PHYS 260 – Programming Assignment 3

1. **Integrating data**

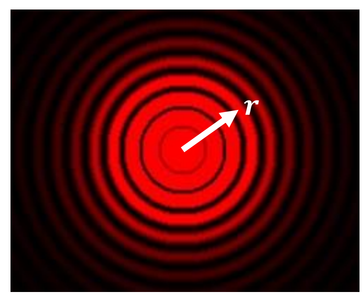
There is a file on Canvas called `forces.txt`, which contains two columns of numbers, the first representing time t in seconds and the second the net force in the x-direction in Newtons of a particle, measured once every 0.1 seconds from time $t = 0$ to $t = 10$. At $t = 0$, the position and velocity of the particle are both zero. The mass of the particle is 1.30 kg. Write a program to do the following:

(a) Read in the data and, using the Trapezoidal Rule, calculate the velocity and position of the particle in the x direction as functions of time.

(b) Make a graph that displays, on the same plot, the acceleration, velocity and position as functions of time.

2. **Diffraction**

Our ability to resolve detail in astronomical observations is limited by the diffraction of light in our telescopes. Light from stars can be treated effectively as coming from a point source at infinity. When such light, with wavelength λ, passes through the circular aperture of a telescope (which we'll assume to have unit radius) and is focused by the telescope in the focal plane, it produces not a single dot, but a circular diffraction pattern consisting of a central spot surrounded by a series of concentric rings. This is shown in the figure to the right. The intensity of the light in this diffraction pattern is given by



$$I(r) = \left(\frac{J_1(kr)}{kr}\right)^2$$

where r is the distance in the focal plane from the center of the diffraction pattern, $k = 2\pi/\lambda$, and $J_1(x)$ is a Bessel function. The Bessel functions $J_m(x)$ are given by

$$J_m(x) = \frac{1}{\pi}\int_0^\pi \cos(m\theta - x\sin\theta)\,d\theta,$$

where m is a nonnegative integer and $x \geq 0$.

(a) Write a Python function `J(m,x)` that calculates the value of $J_m(x)$ using the Trapezoidal Rule or Simpson's Rule with $N = 1000$ points. Use your function in a program to make a plot, on a single graph, of the Bessel functions J_0, J_1, and J_2 as a function of x from $x = 0$ to $x = 20$. When you are confident that your program works, proceed to part (b).

(b) Make a second program that makes a density plot of the intensity of the circular diffraction pattern of a point light source with $\lambda = 500$ nm, in a square region of the focal plane, using the formula given above. Your picture should cover values of r from zero up to about 1 μm. Add axes labels and a scale bar using `xlabel`, `ylabel`, `colorbar` functions from `pylab`. Note that $\lim_{x\to 0} J_1(x)/x = 1/2$.

The central spot in the diffraction pattern is so bright that it may be difficult to see the rings around it on the computer screen. The `imshow` function has an additional argument `vmax` that allows you to set the value that corresponds to the brightest point in the plot. For instance, if you say "`imshow(A, vmax=0.1)`", then elements in `A` with value `0.1`, or any greater value, will produce the brightest (most positive) color on the screen. By lowering the `vmax` value, you can reduce the total range of values between the minimum and maximum brightness, and hence increase the sensitivity of the plot, making subtle details visible. (There is also a `vmin` argument that can be used to set the value that corresponds to the dimmest (most negative) color.) For this exercise a value of `vmax=0.01` with the `hot` color scheme appears to work well.

(c) (optional): Remove the loops that create your density plot and, instead, use the `numpy` function `meshgrid` to produce a matrix of x and y values and then calculate the intensity $I(r)$ by operating on this matrix. This method is significantly faster than using loops and will facilitate production of high resolution images.

3. **Black body radiation**

The Planck theory of thermal radiation tells us that in the angular frequency interval ω to $\omega + d\omega$, a black body of unit area radiates electromagnetically an amount of thermal energy per second equal to $I(\omega)d\omega$, where

$$I(\omega) = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}$$

Here $\hbar$ is Planck's constant over 2π, c is the speed of light, and k_B is Boltzmann's constant.

