PHYS 260 – Programming Assignment 6

1. Maximum efficiency of a light bulb

An incandescent light bulb is a simple device – it contains a filament usually made of tungsten, heated by the flow of electricity until it becomes hot enough to radiate thermally. Essentially all of the power consumed by such a bulb is radiated as electromagnetic energy, but some of the radiation is not in the visible wavelengths, which means it is useless for lighting purposes.

Let us define the efficiency of a light bulb to be the fraction of the radiated energy that falls in the visible band. It's a good approximation to assume that the radiation from a filament at temperature T obeys the Planck radiation law, meaning that the power radiated per unit wavelength λ obeys

$$I(\lambda) = 2\pi A h c^2 \frac{\lambda^{-5}}{e^{hc/\lambda k_B T} - 1}$$

where A is the surface area of the filament, h is Planck's constant, c is the speed of light and k_B is Boltzmann's constant. The visible wavelengths run from $\lambda_1 = 390$ nm to $\lambda_2 = 750$ nm, so the total energy radiated in the visible window is $\int_{\lambda_1}^{\lambda_2} I(\lambda) d\lambda$ and the total energy at all wavelengths is $\int_0^\infty I(\lambda) d\lambda$. Dividing one expression by the other and substituting for $I(\lambda)$ from above, we get an expression for the efficiency η of the light bulb thus:

$$\eta = \frac{\int_{\lambda_1}^{\lambda_2} \frac{\lambda^{-5}}{e^{hc/\lambda k_B T} - 1} d\lambda}{\int_0^\infty \frac{\lambda^{-5}}{e^{hc/\lambda k_B T} - 1} d\lambda}$$

where the leading constants and the area A have canceled out. Making the substitution $x = hc/\lambda k_B T$, this can also be written as

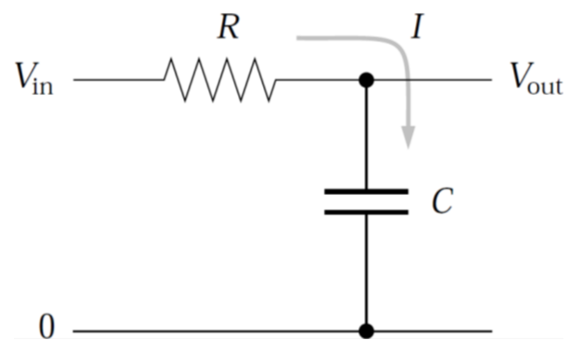
$$\eta = \frac{\int_{hc/\lambda_2 k_B T}^{hc/\lambda_1 k_B T} \frac{x^3}{e^x - 1} d\lambda}{\int_0^\infty \frac{x^3}{e^x - 1} d\lambda} = \frac{15}{\pi^4} \int_{hc/\lambda_2 k_B T}^{hc/\lambda_1 k_B T} \frac{x^3}{e^x - 1} dx$$

where we have made use of the known exact value of the integral in the denominator.

(a) Write a Python function that takes a temperature T as its argument and calculates the value of η for that temperature from the formula above. The integral cannot be done analytically, so compute it numerically with the Trapezoid or Simpson's Rule. Use your function to make a graph of η as a function of temperature between 300 K and 10 000 K. You should see that there is an intermediate temperature where the efficiency is a maximum.

(b) Calculate the temperature of maximum efficiency of the light bulb to within 1 K using golden ratio search. An accuracy of 1 K is the equivalent to a few parts in ten thousand in this case. To achieve this accuracy you'll need values for fundamental constants accurate to several significant figures. Is it practical to run a tungsten-filament light bulb at the temperature you found? If not, why not?

2. Low-pass filter



To the right is a simple electronic circuit with one resistor and one capacitor. This circuit acts as a low-pass filter: you send a signal in on the left and it comes out filtered on the right.

