The Elo Rating System

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This document accompanies the video at youtu.be/inXUp5j107I

Please report errors or other comments to github.com/jack-jjm/elo-video

1. The Elo Rating System

Let's speed through the part you (probably) already know as quickly as possible. The Elo rating system is a method of "measuring" the skill of players at a two player game, most famously chess. A number is assigned to each player, called their "Elo rating". Every player gets the same initial rating, but after each game, the ratings of the two players involved in the game are updated based on the outcome - the winner gaining some points, and the loser losing some points. Once all of the players have played many games, their ratings are supposed to converge to values which somehow represent their "true" skill level, players with higher ratings being more skilled. The Elo system is specific in what it means by "true" skill, in that it provides a formula which, given the ratings of players, predicts the probability that each player wins a game. If the system works as intended, those probability estimates should be accurate once the players' ratings have all converged.

This document aims to give a concise and readable overview of the mathematical motivation for the system. We will not cover convergence issues (see Section 4).

1.1. The Rules

In the interest of having all the rules of the system clearly laid out in one place for reference, let's recap the rules of Elo before getting into the motivation behind them. Suppose a two player game in which the only outcomes are a win or a loss (we'll deal with draws in a minute).

1. Initialize all players with a rating of $R_{\text{init}} \in \mathbb{R}$.
2. Given players i and j with (current) ratings R_i and R_j, we define the *estimated* probability of player i beating player j:

$$\hat{p}_{ij} = \frac{1}{1 + e^{\frac{R_j - R_i}{s}}} \tag{1}$$

   for some $s > 0$.

3. When player i beats player j, we add to player i-s rating the value

$$\Delta R = k\hat{p}_{ji} \tag{2}$$

(note the order of the indices) and subtract the same from player j-s rating, for some $k > 0$. In other words, we compute the estimated probability of the game having had the opposite outcome (given the players' current ratings), multiply by k, and add that value to the winner's rating while subtracting it from the loser's.

The constants R_{init}, s and k are arbitrary. A standard choice is $R_{\text{init}} = 1500$ and $s = \frac{400}{\ln 10}$. For these values, k is generally chosen on the order of 10 or 100 (say between 10 and 40). Larger values of k imply more volatile player ratings.

In order for formula Equation 1 to be a sensible probability formula, it should be the case that $\hat{p}_{ij} = 1 - \hat{p}_{ji}$. You can verify with a bit of algebra that this is indeed the case.

Extension to games with draws

To modify the system for a game with draws (such as chess), we tweak the update formula in the following way. First, we assign "scores" to the players at the end of a game. A winner gets a score of 1, a loser a score of 0, and in the case of a draw, both players score $\frac{1}{2}$. The update formula becomes:

$$\Delta R_i = k\left(s_i - \hat{p}_{ij}\right) \tag{3}$$

where ΔR_i is the amount of we add to player i-s rating, and s_i is player i-s score for the game that was just played. Note that when players i and j play a game, $\Delta R_j = -\Delta R_i$, as in the draw-free case. This is a consequence of the fact that the scoring system has been defined such that $s_i + s_j = 1$, always.

The logic of formula (1) is that rather than thinking of it as a probability, we now think of $\hat{p}_{ij}$ as the expected value of the score of player i when playing player j, based on their current ratings. So the update to a player's rating is actually k times the difference between their actual score and their expected score. From this point of view, the draw-free case becomes a special case of this more general system, since when drawing is impossible, expected score and probability are one and the same. Indeed, if drawing is impossible, (3) becomes (2).

In the next section, when we we introduce the Bradley-Terry model as a motivation for the Elo system, you'll find that if you think about it, the notion of draws really doesn't make sense in that model. We won't worry about this too much. Unless specified otherwise, in this document we will only ever be thinking about the draw-free case. You can think of the extension to games with draws as a purely ad-hoc extension of the system which breaks from the underlying theory somewhat, but happens to work well in practice.

1.2. Mathematical Motivation

Every explanation of the Elo system that I've come across stops here, leaving you wondering: alright, fine, but where did a crazy formula like equation Equation 1 *come from* in the first place? Surely there's some underlying model of a two player game that we're assuming here, and from which we can derive the Elo formulas?

