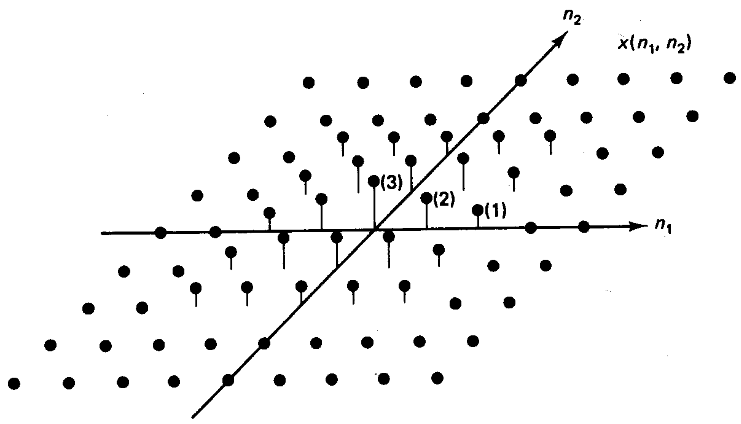
2D Systems

- Images are outputs of 2D systems
- Continuous vs. sampled (discrete) images
- 2D discrete (digital) images are sequences which are functions of two integer arguments
- System – input-output relationship

1

Lim, 1990

2D Sequence $x(n_1,n_2)$



2

Lim, 1990

2D Sequence $x(n_1,n_2)$ (cont')

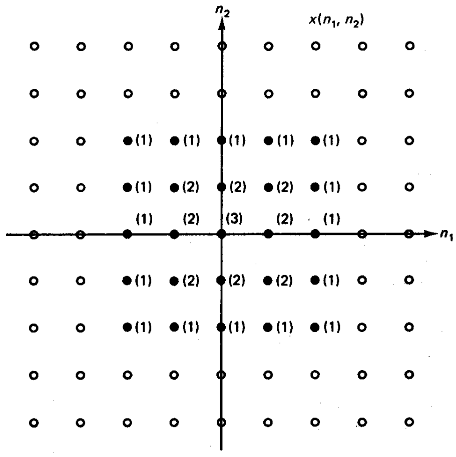


Figure 1.2 Alternate way to sketch the 2-D sequence in Figure 1.1. Open circles represent amplitudes of zero, and filled-in circles represent nonzero amplitudes, with values in parentheses representing the amplitude.

3

Lim, 1990

2D Sequence $x(n_1,n_2)$ (cont')

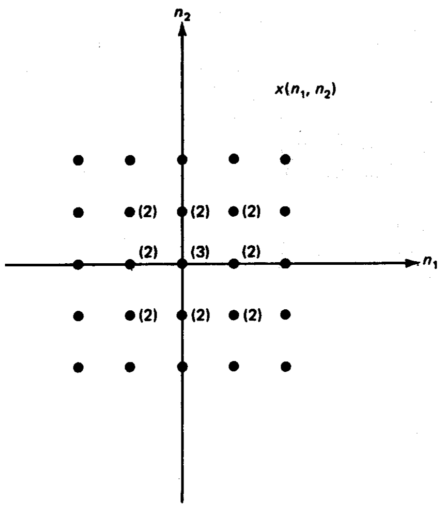


Figure 1.3 Sequence in Figure 1.2 sketched with some simplification. Open circles have been eliminated and filled-in circles with amplitude of 1 have no amplitude specifications.

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2D Sequences (cont')

- Impulses

$$\delta(n_1,n_2)=\begin{cases}1 & n_1=n_2=0\\ 0 & \text{otherwise}\end{cases}$$ impulse (unit sample sequence)

$$\delta_T(n_1)=\begin{cases}1 & n_1=0\\ 0 & \text{otherwise}\end{cases}$$ line impulse (T indicates 2D sequence of one variable)

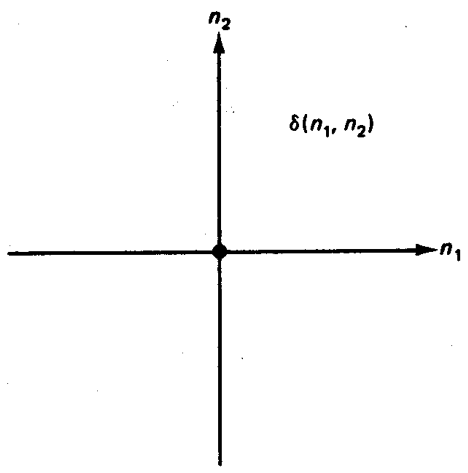
$$u(n_1,n_2)=\begin{cases}1 & n_1,n_2\geq 0\\ 0 & \text{otherwise}\end{cases}$$ unit step sequence

$$x(n_1,n_2)=f(n_1)g(n_2)$$ separable sequence

5

Lim, 1990

Impulse $\delta(n_1,n_2)$



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Sequence Representation as Linear Combination of Shifted Impulses