Using Ohm's law and the capacitor law and assuming that the output load has very high impedance, so that a negligible amount of current flows through it, we can write down the equations governing this circuit as follows. Let I be the current that flows through R and into the capacitor, and let Q be the charge on the capacitor. Then:

$$IR = V_{\text{in}} - V_{\text{out}}, \qquad Q = CV_{\text{out}}, \qquad I = \frac{dQ}{dt}.$$

Substituting the second equation into the third, then substituting the result into the final equation, we find that $V_{\text{in}} - V_{\text{out}} = RC(dV_{\text{out}}/dt)$, or equivalently

$$\frac{dV_{\text{out}}}{dt} = \frac{1}{RC}(V_{\text{in}} - V_{\text{out}})$$

(a) Write a program to solve this equation for $V_{\text{out}}(t)$ using Euler's method when the input signal is a square-wave with frequency 1 and amplitude 1:

$$V_{\text{in}}(t) = \begin{cases} 1 & \text{if } \lfloor 2t \rfloor \text{ is even} \\ -1 & \text{if } \lfloor 2t \rfloor \text{ is odd} \end{cases}$$

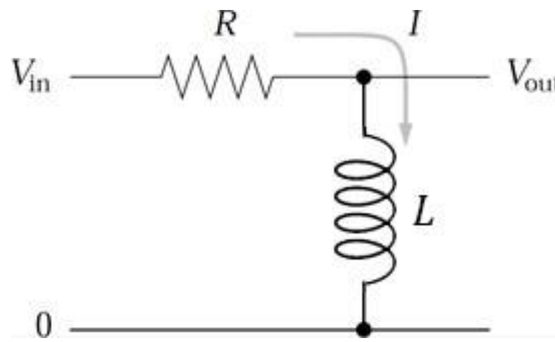
Where $\lfloor x \rfloor$ is the floor function, which returns the value of x rounded down to the next lowest integer. Use the program to make plots of both the input V_{in} and output V_{out} of the filter circuit from $t = 0$ to $t = 5$ when $RC = 0.01, 0.1$, and 1, with initial condition $V_{\text{out}}(0) = 0$. Plot all values on the same graph. You will have to make a decision about what value of h to use in your calculation. Small values give more accurate results, but the program will take longer to run. Try a variety of different values and choose one for your final calculations that seems sensible to you.

(b) Repeat (a) with $V_{\text{in}}(t) = \sin(t) + \sin(10t) + \sin(100t)$.

(c) Based on the graphs produced by your program, describe what you see and explain what the circuit is doing.

A program similar to the one you wrote is running inside most stereos and music players, to create the effect of the "bass" control. In the old days, the bass control on a stereo would have been connected to a real electronic low-pass filter in the amplifier circuitry, but these days there is just a computer processor that simulates the behavior of the filter in a manner similar to your program.

3. High-pass Filter



To the right is a circuit similar to the low-pass filter, but the capacitor has been replaced with an inductor L. In this case, the circuit acts as a high-pass filter, filtering out low frequencies of the input signal.

Using Ohm's law and Faraday's law and assuming that the output load has very high impedance, so that a negligible amount of current flows through it, we can write down the equations governing this circuit as follows. Let I be the current that flows through R and into the inductor, and let dI/dt be the rate at which the current is changing. Applying Faraday's Law to a closed loop gives:

$$\oint_{\mathcal{P}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\Downarrow$$

$$-V_{\text{in}} + IR = -L\frac{dI}{dt}$$

Substituting $I = (V_{\text{in}} - V_{\text{out}})/R$ and rearranging gives the differential equation for V_{out}:

$$\frac{dV_{\text{out}}}{dt} = \frac{dV_{\text{in}}}{dt} - \frac{R}{L}V_{\text{out}}$$

(a) Make a copy of your low-pass filter program and modify it so that it solves the differential equation above for the high-pass filter. Create an additional function `Vinprime(t)` that computes dV_{in}/dt numerically using a forward difference. Starting with the square-wave input described in the previous problem, use the program to make plots of both the input V_{in} and output V_{out} of the filter circuit from $t = 0$ to $t = 5$ when $R/L = 0.1, 10,$ and 100, with initial condition $V_{\text{out}}(0) = 0$. Plot all values on the same graph.

(b) Repeat (a) with $V_{\text{in}}(t) = \sin(t) + \sin(10t) + \sin(100t)$.

(c) Based on the graphs produced by your program, describe what you see and explain what the circuit is doing.