There is, and that model is the *Bradley-Terry model*[1]. We assume that every player has a secret, strictly positive number known as their *strength*. A player's strength never changes and, as the name implies, represents their skill at the game. The game these players are playing is very simple. When two players with strengths A and B play a game, the winner is simply randomly decided. The probability of the first player winning is

$$\frac{A}{A+B} \tag{4}$$

while the probability of the second player winning is $\frac{B}{A+B}$. In other words, it's as if each player puts a number of tickets with their name on it into a hat, the number of tickets being proportional to their strength. One ticket is drawn, and the name on it is the winner. There's no profound justification as to why this is a good model of real games such as chess - it's just mathematically very simple.

Odds

Odds are an alternative way of quantifying probability. If the probability of an event is the ratio of the number of ways the event can happen to the total number of things that can happen, the odds of an event are the ratio of the number of ways the event can happen to the number of ways its complement can happen. For example, the odds of getting a six on a single dice roll are $\frac{1}{5}$. More formally, we can define odds by the following formula:

$$\text{Odds} = \frac{\text{Probability}}{1 - \text{Probability}} \tag{5}$$

Described in terms of odds, the Bradley-Terry model states that the odds a player with strength A beats a player with strength B are $\frac{A}{B}$.

Taking a Logarithm

To take two players' strengths and convert them into actual information about how much better one player is than the other (i.e. a win probability), we have to take a ratio. Taking differences is generally more convenient than taking ratios, so instead of talking about player strengths, we can talk about log strengths. That way, the log-odds are simply the difference of the log strengths:

$$\ln A - \ln B = \ln\left(\frac{A}{B}\right) = \ln \text{Odds(Player 1 wins)} \tag{6}$$

Since the probability of player one winning is ultimately some function of the log-odds, this means that the gap between two players' strength on its own encodes all of the relevant information about the skill difference between the two players. If one player has a log strength 100 points greater than another player, this always means the same thing for the probability that the former player would beat the latter.

It may happen that the log strengths of the players are very small numbers. In that case, it can be convenient to multiply all of the log strengths by some fixed, large positive number s and deal instead in terms of the *scaled* log strengths $s \ln A$, and so on. This is equivalent to adopting a smaller unit of measurement to express the log strengths.

Suppose two players have scaled log-strengths R_1 and R_2. The odds of player one winning their game can be found from the scaled log strengths as:

[1]Called the *Zermelo model* for most of the accompanying video.

$$\text{Odds(Player 1 wins)} = e^{\frac{R_1 - R_2}{s}} \tag{7}$$

We can invert formula Equation 5 to obtain the formula for probability in terms of odds[2]: $p = \frac{1}{1+\frac{1}{\text{odds}}}$. From this we can obtain the probability of player one winning:

$$p = \frac{1}{1 + e^{\frac{R_2 - R_1}{s}}} \tag{8}$$

And we have recovered formula Equation 1.

This is the origin of the Elo probability formula. When you model a game using the Elo rating system, what you're really doing is assuming that the game works according to the Bradley-Terry model, and the "ratings" you're looking for are the logarithms of the player strengths (which you cannot observe directly), scaled by the constant s. The one caveat to this is that there are actually *multiple* sets of Elo ratings equivalent to the same underlying Bradley-Terry model. We'll discuss the equivalence of Elo and Bradley-Terry more precisely in the next section.

1.3. Properties and Intuitions

In this section, we'll establish a few basic properties of the Elo system (both the model and the update algorithm). All properties stated here apply to both the draw-free case and the extension to games with draws.

A very basic property is that the Elo system always gives the same number of rating points to the winner of a game as it removes from the loser. This amount is between 0 and k. We can think of this as the winner "taking" points from the loser. This implies that the total Elo among all players is fixed.

> **Theorem 1.1** The sum of the Elo ratings of all players always equals nR_{init}, where n is the number of players. The average Elo of all players is therefore R_{init}.

Remember that the Elo system is *additively invariant* - adding the same constant to all ratings does not change the predicted win probabilities. It is therefore unclear, in principle, to which set of ratings the Elo algorithm will converge (assuming it converges at all). Essentially, the requirement that the final ratings predict the correct win probabilities determines only the "difference matrix" between the ratings. The above fact adds a constraint on the average rating, which along with the difference matrix, uniquely determines one set of ratings.

It can be nice to have a clear picture of how changing the initial rating and the scale affect the rating system.

> **Theorem 1.2** Suppose an initial rating of R_{init} is chosen, and a set of players play some number of games together. Afterwards, for all i, player i ends up with a final rating of R_i. Now suppose we "rewind the clock" and instead use an inital rating of $R_{\text{init}} + a$. Then the players play the exact same sequence of games, which all have exactly the same outcomes. Then the new final rating of the i-th player will be $R_{i'} = R_i + a$.

