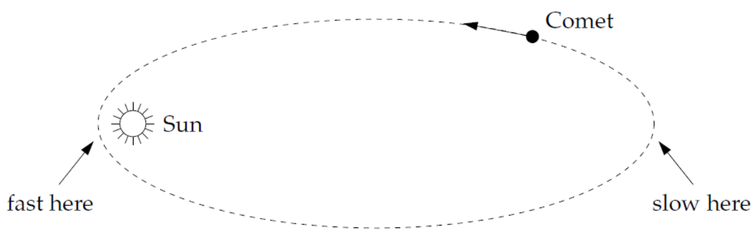
PHYS 260 – Programming Assignment 7

1. Cometary Orbits

Many comets travel in highly elongated orbits around the Sun. For much of their lives they are far out in the solar system, moving very slowly, but on rare occasions their orbit brings them close to the Sun for a fly-by and for a brief period of time they move very fast indeed:



This is a classic case where an adaptive step size method is useful, because for the large periods of time when the comet is moving slowly we can use long time-steps, so that the program runs quickly, but short time-steps are crucial in the brief but fast-moving period close to the Sun.

The differential equation obeyed by a comet is straightforward to derive. The force between the Sun, with mass M at the origin, and a comet of mass m with position vector $\vec{r}$ is GMm/r^2 in direction of $-\vec{r}\,/r$ (i.e. the direction towards the Sun), and hence Newton's second law tells us that

$$m\frac{d^2\vec{r}}{dt^2} = -\left(\frac{GMm}{r^2}\right)\frac{\vec{r}}{r}$$

Canceling the m and taking the x component we have

$$\frac{d^2x}{dt^2} = -GM\frac{x}{r^3}$$

and similarly for the other two coordinates. We can, however, throw out one of the coordinates because the comet stays in a single plane as it orbits. If we orient our axes so that this plane is perpendicular to the z-axis, we can forget about the z coordinate and we are left with just two second-order equations to solve:

$$\frac{d^2x}{dt^2} = -GM\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -GM\frac{y}{r^3}$$

where $r = \sqrt{x^2 + y^2}$.

(a) Turn these two second-order equations into four first-order equations, using the methods you have learned. Write a program to solve your equations using the fourth-order Runge-Kutta method with a *fixed* step size. Use $M_{\text{sun}} = 1.989 \times 10^{30}$ kg and $G = 6.674 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$. As an initial condition, take a comet at coordinates $x = 4$ billion kilometers and $y = 0$ (which is somewhere out around the orbit of Neptune) with initial velocity $v_x = 0$ and $v_y = 500$ m/s. Make a graph showing the trajectory of the comet (i.e. a plot of y against x).

Choose a fixed step size h that allows you to accurately calculate at least two full orbits of the comet. Since orbits are periodic, a good indicator of an accurate calculation is that successive orbits of the comet lie on top of one another on your plot. If they do not then you need a smaller value of h. Give a short description of your findings. What value of h did you use? What did you observe in your simulation? How long did the calculation take? Recall that you can time your program by calling the function `time.time()` via `import time`.

(b) Make a copy of your program and modify the copy to do the calculation using an adaptive step size. Set a target accuracy of $\delta = 1$ kilometer per year in the position of the comet and again plot the trajectory. What do you see? How do the speed, accuracy, and step size of the calculation compare with those in part (b)? Modify your program to place dots on your graph showing the position of the comet at each Runge-Kutta step around a single orbit. You should see the steps getting closer together when the comet is close to the Sun and further apart when it is far out in the solar system.

Calculations like this can be extended to cases where we have more than one orbiting body. We can include planets, moons, asteroids and others. Analytic calculations are impossible for such complex systems, but with careful numerical solution of differential equations we can calculate the motions of objects throughout the entire solar system.