$$\begin{aligned}x(n_1,n_2) = \cdots &+ x(-1,-1)\,\delta(n_1+1,n_2+1) + x(0,-1)\,\delta(n_1,n_2+1)\\ &+ x(1,-1)\,\delta(n_1-1,n_2+1) + \cdots + x(-1,0)\,\delta(n_1+1,n_2)\\ &+ x(0,0)\,\delta(n_1,n_2) + x(1,0)\,\delta(n_1-1,n_2)\\ &+ \cdots + x(-1,1)\,\delta(n_1+1,n_2-1)\\ &+ x(0,1)\,\delta(n_1,n_2-1) + x(1,1)\,\delta(n_1-1,n_2-1) + \cdots\\ = \sum_{k_1=-\infty}^{\infty}&\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\,\delta(n_1-k_1,n_2-k_2).\end{aligned}$$

sequence

value

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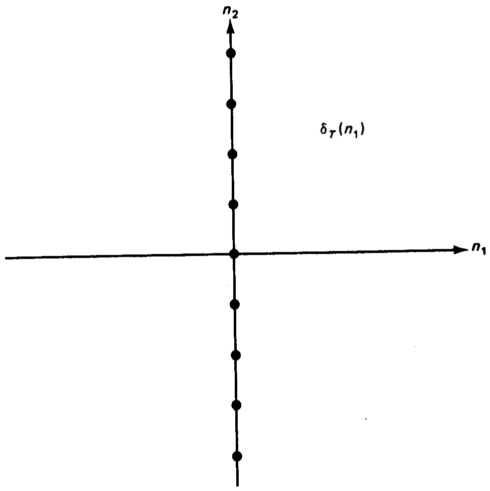
Lim, 1990

Line Impulse $\delta_T(n_1)$



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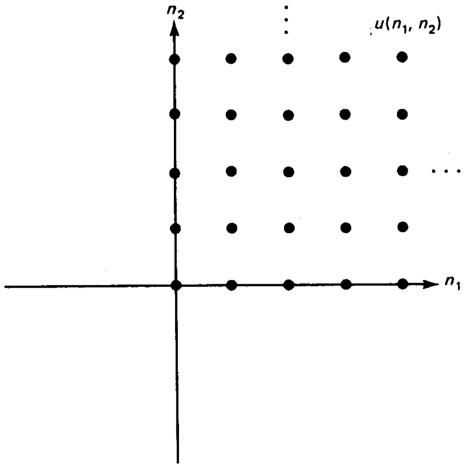
Lim, 1990

Unit Step Sequence $u(n_1,n_2)$



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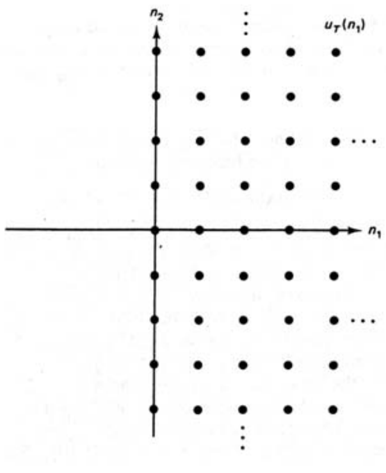
Step Sequence $u_T(n_1)$



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Jain, 1989

2D Separable Functions

$f(x_1,x_2)= f_1(x_1)\, f_2(x_2)$

TABLE 2.1 Some Special Functions

Function	Definition	Function	Definition
Dirac delta	$\delta(x) = 0, x \neq 0$	*Rectangle*	$\operatorname{rect}(x) = \begin{cases} 1, & \lvert x \rvert \leq \frac{1}{2} \\ 0, & \lvert x \rvert > \frac{1}{2} \end{cases}$
	$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \delta(x)\, dx = 1$	*Signum*	$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$
Sifting property	$\int_{-\infty}^{\infty} f(x')\,\delta(x - x')\, dx' = f(x)$	*Sinc*	$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$
Scaling property	$\delta(ax) = \frac{\delta(x)}{\lvert a \rvert}$	*Comb*	$\operatorname{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$
Kronecker delta	$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$	*Triangle*	$\operatorname{tri}(x) = \begin{cases} 1 - \lvert x \rvert, & \lvert x \rvert \leq 1 \\ 0, & \lvert x \rvert > 1 \end{cases}$
Sifting property	$\sum_{m=-\infty}^{\infty} f(m)\,\delta(n - m) = f(n)$		

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Periodic Sequence

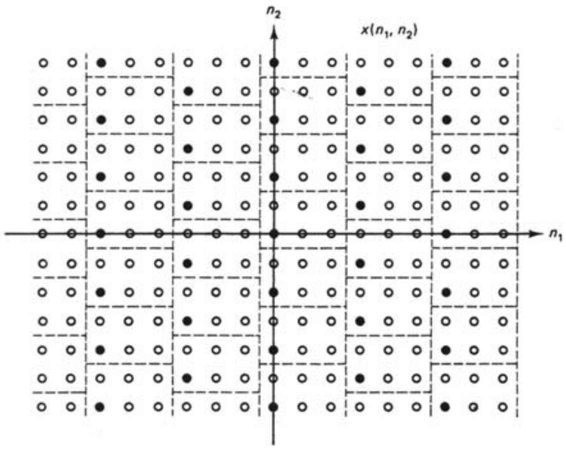

Figure 1.8 Periodic sequence with a period of 6 × 2.

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Linear Systems

- Input-output relationship is called a system if there is a unique output for any given input

$$y(n_1, n_2) = T\left[x(n_1, n_2)\right].$$

- $Linearity \Leftrightarrow T\left[ax_1(n_1, n_2) + bx_2(n_1, n_2)\right] = ay_1(n_1, n_2) + by_2(n_1, n_2)$

which is the *principle of superposition*.