[2]Note that the calculation will at one point involve the possibly slightly unobvious identity $\frac{x}{1+x} = \frac{1}{1+\frac{1}{x}}$

Proof. Let $R_{i,t}$ denote the rating of the i-th player after t games have been played in the *first* series of games. Let $R_{(i,t)'}$ denote the same player's rating after t games in the *second* series of games. We will prove by induction that $R'_{i,t} = R_{i,t} + a$ for all t, in particular implying it holds after all of the games have been played. The condition holds at time $t = 0$, when $R'_{i,t} = R_{\text{init}} + a = R_{i,t} + a$. The $(t+1)$-st game will add the same value to $R'_{i,t}$ as to $R_{i,t}$, so if the condition holds after t games, it will hold after $t + 1$. ■

> **Theorem 1.3** In the same sense as the previous theorem, if we retroactively change the scale of the system from s to as, and change the step-size parameter k to ak, the new final rating of the i-th player will be $R_{i'} = a(R_i - R_{\text{init}}) + R_{\text{init}}$.

Proof. Obviously the condition holds at $t = 0$. Suppose it holds at time t. If the next game is between players i and j, then in the *second* series of games, the estimated probability of player i winning will be $p = \frac{1}{1+e^d}$, where

$$\begin{aligned} d &= \frac{R_{(i,t)'} - R_{(j,t)'}}{as} \\ &= \frac{a(R_i - R_{\text{init}}) + R_{\text{init}} - \big(a\big(R_j - R_{\text{init}}\big) + R_{\text{init}}\big)}{as} \\ &= \frac{R_{i,t} - R_{j,t}}{s} \end{aligned} \tag{9}$$

which is exactly the estimated probability of the i-th player winning that game in the first series of games. Therefore, the rating update will be kp in the first series of games and akp in the second, and we have

$$\begin{aligned} R_{(i,t+1)'} &= R_{(i,t)'} + akp \\ &= a\big(R_{i,t} + kp - R_{\text{init}}\big) + R_{\text{init}} \\ &= a\big(R_{i,t+1} - R_{\text{init}}\big) + R_{\text{init}} \end{aligned} \tag{10}$$

as required. ■

The intuition as to why the previous theorem requires multiplying k as well as s is that k and s are essentially in the same "units". The value of k has no meaning on its own, only its ratio to s is meaningful.

2. The Bradley-Terry Model

In the last section we showed how the Elo rating system is essentially equivalent to the Bradley-Terry model. In this section we'll be a little more fastidious about what exactly this means. We begin with a generic definition for the notion of a model of a game, to act as a common language with which to compare different models of games.

> **Definition 2.1** A *game* is a set Π called the set of *players*, and a function p such that for any two distinct players x and y, $0 \leq p(x, y) \leq 1$, and $p(x, y) = 1 - p(y, x)$. We think of $p(x, y)$ as the probability that x beats y in the game.

This is about the most assumption-free model of a two player game possible. All we're saying is that for any two players, it is meaningful to talk about the probability that one beats the other, making no further assumptions about those probabilities.

Now we formally define the two specific models of games we've been discussing so far.

> **Definition 2.2** A *Bradley-Terry model* (BTM) for a game (Π, p) is a function $f : \Pi \to \mathbb{R}_{>0}$ such that for any two distinct players x and y:
>
> $$p(x, y) = \frac{f(x)}{f(x) + f(y)} \tag{11}$$
>
> We call f the *strength function.* Note that several distinct Bradley-Terry models might exist for a given game (i.e. several assignments of strength might yield the same probabilities). If at least one Bradley-Terry model exists for a given game, we say that the game *is Bradley-Terry.*

It is absolutely not the case that all games are Bradley-Terry, if only we choose the right strength function. For one thing, in a BTM, no player can have probability zero of beating another (merely because f is required to be strictly positive).

> **Definition 2.3** A *logistic model* for a game (Π, p) is a function $R : \Pi \to \mathbb{R}$ such that, for some constant $s > 0$:
>
> $$p(x, y) = \frac{1}{1 + e^{\frac{R(y) - R(x)}{s}}} \tag{12}$$
>
> We call R the (true) *rating function.* Note that the constant s is unique (for given functions R and p, only one s will make the above identity true), and we call ithe *scale* of the model. If at least one logistic model exists for a game, we say that the game *is logistic.*

We call this a *logistic* and not an *Elo* model as you might expect, to avoid confusion with Thurstone models, which as we discuss when we introduce them in Section 3, have at least as good a claim as the logistic model to the name of Elo.

