PHYS 260 – Programming Assignment 7

1. The Lorenz equations

One of the most celebrated sets of differential equations in physics is the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = \rho x - y - xz, \quad \frac{dz}{dt} = xy - \beta z,$$

where σ, ρ, and β are constants.

These equations were first studied by Edward Lorenz in 1963, who derived them from a simplified model of weather patterns. The reason for their fame is that they were one of the first incontrovertible examples of *deterministic chaos*, the occurrence of apparently random motion even though there is no randomness built into the equations.

(a) Write a program to solve the Lorenz equations for the case $\sigma = 10, \rho = 28$, and $\beta = 8/3$ in the range from $t = 0$ to $t = 50$ with initial conditions $(x, y, z) = (0,1,0)$. Have your program make a plot of y as a function of time. Note the unpredictable nature of the motion.

(b) Now make a plot of z against x. You should see a picture of the famous "strange attractor" of the Lorenz equations, a lop-sided butterfly-shaped plot that never repeats itself.

2. Period of a nonlinear pendulum

(a) Use the pendulum program you wrote in class to calculate the period of the nonlinear pendulum with $g = l = 1.0$ and $m = 1.0$.

(b) Make a plot of the period T as a function of θ_0 where θ_0 is the initial angular position of the pendulum bob (θ_0 is also the amplitude of the oscillation). You can generate data for T vs. θ_0 either by adding a loop to your program in (a) or simply just running (a) several times for various θ_0 values. Use at least ten values θ_0 that range from 0.1π to 0.99π. What happens to the period as θ_0 increases? How does this compare with a pendulum exhibiting small angle oscillations? Recall that a pendulum undergoing small angle oscillations obeys the differential equation given by

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta$$

and has with a period $T = 2\pi\sqrt{l/g}$.

3. Damped-driven nonlinear pendulum and chaos

(a) Extend your pendulum program to include an animation of the pendulum with VPython. Use a sphere for the pendulum bob, a cylinder for the arm and another thin cylinder for the pivot point. Add a graph that plots θ as a function of time.

(b) A damped-driven pendulum experiences an external driving force and a velocity-dependent damping force. Suppose that the damping force is proportional to the angular velocity of the pendulum and driving force is sinusoidal with amplitude A and angular frequency f. The differential equation describing the motion of this pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - c\frac{d\theta}{dt} + A\sin ft$$

Modify your pendulum program to include these new terms and perform the following simulations of the pendulum with the following conditions and describe what happens in each case.

1) $c = 0.5, A = 0$

2) $c = 0.5, A = 0.5, f = 1.0$ (Note that $f = 1.0$ is also equal to $\sqrt{g/l}$ for this pendulum)

3) $c = 0.0, A = 0.1, f = 1.0$

4) $c = 0.5, A = 1.2, f = 2/3$

Resonance occurs when a system is driven by a sinusoidal force with a frequency that matches the natural frequency of oscillation of the system. For systems that exhibit simple harmonic oscillation, such as a mass on a spring, resonance causes the amplitude of oscillation to increase and, if there is no friction or damping present, become infinitely large. Did this occur in case 2) and 3) above? If not, why not?

(c) Simulation 4) is a classic example of chaotic motion. To better understand the nature of chaos, add a graph that plots the phase-plane diagram of the pendulum. The phase-plane diagram plots $\omega = d\theta/dt$ on the vertical axis and θ on the horizontal axis. Set the limits on the horizontal axis to $\pm\pi$. To make sure that all the θ data fits on the diagram, you will have to shift θ by $\pm 2\pi$ whenever θ becomes smaller or larger (respectively) than $\pm\pi$ (obviously, this occurs whenever the pendulum flips over at the top of its swing). Repeat the four simulations described above and observe the phase-plane diagram for each. What is similar about the phase diagrams for simulations 1-3 that is different for 4?

4. The Butterfly Effect

(a) Another property of chaotic motion is sensitivity to initial conditions, popularly known as the "butterfly effect." To observe this, we will simulate the motion of two identical pendulums that start from very similar, but not identical, initial conditions. Make a copy of your program. In the new copy add a second pendulum to the visualization canvas that has the same position as the first; the two pendulums should be on top of each other. Give the second pendulum bob a different color so you can keep track of each of them as they move. Add lines of code that will calculate the motion of the second pendulum. Also include the motion of the second pendulum on your phase plot. Now perform simulation 4 for both pendulums using the initial conditions $\theta_0 = 0$, $\omega_0 = 0$ for pendulum 1 and $\theta_0 = 0.001$, $\omega_0 = 0$ for pendulum 2. Initially, the pendulums will follow each other very closely. However, after a while, the trajectories of the two pendulums will become should appear completely uncorrelated.

(b) To measure the degree to which the trajectories diverge, add a graph that plots $\ln|\theta_2(t) - \theta_1(t)|$ as a function of time. After a while, you should observe that this quantity is approximately linear in time, which suggests that

$$\theta_2(t) - \theta_1(t) \sim e^{\lambda t}$$

where λ is a positive constant known as the Lyapunov exponent. Estimate λ from your graph. Systems with $\lambda > 0$ exhibit chaotic motion.