camera gets behind another

1) Pure rotations-arbitrary scene

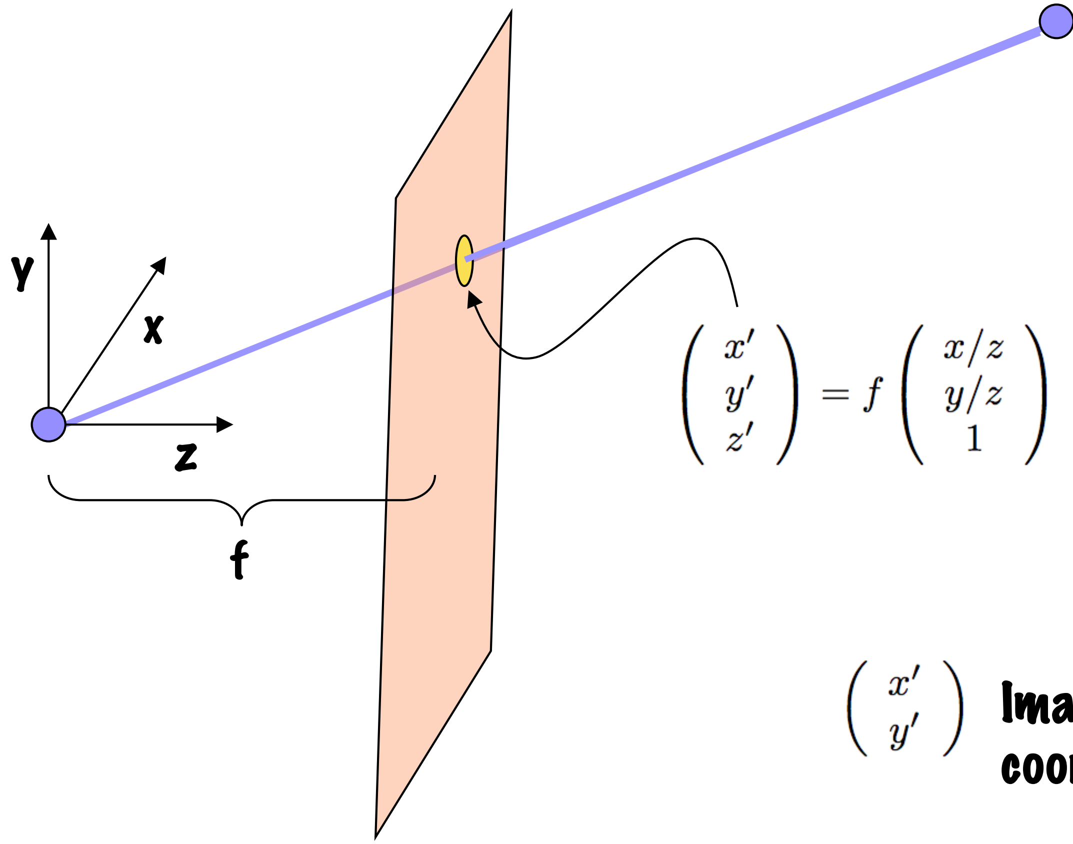


2) Arbitrary views of planar surfaces



3D Perspective and Projection

- Camera model



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = f \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix}$$

$\begin{pmatrix} x' \\ y' \end{pmatrix}$ **Image coordinates**

Perspective Projection Properties

- Lines to lines (linear)
- Conic sections to conic sections
- Convex shapes to convex shapes
- Foreshortening

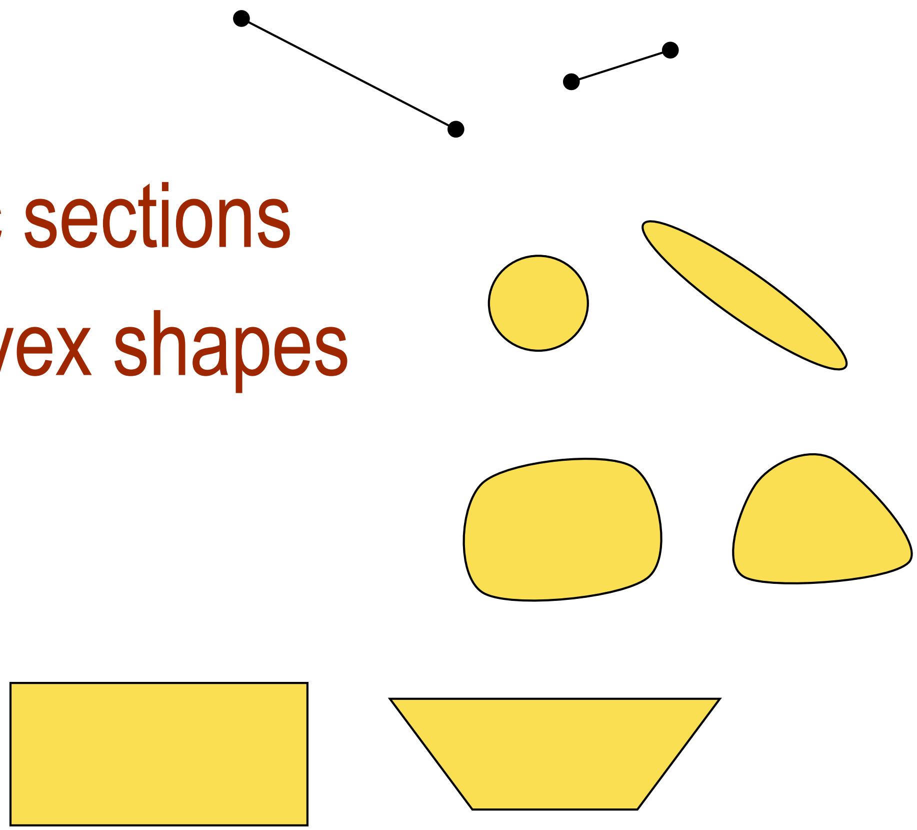


Image Homologies

- Images taken under cases 1,2 are perspectively equivalent to within a linear transformation
  - Projective relationships – equivalence is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv \begin{pmatrix} d \\ e \\ f \end{pmatrix} \Longleftrightarrow \begin{pmatrix} a/c \\ b/c \\ 1 \end{pmatrix} = \begin{pmatrix} d/f \\ e/f \\ 1 \end{pmatrix}$$

Transforming Images To Make Mosaics

Linear transformation with matrix P

$$\bar{x}^* = P\bar{x} \qquad P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & 1 \end{pmatrix} \qquad \begin{aligned} x^* &= p_{11}x + p_{12}y + p_{13} \\ y^* &= p_{21}x + p_{22}y + p_{23} \\ z^* &= p_{31}x + p_{32}y + 1 \end{aligned}$$

Perspective equivalence

$$\begin{aligned} x' &= \frac{p_{11}x+p_{12}y+p_{13}}{p_{31}x+p_{32}y+1} \\ y' &= \frac{p_{21}x+p_{22}y+p_{23}}{p_{31}x+p_{32}y+1} \end{aligned}$$

Multiply by denominator and reorganize terms

$$\begin{aligned} p_{31}xx' + p_{32}yx' - p_{11}x - p_{12}y - p_{13} &= -x' \\ p_{31}xy' + p_{32}yy' - p_{21}x - p_{22}y - p_{23} &= -y' \end{aligned}$$