(a) Integrate the expression over all possible values of ω to show that the total energy per unit area radiated by the black body is

$$W = \frac{k_B^4 T^4}{4\pi^2 c^2 \hbar^3} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

(b) Write a program to evaluate the integral in the expression using either the Trapezoidal Rule or Simpson's Rule. Include an error estimate. Equations 5.28-5.29 in the book will be useful.

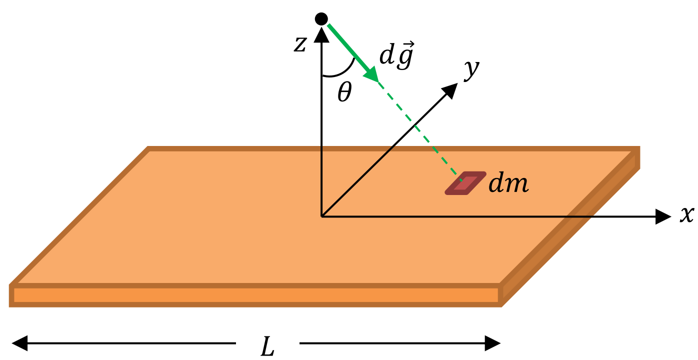
(c) Even before Planck gave his theory of thermal radiation around the turn of the 20th century, it was known that the total energy W given off by a black body per unit area per second followed Stefan's law: $W = \sigma T^4$, where σ is the Stefan-Boltzmann constant. Use your value for the integral above to compute a value for the Stefan-Boltzmann constant (in SI units) to three significant figures. Check your result against the known value. You should get good agreement.

(d) The visible spectrum extends over a frequency range from about $f_1 = 400$ THz to $f_2 = 790$ THz where 1 THz $= 10^{12}$ Hz. Recall that the relationship between angular frequency and frequency is $\omega = 2\pi f$. Under normal operation, an incandescent light bulb filament is at a temperature of about 2800 K. Assuming that the filament behaves as a black body, calculate the total power emitted by a black body in the visible spectrum. Compare this value to the total power output, thereby determining the efficiency η of the light bulb:

$$\eta = \frac{W_{\text{visible}}}{W_{\text{total}}} = \frac{\frac{\hbar}{4\pi^2 c^2}\int_{\omega_1}^{\omega_2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} d\omega}{\sigma T^4}$$

4. **Gravitational field of a spiral galaxy**

Most of the mass in a spiral galaxy (such as our own Milky Way) lies in a thin plane or disk passing through the galactic center. The gravitational pull exerted by that plane on bodies outside the galaxy can be determined using a simplified model that we will employ here. Recall that the gravitational field $\vec{g}$ of a particle m is given by



$$\vec{g} = \frac{Gm}{r^2}\hat{r}$$

where G is Newton's gravitational constant, r is the distance from the particle and $\hat{r}$ is a unit vector that points toward the particle. To derive the gravitational field of a galaxy, we model the galaxy as a thin square with side length L with mass uniformly distributed over the area of the square. The thickness of the square is negligible. We can define a mass per unit area as $\sigma = M/L^2$ where M is the total mass of the galaxy. We can derive an equation for the gravitational field by applying the above equation for a point mass and then integrating over the whole square. For simplicity, let's calculate the gravitational field at a point a distance z above the center of the square. By symmetry, the x and y components of the gravitational field will be zero at this location. Imagine that the square is composed of infinitesimal point particles of mass dm. We can write dm in terms of the mass density as $dm = \sigma\, dx\, dy$. The magnitude of the z-component of the gravitational field of dm is given by

$$dg_z = \frac{Gdm}{r^2}\cos\theta = \frac{G\sigma dxdy}{x^2 + y^2 + z^2}\frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{G\sigma z}{(x^2 + y^2 + z^2)^{3/2}}dxdy$$

The gravitational field of the entire galaxy is then

$$g_z = G\sigma z \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \frac{dx\, dy}{(x^2 + y^2 + z^2)^{3/2}}$$

Working in units where $G = \sigma = L = 1$, write a program to calculate and plot g_z as a function of z from $z = 0$ to $z = 1$. For the double integral use the Trapezoid Rule with 100 sample points along each axis. You should see a smooth curve, except at very small values of z, where the force increases suddenly. This is not a real effect, but an artifact of our calculation. Explain where this artifact comes from and suggest a strategy to remove it, or at least to decrease its size. Hint: Try increasing the sample points for small z.