2. Orbit of the Moon

Use the Verlet method to calculate the orbit of the Moon around the Earth. The equations of motion for the position $\vec{r} = (x, y)$ of the planet in its orbital plane are given by

$$\frac{d^2\vec{r}}{dt^2} = -GM\frac{\vec{r}}{r^3}$$

where $G = 6.67408 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is Newton's gravitational constant and $M = 5.972 \times 10^{24}$ kg is the mass of Earth.

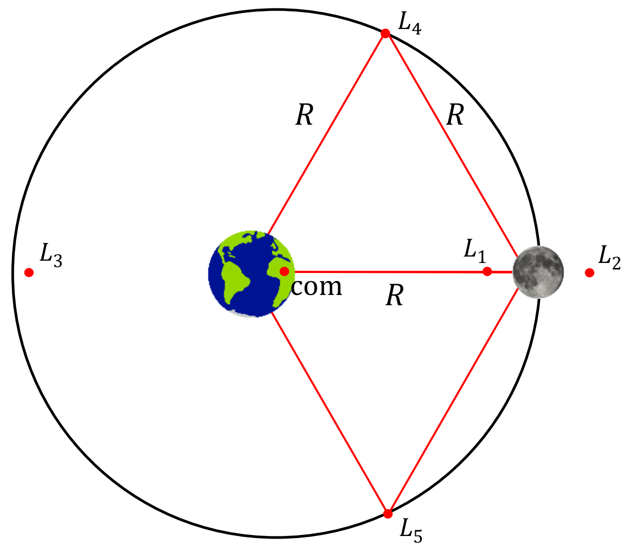
The orbit of the Moon is not perfectly circular, the planet being sometimes closer to and sometimes further from the Earth. When it is at its closest point, or *perigee*, it is moving precisely tangentially (i.e. perpendicular to the line between itself and the Earth) and it has distance 3.626×10^8 m from the Earth and linear velocity 1.077×10^3 m/s.

(a) Write a program to calculate the orbit of the Moon using the Verlet method, with a time-step of $h = 1$ hour. Make a plot of the orbit, showing several complete revolutions about the Earth. The orbit should be very slightly, but visibly, non-circular.

(b) The gravitational potential energy of the Moon is $-GMm/r$, where $m = 7.348 \times 10^{22}$ kg is the mass of the Moon, and its kinetic energy is $\frac{1}{2}mv^2$ as usual. Modify your program to calculate both of these quantities at each step, along with their sum (which is the total energy), and make a plot showing all three as a function of time on the same axes. You should find that the potential and kinetic energies vary visibly during the course of an orbit, but the total energy remains constant.

(c) Now plot the total energy alone without the others and you should be able to see a slight variation over the course of an orbit. Because you're using the Verlet method, however, which conserves energy in the long term, the energy should always return to its starting value at the end of each complete orbit.

3. Three-body problem and stability of orbits



The three-body problem involves calculating the trajectories of three objects that all interact with each other via gravity. Unlike the two-body problem, there is no closed form solution and numerical methods are required to calculate the trajectories. As a simplified case, we will choose the mass of the third body to be small compared to the others. Thus, the third body will have little effect on the motion of the two massive bodies. In Assignment 5, you calculated Lagrange points: points near two massive bodies where a third body can undergo an orbit such that its position relative to the massive bodies does not change with time. The points L_1, L_2, and L_3 all lie on the axis of the two massive bodies. If you continued with the optional part of the assignment, you determined the positions of L_4 and L_5, which turned out to lie at the top vertex of an equilateral triangle with the two massive bodies at the other vertices. The positions of the Lagrange points for the Earth and Moon are shown to the right.

