PHYS 260 – Programming Assignment 5

1. Wien displacement constant

Planck's radiation law tells us that the intensity of radiation per unit area and per unit wavelength λ from a black body at temperature T is

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda k_B T} - 1},$$

where h is Planck's constant. c is the speed of light, and k_B is Boltzmann's constant.

(a) Show by differentiating that the wavelength λ at which the emitted radiation is strongest is the solution of the equation

$$5e^{-hc/\lambda k_B T} + \frac{hc}{\lambda k_B T} - 5 = 0$$

Make the substitution $x = hc/\lambda k_B T$ and hence show that the wavelength of maximum radiation obeys the *Wien displacement law:*

$$\lambda = \frac{b}{T}$$

where the *Wien displacement constant* is $b = hc/k_B x$, and x is the solution to the nonlinear equation

$$5e^{-x} + x - 5 = 0$$

(b) Write a program to solve this equation to an accuracy of $\epsilon = 10^{-6}$ using the relaxation method, and hence find a value for the displacement constant.

(c) The displacement law is the basis for the method of *optical pyrometry* a method for measuring the temperatures of objects by observing the color of the thermal radiation they emit. The method is commonly used to estimate the surface temperatures of astronomical bodies, such as the Sun. The wavelength peak in the Sun's emitted radiation falls at $\lambda = 502$ nm. From the equations above and your value of the displacement constant, estimate the surface temperature of the Sun.

2. Roots of a polynomial

Consider the sixth-order polynomial

$$P(x) = 924x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1$$

There is no general formula for the roots of a sixth-order polynomial, but one can find them easily enough using a computer.

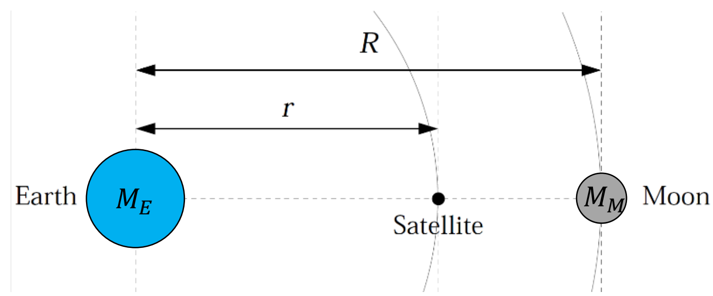
(a) Make a plot of $P(x)$ from $x = 0$ to $x = 1$ and by inspecting it find rough values for the six roots of the polynomial – the points at which the function is zero.

(b) Write a program to solve for the positions of all six roots to at least ten decimal places of accuracy, using Newton's method

The polynomial in this example is the sixth Legendre polynomial (mapped onto the interval from zero to one). Legendre polynomials frequently appear in physics. In particular, it shows up in electricity and magnetism when solving Laplace's equation in spherical coordinates.

3. Lagrange points L_1, L_2 and L_3

There is a special point between the Earth and the Moon, called the L_1 Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit.



(a) Assuming circular orbits, and assuming that the Earth is much more massive than either the Moon or the satellite, show that the distance r from the center of the Earth to the L_1 point satisfies

$$\frac{GM_E}{r^2} - \frac{GM_M}{(R-r)^2} = \omega^2 r,$$

where R is the distance from the Earth to the Moon, M_E and M_M are the Earth and Moon masses, G is Newton's gravitational constant, and ω is the angular velocity of both the Moon and the satellite.

(b) The equation above is a fifth-order polynomial equation in r (also called a quintic equation). Write a program that uses the secant method to solve for the distance r from the Earth to the L_1 point. Compute a solution accurate to at least four significant figures. The values of the various parameters are:

$$G = 6.674 \times 10^{-11}\ \mathrm{kg} \cdot \mathrm{m}^3/\mathrm{s}^2$$

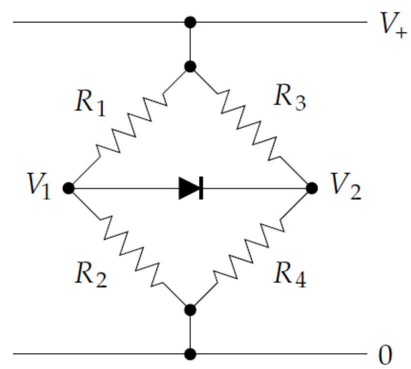
$$M_E = 5.974 \times 10^{24}\ \mathrm{kg} \quad M_M = 7.348 \times 10^{22}\ \mathrm{kg} \quad R = 3.844 \times 10^8\ \mathrm{m} \quad \omega = 2.662 \times 10^{-6}\ 1/\mathrm{s}$$

You will also need to choose two starting values for r. Note that the secant method has the advantage over Newton's method here because you do not need to compute the analytical derivative of a function, which would be a little messy in this case.

(c) There are two other Lagrange points on the Earth-Moon axis: L_2, which is to the right of the Moon in the figure, and L_3, which is to the left of Earth in the figure. Using similar methods as in (a) and (b), solve for the locations of L_2 and L_3.

4. Nonlinear circuits

Resistors are linear – current is proportional to voltage – and the resulting equations we need to solve are therefore also linear and can be solved by standard matrix methods. Real circuits, however, often include nonlinear components. To solve for the behavior of these circuits we need to solve nonlinear equations.



Consider the sample circuit on the right, a variation on the classic Wheatstone bridge. The resistors obey the normal Ohm law, but the diode obeys the diode equation:

$$I = I_0(e^{V/V_T} - 1),$$

where V is the voltage across the diode and I_0 and V_T are constants.