Linear system, solve for P

$$\begin{pmatrix} -x_1 & -y_1 & -1 & 0 & 0 & 0 & x_1x'_1 & y_1x'_1 \\ -x_2 & -y_2 & -1 & 0 & 0 & 0 & x_2x'_2 & y_2x'_2 \\ & & & \vdots & & & & \\ -x_N & -y_N & -1 & 0 & 0 & 0 & x_Nx'_N & y_Nx'_2 \\ 0 & 0 & 0 & -x_1 & -y_1 & -1 & x_1y'_1 & y_1y'_1 \\ 0 & 0 & 0 & -x_2 & -y_2 & -1 & x_2y'_2 & y_2y'_2 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & -x_N & -y_N & -1 & x_Ny'_N & y_Ny'_N \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{23} \\ p_{23} \\ p_{31} \\ p_{32} \end{pmatrix} = \begin{pmatrix} -x'_1 \\ -x'_2 \\ \vdots \\ -x'_N \\ -y'_1 \\ -y'_2 \\ \vdots \\ -y'_N \end{pmatrix}$$

Image Mosaicing

4 Correspondences

5 Correspondences

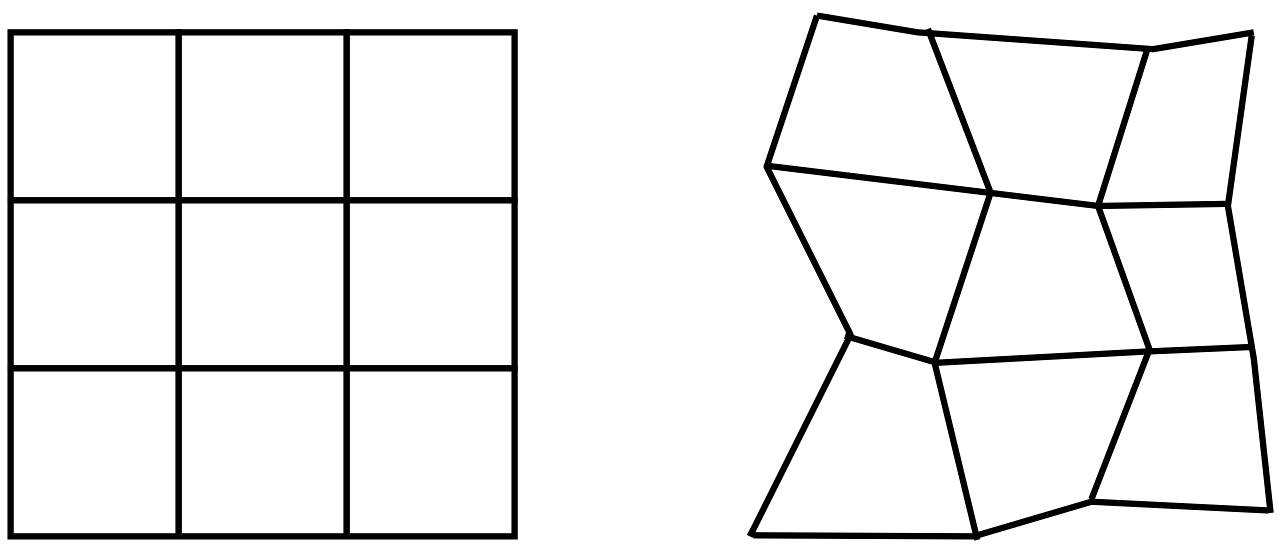
6 Correspondences

Mosaicing Issues

- Need a canvas (adjust coordinates/origin)
- Blending at edges of images (avoid sharp transitions)
- Adjusting brightnesses
- Cascading transformations

Specifying Warps – Another Strategy

- Let the # DOFs in the warp equal the # of control points (x1/2)
  - Interpolate with some grid-based interpolation
    - E.g. binlinear, splines



Landmarks Not On Grid

- Landmark positions driven by application
- Interpolate transformation at unorganized correspondences
  - Scattered data interpolation
- How do we do scattered data interpolation?
  - Idea: use kernels!
- Radial basis functions
  - Radially symmetric functions of distance to landmark

RBFs – Formulation

- Represent f as weighted sum of basis functions

$$\underbrace{f(\bar{x}) = \sum_{i=1}^{N} k_i \phi_i(\bar{x})}_{\text{Sum of radial basis functions}} \qquad \phi_i(\bar{x}) = \phi\left(\underbrace{||\bar{x} - \bar{x}_i||}_{\text{Basis functions centered at positions of data}}\right)$$

- Need interpolation for vector-valued function, T:

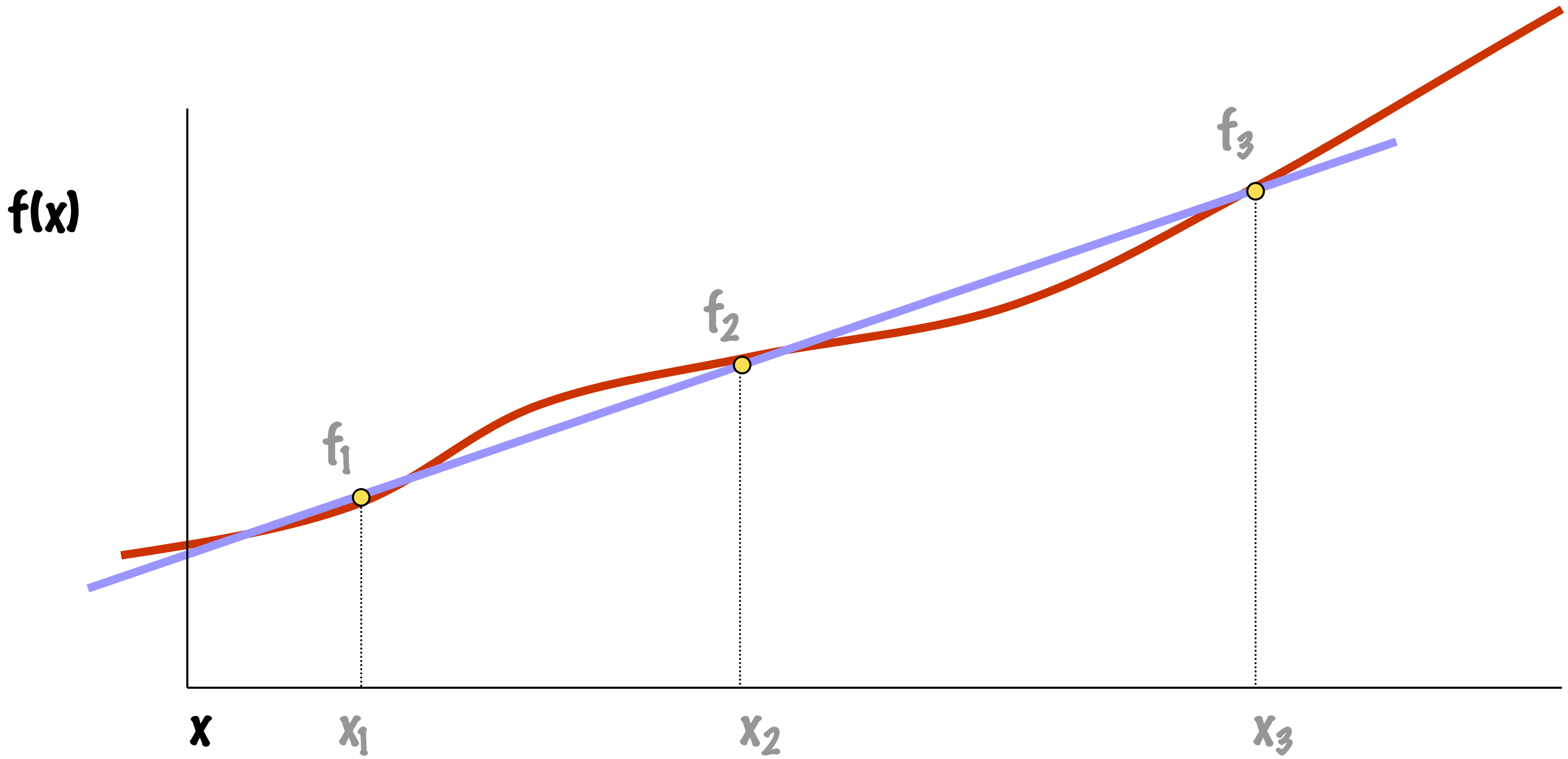
$$\begin{aligned} T^x(\bar{x}) &= \textstyle\sum_{i=1}^{N} k_i^x \phi_i(\bar{x}) \\ T^y(\bar{x}) &= \textstyle\sum_{i=1}^{N} k_i^y \phi_i(\bar{x}) \end{aligned}$$