As hinted at in the definitions, several BTMs may exist for a given game. This is just a roundabout way of saying that for a given set of probabilities, the player strengths are not unique - indeed, if you multiply every player strength by the same positive number, none of the ratios change, and so all of the probabilities remain the same. This is in fact the *only* way to obtain an equivalent BTM:

> **Theorem 2.1** If f is a Bradley-Terry model for a game, then the set of all Bradley-Terry models for that game is af, for all $a \in \mathbb{R}_{>0}$.

Similarly, we can *add* a constant to every true player rating in a logistic model without changing the probabilities. This is obvious enough from the logistic probability function itself: the probability depends only on the difference in ratings between two players. It also makes sense in view of our discussion that the ratings are essentially the log strengths. Multiplying the strengths by a constant is the same as adding a constant to the log strengths.

> **Theorem 2.2** If R is a logistic model for a game, then the set of all logistic models for that game is $aR + b$, for all $a \in \mathbb{R}, b \in \mathbb{R}_{>0}$.

The reason for the extra degree of freedom here is simply because of the scaling constant s that we allow for in the definition of a logistic model, while there is no such arbitrary constant in the definition of a BTM. Thus there are two ways of changing the ratings of players without changing the resulting win probabilities. We can add a constant to all ratings (which leaves the probabilities unchanged because the logistic formula is inherently additively invariant) or we can multiply the ratings by a positive constant (which leaves the probabilities unchanged because that constant will get cancelled out in the probability formula when dividing by the scale). Multiplying the ratings by a constant changes the scale in the way you would predict:

> **Theorem 2.3** If R is a logistic model of scale s for a game and $a > 0$, then $aR + b$ is a logistic model of scale as for that same game.

This raises an interesting question which we won't answer straight away. If there are several equally "valid" rating functions for a given game, which set of ratings does the Elo algorithm actually converge to?

Finally, our discussion in the previous section on deriving the Elo formulas from the Bradley-Terry model becomes the following theorem.

> **Theorem 2.4** A game is Bradley-Terry if and only if it is logistic. If f is a Bradley-Terry model for a game, then for any constant $s > 0$, the function $R(x) = s\ln(x)$ is a logistic model. If R is a logistic model for a game, then there is a (unique) constant $s > 0$ such that the function $f(x) = e^{\frac{R(x)}{s}}$ is a Bradley-Terry model for the game. The unique constant is equal to the scale of f.

Odds Transitivity

A property which you may see references to in the literature is *odds transitivity*. For example, it appears in [1] and (referencing that paper) in [2]. We can state the property in our terminology like this:

> **Definition 2.4** A game p is *odds-transitive* if for any three distinct players x, y and z:
>
> $$\omega(x, y)\omega(y, z) = \omega(x, z) \tag{13}$$
>
> where $\omega(x, y)$ refers to the odds that x beats y, i.e. $\omega(x, y) = \frac{p(x,y)}{p(y,x)}$.

The relevance of this property for us is the following:

Theorem 2.5 Given a game, the following three properties are equivalent:

1. The game is Bradley-Terry.
2. The game is logistic.
3. The game is odds-transitive and no player has probability 1 of beating any other player.

Proof. The simplest direction is to show that a Bradley-Terry game is odds-transitive, indeed, this is really the direction that motivates defining the property in the first place. This direction follows immediately by writing out the odds transitivity equation using the Bradley-Terry formula for odds, and cancelling like terms. That no player beats another with probability 1 follows from the fact that a BTM strength function is not allowed to be zero.

The slightly more complicated direction is to show that any odds-transitive game in which no player beats another with probability 1 is also Bradley-Terry. To show this, choose some arbitrary player x_0, and define the following function:

$$\begin{gathered} f(x) = \omega(x, x_0) \text{ for } x \neq x_0 \\ f(x_0) = 1 \end{gathered} \tag{14}$$

I claim that this is a Bradley-Terry model for the game. To prove this, we first need to show that f is always positive and non-zero, since this is a requirement for a BTM.