Linear vs. Nonlinear System

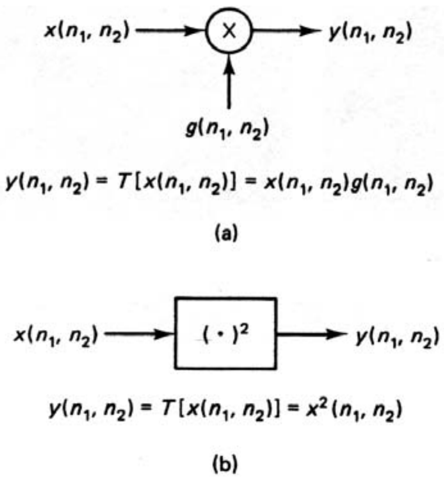

Figure 1.12 (a) Example of a linear shift-variant system; (b) example of a nonlinear shift-invariant system.

2D Convolution (cont')

$$y(n_1,n_2) = x(n_1,n_2) * h(n_1,n_2) \equiv$$

$$\sum_{k_1'=-\infty}^{\infty} \sum_{k_2'=-\infty}^{\infty} x(k_1,k_2) h(n_1-k_1,n_2-k_2)$$

where

$x(n_1,n_2)$ is the input and

$h(n_1-k_1,n_2-k_2)$ the impulse response

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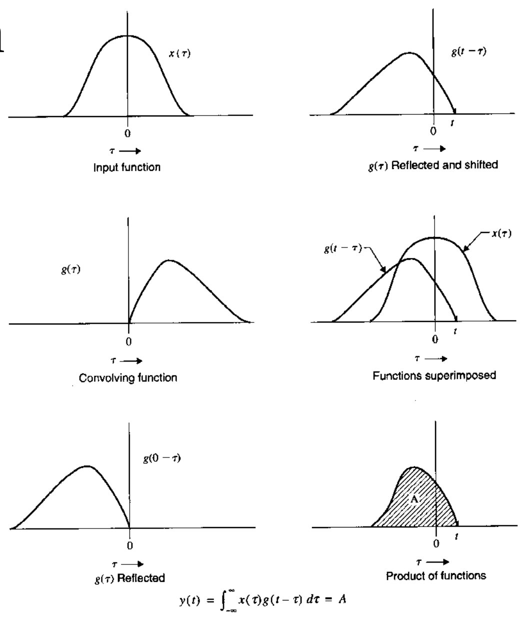
Castleman, 1996

1D Convolution by Castleman



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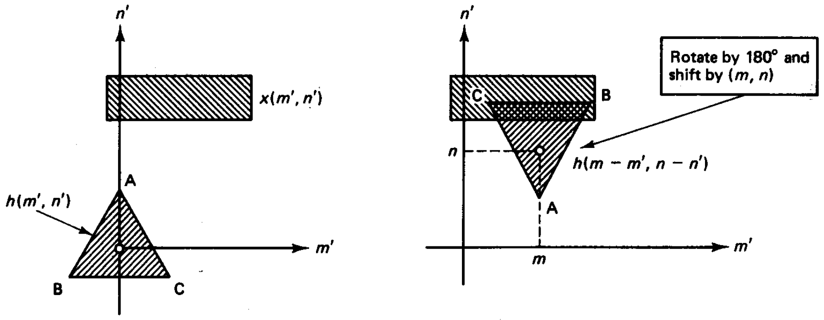
Jain, 1989

2D Convolution by Jain (1)



(a) impulse response

(b) output at location (m, n) is the sum of product of quantities in the area of overlap.

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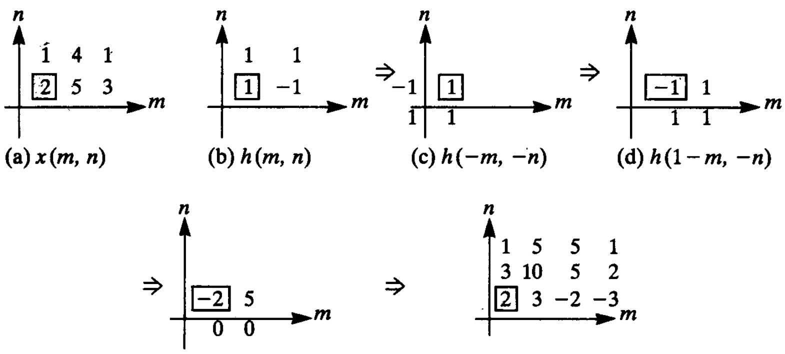
Jain, 1989

2D Convolution by Jain (2)