(a) Modify your Moon orbit code from the previous problem so that the Earth can move as well. Place the origin at the center of mass of the Earth-Moon system, which is given by

$$\vec{r}_{\text{com}} = \frac{M_E\vec{r}_E + M_M\vec{r}_M}{M_E + M_M}$$

where $M_E = 5.972 \times 10^{24}$ kg, $M_M = 7.348 \times 10^{22}$ kg are the masses of the Earth and Moon, respectively. For simplicity, we will assume that both Earth and Moon perform circular orbits about the center of mass. Initialize both bodies on the x-axis and, using $R = 3.844 \times 10^8$ m for the distance between the Earth and Moon and the above equation for $\vec{r}_{\text{com}}$, solve for the initial positions of the Earth and Moon in terms of M_E, M_M and R. Using these values, sum forces in the radial direction for the moon and show that the angular velocity ω of the Earth-Moon system can be written

$$\omega = \sqrt{\frac{G(M_E + M_M)}{R^3}}$$

Using this value of ω, give appropriate initial velocities and calculate their trajectories using the Verlet algorithm. Make an animation of the orbit in VPython.

(b) Now you will add a third object, a satellite of mass $m = 1.00 \times 10^{10}$ kg, located at L_1. When you calculated the location of L_1 in Assignment 5, you assumed that Earth was stationary. For this problem, everything rotates about the center of mass of the system and the position of L_1 will be slightly different. For this problem, the initial location of L_1 ends up being $(x, y) = (3.2170270438360560 \times 10^8\ \text{m}, 0)$. Place the satellite at L_1 and give it an initial velocity of $\vec{v}_0 = r\omega\,\hat{y}$ where r is the distance between L_1 and the origin (i.e. the Earth-Moon center of mass) and

ω is the angular velocity of the system's orbit. Compute the forces between all objects and allow all three objects to move. What happens to the satellite? Does it undergo a stable circular orbit? What do you conclude about the stability of satellites at L_1?

(c) Now perform the same calculation with the satellite at L_4. You will have to figure out the appropriate values for the components of the satellite's initial velocity vector $\vec{v}_0 = v_{0x}\hat{x} + v_{0y}\hat{y}$. Be sure to remember that all three bodies rotate about the system's center of mass. You should observe the satellite moving in a stable circular orbit this time.

(d) Add code that calculates the distance between the satellite and the Earth as well as the distance between the satellite and the Moon at each time step. Add a graph that displays these two distances as a function of time. Now reinitialize the satellite as in (c), but with a velocity vector $\vec{v}_0 = v_{0x}\hat{x} + 0.99v_{0y}\hat{y}$ where v_{0x} and v_{0y} are the values used in (c). You should see the distances to the Earth and Moon vary with time, but oscillate about a constant value. Try other initial conditions. What do you conclude about the stability of a satellite at L_4?

4. Vibration in one-dimensional system (optional)

Suppose we have a system of N identical masses (in zero gravity) joined by identical linear springs like this:

If we jostle the system the masses will vibrate relative to one another under the action of the springs. The motions of the system could be used as a model of the vibration of atoms in a solid, which can be represented with reasonable accuracy in exactly this way. The horizontal displacements ξ_i of masses $i = 1 \ldots N$ satisfy equations of motion

$$m\frac{d^2\xi_1}{dt^2} = k(\xi_2 - \xi_1) + F_1$$

$$m\frac{d^2\xi_i}{dt^2} = k(\xi_{i+1} - \xi_i) + k(\xi_{i-1} - \xi_i) + F_i$$

$$m\frac{d^2\xi_N}{dt^2} = k(\xi_{N-1} - \xi_N) + F_N$$

where m is the mass, k is the spring constant, and F_i is the external force on mass i.

(a) Write a program to solve for the motion of the masses using the fourth-order Runge-Kutta method for the case we studied previously where $m = 1$ and $k = 6$, and the driving forces are all zero except for $F_1 = \cos\omega t$ with $\omega = 2$. Plot your solutions for the displacements ξ_i of all the masses as a function of time from $t = 0$ to $t = 20$ on the same plot. Write your program to work with general N, but test it out for small values - $N = 5$ is a reasonable choice.

You will need first of all to convert the N second-order equations of motion into $2N$ first-order equations. Then combine all of the dependent variables in those equations into a single large vector $\vec{r}$ to which you can apply the Runge-Kutta method in the standard fashion.

(b) Create an animation of the masses using VPython.