Solve For k's With Landmarks as Constraints

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} k_1^x \\ k_2^x \\ \vdots \\ k_N^x \\ k_1^y \\ k_2^y \\ \vdots \\ k_N^y \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_N \\ y'_1 \\ y'_2 \\ \vdots \\ y'_N \end{pmatrix}$$

$$B = \begin{pmatrix} \phi_1(\bar{x}_1) & \phi_2(\bar{x}_1) & \dots & \phi_N(\bar{x}_1) \\ \phi_1(\bar{x}_2) & \phi_2(\bar{x}_2) & \dots & \phi_N(\bar{x}_2) \\ \vdots & & & \\ \phi_1(\bar{x}_N) & \phi_2(\bar{x}_N) & \dots & \phi_N(\bar{x}_N) \end{pmatrix}$$

Issue: RBFs Do Not Easily Model Linear Trends



RBFs – Formulation w/Linear Term

- Represent f as weighted sum of basis functions

$$\underbrace{f(\bar{x}) = \sum_{i=1}^{N} k_i \phi_i(\bar{x})}_{\text{Sum of radial basis functions}} + \underbrace{p_2 y + p_1 x + p_o}_{\text{Linear part of transformation}} \qquad \phi_i(\bar{x}) = \underbrace{\phi\left(||\bar{x} - \bar{x}_i||\right)}_{\text{Basis functions centered at positions of data}}$$

- Need interpolation for vector-valued function, T:

$$
\begin{aligned}
T^x(\bar{x}) &= \textstyle\sum_{i=1}^{N} k_i^x \phi_i(\bar{x}) + p_2^x y + p_1^x x + p_o^x \\
T^y(\bar{x}) &= \textstyle\sum_{i=1}^{N} k_i^y \phi_i(\bar{x}) + p_2^y y + p_1^y x + p_o^y
\end{aligned}
$$

RBFs – Solution Strategy

- Find the k's and p's so that f() fits at data points
  - The k's can have no linear trend (force it into the p's)
- Constraints -> linear system

$$T^x(\bar{x}_i) = x'_i \qquad T^y(\bar{x}_i) = y'_i$$

Correspondences must match

$$\sum_{i=1}^{N} k_i^x = 0 \qquad \sum_{i=1}^{N} k_i^y = 0$$

$$\sum_{i=1}^{N} k_i^x \bar{x}_i = \bar{0} \qquad \sum_{i=1}^{N} k_i^y \bar{x}_i = \bar{0}$$

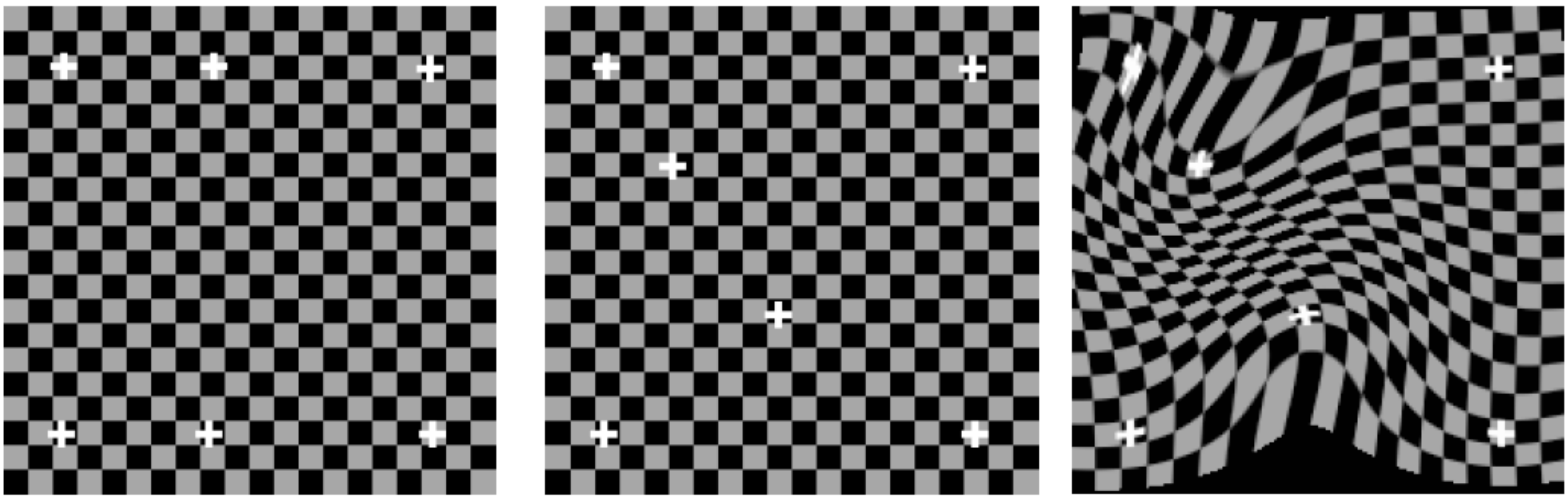
Keep linear part separate from deformation

RBFs – Linear System

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} k_1^x \\ k_2^x \\ \vdots \\ k_N^x \\ p_2^x \\ p_1^x \\ p_0^x \\ k_1^y \\ k_2^y \\ \vdots \\ k_N^y \\ p_2^y \\ p_1^y \\ p_0^y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x'_1 \\ x'_2 \\ \vdots \\ x'_N \\ 0 \\ 0 \\ 0 \\ y'_1 \\ y'_2 \\ \vdots \\ y'_N \end{pmatrix}$$