(a) $x(m, n)$ (b) $h(m, n)$ (c) $h(-m, -n)$ (d) $h(1-m, -n)$

(e) $y(1, 0) = -2 + 5 = 3$ (f) $y(m, n)$

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Lim, 1990

…and 2D convolution by Lim

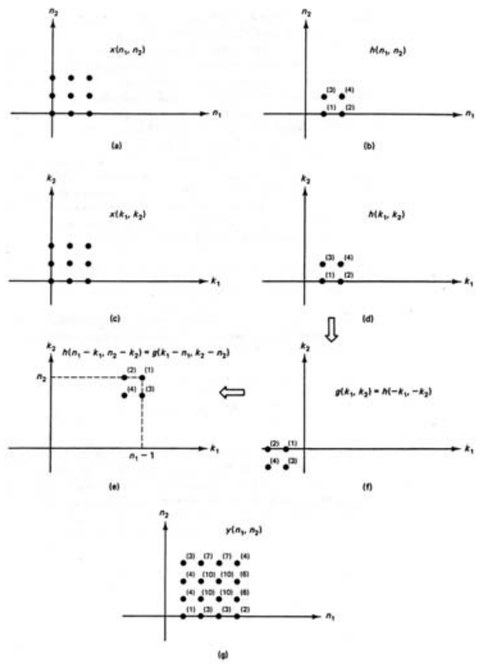

Figure 1.13 Example of convolving two sequences.

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2D Convolution

- Applications
  - ❑ Deconvolution (deblurring due to deformation, motion, atmosphere)
  - ❑ Noise removal
    - Estimating original signal
    - Detecting known feature embedded in a noisy background
  - ❑ Feature enhancement (edges, spots)

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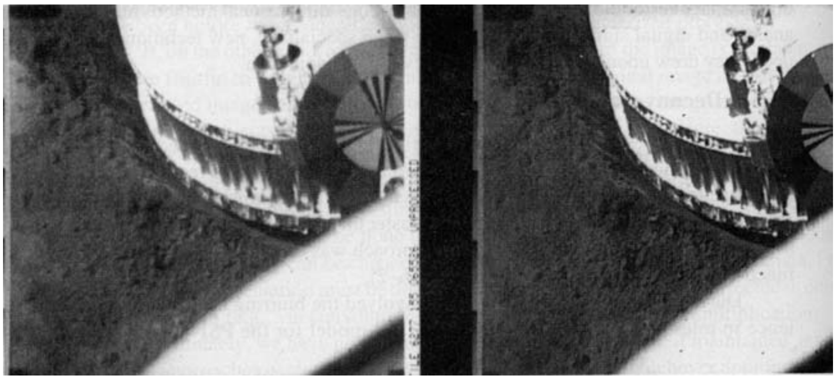
Castleman, 1996

Applications 1: Deconvolution



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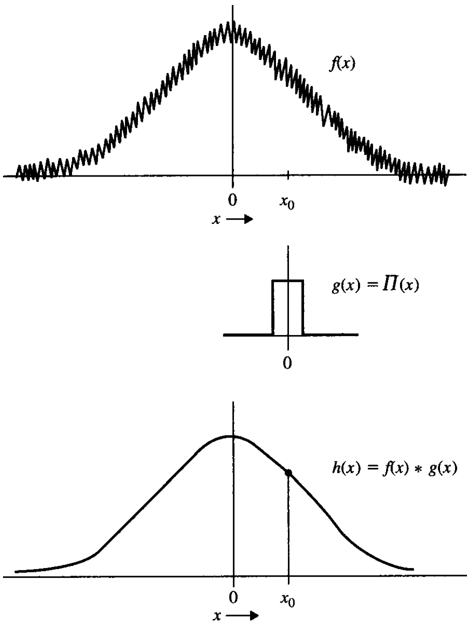
Castleman, 1996

Applications 2: Smoothing



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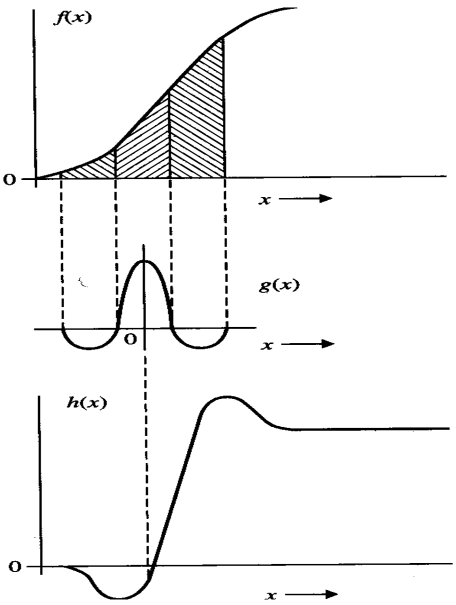
Castleman, 1996

Applications 3: Edge Enhancement



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Lim, 1990

Region of Convolution

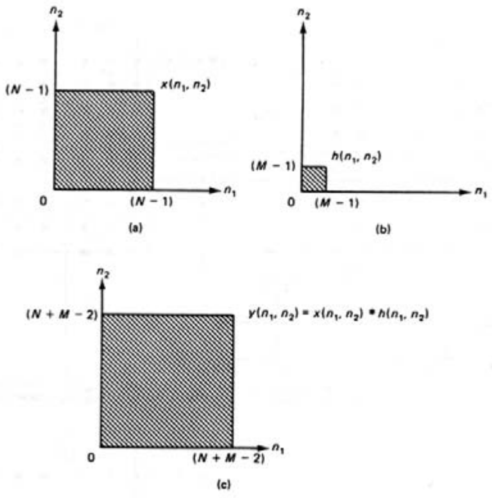

Figure 1.14 Regions of (n_1, n_2) where $x(n_1, n_2)$, $h(n_1, n_2)$, and $y(n_1, n_2) = x(n_1, n_2) * h(n_1, n_2)$ can have nonzero amplitude.