5. **Integrating a singular integral (optional)**

A singular integral is an integral that contains an integrand that diverges (reaches infinity) at one or more values over the domain of integration. Even though the integrand reaches infinity, the integral may still have a finite value. Integrals of this type are often encountered when calculating the period of oscillation of an anharmonic oscillator (see the next problem). Consider the integral

$$I = \int_0^1 \frac{dx}{\sqrt{1 - x^4}} \approx 1.31103$$

Even though the integrand contains a singularity at $x = 1$, it turns out that the integral is finite. However, this integral is still difficult to solve analytically and cannot be expressed in terms of elementary functions. We can turn to numerical methods to solve this integral. However, they must be applied with great care. This problem will guide you through a method for solving this integral.

(a) Let's try to solve this using the Trapezoidal Rule. The computer will complain when $x = 1$, so instead, let's calculate the integral

$$\int_0^{1-\epsilon} \frac{dx}{\sqrt{1 - x^4}}$$

where ϵ is a small number. This avoids the singularity at $x = 1$ and it might be possible to get a good approximate solution if ϵ is small enough. Use the Traepzoidal Rule with $N = 100$ slices and $\epsilon = 10^{-6}$ to calculate the above integral. You should obtain a value of 3.7379, which is not very close to the correct value. You can experiment with other values of ϵ and N and you can get better results by increasing N. However, more elegant methods exist for dealing with singularities.

(b) In order to better understand the behavior of the integrand close to the singularity, let's factor the denominator so we can rewrite the integral as

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \int_0^1 \frac{dx}{\sqrt{(1 - x)(1 + x)(1 + x^2)}}$$

When $x \to 1$, the integrand $f(x)$ will approximately be equal to

$$f(x) \approx \frac{1}{2\sqrt{1 - x}}$$

Make a plot of $1/\sqrt{1-x^4}$ and $1/(2\sqrt{1-x})$ on the same graph from $x = 0$ to $x = 0.999$ (thus avoiding the singularity). You should see that the two functions become identical when x approaches 1.

(c) We can now deal with the singularity by "subtracting" it off. We can rewrite the integral as

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \left(\frac{1}{\sqrt{1-x^4}} - \frac{1}{2\sqrt{1-x}} \right) dx + \int_0^1 \frac{dx}{2\sqrt{1-x}}$$

Notice that all we have done was add zero. However, doing this allows us to make headway. Notice that the last integral can be solved analytically and is equal to

$$\int_0^1 \frac{dx}{2\sqrt{1-x}} = -\sqrt{1-x}\Big|_0^1 = 1$$

The first integral on the right side of the equation can be solved numerically.

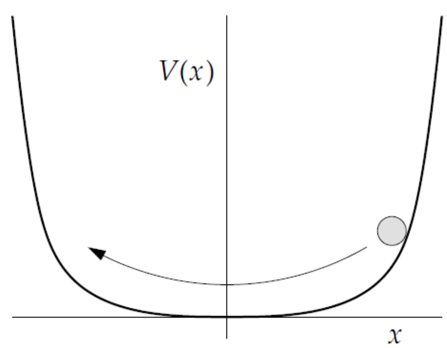
$$I' = \int_0^1 \left(\frac{1}{\sqrt{1-x^4}} - \frac{1}{2\sqrt{1-x}} \right)$$

Since the two terms of the integrand have the same behavior at the singularity, this ensures that the value of the integrand is exactly zero when $x = 1$.

Solve the integral I' numerically using the Trapezoid Rule with $N = 1000$ slices. The computer will still complain about division by zero when $x = 1$. However, this is not so concerning because we know from the graphs in part (b) that the integrand should be exactly equal to zero when $x = 1$. Use an `if` statement to return a value of zero for the integrand when $x = 1$.

6. **Period of anharmonic oscillator (optional)**

The simple harmonic oscillator crops up in many places. Its behavior can be studied readily using analytic methods and it has the important property that its period of oscillation is a constant, independent of its amplitude, making it useful, for instance, for keeping time in watches and clocks.



Frequently in physics, however, we also come across anharmonic oscillators, whose period varies with amplitude and whose behavior cannot usually be calculated analytically. A general classical oscillator can be thought of as a particle in a concave potential well. When disturbed, the particle will rock back and forth in the well as shown in the figure.