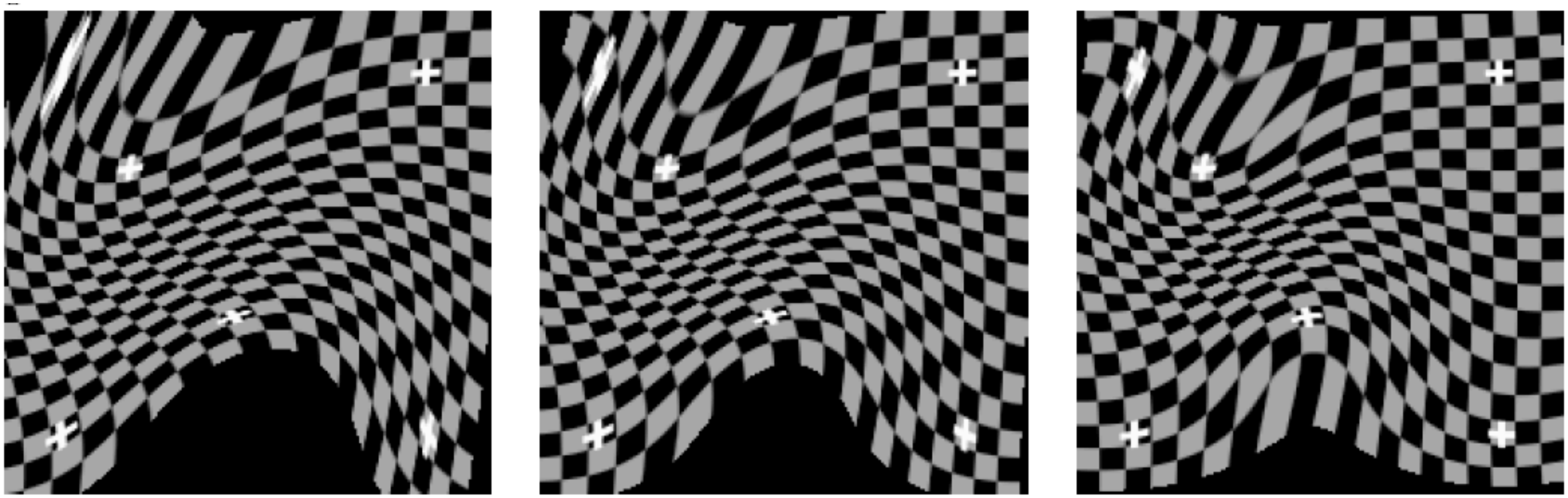
$$B = \begin{pmatrix} x_1 & x_2 & \dots & x_N & 0 & 0 & 0 \\ y_1 & y_2 & \dots & y_N & 0 & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & 0 \\ \phi_{11} & \phi_{12} & \dots & \phi_{1N} & y_1 & x_1 & 1 \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} & y_2 & x_2 & 1 \\ \vdots & & & & & & \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} & y_N & x_N & 1 \end{pmatrix}$$

RBF Warp – Example



What Kernel Should We Use

- Gaussian
  - Variance is free parameter – controls smoothness of warp



$\sigma = 2.5$ $\sigma = 2.0$ $\sigma = 1.5$

From: Arad et al. 1994

RBFs – Aligning Faces

Mona Lisa – Target

Venus – Source

Venus – Warped

RBFs – Special Case: Thin Plate Splines

- A special class of kernels

$$\phi_i(x) = ||x - x_i||^2 \lg\left(||x - x_i||\right)$$

- Minimizes the distortion function (bending energy)

$$\int \left[\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 \right] dxdy.$$

  - No scale parameter. Gives smoothest results
  - Bookstein, 1989

Application: Image Morphing

- Combine shape and intensity with time parameter t
  - Just blending with amounts t produces "fade"

$$I(t) = (1-t)I_1 + tI_2$$

  - Use control points with interpolation in t

$$\bar{c}(t) = (1-t)\bar{c}_1 + t\bar{c}_2$$

  - Use c1, c(t) landmarks to define T1, and c2,c(t) landmarks to define T2

Image Morphing

- Create from blend of two warped images

$$I_t(\bar{x}) = (1-t)I_1\left(T_1(\bar{x})\right) + tI_2\left(T_2(\bar{x})\right)$$

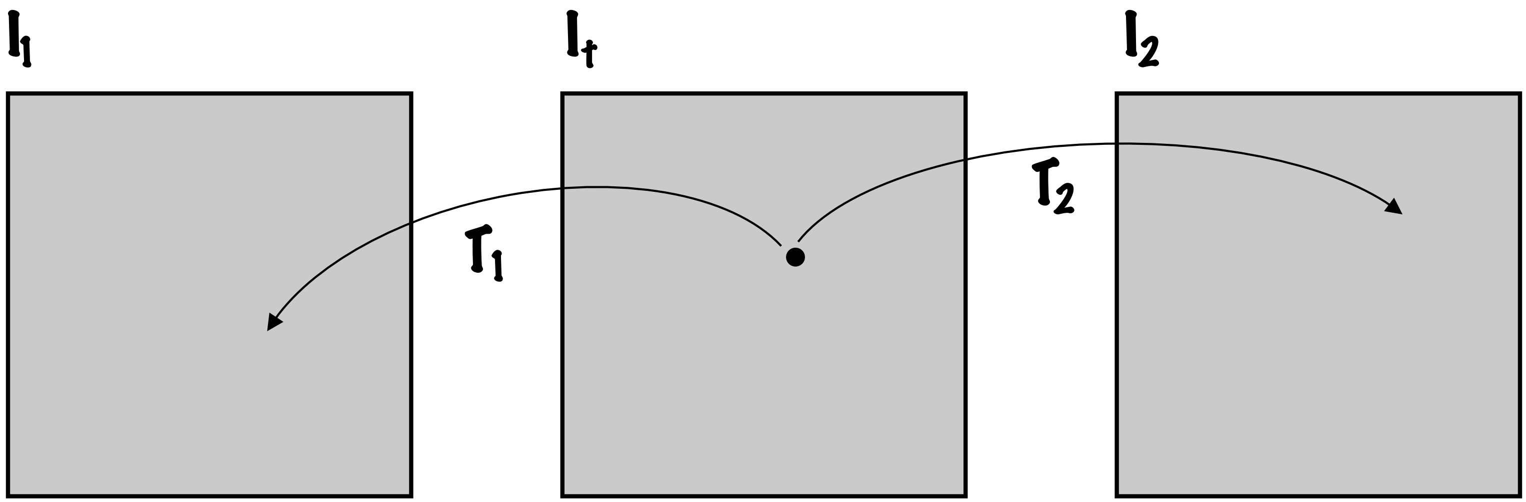


Image Morphing

Application: Image Templates/Atlases

- Build image templates that capture statistics of class of images
  - Accounts for shape and intensity
  - Mean and variability
- Purpose
  - Establish common coordinate system (for comparisons)
  - Understand how a particular case compares to the general population

Templates – Formulation

- N landmarks over M different subjects/samples

Correspondences

Images

$I^j(\bar{x})$ $\quad$ $\bar{c}_i^j$

$$\begin{pmatrix} \bar{c}_1^1 & \dots & \bar{c}_N^1 \\ \vdots & & \vdots \\ \bar{c}_1^M & \dots & \bar{c}_N^M \end{pmatrix}$$