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Lim, 1990

Convolution with a Separable Sequence

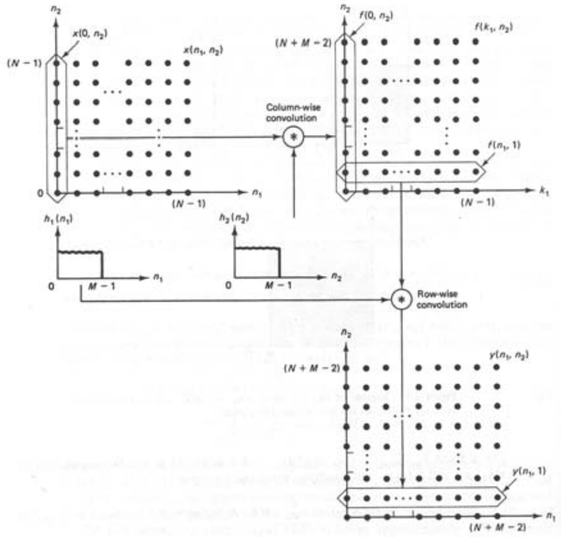

Figure 1.15 Convolution of $x(n_1, n_2)$ with a separable sequence $h(n_1, n_2)$.

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The Fourier Transform of a 2D Sequence $x(m,n)$

- The *Fourier Transform Pair*

$$X(\omega_1, \omega_2) \equiv \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m,n) \exp\{-j(\omega_1 m + \omega_2 n)\}, \ -\pi \le \omega_1, \omega_2 < \pi$$

$$x(m,n) = \frac{1}{(2\pi)^2} \int_{\omega_1=-\pi}^{\pi} \int_{\omega_2=-\pi}^{\pi} X(\omega_1, \omega_2) \exp\{j(\omega_1 m + \omega_2 n)\} d\omega_1 d\omega_2$$

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Lim, 1990

Properties of the Fourier Transform by Lim

$$x(n_1, n_2) \longleftrightarrow X(\omega_1, \omega_2)$$
$$y(n_1, n_2) \longleftrightarrow Y(\omega_1, \omega_2)$$

Property 1. *Linearity*
$$ax(n_1, n_2) + by(n_1, n_2) \longleftrightarrow aX(\omega_1, \omega_2) + bY(\omega_1, \omega_2)$$

Property 2. *Convolution*
$$x(n_1, n_2) * y(n_1, n_2) \longleftrightarrow X(\omega_1, \omega_2)Y(\omega_1, \omega_2)$$

Property 3. *Multiplication*
$$x(n_1, n_2)y(n_1, n_2) \longleftrightarrow X(\omega_1, \omega_2) \circledast Y(\omega_1, \omega_2)$$
$$= \frac{1}{(2\pi)^2} \int_{\theta_1=-\pi}^{\pi} \int_{\theta_2=-\pi}^{\pi} X(\theta_1, \theta_2)Y(\omega_1 - \theta_1, \omega_2 - \theta_2)\, d\theta_1\, d\theta_2$$

Property 4. *Separable Sequence*
$$x(n_1, n_2) = x_1(n_1)x_2(n_2) \longleftrightarrow X(\omega_1, \omega_2) = X_1(\omega_1)X_2(\omega_2)$$

Property 5. *Shift of a Sequence and a Fourier Transform*
(a) $x(n_1 - m_1, n_2 - m_2) \longleftrightarrow X(\omega_1, \omega_2)e^{-j\omega_1 m_1}e^{-j\omega_2 m_2}$
(b) $e^{j\nu_1 n_1}e^{j\nu_2 n_2}x(n_1, n_2) \longleftrightarrow X(\omega_1 - \nu_1, \omega_2 - \nu_2)$

Property 6. *Differentiation*
(a) $-jn_1x(n_1, n_2) \longleftrightarrow \dfrac{\partial X(\omega_1, \omega_2)}{\partial \omega_1}$
(b) $-jn_2x(n_1, n_2) \longleftrightarrow \dfrac{\partial X(\omega_1, \omega_2)}{\partial \omega_2}$

Property 7. *Initial Value and DC Value Theorem*
(a) $x(0, 0) = \dfrac{1}{(2\pi)^2} \displaystyle\int_{\omega_1=-\pi}^{\pi} \int_{\omega_2=-\pi}^{\pi} X(\omega_1, \omega_2)\, d\omega_1\, d\omega_2$
(b) $X(0, 0) = \displaystyle\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2)$

Property 8. *Parseval's Theorem*
(a) $\displaystyle\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2)y^*(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{\omega_1=-\pi}^{\pi} \int_{\omega_2=-\pi}^{\pi} X(\omega_1, \omega_2)Y^*(\omega_1, \omega_2)\, d\omega_1\, d\omega_2$
(b) $\displaystyle\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |x(n_1, n_2)|^2 = \frac{1}{(2\pi)^2} \int_{\omega_1=-\pi}^{\pi} \int_{\omega_2=-\pi}^{\pi} |X(\omega_1, \omega_2)|^2\, d\omega_1\, d\omega_2$