The harmonic oscillator corresponds to a quadratic potential $V(x) \propto x^2$. Any other form gives an anharmonic oscillator. (Thus there are many different kinds of anharmonic oscillator, depending on the exact form of the potential.)

One way to calculate the motion of an oscillator is to write down the equation for the conservation of energy in the system. If the particle has mass m and position x, then the total energy is equal to the sum of the kinetic and potential energies thus:

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x)$$

Since the energy must be constant over time, this equation is effectively a (nonlinear) differential equation linking x and t.

Let us assume that the potential $V(x)$ is symmetric about $x = 0$ and let us set our anharmonic oscillator going with amplitude A. That is, at $t = 0$ we release it from rest at position $x = A$ and it swings back towards the origin. Then at $t = 0$ we have $dx/dt = 0$ and the equation above reads $E = V(A)$, which gives us the total energy of the particle in terms of the amplitude. When the particle reaches the origin for the first time, it has gone through one quarter of a period of the oscillator. We can rearrange the above equation as follows. Solving for dx/dt gives

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\big(V(A) - V(x)\big)}$$

Taking the reciprocal of each side gives

$$\frac{dt}{dx} = \frac{1}{\sqrt{\frac{2}{m}\big(V(A) - V(x)\big)}}$$

Rearranging and integrating gives

$$\int_0^{T/4} dt = \int_0^A \frac{dx}{\sqrt{\frac{2}{m}\big(V(A) - V(x)\big)}}$$

where T is the period of oscillation. Finally we can express the period of oscillation as

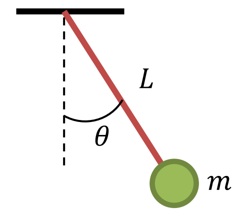
$$T = \sqrt{8m} \int_0^A \frac{dx}{\sqrt{V(A) - V(x)}}$$

(a) Now suppose that $V(x) = x^4$, show that you can write the period as

$$T = \frac{\sqrt{8m}}{A} \int_0^1 \frac{dz}{\sqrt{1 - z^4}}$$

(b) Make a graph of the period as a function of A. You can solve the integral using the method from the previous problem.

7. **Period of a nonlinear (real) pendulum (optional)**



A simple pendulum consists of a point mass m suspended from a pivot point by a massless rod of length L. The position of the pendulum at any point in time is given by the angle θ. Summing torques about the pivot point yields the nonlinear differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This differential equation cannot be solved analytically, but later this semester we will solve it numerically. In previous classes, you solved it analytically by using the small angle approximation $\sin\theta \approx \theta$. Under this approximation, the period of oscillation is $T = 2\pi\sqrt{L/g}$ and is independent of amplitude. For larger amplitudes of motion, where the small angle approximation is not valid, this is not the case. From the previous problem, the period of the pendulum can be written as

$$T = \sqrt{8m}\int_0^A \frac{dx}{\sqrt{V(A) - V(x)}}$$

Here, $V(x) = mg(L - L\cos\theta)$, where $L - L\cos\theta$ is the height of the mass above its equilibrium position.

$$T = \sqrt{\frac{8L}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

We will use methods similar to problem 4 to solve this integral. However, before we do, it is helpful to make some substitutions.

(a) Use the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$ to rewrite the integral as

$$T = \left(\frac{2\sqrt{L/g}}{\sin\frac{\theta_0}{2}}\right)\int_0^{\theta_0} \frac{d\theta}{\sqrt{1 - \left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right)^2}}$$

(b) Now let $z = \frac{\sin\theta/2}{\sin\theta_0/2}$ and show that the period can be written

$$T = 4\sqrt{\frac{L}{g}}\int_0^1 \frac{dz}{\sqrt{(1 - z^2)\left(1 - \sin^2\left(\frac{\theta_0}{2}\right)z\right)}}$$

(c) The integral above can be solved numerically by subtracting off the singularity at $z = 1$. Use this numerical solution and plot the period of oscillation when $L = 1.0$ m and for θ_0 ranging from 0 to 0.99π. You should find the period of oscillation when $\theta_0 = \pi/4$ is about 2.09 s. Explain your graph and why the small angle approximation is not valid for larger values of θ_0.