Mean of correspondences (template)

$$\hat{c}_i = \frac{1}{M}\sum_{j=1}^{M} \bar{c}_i^j$$

Transformations from mean to subjects

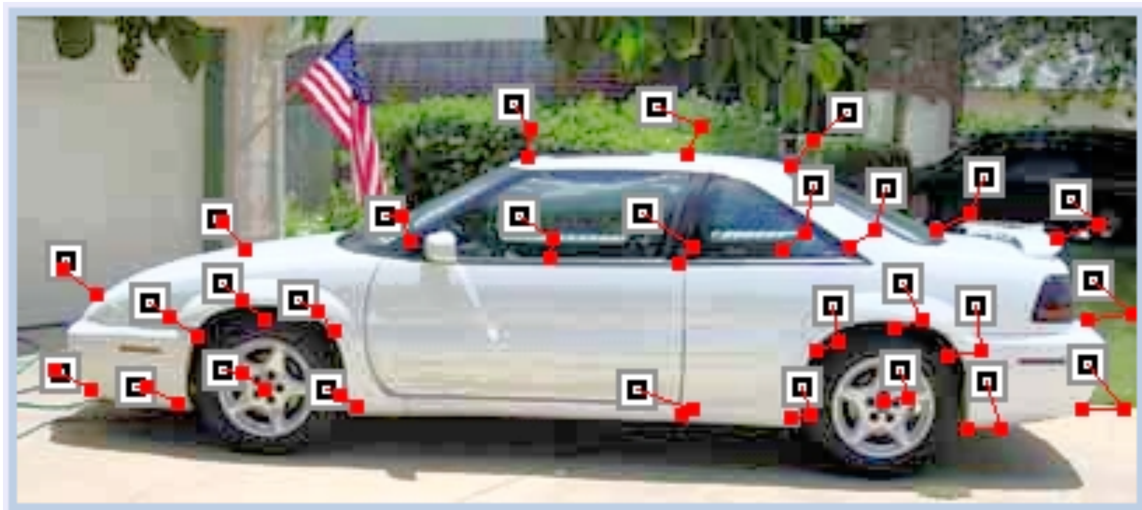
$$\bar{c}_i^j = T^j(\hat{c}_i)$$

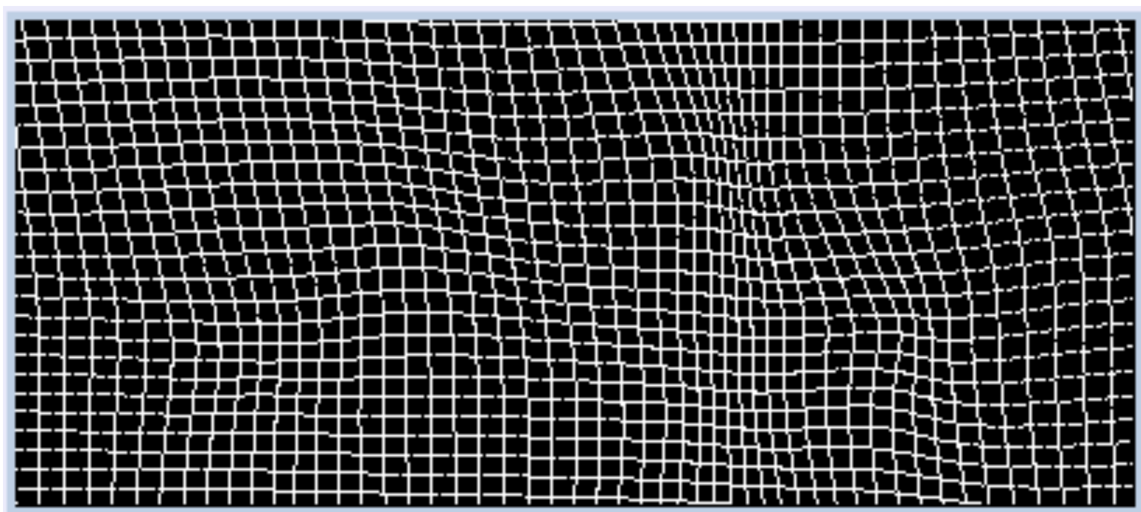
Templated image

$$\hat{I}(\bar{x}) = \frac{1}{M}\sum_j I^j\left(T^j(\bar{x})\right)$$

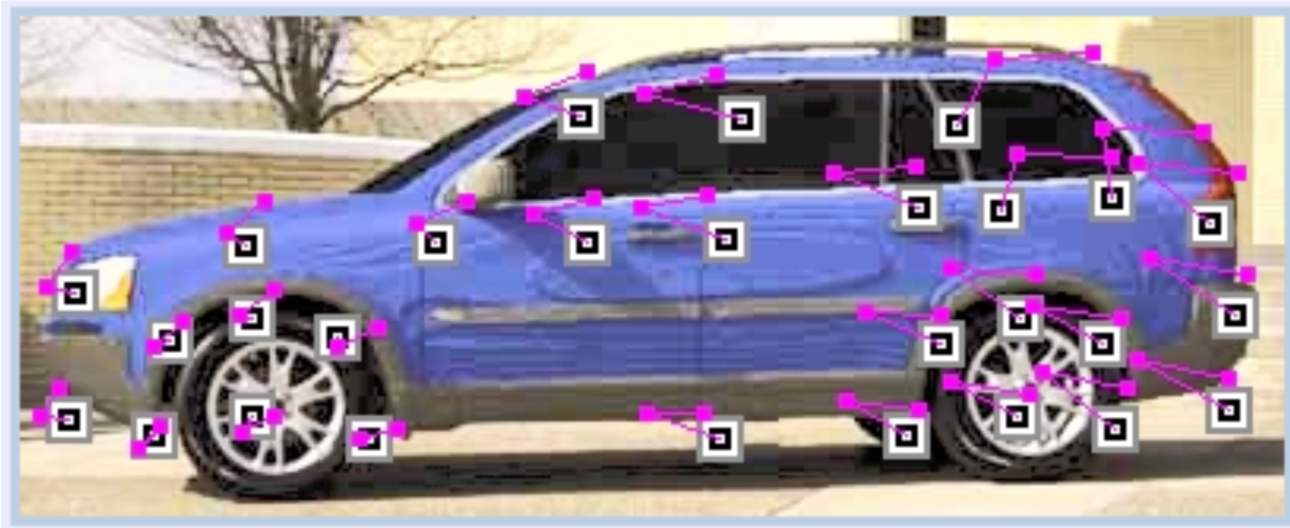
Cars

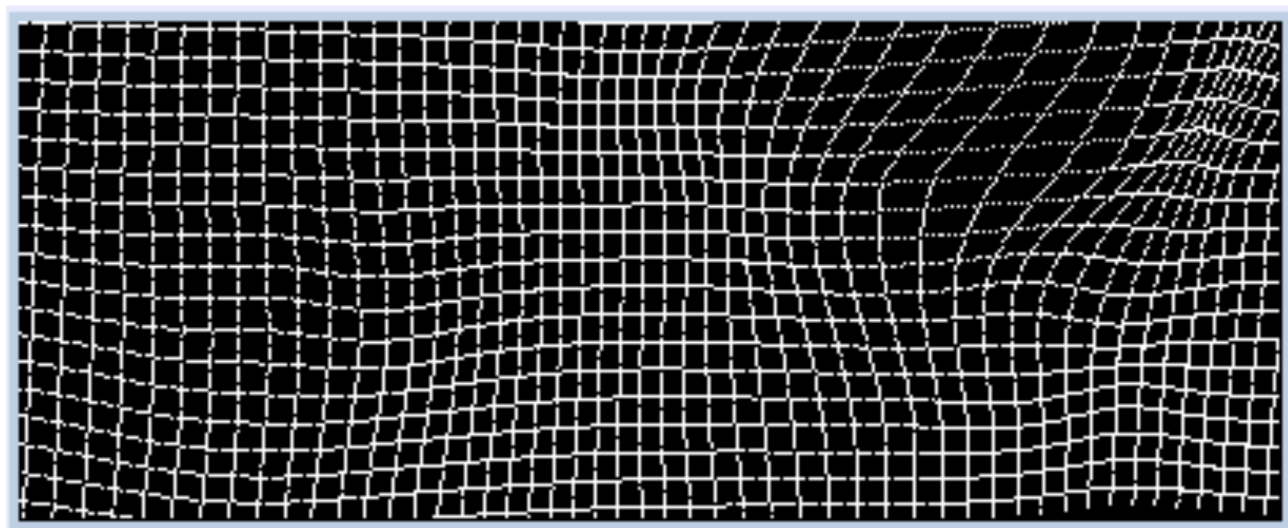