Property 9. *Symmetry Properties*
(a) $x(-n_1, n_2) \longleftrightarrow X(-\omega_1, \omega_2)$
(b) $x(n_1, -n_2) \longleftrightarrow X(\omega_1, -\omega_2)$
(c) $x(-n_1, -n_2) \longleftrightarrow X(-\omega_1, -\omega_2)$
(d) $x^*(n_1, n_2) \longleftrightarrow X^*(-\omega_1, -\omega_2)$
(e) $x(n_1, n_2)$: real $\longleftrightarrow X(\omega_1, \omega_2) = X^*(-\omega_1, -\omega_2)$
$X_R(\omega_1, \omega_2)$, $|X(\omega_1, \omega_2)|$: even (symmetric with respect to the origin)
$X_I(\omega_1, \omega_2)$, $\theta_x(\omega_1, \omega_2)$: odd (antisymmetric with respect to the origin)
(f) $x(n_1, n_2)$: real and even $\longleftrightarrow X(\omega_1, \omega_2)$: real and even
(g) $x(n_1, n_2)$: real and odd $\longleftrightarrow X(\omega_1, \omega_2)$: pure imaginary and odd

Property 10. *Uniform Convergence*
For a stable $x(n_1, n_2)$, the Fourier transform of $x(n_1, n_2)$ uniformly converges.

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Castleman, 1996

Properties of the Fourier Transform by Castleman

Property	Spatial domain	Frequency domain
Addition theorem	$f(x, y) + g(x, y)$	$F(u, v) + G(u, v)$
Similarity theorem	$f(ax, by)$	$\dfrac{1}{\lvert ab\rvert}F\left(\dfrac{u}{a}, \dfrac{v}{b}\right)$
Shift theorem	$f(x-a, y-b)$	$e^{-j2\pi(au+bv)}F(u, v)$
Convolution theorem	$f(x, y) * g(x, y)$	$F(u, v)G(u, v)$
Separable product	$f(x)g(y)$	$F(u)G(v)$
Differentiation	$\left(\dfrac{\partial}{\partial x}\right)^m\left(\dfrac{\partial}{\partial y}\right)^n f(x, y)$	$(j2\pi u)^m(j2\pi v)^n F(u, v)$
Rotation	$f(x\cos\theta + y\sin\theta,$ $-x\sin\theta + y\cos\theta)$	$F(u\cos\theta + v\sin\theta,$ $-u\sin\theta + v\cos\theta)$
Laplacian	$\nabla^2 f(x, y) = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)f(x, y)$	$-4\pi^2(u^2 + v^2)F(u, v)$

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Properties of the Fourier Transform

- The addition theorem (addition in time/spatial domain corresponds to addition in frequency)
- The shift theorem (shifting a function causes only to phase shift)
- The convolution theorem (convolution is equivalent to multiplication in the other domain)
- …

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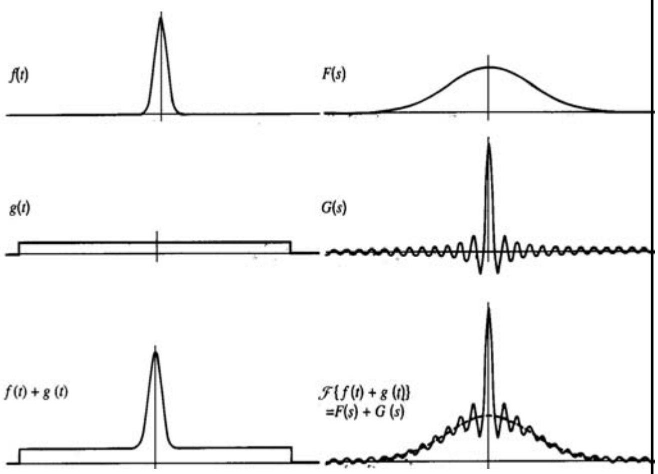
Castleman, 1996

The Addition Theorem



$$\Phi\{f(t)+g(t)\}=$$

$$\int_{-\infty}^{\infty}[f(t)+g(t)]e^{-2j\pi st}dt$$

$$\int_{-\infty}^{\infty} f(t)e^{-2j\pi st}dt+\int_{-\infty}^{\infty} g(t)e^{-2j\pi st}dt=$$

$$F(s)+G(s)$$

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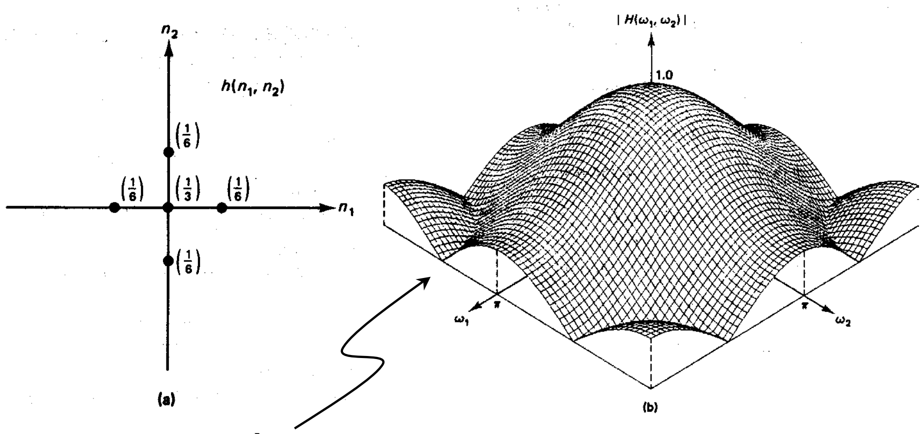
Lim, 1990

The Fourier Transform – Example 1



low-pass→blur

Figure 1.19 (a) 2-D sequence $h(n_1, n_2)$; (b) Fourier transform magnitude $|H(\omega_1, \omega_2)|$ of $h(n_1, n_2)$ in (a).