With that done, we now have to show the main property of a BTM: that for any two distinct players x, y, the odds $\omega(x, y)$ that x beats y is equal to $\frac{f(x)}{f(y)}$. We consider three cases: when neither x nor y equals x_0, when $x = x_0$, and when $y = x_0$.

In the first case, we have by odds transitivity $\omega(x, y) = \omega(x, x_0)\omega(x_0, y)$. A general property of odds is that the odds of any event are equal to the inverse of the odds of the complementary event. In our case this implies $\omega(x_0, y) = \frac{1}{\omega(y, x_0)}$, and therefore $\omega(x, y) = \frac{f(x)}{f(y)}$, as required.

If $x = x_0$, we have

$$\omega(x, y) = \omega(x_0, y) = \frac{1}{f(y)} = \frac{f(x)}{f(y)} \tag{15}$$

and if $y = x_0$, we have

$$\omega(x, y) = \omega(x, x_0) = \frac{f(x)}{1} = \frac{f(x)}{f(y)} \tag{16}$$

■

Note that the requirement, in condition (3), that no player have a win probability of 1 against any other, is very minor. In an odds-transitive game, such a player would have to have a win probability of 1 against every other, a “perfect player”. If two such perfect players existed, then odds-transitivity would not make sense, since applying it with both x and z perfect players would require multiplying zero by infinity. Therefore, in the worst case, in order to apply the equivalence theorem to an odds-transitive game, we would only need to exclude a single perfect player.

3. The Thurstone Model

Consider the following model of a two player game, with no draws. When two players play, they each generate a random number independently from a normal distribution. Whoever generates the larger number wins the game. The variances of these normal distributions are assumed to be fixed and the same for all players, but the mean of each player's normal distribution is unique to them. This is called the *Thurstone model*.

> **Definition 3.1** A *Thurstone model* for a game (Π, p) is a function $\mu : \Pi \to \mathbb{R}$ such that for any two players x and y:
>
> $$p(x, y) = P(X > Y) \tag{17}$$
>
> where $X \sim \mathcal{N}(\mu(x), \sigma^2)$ and $Y \sim \mathcal{N}(\mu(y), \sigma^2)$ are independent, σ^2 is a non-negative constant independent of x and y, and $p(x, y)$ is the probability that x beats y.
>
> If at least one Thurstone model exists for a game, we will say that the game *is Thurstone*.

You may have come across Thurstone models before if you've tried to read up on the Elo system. The two are sometimes conflated together, or discussed in close proximity to one another in a way that can be hard to tease apart for a learner. Our main purpose in this section is to clarify that the Thurstone model and the Bradley-Terry model are *not* equivalent. Specifically, we will prove that there exist games that are Thurstone but not Bradley-Terry (the converse is probably true as well, but we won't prove it here).

One reason why Thurstone and B-T models are sometimes conflated is historical. The original Elo system, from the late 1950s, was Thurstone. The logistic model was introduced later, and at some point became the model most strongly associated with the name "Elo". If you read Arpad Elo's book from 1978, the system that the book is essentially all about is the Thurstone - the logistic model discussed only in a short section in chapter 8.

General form

Another reason is that the two models are mathematically more similar than they first appear. In a Thurstone model, we have $p(x, y) = P(X > Y)$, which can also be written as $P(X - Y > 0)$. Since the difference of two normal distributions is also normally distributed, $X - Y$ is itself a normal random variable of mean $\mu(x) - \mu(y)$, and so $p(x, y)$ is equal to the probability that such a normal variable is greater than zero. We won't actually write out the integral, but the point is that $p(x, y)$ is a function of the difference in "true ratings" $\mu(x) - \mu(y)$. The same is true of the Bradley-Terry model: $p(x, y)$ is a function of the difference in true ratings $R(x) - R(y)$. In other words, both models are of the form

$$p(x, y) = f(\Delta_{xy}) \tag{18}$$

Where Δ_{xy} is the rating difference. From this perspective, the only difference between the two systems is the function f used to transform a difference in ratings into a probability.

In the Bradley-Terry model this function is a logistic function, and in the Thurstone model it it is the cumulative distribution function of a normal distribution. The two curves are numerically quite close to one another. For this reason, explanations of the Elo system sometimes discuss the difference between the Thurstone and logistic models as though it's a technicality, merely a matter of choosing a curve that fit the data slightly more optimally. Numerically this may be the case, but conceptually and pedagogically, the Thurstone and Bradley-Terry models have very different underpinnings.