Car Landmarks and Warp



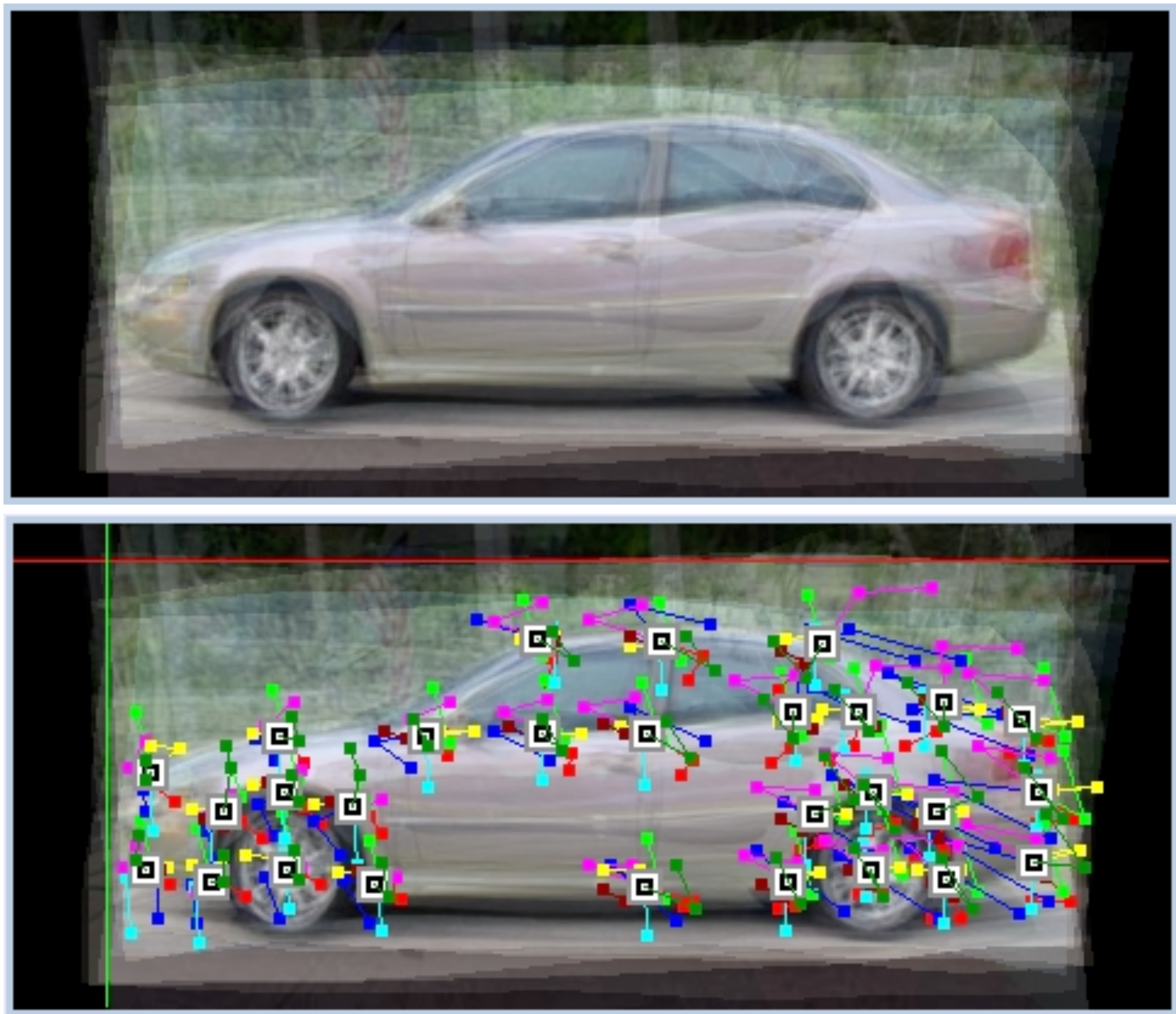


Car Landmarks and Warp

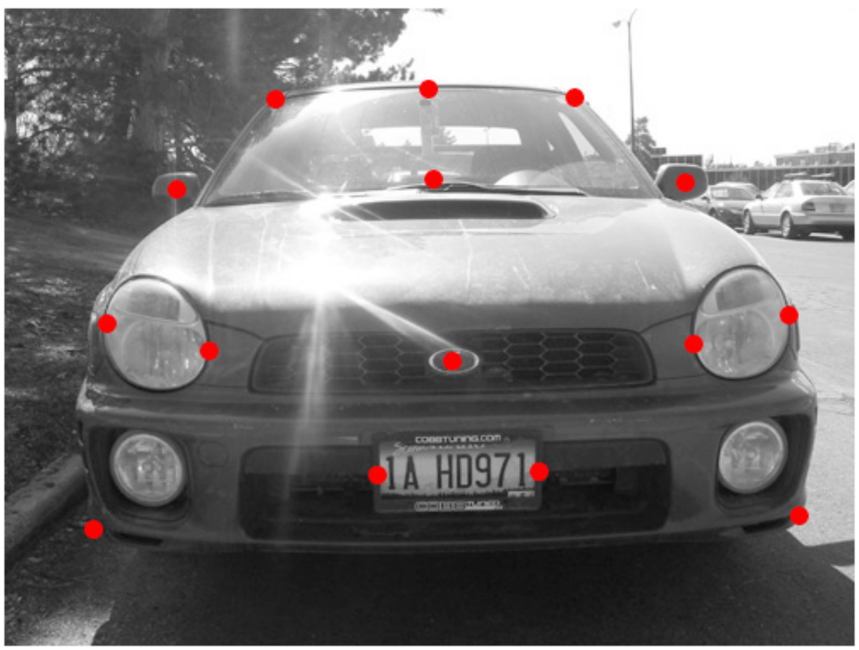




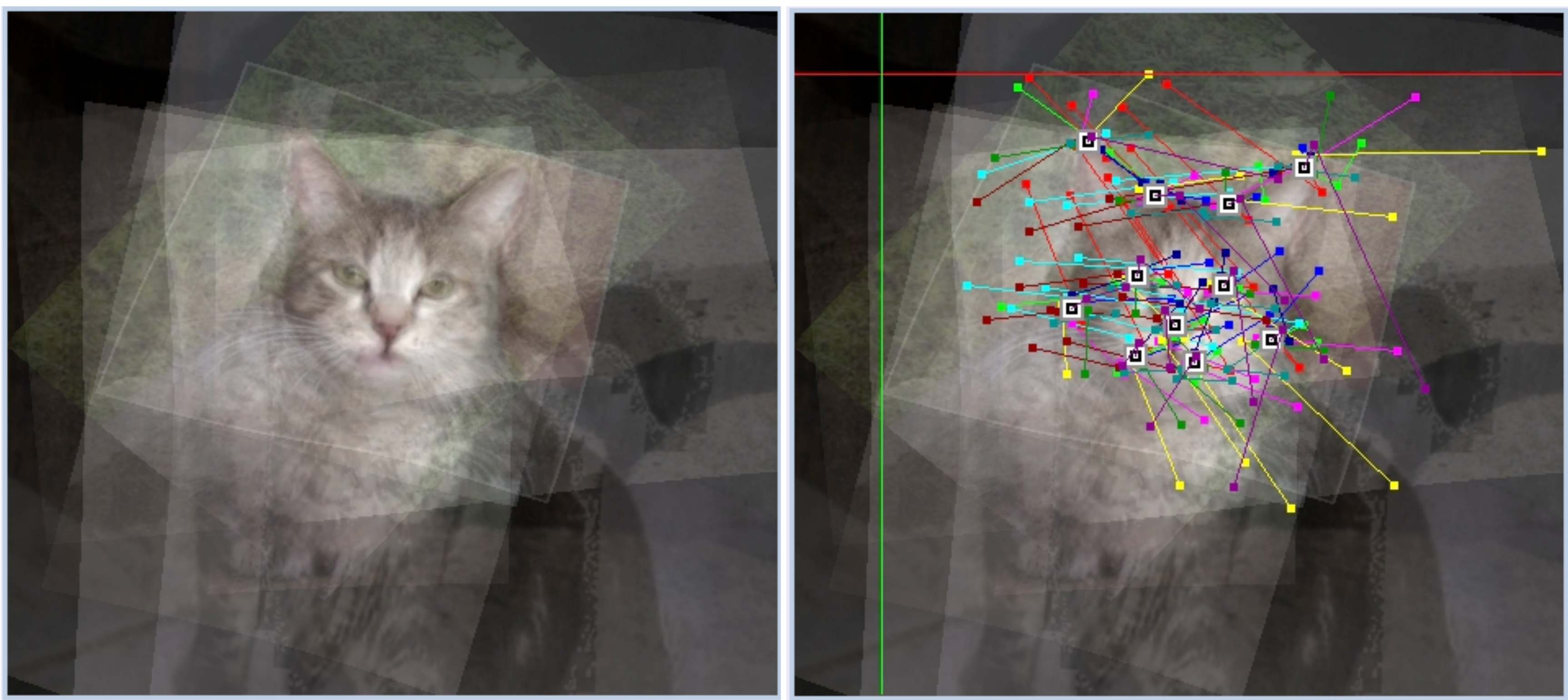
Car Mean



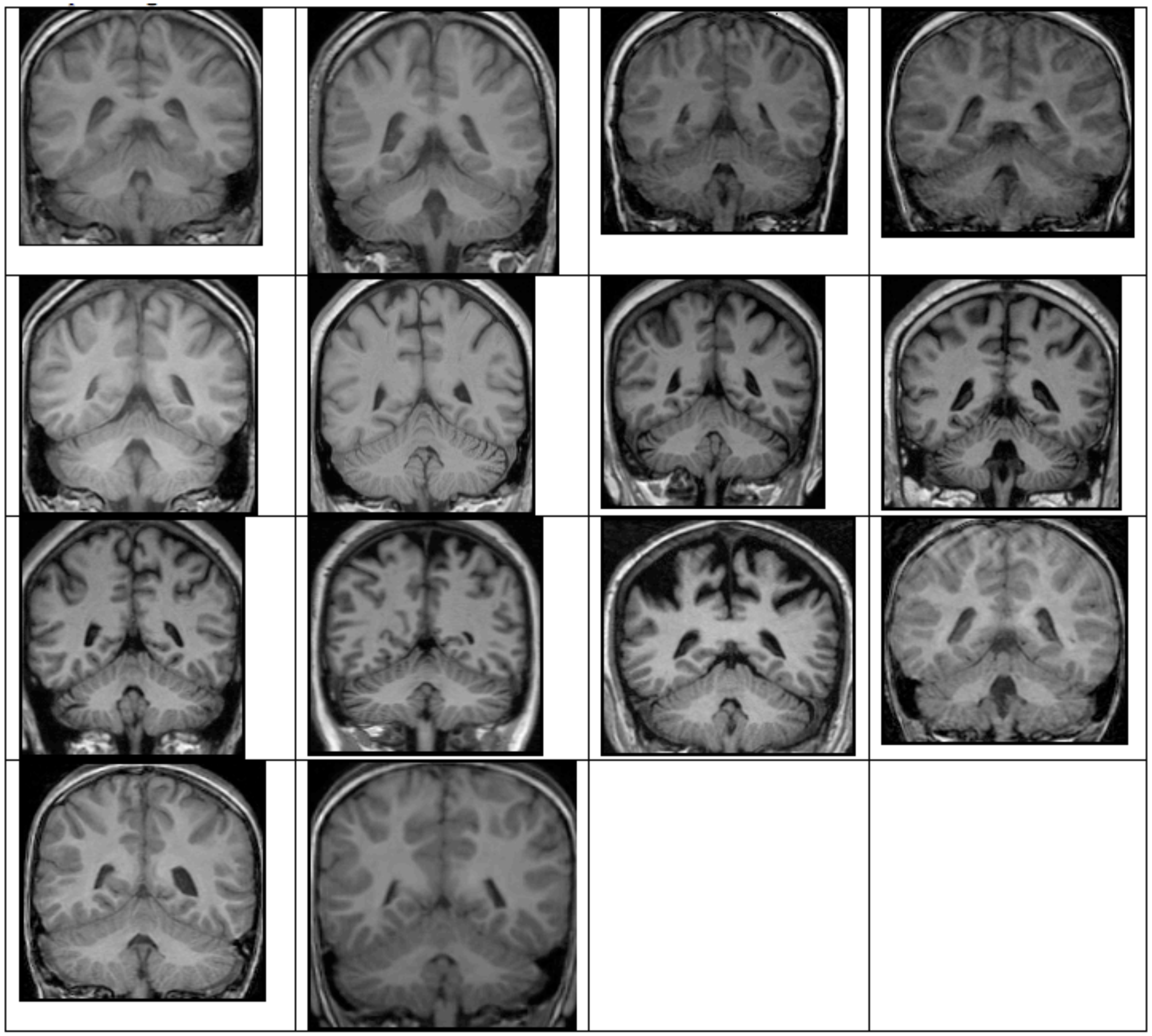
Cars



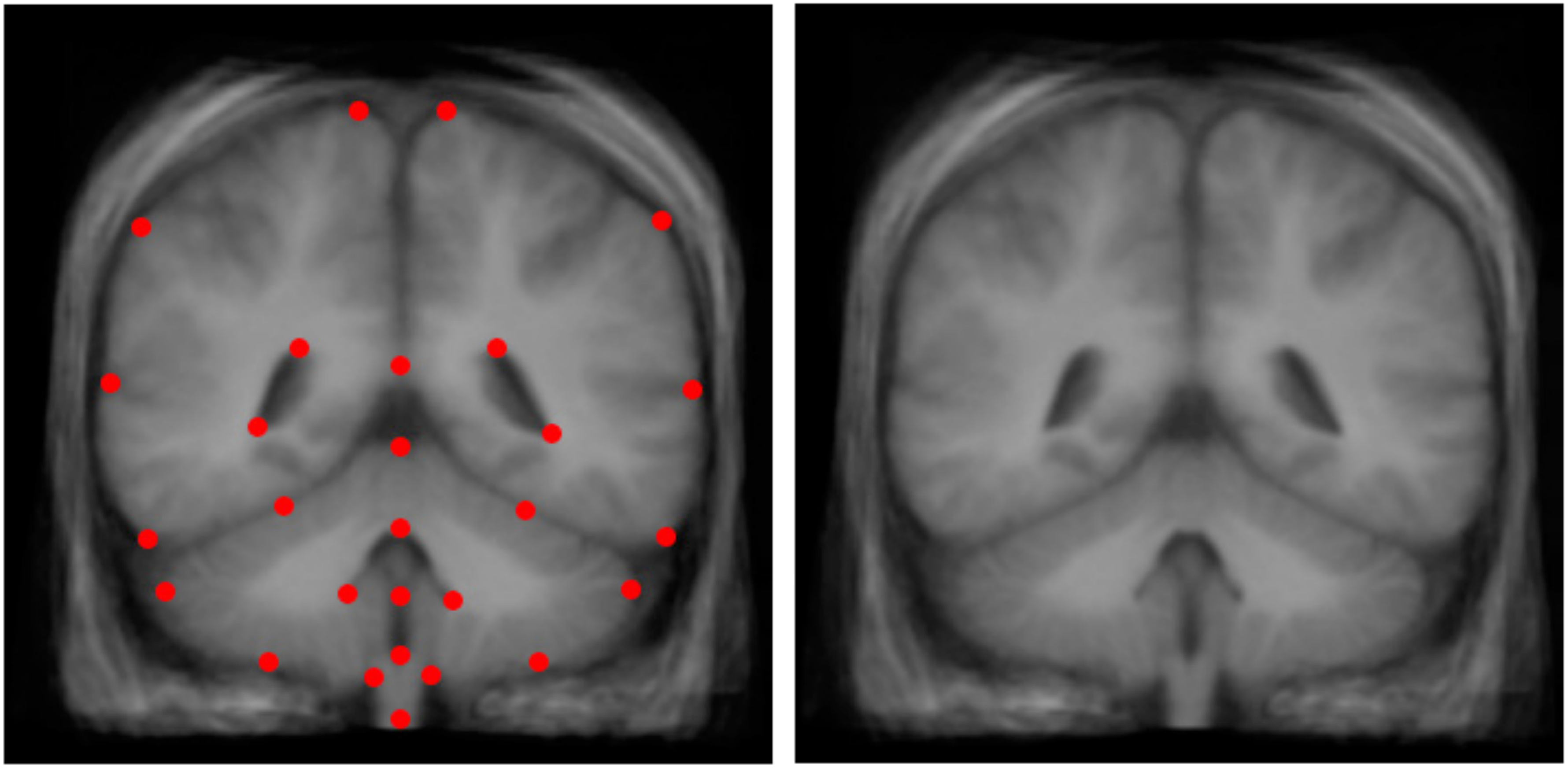
Cats



Brains



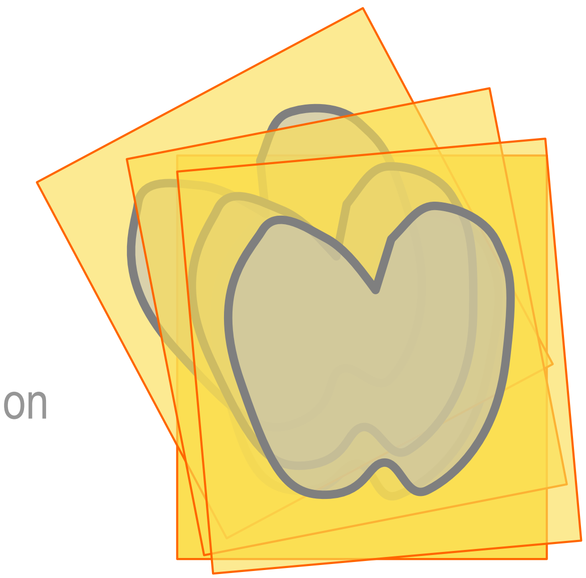
Brain Template



Automatic Registration of Images/Objects