(a) The Kirchoff current law says that the total net current flowing into or out of every point in a circuit must be zero. Applying the law to voltage V_1 in the circuit above we get

$$\frac{V_1 - V_+}{R_1} + \frac{V_1}{R_2} + I_0[e^{(V_1-V_2)/V_T} - 1] = 0.$$

Derive the corresponding equations for voltage V_2.

(b) Solve the two nonlinear equations for the voltages V_1 and V_2 with the conditions

$V_+ = 5\text{ V}$

$R_1 = 1\text{ k}\Omega \quad R_2 = 4\text{ k}\Omega \quad R_3 = 3\text{ k}\Omega \quad R_4 = 2\text{ k}\Omega$

$I_0 = 3\text{ nA} \quad V_T = 0.05\text{ V}$

Use your `solvematrix` program and use Newton's method to solve the equations.

(c) The electronic engineer's rule of thumb for diodes is that the voltage across a (forward biased) diode is always about 0.6 V. Confirm that your results agree with this rule.

5. Rotating reference frames and Lagrange points L_4 and L_5 (optional)

It can be shown that L_1, L_2 and L_3 are all unstable equilibrium points. As such, a satellite placed at these locations will not reside there for long if perturbed by another force. However, there are two more Lagrange points that are found outside of the Earth-Moon axis called L_4 and L_5. It turns out that these are stable equilibrium points and there are often natural satellites, called trojan satellites, found at these locations. In 2010, Earth's first trojan was discovered at L_4.

The easiest way to determine the locations of L_4 and L_5 is to write out Newton's 2nd Law in a reference frame that rotates with angular velocity ω. Since this rotating reference frame is not an inertial reference frame, this leads to the introduction of fictitious-forces. It can be shown that Newton's 2nd Law in a reference frame that rotates with *constant* angular velocity $\vec{\omega}$ is given by

$$\vec{F}_{\mathrm{r}} = \vec{F}_{\mathrm{i}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_{\mathrm{r}}$$

Here, $\vec{F}_{\mathrm{i}}$ are physical forces (such as the gravitational attraction between the satellite and Earth), $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the centrifugal force, and $-2m\vec{\omega} \times \vec{v}_{\mathrm{r}}$ is the Coriolis force. In these equations, $\vec{r}$ and $\vec{v}_{\mathrm{r}}$ are the position and velocity of the object relative to the *rotating* reference frame. Since we are interested in positions where a satellite appears to be stationary relative to the Earth and moon, we investigate the case $\vec{v}_{\mathrm{r}} = \vec{0}$. Let us place the origin of the rotating reference frame at the center of mass of the Earth-Moon system. The Earth, Moon and satellite will orbit about this point with angular velocity ω. Writing Newton's 2nd Law for the satellite gives

$$\vec{F}_{\mathrm{r}} = \vec{F}_{\mathrm{Earth}} + \vec{F}_{\mathrm{Moon}} - m\omega^2 r\, \hat{r}$$

where $\vec{F}_{\mathrm{Earth}}$ and $\vec{F}_{\mathrm{Moon}}$ are the gravitational force between the satellite and the Earth and Moon, respectively and $\hat{r}$ is a unit vector that points from the origin to the location of the satellite. The locations of all Lagrange Points can be found by setting $\vec{F}_{\mathrm{r}} = \vec{0}$ and solving for r. In fact, this is exactly the equation that we solved in (b) and (c). However, this equation becomes quite messy when we investigate points that lie off the Earth-Moon axis. We will further simplify the picture by noticing that we can express the centrifugal force as the gradient of a potential. That is, we can write

$$-\omega^2 r\, \hat{r} = -\vec{\nabla}\left(\frac{1}{2}\omega^2 r^2\right)$$

Here, $V_{\mathrm{centrifugal}} = \frac{1}{2}\omega^2 r^2$ is the centrifugal potential. Now we can write the total potential at location (x, y) in the rotating reference frame as

$$V = -\frac{GM_E}{\sqrt{(x - x_E)^2 + (y - y_E)^2}} - \frac{GM_M}{\sqrt{(x - x_M)^2 + (y - y_M)^2}} - \frac{1}{2}\omega^2(x^2 + y^2)$$

where (x_E, y_E) and (x_M, y_M) are the positions of the Earth and Moon, respectively. We can now identify the locations of Lagrange points by making a contour plot of V and looking for saddle points as well as local extrema. These points are characterized by $\partial V/\partial x = \partial V/\partial y = 0$.

(a) Place the origin at the location of the Earth-Moon center of mass. Calculate the total potential $V(x, y)$ on a $3R \times 3R$ square grid and store the values in a matrix `V`. Visualize the potential with a density plot. Since the potential diverges at positions close to the Earth and Moon, it is more useful to plot the density plot of `log(V)` instead. Remember that you can adjust the coloring by adjusting values of `vmin` and `vmax` inside of the function call to `imshow`.

(b) Now make a two contour plots with

```
contour(-log(abs(V)), levels, vmin=logmin, vmax=logmax,linewidths=0.7,
colors="k", linestyles="solid")
```

```
contourf(-log(abs(V)), levels, vmin=logmin, vmax=logmax,cmap="plasma")
```

Here, `levels` is an array of "heights" (`log(V)` values in this case). This tells the plotting program to draw a contour line at each of the heights listed in `levels`. It seems that `logmin = -14.35` and `logmax = -14.25` along with `levels = linspace(logmin,logmax,50)` produces a plot that enables identification of all the Lagrange points. Experiment with other values so you can learn what these commands are doing.

(c) Mark the positions of L_1, L_2 and L_3 on your contour plot (you can do this in Word or Photoshop). Where are the approximate locations of L_4 and L_5, the off-axis Lagrange Points? It can be shown analytically that L_4 (and similarly L_5) resides on a vertex of an equilateral triangle of side length R with the Earth and Moon being the other vertices. Confirm that this is the case using your contour plot.