Hyper-geometric Distribution

Actually, it is possible to give an equivalent definition of the Bradley-Terry model which makes it very similar to the one we gave of the Thurstone model. If we assume that when two players play a game, they each generate a random number, and whoever generates the larger one wins, this can be made equivalent to a Bradley-Terry model - provided that we assume the numbers are drawn from a relatively obscure kind of probability distribution called a *hypergeometric distribution*. [3] Again, this is interesting, but pedagogically probably not the best definition of the Thurstone model for someone encountering it for the first time.

Non-equivalence

We now move on to our main purpose here, checking that the Thurstone and Bradley-Terry models are not equivalent. This may seem obvious since, as discussed above, both models are of the form Equation 18, and the function f is different in each case. But remember that there's no reason to use the same ratings in each model. If we have some Thurstone with a certain rating function, and we simply swap out the probability function for that of the Bradley-Terry model while using the same ratings, of course the win probabilities will change. But could there not exist an alternate set of ratings such that the resulting BTM produces the same probabilities as the initial Thurstone model? To rule out this kind of thing, we have to actually prove that there exist Thurstone games for which *no* Bradley-Terry model exists.

Our approach will be to construct a game which is Thurstone, but does not satisfy odds transitivity. Our first step is to establish that odds transitivity in some sense characterizes the logistic function. In any model satisfying Equation 18, odds transitivity can be stated as:

$$\frac{f(\Delta_{xy})}{1-f(\Delta_{xy})}\frac{f(\Delta_{yz})}{1-f(\Delta_{yz})}=\frac{f(\Delta_{xz})}{1-f(\Delta_{xz})} \tag{19}$$

which is basically a functional equation for f. In this light, odds transitivity becomes a property of a function, not a game. The theorem below establishes that there is essentially only one family of solutions to this functional equation, the sigmoid functions of the form $f(x)=\sigma\left(\frac{x}{s}\right)$, where

$$\sigma(x)=\frac{1}{1+e^{-x}} \tag{20}$$

Theorem 3.1 Let $f:\mathbb{R}\to(0,1)$ be an increasing function satisfying, for any two real numbers x, y:

$$\frac{f(x)}{1-f(x)}\frac{f(y)}{1-f(y)}=\frac{f(x+y)}{1-f(x+y)} \tag{21}$$

Then for some $s>0$, $f(x)=\sigma\left(\frac{x}{s}\right)$ for all x.

Proof. Define $g(x) = \frac{f(x)}{1-f(x)}$, then we have

$$g(x+y) = g(x)g(y) \tag{22}$$

Furthermore, g is increasing (this follows from the fact that f is increasing) and $g(0) = 0$. It is well known that a function satisfying these conditions must be an exponential function, thus we must have $g(x) = e^{\frac{x}{s}}$ for some $s > 0$. Therefore

$$f(x) = e^{\frac{x}{s}}(1 - f(x)) \tag{23}$$

which can be solved for $f(x)$ to obtain the desired result. ■

This immediately produces:

> **Theorem 3.2** There exists at least one game which is Thurstone but not Bradley-Terry.

Proof. Since the function f in Equation 18 for the Thurstone model is *not* a sigmoid[3], there must exist two real numbers x and y such that the functional equation of the previous theorem is not verified. If we construct a Thurstone game with three players, such that the difference in means between the first two players is x, and the difference in means between the second two players is y, this game will therefore not be Bradley-Terry. ■

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[3]Strictly speaking this needs to be proven. See Section 4.

4. Addenda

Pairwise comparison theory

Bradley-Terry and Thurstone models belong to the mathematical field of **pairwise comparison**, if you're interested in reading about other approaches to this kind of problem.

Does anyone still use Elo?

The logistic "Elo" model superseded the older Thurstone model as it was found to model chess skill better, but nowadays even the logistic model is largely being replaced by even newer systems. Glicko and Glicko-2 are some of the most popular ones - most popular online chess servers use one of these. People nowadays may call systems such as Glicko "Elo", adding to the existing terminological confusion. I believe the online game *Age of Empires II: Definitive Edition* uses Elo, i.e. a logistic model.