- Within a (limited) class of transformations find the one that best matches the image
  - E.g. find rotation, translation to optimize the correlation (or squared difference) between two images

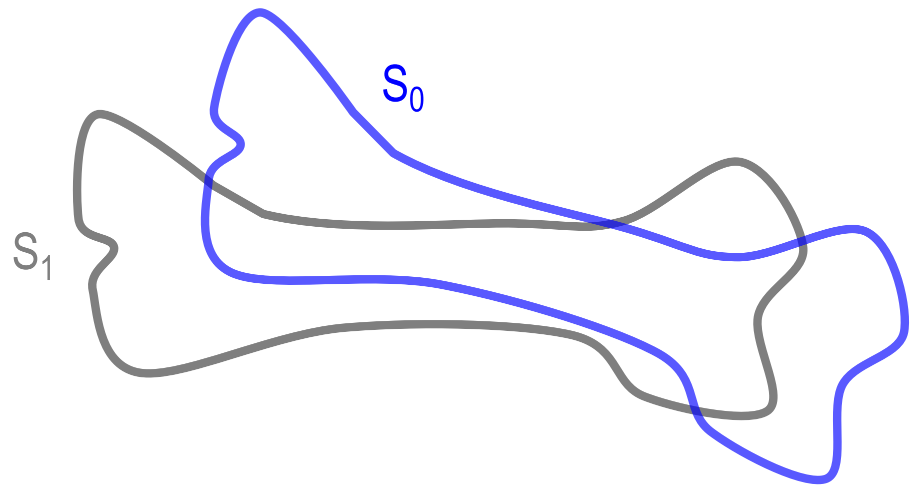
$$E(T,\theta) = \int |I_0(R(\theta)x - T) - I_1(x)|^2 \, dx$$



- How to solve:
  - Gradient descent
    - Derivatives of energy with respect to transformation
    - Includes derivatives of image
    - Can get stuck in local minima
  - Search
    - E.g. brute force (all angles and translations), or be smart

Rigid Registration of Objects/Point Sets

- Optimize squared distance to nearest point in other set



- Algorithm (iterative closest point, Besl 1992)
  1. For every point in S_0 find nearest point on S_1
  2. Find best rigid alignment of these correspondences
     - Closed form solution, Horn 1987
  3. Repeat until converged