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Lim, 1990

The Fourier Transform – Example 1

$$H(\omega_1,\omega_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} h(n_1,n_2)\exp\{-j(\omega_1 n_1 + \omega_2 n_2)\}$$

$$= \frac{1}{3} + \frac{1}{6}e^{-j\omega_1} + \frac{1}{6}e^{-j\omega_2} + \frac{1}{6}e^{j\omega_1} + \frac{1}{6}e^{j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{3}\cos(\omega_1) + \frac{1}{3}\cos(\omega_2)$$

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Lim, 1990

The Fourier Transform – Example 1

Figure 1.20 (a) Image of 256 × 256 pixels; (b) image processed by filtering the image in (a) with a lowpass filter whose impulse response is given by $h(n_1, n_2)$ in Figure 1.19 (a).

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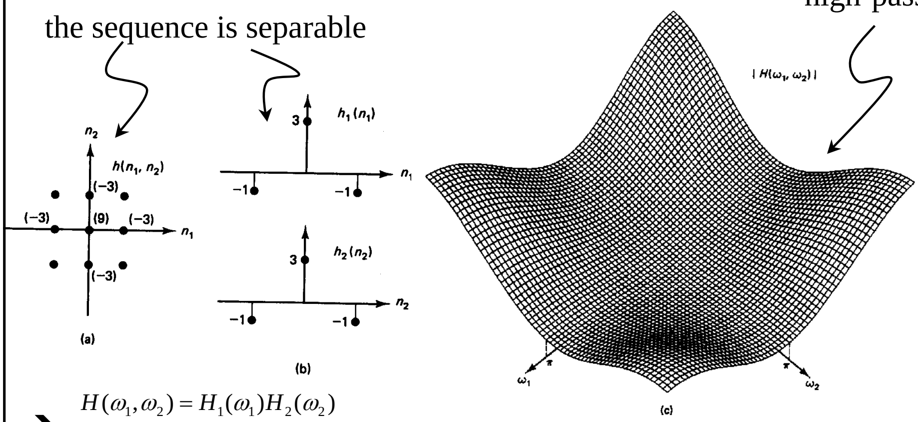
Lim, 1990

The Fourier Transform – Example 2

high-pass

the sequence is separable



➔ $$H(\omega_1,\omega_2) = H_1(\omega_1)H_2(\omega_2)$$
$$= (3 - 2\cos\omega_1)(3 - 2\cos\omega_2)$$

Figure 1.21 (a) 2-D sequence $h(n_1, n_2)$; (b) possible choice of $h_1(n_1)$ and $h_2(n_2)$ where $h(n_1, n_2) = h_1(n_1)h_2(n_2)$; (c) Fourier transform magnitude $|H(\omega_1, \omega_2)|$ of $h(n_1, n_2)$ in (a).

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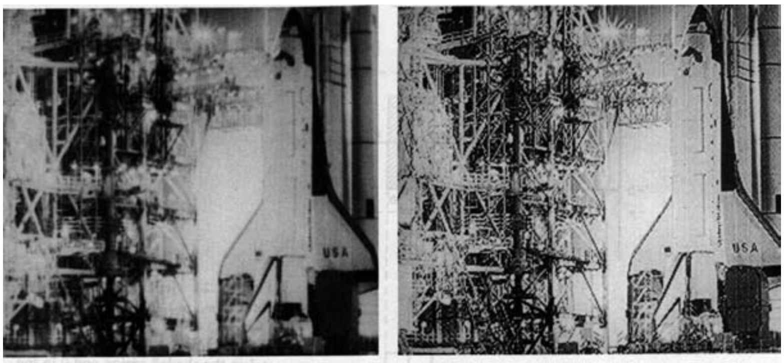
Lim, 1990

The Fourier Transform – Example 2



(a) (b)

Figure 1.22 (a) Image of 256 × 256 pixels; (b) image obtained from filtering the image in (a) with a highpass filter whose impulse response is given by $h(n_1, n_2)$ in Figure 1.21(a).