The question of convergence

This document was originally intended to at least superficially cover the topic of convergence, i.e. why the update algorithm actually produces correct ratings, but I dropped the section because the topic turned out to be more of a rabbit-hole than I thought. The basic intuition is that correct ratings are the only stable equilibria of the system, in the sense of the **expected** rating change. If two players play a game, and the estimated win probability of one player is $\hat{p}$, while their true win probability is p, then their expected rating change after the game is $k(p - \hat{p})$. Therefore, a player's rating is only stable *on average* if their rating is correct. The plan to prove convergence from here had originally been to find a loss function whose gradient is equal to that average update, and thereby reduce the problem to stochastic gradient descent.

The trouble starts when we introduce more than two players. If we have n players, at at each timestep we choose two of them to randomly play a game, then the expected rating change of the i-th player at each timestep is

$$\frac{1}{n}\frac{k}{n-1}\sum_{j\neq -i}(p_{ij} - \hat{p}_{ij}) \tag{24}$$

Where p_{ij} and $\hat{p}_{ij}$ are the true and estimated probabilities of i beating j, and n is the number of players. It is not obvious that the only way to make this zero for all i is to set $\hat{p}_{ij} = p_{ij}$ for all i, j, and fact, it is false. We have on the order of n^2 unknowns $\hat{p}_{ij}$ and only n linear constraints on them. For a simple example, suppose we have three players, and each player wins against the other with true probability $\frac{1}{2}$. If we choose estimated probabilities of player 1 beating player 2 with probability 0.6, player 2 beating player 3 with probability 0.6, and player 3 beating player 1 with probability 0.6, then the expected rating change above is indeed zero.

The obvious problem with this spurious solution is that it is not odds transitive: we have a cycle of player 1 being strong against player 2, player 2 strong against player 3, and player 3 strong against player 1. Since the logistic model is only capable of representing odds-transitive games, the solution here is presumably (this is a conjecture) to show that odds transitivity (which is a non linear constraint on the estimated probabilities) implies that there is indeed only one solution to make the expected rating change zero for all players.

If anyone would like to contribute an argument along these lines, or another simple proof of convergence, please see the first page of this document for contact information.

Full proof of non-equivalence

In the proof of Theorem 4.1, we asserted without proof that in Equation 18, the function f corresponding to the Thurstone model is a different function (is not identically equal to) the function f corresponding to the Bradley-Terry model. Let's prove that now. For the Thurstone model, $f(x)$ is the probability that a normally distributed random variable of mean x and variance σ^2 is greater than zero. This can also be written as the probability that a normally distributed random variable of mean 0 and variance σ^2 is greater than x:

$$f(x) = 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{t^2}{2\sigma^2}} dt \tag{25}$$

We must prove that this function not identically equal to the function

$$\frac{1}{1 + e^{-\frac{x}{s}}} \tag{26}$$

no matter what we choose for $\sigma > 0$ and $s > 0$.

To do this, we can differentiate both functions and show that the derivatives are not identically equal:

$$\frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{s} \frac{e^{\frac{x}{s}}}{\left(1 + e^{\frac{x}{s}}\right)^2} \tag{27}$$

where without loss of generality we've dropped the factor of 2π. This can be rewritten as

$$e^{-\frac{x}{s}} + e^{\frac{x}{s}} + 2 - \frac{\sigma}{s} e^{\frac{x^2}{2\sigma^2}} = 0 \tag{28}$$

But the left hand side is dominated by the quadratic term and tends to $-\infty$ as $x \to \infty$.

Further reading

The main three papers I'm aware of on the Bradley-Terry model are [4], [1], and [5]. [4] is the earliest paper on the subject that I'm aware of, is explicitly about chess, and defines the BTM in the same way we do in Definition 4.1. An English translation of this paper can be found in **Annotated Readings in the History of Statistics** from Springer. [3] is also a useful overview of the theory. Finally, you can also read [2], but the motivation behind the model is not covered in any real depth.

5. References

[1] I. J. Good, “On the Marking of Chess-Players,” 1955. [Online]. Available: https://sci-hub.se/https://www.jstor.org/stable/3608567

[2] A. Elo, *The Rating of Chess Players, Past and Present*. 1978. [Online]. Available: https://archive.org/details/ratingofchesspla00unse

[3] M. E. Glickman, “A Comprehensive Guide to Chess Ratings.” [Online]. Available: http://www.glicko.net/research/acjpaper.pdf

[4] E. Zermelo, “Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung,” 1929.

[5] R. A. Bradley and M. E. Terry, “Rank Analysis of Incomplete Block Designs,” 1952. [Online]. Available: https://sci-hub.se/https://www.jstor.org/stable/2334029