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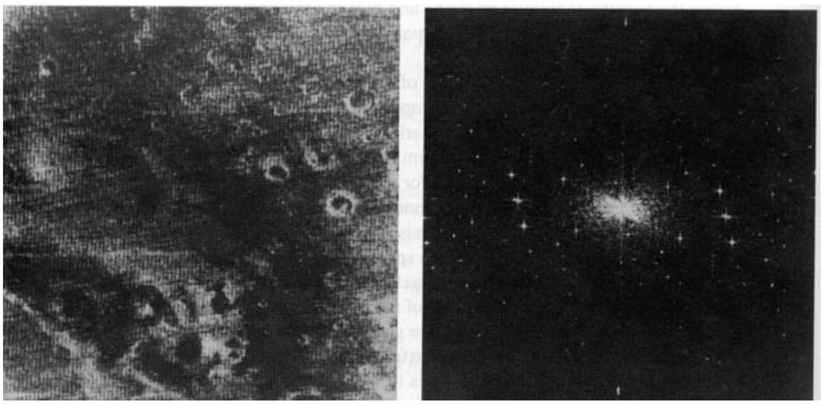
A 2D Fourier Transform



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Lim, 1990

Phase-Only & Magnitude-Only

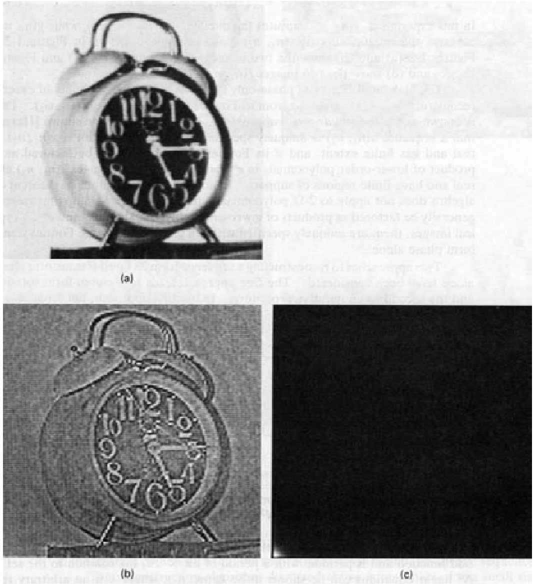

Figure 1.27 Example of phase-only and magnitude-only synthesis. (a) Original image of 128 × 128 pixels; (b) result of phase-only synthesis; (c) result of magnitude-only synthesis.

Lim, 1990

Image Synthesis

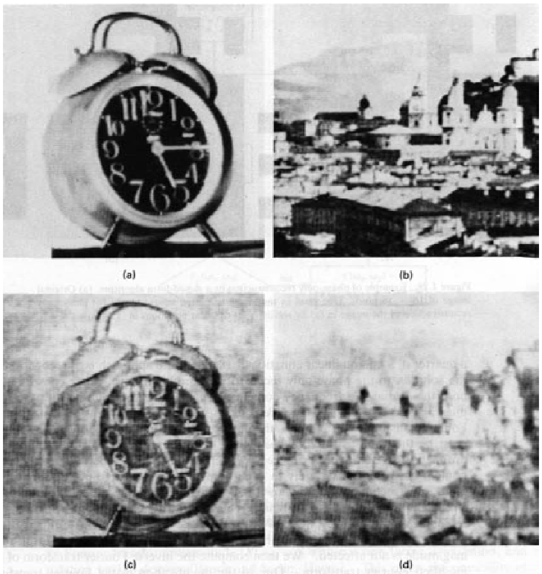

Figure 1.28 Example of image synthesis from the Fourier transform phase of one image and the Fourier transform magnitude of another image. (a) Original image $x(n_1, n_2)$ of 128 × 128 pixels; (b) original image $y(n_1, n_2)$ of 128 × 128 pixels; (c) result of synthesis from $\theta_x(\omega_1, \omega_2)$ and $|Y(\omega_1, \omega_2)|$; (d) result of synthesis from $\theta_y(\omega_1, \omega_2)$ and $|X(\omega_1, \omega_2)|$.

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Lim, 1990

Iterative Phase-Only

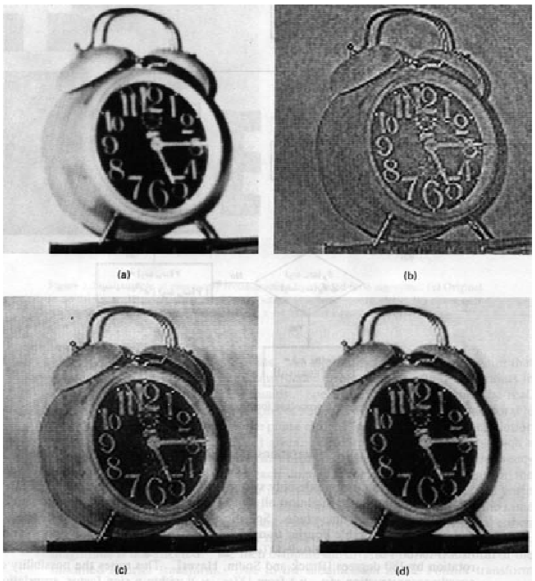

Figure 1.31 Example of phase-only reconstruction by an iterative algorithm. (a) Original image of 128 × 128 pixels; (b) result of phase-only reconstruction of the image in (a) after one iteration of the iterative procedure in Figure 1.30. Since the initial estimate used is $\delta(n_1, n_2)$, this is the same as the phase-only synthesis of (1.39); (c) result after 10 iterations; (d) result after 50 iterations.

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The 2D DFT

- If $g(i,k)$ is an NxN array, then the 2D discrete Fourier transform pair is given by

$$G(m,n) = \frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1} g(i,k)e^{-j2\pi(m\frac{i}{N}+n\frac{k}{N})}$$

$$g(i,k) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} G(m,n)e^{j2\pi(i\frac{m}{N}+k\frac{n}{N})}$$

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